Introduction to EARS models

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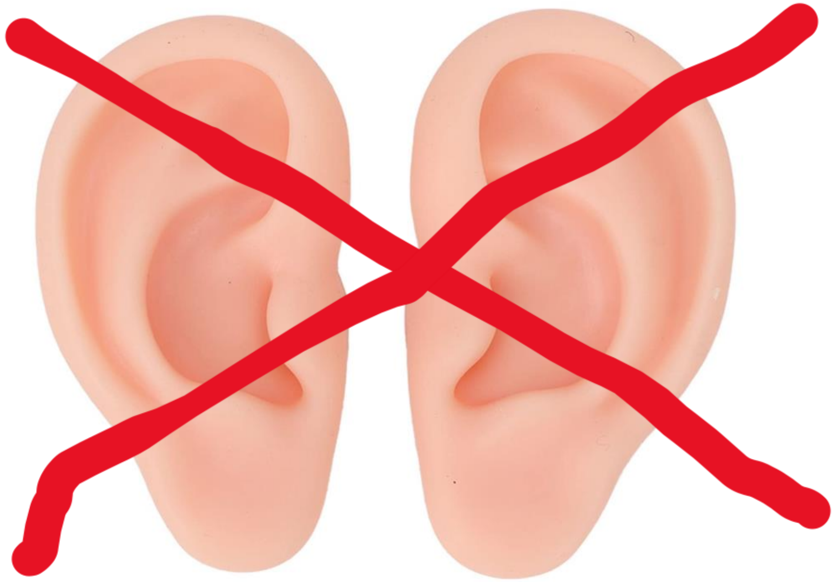
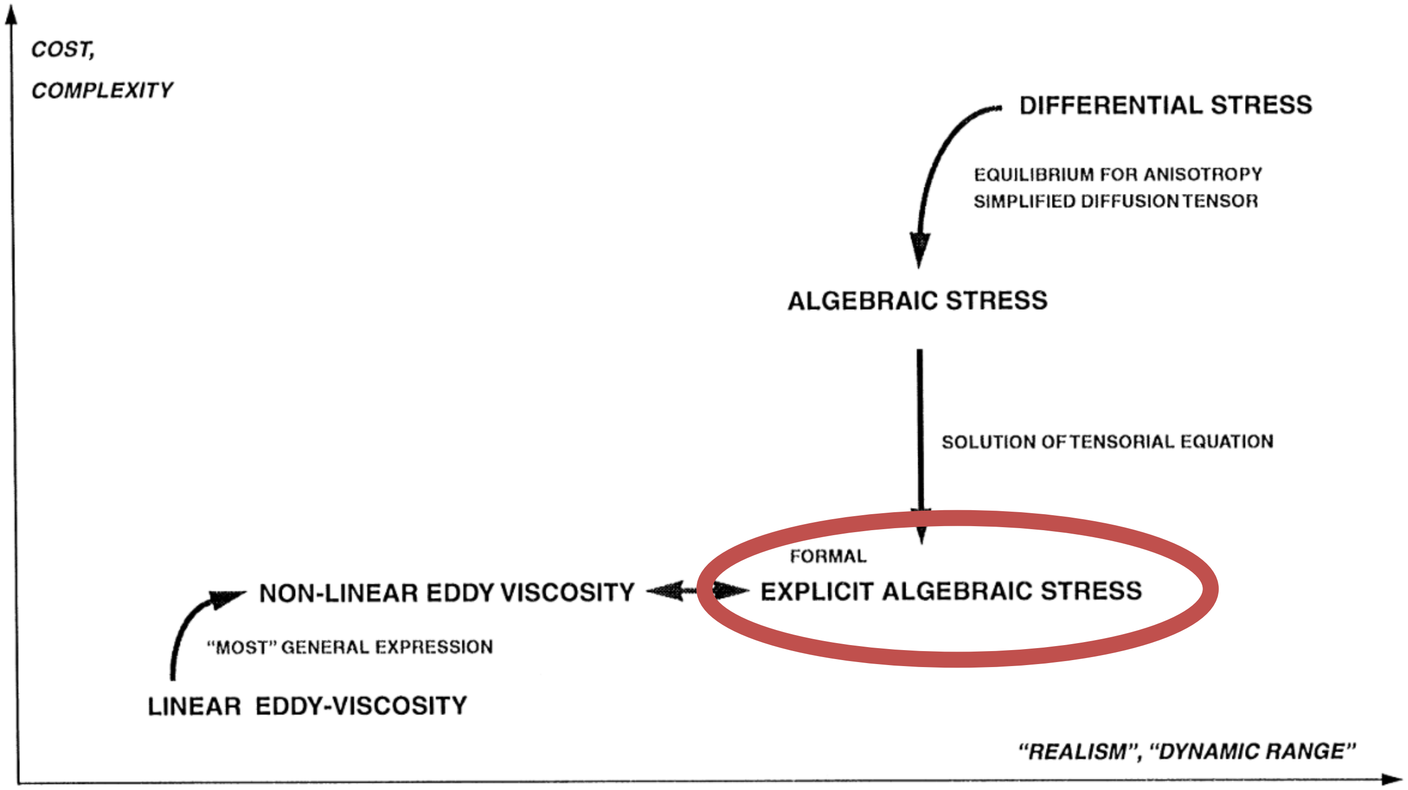
EARS



Explicit **A**lgebraic **R**eynolds **S**tress models

What is an EARS model?

- A class of turbulence models for RANS (and LES)



Gatski & Jongen (2000)

Example of an EARS model

- Model 2 of Wallin & Johansson (2000) with k-eps.

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \underbrace{-\overline{u_i' u_j'} \frac{\partial U_i}{\partial x_j}}_{\mathcal{P}} - \varepsilon + \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right)}_{\mathcal{D}^{(k)}},$$

$$\frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = (C_{\varepsilon 1}\mathcal{P} - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{k} + \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)}_{\mathcal{D}^{(\varepsilon)}}.$$

$$\mathbf{S} = S_{ij} = \frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right),$$

$$\mathbf{\Omega} = \Omega_{ij} = \frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right),$$

$$II_S = S_{ij}S_{ji} \qquad \mathbf{T}^{(1)} = \mathbf{S}$$
$$II_\Omega = \Omega_{ij}\Omega_{ji} \qquad \mathbf{T}^{(4)} = \mathbf{S}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{S}$$

$$P_1 = \left(\frac{1}{27}c_1'^2 + \frac{9}{20}II_S - \frac{2}{3}II_\Omega\right)c_1',$$

$$P_2 = P_1^2 - \left(\frac{1}{9}c_1'^2 + \frac{9}{10}II_S + \frac{2}{3}II_\Omega\right)^3.$$

$$N = \begin{cases} \frac{c_1'}{3} + \left(P_1 + \sqrt{P_2}\right)^{1/3} + \operatorname{sign}\left(P_1 - \sqrt{P_2}\right)\left|P_1 - \sqrt{P_2}\right|^{1/3}, & P_2 \geq 0 \\ \frac{c_1'}{3} + 2\left(P_1^2 - P_2\right)^{1/6} + \cos\left(\frac{1}{3}\left(\frac{P_1}{\sqrt{P_1^2 - P_2}}\right)\right), & P_2 < 0 \end{cases}$$

$$\beta_1 = -\frac{6}{5}\frac{N}{N^2 - 2II_\Omega}, \qquad \beta_4 = -\frac{6}{5}\frac{1}{N^2 - 2II_\Omega}.$$

$$a_{ij} = \beta_1 T_{ij}^{(1)} + \beta_4 T_{ij}^{(4)}$$

$$\overline{u_i' u_j'} = k a_{ij} + \frac{2}{3}k\delta_{ij}$$

How I got interested in EARSMs

FLOW centre
KTH, Stockholm, Sweden

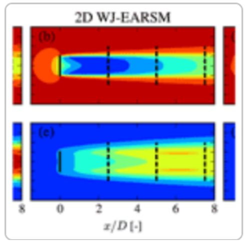
Visited dr. Stefan Wallin in Autumn 2021.

EARSM for wind applications

Research article |

10 Oct 2022

Wind turbine wake simulation with explicit algebraic Reynolds stress modeling



Mads Baungaard, Stefan Wallin, Maarten Paul van der Laan, and Mark Kelly

PAPER • OPEN ACCESS

RANS simulation of a wind turbine wake in the neutral atmospheric pressure-driven boundary layer

M Baungaard[1], M P van der Laan[1], S Wallin[2] and M Abkar[3]

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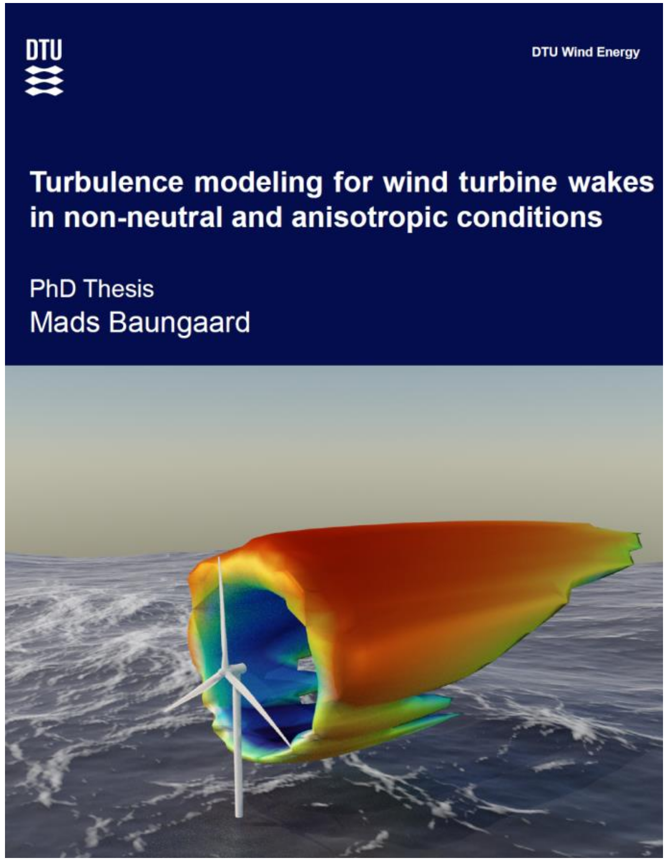


Outline

Part 1:
Basics of turbulence modelling

Part 2:

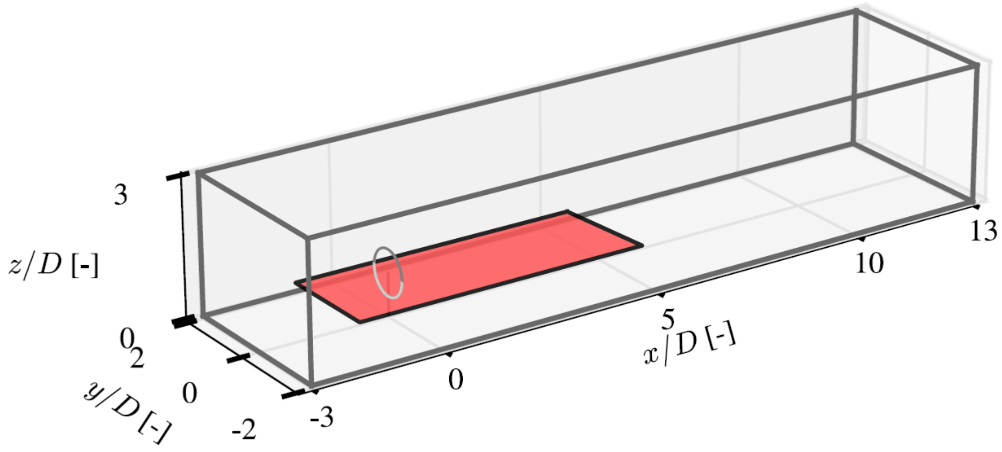
RANS equations

$$\boxed{\frac{\partial U_i}{\partial t} + \frac{\partial \left(U_j U_i\right)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \boxed{\overline{u'_i u'_j}}\right) + F_i}$$

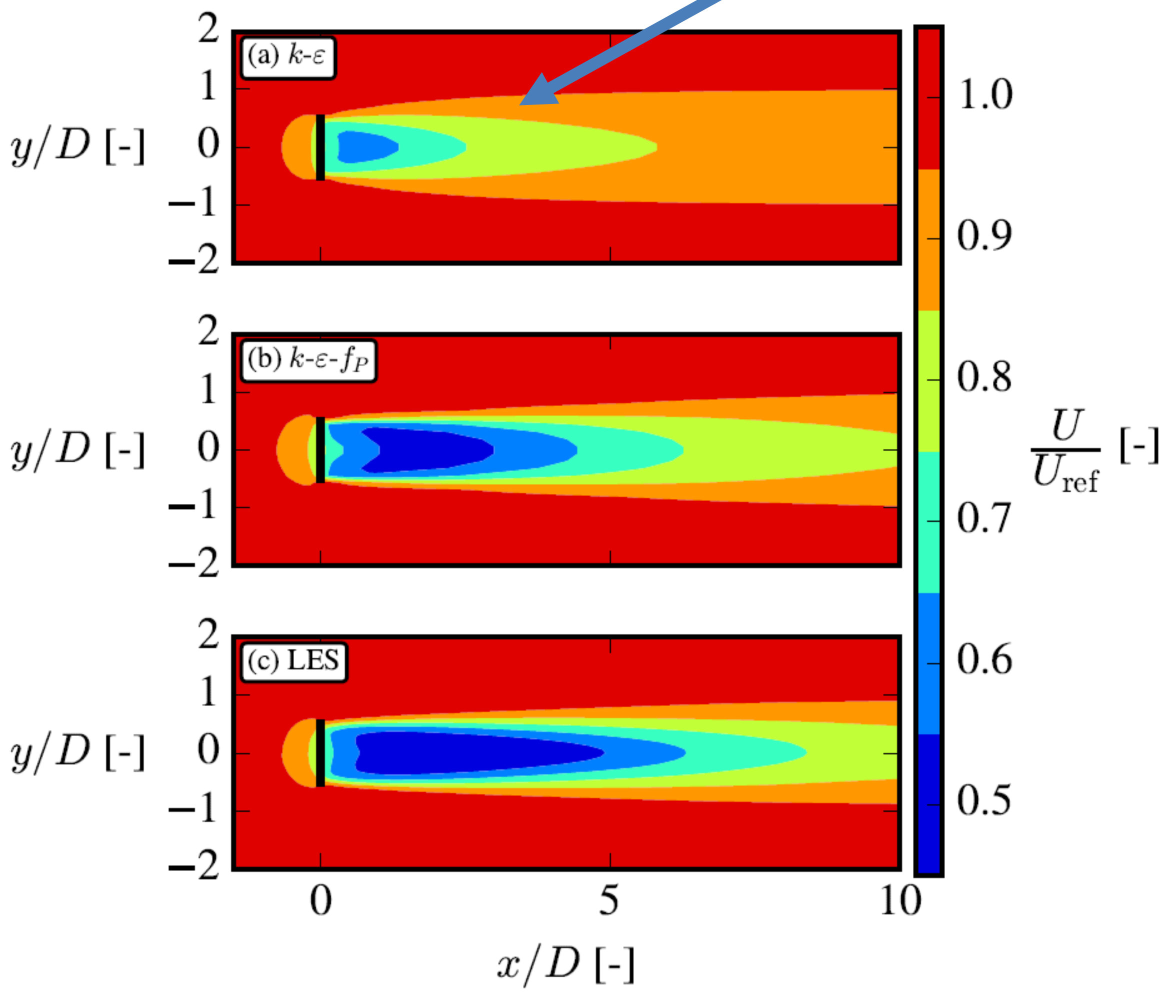
Very important for getting good results!

Turbulence modelling: *"How do we get the Reynolds stress tensor?"*

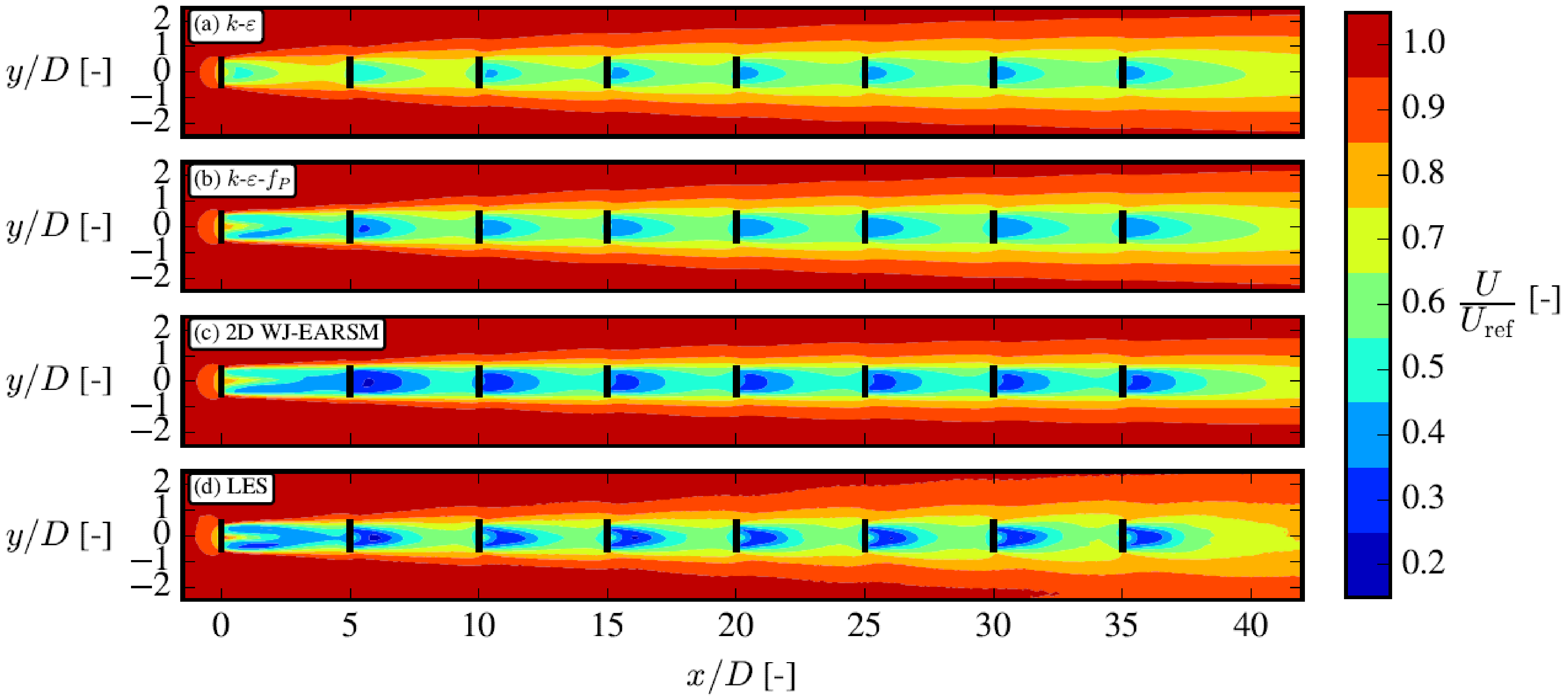
Example 1: a wind turbine wake



Standard k-eps is over-diffusive.



Example 2: a row of turbines



How to model uiuj?

- A natural starting point could be the exact equation

$$\frac{D\overline{u_i'u_j'}}{Dt} = \underbrace{-\overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k} - \overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k}}_{\mathcal{P}_{ij}} \underbrace{-\frac{\partial \overline{u_j'u_i'u_k'}}{\partial x_k} + \nu\frac{\partial^2 \overline{u_i'u_j'}}{\partial x_k^2}}_{\mathcal{D}_{ij}^t + \mathcal{D}_{ij}^v} \underbrace{-\frac{1}{\rho}\overline{u_j'\frac{\partial p'}{\partial x_i}} - \frac{1}{\rho}\overline{u_i'\frac{\partial p'}{\partial x_j}}}_{vel\ pgrad\ corr} \underbrace{-2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}}_{\varepsilon_{ij}}$$

Terms in red need to be modelled

A more compact form of the uiuj equation

- A typical decomposition

$$\underbrace{-\frac{1}{\rho}\overline{u_j'\frac{\partial p'}{\partial x_i}}-\frac{1}{\rho}\overline{u_i'\frac{\partial p'}{\partial x_j}}}_{vel\ pgrad\ corr}=\underbrace{\frac{1}{\rho}\overline{p'\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)}}_{\Pi_{ij}}\underbrace{-\frac{1}{\rho}\left(\frac{\partial\overline{p'u_j'}}{\partial x_i}+\frac{\partial\overline{p'u_i'}}{\partial x_j}\right)}_{\mathcal{D}_{ij}^{p}}$$

and a typical model assumption

$$\varepsilon_{ij}=\frac{2}{3}\varepsilon\delta_{ij},$$

allow the uiuj equation to be written as:

$$\boxed{\frac{D\overline{u_i'u_j'}}{Dt}=\mathcal{P}_{ij}+\mathcal{D}_{ij}+\Pi_{ij}-\frac{2}{3}\varepsilon\delta_{ij}}.$$

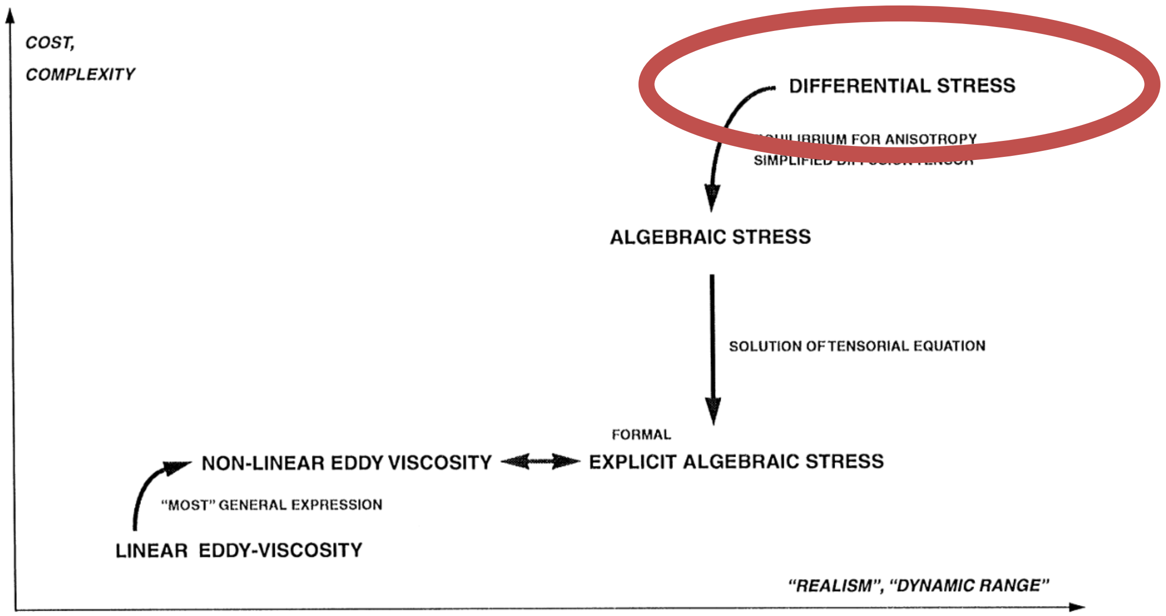
Differential Reynolds stress models (DRSMs)

$$\boxed{\frac{D\overline{u_i'u_j'}}{Dt} = \mathcal{P}_{ij} + \mathcal{D}_{ij} + \Pi_{ij} - \frac{2}{3}\varepsilon\delta_{ij}}.$$

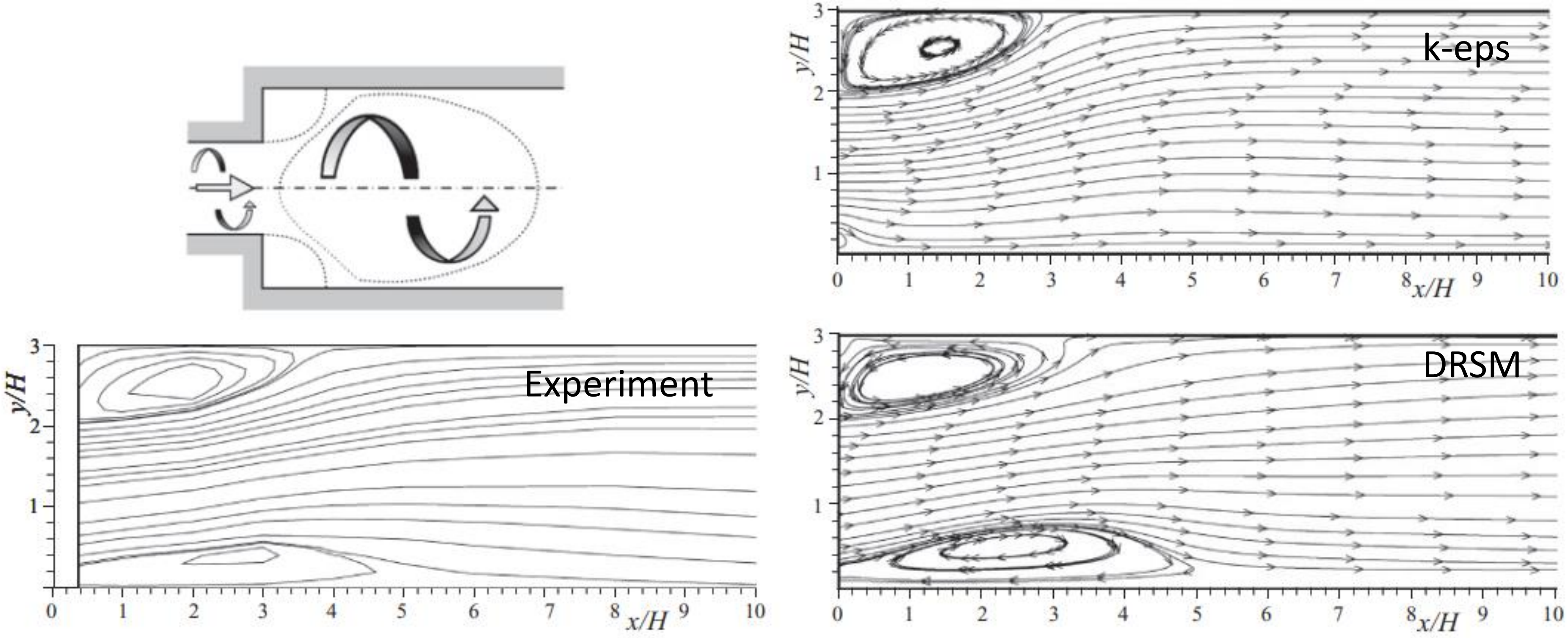
- 7 equations (6 for uiuj + 1 for eps)
- Need models for

$$\Pi_{ij} \qquad \mathcal{D}^p_{ij} \qquad \mathcal{D}^t_{ij}$$

- Complicated, but possible. Two models (LRR and SSG) are available in OpenFOAM.



Example: swirling flow in an expanding pipe



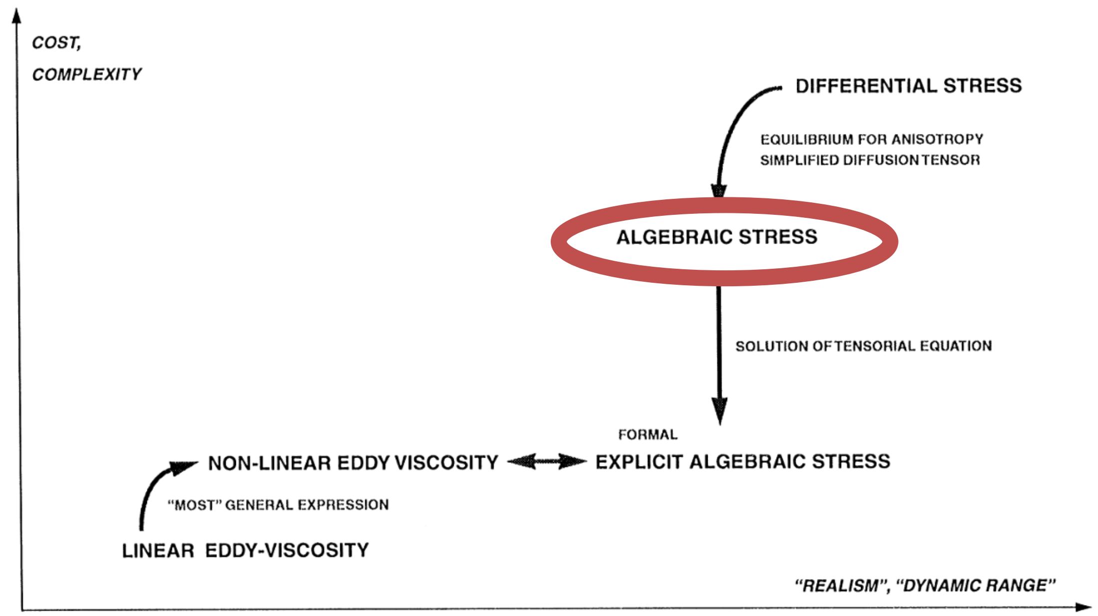
Hanjalic & Launder (2011, p.81)

Motivation for the algebraic Reynolds stress (ARSM) equations

Disadvantages of DRSMs:

- More expensive (7 eqs)
- Difficult to implement
- Numerical robustness

Idea by Rodi (1972,1976):
“Transform the DRSM equations into a set of algebraic equations.”



Derivation of the ARSM equations 1/2

- It is convenient to use the anisotropy tensor

$$\mathbf{a} = a_{ij} \equiv \frac{\overline{u_i' u_j'}}{k} - \frac{2}{3}\delta_{ij},$$

Dimensionless
Symmetric
Traceless → 5 unique elements

- Rewrite DRSM equations to

$$\frac{Da_{ij}}{Dt} = \frac{1}{k}\left(\mathcal{P}_{ij} + \Pi_{ij} + \mathcal{D}_{ij} - \frac{2}{3}\varepsilon\delta_{ij} - \left(\frac{\overline{u_i' u_j'}}{k}\right)\left(\mathcal{P} + \mathcal{D}^{(k)} - \varepsilon\right)\right)$$

and rearrange

“Weak-equilibrium assumption”

$$\Rightarrow \frac{Da_{ij}}{Dt} - \underbrace{\cancelto{0}{\left(\frac{\mathcal{D}_{ij}}{k} - \frac{\overline{u_i' u_j'}}{k^2}\mathcal{D}^{(k)}\right)}}_{\mathcal{D}^{(a)}_{ij}} = \frac{1}{k}\left(\mathcal{P}_{ij} + \Pi_{ij} - \frac{2}{3}\varepsilon\delta_{ij} - \left(\frac{\overline{u_i' u_j'}}{k}\right)(\mathcal{P} - \varepsilon)\right)$$

Derivation of the ARSM equations 2/2

- The "weak-equilibrium assumption" gives an algebraic set of equations

$$0 = \mathcal{P}_{ij} + \Pi_{ij} - \frac{2}{3}\varepsilon\delta_{ij} - \left(\frac{\overline{u_i'u_j'}}{k}\right)(\mathcal{P} - \varepsilon)$$

- Rewrite using definition of the anisotropy tensor

$$a_{ij}\left(\frac{\mathcal{P}}{\varepsilon} - 1\right) = \frac{\mathcal{P}_{ij}}{\varepsilon} + \frac{\Pi_{ij}}{\varepsilon} - \frac{2}{3}\frac{\mathcal{P}}{\varepsilon}\delta_{ij}.$$

Anatomy of the Reynolds stress tensor

$$\overline{u_i'u_j'} = \begin{pmatrix} \overline{u_1'u_1'} & \overline{u_1'u_2'} & \overline{u_1'u_3'} \\ \overline{u_2'u_1'} & \overline{u_2'u_2'} & \overline{u_2'u_3'} \\ \overline{u_3'u_1'} & \overline{u_3'u_2'} & \overline{u_3'u_3'} \end{pmatrix}$$

Unit: m^2/s^2

- Symmetric → six unique elements
- Galilean invariant
- Half of the trace defined as the “turbulent kinetic energy” (TKE)

$$k = \frac{1}{2}\overline{u_i'u_i'} = \frac{1}{2}\left(\overline{u_1'u_1'} + \overline{u_2'u_2'} + \overline{u_3'u_3'}\right)$$

Limits of the Reynolds stress tensor

Rule 1

Rule 2

$$0 \leq \overline{u'_\alpha u'_\alpha} \leq 2k$$

Greek indices imply that there is no summation over repeated indices!

$$\left(\overline{u'_\alpha u'_\beta}\right)^2 \leq \overline{u'_\alpha u'_\alpha}\ \overline{u'_\beta u'_\beta} \leq k^2$$

Rule 3

Rule 4

- Rule 1: Each normal stress must be positive.
- Rule 2: From the definition of TKE.
- Rule 3: Each shear stress must satisfy the Cauchy-Schwarz inequality.
- Rule 4: Comes from rule 2.

Motivation: Wind farm aerodynamics

Lots of parameters!

- Atmospheric conditions
- Oceanic conditions
- Turbine layout
- Turbine design
- Turbine operating conditions

Wind turbine aerodynamics

Wind farm aerodynamics

Boundary-layer meteorology

Cause difficulties in…

- Modelling (wakes, blockage)
- Optimisation (AEP, LCoE, etc.)

Question:

- *What is the best way to tackle this **very high-dimensional** optimisation problem?*

Test

Test