Eigensystems for Symmetric Matrices

Learning Goals: students see that the eigenvalues of symmetric matrices are all real, and that they have a complete basis worth of eigenvectors, which can be chosen to be orthonormal.

Since they appear quite often in both application and theory, lets take a look at symmetric matrices in light of eigenvalues and eigenvectors. One special case is projection matrices. We know that a matrix is a projection matrix if and only if $P = P^2 = P^T$. So they are symmetric.

They also have only very special eigenvalues and eigenvectors. The eigenvalues are either 0 or 1. Why? If a vector is not in the column space, it gets sent to one that is, so the only hope of it being an eigenvector is if it gets sent to zero (hence is in the nullspace, and is an eigenvector with eigenvalue 0). If a vector is in the column space, it gets sent to itself, so is an eigenvector with eigenvalue 1. Notice that the "eigenspaces" are the column space and the nullspace. And notice that they are orthogonal. Usually, the nullspace is orthogonal to the row space, but since $P = P^T$ those two are the same!

It turns out this same thing happens for *any* symmetric matrix:

Example: find the eigensystem of $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. We easily calculate the characteristic polynomial to be $-\lambda^3 + 5\lambda^2 - 6\lambda$. The eigenvalue-eigenvector pairs are thus $(0, (-1, 2, -1))$, $(3, (1, 1, 1))$, and $(2, (1, 0, -1))$. Note that the three eigenvectors are actually orthogonal.

If we assume that the matrix has a full set of eigenvectors, we could diagonalize it as $A = S\Lambda S^{-1}$. Note that A^T has this same factorization because A is symmetric. But A^T can also be written $(S^{-1})^T \Lambda S^T$ which suggests that maybe $S^{-1} = S^T$—making S an orthogonal matrix (that is, its columns are actually orthogonal!). Our example above didn't have unit vectors for the eigenvectors—but it could have. We can change the lengths of eigenvectors at will.

Another subtle point that may have escaped your attention: all the eigenvalues of these matrices are *real*. Symmetric matrices can never have complex eigenvalues.

Let's prove some of these facts:

1) Eigenvalues of a real symmetric matrix are real. For $A\mathbf{x} = \lambda\mathbf{x}$. If we use complex conjugates, $\overline{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$. But A is real, so it equals its own conjugate, and hence $A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$. If we transpose this, we find that $\overline{\mathbf{x}}^T A = \overline{\lambda}\overline{\mathbf{x}}^T$ (note that $A = A^T$). Multiply the first equation on the left by $\overline{\mathbf{x}}^T$ and the second equation on the right by $\mathbf{x}$. Then we get $\overline{\lambda}\overline{\mathbf{x}}^T\mathbf{x} = \overline{\mathbf{x}}^T A\mathbf{x} = \lambda\overline{\mathbf{x}}^T\mathbf{x}$. Now, $\overline{\mathbf{x}}^T\mathbf{x}$ is real and positive (just being non-zero would be OK) because it is the sum of squares of moduli of the entries of $\mathbf{x}$. So we can divide by it to find that $\overline{\lambda} = \lambda$.

2) A real symmetric matrix has real eigenvectors. For solving $A - \lambda I = \mathbf{0}$ need not leave the real domain.

3) Eigenvectors corresponding to different eigenvalues of a real symmetric matrix are orthogonal. For if $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$ with $\lambda \neq \mu$, then $\mathbf{y}^T A\mathbf{x} = \lambda\mathbf{y}^T\mathbf{x} = \lambda(\mathbf{x}\cdot\mathbf{y})$. But numbers are always their own transpose, so $\mathbf{y}^T A\mathbf{x} = \mathbf{x}^T A\mathbf{y} = \mathbf{x}^T\mu\mathbf{y} = \mu(\mathbf{x}\cdot\mathbf{y})$. So $\lambda = \mu$ or $\mathbf{x}\cdot\mathbf{y} = 0$, and it isn't the former, so $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

These orthogonal eigenvectors can, of course, be made into unit vectors giving us orthonormal vectors. Heck, eigenvectors corresponding to the *same* eigenvalue can also be made orthonormal via Gram-Schmidt, without changing the fact that they are eigenvectors (if $\mathbf{x}_1$ and $\mathbf{x}_2$ are both eigenvectors corresponding to eigenvalue λ, then $A(a\mathbf{x}_1 + b\mathbf{x}_2) = \lambda(a\mathbf{x}_1 + b\mathbf{x}_2)$ so linear combinations of eigenvectors with the same eigenvalue remain eigenvectors).
In fact, we might as well push it all the way:

Theorem: a matrix has all real eigenvalues and n orthonormal real eigenvectors if and only if it is real symmetric.

Proof: Let Q be the matrix of eigenvectors. Note that it is an orthogonal matrix, so deserves to be called Q. Now $A = Q\Lambda Q^T$ because $Q^T = Q^{-1}$. A is real because Q and Λ are. And just check that $A^T = (Q^T)^T\Lambda^T Q^T$. Of course $Q^{TT} = Q$ and the diagonal matrix is its own transpose, so A is symmetric.

To prove the converse, we now need to show that a symmetric matrix has n linearly independent eigenvectors, for we have already shown that we can make them orthonormal. This is simple if all the eigenvalues of A are different, because then each eigenvalue has its own eigenvector. If there are repeated eigenvalues there may be a problem, though.

The solution is to break up the repeats. We can "perturb" the matrix A by adding a small number to one of its diagonal entries, or symmetrically placed off-diagonal entries. This will change the characteristic polynomial by a small amount, which will move its roots by a small amount (this all requires some pretty heavy duty calculus!). But now all the roots are not the same, so we have n independent *orthogonal* eigenvectors. But everything is continuous, so as this perturbation goes to zero, the eigenvectors will *remain orthogonal* so there must remain n of them.

(By contrast, look at $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ which is non-symmetric and missing an eigenvector. If we perturb it to $\begin{bmatrix} 1 & 1 \\ 0 & 1+\varepsilon \end{bmatrix}$, the eigenvalues-eigenvector pairs are $(1, (1, 0))$ and $(1+\varepsilon, (1,\varepsilon))$. Now as ε goes to zero, the two eigenvectors coalesce. They can't do that in the symmetric case because they must stay orthogonal.)

Another proof of the fact that we always have enough eigenvectors can be obtained by noting that $A - \lambda I$ is also symmetric and has exactly the same eigenvectors as A with its eigenvalues shifted by λ.

So let λ be an eigenvalue of A of algebraic multiplicity m, and let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ be associated eigenvectors (might as well make them orthonormal). Finish this out to an orthonormal basis for $\mathbf{R}^n$ and put these into the columns of Q.

Look at $Q^{-1}AQ$. Since Q is orthogonal, $Q^{-1} = Q^T$ and

$$Q^{-1}AQ = \begin{bmatrix} \cdots & \mathbf{x}_1^T & \cdots \\ & \vdots & \\ \cdots & \mathbf{x}_r^T & \cdots \\ \cdots & ? & \cdots \\ & \vdots & \\ \cdots & ? & \cdots \end{bmatrix} \begin{bmatrix} \vdots & & \vdots & \vdots & & \vdots \\ \lambda\mathbf{x}_1 & \cdots & \lambda\mathbf{x}_r & ? & \cdots & ? \\ \vdots & & \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \lambda & & & ? & & ? \\ & \lambda & & ? & \cdots & ? \\ & & \lambda & ? & & ? \\ 0 & 0 & 0 & & & \\ & \vdots & & & A' & \\ 0 & 0 & 0 & & & \end{bmatrix}$$ where the

bottom $n - r$ rows start with r zeroes by othogonality. But by symmetry, the last entries of the top r rows must also be zero, because Q^TAQ is symmetric.

Now the characteristic polynomials of A and this block matrix are the same, so if $r < m$ that matrix A' has an eigenvalue of λ, hence it has some eigenvector $\mathbf{y}^*$ corresponding to this eigenvalue. Then $\mathbf{y} = (0, \ldots, 0, \mathbf{y}^*)$ will be an eigenvector of the block matrix with eigenvalue λ. Note how important those zeroes are above A'.

Thus $Q^{-1}AQ\mathbf{y} = \lambda\mathbf{y}$, or $A(Q\mathbf{y}) = \lambda(Q\mathbf{y})$, so $Q\mathbf{y}$ is an eigenvector of A with eigenvalue λ. Furthermore, because of the zeroes at the front of $\mathbf{y}$, $Q\mathbf{y}$ is *not* a linear combination of $\mathbf{x}_1$ through $\mathbf{x}_r$. So if $r < m$ we have found a new eigenvector of A.

Note that the reason this worked—the fact that Q^TAQ had lots of zeroes in it—is only true when A is symmetric (OK, skew-symmetric will work, too, as well as some other special kinds of matrices), so this proof *won't* give more eigenvectors for general matrices!