THE APPROACH OF REYNOLDS AVERAGED NAVIER-STOKES EQUATIONS TO TURBULENCE MODELING

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by

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Abstract

The method of Reynolds-Averaged Navier-Stokes equations (*RANS*) for the modeling of turbulent flows is reviewed. The subject is considered in the limit of incompressible fluids with constant properties.

After the introduction of the concept of Reynolds decomposition and averaging, different classes of *RANS* turbulence closure models are presented and in particular zero-equation models, one-equation models, two-equation models and stress-equation models. For each of the above-mentioned classes of models, the most used modeling techniques and turbulence closures are described, toghether with the methods for their derivation.

Concluding remarks are at the end, followed by an Appendix containing the most used formulations of the fundamental equations of incompressible fluid dynamics.

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Nomenclature

Roman Symbols (Upper Case)

C = nondimensional constant in the law of the wall

C^* = nondimensional constant in the modeled kinetic-energy transport equation

C_D, C_E = nondimensional constants in nonlinear two-equation closure models

C_s = nondimensional constant in diffusion correlation model

C_1 = nondimensional constant in pressure-strain correlation models (Rotta constant)

C_2, C_3, C_4 = nondimensional constants in pressure-strain correlation models

$C_{\varepsilon 1}$, $C_{\varepsilon 2}$ = nondimensional constants in the modeled dissipation-rate transport equation

C_μ = nondimensional constant in two-equation closure models

C_{ijk} = diffusion correlation in the Reynolds-stress transport equation

D_i = diffusion term in the kinetic-energy transport equation

D_ε = diffusion term in the dissipation-rate transport equation

D_{ijk} = diffusion term in the Reynolds-stress transport equation

E = kinetic energy, dimensional and nondimensional

F_i, F_{ij} = functionals

II, III = invariants of the Reynolds-stress anisotropy tensor b_{ij}

K = turbulent kinetic energy

L = reference length, also operator

L_0 = length scale of the mean flow

N = number of repeated experiments in ensemble averaging

P_K = production term in the kinetic-energy transport equation

P_ε = production term in the dissipation-rate transport equation

P_{ij} = production term in the Reynolds-stress transport equation

R'_{ij} = two-point fluctuating velocity correlation tensor

$Re =$ Reynolds number

$Re_T =$ turbulence Reynolds number

$Ro =$ rotation number

$S =$ constant shear rate in homogeneous shear flow

$S_{ij} =$ strain rate tensor

$\overline{S}_{ij} =$ mean strain rate tensor

$S'_{ij} =$ fluctuating strain rate tensor

$\overset{\circ}{\overline{S}}_{ij} =$ frame-indifferent Oldroyd derivative of $\overline{S}_{ij}$

$S_K =$ generalized skewness

$T =$ reference time, also time interval

$T_{ij} =$ stress tensor ($\boldsymbol{T}$)

$\overline{T}_{ij} =$ mean stress

$T_{ij}^{TOT} =$ total mean stress

$T_0 =$ time scale of the mean flow

U = reference velocity

V = fluid volume

$\overline{W}_{ij} =$ mean-vorticity tensor

Roman Symbols (Lower Case)

$b_{ij} =$ Reynolds-stress anisotropy tensor ($\boldsymbol{b}$)

$c_{b1}, c_{b2}, c_{w1}, c_{w2}, c_{w3} =$ constants in modeled eddy-viscosity transport equation

$d_{ij} =$ dissipation-rate anisotropy tensor

$\boldsymbol{e}_i =$ unit vectors ($\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$)

$e_{ijk} =$ alternating unit tensor (permutation tensor)

f = scalar function

$g_x, g_y, g_z =$ components of the body force in rectangular coordinates

$g_r, g_\theta, g_z =$ components of the body force in cylindrical polar coordinates

$g_r, g_\theta, g_\varphi =$ components of the body force in spherical polar coordinates

$h_i =$ scale factors (h_1, h_2, h_3)

$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} =$ unit vectors along x, y, z

$l =$ turbulent length scale, also integral length scale

$l_0 =$ turbulent length scale

$l_m =$ mixing length (Prandtl)

$l_\eta =$ Kolmogorov length scale

$p =$ pressure, dimensional, modified and nondimensional

$\overline{p} =$ mean pressure

$p' =$ fluctuating pressure

$q =$ turbulent velocity scale

$r, \theta, z =$ cylindrical polar coordinates

$r, \theta, \varphi =$ spherical polar coordinates

$t =$ time, dimensional and nondimensional

$u_i =$ velocity (u_1, u_2, u_3) or (u, v, w), dimensional and nondimensional ($\boldsymbol{u}$).

$\overline{u}_i =$ mean velocity ($\overline{u}_1, \overline{u}_2, \overline{u}_3$) or ($\overline{u}$, $\overline{v}$, $\overline{w}$), dimensional and nondimensional ($\overline{\boldsymbol{u}}$)

$u_i' =$ fluctuating velocity (u_1', u_2', u_3') or (u', v', w')

$u_\tau =$ friction velocity

$u^+ =$ mean x-velocity in wall units

$u_x, u_y, u_z =$ velocity components in rectangular coordinates, also (u, v, w)

$u_r, u_\theta, u_z =$ velocity components in cylindrical polar coordinates

$u_r, u_\theta, u_\varphi =$ velocity components in spherical polar coordinates

$x_i =$ spatial coordinates (x_1, x_2, x_3) or (x, y, z), dimensional and nondimensional ($\boldsymbol{x}$)

$y^+ =$ y-coordinate in wall units

Greek Symbols (Upper Case)

$\Pi_{ij} =$ pressure-strain correlation in the Reynolds-stress transport equation

$\Pi_{ij}^{S} =$ slow pressure-strain correlation

$\Pi_{ij}^{R} =$ rapid pressure-strain correlation

Φ_{ε} = destruction term in the dissipation-rate transport equation

$\Omega =$ enstrophy, dimensional and nondimensional

$\Omega_{j} =$ rotation rate of reference frame

Greek Symbols (Lower Case)

α_1, α_2, α_3, α_4 = nondimensional constants in nonlinear two-equation closure models

β_1, β_2, β_3, β_4 = empirical constants in the modeled transport equation for l

γ_1, $\gamma_2 =$ constants in the modeled transport equation for ω

$\gamma^{*} =$ constant in the eddy-viscosity form for two-equation $K-\omega$ closure model

$\delta_{ij} =$ Kronecker's delta

$\varepsilon =$ dissipation rate of mean turbulent kinetic energy (scalar dissipation rate) [L^2T^{-3}]

$\varepsilon_{ij} =$ dissipation-rate correlation tensor

$\varepsilon_{ij}^{I} =$ isotropic part of the dissipation-rate correlation tensor

$\varepsilon_{ij}^{D} =$ deviatoric part of the dissipation-rate correlation tensor

$\kappa =$ von Kármán constant, also wavenumber

μ = fluid dynamic viscosity

$\nu =$ fluid kinematic viscosity

$\nu_T =$ eddy viscosity [L^2T^{-1}]

ρ = fluid density

σ_K = nondimensional constant in the modeled kinetic-energy transport equation

σ_ε = nondimensional constant in the modeled dissipation-rate transport equation

σ_ω = constant in the modeled transport equation for ω

τ = transformed dimensionless time

τ_{ij} = viscous-stress tensor ($\boldsymbol{\tau}$)

τ'_{ij} = Reynolds-stress tensor (divided by the density)

τ'^{I}_{ij} = isotropic part of the Reynolds-stress tensor

τ'^{D}_{ij} = deviatoric part of the Reynolds-stress tensor

τ_0 = turbulent time scale

τ_w = mean shear stress at the wall

ϕ = generic flow variable

$\overline{\phi}$ = mean value of generic flow variable ϕ

ϕ' = fluctuating value of generic flow variable ϕ

ψ = generic flow variable

$\overline{\psi}$ = mean value of generic flow variable ψ

ψ' = fluctuating value of generic flow variable ψ

ω = reciprocal turbulent time scale (also vorticity)

ω_i = vorticity vector, dimensional and nondimensional ($\boldsymbol{\omega}$)

$\overline{\omega}_i$ = mean-vorticity vector

ω'_i = fluctuating-vorticity vector

Mathematical Symbols

$\cdot$ = scalar (dot) product

$\times$ = vector (cross) product

∇ = gradient operator

$\nabla\cdot$ = divergence operator

$\nabla\times$ = curl operator

∇^2 = laplacian operator

$| \; |$ = modulus

$\| \; \|$ = norm

∂ = partial derivative, also boundary

^{T} = time average, also transposed

^{V} = volume average

^{E} = ensemble average

cot = cotangent

exp = exponential

lim = limit

ln = logarithm (base *e*)

sin = sine

tr = trace

Acronyms (Upper Case)

ASM = Algebraic Stress Model (Rodi 1976)

CFD = Computational Fluid Dynamics

DIA = Direct Interaction Approximation theory

DNS = Direct Numerical Simulation of turbulence

FLT = Fu, Launder & Tselepidakis (1987) model

IP = simplified Launder, Reece & Rodi (1975) model

LES = Large Eddy Simulation of turbulence

LRR = Launder, Reece & Rodi (1975) model

MFI = Material Frame Indifference

MRS = Mean Reynolds Stress closures

MTE = Mean Turbulent Energy closures

MTF = Mean Turbulent Field closures

MVF = Mean Velocity Field closures

RANS = Reynolds Averaged Navier-Stokes equations

RDT = Rapid Distortion Theory

RNG = Renormalization Group theory

SL = Shih & Lumley (1985) model

SSG = Speziale, Sarkar & Gatski (1991) model

Acronyms (Lower Case)

lhs = left-hand side

rhs = right-hand side

pde = partial differential equation

pdf = probability density function

Chapter 1

Introduction

Turbulence modeling is an attempt to devise a set of partial differential equations for the calculation of turbulent flows based on physically-consistent approximations of the Navier-Stokes equations. In the approach of Reynolds-Averaged Navier-Stokes equations (*RANS*) the starting point is the Reynolds decomposition of the flow variables (velocity and pressure in the case of an incompressible fluid) into mean and fluctuating parts. The introduction of the Reynolds decomposition in the nonlinear Navier-Stokes equations followed by an ensemble averaging of the equations themselves, gives rise to an unknown term – the Reynolds-stress tensor – that has to be modeled in order for the Reynolds-Averaged Navier-Stokes equations (the governing equations formulated in terms of turbulence statistical moments) to be solved. In this operation essentially consists the problem of the closure of the system of the Navier-Stokes equations.

The models that may be devised for the Reynolds-stress term must ensure that the numerical solutions of the resulting governing equations are characterized by some requisites:

i) consistency with observations (of both experimental and *DNS* nature) of the mean velocity, turbulence intensities and other low-order statistical quantities in fundamental turbulent-flow cases;

ii) consistency with results obtained in various limiting cases;

iii) ability to perform parametric computational studies for analysis and design in engineering applications involving turbulent flows.

1.1 Historical Synthesis

Osborne Reynolds (1895) first introduced the idea of Reynolds decomposition and averaging of the Navier-Stokes equations. Specific concepts like that of the eddy viscosity (Boussinesq 1877) as a basis for a simple model for the Reynolds-stress tensor

($-\rho\overline{u_i'u_j'}$) were put forward to arrive to the algebraic closure of the system of the governing equations. The first successful calculation of a practical turbulent flow based on the modeled pde-system of the Reynolds-Averaged Navier-Stokes equations was achieved after the years 1920s, mainly due to the work of Prandtl (1925) who introduced the concept of the *mixing length* as a basis for the determination of the eddy viscosity. The mixing-length model led to solutions for the turbulent pipe and channel flows in an acceptable agreement with existing experimental data. Von Karman (1930, 1948) made further contributions to the mixing-length approach that continued to be an active research field until the years 1950s. By these times it was clear that the basic assumption behind the mixing-length approach (the direct analogy between the turbulent transport process and the molecular transport process) was not realistic in the sense that turbulent flows do not have a clear-cut separation of scales.

Transport equations for single-point moments of the velocity field have been used to obtain modeled equations for the length and time scales of the fluctuations (or any linearly independent combinations of them, such as velocity and length). Kolmogorov was one of the first to suggest the use of two distinct transport equations as a minimal set of expressions for turbulence modeling. Aiming to develop more general models, Prandtl (1945) tied the eddy viscosity to the turbulent kinetic energy through the development of a separate modeled transport equation, opening the way to the one-equation class of turbulence models, where the turbulent length scale l is specified empirically and the turbulent kinetic energy K is obtained from a modeled transport equation.

More popular are the two-equation turbulence models and, among them, the $K-\varepsilon$ model introduced by Hanjalić & Launder (1972) (see also Launder & Spalding 1974) where transport equations for the turbulent kinetic energy K and for the scalar energy-dissipation rate ε (from which the turbulent length and time scales are built up) are used. Because of the relatively low computational effort required, the $K-\varepsilon$ model has been for a long time one of the most commonly used turbulence models for the solution of practical engineering problems.

Almost all the above-mentioned turbulence closure models rely on the concept of eddy viscosity (Boussinesq 1877) and suffer from the drawbacks intrinsic to all the eddy-viscosity models, namely the inability to properly account for streamline curvature, body forces and history effects on the individual components of the Reynolds-stress tensor. In general these elements act to augment or suppress individual components of the Reynolds-stress tensor, an effect that can not be easily embodied in eddy-viscosity formulations.

Rotta (1951) gave a fundamental contribution to a full Reynolds-stress turbulence closure. The approach of Rotta (today referred to as second-order or second-moment closure) is based on the Reynolds-stress transport equation. On the basis of ideas of Kolmogorov of the years 1940s and introducing entirely new concepts, Rotta succeded in closing the Reynolds-stress transport equation. The Reynolds-stress closure was able to account for both history and nonlocal effects on the evolution of the components of the Reynolds-stress tensor. However, due to the fact that the approach requires the solution of six additional transport equations for the individual independent components of the Reynolds-stress tensor, it was not computationally feasible for the next few decades to solve complex engineering flows based on a full second-order closure. By the years 1970s with the initial availability of high-performance computing systems, the first works of development and implementation of second-order closure models started to appear (Daly & Harlow 1970, Donaldson 1972).

Launder, Reece & Rodi (1975) developed a new second-order closure model (the *LRR* model) that improved significantly the work of Rotta (1951). Systematic models for the pressure-strain correlation and turbulent transport terms were derived and a modeled transport equation for the turbulent dissipation rate was solved in conjunction with the Reynolds-stress model. Launder, Reece & Rodi (1975) showed how second-order closure models could be calibrated and applied to the solution of practical turbulent flows (Speziale 1991). It has to be noted that when the Launder, Reece & Rodi (1975) model is contracted and supplemented with an eddy-viscosity representation for the Reynolds stress, a two-equation $K-\varepsilon$ model is obtained, almost identical to that derived by Hanjalić & Launder (1972).

After the work of Launder, Reece & Rodi (1975) the effort in the research on second-order closure models continued. Lumley (1978) implemented the constraint of realizability and made significant contributions to the modeling of the pressure-strain correlation and buoyancy effects. The group of Launder continued the work of refinement and application of second-order closure models to problems of engineering interest (Launder 1990). Speziale (1985,1987*a*) exploited invariance arguments and consistency conditions for solutions of the system of the Navier-Stokes equations in a rapidly rotating frame, to develop new models for the rapid pressure-strain correlation. Haworth & Pope (1986) developed a second-order closure model starting from the pdf-based Langevian equation.

1.2 Classification

Overall, there are different classes of one-point closure models within the framework of the *RANS* approach to the modeling of turbulent flows.

1. *Zero-equation models*, in which only a system of modeled partial differential equations for the mean field ($\overline{u}_i$, $\overline{p}$) is solved and no other pdes are used.

These models are also called Mean Velocity Field (*MVF*) closures by Mellor & Herring (1973).

2. *One-equation models*, that involve, with respect to Class 1, an additional modeled transport equation for the calculation of the turbulent velocity scale. In the most popular models this equation is cast in terms of the average turbulent kinetic energy K;

3. *Two-equation-models*, that involve, with respect to Class 2, an additional modeled transport equation for the calculation of the turbulent length scale. In the most popular models (the $K-\varepsilon$ models) this equation is cast in terms of the scalar rate-of-dissipation of turbulent kinetic energy ε.

The last two classes of models are also called Mean Turbulent Energy (*MTE*) closures, representing a subset of the wider category of Mean Turbulent Field (*MTF*) closures, by Mellor & Herring (1973).

4. *Stress-equation models*, that involve, with respect to Class 1, additional modeled transport equations for the Reynolds-stress tensor τ'_{ij} and for the dissipation rate ε. For this reason models of Class 4 are named $\tau'_{ij} - \varepsilon$ models.

These models are also called Mean Reynolds Stress (*MRS*) closures, representing a subset of the wider category of Mean Turbulent Field (*MTF*) closures – the latter including models of Class 2, 3 and 4 – by Mellor & Herring (1973).

Under the viewpoint of the conceivement of the models, there are two main approaches to the development of *RANS* closure models:

i) the continuum mechanics approach, typically based on a Taylor series expansion. Invariance constraints, together with consistency conditions like *RDT* (Rapid Distortion Theory) and realizability, are used for appropriate simplifications of the models. Additional constants are evaluated with reference to benchmark physical experiments;

ii) the statistical mechanics approach, typically based on the construction of an asymptotic expansion. In this case the constants of the model are calculated explicitly. Two primary examples of this approach are the two-scale Direct Interaction Approximation (*DIA*) model of Yoshizawa (1984) and the Renormalization Group theory (*RNG*) model of Yakhot & Orszag (1986).

Chapter 2

Reynolds Decomposition and Averaging

The flow of a viscous incompressible fluid with costant properties is governed – besides continuity – by the Navier-Stokes equation (Einstein summation convention applies to repeated indices):

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} . \tag{2.1}$$

Equation (2.1) can be written in conservative form (see Appendix) and the complete system of the Navier-Stokes equations assume the form:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i u_j\right) = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{2.2}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.3}$$

where u_i is the fluid velocity, p is the modified pressure (divided by the density ρ) and ν is the fluid kinematic viscosity. Note that p is a solution of the Poisson pressure equation obtained by taking the divergence of (2.1/2.2) (see Appendix):

$$\frac{\partial^2 p}{\partial x_i \partial x_i} = -\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} . \tag{2.4}$$

Body forces do not appear explicitly and they can be included into the modified pressure in terms of their potential.

2.1 Reynolds-Averaged Navier-Stokes Equations

According to the concept of Reynolds decomposition, the dependent variables of system (2.2-2.3) are decomposed into mean and fluctuationg parts:

$$u_i = \overline{u}_i + u_i' \tag{2.5}$$

$$p = \overline{p} + p' \tag{2.6}$$

and it is assumed that the following properties of the average operator hold (Tennekes & Lumley 1972, Reynolds 1976, Speziale 1991, Speziale 1996):

$$\overline{\phi'} = \overline{\psi'} = 0 \tag{2.7}$$

$$\overline{\phi\psi} = \overline{\phi}\,\overline{\psi} + \overline{\phi'\psi'} \tag{2.8}$$

$$\overline{\overline{\phi}\phi'} = \overline{\overline{\psi}\psi'} = \overline{\overline{\phi}\psi'} = \overline{\overline{\psi}\phi'} = 0 \tag{2.9}$$

$$\overline{\phi^2} = \overline{\phi}^{\,2} + \overline{\phi'^2} \tag{2.10}$$

$$\overline{\psi^2} = \overline{\psi}^{\,2} + \overline{\psi'^2} \tag{2.11}$$

$$\overline{\frac{\partial \phi}{\partial x_i}} = \frac{\partial \overline{\phi}}{\partial x_i} \tag{2.12}$$

$$\overline{\frac{\partial \phi}{\partial t}} = \frac{\partial \overline{\phi}}{\partial t}, \tag{2.13}$$

where ϕ and ψ are two generic flow variables.

The mean value of any generic flow variable ϕ can be calculated in a *turbulent statistically steady state* as a time average:

$$\overline{\phi} = \overline{\phi}^T(\boldsymbol{x}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \phi(\boldsymbol{x},t)dt\,, \tag{2.14}$$

in a *turbulent spatially homogeneous* flow as a volume average:

$$\overline{\phi} = \overline{\phi}^V(t) = \lim_{V\to\infty} \frac{1}{V}\int_{V} \phi(\boldsymbol{x},t)d^3x\,, \tag{2.15}$$

while in a *turbulent flow of more general nature* (neither statistically steady nor homogeneous), the mean value can be computed as an ensemble average:

$$\overline{\phi} = \overline{\phi}^E(\boldsymbol{x},t) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \phi^k(\boldsymbol{x},t)\,, \tag{2.16}$$

an average taken over N repeated experiments. The concept of ensemble average is borrowed from statistical mechanics. One imagines a set of different flows that are identical in all the variables that can be controlled but different in the details, and averages over them.

On the basis of the *ergodic hypothesis* it is assumed that in a turbulent statistically steady state and in a turbulent spatially homogeneous flow:

$$\overline{\phi}^T = \overline{\phi}^E \tag{2.17}$$

$$\overline{\phi}^V = \overline{\phi}^E\,, \tag{2.18}$$

respectively.

By substituting expressions (2.5-2.6) into the system of the Navier-Stokes equations (2.2-2.3), taking an ensemble average and enforcing properties (2.7-2.13), one obtains a system of partial differential equations that governs the mean-velocity and pressure fields of the turbulent flow of a viscous incompressible fluid:

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u_i u_j}\right) = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} \tag{2.19}$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 . \tag{2.20}$$

Owing to the nonlinearity of the convective term of the Navier-Stokes equation, one has:

$$\overline{u_i u_j} = \overline{\left(\overline{u}_i + u_i'\right)\left(\overline{u}_j + u_j'\right)} = \overline{\overline{u}_i \overline{u}_j} + \overline{\overline{u}_i u_j'} + \overline{u_i' \overline{u}_j} + \overline{u_i' u_j'} = \overline{u}_i \overline{u}_j + \overline{u_i' u_j'} \tag{2.21}$$

that gives:

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u}_i \overline{u}_j\right) = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right), \tag{2.22}$$

so that (Tennekes & Lumley 1972, Hinze 1975) the *Reynolds-Averaged Navier-Stokes equations* (*RANS*) are obtained (the convective term is now written in nonconservative form, see Appendix):

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}'}{\partial x_j} \tag{2.23}$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 . \tag{2.24}$$

Equation (2.23) is an equation of balance of mean linear momentum, equation (2.24) is the mean continuity equation, $\overline{u}_i$ and $\overline{p}$ are the mean velocity and pressure and:

$$\tau'_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j = \overline{u'_i u'_j} \tag{2.25}$$

is the Reynolds-stress term (actually $\tau'_{ij} = \overline{u'_i u'_j}$ is the Reynolds-stress tensor divided by the density, a not so relevant detail in constant-density fluid flows) that incorporates the effect of turbulent motion on the mean stress. The Reynolds-stress tensor is symmetric ($\tau'_{ij} = \tau'_{ji}$), the diagonal components are normal stresses (negative pressures) and the off-diagonal components are shear stresses.

System (2.23-2.24) is not a closed system for the calculation of the four dependent variables $\overline{u}_i$ and $\overline{p}$ in the sense that the Reynolds-stress tensor contains six additional independent unknowns. The problem of the closure of the system of the Navier-Stokes equations consists in expressing, through models, the Reynolds-stress tensor as a function of the mean-field and/or other variables (see subsequent Chapters).

Equation (2.23) can be written as:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\nu \overline{S}_{ij} - \tau'_{ij}\right) = \frac{\partial T_{ij}^{TOT}}{\partial x_j} \tag{2.26}$$

where τ'_{ij} is the Reynolds-stress term (2.25), T_{ij}^{TOT} is the total mean stress (divided by the density):

$$T_{ij}^{TOT} = \overline{T}_{ij} - \tau'_{ij} = -\overline{p}\delta_{ij} + 2\nu \overline{S}_{ij} - \overline{u'_i u'_j} \tag{2.27}$$

and:

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \tag{2.28}$$

is the mean rate-of-strain tensor. Note that $\overline{S}_{ii} = 0$ because of continuity.

More in general, turbulence models represent high-order moments of the fluctuating velocity components (the components of the Reynolds stress tensor $\overline{u_i'u_j'}$) expressed in terms of lower-order moments. This can be done directly as in the case of the eddy-viscosity models, or indirectly, as in the case of models based on the solution of partial differential equations (see subsequent Chapters). The assumption that high-order quantities can be computed from low-order quantities is not exactly correct and almost all the observations about the drawbacks of *RANS* turbulence models are related to this point. For example, in many flows the mean-velocity profile is established well before the high-order moments, meaning that the high-order moments can not be uniquely determined from the mean velocity.

2.2 Fluctuating-Velocity Transport Equation

By subtracting equation (2.23) from (2.2) after the introduction of decomposition (2.5-2.6) and equation (2.24) from (2.3), one obtains a system of partial differential equations that governs the fluctuating field of a viscous incompressible fluid:

$$\frac{\partial u_i'}{\partial t} + \overline{u}_j \frac{\partial u_i'}{\partial x_j} = -u_j' \frac{\partial u_i'}{\partial x_j} - u_j' \frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij}'}{\partial x_j} \tag{2.29}$$

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{2.30}$$

[equation (2.29) is nonlinear in the velocity fluctuations]. Moreover, by taking the divergence of equation (2.29), one obtains the Poisson equation for the pressure fluctuations:

$$\frac{\partial^2 p'}{\partial x_i \partial x_i} = -\frac{\partial^2}{\partial x_j \partial x_j}\left(u_i' \overline{u}_j + \overline{u}_i u_j' + u_i' u_j' - \tau_{ij}'\right). \qquad (2.31)$$

An analysis of the mathematical properties of system (2.29-2.30) with particular reference to the form of the solution for the fluctuating velocity u_i', shows that the fluctuating velocity has the general mathematical form:

$$u'(\boldsymbol{x},t) = F_i\left[\overline{\boldsymbol{u}}(\boldsymbol{y},s), \boldsymbol{u}'(\boldsymbol{y},0), \boldsymbol{u}'(\boldsymbol{y},s)\big|_{\partial V}, \boldsymbol{x}, t\right], \qquad \boldsymbol{y} \in V, \qquad s \in (-\infty, t) \qquad (2.32)$$

where F_i denotes a functional, V is the fluid volume and ∂V is its bounding surface. This means that the fluctuating velocity is a functional of the global history of the mean-velocity field, with an implicit dependence on initial and boundary conditions (Speziale 1991). This means that also τ_{ij}' is a functional of the global history of the mean-velocity field and also depends on the initial and boundary conditions. Lumley (1970) noted that it is not possible to obtain an acceptable Reynolds-stress closure if there is a detailed dependence on such initial and boundary conditions.

In the case of turbulent flows sufficiently far from the boundaries and sufficiently ahead in their temporal evolution, it is reasonable to assume that the initial and boundary conditions on the fluctuating velocity – besides those on the Reynolds-stress tensor – simply set the length and time scales of the turbulent phenomenon.

This assumption brings to the conclusion that:

$$\tau_{ij}'(\boldsymbol{x},t) = F_{ij}\left[\overline{\boldsymbol{u}}(\boldsymbol{y},s), l_0(\boldsymbol{y},s), \tau_0(\boldsymbol{y},s), \boldsymbol{x}, t\right], \quad \boldsymbol{y} \in V, \quad s \in (-\infty, t) \qquad (2.33)$$

where l_0 is the turbulent length scale, τ_0 is the turbulent time scale and F_{ij} is a functional that depends on the initial and boundary conditions for τ'_{ij}. Equation (2.33) represents a fundamental expression for what Reynolds-stress modeling is concerned.

2.3 Eddy Viscosity

The Reynolds-stress tensor τ'_{ij} can be divided into isotropic and deviatoric parts:

$$\tau'_{ij} = \tau'^{I}_{ij} + \tau'^{D}_{ij} . \tag{2.34}$$

Eddy-viscosity models for the deviatoric part of the Reynolds-stress tensor that have the form:

$$\tau'^{D}_{ij} = -2\nu_T \overline{S}_{ij} = -\nu_T \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) \tag{2.35}$$

where $\overline{S}_{ij}$ is the mean rate-of-strain tensor (2.28) and:

$$\nu_T \propto \frac{l_0^2}{\tau_0} \tag{2.36}$$

is the turbulent or eddy viscosity (Boussinesq 1877), represent one of the simplest examples of possible forms for (2.33) (note that the eddy viscosity has dimensions $L^2 T^{-1}$).
The assumption that the Reynolds-stress tensor is characterized by a single length and time scale – or a single velocity and length scale – constitutes a considerable simplification. Turbulent flows in their real development exhibit a wide range of excited scales in both space and time.

The fluctuating momentum equation (2.29) can be written in operator form as follows (Speziale 1991):

$$Lu_i' = 0\,. \tag{2.37}$$

Different moments (2.37) give rise, respectively, to a partial differential equation that governs the evolution of the turbulent dissipation rate (Chapter 5) and to the Reynolds-stress transport equation (Chapter 6).
The constraints of realizability and frame invariance are, among others requisites, of remarkable importance in the process of formulation of Reynolds-stress models.

2.4 Realizability

Realizability was first posed by Schumann (1977) and then rigorously introduced by Lumley (1978, 1983). Essentially it requires that the first and second moments resulting from solutions of appropriate modeled equations, are consistent in all cases with a physical realization of a turbulent flow, i.e. they are the moments of a physically possible probability-density function. This implies that, for any turbulent flow (also in the limit of two- or one-dimensional turbulence), a Reynolds-stress model yield positive component energies:

$$\tau'_{\alpha\alpha} \geq 0 \tag{2.38}$$

($\alpha = 1,2,3$) where greek indices indicate that there is no summation. Inequality (2.38) is a direct consequence of the definition of the Reynolds-stress tensor (2.25). It has been shown that realizability could be satisfied identically in homogeneous turbulent flows by Reynolds-stress transport models. This is accomplished by requiring that whenever a component energy $\tau'_{\alpha\alpha}$ vanishes, the time rate $\dot{\tau}'_{\alpha\alpha}$ also vanishes.

2.5 Frame Invariance

Donaldson was probably the first that addressed the issue of unequivocal use of coordinate invariance in turbulence modeling (Donaldson & Rosenbaum 1968). The approach, termed by Donaldson as *invariant modeling*, was based on the Reynolds-stress transport equation and required that all modeled terms be cast in tensor form uniquely. The question of frame invariance, where time-dependent rotations and translations of the reference frame are accounted for, was first considered by Lumley (1970) and more extensively by Speziale (1989).

In a general noninertial reference frame which can undergo arbitrary time-dependent rotations and translations relative to an inertial frame, the fluctuating momentum equation (2.29) takes the form:

$$\frac{\partial u_i'}{\partial t} + \overline{u}_j \frac{\partial u_i'}{\partial x_j} = -u_j' \frac{\partial u_i'}{\partial x_j} - u_j' \frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij}'}{\partial x_j} - 2 e_{ijk} \Omega_j u_k' \tag{2.39}$$

where e_{ijk} is the permutation tensor and Ω_j is the rotation rate of the reference frame, relative to the inertial frame (Speziale 1989). From equation (2.39) it appears that the evolution of the fluctuating velocity only depends directly on the motion of the reference frame though the Coriolis acceleration. Translational, centrifugal and angular accelerations only have a indirect effect through the changes that they induce in the mean-velocity field. As a consequence, closure models for the Reynolds-stress tensor must be form invariant under the extended Galileian group of transformations:

$$\boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{c}(t) \tag{2.40}$$

that allows for an arbitrary translational acceleration $\ddot{\boldsymbol{c}}$ of the reference frame relative to an inertial frame $\boldsymbol{x}$.

In the limit of two-dimensional turbulence (or turbulence where the ratio of the fluctuating to mean time scales $\tau_0/T_0 << 1$), the Coriolis acceleration is derivable from a scalar potential that can be absorbed into the fluctuating pressure (or neglected), giving complete frame indifference (Speziale 1981, 1983). This invariance under arbitrary time-dependent rotations and translations of the reference frame, specified by:

$$\boldsymbol{x}^* = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{c}(t) \tag{2.41}$$

where $\boldsymbol{Q}(t)$ is any time-dependent proper orthogonal rotation tensor, is referred to as Material Frame Indifference (*MFI*), the term that has been traditionally used for the analogous manifest invariance of constitutive equations in modern continuum mechanics.

For general three-dimensional turbulent flows where the ratio of the fluctuating to the mean time scales $\tau_0/T_0 = O(1)$, *MFI* does not apply as a result of Coriolis effects (Lumley 1970).

The Coriolis acceleration in (2.39) can be combined with the mean velocity in such a way that frame dependence enters exclusively through the appearance of the *intrinsic* or *absolute* mean-vorticity tensor (Speziale 1989):

$$\overline{W}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i}\right) + e_{mji}\Omega_m , \tag{2.42}$$

a more general definition of the mean-vorticity tensor in inertial reference frame:

$$\overline{W}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i}\right). \tag{2.43}$$

This result, in conjunction with the *MFI* constraint in the two-dimensional limit, considerably restricts the possible forms of *RANS* turbulence models.

2.6 Robustness

A further constraint is that the complete model should be well posed in the sense of being relatively insensitive to small perturbations in the turbulence structure. Turbulence is observed to be a relatively stable phenomenon in that mean values (the first and second moments at least) are relatively insensitive to small local disturbances. This aspect is particularly important if the model is to be used for complex and practically relevant engineering flows, but it appears difficult to analyze. Computational experiments probably represent the only feasible approach to the investigation of the dynamic behavior of any turbulence closure model.

Chapter 3

Zero-Equation Models

On the basis of equation (2.33), invariance under the extended Galileian group of transformations brings to the following general form of Reynolds-stress models (Speziale 1991):

$$\tau'_{ij}(\boldsymbol{x},t) = F_{ij}\left[\overline{\boldsymbol{u}}(\boldsymbol{y},s) - \overline{\boldsymbol{u}}(\boldsymbol{x},s), l_0(\boldsymbol{y},s), \tau_0(\boldsymbol{y},s), \boldsymbol{x}, t\right], \quad \boldsymbol{y} \in V\ , \quad s \in (-\infty, t). \qquad (3.1)$$

The variables in (3.1) can be expanded in Taylor series:

$$\overline{\boldsymbol{u}}(\boldsymbol{y},s) - \overline{\boldsymbol{u}}(\boldsymbol{x},s) = (y_i - x_i)\frac{\partial \overline{\boldsymbol{u}}}{\partial x_i} + \frac{(y_i - x_i)(y_j - x_j)}{2!}\frac{\partial^2 \overline{\boldsymbol{u}}}{\partial x_i \partial x_j}$$

$$+ (s-t)(y_i - x_i)\frac{\partial^2 \overline{\boldsymbol{u}}}{\partial t \partial x_i} + \cdots \qquad (3.2)$$

$$l_0(\boldsymbol{y},s) = l_0 + (y_i - x_i)\frac{\partial l_0}{\partial x_i} + (s-t)\frac{\partial l_0}{\partial t} + \frac{(s-t)^2}{2!}\frac{\partial^2 l_0}{\partial t^2}$$

$$+ \frac{(y_i - x_i)(y_j - x_j)}{2!}\frac{\partial^2 l_0}{\partial x_i \partial x_j} + (s-t)(y_i - x_i)\frac{\partial^2 l_0}{\partial t \partial x_i} + \cdots \qquad (3.3)$$

$$\tau_0(\boldsymbol{y},s) = \tau_0 + (y_i - x_i)\frac{\partial \tau_0}{\partial x_i} + (s-t)\frac{\partial \tau_0}{\partial t} + \frac{(s-t)^2}{2!}\frac{\partial^2 \tau_0}{\partial t^2}$$

$$+ \frac{(y_i - x_i)(y_j - x_j)}{2!}\frac{\partial^2 \tau_0}{\partial x_i \partial x_j} + (s-t)(y_i - x_i)\frac{\partial^2 \tau_0}{\partial t \partial x_i} + \cdots \qquad (3.4)$$

where the mean velocity field $\overline{\boldsymbol{u}}$, the turbulent length scale l_0 and the turbulent time scale τ_0 on the right-hand side of equations (3.2-3.4) are evaluated at $\boldsymbol{x}$ and t.

By splitting τ'_{ij} into isotropic and deviatoric parts [equation (2.33)] and making use of dimensional analysis, one obtains:

$$\tau'_{ij} = \tau'^{I}_{ij} + \tau'^{D}_{ij} = \frac{2}{3}K\delta_{ij} - \frac{l_0^2}{\tau_0^2}\hat{F}_{ij}\left[\overline{\boldsymbol{u}}(\boldsymbol{y},s) - \overline{\boldsymbol{u}}(\boldsymbol{x},s),\boldsymbol{x},t\right], \quad \boldsymbol{y} \in V\,, \quad s \in (-\infty,t) \qquad (3.5)$$

(δ_{ij} is the Kronecker's delta), where:

$$K = \frac{1}{2}\overline{u'_i u'_i} = \frac{1}{2}\tau'_{ii} \qquad (3.6)$$

is the average turbulent kinetic energy and:

$$\overline{\boldsymbol{u}} = \frac{\tau_0 \overline{\boldsymbol{u}}_{dim}}{l_0} \qquad (3.7)$$

is the nondimensional mean velocity (the symbol has not been altered with respect to the dimensional form). $\hat{F}_{ij}$ is a traceless and dimensionless functional of its arguments.

Making use of the Taylor series expansions (3.2-3.4), one obtains:

$$\overline{u}_i(\boldsymbol{y},s) - \overline{u}_i(\boldsymbol{x},s) = \frac{\tau_0}{T_0}\left(y_j^* - x_j^*\right)\left(\frac{\partial \overline{u}_i}{\partial x_j}\right)^* + O\left(\frac{\tau_0^2}{T_0^2}\right) \qquad (3.8)$$

where:

$$y_i^* - x_i^* = \frac{y_i - x_i}{l_0} \qquad (3.9)$$

and:

$$\left(\frac{\partial \overline{u}_i}{\partial x_j}\right)^* = T_0 \frac{\partial \overline{u}_i}{\partial x_j} \tag{3.10}$$

are nondimensional variables of $O(1)$ and T_0 is the time scale of the mean flow.

If, in analogy with the molecular fluctuations of continuum flows, the complete separation of scales is assumed (L_0 is the length scale of the mean flow):

$$\frac{\tau_0}{T_0} << 1 \tag{3.11}$$

$$\frac{l_0}{L_0} << 1 \tag{3.12}$$

equation (3.5) can be better defined(*).

Making use of equations (3.8-3.12), equation (3.5) assumes the approximate form:

$$\tau'_{ij} = \tau'^{I}_{ij} + \tau'^{D}_{ij} = \frac{1}{3}\tau'_{kk}\delta_{ij} - \frac{l_0^2}{\tau_0^2} G_{ij}(\overline{u}_{kl}) = \frac{2}{3}K\delta_{ij} - \frac{l_0^2}{\tau_0^2} G_{ij}(\overline{u}_{kl}) \tag{3.13}$$

where [equation (3.10)]:

(*) Note that assumptions (3.11-3.12) represent a considerable simplification, in the sense that the molecolar fluctuations in continuum flows are such that $\tau_0/T_0 \le 10^{-6}$ and with turbulent fluctuations τ_0/T_0 can be of $O(1)$.

$$\overline{u}_{kl} = \frac{\tau_0}{T_0}\left(\frac{\partial \overline{u}_k}{\partial x_l}\right)^* = \frac{\tau_0}{T_0}\left(T_0 \frac{\partial \overline{u}_k}{\partial x_l}\right) \tag{3.14}$$

is the nondimensional mean-velocity gradient.

3.1 Eddy-Viscosity Model

Being the tensor function G_{ij} symmetric and traceless, to the first order in τ_0/T_0, equation (3.13) becomes (Smith 1971):

$$\tau'_{ij} = \tau'^{I}_{ij} + \tau'^{D}_{ij} = \frac{2}{3}K\delta_{ij} - 2\nu_T \overline{S}_{ij} = \frac{2}{3}K\delta_{ij} - \nu_T\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \tag{3.15}$$

where $\overline{S}_{ij}$ is the strain-rate tensor of the mean field (2.27):

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

and:

$$\nu_T = \frac{l_0^2}{\tau_0} \tag{3.16}$$

is the eddy viscosity.

Equation (3.15) is the *standard eddy-viscosity model* for the Reynolds-stress tensor and is the result of a derivation process in which in (3.8) only first-order terms in τ_0/T_0 are considered. In the case in which also second-order terms are mantained, the procedure of derivation brings to the anisotropic eddy-viscosity group of models (see subsequent Chapters).

According to (3.16), in the simplest description of the phenomenon one can imagine, turbulence can be characterized by a single length and time scale (or by a single velocity and length scale).

The eddy-viscosity model for the Reynolds-stress tensor (3.15) is not closed until indications are given on the turbulent length and time scales in (3.16). In zero-equation models l_0 and τ_0 are given algebraically by empirical means (Section 3.2).

With reference to the problem of the closure of the system of the Reynolds-Averaged Navier-Stokes equations, the system of equations to be solved in the zero-equation class of models for the calculation of $\overline{u}_i$ and $\overline{p}$, becomes:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau'_{ij}}{\partial x_j} \tag{3.17}$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{3.18}$$

$$\tau'_{ij} = \frac{2}{3} K \delta_{ij} - \nu_T \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right). \tag{3.19}$$

A closed system of four partial differential equations [equations (3.17-3.18)] for the determination of $\overline{u}_i$ and $\overline{p}$ is obtained when the form of the eddy viscosity is defined(*).

(*) Note that in incompressible flows, the isotropic part of τ'_{ij} [the first term on the right-hand side of equation (3.19)] can be absorbed into the mean-pressure term of (3.17) (compared to which is practically negligible) so that K need not to be calculated explicitly.

3.2 Eddy-Viscosity Forms

The earliest example of zero-equation model is represented by the Prandtl's mixing-length theory (Prandtl 1925). Based on an analogy between the turbulent length scale and the mean free path in kinetic theory of gases, Prandtl for a plane shear flow, argued that:

$$\nu_T = l_m^2 \left| \frac{d\overline{u}}{dy} \right| \tag{3.20}$$

where the mean velocity has the form $\overline{\boldsymbol{u}} = \overline{u}(y)\boldsymbol{i}$ and $l_m = l_0$ is the *mixing length*, representing the distance traversed by a small lump of fluid before losing its momentum. Near a plane solid boundary it has further assumed that:

$$l_m = \kappa y \tag{3.21}$$

where κ is the von Kármán constant. With the additional assumption that the shear stress is approximately constant in the near-wall region, from (3.20) and (3.21) the *law of the wall* is obtained:

$$u^+ = \frac{1}{\kappa} ln\, y^+ + C \tag{3.22}$$

with:

$$u^+ = \frac{\overline{u}}{u_\tau} \tag{3.23}$$

$$y^+ = \frac{y u_\tau}{\nu} \ , \tag{3.24}$$

and where κ is the Von Karman constant, C is a nondimensional constant, u_τ is the friction velocity ($u_\tau = \sqrt{\tau_w/\rho}$, $\tau_w = \mu(\partial\overline{u}/\partial y)_{wall}$) and y^+, the nondimensional y-coordinate in local wall units, is measured normally with respect to the wall. The law of the wall (3.22) with $\kappa = 0.41$ and $C = 5.0$ has shown to be in remarkably good agreement with experimental data of turbulent channel and pipe flows with $30 \le y^+ \le 1000$ (Schlichting 1968).

In the last decades, with the particularly fast development of Computational Fluid Dynamics (*CFD*), efforts have been made to generalize mixing-length models to three-dimensional turbulent flows. The mixing-length theory of Prandtl (3.20) has two direct generalizations to three-dimensional flows.

3.2.1 Strain-Rate Form

The generalized strain-rate form of model equation (3.20) is:

$$\nu_T = l_m^2\left(2\overline{S}_{ij}\overline{S}_{ij}\right)^{1/2} \tag{3.25}$$

where $\overline{S}_{ij}$ is the mean rate-of-strain tensor (2.27). This model, due to Smagorinsky (1963), has been primarily used as a subgrid-scale model within the Large Eddy Simulation (*LES*) approach to the modeling of turbulent flows.

3.2.2 Vorticity Form

The generalized vorticity form of model equation (3.20) is:

$$\nu_T = l_m^2\left(\overline{\omega}_i\overline{\omega}_i\right)^{1/2} \tag{3.26}$$

where:

$$\overline{\omega}_i = e_{ijk} \frac{\partial \overline{u}_k}{\partial x_j} \tag{3.27}$$

is the mean-vorticity vector. This model, due to Baldwin & Lomax (1978), has been extensively used for *RANS* calculations in aerodynamics (Wilcox 1993).

3.3 Summary

The zero-equation models only allow the calculation of the mean-velocity and pressure fields. The primary advantage of these models is that they can be used rather easily.
The disadvantages are:
i) they need a specific prescription for the turbulent length scale in each problem considered;
ii) they completely neglect history effects.
Overall, they are not accurate when strong surface curvatures are present, in cases of free-stream turbulence, near separation points and in boundary layers subjected to strong accelerations. Moreover, they do not compute the turbulent kinetic energy that is an essential measure of the intensity of the turbulent fluctuations.

Chapter 4

One-Equation Models

One-equation closure models have been developed to eliminate some of the deficiencies of the zero-equation models, i.e. to provide the calculation of the turbulent kinetic energy K (or equivalently of the turbulent velocity scale, often in this context referred to as q, where $q = \sqrt{2K}$) and to account for some nonlocal and history effects in the definition of the eddy viscosity.

4.1 Kinetic-Energy Transport Equation

The exact form of the turbulent kinetic-energy transport equation (Tennekes & Lumley 1972) is obtained by contraction(*) of the Reynolds-stress transport equation (Chapter 6). One has:

$$\frac{\partial K}{\partial t} + \overline{u}_i \frac{\partial K}{\partial x_i} = -\tau'_{ij} \frac{\partial \overline{u}_i}{\partial x_j} - \varepsilon - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\overline{u'_k u'_k u'_i} + \overline{p' u'_i} \right) + \nu \frac{\partial^2 K}{\partial x_i \partial x_i} \tag{4.1}$$

where K is the average turbulent kinetic energy [the mean kinetic energy of the fluctuations, see also equation (3.6)]:

$$K = \frac{1}{2}\tau'_{ii} . \tag{4.2}$$

and ε $\left[L^2/T^3\right]$ is the dissipation rate (per unit mass) of the mean kinetic energy of turbulent fluctuations (scalar dissipation rate).

(*) Contraction is obtained by setting $i = j$, summing from one to three and dividing the result by two.

Equation (4.1) can be written in the form:

$$\frac{\partial K}{\partial t}+\overline{u}_i\frac{\partial K}{\partial x_i}=P_K-\varepsilon-\frac{\partial D_i}{\partial x_i} \tag{4.3}$$

stating that the mean of turbulent kinetic energy fluctuations is balanced by the production, the dissipation and the diffusion of this energy.

In (4.3) the term D_i:

$$D_i=\frac{1}{2}\overline{u'_k u'_k u'_i}+\overline{p' u'_i}-\nu\frac{\partial K}{\partial x_i} \tag{4.4}$$

is the diffusive flux of turbulent kinetic energy.

More in particular, equation (4.3) can be written as:

$$\frac{\partial K}{\partial t}+\overline{u}_i\frac{\partial K}{\partial x_i}=P_K-\varepsilon-\frac{\partial}{\partial x_i}\left(\frac{1}{2}\overline{u'_k u'_k u'_i}+\overline{p' u'_i}\right)+\nu\frac{\partial^2 K}{\partial x_i \partial x_i} \tag{4.5}$$

representing a balance between the temporal derivative of the turbulent kinetic energy and the convective transport term on the left-hand side of equation (4.5), and the following terms on the right-hand side:

i) the term P_K, representing the rate of production of turbulent kinetic energy:

$$P_K=-\tau'_{ij}\frac{\partial \overline{u}_i}{\partial x_j} \tag{4.6}$$

that does not require any modeling in the formulation of one-equation closure models;

ii) the term $\varepsilon=\varepsilon_{ii}/2$, representing the turbulent rate of energy dissipation that, in the hypothesis of isotropy of the small-scale turbulence (high Reynolds numbers), has the form of the isotropic dissipation:

$$\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}; \tag{4.7}$$

iii) the gradient $\partial/\partial x_i$ of the turbulent transport term, where:

$$\frac{1}{2}\overline{u_k' u_k' u_i'} + \overline{p' u_i'} \tag{4.8}$$

is the turbulent transport term;
iv) the term representing the viscous transport:

$$\nu \frac{\partial^2 K}{\partial x_i \partial x_i}, \tag{4.9}$$

that does not require any modeling in the process of formulation of one-equation closure models(*).

4.1.1 Low-Reynolds-Numbers Formulation

It is possible to implement a low-Reynolds-numbers version of (4.1) (Norris & Reynolds 1975, Reynolds 1976) by using a *nonisotropic* rate of energy dissipation (local isotropy is not invoked):

$$\varepsilon = 2\nu \overline{S_{ij}' S_{ij}'} \tag{4.10}$$

(*) Equation (4.1) can be formulated in terms of the turbulent length scale $q = \sqrt{2K}$, as follows:

$$\frac{\partial (q^2)}{\partial t} + \overline{u}_i \frac{\partial (q^2)}{\partial x_i} = -2\tau_{ij}' \frac{\partial \overline{u}_i}{\partial x_j} - 2\varepsilon - \frac{\partial}{\partial x_i}\left(\overline{u_k' u_k' u_i'} + 2\overline{p' u_i'}\right) + \nu \frac{\partial^2 (q^2)}{\partial x_i \partial x_i}.$$

and a diffusive flux of turbulent kinetic energy in the form:

$$D_i = \frac{1}{2}\overline{u'_k u'_k u'_i} + \overline{p' u'_i} - \nu \overline{S'_{ij} \frac{\partial u'_k}{\partial x_i}} \tag{4.11}$$

where:

$$S'_{ij} = \frac{1}{2}\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right) \tag{4.12}$$

is the rate-of-strain tensor of the fluctuating field.
The result is:

$$\frac{\partial K}{\partial t} + \overline{u}_i \frac{\partial K}{\partial x_i} = -\tau'_{ij} \frac{\partial \overline{u}_i}{\partial x_j} - 2\nu \overline{S'_{ij} S'_{ij}} + \frac{1}{2}\overline{u'_k u'_k u'_i} + \overline{p' u'_i} - \nu \overline{S'_{ij} \frac{\partial u'_k}{\partial x_i}}. \tag{4.13}$$

4.2 Modeled Kinetic-Energy Transport Equation

Equation (4.1/4.5) can be closed if models for the dissipation term (4.7) and for the turbulent transport term (4.8) are provided.

4.2.1 Dissipation Term

Making use of scaling arguments similar to those used by Kolmogorov for high-Reynolds-number turbulent flows, the turbulent scalar dissipation rate ε (4.7) is modeled as follows:

$$\varepsilon = C^* \frac{K^{3/2}}{l} \tag{4.14}$$

where $l = l_0$ is the turbulent length scale and $C^* = 0.166$ is a non dimensional constant.

4.2.2 Turbulent Transport Term

Consistently with assumptions (3.11-3.12) of clear-cut separation of scales (the turbulent transport processes emulate the molecular transport processes) the turbulent transport term (4.8) is modeled by assuming a gradient trasport hypotesis:

$$\frac{1}{2}\overline{u'_k u'_k u'_i} + \overline{p' u'_i} = -\frac{\nu_T}{\sigma_K}\frac{\partial K}{\partial x_i} \tag{4.15}$$

where ν_T is the eddy viscosity and $\sigma_K \approx 1.0.$ is a nondimensional constant.

4.3 One-Equation Turbulence Models

By substituting (4.14) and (4.15) in (4.1/4.5) one obtains the modeled form – for one-equation closure models – of the turbulent kinetic-energy transport equation.

Thus, the system of equations to be solved in one-equation class of models for the calculation of $\overline{u}_i$, $\overline{p}$ and K becomes:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau'_{ij}}{\partial x_j} \tag{4.16}$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{4.17}$$

$$\frac{\partial K}{\partial t} + \overline{u}_i \frac{\partial K}{\partial x_i} = -\tau'_{ij}\frac{\partial \overline{u}_i}{\partial x_j} - C^* \frac{K^{3/2}}{l} + \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_K}\frac{\partial K}{\partial x_i}\right) + \nu \frac{\partial^2 K}{\partial x_i \partial x_i} \tag{4.18}$$

$$\tau'_{ij} = \frac{2}{3}K\delta_{ij} - \nu_T\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \tag{4.19}$$

$$\nu_T = K^{1/2} l \tag{4.20}$$

($l = l_0$ is the turbulent length scale) where:

$$C^* = 0.166\,, \quad \sigma_K \approx 1.0 \tag{4.21}$$

The form (4.20) that the eddy viscosity assumes in one-equation class of models has been independently proposed by Kolmogorov (1942) and Prandtl (1945).

A closed system of five partial differential equations [equations (4.16-4.18)] for the determination of $\overline{u}_i$, $\overline{p}$ and K is obtained when the turbulent length scale l is defined empirically(*).

The modeled transport equation (4.18) can not be integrated to solid boundaries, so that wall functions have to be used to overcome this problem.

Bradshaw *et al.* (1967) proposed an alternative one-equation turbulence model based on a modeled trasport equation for the Reynolds shear stress $\overline{u'v'}$, that gave acceptable results for turbulent boundary-layer calculations. For results concerning the application of one-equation closure models one can refer to Cebeci & Smith (1974), Rodi (1980) and Bradshaw *et al.* (1981).

4.4 Modeled Eddy-Viscosity Transport Equation

Baldwin & Barth (1990) and Spalart & Allmaras (1992, 1994) proposed one-equation closure models where the Reynolds-Averaged Navier-Stokes equations (4.16-4.17) with the eddy-viscosity model for the Reynolds-stress tensor (4.19) are solved in conjunction with a modeled transport equation for the eddy viscosity ν_T.

(*) In incompressible flows, the first term on the right-hand side of (4.19) can be absorbed in the mean-pressure term of (4.16).

The model of Spalart & Allmaras (1992, 1994) has four nested versions from the simplest, applicable only to free shear flows, to the most complete, applicable to viscous flows past solid bodies and with laminar regions.
A typical form of the modeled eddy-viscosity transport equation (Spalart & Allmaras 1994) is:

$$\frac{\partial \nu_T}{\partial t} + \overline{u}_i \frac{\partial \nu_T}{\partial x_i} = c_{b1} S \nu_T + \frac{1}{\sigma}\left[\nabla \cdot \left(\nu_T \nabla \nu_T\right) + c_{b2}\left(\nabla \nu_T\right)^2\right] - c_{w1} f_w \left[\frac{\nu_T}{d}\right]^2 \tag{4.23}$$

where:

$$c_{b1} = 0.1355\,, \quad c_{b2} = 0.622 \tag{4.24}$$

$$c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{\left(1 + c_{b2}\right)}{\sigma}\,, \quad c_{w2} = 0.3\,, \quad c_{w3} = 2 \tag{4.25}$$

$$f_w = g\left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right]^{1/6} \tag{4.26}$$

$$g = r + c_{w2}\left(r^6 - r\right) \tag{4.27}$$

$$r = \frac{\nu_T}{S \kappa^2 d^2} \tag{4.28}$$

($\kappa = 0.41$ is the von Kármán constant and $\sigma = 2/3$) and where $S = |\omega|$ is the magnitude of the vorticity (chosen a measure of the deformation tensor $\partial \overline{u}_i / \partial x_j$ in the production term) and d is the distance from the closest wall.

Equation (4.23) represents a balance between the temporal derivative of the eddy viscosity and the convective term on the left-hand side and the following terms on the right-hand side:

i) the modeled production term [the first on the rhs of (4.23), the only term that survives in the case of *homogeneous turbulence*][(*)].

ii) the modeled diffusion term [the second on the rhs of (4.23)];

iv) the destruction term [the third on the rhs of (4.23)];

Corrections can be made to (4.23) and a viscous-diffusion term can be added in the case of finite Reynolds numbers. Moreover, a trip term and an additional term for laminar regions can be added to (4.23) to obtain the complete version of the model (Spalart & Allmaras 1994).

4.5 Summary

One-equation models are superior to zero-equation models in that the time scale of the eddy viscosity is built up from turbulence statistics rather than from the mean-velocity gradients.

The main deficiencies of one-equation models are:

i) the use of the eddy viscosity concept;

ii) the need to provide empirically a particular definition of the turbulent length scale. This is virtually impossible to do in complex three-dimensional turbulent flows. The turbulent length and time scales are not universal. They strongly depend on the particular flow case considered.

The latter point makes both zero-equation and one-equation models incomplete. The two-equation models (Chapter 5) are actually the first complete turbulence models in

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(*) Note that in the case of *isotropic turbulence* equation (4.23) becomes:

$$\frac{\partial \tilde{\nu}}{\partial t} + \bar{u}_i \frac{\partial \tilde{\nu}}{\partial x_i} = 0$$

the sense that only require the specification of initial and boundary conditions for their application.

In spite of their deficiencies, one-equation models (and also zero-equation models) have remarkably contributed to the computation of practical engineering turbulent flows, in particular in the field of complex aerodynamic calculations (see Cebeci & Smith 1968 and Johnson & King 1984).

Chapter 5

Two-Equation Models

Two-equation models are among the most popular *RANS* turbulence models for scientific and engineering calculations. What distinguishes two-equation models from the classes of turbulence models considered in the previous Chapters is the fact that two separated modeled transport equations are solved for two independent turbulent quantities, directly related to the turbulent length and time scales (or any two linearly independent combinations of them, such as velocity and length). In the $K-\varepsilon$ model, the most popular two-equation model, the length and time scales are built up from the turbulent kinetic energy K and the turbulent dissipation rate ε, as follows:

$$l_0 \propto \frac{K^{3/2}}{\varepsilon} \tag{5.1}$$

$$\tau_0 \propto \frac{K}{\varepsilon}. \tag{5.2}$$

Thus, separate modeled transport equations are solved for the turbulent kinetic energy and the turbulent dissipation rate. This fact obviates the need to specify the turbulent scales specifically for any different flow case.

5.1 Kinetic-Energy Transport Equation

For what K is concerned, in order to close the exact transport equation for the turbulent kinetic energy [equation (4.1/4.5)], only a model for the turbulent transport term (4.8) on the right hand-side of equation (4.1/4.5) is needed.
On the basis of assumptions (3.11-3.12) the gradient transport model (4.15) is used:

$$\frac{1}{2}\overline{u'_k u'_k u'_i} + \overline{p' u'_i} = -\frac{\nu_T}{\sigma_K}\frac{\partial K}{\partial x_i}$$

($\sigma_K \approx 1.0$). By performing the substitution, the modeled form – for two-equation models – of the turbulent kinetic-energy transport equation is obtained.

5.2 Dissipation-Rate Transport Equation

By recalling the fluctuating momentum equation in operator form (Speziale 1991), equation (2.36):

$$Lu'_i = 0$$

the exact turbulent rate-of-dissipation transport equation is obtained from the first-order moment:

$$2\nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial}{\partial x_j}\left(Lu'_i\right)} = 0 \tag{5.3}$$

that in explicit form, becomes (Hinze 1975):

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i\frac{\partial \varepsilon}{\partial x_i} =$$

$$-2\nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_k}{\partial x_j}}\frac{\partial \overline{u}_i}{\partial x_k} - 2\nu\overline{\frac{\partial u'_j}{\partial x_i}\frac{\partial u'_j}{\partial x_k}}\frac{\partial \overline{u}_i}{\partial x_k} - 2\nu\overline{u'_k\frac{\partial u'_i}{\partial x_j}}\frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j} - 2\nu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_k}{\partial x_m}}$$

$$-\nu\frac{\partial}{\partial x_k}\left(\overline{u'_k\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_i}{\partial x_m}}\right) - 2\nu\frac{\partial}{\partial x_k}\left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u'_k}{\partial x_m}}\right)$$

$$-2\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m} \frac{\partial^2 u_i'}{\partial x_k \partial x_m}}$$

$$+\nu \frac{\partial^2 \varepsilon}{\partial x_i \partial x_i} \tag{5.4}$$

where $\varepsilon = \varepsilon_{ii}/2$ is the scalar turbulent disssipation rate in its isotropic form [see also equation (4.7)]:

$$\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} . \tag{5.5}$$

The seven higher-order correlations on the right-hand side of equation (5.4) correspond to three physical effects.

The first four terms represent the production of dissipation and in particular:

i) the first two terms on the right-hand side of equation (5.4) represent the production of dissipation by mean strains;

ii) the third term on the rhs of (5.4) is a inhomogeneous production term;

iii) the fourth term on the rhs of (5.4) represents the production of dissipation by vortex stretching.

The second two terms on the right-hand side of equation (5.4) represent the turbulent diffusion of dissipation and the following term the turbulent destruction of dissipation by viscous diffusion.

The last term on the right-hand side of equation (5.4) represents the effects of molecular diffusion.

More in particular equation (5.4) can be written as:

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} = P_\varepsilon + D_\varepsilon - \Phi_\varepsilon + \nu \frac{\partial^2 \varepsilon}{\partial x_i \partial x_i} , \tag{5.6}$$

representing a balance between the temporal derivative of the turbulent dissipation rate and the convective transport term on the lhs of equation (5.6) and the following terms on the rhs:

i) the term P_ε, representing the production of dissipation:

$$P_\varepsilon = -2\nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}}\frac{\partial \overline{u}_i}{\partial x_k} - 2\nu \overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_k}}\frac{\partial \overline{u}_i}{\partial x_k} - 2\nu \overline{u_k'\frac{\partial u_i'}{\partial x_j}}\frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j} - 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}; \quad (5.7)$$

ii) the term D_ε, representing the turbulent diffusion of dissipation:

$$D_\varepsilon = -\nu \frac{\partial}{\partial x_k}\left(\overline{u_k'\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right) - 2\nu \frac{\partial}{\partial x_k}\left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}\right); \quad (5.8)$$

iii) the term Φ_ε, representing the turbulent destruction of dissipation:

$$\Phi_\varepsilon = -2\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m}\frac{\partial^2 u_i'}{\partial x_k \partial x_m}}; \quad (5.9)$$

iv) the term representing the viscous transport:

$$\nu \frac{\partial^2 \varepsilon}{\partial x_i \partial x_i} \quad (5.10)$$

that does not require any modeling in the formulation of two-equation closure models.

5.3 Modeled Dissipation-Rate Transport Equation

In order to develop the two-equation model, the terms P_ε, D_ε and Φ_ε in (5.6) have to be modeled.

5.3.1 Production Term

The production term P_ε is modeled by thinking that the production of dissipation is governed by the level of anisotropy in the Reynolds-stress tensor and by the mean-velocity gradients (scaled by K and ε, that determine the turbulent length and time scales).

One has:

$$P_\varepsilon = P_\varepsilon\left(b_{ij}, \frac{\partial \overline{u}_i}{\partial x_i}, K, \varepsilon \right) \tag{5.11}$$

where:

$$b_{ij} = \frac{1}{2K}\left(\tau'_{ij} - \frac{2}{3}K\delta_{ij} \right) \tag{5.12}$$

is the (nondimensional) anisotropy (tensor) of the Reynolds-stress tensor (*).

Coordinate invariance and dimensional analysis give:

$$P_\varepsilon = -2C_{\varepsilon 1}\varepsilon b_{ij}\frac{\partial \overline{u}_i}{\partial x_j} = -C_{\varepsilon 1}\frac{\varepsilon}{K}\tau'_{ij}\frac{\partial \overline{u}_i}{\partial x_j} \tag{5.13}$$

as the leading term of the Taylor series expansion of (5.11) with the assumption that $\|\boldsymbol{b}\|$ and τ_0/T_0 are small. $C_{\varepsilon 1}$ is a nondimensional constant. Equation (5.13) has been originally formulated by simply thinking that the production of dissipation should be proportional to the production of turbulent kinetic energy (Hanjalić & Launder 1972).

(*) $b_{ij} = \dfrac{\overline{u'_i u'_j}}{\overline{u'_k u'_k}} - \dfrac{1}{3}\delta_{ij} = \dfrac{1}{2K}\left(\tau'_{ij} - \dfrac{2}{3}K\delta_{ij} \right)$.

5.3.2 Diffusion Term

On the basis of assumptions (3.11-3.12) a gradient transport hypothesis is used to model the turbulent diffusion term D_ε.

One has:

$$D_\varepsilon = \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) \tag{5.14}$$

where ν_T is the eddy viscosity and σ_ε is a nondimensional constant.

5.3.3 Destruction Term

The destruction term Φ_ε is modeled by thinking that the destruction of dissipation is determined by the turbulent length and time scales alone (assumption of isotropic turbulence), so that [expressions (5.1-5.2)]:

$$\Phi_\varepsilon = \Phi_\varepsilon(K, \varepsilon). \tag{5.15}$$

Dimensional analysis gives:

$$\Phi_\varepsilon = C_{\varepsilon 2}\frac{\varepsilon^2}{K} \tag{5.16}$$

where $C_{\varepsilon 2}$ is a nondimensional constant.

5.4 $K-\varepsilon$ Model

By substituting (5.13), (5.14) and (5.16) into (5.6) one obtains the modeled form of the dissipation-rate transport equation.

Thus the system of equations to be solved in the two-equation standard $K-\varepsilon$ model for the calculation of $\overline{u}_i$, $\overline{p}$, K and ε (see Launder & Spalding 1974) becomes:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau'_{ij}}{\partial x_j} \tag{5.17}$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{5.18}$$

$$\frac{\partial K}{\partial t} + \overline{u}_i \frac{\partial K}{\partial x_i} = -\tau'_{ij} \frac{\partial \overline{u}_i}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_K} \frac{\partial K}{\partial x_i} \right) + \nu \frac{\partial^2 K}{\partial x_i \partial x_i} \tag{5.19}$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} = -C_{\varepsilon 1} \frac{\varepsilon}{K} \tau'_{ij} \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_i} \right) - C_{\varepsilon 2} \frac{\varepsilon^2}{K} + \nu \frac{\partial^2 \varepsilon}{\partial x_i \partial x_i} \tag{5.20}$$

$$\tau'_{ij} = \frac{2}{3} K \delta_{ij} - \nu_T \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) \tag{5.21}$$

$$\nu_T = C_\mu \frac{K^2}{\varepsilon}. \tag{5.22}$$

The system of the six partial differential equations (5.17-5.20) is a closed system for the determination of $\overline{u}_i$, $\overline{p}$, K and ε (*). The form (5.22) that the eddy viscosity assumes in the $K-\varepsilon$ model is based on Launder & Spalding (1974).

On the basis of comparisons with physical experiments, the constants assume the values:

(*) In incompressible flows the first term on the right-hand side of (5.21) can be incorporated in the mean-pressure term of (5.17).

$$C_{\varepsilon 1} = 1.44\,, \quad C_{\varepsilon 2} = 1.92\,, \quad C_{\mu} = 0.09\,, \quad \sigma_K = 1.0\,, \quad \sigma_{\varepsilon} = 1.3 \tag{5.23}$$

where the value of C_{μ} is chosen to give results consistent with the law of the wall (3.22). The standard $K-\varepsilon$ can not be integrated to a solid boundary. Wall functions or damping methods have to be implemented for this scope (see Patel *et al.* 1985).

The $K-\varepsilon$ model has been used for calculation and comparison in a variety of different turbulent flows.

In *isotropic turbulence*, where:

$$\tau'_{ij} = \tau''^{I}_{ij} = \frac{2}{3} K(t)\delta_{ij} \tag{5.24}$$

$$\varepsilon_{ij} = \varepsilon^{I}_{ij} = \frac{2}{3}\varepsilon(t)\delta_{ij}\,, \tag{5.25}$$

the standard $K-\varepsilon$ model predicts the following rate of decay of the turbulent kinetic energy (Reynolds 1987):

$$K(t) = K_0\left[1 + \frac{(C_{\varepsilon 2}-1)\varepsilon_0 t}{K_0}\right]^{-1/(C_{\varepsilon 2}-1)}, \tag{5.26}$$

a results that is in a reasonable agreement with the physical experiments performed by Comte-Bellot & Corrsin (1971).

In *homogeneous shear flow*, where an initially isotropic turbulence is subjected to a constant shear rate S with mean-velocity gradients:

$$\frac{d\bar{u}}{dy} = S \tag{5.27}$$

or, more in general:

$$\frac{\partial \overline{u}_i}{\partial x_j} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.28}$$

the standard $K-\varepsilon$ model gives results in terms of temporal evolution of the turbulent kinetic energy in an acceptable agreement with the experimental data of Tavoularis & Corrsin (1981).

In a *nonhomogeneous turbulent flow*, the flow past a backward-facing step at $Re \approx 100{,}000$, the standard $K-\varepsilon$ model has been used by Speziale & Ngo (1988) and the results compared with the experimental data of Kim *et al.* (1980). The comparisons show a discrepacy between computed and experimental results for the reattachment point of the mean-flow streamlines and turbulent intensities of about 20%. Avva *et al.* (1988) reported improved predictions for the reattachment point by using a double-layer logarithmic wall region.

Yakhot & Orszag (1986) derived a $K-\varepsilon$ model making use of the Renormalization Group theory (*RNG*). An expansion is made about an equilibrium state with known Gaussian statistics making use of the correspondence principle. Bands of high wavenumbers (small scales) are systematically removed and space is rescaled. The equations for the renormalized velocity field (large scales) account for the effect of the small scales that have been removed through the presence of an eddy viscosity. The removal of only the smallest scales gives rise to subgrid-scale closures for the Large Eddy Simulation (*LES*) approach to turbulence modeling. The removal of larger scales give rise to Reynolds-stress models. In the limit of high Reynolds numbers, the *RNG*-based $K-\varepsilon$ model of Yakhot & Orszag (1986) coincides with the standard $K-\varepsilon$ model with the difference that, on the basis of the theory, the constants of the model are calculated explicitly. Their values are:

$$C_{\varepsilon 1} = 1.063\,,\quad C_{\varepsilon 2} = 1.7215\,,\quad C_\mu = 0.0837\,,\quad \sigma_K = 0.7179\,,\quad \sigma_\varepsilon = 0.7179\,. \tag{5.29}$$

In the *RNG*-based $K-\varepsilon$ model of Yakhot & Orszag (1986) wall functions or Van Driest dumping are not required when approaching a solid wall. Problems related to the values of the constants (5.29) are reported by Speziale & Mac Giolla Mhuiris (1989*a*) and Speziale *et al.* (1989).

One of the major deficiencies of the standard $K-\varepsilon$ model lies in the fact that an eddy-viscosity model for the Reynolds-stress tensor is used. Eddy-viscosity models for the Reynolds-stress tensor have the following main problems:

i) they are purely dissipative and thus they can not account for the relaxation effects of the Reynolds stress;

ii) they are oblivious to the presence of rotational strains, i.e. they are not able to distinguish between physically different cases such as plane shear, plane strain or rotating plane shear.

iii) they cannot properly describe flow structures generated by normal Reynolds stress anisotropies (e.g. secondary flows in noncircular ducts).

Thus, the adoption of the Boussinesq hypothesis (5.21) in the model of the Reynolds-stress tensor gives predictions with limited success because:

i) the assumption of isotropy in the Reynolds stress tensor (i.e. in the modeling of $\tau_{ij}'^{D}$, see Section 5.7);

ii) the linear stress-strain relationship.

In an attempt to overcome the aforementioned deficiencies, properly invariant anisotropic generalizations of eddy-viscosity models for the Reynolds-stress tensor, nonlinear in the mean-velocity gradients, have been developed (Section 5.7).

5.5 $K-l$ Model

The $K-l$ model (Mellor & Herring 1973) is based [besides on equations (5.17-5.18)] on the solution of two modeled transport equations, one for the turbulent kinetic energy K in the form of one-equation closure models, and another for the integral length scale l defined as:

$$l(\boldsymbol{x},t)=\frac{1}{2K}\int_{-\infty}^{+\infty}\frac{R'_{ii}(\boldsymbol{x},\boldsymbol{r},t)}{4\pi|\boldsymbol{r}|^2}d^3r \qquad (5.30)$$

where:

$$R'_{ij}=\overline{u'_i(\boldsymbol{x},t)u'_j(\boldsymbol{x}+\boldsymbol{r},t)} \qquad (5.31)$$

is the two-point fluctuating velocity-correlation tensor.
The typical form of the $K-l$ model and in particular of the modeled transport equation for l [equation (5.33)] is:

$$\frac{\partial K}{\partial t}+\overline{u}_i\frac{\partial K}{\partial x_i}=-\tau'_{ij}\frac{\partial\overline{u}_i}{\partial x_j}-C^*\frac{K^{3/2}}{l}+\frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_K}\frac{\partial K}{\partial x_i}\right)+\nu\frac{\partial^2 K}{\partial x_i\partial x_i} \qquad (5.32)$$

$$\frac{\partial(Kl)}{\partial t}+\overline{u}_i\frac{\partial(Kl)}{\partial x_i}=\frac{\partial}{\partial x_i}\left[\left(\nu+\beta_1K^{1/2}l\right)\frac{\partial}{\partial x_i}(Kl)+\beta_2K^{3/2}l\frac{\partial l}{\partial x_i}\right]-\beta_3 l\tau'_{ij}\frac{\partial\overline{u}_i}{\partial x_i}-\beta_4K^{3/2} \qquad (5.33)$$

$$\tau'_{ij}=\frac{2}{3}K\delta_{ij}-\nu_T\left(\frac{\partial\overline{u}_i}{\partial x_j}+\frac{\partial\overline{u}_j}{\partial x_i}\right) \qquad (5.34)$$

$$\nu_T=K^{1/2}l \qquad (5.35)$$

where C^*, β_1, β_2, β_3 and β_4 are empirical constants. Equation (5.33) is derived by integrating the contracted form of a modeled transport equation for the two-point velocity-correlation tensor R'_{ij} (see Wolfshtein 1970). The Mellor's group utilized the $K-l$ model for the solution of a variety of engineering and geophysical fluid-dynamics problems (see Mellor & Herring 1973, Mellor & Yamada 1974).

The turbulent macroscale is based on the integral length scale (5.30) rather than on the turbulent dissipation rate, that only formally determines the turbulent microscale.
For homogeneous flows it can be shown that the $K-l$ model is equivalent to a $K-\varepsilon$ model with slightly different values of the constants C_μ, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ (Speziale 1990). Moreover, the modeled transport equation (5.33) for l has problems similar to the modeled transport equation (5.20) for ε for what integration to solid boundaries is concerned.

5.6 $K-\omega$ Model

The $K-\omega$ model (Wilcox & Traci 1976, Wilcox 1988) is based [besides on equations (5.17-5.18)] on the solution of two modeled transport equations, one for the turbulent kinetic energy K in the form of one-equation closure models, and another for the reciprocal turbulent time scale ω:

$$\omega = \frac{\varepsilon}{K}. \tag{5.36}$$

The typical form of the $K-\omega$ model and in particular of the modeled transport equation for ω [equation (5.38)] is:

$$\frac{\partial K}{\partial t} + \overline{u}_i \frac{\partial K}{\partial x_i} = -\tau'_{ij} \frac{\partial \overline{u}_i}{\partial x_j} - C^* \frac{K^{3/2}}{l} + \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_K} \frac{\partial K}{\partial x_i} \right) + \nu \frac{\partial^2 K}{\partial x_i \partial x_i} \tag{5.37}$$

$$\frac{\partial \omega}{\partial t} + \overline{u}_i \frac{\partial \omega}{\partial x_i} = -\gamma_1 \frac{\omega}{K} \tau'_{ij} \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_\omega} \frac{\partial \omega}{\partial x_i} \right) - \gamma_2 \omega^2 + \nu \frac{\partial^2 \omega}{\partial x_i \partial x_i} \tag{5.38}$$

$$\tau'_{ij} = \frac{2}{3} K \delta_{ij} - \nu_T \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) \tag{5.39}$$

$$\nu_T = \gamma^* \frac{K}{\omega}, \tag{5.40}$$

where γ_1, γ_2, γ^* and σ_ω are constants. Equation (5.38) is obtained by making the same kind of assumptions for the modeling the exact transport equation for ω that have been made for the derivation the modeled transport equation (5.20) for ε.

In *homogeneous turbulent flows* there are some differences between the $K-\varepsilon$ and $K-\omega$ models, in particular for what the treatment of the transport terms is concerned. The $K-\varepsilon$ model is based on a gradient transport hypothesis for ε, while the $K-\omega$ model uses the same hypothesis for ω instead. Moreover, in the integration of turbulence models to a wall, in spite of the fact that ω is singular at a solid boundary, the $K-\omega$ model requires less empirical damping (Wilcox 1988).

5.7 Nonlinear Eddy-Viscosity Models

In an attempt to overcome the deficiencies and limits of the standard $K-\varepsilon$ model outlined at end of Section 5.4, models for the Reynolds-stress tensor nonlinear in the mean strains, have been developed.

By also considering the second-order terms in the Taylor series expansions (3.2-3.4) subjected to invariance under the extended group of transformations (2.39), a more general representation of the Reynolds-stress tensor is obtained (Speziale 1991):

$$\tau'_{ij} = \frac{2}{3}K\delta_{ij} - 2\frac{l_0^2}{\tau_0}\overline{S}_{ij} + \alpha_1 l_0^2\left(\overline{S}_{ik}\overline{S}_{kj} - \frac{1}{3}\overline{S}_{mn}\overline{S}_{mn}\delta_{ij}\right)$$

$$+\alpha_2 l_0^2\left(\overline{W}_{ik}\overline{W}_{kj} - \frac{1}{3}\overline{W}_{mn}\overline{W}_{mn}\delta_{ij}\right) + \alpha_3 l_0^2\left(\overline{S}_{ik}\overline{W}_{jk} + \overline{S}_{jk}\overline{W}_{ik}\right)$$

$$+\alpha_4 l_0^2\left(\frac{\partial \overline{S}_{ij}}{\partial t} + \overline{\boldsymbol{u}}\cdot\nabla\overline{S}_{ij}\right) \tag{5.41}$$

where $\overline{S}_{ij}$ is the mean strain-rate tensor (2.27):

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

and $\overline{W}_{ij}$ is the mean-vorticity tensor (2.42):

$$\overline{W}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i}\right).$$

The constants α_1, α_2, α_3 and α_4 are dimensionless and for $\alpha_i \to 0$ equation (5.41) collapses to the standard model (3.13-3.15)(*).

When $\alpha_4 = 0$ the deviatoric part of τ'_{ij} in (5.41) is of the general form:

$$\tau'^{D}_{ij} = A_{ijkl}\frac{\partial \overline{u}_k}{\partial x_l} \tag{5.42}$$

where A_{ijkl} depends algebraically on the mean-velocity gradients. Thus, the term *anisotropic eddy-viscosity model* for the Reynolds-stress tensor started to be used in the literature.

(*) Note that the first term on the rhs of model equation (5.41) is the isotropic part τ'^{I}_{ij} of τ'_{ij}, while the second term on the rhs of (5.41) is the deviatoric part of τ'_{ij} that, according to the Boussinesq hypothesis (3.15) is:

$$\tau'^{D}_{ij} = -2\frac{l_0^2}{\tau_0}\overline{S}_{ij} = -\nu_T\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right).$$

In the limit as A_{ijkl} becomes isotropic, the Boussinesq hypothesis is recovered, so that equation (3.15/5.21) is also named *isotropic eddy viscosity model*.

The most proper characterization of the structure of the above models is probably that of *nonlinear* or *viscoelastic* corrections to eddy-viscosity models (Speziale 1991).

Lumley (1970) was among the first to work on the development of this kind of models (with $\alpha_4 = 0$) building up the length and time scales from the turbulent kinetic energy, the turbulent dissipation rate and the invariants of $\overline{S}_{ij}$ and $\overline{W}_{ij}$.

Saffman (1977) proposed a class of anisotropic models to be solved in conjunction with modeled transport equations for K and ω^2 [where ω is the inverse time scale (5.36)].

Pope (1975) and Rodi (1976) developed alternative anisotropic eddy-viscosity models where the models for the Reynolds-stress tensor are derived from an analysis of the Reynolds-stress transport equation by making an equilibrium hypothesis (Chapter 7).

Yoshizawa (1984, 1987) derived a complete two-equation model with a nonlinear correction to the eddy viscosity in the full form (5.41) by means of a two-scale Direct Interaction Approximation (*DIA*) method. To derive this model the Kraichnan's *DIA* method (Kraichnan 1964) is combined with a scale-expansion technique where the slow variations of the mean field are distinguished from the fast variations of the fluctuating field with the use of a scale parameter. The length and time scales are built up from the turbulent kinetic energy and the turbulent dissipation rate. The derived modeled transport equations are identical to (5.19) and (5.20), except for the fact that additional higher-order cross-diffusion terms are added. The values of the constants are calculated directly from the theory, as happened with the *RNG*-derived $K-\varepsilon$ model. Problems related to the values of the constants are reported by Nisizima & Yoshizawa (1987).

Speziale (1987*b*) developed the so-called *nonlinear* $K-\varepsilon$ *model*, based on a simplified version of (5.41) obtained by invoking the constraint of *MFI* in the limit of two-dimensional turbulence. In this model, where length and time scales are built up from the turbulent kinetic energy and the dissipation rate, the Reynolds-stress tensor assumes the form:

$$\tau'_{ij} = \frac{2}{3}K\delta_{ij} - 2C_\mu \frac{K^2}{\varepsilon}\overline{S}_{ij} - 4C_D C_\mu^2 \frac{K^3}{\varepsilon^2}\left(\overline{S}_{ik}\overline{S}_{kj} - \frac{1}{3}\overline{S}_{mn}\overline{S}_{mn}\delta_{ij}\right)$$

$$- 4C_E C_\mu^2 \frac{K^3}{\varepsilon^2}\left(\overset{\circ}{\overline{S}}_{ij} - \frac{1}{3}\overset{\circ}{\overline{S}}_{mm}\delta_{ij}\right) \tag{5.43}$$

where:

$$\overset{\circ}{\overline{S}}_{ij} = \frac{\partial \overline{S}_{ij}}{\partial t} + \overline{u}_k \frac{\partial \overline{S}_{ij}}{\partial x_k} - \frac{\partial \overline{u}_i}{\partial x_k}\overline{S}_{kj} - \frac{\partial \overline{u}_j}{\partial x_k}\overline{S}_{ki} \tag{5.44}$$

is the frame-indifferent Oldroyd derivative of $\overline{S}_{ij}$ and the constants assume the values:

$$C_\mu = 0.09\,, \quad C_D = 1.68\,, \quad C_E = 1.68\,. \tag{5.45}$$

Equation (5.43) can be seen as an approximation for turbulent flows where $\tau_0/T_0 << 1$ since *MFI* becomes exact in the limit $\tau_0/T_0 \to 0$.

Speziale (1987*b*) and Speziale & Mac Giolla Mhuiris (1989*a*) showed that this model gives rather accurate predictions for the normal Reynolds-stress anisotropies in turbulent channel flow and homogeneous shear flow, where the standard $K-\varepsilon$ model erroneously predicts that $\tau_{xx} = \tau_{yy} = \tau_{zz} = 2K/3$. As a result, the nonlinear $K-\varepsilon$ model is able to predict turbulent secondary flows in noncircular ducts unlike the standard $K-\varepsilon$ model that erroneously predicts a unidirectional mean turbulent flow (see also Chapter 8). Comparably good predictions of turbulent secondary flows in a rectangular duct have been obtained earlier by Launder & Ying (1972), Gessner & Po (1976) and Demuren & Rodi (1984) using the nonlinear algebraic Reynolds-stress model of Rodi (1976) (Chapter 7). It must be mentioned that, in spite of some noticeable improvements, the nonlinear $K-\varepsilon$ model suffer of some of the same deficiencies of the simpler two-equation models.

5.8 Summary

Overall, in spite of the fact that two-equation closure models constitute a simple and complete class of turbulence models, they still present significant deficiencies, so that the results of their applications in complex turbulent flows remain rather unsatisfactory. The two-equation closure models that incorporate the eddy-viscosity concept suffer of the following major deficiencies:

i) they are not able to properly account for streamline curvature, rotational strains and other body-force effects;

ii) they neglect non local and history effects on Reynolds-stress anisotropies.

The majority of these deficiencies are related to the assumption that there is a clear-cut separation of scales at the second-moment level (that of the Reynolds-stress tensor). As an example, in homogeneous shear flow, the equilibrium value of the ratio between the fluctuating and the mean time scale is for the $K-\varepsilon$ model:

$$\frac{\tau_0}{T_0} = \frac{SK}{\varepsilon} \cong 4.8\,, \tag{5.46}$$

a value remarkably different from assumption (3.11):

$$\frac{\tau_0}{T_0} << 1$$

that constitutes one of the basis for the derivation of the $K-\varepsilon$ model.

Some of the above-mentioned deficiencies of two-equation closure models can be in part overcome with the use of two-equation models with nonlinear algebraic corrections to the eddy-viscosity representation, derived from the Reynolds-stress transport equation (Chapter 7). Major improvements can be achieved by implementing full high-order closures, the simplest of which are second-order closure models (Chapter 6).

Chapter 6

Stress-Equation Models

More sophisticated one-point turbulence models are the second-order (or second-moment) closure models, that involve the solution of modeled equations for the fully inhomogeneous Reynolds-stress transport equation and for the dissipation-rate transport equation. For this reason this class of turbulence models is called $\tau'_{ij} - \varepsilon$ models. No approximations to the Reynolds-stress term on the right-hand side of equation (2.23) are required in this case.

6.1 Dissipation-Rate Transport Equation

For what ε is concerned, the turbulent dissipation rate is typically taken to be a solution of the modeled transport equation:

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} = -C_{\varepsilon 1} \frac{\varepsilon}{K} \tau'_{ij} \frac{\partial \overline{u}_i}{\partial x_j} + C_\varepsilon \frac{\partial}{\partial x_i} \left(\frac{K}{\varepsilon} \tau'_{ij} \frac{\partial \varepsilon}{\partial x_j} \right) - C_{\varepsilon 2} \frac{\varepsilon^2}{K} + \nu \frac{\partial^2 \varepsilon}{\partial x_i \partial x_i} \tag{6.1}$$

where:

$$C_{\varepsilon 1} = 1.44, \quad C_\varepsilon = 0.15, \quad C_{\varepsilon 2} = 1.92 \tag{6.2}$$

Equation (6.1) is identical to the ε-transport equation (5.20) used in the standard $K - \varepsilon$ model except for the fact that the model for the turbulent diffusion-of-dissipation term is anisotropic.

6.2 Reynolds-Stress Transport Equation

Recalling the fluctuating momentum equation in operator form (Speziale 1991), equation (2.36):

$$Lu_i' = 0,$$

the Reynolds-stress transport equation is obtained from the second moment:

$$\overline{u_i' L u_j' + u_j' L u_i'} = 0 . \tag{6.3}$$

More explicitly, equation (6.3) takes the form (Hinze 1975):

$$\frac{\partial \tau_{ij}'}{\partial t} + \overline{u}_k \frac{\partial \tau_{ij}'}{\partial x_k} = -\tau_{ik}' \frac{\partial \overline{u}_j}{\partial x_k} - \tau_{jk}' \frac{\partial \overline{u}_i}{\partial x_k}$$

$$+ \overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} - 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} - \frac{\partial}{\partial x_k}\left(\overline{u_i' u_j' u_k'} + \overline{p' u_i'}\delta_{jk} + \overline{p' u_j'}\delta_{ik}\right) + \nu \frac{\partial^2 \tau_{ij}'}{\partial x_k \partial x_k} . \tag{6.4}$$

Equation (6.4) can be written in the form:

$$\frac{\partial \tau_{ij}'}{\partial t} + \overline{u}_k \frac{\partial \tau_{ij}'}{\partial x_k} = P_{ij} + \Pi_{ij} - \varepsilon_{ij} - \frac{\partial D_{ijk}}{\partial x_k} \tag{6.5}$$

where the term D_{ijk}:

$$D_{ijk} = C_{ijk} - \nu \frac{\partial \tau_{ij}'}{\partial x_k} \tag{6.6}$$

of diffusive nature (equal to zero for homogeneous turbulence), is responsible for the turbulent energy transfer from regions of higher to lower turbulence intensity.

More in particular, equation (6.5) can be written as:

$$\frac{\partial \tau'_{ij}}{\partial t} + \overline{u}_k \frac{\partial \tau'_{ij}}{\partial x_k} = P_{ij} + \Pi_{ij} - \varepsilon_{ij} - \frac{\partial C_{ijk}}{\partial x_k} + \nu \frac{\partial^2 \tau'_{ij}}{\partial x_k \partial x_k}, \tag{6.7}$$

representing a balance between the temporal derivative of the Reynolds-stress tensor and the convective transport term on the left-hand side of equation (6.7) and the following terms on the right-hand side:

i) the Reynols-stress production term P_{ij}:

$$P_{ij} = -\tau'_{ik} \frac{\partial \overline{u}_j}{\partial x_k} - \tau'_{jk} \frac{\partial \overline{u}_i}{\partial x_k} \tag{6.8}$$

that does not require any modeling in the formulation of stress-equation models;

ii) the pressure-strain correlation tensor Π_{ij}:

$$\Pi_{ij} = \overline{p' \left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right)}, \tag{6.9}$$

consisting of a one-point correlation between the fluctuating pressure and the fluctuating strain-rate tensor;

iii) the dissipation-rate correlation tensor ε_{ij}:

$$\varepsilon_{ij} = 2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}}, \tag{6.10}$$

viscous in origin and consequently dominated by the small, dissipative scales of turbulence;

iv) the gradient of the turbulent diffusion correlation $\partial C_{ijk} / \partial x_k$, where the turbulent diffusion is:

$$C_{ijk} = \overline{u_i' u_j' u_k'} + \overline{p' u_i'} \delta_{jk} + \overline{p' u_j'} \delta_{ik} ; \tag{6.11}$$

v) the term representing the viscous transport:

$$\nu \frac{\partial^2 \tau_{ij}'}{\partial x_k \partial x_k} , \tag{6.12}$$

that does not require any modeling in the formulation of stress-equation models.

Full second-order closure models are based on the fully inhomogeneous Reynolds-stress transport equation (6.4), that actually represents six partial differential equations for the independent components of the Reynolds-stress tensor τ_{ij}'.

Key physical processes are naturally present in the Reynolds-stress transport equation. Relaxation effects, with possibly different characteristic times for the different components of the Reynolds stresses in the response of turbulence to external forcing or changes in geometry, are allowed. Equation (6.4) incorporates terms for the convection and diffusion of Reynolds stresses, so that second-order closure models are able to account for nonlocal and history effects. Moreover, equation (6.4) incorporates explicit convection and production terms (unmodeled in the formulation of stress-equation models) that automatically adapt themselves in flows with streamline curvatures or system rotations by adding scale factors or Coriolis terms. Thus, second-order closure models facilitate the computation of flows with extra strains imposed on the primary flow, giving in general better results in complex turbulent flows with respect to other classes of models. The creation of turbulence anisotropy and the consequence of such anisotropy on the behavior of those components of the Reynolds-stress tensor that dominate the evolution of the mean flow, are accounted for.

In the case of noninertial reference frame, equation (6.4/6.7) assumes the form:

$$\frac{\partial \tau'_{ij}}{\partial t} + \overline{u}_k \frac{\partial \tau'_{ij}}{\partial x_k} = -\tau'_{ik} \frac{\partial \overline{u}_j}{\partial x_k} - \tau'_{jk} \frac{\partial \overline{u}_i}{\partial x_k} + \Pi_{ij} - \varepsilon_{ij} - \frac{\partial C_{ijk}}{\partial x_k} + \nu \frac{\partial^2 \tau'_{ij}}{\partial x_k \partial x_k}$$

$$-2e_{mki}\Omega_m \tau'_{jk} - 2e_{mkj}\Omega_m \tau'_{ik} \qquad (6.13)$$

where Ω_m is the rate-of-rotation of the reference frame.

6.3 Modeled Reynolds-Stress Transport Equation

In order to develop second-order closure models the higher-order correlation terms Π_{ij}, ε_{ij} and C_{ijk} in (6.7), have to be modeled. Far from solid walls, these terms are usually modeled as follows:

i) the pressure-strain correlation Π_{ij} and the dissipation-rate correlation ε_{ij} are usually modeled (even for nonhomogeneous turbulent-flow cases) on the basis of the concept of homogeneous turbulence, where the departures from the isotropy condition (i.e. departures of $\overline{u'_i u'_j}$ from the isotropic form $u'^2 \delta_{ij}$) are assumed to be small enough to allow a Taylor-series expansion about a state of isotropic turbulence;

ii) the third-order transport term C_{ijk} is modeld by using a gradient transport hypothesis, based on assumptions (3.11-3.12) that there is a clear-cut separation of scales between mean and fluctuating fields.

Near solid walls, wall functions or damping methods have to be used.

It has to be noted that, in the development of second-moment closure models the assumption of separation of scales is made only at the third-moment level. This fact probably supports the main reason for the second-order closure models to exist i.e.,

since rough approximations for the second moments ($\overline{u_i'u_j'}$) in eddy-viscosity models give acceptable approximations for the first-order moments ($\overline{u}_i$ and $\overline{p}$), it may happen that rough approximations for the third-order moments give acceptable approximations for second-order moments in Reynolds-stress transport models.
Closures belonging to the stress-equation class of models based on the closure of the three higher-order correlation terms in the Reynolds-stress transport equation (6.7), are referred to as $\tau_{ij}' - \varepsilon$ models. In addition to equations (2.23-2.24) describing the evolution of the mean field and a modeled version of equation (6.7) for τ_{ij}', it includes a modeled transport equation for ε [equation (6.1)] analogous to that used in the two-equation $K-\varepsilon$ model.

6.3.1 Pressure-Strain Correlation

The pressure-strain correlation has zero trace in the case of incompressible fluid and thus leads to a redistribution between various components of the Reynolds-stress tensor without any direct change in the turbulent kinetic energy.
From equation (2.30) the following equation for the fluctuating pressure in incompressible flow is obtained:

$$\frac{\partial^2 p'}{\partial x_i \partial x_i} = -\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i} - 2\frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial u_j'}{\partial x_i}. \tag{6.14}$$

Almost all the models for the pressure-strain correlation that have been incorporated in second-order closure models are based on the assumption of *local homogeneity*.
The formal solution of (6.14), based on Green's functions in unbounded domain, in the case of homogeneous turbulent flow gives:

$$\Pi_{ij} = A_{ij} + M_{ijkl}\frac{\partial \overline{u}_k}{\partial x_l} = \Pi_{ij}^S + \Pi_{ij}^R \tag{6.15}$$

where:

$$A_{ij} = \frac{1}{4\pi}\overline{\int\int\limits_{-\infty}^{+\infty}\int \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\frac{\partial^2 u'_k u'_l}{\partial y_k \partial y_l}\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)d^3 y} \tag{6.16}$$

$$M_{ijkl} = \frac{1}{2\pi}\overline{\int\int\limits_{-\infty}^{+\infty}\int \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\frac{\partial u'_l}{\partial y_k}\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)d^3 y}\,. \tag{6.17}$$

In equation (6.15) the first term on the right-hand side is referred to as the *slow pressure-strain* Π_{ij}^S, while the second term on the rhs of (6.15) is the *rapid pressure-strain* Π_{ij}^R. Nondimensional quantities [on the rhs of (6.18-6.19)] can be defined as follows:

$$A_{ij} = \varepsilon \mathrm{A}_{ij} \tag{6.18}$$

$$M_{ijkl} = K\mathrm{M}_{ijkl}\,. \tag{6.19}$$

It has been shown that A_{ij} and M_{ijkl} are functionals, in time and wavenumber space, of the energy-spectrum tensor (Weinstock 1981, Reynolds 1987).
Thus, in one-point closures, algebraic models for A_{ij} and M_{ijkl} are adopted, that are functionals of the Reynolds-stress anisotropy tensor $\boldsymbol{b}$ (5.12):

$$b_{ij} = \frac{1}{2K}\left(\tau'_{ij} - \frac{2}{3}K\delta_{ij}\right)$$

and of the turbulent dissipation rate, obtained by using dimensional arguments and the fact that Π_{ij} vanishes in the limit of isotropic turbulence.
The pressure-strain correlation model becomes:

$$\Pi_{ij} = \varepsilon \mathrm{A}_{ij}(\boldsymbol{b}) + K \mathrm{M}_{ijkl}(\boldsymbol{b}) \frac{\partial \overline{u}_k}{\partial x_l}. \tag{6.20}$$

In practice, all the models for the pressure-strain correlation that have been used in second-order closure models are of the type (6.20).

The first and second terms on the right-hand side of model equation (6.20) represent models for the the slow pressure-strain and for the rapid pressure-strain terms, respectively.

The slow part (equal to zero in isotropic turbulence) is thought as tending to reduce turbulence anisotropies and corresponds to the nonlinear term:

$$-\frac{\partial u_i'}{\partial x_j} \frac{\partial u_j'}{\partial x_i} \tag{6.21}$$

on the right-hand side of the Poisson equation for the pressure fluctuations (6.14).

The rapid part corresponds to the linear term:

$$-2 \frac{\partial \overline{u}_i}{\partial x_j} \frac{\partial u_j'}{\partial x_i} \tag{6.22}$$

in (6.14) that represents the effects of the mean-flow distorsion of turbulence. The presence of a linear term in (6.14) is attractive in terms of possible analytical simplifications in the operations of modeling.

Π_{ij}^S is the limit of the pressure-strain correlation in the limit of the relaxational flows, while Π_{ij}^R is the Rapid Distortion Theory (*RDT*) limit of the pressure-strain correlation. However, the velocity fluctuation u_i' is, in general, a functional of the mean velocity gradient in homogeneous turbulence, a fact that leads to an explicit dependence of both Π_{ij}^S and Π_{ij}^R on the mean velocity gradient history and a consequent ambiguity in the decomposition into slow and rapid parts (see Speziale *et al.* 1992).

Rotta (1951) was probably the first to develop a model for the pressure-strain correlation. He ignored the rapid part and noted that the slow part Π_{ij}^{S} contributed to the return to isotropy in turbulent flows where the mean strains are suddenly removed. This brought to the simple return-to-isotropy pressure-strain model:

$$\Pi_{ij} = -C_1 \varepsilon b_{ij} \tag{6.23}$$

where ε is the turbulent dissipation rate, b_{ij} is the anisotropy of the Reynolds-stress tensor (5.12) and C_1 has been named the Rotta constant. The relaxation of Reynolds stresses in homogeneous turbulent flows when the mean strain is removed, is an ideal problem useful for the evaluation of models for Π_{ij}^{S}. The Rotta model predicts that the isotropy decays to zero according to a power law, giving a return to isotropy as seen in experiments with:

$$C_1 \approx 2.8\,. \tag{6.24}$$

Later, Rotta (1972) introduced a model for the rapid term where M_{ijkl} is taken to be an isotropic tensor (the lowest-order contribution to this term). This brought to a more complete pressure-strain model (Rotta 1972):

$$\Pi_{ij} = -C_1 \varepsilon b_{ij} + C_2 K \overline{S}_{ij} \tag{6.25}$$

where K is the average turbulent kinetic energy, C_2 is also a constant and $\overline{S}_{ij}$ is the mean strain-rate tensor (2.27):

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right).$$

Model (6.25) has been used for the calculation of a number of engineering and geophysical flows by Mellor & Herring (1973) and Mellor & Yamada (1974).

Launder, Reece & Rodi (1975) were the first to obtain the more general form of equation (6.20), linear in b_{ij}. Using additional symmetry and normalization constraints they obtained:

$$\mathrm{M}_{iikl} = \mathrm{M}_{ijkk} = 0 \tag{6.26}$$

$$\mathrm{M}_{ikkj} = \mathrm{M}_{ikjk} = 2\tau'_{ij}\,, \tag{6.27}$$

that follow rigorously from the continuity equation and the properties of the two-point double velocity correlation tensor. As a consequence, the rapid pressure-strain model involves only one free coefficient.

The model derived by Launder, Reece & Rodi (1975) (the *LRR* model) for the pressure-strain correlation is:

$$\Pi_{ij} = -C_1 \varepsilon b_{ij} + \frac{4}{5} K \overline{S}_{ij} + \frac{(18C_2 + 12)}{11} K \left(b_{ik}\overline{S}_{jk} + b_{jk}\overline{S}_{ik} - \frac{2}{3} b_{kl}\overline{S}_{kl}\delta_{ij} \right)$$

$$+ \frac{(20 - 14C_2)}{11} K \left(b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik} \right) \tag{6.28}$$

where $\overline{W}_{ij}$ is the mean-vorticity tensor (2.42):

$$\overline{W}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i} \right)$$

and, based on homogeneous turbulence experiments:

$$C_1 = 3.0\,, \quad C_2 = 0.4\,. \tag{6.29}$$

A simplified form of model (6.28) has also been proposed by Launder, Reece & Rodi (1975) (the *IP* model) as follows:

$$\Pi_{ij} = -C_1 \varepsilon b_{ij} + C_2 \left(P_{ij} - \frac{1}{3} P_{kk} \delta_{ij} \right) \qquad (6.30)$$

where P_{ij} is the Reynolds-stress production tensor (6.8):

$$P_{ij} = -\tau'_{ik} \frac{\partial \overline{u}_j}{\partial x_k} - \tau'_{jk} \frac{\partial \overline{u}_i}{\partial x_k}$$

and the constants assume the values:

$$C_1 = 3.6\,, \quad C_2 = 0.6\,. \qquad (6.31)$$

The term proportional to $4K\overline{S}_{ij}/5$ in (6.28) governs the initial response of isotropic turbulence to a mean strain (Crow 1968).

Models for the pressure-strain correlation linear in b_{ij}, were found soon to be incapable of describing the behavior of a broad number of turbulent flows, so that nonlinear models for the pressure-strain correlation term have been developed (Section 6.5).

6.3.2 Dissipation-Rate Correlation

The modeling of the rate-of-dissipation tensor, divided into isotropic and deviatoric parts:

$$\varepsilon_{ij} = \varepsilon^I_{ij} + \varepsilon^D_{ij} \qquad (6.32)$$

at high-Reynolds-number turbulent flows, is usually based on the Kolmogorov hypothesis of *local isotropy*, so that:

$$\varepsilon_{ij}^{D} = 0 \tag{6.33}$$

and the dissipation-rate tensor reduces to the isotropic part:

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} \tag{6.34}$$

where the scalar dissipation rate $\varepsilon = \varepsilon_{ii}/2$ has the form (5.5):

$$\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}.$$

Thus, the modeled transport equation (6.1) is used, to be solved for the scalar dissipation rate ε.

Near solid boundaries, anisotropic corrections to (6.34) have been proposed, typically of the algebraic form (Hanjalić & Launder 1976):

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} + 2\varepsilon f_s b_{ij} \tag{6.35}$$

where f_s is a function of the turbulence Reynolds number Re_T:

$$Re_T = \frac{K^2}{\nu\varepsilon}. \tag{6.36}$$

Equation (6.35), that can be seen as a first-order Taylor-series expansion about a state of isotropic turbulence, is solved in conjunction with equation (6.1), where the coefficients are taken to be functions of Re_T as a solid boundary is approached (Hanjalić & Launder 1976).

6.3.3 Diffusion Correlation

A model for the third-order diffusion correlation term C_{ijk} is also needed. C_{ijk} is a third-order moment and the simplifying assumpion of gradient transport is typically made. All the commonly used second-order closures are based on models of C_{ijk} of the form:

$$C_{ijk} = D_{ijklmn} \frac{\partial \tau'_{lm}}{\partial x_n} \tag{6.37}$$

where the diffusion tensor D_{ijklmn} can depend anisotropically on τ'_{ij}.

For many incompressible turbulent flows, the pressure-diffusion terms:

$$\overline{p'u'_i}\delta_{jk}, \ \overline{p'u'_j}\delta_{ik}$$

in C_{ijk} [equation (6.11)] can be neglected with respect to the triple velocity correlation:

$$\overline{u'_i u'_j u'_k}.$$

Then, the symmetry of C_{ijk} under an interchange of any of its three indices, give the form (Hanialić & Launder 1972):

$$C_{ijk} = -C_s \frac{K}{\varepsilon}\left(\tau'_{im} \frac{\partial \tau'_{jk}}{\partial x_m} + \tau'_{jm} \frac{\partial \tau'_{ik}}{\partial x_m} + \tau'_{km} \frac{\partial \tau'_{ij}}{\partial x_m} \right). \tag{6.38}$$

Equation (6.38) is sometimes used in the isotropized form (Mellor & Herring 1973):

$$C_{ijk} = -\frac{2}{3} C_s \frac{K^2}{\varepsilon}\left(\frac{\partial \tau'_{jk}}{\partial x_i} + \frac{\partial \tau'_{ik}}{\partial x_j} + \frac{\partial \tau'_{ij}}{\partial x_k} \right) \tag{6.39}$$

where the constant C_s (Launder, Reece & Rodi 1975) is:

$$C_s \approx 0.11\,. \tag{6.40}$$

Similar models for C_{ijk} have been also derived by Lumley (1978).

6.4 Stress-Equation Turbulence Models

Overall, a typical form of the system of the eleven partial differential equations to be solved in the stress-equation class of models for the calculation of $\overline{u}_i$, $\overline{p}$, τ'_{ij} and ε is:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau'_{ij}}{\partial x_j} \tag{6.41}$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{6.42}$$

$$\frac{\partial \tau'_{ij}}{\partial t} + \overline{u}_k \frac{\partial \tau'_{ij}}{\partial x_k} = -\tau'_{ik} \frac{\partial \overline{u}_j}{\partial x_k} - \tau'_{jk} \frac{\partial \overline{u}_i}{\partial x_k}$$

$$+ A_{ij} + M_{ijkl} \frac{\partial \overline{u}_k}{\partial x_l} - \frac{2}{3}\varepsilon \delta_{ij} - \varepsilon_{ij}^{D} - \frac{\partial}{\partial x_k}\left(D_{ijklmn} \frac{\partial \tau'_{lm}}{\partial x_n} \right) + \nu \frac{\partial^2 \tau'_{ij}}{\partial x_k \partial x_k} \tag{6.43}$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} = -C_{\varepsilon 1} \frac{\varepsilon}{K} \tau'_{ij} \frac{\partial \overline{u}_i}{\partial x_j} + C_{\varepsilon} \frac{\partial}{\partial x_i}\left(\frac{K}{\varepsilon} \tau'_{ij} \frac{\partial \varepsilon}{\partial x_j} \right) - C_{\varepsilon 2} \frac{\varepsilon^2}{K} + \nu \frac{\partial^2 \varepsilon}{\partial x_i \partial x_i} \tag{6.44}$$

where the modeled terms in equation (6.43) can assume different specific forms according to the discussion made above. The constants in (6.44) assume the values given in (6.2):

$$C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon} = 0.15, \quad C_{\varepsilon 2} = 1.92.$$

6.5 Nonlinear Pressure-Strain Correlation Models

Models for the pressure-strain correlation (6.9) of the type (6.20) linear in b_{ij}, have demonstrated to be not adequate in describing turbulent flows in a remarkable number of cases (see also Chapter 8).

In the return-to-isotropy problem (a problem related to the slow pressure-strain) the linear models can lead to substantial errors in the rate of return and in the trajectory of return in the phase space described by the invariants II and III of b_{ij}.

In particular the linear model for the pressure-strain correlation term, in the return-to-isotropy problem predicts that:

i) each component of the anisotropy tensor relaxes at the same rate;

ii) the decay rate is independent of the initial state of anisotropy.

In disagreement with these properties of the linear model, it is experimentally observed that the rate of return to isotropy decreases with increasing values of the third invariant III, and that different components have different rates of relaxation.

Lumley (1978) developed a general form of the slow pressure-strain correlation based on the Cayley-Hamilton theorem:

$$\Pi_{ij}^{S} = -C_1 \varepsilon b_{ij} + C_2 \varepsilon \left(b_{ik} b_{kj} - \frac{1}{3} b_{kl} b_{kl} \delta_{ij} \right) \tag{6.45}$$

where the coefficients C_1 and C_2 can be nonlinear functions of the invariants II ($II = b_{ik} b_{ik}$) and III ($III = b_{ik} b_{kj} b_{ji}$) of b_{ij}.

Sarkar & Speziale (1990) showed that the quadratic model:

$$\Pi_{ij}^{S} = -C_1 \varepsilon b_{ij} + C_2 \varepsilon \left(b_{ik} b_{kj} - \frac{1}{3} b_{kl} b_{kl} \delta_{ij} \right) \tag{6.46}$$

with:

$$C_1 = 3.4\,, \quad C_2 = 3(C_1 - 2) = 4.2 \tag{6.47}$$

is able to capture the trends in a broad number of return-to-isotropy experiments.

Linear models for the pressure-strain correlation were also found to be incapable of describing turbulent flows with a broad selection of mean velocity gradients (the rapid term is involved in this case). The recognition of this fact and the attempt to satisfy the realizability constraint, stimulated the development of nonlinear models for the rapid pressure-strain term. Realizability was introduced by Schumann (1977) and its use as a constraint on turbulence models has been developed by Lumley (1978, 1983). Realizability requires that the predicted normal components of the Reynolds-stress tensor be nonnegative in all situations (Section 2.4).

Two models have been developed on the basis of realizability constraints [$\overline{S}_{ij}$ is the mean strain-rate tensor (2.27), $\overline{W}_{ij}$ is the mean-vorticity tensor (2.42)].

The model of Shih & Lumley (1985) (the *SL* model):

$$\Pi_{ij} = -C_1 \varepsilon b_{ij} + \frac{4}{5} K \overline{S}_{ij} + 12\alpha_5 K \left(b_{ik}\overline{S}_{jk} + b_{jk}\overline{S}_{ik} - \frac{2}{3} b_{kl}\overline{S}_{kl}\delta_{ij} \right)$$

$$+ \frac{4}{3}(2 - 7\alpha_5) K \left(b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik} \right)$$

$$+ \frac{4}{5} K \left(b_{il} b_{lm}\overline{S}_{jm} + b_{jl} b_{lm}\overline{S}_{im} - 2 b_{ik}\overline{S}_{kl} b_{lj} - 3 b_{kl}\overline{S}_{kl} b_{ij} \right)$$

$$+ \frac{4}{5} K \left(b_{il} b_{lm}\overline{W}_{jm} + b_{jl} b_{lm}\overline{W}_{im} \right) \tag{6.48}$$

where:

$$C_1 = 2 + \frac{F}{9} exp\left(-7.77 \big/ \sqrt{Re_T}\right)\left\{72 \big/ \sqrt{Re_T} + 80.1 ln\left[1 + 62.4\left(-II + 2.3III\right)\right]\right\} \tag{6.49}$$

$$F = 1 + 9II + 27III \tag{6.50}$$

$$II = -\frac{1}{2} b_{ij} b_{ij}, \qquad III = \frac{1}{3} b_{ij} b_{jk} b_{ki} \tag{6.51}$$

$$Re_T = \frac{4}{9} \frac{K^2}{\nu\varepsilon} \tag{6.52}$$

$$\alpha_5 = \frac{1}{10}\left(1 + \frac{4}{5} F^{1/2}\right). \tag{6.53}$$

The model of Fu, Launder & Tselepidakis (1987) (the *FLT* model):

$$\Pi_{ij} = -C_1 \varepsilon b_{ij} + C_2 \varepsilon \left(b_{ik} b_{kj} - \frac{1}{3} b_{kl} b_{kl} \delta_{ij} \right)$$

$$+ \frac{4}{5} K \overline{S}_{ij} + 1.2K \left(b_{ik} \overline{S}_{jk} + b_{jk} \overline{S}_{ik} - \frac{2}{3} b_{kl} \overline{S}_{kl} \delta_{ij} \right)$$

$$+ \frac{26}{15} K \left(b_{ik} \overline{W}_{jk} + b_{jk} \overline{W}_{ik} \right)$$

$$+ \frac{4}{5} K \left(b_{ik} b_{kl} \overline{S}_{jl} + b_{jk} b_{kl} \overline{S}_{il} - 2 b_{ik} \overline{S}_{kl} b_{lj} - 3 b_{kl} \overline{S}_{kl} b_{ij} \right)$$

$$+ \frac{4}{5} K \left(b_{ik} b_{kl} \overline{W}_{jl} + b_{jk} b_{kl} \overline{W}_{il} \right)$$

$$-\frac{14}{5}K\left[8II\left(b_{ik}\overline{W}_{jk}+b_{jk}\overline{W}_{ik}\right)+12\left(b_{ik}b_{kl}\overline{W}_{lm}b_{mj}+b_{jk}b_{kl}\overline{W}_{lm}b_{mi}\right)\right] \tag{6.54}$$

where:

$$C_1=-120II\sqrt{F}-2\sqrt{F}+2\,,\qquad C_2=144II\sqrt{F}\,. \tag{6.55}$$

These models reduce to the linear form of the *LRR* model in the limit of small Reynolds-stress anisotropies.

There is a fundamental inconsistency with general expressions for the rapid pressure-strain term nonlinear in b_{ij}, that bring these models to failure. The rapid pressure-strain correlation is a linear functional of the energy spectrum tensor and this property is violated by models that are nonlinear in b_{ij}. From its definition [equation (6.19)]:

$$\mathrm{M}_{ijkl}\approx\iiint_{-\infty}^{+\infty}\frac{\kappa_i\kappa_j}{\kappa^2}E_{kl}(\boldsymbol{\kappa},t)d^3\kappa \tag{6.56}$$

that is linear in the energy spectrum tensor $E_{kl}(\boldsymbol{\kappa},t)$ (the Fourier transform of the two-point double velocity correlation tensor). Being the anisotropy of the Reynolds-stress tensor (5.12):

$$b_{ij}=\frac{1}{2K}\left(\tau'_{ij}-\frac{2}{3}K\delta_{ij}\right)$$

where:

$$\tau'_{ij}=\iiint_{-\infty}^{+\infty}E_{ij}(\boldsymbol{\kappa},t)d^3\kappa\,, \tag{6.57}$$

it follows that models for M_{ijkl} that are nonlinear in b_{ij} are also nonlinear in E_{ij}. Flows where *RDT* theory is a good approximation (that are linear) are examples where nonlinear pressure-strain models would fail because the principle of superposition is violated by such models. Moreover, according to Durbin & Speziale (1994), nonlinearity is not needed to avoid realizability violations. It has to be also noted that the full pressure-strain correlation is not linear in the energy spectrum tensor and there is no formal inconsistency in having some terms that are nonlinear in b_{ij}.

The most general nonlinear form of the pressure-strain correlation based on equation (6.20) is a simplified version of the general form originally derived by Lumley (1978) and Reynolds (1987) (see Speziale 1991). It can be shown that invariance under a change of coordinates in conjunction with the assumption of analyticity about the isotropic state $b_{ij} = 0$ brings (6.20) to assume the form:

$$\Pi_{ij} = a_0 \varepsilon b_{ij} + a_1 \varepsilon \left(b_{ik} b_{kj} - \frac{1}{3} II \delta_{ij} \right) + a_2 K \overline{S}_{ij}$$

$$+ \left(a_3 tr\boldsymbol{b} \cdot \overline{\boldsymbol{S}} + a_4 tr\boldsymbol{b}^2 \cdot \overline{\boldsymbol{S}} \right) K b_{ij} + \left(a_5 tr\boldsymbol{b} \cdot \overline{\boldsymbol{S}} + a_6 tr\boldsymbol{b}^2 \cdot \overline{\boldsymbol{S}} \right) K \left(b_{ik} b_{kj} - \frac{1}{3} II \delta_{ij} \right)$$

$$+ a_7 K \left(b_{ik} \overline{S}_{jk} + b_{jk} \overline{S}_{ik} - \frac{2}{3} tr\boldsymbol{b} \cdot \overline{\boldsymbol{S}} \delta_{ij} \right) + a_8 K \left(b_{ik} b_{kl} \overline{S}_{jl} + b_{jk} b_{kl} \overline{S}_{il} - \frac{2}{3} tr\boldsymbol{b}^2 \cdot \overline{\boldsymbol{S}} \delta_{ij} \right)$$

$$a_9 K \left(b_{ik} \overline{W}_{jk} + b_{jk} \overline{W}_{ik} \right) + a_{10} K \left(b_{ik} b_{kl} \overline{W}_{jl} + b_{jk} b_{kl} \overline{W}_{il} \right) \tag{6.58}$$

where:

$$a_i = a_i \left(II, III \right), \quad i = 0,1 \ldots 10 \tag{6.59}$$

$$II = b_{ij} b_{ij}, \quad III = b_{ik} b_{kl} b_{li}, \tag{6.60}$$

(tr is trace). The eigenvalues $b^{(\alpha)}$ of b_{ij} are bounded:

$$-\frac{1}{3} \le b^{(\alpha)} \le \frac{2}{3}, \quad \alpha = 1,2,3. \tag{6.61}$$

The model equation (6.58) can be viewed as a Taylor-series expansion in b_{ij} and, since the norm of b_{ij} can be small, nonlinearities are often limited to quadratic or cubic terms. For many engineering flows:

$$\|\boldsymbol{b}\|_2 = \left|b^{(\alpha)}\right|_{max} < 0.25 \tag{6.62}$$

so that a low-order Taylor-series truncation of (6.58) can provide an appropriate approximation.
To the first order in b_{ij}, one has(*):

$$\Pi_{ij} = -C_1\varepsilon b_{ij} + C_2K\overline{S}_{ij} + C_3K\left(b_{ik}\overline{S}_{jk} + b_{jk}\overline{S}_{ik} - \frac{2}{3}b_{mn}\overline{S}_{mn}\delta_{ij}\right) + C_4K\left(b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik}\right) \tag{6.63}$$

that is the form of the *LRR* model equation (6.28) with:

$$C_1 = 3.6, \quad C_2 = 0.8, \quad C_3 = 0.6, \quad C_4 = 0.6. \tag{6.64}$$

(*) Note that the representation of the slow pressure-strain correlation in model equation (6.63) is the Rotta (1951) return-to-isotropy model with the Rotta costant C_1 adjusted from 2.8 to 3.6.
Moreover from model equation (6.63) with $C_3 = C_4 = 0$, model (6.25) is recovered.

With reference to the general expression (6.58) Speziale, Sarkar & Gatski (1991) showed that two-dimensional mean turbulent flows have a special structure, implying that cubic and quadratic nonlinearities in the rapid pressure-strain correlation do not add any information that is not already present in the linear form. In such flows, the equilibrium values of II, III, b_{33} and P_K/ε [P_K is the turbulent kinetic-energy production term (4.6)] are invariant with respect to the values of the mean strain and rotation (the equilibrium values of b_{11}, b_{22} and b_{12} depend on the characteristics of the flow). The higher-order terms that contain the coefficients a_8 and a_{10} can be written as a linear combination of the lower-order terms in the expansion, with the multipliers in this combination being completely determined by b_{33} and III.

Therefore, the most general form in the case of two-dimensional flows in equilibrium with constant mean-velocity gradients, reduces to the simplified expression:

$$\Pi_{ij} = -C_1 \varepsilon b_{ij} + C_2 \varepsilon \left(b_{ik} b_{kj} - \frac{1}{3} b_{kl} b_{kl} \delta_{ij} \right)$$

$$+ C_3 K \overline{S}_{ij} + C_4 K \left(b_{ik} \overline{S}_{jk} + b_{jk} \overline{S}_{ik} - \frac{2}{3} b_{kl} \overline{S}_{kl} \delta_{ij} \right)$$

$$+ C_5 K \left(b_{ik} \overline{W}_{jk} + b_{jk} \overline{W}_{ik} \right). \tag{6.65}$$

The elimination of the higher-order terms facilitates the analytical development of models for plane, homogeneous flows. However, since the above-mentioned invariants have not been experimentally proven to be universal in these flows, there are some deficiencies in the group of models (6.65).

Speziale, Sarkar & Gatski (1991) developed a model (the *SSG* model) based on model equation (6.65) that is appropriate for flows with *mild to moderate nonequilibrium effects* (mild departures from equilibrium):

$$\Pi_{ij} = -\left(C_1\varepsilon + C_1^* P_K\right)b_{ij} + C_2\varepsilon\left(b_{ik}b_{kj} - \frac{1}{3}b_{kl}b_{kl}\delta_{ij}\right)$$

$$+\left(C_3 - C_3^* II^{1/2}\right)K\overline{S}_{ij} + C_4 K\left(b_{ik}\overline{S}_{jk} + b_{jk}\overline{S}_{ik} - \frac{2}{3}b_{kl}\overline{S}_{kl}\delta_{ij}\right)$$

$$+ C_5 K\left(b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik}\right) \tag{6.66}$$

where:

$$C_1 = 3.4\,, \quad C_1^* = 1.80\,, \quad C_2 = 4.2\,, \quad C_3 = \frac{4}{5} \tag{6.67}$$

$$C_3^* = 1.30\,, \quad C_4 = 1.25\,, \quad C_5 = 0.4\,, \quad II = b_{ij}b_{ij}\,. \tag{6.68}$$

This model has been obtained from a dynamical-system approach where the constants have been evaluated from the equilibrium states of homogeneous shear flow while mantaining the Crow (1968) constraint.

Some of the linear models, such as the *LRR* model and the *IP* model (Section 6.3.1), can be obtained from the *SSG* model with an appropriate choice of the constants.

The *LRR* model is recovered with:

$$C_1 = 3.0\,, \quad C_1^* = 0\,, \quad C_2 = 0\,, \quad C_3 = \frac{4}{5}$$

$$C_3^* = 0\,, \quad C_4 = 1.75\,, \quad C_5 = 1.31\,. \tag{6.69}$$

The *IP* model is recovered with:

$$C_1 = 3.6\,, \quad C_1^* = 0\,, \quad C_2 = 0\,, \quad C_3 = \frac{4}{5}$$

$$C_3^* = 0\,, \quad C_4 = 1.2\,, \quad C_5 = 1.2\,. \tag{6.70}$$

Speziale *et al.* (1991) showed that, with better calibration, quadratic models can give good results in homogeneous plane flows such as shear flow with and without system rotation, plane strain flow and axisymmetric expansion.

6.6 Summary

Full second-order closures account for more turbulence physics than lower-order closures, due to the explicit presence of relaxation effects, body-force terms and explicit anisotropy effects in the mean production term.

Note that all existing second-order closures are still limited in not being universally valid for all turbulent flows.

Chapter 7

Algebraic-Stress Models

Additional two-equation turbulence closure models can be obtained starting from the Reynolds-stress transport equation (Chapter 6). This group of models is characterized by the fact that nonlinear anisotropic eddy-viscosity models for the Reynolds-stress tensor are derived algebraically from an analysis of the Reynolds-stress transport equation. It is assumed that the turbulence is locally *homogeneous* and *in equilibrium*.

7.1 Reynolds-Stress Equation

For homogeneous turbulence, the Reynolds-stress transport equation (6.4/6.7) reduces to:

$$\frac{\partial \tau'_{ij}}{\partial t} = -\tau'_{ik}\frac{\partial \overline{u}_j}{\partial x_k} - \tau'_{jk}\frac{\partial \overline{u}_i}{\partial x_k} + \Pi_{ij} - \varepsilon_{ij} \tag{7.1}$$

(Π_{ij} is the pressure-strain correlation, ε_{ij} is the dissipation-rate correlation) and the modeled form (6.43) to:

$$\frac{\partial \tau'_{ij}}{\partial t} = -\tau'_{ik}\frac{\partial \overline{u}_j}{\partial x_k} - \tau'_{jk}\frac{\partial \overline{u}_i}{\partial x_k} + A_{ij} + M_{ijkl}\frac{\partial \overline{u}_k}{\partial x_l} - \varepsilon_{ij}\,. \tag{7.2}$$

By assuming the form of the deviatoric part of the dissipation-rate tensor as discussed in Durbin & Speziale (1991) (Kolmogorov local isotropy is not invoked):

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} + 2\varepsilon d_{ij}\,, \tag{7.3}$$

and using equation (6.20), one obtains:

$$\frac{\partial \tau'_{ij}}{\partial t} = -\tau'_{ik}\frac{\partial \overline{u}_j}{\partial x_k} - \tau'_{jk}\frac{\partial \overline{u}_i}{\partial x_k} + \varepsilon \mathrm{A}_{ij} + K\mathrm{M}_{ijkl}\frac{\partial \overline{u}_k}{\partial x_l} - \frac{2}{3}\varepsilon\delta_{ij} - 2\varepsilon d_{ij} \tag{7.4}$$

where A_{ij} is the slow pressure strain (6.16), M_{ijkl} is the rapid pressure strain (6.17), $\mathrm{A}_{ij}(\boldsymbol{b})$ is given by equation (6.18), $\mathrm{M}_{ijkl}(\boldsymbol{b})$ is given by equation (6.19), b_{ij} is the anisotropy of the Reynolds-stress tensor (5.12):

$$b_{ij} = \frac{1}{2K}\left(\tau'_{ij} - \frac{2}{3}K\delta_{ij}\right)$$

and d_{ij} is the anisotropy of the dissipation-rate tensor:

$$d_{ij} = \frac{1}{2\varepsilon}\left(\varepsilon_{ij} - \frac{2}{3}\varepsilon\delta_{ij}\right). \tag{7.5}$$

It is assumed that an equilibrium state is reached, where b_{ij}, d_{ij}, A_{ij}, M_{ijkl} and K/ε achieve constant values, independent of the initial conditions.

7.1.1 Pressure-Strain Correlation

It can be seen that, for two-dimensional mean turbulent flows in equilibrium (Speziale, Sarkar & Gatski 1991) the pressure-strain correlation has the form:

$$\Pi_{ij} = -C_1\varepsilon b_{ij} + C_1^*\varepsilon\left(b_{ik}b_{kj} - \frac{1}{3}b_{kl}b_{kl}\delta_{ij}\right)$$

$$+ C_2K\overline{S}_{ij} + C_3K\left(b_{ik}\overline{S}_{jk} + b_{jk}\overline{S}_{ik} - \frac{2}{3}b_{kl}\overline{S}_{kl}\delta_{ij}\right)$$

$$+C_4K\left(b_{ik}\overline{W}_{jk}+b_{jk}\overline{W}_{ik}\right) \qquad (7.6)$$

where b_{ij} is the anisotropy of the Reynolds-stress tensor (5.12) and:

$$\overline{S}_{ij}=\frac{1}{2}\left(\frac{\partial\overline{u}_i}{\partial x_j}+\frac{\partial\overline{u}_j}{\partial x_i}\right)$$

$$\overline{W}_{ij}=\frac{1}{2}\left(\frac{\partial\overline{u}_i}{\partial x_j}-\frac{\partial\overline{u}_j}{\partial x_i}\right)$$

are the mean strain-rate tensor (2.27) and the mean-vorticity tensor (2.42), respectively(*). In equations (7.4) and (7.6) the anisotropy of the dissipation-rate tensor d_{ij} and the quadratic return term containing C_1^* ($<<1$) are often neglected. In many cases this assumpion is justified by invoking the Kolmogorov hypothesis of local isotropy for high-Reynolds-number turbulent flows (though somewhat debatable). The coefficients C_1-C_4 have been obtained by Speziale, Sarkar & Gatski (1991) based on calibration with homogeneous shear-flow experiments.

7.1.2 General Solution

In the case of homogeneous turbulence in equilibrium with constant mean velocity gradients, b_{ij} reaches equilibrium values that are independent of the initial contitions. Thus:

$$\frac{\partial b_{ij}}{\partial t}=0 \qquad (7.7)$$

(*) Note that equation (7.6) is the equilibrium-limit form assumed by the *SSG* model of Speziale, Sarkar & Gatski (1991) [equation (6.68)].

and, from the definition of b_{ij}, one has:

$$\frac{\partial \tau'_{ij}}{\partial t} = 2(P_K - \varepsilon)b_{ij} + \frac{2}{3}(P_K - \varepsilon)\delta_{ij} \tag{7.8}$$

where P_K is the turbulence production term (4.6):

$$P_K = -\tau'_{ij}\frac{\partial \overline{u}_i}{\partial x_j}.$$

It then follows from model equation (7.6) that the Reynolds-stress transport equation has the equilibrium form (Gatski & Speziale 1993):

$$b^*_{ij} = -\overline{S}^*_{ij} - b^*_{ik}\overline{S}^*_{jk} - b^*_{jk}\overline{S}^*_{ik} + \frac{2}{3}b^*_{kl}\overline{S}^*_{kl}\delta_{ij} + b^*_{ik}\overline{W}^*_{kj} + b^*_{jk}\overline{W}^*_{ki} \tag{7.9}$$

where:

$$\overline{S}^*_{ij} = \frac{1}{2}g\frac{K}{\varepsilon}(2 - C_3)\overline{S}_{ij} \tag{7.10}$$

$$\overline{W}^*_{ij} = \frac{1}{2}g\frac{K}{\varepsilon}(2 - C_4)\overline{W}_{ij} \tag{7.11}$$

$$b^*_{ij} = \left(\frac{C_3 - 2}{C_2 - \frac{4}{3}}\right)b_{ij} \tag{7.12}$$

$$g = \left(\frac{C_1}{2} + \frac{P_K}{\varepsilon} - 1\right)^{-1}. \tag{7.13}$$

Model equation (7.9) actually constitutes a generalization of the *implicit* Algebraic Stress Model (*ASM*) of Rodi (1976), developed on the basis of the *LRR* second-order closure and written in a more compact form, based on the normalized variables defined above. With the use of integrity-bases methods from linear algebra, Pope (1975) showed that the general solution to the implicit algebraic-stress equation (7.9) has the form:

$$\boldsymbol{b}^{*} = \sum_{\lambda=1}^{10} G^{(\lambda)} \boldsymbol{T}^{(\lambda)} \tag{7.14}$$

where:

$$\boldsymbol{T}^{(1)} = \overline{\boldsymbol{S}}^{*}, \quad \boldsymbol{T}^{(2)} = \overline{\boldsymbol{S}}^{*}\overline{\boldsymbol{W}}^{*} - \overline{\boldsymbol{W}}^{*}\overline{\boldsymbol{S}}^{*}, \quad \boldsymbol{T}^{(3)} = \overline{\boldsymbol{S}}^{*2} - \frac{1}{3} tr\left(\overline{\boldsymbol{S}}^{*2}\right)\boldsymbol{I} \tag{7.15a}$$

$$\boldsymbol{T}^{(4)} = \overline{\boldsymbol{W}}^{*2} - \frac{1}{3} tr\left(\overline{\boldsymbol{W}}^{*2}\right)\boldsymbol{I}, \quad \boldsymbol{T}^{(5)} = \overline{\boldsymbol{W}}^{*}\overline{\boldsymbol{S}}^{*2} - \overline{\boldsymbol{S}}^{*2}\overline{\boldsymbol{W}}^{*} \tag{7.15b}$$

$$\boldsymbol{T}^{(6)} = \overline{\boldsymbol{W}}^{*2}\overline{\boldsymbol{S}}^{*} + \overline{\boldsymbol{S}}^{*}\overline{\boldsymbol{W}}^{*2} - \frac{2}{3} tr\left(\overline{\boldsymbol{S}}^{*}\overline{\boldsymbol{W}}^{*2}\right)\boldsymbol{I}, \quad \boldsymbol{T}^{(7)} = \overline{\boldsymbol{W}}^{*}\overline{\boldsymbol{S}}^{*}\overline{\boldsymbol{W}}^{*2} - \overline{\boldsymbol{W}}^{*2}\overline{\boldsymbol{S}}^{*}\overline{\boldsymbol{W}}^{*} \tag{7.15c}$$

$$\boldsymbol{T}^{(8)} = \overline{\boldsymbol{S}}^{*}\overline{\boldsymbol{W}}^{*}\overline{\boldsymbol{S}}^{*2} - \overline{\boldsymbol{S}}^{*2}\overline{\boldsymbol{W}}^{*}\overline{\boldsymbol{S}}^{*}, \quad \boldsymbol{T}^{(9)} = \overline{\boldsymbol{W}}^{*2}\overline{\boldsymbol{S}}^{*2} + \overline{\boldsymbol{S}}^{*2}\overline{\boldsymbol{W}}^{*2} - \frac{2}{3} tr\left(\overline{\boldsymbol{S}}^{*2}\overline{\boldsymbol{W}}^{*2}\right)\boldsymbol{I} \tag{7.15d}$$

$$\boldsymbol{T}^{(10)} = \overline{\boldsymbol{W}}^{*}\overline{\boldsymbol{S}}^{*2}\overline{\boldsymbol{W}}^{*2} - \overline{\boldsymbol{W}}^{*2}\overline{\boldsymbol{S}}^{*2}\overline{\boldsymbol{W}}^{*} \tag{7.15e}$$

are the integrity bases and tr denotes trace.

The general three-dimensional solution for $G^{(\lambda)}$ has been first provided by Gatski & Speziale (1993), as follows:

$$G^{(1)} = -\frac{1}{2}\left(6 - 3\eta_1 - 21\eta_2 - 2\eta_3 + 30\eta_4\right)\Big/D \tag{7.16a}$$

$$G^{(2)} = -\left(3 + 3\eta_1 - 6\eta_2 + 2\eta_3 + 6\eta_4\right)/D \tag{7.16b}$$

$$G^{(3)} = \left(6 - 3\eta_1 - 12\eta_2 - 2\eta_3 - 6\eta_4\right)/D \tag{7.16c}$$

$$G^{(4)} = -3\left(3\eta_1 + 2\eta_3 + 6\eta_4\right)/D \tag{7.16d}$$

$$G^{(5)} = -9/D \ , \quad G^{(6)} = -9/D \ , \quad G^{(7)} = 9/D \tag{7.16e}$$

$$G^{(8)} = 9/D \ , \quad G^{(9)} = 18/D \ , \quad G^{(10)} = 0 \ , \tag{7.16f}$$

where the denominator D is given by:

$$D = 3 - \frac{7}{2}\eta_1 + \eta_1^2 - \frac{15}{2}\eta_2 - 8\eta_1\eta_2 + 3\eta_2^2 - \eta_3 + \frac{2}{3}\eta_1\eta_3$$

$$- 2\eta_2\eta_3 + 21\eta_4 + 24\eta_5 + 2\eta_1\eta_4 - 6\eta_2\eta_4 \tag{7.17}$$

and:

$$\eta_1 = tr\left(\overline{\boldsymbol{S}}^{*2}\right),\ \eta_2 = tr\left(\overline{\boldsymbol{W}}^{*2}\right),\ \eta_3 = tr\left(\overline{\boldsymbol{S}}^{*2}\right),\ \eta_4 = tr\left(\overline{\boldsymbol{S}}^{*}\overline{\boldsymbol{W}}^{*2}\right),\ \eta_5 = tr\left(\overline{\boldsymbol{S}}^{*2}\overline{\boldsymbol{W}}^{*2}\right). \tag{7.18}$$

Taulbee (1992) derived a simplified three-dimensional solution of (7.9) for the degenerate case where $C_3 = 2$.

Algebraic-stress models such as that of Rodi (1976) (7.9) include anisotropic effects but are not easy to use due to the implicit stress-strain relationship.

Gatski & Speziale (1993), on the basis of the *SSG* second-order closure, obtained the following *explicit* solution of (7.9) for two-dimensional mean flows. The solution is rather simplified since only the first three integrity bases are linearly independent:

$$b_{ij}^{*}=-\frac{3}{3-2\eta^{2}+6\xi^{2}}\left[\overline{S}_{ij}^{*}+\overline{S}_{ik}^{*}\overline{W}_{kj}^{*}+\overline{S}_{jk}^{*}\overline{W}_{ki}^{*}-2\left(\overline{S}_{ik}^{*}\overline{S}_{kj}^{*}-\frac{1}{3}\overline{S}_{kl}^{*}\overline{S}_{kl}^{*}\delta_{ij}\right)\right] \tag{7.19}$$

where:

$$\eta=\left(\overline{S}_{ij}^{*}\overline{S}_{ij}^{*}\right)^{1/2},\qquad \xi=\left(\overline{W}_{ij}^{*}\overline{W}_{ij}^{*}\right)^{1/2}. \tag{7.20}$$

Equation (7.19) in terms of τ_{ij}' becomes:

$$\tau_{ij}'=\frac{2}{3}K\delta_{ij}-\frac{3}{3-2\eta^{2}+6\xi^{2}}\left[\alpha_{1}\frac{K^{2}}{\varepsilon}\overline{S}_{ij}+\alpha_{2}\frac{K^{3}}{\varepsilon^{2}}\left(\overline{S}_{ik}\overline{W}_{kj}+\overline{S}_{jk}\overline{W}_{ki}\right)\right.$$

$$\left.-\alpha_{3}\frac{K^{3}}{\varepsilon^{2}}\left(\overline{S}_{ik}\overline{S}_{kj}-\frac{1}{3}\overline{S}_{kl}\overline{S}_{kl}\delta_{ij}\right)\right] \tag{7.21}$$

where α_1, α_2 and α_3 are *not constant* but are rather related to the coefficients C_1-C_4 and g. In mathematical terms they are projections of the fixed points of A_{ij} and M_{ijkl} onto the fixed points of b_{ij}, thar are quantities that can vary from one flow to another. However, for two-dimensional mean turbulent flows, it appears that C_1-C_4 can be approximated by constants due to the linear dependence on b_{ij}, that allows the use of the principle of superposition.

Equation (7.21) has been named *explicit* Algebraic Stress Model since it is the explicit solution to the implicit Algebraic Stress Model (*ASM*) given by equation (7.9). It has the form of an anisotropic eddy-viscosity model [the coefficient of equation (7.21)] with strain-dependent coefficients.

The anisotropic eddy-viscosity models for the Reynolds-stress tensor obtained via the algebraic-stress approximation need to be regularized before they are applied to general complex turbulent flows. In fact the denominator of the eddy-viscosity coefficient in equation (7.21) can vanish, giving singular behavior. The regularization has been accomplished by Gatski & Speziale (1993) via a Padé type approximation, as follows:

$$\frac{3}{3-2\eta^2+6\xi^2} \approx \frac{3\left(1+\eta^2\right)}{3+\eta^2+6\xi^2\eta^2+6\xi^2}\,. \tag{7.22}$$

With the use of expression (7.22) regular for general turbulent flows, a good approximation is obtained for near-equilibrium flows and the problem of a singular or negative eddy viscosity for sufficiently large values of η (a feature that can lead to divergent computations) is resolved.

Yoshizawa (1984), Speziale (1987*b*), Rubinstein & Burton (1990) and Zhou *et al.* (1994) derived models of this general form (tensorially quadratic with constant coefficients) via expansion techniques based respectively on two-scale *DIA*, continuum mechanics, $\varepsilon - RNG$ (see Yakhot & Orszag 1986 and Yakhot *et al.* 1992) and recursion *RNG* techniques (see also Chapter 6). However, due to the constants coefficients, these models are not consistent with second-order closures and they have dispersive terms that grow unbounded with the strain rate, a feature that can destabilize the computations. Shih *et al.* (1993) and Craft *et al.* (1993) also proposed models similar to (7.21). Other regularization schemes based on Padé approximation have been developed by Speziale & Xu (1995) using results from Rapid Distortion Theory (*RDT*). Regularized approximations derived from Reynolds-stress closures of the type (7.22) may be the way to obtain new improved nonlinear $K-\varepsilon$ models.

The above arguments explain why previous anisotropic corrections to eddy-viscosity models (Chapter 5) have not fully succeeded:

i) the coefficients should depend nonlinearly on the invariants of the rotational and irrotational strain rates;

ii) only the traditional algebraic-stress models – based on the solution of the implicit algebraic equation (7.9) – exhibit such a dependence and they are ill-behaved. This problem can be ovecome by means of a regularization procedure based on Padé type of approximations as discussed above.

In the case of rotating reference frames, Coriolis terms have to be added on the right-hand side of equation (7.1) as in the case of the complete equation (6.13), and the mean-vorticity tensor must be replaced by the absolute mean-vorticity tensor (2.41):

$$\overline{W_{ij}} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i}\right) + e_{mji}\Omega_m .$$

Gatski & Speziale (1993) showed that this analysis accounts for such noninertial effects in rotating frames if the extended definition $\overline{W_{ij}^*}$ is used:

$$\overline{W_{ij}^*} = \frac{1}{2} g \frac{K}{\varepsilon}(2 - C_4)\left[\overline{W_{ij}} + \left(\frac{C_4 - 4}{C_4 - 2}\right) e_{mji}\Omega_m\right] \tag{7.23}$$

where Ω_m is the angular velocity of the reference frame.

If separation of scales holds ($\eta, \xi << 1$), the linear limit of equation (7.21) can be taken and the eddy-viscosity model:

$$\tau'_{ij} = \frac{2}{3} K \delta_{ij} - 2 C_\mu^* \frac{K^2}{\varepsilon} \overline{S}_{ij} \tag{7.24}$$

is recovered, that forms the basis for the standard $K - \varepsilon$ model of Launder & Spalding (1974). In practical turbulent flows one has not separation of scales and η, ξ are not negligible with respect to unity, but rather they are of $O(1)$. Nonetheless, for two-dimensional turbulent shear flows in equilibrium, the explicit Algebraic Stress Model (*ASM*) (7.21) derived by Gatski & Speziale (1993) gives:

$$\tau'_{xy} = -C^*_\mu \frac{K^2}{\varepsilon} \frac{\partial \bar{u}}{\partial y} \qquad (7.25)$$

with:

$$C^*_\mu \approx 0.094 \, , \qquad (7.26)$$

a value rather close to $C_\mu = 0.09$ of the standard $K-\varepsilon$ model.

7.2 Kinetic-Energy Equation

In *homogeneous turbulence*, the turbulent kinetic-energy K is a solution of a reduced form of the exact kinetic-energy transport equation (4.1/4.5):

$$\frac{\partial K}{\partial t} = -\tau'_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \varepsilon \, . \qquad (7.27)$$

Thus, a closure is reached if a transport equation for ε is provided in terms of τ'_{ij} and $\partial \bar{u}_i / \partial x_j$ (*).

(*) Note that, in the case of *weakly inhomogeneous turbulence* near equilibrium, a modeled kinetic-energy transport equation can be obtained by including in equation (7.27) the convective term and adding a gradient transport term on the right-hand side as a model for the turbulent transport term [equation (4.15)]:

$$\frac{\partial K}{\partial t} + \bar{u}_i \frac{\partial K}{\partial x_i} = -\tau'_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_K} \frac{\partial K}{\partial x_i} \right)$$

where typically $\sigma_K = 1.0$.

7.3 Dissipation Equations

In the case of *homogeneous turbulence* one has:

$$\varepsilon = 2\nu \overline{\omega_i' \omega_i'} \tag{7.28}$$

where ω_i' is the fluctuating vorticity vector:

$$\omega_i' = e_{ijk} \frac{\partial u_k'}{\partial x_j}. \tag{7.29}$$

When the vorticity equation is used to write a transport equation for the fluctuating enstrophy $\overline{\omega_i' \omega_i'}$ (see Appendix), an alternative form [with respect to form (5.4)] of the exact dissipation-rate equation for homogeneous turbulence, results:

$$\frac{\partial \varepsilon}{\partial t} = -\varepsilon_{ij} \frac{\partial \overline{u}_i}{\partial x_j} - \varepsilon_{ij}^C \frac{\partial \overline{u}_i}{\partial x_j} + 2\nu \overline{\omega_i' \omega_j' \frac{\partial u_i'}{\partial x_j}} - 2\nu^2 \overline{\frac{\partial \omega_i'}{\partial x_j} \frac{\partial \omega_i'}{\partial x_j}} \tag{7.30}$$

where:

$$\varepsilon_{ij}^C = 2\nu \overline{\frac{\partial u_k'}{\partial x_i} \frac{\partial u_k'}{\partial x_j}} \tag{7.31}$$

is the complementary dissipation rate.
Equation (7.30) can be written in the form:

$$\frac{\partial \varepsilon}{\partial t} = -2\varepsilon \left(d_{ij} + d_{ij}^C \right) \frac{\partial \overline{u}_i}{\partial x_j} + \frac{7}{3\sqrt{15}} S_K \, Re_T^{1/2} \frac{\varepsilon^2}{K} - 2\nu^2 \overline{\frac{\partial \omega_i'}{\partial x_j} \frac{\partial \omega_i'}{\partial x_j}}, \tag{7.32}$$

where the generalized skewness S_K is:

$$S_K = \frac{6\sqrt{15}}{7} \frac{\overline{\omega_i' \omega_j' \dfrac{\partial u_i'}{\partial x_j}}}{\left(\overline{\omega_k' \omega_k'}\right)^{3/2}}, \tag{7.33}$$

the turbulence Reynolds number Re_T is:

$$Re_T = \frac{K^2}{\nu \varepsilon}, \tag{7.34}$$

the anisotropy of the dissipation-rate tensor d_{ij} (7.5) is:

$$d_{ij} = \frac{1}{2\varepsilon}\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right)$$

and the anisotropy of the complementary dissipation rate d_{ij}^C is:

$$d_{ij}^C = \frac{1}{2\varepsilon}\left(\varepsilon_{ij}^C - \frac{1}{3}\varepsilon_{kk}^C\delta_{ij}\right). \tag{7.35}$$

Models for the anisotropic effects in the dissipation rate, i.e. models of the dissipation-rate tensor d_{ij} and of the complementary dissipation-rate tensor d_{ij}^C [see equation (7.45)] can be obtained from an analysis of the transport equation for the tensor dissipation that, in the case of *homogeneous turbulence*, is given by (Durbin & Speziale 1991):

$$\frac{\partial \varepsilon_{ij}}{\partial t} =$$

$$2\nu \overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right) \frac{\partial u_l'}{\partial x_k} \frac{\partial u_k'}{\partial x_l}} - 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_l'}{\partial x_k} \frac{\partial u_j'}{\partial x_l}} - 2\nu \overline{\frac{\partial u_j'}{\partial x_k} \frac{\partial u_l'}{\partial x_k} \frac{\partial u_i'}{\partial x_l}} - 4\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m} \frac{\partial^2 u_j'}{\partial x_k \partial x_m}}$$

$$- \varepsilon_{ik} \frac{\partial \overline{u}_j}{\partial x_k} - \varepsilon_{jk} \frac{\partial \overline{u}_i}{\partial x_k}$$

$$+ 4\nu \left(\overline{\frac{\partial u_j'}{\partial x_i} \frac{\partial u_l'}{\partial x_k}} + \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_l'}{\partial x_k}} - \overline{\frac{\partial u_i'}{\partial x_l} \frac{\partial u_j'}{\partial x_k}} \right) \frac{\partial \overline{u}_k}{\partial x_l}. \tag{7.36}$$

Equation (7.36) can be written in the form:

$$\frac{\partial \varepsilon_{ij}}{\partial t} = N_{ij} - \varepsilon_{ik} \frac{\partial \overline{u}_j}{\partial x_k} - \varepsilon_{jk} \frac{\partial \overline{u}_i}{\partial x_k} + A_{ijkl}^{\varepsilon} \frac{\partial \overline{u}_k}{\partial x_l} \tag{7.37}$$

where, more in particular:

i) the term N_{ij} represents the production by vortex stretching – destruction by viscous diffusion:

$$N_{ij} = 2\nu \overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right) \frac{\partial u_l'}{\partial x_k} \frac{\partial u_k'}{\partial x_l}} - 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_l'}{\partial x_k} \frac{\partial u_j'}{\partial x_l}}$$

$$- 2\nu \overline{\frac{\partial u_j'}{\partial x_k} \frac{\partial u_l'}{\partial x_k} \frac{\partial u_i'}{\partial x_l}} - 4\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m} \frac{\partial^2 u_j'}{\partial x_k \partial x_m}}; \tag{7.38}$$

ii) the term A_{ijkl}^{ε} is a structure and redistribution term:

$$A^{\varepsilon}_{ijkl} = 4\nu\left(\overline{\frac{\partial u'_j}{\partial x_i}\frac{\partial u'_l}{\partial x_k}} + \overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_l}{\partial x_k}} - \overline{\frac{\partial u'_i}{\partial x_l}\frac{\partial u'_j}{\partial x_k}}\right). \tag{7.39}$$

7.4 Modeled Dissipation-Rate Equation

In order to develop the two-equation model, the destruction term in equation (7.30/7.32) [and the first term on the rhs of equation (7.45)] have to be modeled.

7.4.1 Destruction Term

In the case of *isotropic turbulence* one has:

$$2\nu^2 \overline{\frac{\partial \omega'_i}{\partial x_j}\frac{\partial \omega'_i}{\partial x_j}} \propto \nu^2 \int_0^\infty \kappa^4 E(\kappa,t)d\kappa \tag{7.40}$$

where $E(\kappa,t)$ is the energy spectrum. The major contributions to the integral in (7.40) occur at the high wavenumbers, where:

$$E(\kappa,t) \propto E(\kappa,l_\eta) \tag{7.41}$$

where l_η is the Kolmogorov length scale.

$$l_\eta = \frac{\nu^{3/4}}{\varepsilon^{1/4}}. \tag{7.42}$$

Kolmogorov scaling gives (Bernard & Speziale 1992):

$$2\nu^2 \overline{\frac{\partial \omega'_i}{\partial x_j}\frac{\partial \omega'_i}{\partial x_j}} \propto \left[Re_T^{1/2} + O(1)\right]\frac{\varepsilon^2}{K} = \frac{7}{3\sqrt{15}} G_K \, Re_T^{1/2}\frac{\varepsilon^2}{K} + C_{\varepsilon 2}\frac{\varepsilon^2}{K}, \tag{7.43}$$

and equation (7.32) becomes:

$$\frac{\partial \varepsilon}{\partial t} = -2\varepsilon\left(d_{ij} + d_{ij}^C\right)\frac{\partial \overline{u}_i}{\partial x_j} + \frac{7}{3\sqrt{15}}\left(S_K - G_K\right)Re_T^{1/2}\frac{\varepsilon^2}{K} + C_{\varepsilon 2}\frac{\varepsilon^2}{K}. \tag{7.44}$$

On the basis of the standard high-Reynolds-numbers equilibrium hypothesis, $S_K = G_K$ and equation (7.44) becomes:

$$\frac{\partial \varepsilon}{\partial t} = -2\varepsilon\left(d_{ij} + d_{ij}^C\right)\frac{\partial \overline{u}_i}{\partial x_j} + C_{\varepsilon 2}\frac{\varepsilon^2}{K} \tag{7.45}$$

($C_{\varepsilon 2}$ is a constant).

(*) Note that, if in (7.44) the anisotropy of the dissipation-rate tensor is assumed to be proportional to the anisotropy of the Reynolds-stress tensor ($d_{ij} \propto b_{ij}$ and $d_{ij}^C \propto b_{ij}$), equation (5.20) for homogeneous flows is recovered:

$$\frac{\partial \varepsilon}{\partial t} = -C_{\varepsilon 1}\frac{\varepsilon}{K}\tau'_{ij}\frac{\partial \overline{u}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K},$$

where the values of the constants are given in (5.23), $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$. Moreover, in the case of *weakly inhomogeneous turbulence* near equilibrium, a modeled dissipation-rate transport equation can be obtained by including in the above equation the convective term and adding a gradient transport term on the right-hand side for the diffusion term [equation (5.14)]:

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i\frac{\partial \varepsilon}{\partial x_i} = -C_{\varepsilon 1}\frac{\varepsilon}{K}\tau'_{ij}\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) - C_{\varepsilon 2}\frac{\varepsilon^2}{K}$$

where typically $\sigma_\varepsilon = 1.3$.

7.4.2 Production Term

Speziale & Gatski (1992, 1995), working on equation (7.36), developed the following models for d_{ij} and d_{ij}^C based on an expansion technique that makes use of tensor invariance, symmetry properties and the fact that d_{ij} is small, so that only linear terms need to be mantained:

$$A_{ijkl}^{\varepsilon} \frac{\partial \overline{u}_k}{\partial x_l} = \frac{16}{15} \varepsilon \overline{S}_{ij}$$

$$+ \left(\frac{30}{11} \alpha_3 + \frac{20}{11} \right) \varepsilon \left(d_{ik} \overline{S}_{jk} + d_{jk} \overline{S}_{ik} - \frac{2}{3} d_{kl} \overline{S}_{kl} \delta_{ij} \right)$$

$$- \left(\frac{14}{11} \alpha_3 - \frac{20}{11} \right) \varepsilon \left(d_{ik} \overline{W}_{jk} + d_{jk} \overline{W}_{ik} \right)$$

$$- \left(\frac{14}{11} \alpha_3 - \frac{16}{33} \right) \varepsilon d_{kl} \overline{S}_{kl} \delta_{ij} \tag{7.46}$$

$$N_{ij} = \frac{2}{3} N \delta_{ij} + N_{ij}^D \tag{7.47}$$

$$N = \frac{\varepsilon}{K} P_K - C_{\varepsilon 2} \frac{\varepsilon^2}{K} \tag{7.48}$$

$$N_{ij}^D = -C_{\varepsilon 5} \frac{\varepsilon}{K} \left(\varepsilon_{ij} - \frac{2}{3} \varepsilon \delta_{ij} \right) \tag{7.49}$$

where P_K is the turbulence production term (4.6):

$$P_K = -\tau'_{ij} \frac{\partial \overline{u}_i}{\partial x_j}$$

and $\overline{S}_{ij}$ and $\overline{W}_{ij}$ are the mean strain-rate tensor (2.27) and the mean-vorticity tensor (2.42), respectively. The standard equilibrium hypothesis is made, where:

$$\frac{\partial d_{ij}}{\partial t} = 0 . \tag{7.50}$$

This brings to an algebraic model for the dissipation-rate anisotropy tensor formed by a system of equations analogous to that obtained in the algebraic-stress approximation (Section 7.1.2). For two-dimensional mean turbulent flows, the exact solution is (Speziale & Gatski 1992, 1995):

$$d_{ij} = -2C_{\mu\varepsilon}\left[\overline{S}^*_{ij} + \left(\frac{\frac{7}{11}\alpha_3 + \frac{1}{11}}{C_{\varepsilon 5} + \frac{P_K}{\varepsilon} - 1}\right)\left(\overline{S}^*_{ik}\overline{W}^*_{jk} + \overline{S}^*_{jk}\overline{W}^*_{ik}\right)\right.$$

$$\left. + \left(\frac{\frac{30}{11}\alpha_3 - \frac{2}{11}}{C_{\varepsilon 5} + \frac{P_K}{\varepsilon} - 1}\right)\left(\overline{S}^*_{ik}\overline{S}^*_{jk} - \frac{1}{3}\overline{S}^*_{mn}\overline{S}^*_{mn}\delta_{ij}\right)\right] \tag{7.51}$$

where:

$$C_{\mu\varepsilon} = \frac{1}{15\left(C_{\varepsilon 5} + \frac{P_K}{\varepsilon} - 1\right)}\left[1 + 2\overline{W}^*_{ij}\overline{W}^*_{ij}\left(\frac{\frac{7}{11}\alpha_3 + \frac{1}{11}}{C_{\varepsilon 5} + \frac{P_K}{\varepsilon} - 1}\right)^2\right.$$

$$-\frac{2}{3}\left(\frac{\frac{15}{11}\alpha_3-\frac{1}{11}}{C_{\varepsilon 5}+\frac{P_K}{\varepsilon}-1}\right)\overline{S}_{ij}^*\overline{S}_{ij}^*\Bigg]^{-1} \tag{7.52}$$

$$\overline{S}_{ij}^* = \overline{S}_{ij}\frac{K}{\varepsilon} \tag{7.53}$$

$$\overline{W_{ij}}^* = \overline{W_{ij}}\frac{K}{\varepsilon}\,. \tag{7.54}$$

The substitution of the above algebraic equations into the contraction of the ε_{ij}-transport equation (7.36/7.37) gives the following scalar dissipation-rate equation for homogeneous flow:

$$\frac{\partial \varepsilon}{\partial t} = -C_{\varepsilon 1}^*\frac{\varepsilon}{K}\tau'_{ij}\frac{\partial \overline{u}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K} \tag{7.55}$$

where $C_{\varepsilon 1}^*$ is *not constant* and has the form:

$$C_{\varepsilon 1}^* = 1+\frac{2(1+\alpha)}{15C_\mu}\left[\frac{C_{\varepsilon 5}+C_\mu\eta^2-1}{\left(C_{\varepsilon 5}+C_\mu\eta^2-1\right)^2+\beta_1^2\xi^2-\frac{1}{3}\beta_2^2\eta^2}\right] \tag{7.56}$$

$$\eta = \left(2\overline{S}_{ij}^*\overline{S}_{ij}^*\right)^{1/2} \tag{7.57}$$

$$\xi = \left(2\overline{W_{ij}}^*\overline{W_{ij}}^*\right)^{1/2} \tag{7.58}$$

$$\alpha = \frac{3}{4}\left(\frac{14}{11}\alpha_3 - \frac{16}{33}\right) \tag{7.59}$$

$$\beta_1 = \frac{7}{11}\alpha_3 + \frac{1}{11} \tag{7.60}$$

$$\beta_2 = \frac{15}{11}\alpha_3 - \frac{1}{11} \tag{7.61}$$

$$C_{\varepsilon 5} \approx 5 \tag{7.62}$$

$$\alpha_3 \approx 0.6\ .^{(*)} \tag{7.63}$$

Thus, in model equation (7.55) the coefficient of the first term on the right-hand side is not constant, but is formally a function of the nondimensional strains η and ξ.

The constants α_3 (7.63) and $C_{\varepsilon 5}$ (7.62) have been evaluated using *DNS* results for homogeneous shear flow (Rogers, Moin & Reynolds 1986). The constant $C_{\varepsilon 2}$ in (7.55) is typically evaluated so as to predict the decay rate of isotropic turbulence. The above dissipation-rate model predicts that the turbulent kinetic energy decays according to the standard power law in isotropic turbulence (Speziale 1991):

$$K \approx t^{-1/(C_{\varepsilon 2}-1)}\ . \tag{7.64}$$

(*) Note that the contraction of equation (7.46) gives:

$$d_{ij}^{C}\frac{\partial \overline{u}_i}{\partial x_j} = \alpha d_{ij}\frac{\partial \overline{u}_i}{\partial x_j}$$

that has been used to obtain the above results.

The value of $C_{\varepsilon 2}$ can be taken:

$$C_{\varepsilon 2} = 1.83 \tag{7.65}$$

that gives an average decay exponent of approximately 1.2, in good agreement with the most invoked experimental data (see Comte-Bellot & Corrsin 1971). In the standard $K-\varepsilon$ model a value of $C_{\varepsilon 2} = 1.92$ is chosen by using the same procedure.

For two-dimensional *turbulent shear flows* in equilibrium:

$$C_{\varepsilon 1}^{*} \approx 1.4, \tag{7.66}$$

a value remarkably close to the constant value $C_{\varepsilon 1} = 1.44$, traditionally used in the standard $K-\varepsilon$ model.

For more general two-dimensional *homogeneous turbulent flows*, model equation (7.55) takes the form:

$$\frac{\partial \varepsilon}{\partial t} = C_{\varepsilon 1}^{*} C_{\mu} \frac{\varepsilon^2}{K} \eta^2 - C_{\varepsilon 2} \frac{\varepsilon^2}{K} \tag{7.67}$$

where:

$$C_{\varepsilon 1}^{*} = C_{\varepsilon 1}^{*}(\eta, \xi) \tag{7.68}$$

$$C_{\mu} = C_{\mu}(\eta, \xi). \tag{7.69}$$

Lumley (1992) proposed a model for the dissipation rate for *near-equilibrium turbulent flows* of the form:

$$\frac{\partial \varepsilon}{\partial t} = C_1 \frac{\varepsilon^2}{K}\eta - C_{\varepsilon 2}\frac{\varepsilon^2}{K} \tag{7.70}$$

where C_1 is a *constant*.

Model (7.70) contains less physics than model (7.67) since it does not depend on rotational strains. It has long been recognized that the dissipation rate is dramatically altered by rotations. Anisotropic dissipations as used by Speziale & Gatski (1992, 1995) may be a method to systematically include the influence of rotation on the dissipation rate.

7.5 Algebraic-Stress Turbulence Models

Overall, a typical form the system of equations to be solved in nonlinear algebraic-stress, two-equation models for the calculation of $\overline{u}_i$, $\overline{p}$, K and ε (within the above-mentioned general hypotheses) becomes:

$$\frac{\partial \overline{u}_i}{\partial t} = -\frac{\partial \overline{p}}{\partial x_i} - \frac{\partial \tau'_{ij}}{\partial x_j} \tag{7.71}$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{7.72}$$

$$\frac{\partial K}{\partial t} = -\tau'_{ij}\frac{\partial \overline{u}_i}{\partial x_j} - \varepsilon \tag{7.73}$$

$$\frac{\partial \varepsilon}{\partial t} = -C^*_{\varepsilon 1}\frac{\varepsilon}{K}\tau'_{ij}\frac{\partial \overline{u}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K} \tag{7.74}$$

$$\tau'_{ij} = \frac{2}{3}K\delta_{ij} - \frac{3}{3-2\eta^2+6\xi^2}\left[\alpha_1\frac{K^2}{\varepsilon}\overline{S}_{ij} + \alpha_2\frac{K^3}{\varepsilon^2}\left(\overline{S}_{ik}\overline{W}_{kj} + \overline{S}_{jk}\overline{W}_{ki}\right)\right.$$

$$-\alpha_3 \frac{K^3}{\varepsilon^2}\left(\overline{S}_{ik}\overline{S}_{kj} - \frac{1}{3}\overline{S}_{kl}\overline{S}_{kl}\delta_{ij}\right)\Bigg]. \tag{7.75}$$

where the equations (7.74) and (7.75) can assume different specific forms according to the discussion made above. The system of the six partial differential equations (7.71-7.74) is a closed system for the determination of $\overline{u}_i$, $\overline{p}$, K and ε.

Chapter 8

Results and Comparisons

Illustrative examples are presented to show the capabilities of *RANS* turbulence closure models to predict turbulent flows.

8.1 Two-Equation Models

The capability of two-equation closure models to provide turbulent-flow predictions is presented with reference to four nontrivial turbulent-flow cases.

8.1.1 Homogeneous Shear Flow

The case of homogeneous shear flow in a rotating frame is considered (Figure 8.1). In this flow an initially isotropic turbulence (with turbulent kinetic energy K_0 and turbulent dissipation rate ε_0) is suddenly subjected to a uniform shear with constant

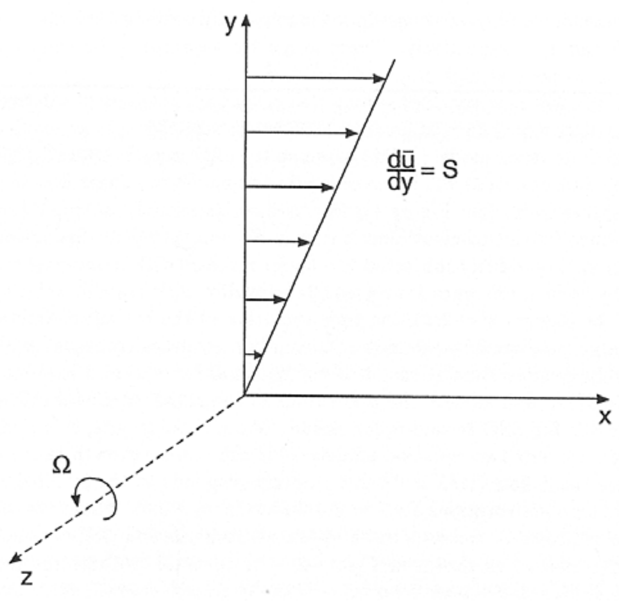


Figure 8.1. Scheme of homogeneous shear flow in a rotating frame (adapted from Speziale 1996).

shear rate $S = d\bar{u}/dy$ in a reference frame steadily rotating with angular velocity Ω (see also Chapter 5). More in particular one has:

$$\frac{\partial \bar{u}_i}{\partial x_j} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8.1}$$

$$\Omega_i = (0,0,\Omega). \tag{8.2}$$

In Figure 8.2 the temporal evolution of the turbulent kinetic energy predicted by the explicit algebraic-stress model (*ASM*) of Gatski & Speziale (1993) is compared with the

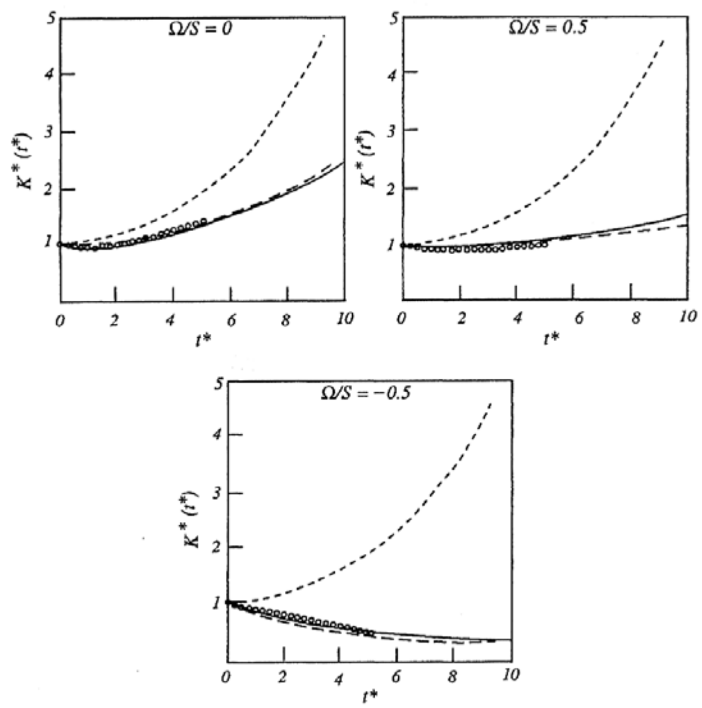


Figure 8.2. Temporal evolution of the turbulent kinetic energy in rotating homogeneous shear flow: (——) explicit *ASM* (Gatski & Speziale 1993); (– – –) *SSG* model; (----) $K-\varepsilon$ model; (○○○) *LES* calculations of Bardina *et al.* (1983) (adapted from Gatski & Speziale 1993).

LES calculations of Bardina, Ferziger & Reynolds (1983), as well as with the predictions of the standard $K-\varepsilon$ model and the full *SSG* second-order closure model. The explicit two-equation model of Gatski & Speziale (1993) gives the correct growth rate for pure shear flow ($\Omega/S=0$) and properly responds to the stabilizing effect of the rotations $\Omega/S=0.5$ and $\Omega/S=-0.5$. The results are close to those obtained from the full *SSG* second-oder closure. In contrast, the standard $K-\varepsilon$ model overpredicts the growth rate of the turbulent kinetic energy in pure shear flow and fails to predict the stabilizing effect of the rotations.

8.1.2 Channel Flow

The problem of the rotating channel flow has been considered by Launder *et al.* (1987). In this problem a turbulent channel flow is subjected to a steady spanwise rotation (Figure 8.3).

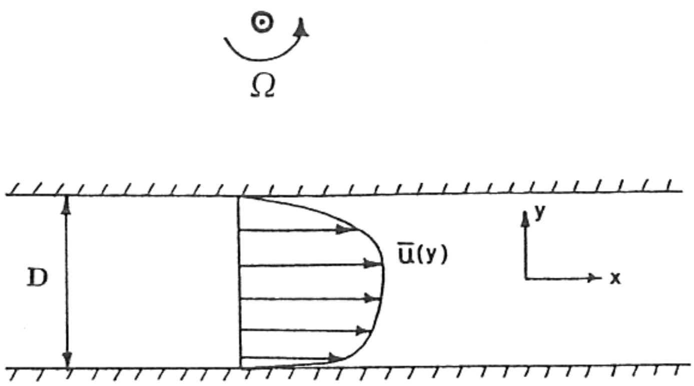


Figure 8.3. Fully developed turbulent channel flow in a rotating frame (adapted from Speziale 1991).

Experiments of both experimental and numerical nature (Johnston *et al.* 1972, Kim 1983) indicate that Coriolis forces arising from a system rotation cause the mean-velocity profile $\bar{u}(y)$ to become asymmetric about the channel centerline.

In Figure 8.4 the predictions of the explicit two-equation model of Gatski & Speziale (1993) for the mean-velocity profile in a rotating channel flow is compared with the experimental data of Johnston, Halleen & Lezius (1972) for a rotation number $Ro = 0.068$. The model correctly predicts that the mean-velocity profile is *asymmetric*, in line with the experimental results. In contrast, the standard $K-\varepsilon$ model incorrectly predicts a symmetric mean-velocity profile, identical to that obtained in an inertial frame.

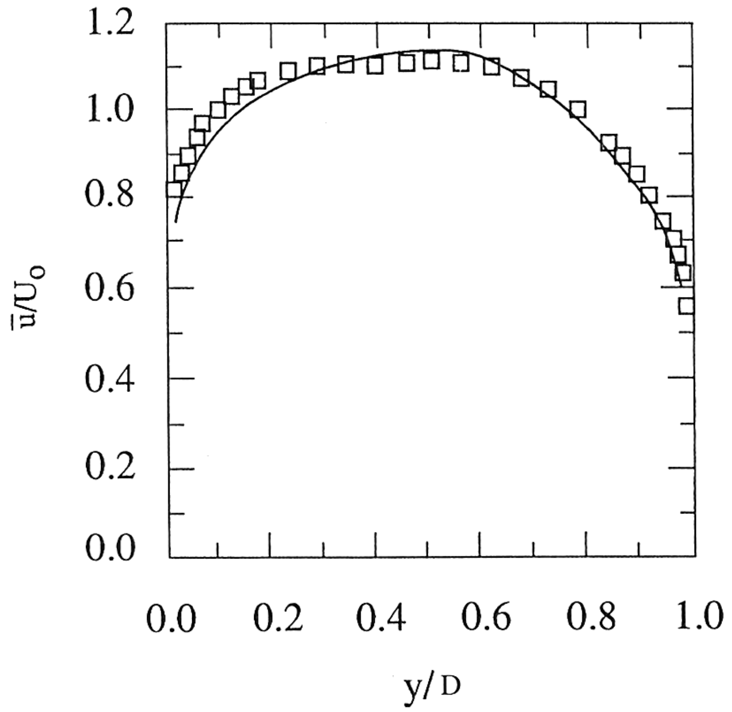


Figure 8.4. Mean-velocity profile in rotating channel flow: (——) explicit *ASM* (Gatski & Speziale 1993); (□□□) experimental data of Johnston *et al.* (1972) (adapted from Speziale 1996).

8.1.3 Square-Duct Flow

In this case effects arising from normal Reynolds-stress differences are present. For turbulent shear flows that are predominantly unidirectional with secondary flows or recirculation zones driven by small Reynolds-stress differences, a quadratic approximation of the anisotropic eddy-viscosity model as that of Gatski & Speziale (1993) collapses into the nonlinear $K-\varepsilon$ model of Speziale (1987*b*) (Chapter 5). In Figure 8.5 it is shown how the nonlinear $K-\varepsilon$ model predicts an eight-vortex secondary flow in a square duct, in line with experimental observations. The standard $K-\varepsilon$ model erroneously predicts that there is no secondary flow. In order to be able to predict secondary flows in noncircular ducts, the axial mean velocity $\bar{u}_z$ must give rise to a nonzero normal Reynolds-stress difference $\tau'_{yy}-\tau'_{xx}$ (see Speziale & Ngo 1988). This requires an *anisotropic* eddy viscosity (of any kind).

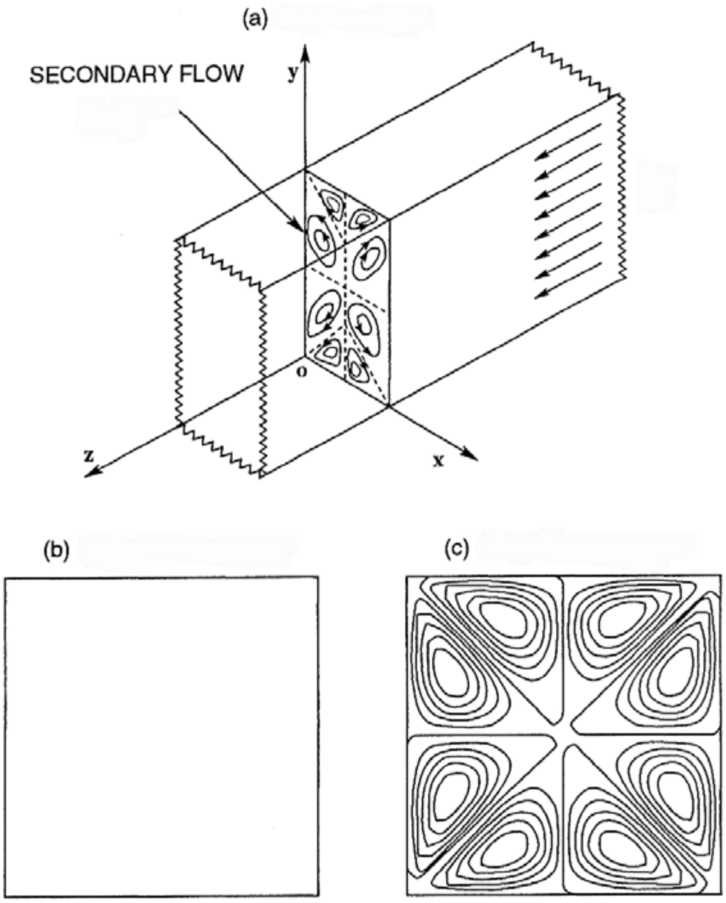
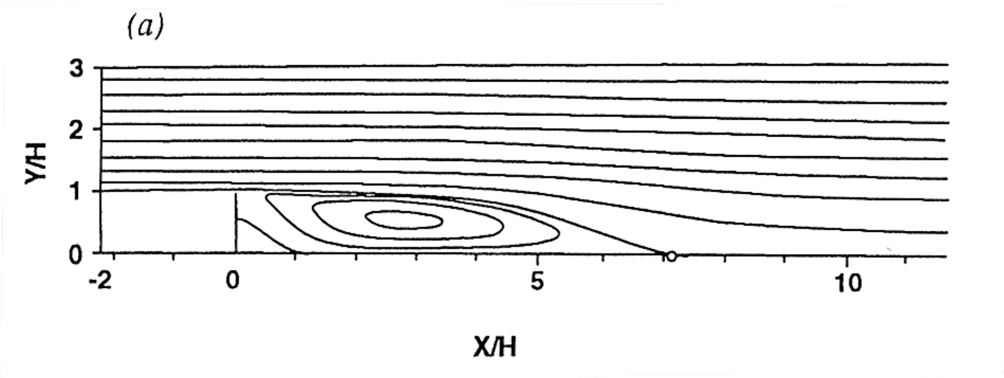


Figure 8.5. Turbulent secondary flow in a rectangular duct: *a*) experiments; *b*) standard $K-\varepsilon$ model; *c*) nonlinear $K-\varepsilon$ model (adapted from Speziale 1996).

8.1.4 Flow Past Backward-Facing Step

In Figure 8.6 results obtained from the nonlinear $K-\varepsilon$ model of Speziale (1987*b*) are compared with the experimental data of Kim, Kline & Johnston (1980) and Eaton & Johnston (1980) for turbulent flow past a backward-facing step. The reattachment is predicted at $x/H \approx 7.0$ in close agreement with the experimental data.



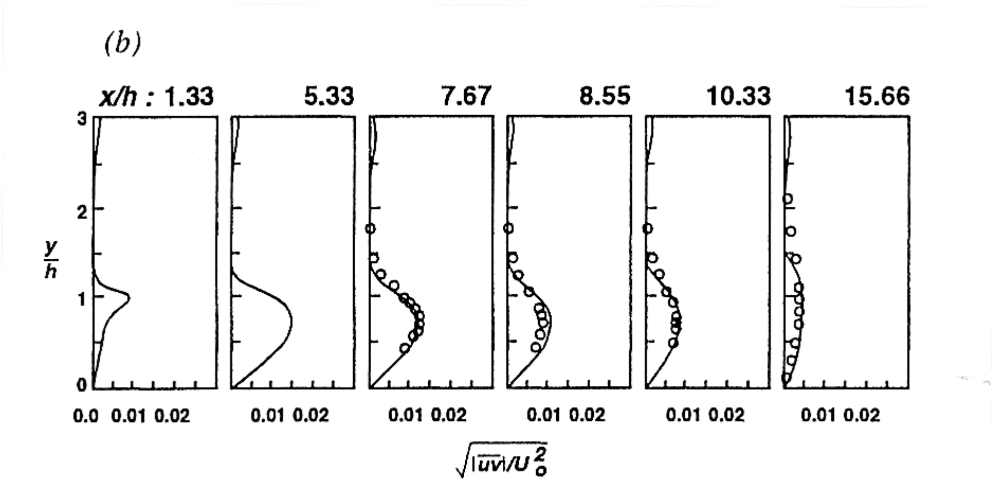


Figure 8.6. Turbulent flow past a backward-facing step: (——) nonlinear $K-\varepsilon$ model, (○○○) experimental data of Kim *et al.* (1980) and Eaton & Johnston (1981); *a*) streamlines (○ is the experimental mean reattachment point); *b*) turbulent shear-stress profile (adapted from Speziale 1996).

8.2 Stress-Equation Models

The capability of stress-equation closure models to provide turbulent-flow predictions is presented with reference to four turbulent-flow cases.

8.2.1 Return-to-Isotropy Problem

The ability of second-order closure models in describing Reynolds-stress relaxation effects can be shown by considering the return-to-isotropy problem. In this problem, an initially anisotropic homogeneous turbulence generated by the application of constant mean-velocity gradients, gradually relaxes to a state of isotropy ($b_{ij} \to 0$ as $t \to \infty$), after the mean velocity gradients (mean strains) are suddenly removed at time $t = 0$. By introducing the transformed dimensionless time τ, where:

$$d\tau = \frac{1}{2K}\varepsilon dt \quad , \tag{8.3}$$

the modeled Reynolds-stress equation for homogeneous flows (7.2) can be written in the equivalent form:

$$\frac{db_{ij}}{d\tau} = 2b_{ij} + \mathrm{A}_{ij} \tag{8.4}$$

where A_{ij} is the dimensionless slow pressure-strain correlation (6.18). Since in this problem the rapid pressure-strain term and the transport term vanish and since the dissipation rate can be absorbed into the dimensionless time τ, only a model for the slow pressure-strain correlation is needed, as shown in (8.4).

In Figure 8.7 predictions of the *SSG* model and of the *LRR* model (with $\mathrm{A}_{ij} = -C_1 b_{ij}$ and the Rotta constant $C_1 = 3.0$) for the temporal evolution of the anisotropy tensor $b_{ij}(\tau)$, are compared with the experimental data of Choi & Lumley (1984) for the relaxation from plane-strain case. It appears that the models reproduce satisfactorily the experimental trends, predicting a gradual return to isotropy consistent with the experimental data. In contrast to these results, two-equation, eddy-viscosity or even nonlinear algebraic-stress models erroneously predict that, for $\tau > 0$, b_{ij} abruptly goes to zero.

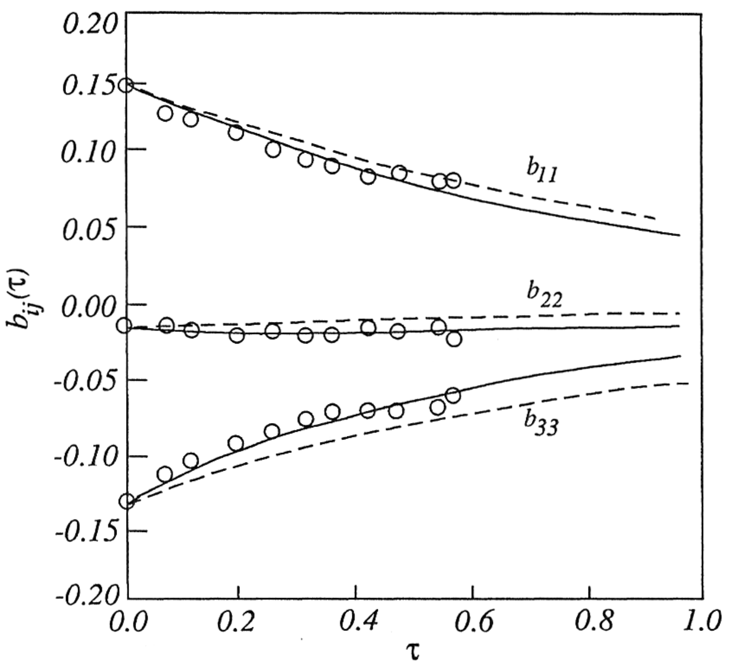


Figure 8.7. Temporal evolution of the anisotropy tensor in the return-to-isotropy problem: (—–) *SSG* model; (– – –) *LRR* model; (○○○) experimental data of Choi & Lumley (1984) (adapted from Speziale *et al.* 1991).

8.2.2 Axisymmetric Expansion

It is worth noting that while the *SSG* model has been derived and calibrated based on near-equilibrium two-dimensional mean turbulent flows, it performs remarkably well also on certain three-dimensional, homogeneously-strained turbulent flows. In Figure 8.8 the predictions of the *SSG* and *LRR* models are compared with the *DNS* calculations of Lee & Reynolds (1985) for the axisymmetric expansion ($t^* = \Gamma t$ where Γ is the strain rate). The agreement of the results in particular of the *SSG* model with the direct numerical simulations is rather satisfactory.

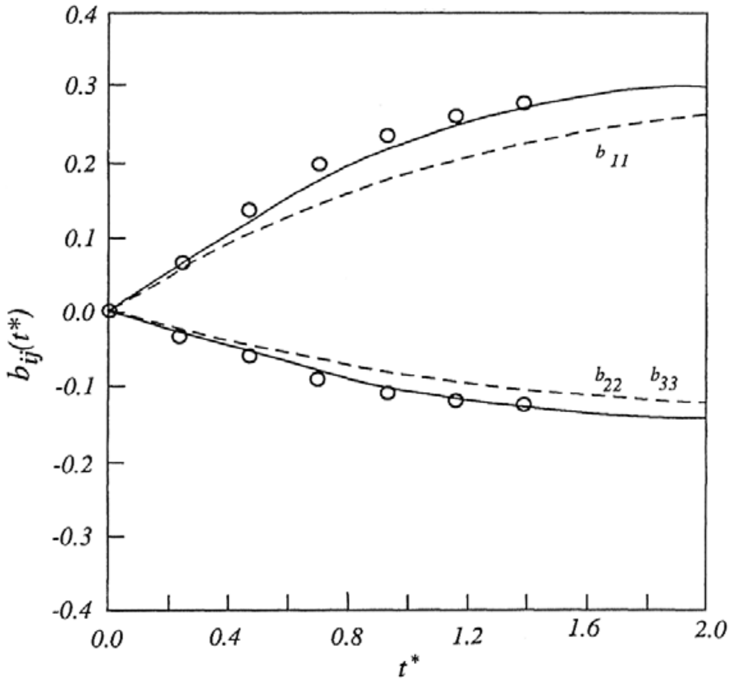


Figure 8.8. Temporal evolution of the anisotropy tensor in the axisymmetric expansion for $\varepsilon_0/\Gamma K_0 = 2.45$: (——) *SSG* model; (– – –) *LRR* model; (○○○) experimental data of Lee & Reynolds (1985) (adapted from Speziale 1996).

8.2.3 Homogeneous Shear Flow

A comparison of the predictions of the standard $K-\varepsilon$ model and the *LRR* model is presented for the problem of homogeneous turbulent shear flow in a rotating frame.

In the case of pure shear flow ($\Omega = 0$) the *LRR* model gives better predictions than the $K-\varepsilon$ model for the equilibrium values (obtained in the limit as $t \to \infty$) of b_{ij} and SK/ε (Table 8.1). Since the standard $K-\varepsilon$ model is frame indifferent, it erroneously gives solutions for rotating shear flow that are independent of Ω. Second-order closure models instead, give rotationally dependent solutions due to the effect of the Coriolis acceleration.

Further comparisons are shown in Table 8.2. Equilibrium values for homogeneous shear flow are reported for the *LRR* model, the *SL* model, the *FLT* model and the *SSG* model. The results of the *SSG* and *FLT* models give results in line with the experimental data of Tavoularis & Karnik (1989). The *LRR* model does not perform as well. The problem

Table 8.1

Comparison of predictions of different models with the experimental data of Tavoularis & Corssin (1981) in homogeneous shear flow.

Equil. values	*$K-\varepsilon$ model*	*LRR model*	*Experiments*
$(b_{11})_\infty$	0	0.193	0.201
$(b_{22})_\infty$	0	-0.096	-0.147
$(b_{12})_\infty$	-0.217	-0.185	-0.150
$(SK/\varepsilon)_\infty$	4.82	5.65	6.08

largely lies in the enforcement of the symmetry and normalization constraints, equations (6.26-6.27) that remove the necessary degrees of freedom from the *LRR* model. If the *LRR* model is thought of instead as an approximation to the general model valid in the two-dimensional equilibrium limit, then equation (6.26-6.27) no longer places any constraints on the model constants.

Table 8.2

Comparison of predictions of different models with the experimental data of Tavoularis & Karnik (1989) in homogeneous shear flow ($P_K/\varepsilon = 1.8$)

Equil. values	*LRR model*	*SL model*	*FLT model*	*SSG model*	*Experiments*
$(b_{11})_\infty$	0.152	0.120	0.196	0.218	0.21
$(b_{22})_\infty$	-0.119	-0.122	-0.136	-0.145	-0.14
$(b_{33})_\infty$	-0.033	0.002	-0.060	-0.073	-0.07
$(b_{12})_\infty$	-0.186	-0.121	-0.151	-0.164	-0.16
$(SK/\varepsilon)_\infty$	4.83	7.44	5.95	5.50	5.0

For any *homogeneous turbulent flow* in a rotating frame (see also equation 6.13), second-order closure models take the form (Speziale 1989):

$$\frac{\partial \tau'_{ij}}{\partial t} = -\tau'_{ik}\frac{\partial \overline{u}_j}{\partial x_k} - \tau'_{jk}\frac{\partial \overline{u}_i}{\partial x_k} + \Pi_{ij} - \varepsilon_{ij} - 2\left(\tau'_{ik}e_{mkj}\Omega_m + \tau'_{jk}e_{mki}\Omega_m\right) \tag{8.5}$$

where the mean-vorticity tensor $\overline{W_{ij}}$ in the model for Π_{ij} (equation 6.63) is replaced by the intrinsic mean-vorticity tensor defined in equation (2.41).

The equations of motion for the *LRR* model are obtained by substituting (8.1-8.2) into (8.5), and in the reduced form for homogeneous flows of the modeled ε- transport equation (5.20) (see also Chapter 7):

$$\frac{\partial \varepsilon}{\partial t} = -C_{\varepsilon 1}\frac{\varepsilon}{K}\tau'_{ij}\frac{\partial \overline{u}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K}, \tag{8.6}$$

that is not directly affected by rotations.

A complete dynamical-systems analysis of these nonlinear ordinary differential equations (that are typically solved for initial conditions that correspond to a state of isotropic turbulence) has been conducted by Speziale & Mac Giolla Mhuiris (1989*a*). It has been found that ε/SK and b_{ij} have bounded equilibrium values that are independent of the initial conditions and only depend on Ω and S through the dimensionless ratio Ω/S .

In the case of *nonequilibrium homogeneous turbulence* the results are remarkably different.

In Figure 8.9 predictions of the *SSG*, *SL* and *FLT* models for the temporal evolution of the turbulent kinetic energy are compared with the *DNS* calculations of Lee, Kim & Moin (1990) in the case of a rapidly distorted turbulent flow that initially is far from equilibrium, being $SK_0/\varepsilon_0 = 50$ (the equilibrium value of SK/ε is approximately 5). All the models perform rather poorly with respect to the *DNS* calculations of Lee *et al.*

(1990). Even the *SSG* model that performs remarkably well in homogeneous shear flows that are not far from equilibrium, strongly overpredicts the growth rate of the turbulent kinetic energy in this strongly nonequilibrium test case.

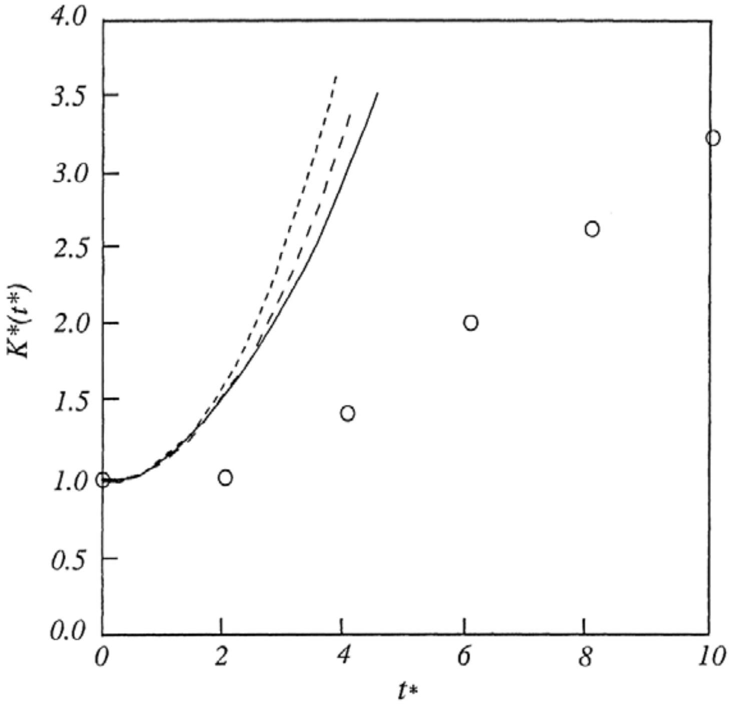


Figure 8.9. Temporal evolution of the turbulent kinetic energy in nonequilibrium homogeneous shear flow ($SK_0/\varepsilon_0 = 50$): (——) *SSG* model; (– – –) *SL* model; (----) *FLT* model; (○○○) *DNS* calculations of Lee *et al.* (1990) (adapted from Speziale *et al.* 1992).

8.2.4 Channel Flow

In Figure (8.10) the mean-velocity profile computed by Launder *et al.* (1987) in the rotating channel flow using the Gibson & Launder (1978) second-order closure model is compared with the results of the $K-\varepsilon$ model and the experimental data of Johnston *et al.* (1972), at Reynolds number $Re = 11500$ and rotation number $Ro = 0.21$. The second-order closure model gives a highly asymmetric mean-velocity profile, well within the range of the experimental data, while the standard $K-\varepsilon$ model erroneously predicts the same symmetric mean-velocity profile as in an inertial frame ($\Omega = 0$).

Comparable improvements in the prediction of curved turbulent shear flows have been obtained by Gibson & Rodi (1981) and Gibson & Younis (1986) using second-order closure models.

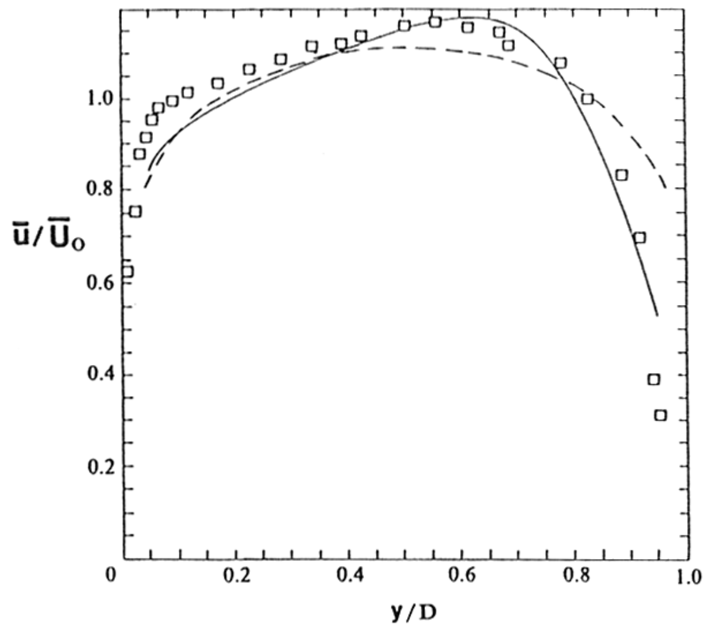


Figure 8.10. Mean-velocity profile in rotating channel flow: (——) second-order closure model; (– – –) $K-\varepsilon$ model; (□□□) experimental data of Johnston *et al.* (1972) (adapted from Launder *et al.* 1987).

Major deficiencies of second-order closure models emerge in *nonhomogeneous wall-bounded turbulent flows.* More in particular:

i) *ad hoc* wall-reflection terms are needed in many pressure-strain models (that depend of the distance y from the wall) in order to mask deficient predictions for the logarithmic region of a turbulent boundary layer;

ii) near-wall models must be typically introduced that depend on the unit normal to the wall, a feature that makes it virtually impossible to systematically integrate second-order closures in complex geometries.

Demuren & Sarkar (1993) and Abid & Speziale (1993) have investigated the prediction of Reynolds stress anisotropies in the logarithmic region of an equilibrium turbulent boundary layer when *ad hoc* wall reflection terms are used.

Table 8.3 compares the predictions of the *LRR* model without wall reflection, the *SL* model, the *FLT* model and the *SSG* model with the experimental data of Laufer (1951) for the log-layer of turbulent channel flow. Most of the models give errors ranging from 30% to 100%. These models are then typically forced into agreement with experimental data by the addition of wall-proximity terms that depend inversely on the distance from the wall, an alteration that compromises the ability to apply the models in complex geometries where the wall distance is not always uniquely defined.

Table 8.3

Comparison of predictions of different models with the experimental data of Laufer (1951) in the logarithmic layer of turbulent channel flow ($P_K/\varepsilon = 1$)

Equil. values	*LRR model*	*SL model*	*FLT model*	*SSG model*	*Experiments*
$(b_{11})_\infty$	0.129	0.079	0.141	0.201	0.22
$(b_{22})_\infty$	-0.101	-0.082	-0.099	-0.127	-0.15
$(b_{33})_\infty$	-0.028	0.003	-0.042	-0.074	-0.07
$(b_{12})_\infty$	-0.178	-0.116	-0.162	-0.160	-0.16
$(SK/\varepsilon)_\infty$	2.80	4.30	3.09	3.12	3.1

Only the *SSG* model gives acceptable results for the logarithmic layer without any wall-reflection term. According to Abid & Speziale (1993) the good performance of the *SSG* model is due to:

i) a careful and accurate calibration of homogeneous shear flow (see Table 8.2);

ii) the use of a Rotta coefficient not too far from the value one.

The significance of these results is demonstrated in Figure 8.11 where full Reynolds-stress computations of turbulent channel flow of Abid & Speziale (1993) are compared with the experimental data of Laufer (1951). The same favorable trends are exhibited in these results, as with those shown in Table 8.3 that have been obtained by a simplified log-layer analysis.

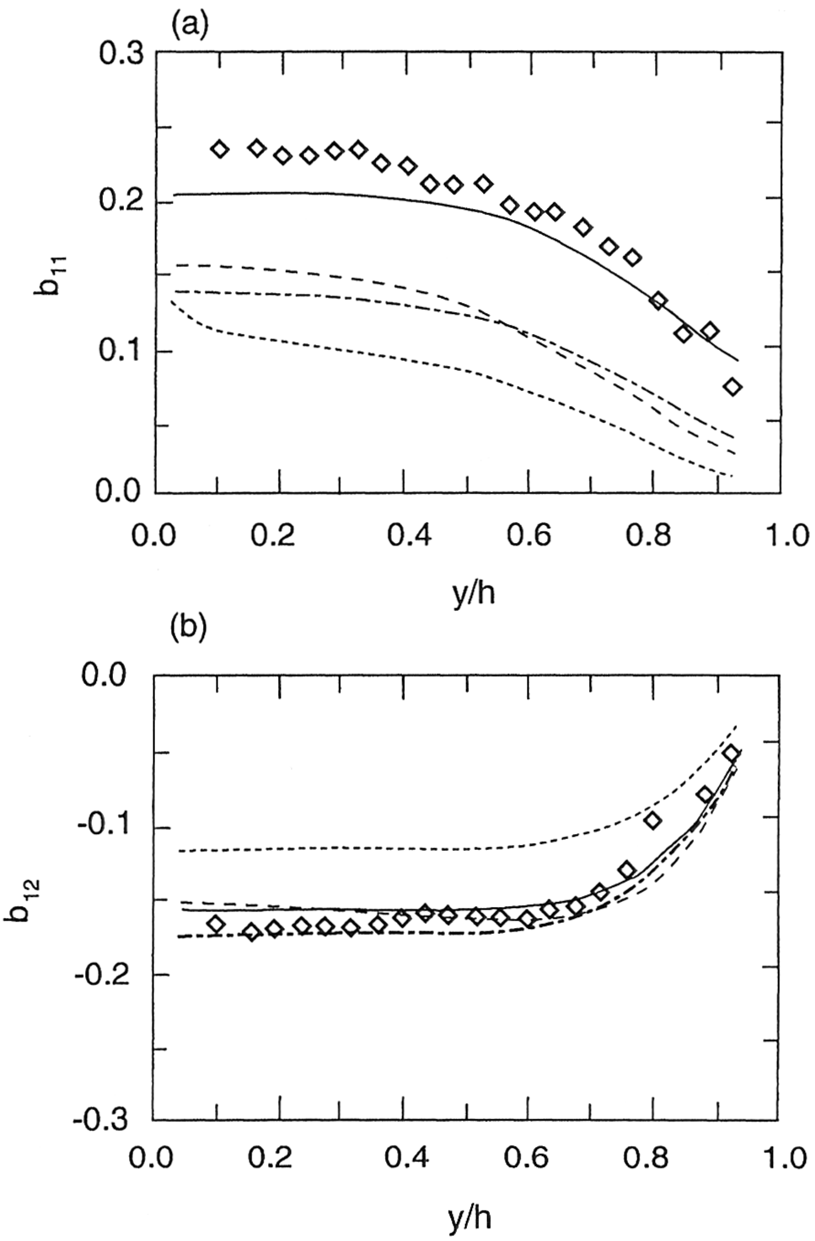


Figure 8.11. Full Reynolds-stress calculations of turbulent channel flow at $Re = 61600$: (——) *SSG* model; (– – –) *FLT* model; (– · – · –) *LRR* model; (----) *SL* model; (◊◊◊) experimental data of Laufer (1951) (adapted from Abid & Speziale 1993).

Demuren & Sarkar (1993) showed that in the logarithmic layer the transport equation for the Reynolds-stress anisotropy b_{ij} reduces to an algebraic equation *independent* of the turbulent dissipation rate. Therefore, the differences in the predictions of equilibrium values of Reynolds-stress anisotropy between any two second-order closures is solely a function of differences in their pressure-strain models.

The near-wall problems largely arise from the use of homogeneous pressure-strain models of the form (6.20) that are only theoretically justified for near-equilibrium homogeneous turbulence. Durbin (1993) developed an elliptic relaxation model that accounts for wall blocking and introduces nonlocal effects in the vicinity of the walls, eliminating the need for *ad hoc* damping functions. While this is a promising new approach, it does not alleviate the problems that the commonly used pressure-strain models have in nonequilibrium homogeneous turbulence.

Chapter 9

Concluding Remarks

In the framework of the *RANS* approach to turbulence modeling, Reynolds-stress models provide at most predictive information about first and second one-point moments (mean velocity, mean pressure, turbulence intensity) that is often what is needed in traditional engineering. Reynolds-stress modeling actually constitutes a low-order one-point closure and thus it can not provide information about flow structures. Moreover, since spectral information needs to be indirectly built into the Reynolds-stress models, a given model can not be expected to perform well in a number of flows where the spectrum of the energy-containing eddies deeply changes.

The appropriate issue related to the formulation of models for the Reynolds-stress term in the Reynolds-Averaged Navier-Stokes equations is whether or not a model of this nature can provide adequate answers for the mean velocity, mean pressure and turbulence intensity in a number of turbulent flows of engineering interest.

The calculation of complex three-dimensional turbulent flows invariably involves a balance between the computational resources required to obtain solutions and the physical realism of the turbulence closure to be used. A large body of results and comparisons with experimental measurements illustrates that there are substantial benefits in using second-order closure models in many cases of applicative relevance (no models for the Reynolds-stress tensor in the Reynolds-Averaged Navier-Stokes equations are needed in this case). Second-order closure models represent the most promising class of *RANS* turbulence models, in the sense that they possess superior predictive capabilities with respect to zero-, one- and two-equation models.

There are some major areas in which additional research efforts are required in order to improve the predictive capabilities of second-order closure models:

i) the introduction of improved transport models for the turbulent length scale, incorporating at least some kind of two-point and directional information through the implementation of some appropriate integral of the two-point velocity correlation tensor

R'_{ij} (a transport model for an anisotropic integral length scale). Also the use of gradient transport models has to be reconsidered;

ii) the introduction of asymptotically consistent low turbulence Reynolds number (Re_T) extensions of already existing models, that can be integrated to a solid boundary. Current models use specifically-developed dumping functions based on the turbulence Reynolds number Re_T that do not allow the treatment of geometrical discontinuities such as those occurrig in a square duct or in back-step kind of problems. Moreover, the representation of nonlinear effects of rotational and irrotational strains in the modeling of near-wall anisotropies in the dissipation, has to be implemented.

In addition, entirely new nonequilibrium models are needed for the pressure-strain correlation and the dissipation-rate tensor. The former should contain nonlinear strain-rate effects and the latter should account for the effects of anisotropic dissipation and nonequilibrium vortex-stretching.

Appendix

Governing Equations

A number of differential equations that govern fluid dynamics are reported in different forms and notations in the case of Newtonian incompressible viscous fluid with constant properties. Some emphasis is given to the fact that the Navier-Stokes equations can be written in different forms, according to dimensionality and to the form in which both convective and diffusive terms are expressed. Corresponding forms can be recognized in a number of other equations, derived from the momentum equations.

A.1 Basic Equations

The principle of mass conservation can be expressed in differential form as follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \qquad \text{(A.1}a\text{)}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \qquad \text{(A.1}b\text{)}$$

where ρ is the fluid density.

The differential form of the momentum equation for a continuum is:

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} \qquad \text{(A.2}a\text{)}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \text{(A.2}b\text{)}$$

where body forces do not appear explicitly (they can be included in the pressure terms in terms of their potential) and τ_{ij} is the viscous stress tensor ($T_{ij} = -p\delta_{ij} + \tau_{ij}$ is the stress tensor).

The dot product of $\boldsymbol{u}$ with the momentum equation (A.2) ($|\boldsymbol{u}|^2 = u_j u_j = u^2$) gives a differential equation that governs the kinetic energy $E = u^2/2$:

$$\frac{\partial}{\partial t}(\rho E) + \nabla \cdot (\rho \boldsymbol{u} E) = -\boldsymbol{u} \cdot \nabla p + \boldsymbol{u} \cdot (\nabla \cdot \boldsymbol{\tau}) \tag{A.3a}$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_i}(\rho u_i E) = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ij}}{\partial x_j}. \tag{A.3b}$$

The following identity can be used:

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(\tau_{ij} u_i) - \tau_{ij} \frac{\partial u_i}{\partial x_j} \tag{A.4}$$

that gives:

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_i}(\rho u_i E) = -u_i \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\tau_{ij} u_i) - \tau_{ij} \frac{\partial u_i}{\partial x_j}. \tag{A.5}$$

A.2 Continuity Equation

In the case of incompressible fluid, the principle of mass conservation is expressed by the continuity equation:

$$\nabla \cdot \boldsymbol{u} = 0 \tag{A.6a}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad (A.6b)$$

in symbolic and index notation, respectively. In equation (A.6) no time dependence appears, so that it has the same form in both steady and unsteady flows. Also in switching from the dimensional to the nondimensional formulation, besides the fact that a nondimensional velocity is involved, equation (A.6) keeps the same form. No dimensionless groups are involved, meaning that the continuity equation has a kinematic character, being not directly influenced by any flow parameter.

A.3 Navier-Stokes Equation

In the case of Newtonian fluid, one has:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (A.7)$$

$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad (A.8)$$

or also:

$$\tau_{ij} = 2\mu S_{ij} \qquad (A.9)$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (A.10)$$

$$\frac{\partial \tau_{ij}}{\partial x_j} = 2\mu\frac{\partial S_{ij}}{\partial x_j}, \qquad (A.11)$$

and equation (A.2), in the case of incompressible fluids, becomes:

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_j}\left(u_i u_j\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+2\nu\frac{\partial S_{ij}}{\partial x_j} \tag{A.12}$$

where S_{ij} is the rate-of-strain tensor.

Thus, the Navier-Stokes equation in primitive variables (velocity-pressure formulation) for a Newtonian incompressible viscous fluid with constant properties is:

$$\frac{\partial \boldsymbol{u}}{\partial t}+\nabla\cdot\left(\boldsymbol{u}\boldsymbol{u}\right)=-\frac{1}{\rho}\nabla p+\nu\nabla^2\boldsymbol{u} \tag{A.13a}$$

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_j}\left(u_i u_j\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j} \tag{A.13b}$$

where body forces do not appear explicitly and μ,ν and ρ are the fluid dynamic and kinematic viscosity and density, respectively [$\nabla^2\boldsymbol{u}=\nabla\cdot\left(\nabla\boldsymbol{u}\right)$].

By considering characteristic reference velocity and length scales U and L, the nondimensional form of equation (A.13) in terms of nondimensional velocity and pressure, is obtained:

$$\frac{\partial \boldsymbol{u}}{\partial t}+\nabla\cdot\left(\boldsymbol{u}\boldsymbol{u}\right)=-\nabla p+\frac{1}{Re}\nabla^2\boldsymbol{u} \tag{A.14a}$$

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_j}\left(u_i u_j\right)=-\frac{\partial p}{\partial x_i}+\frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j\partial x_j} \tag{A.14b}$$

where $Re=UL/\nu$ is the Reynolds number. Variables and operators are nondimensional and, for simplicity, their symbols have not been altered in switching from the dimensional to the dimensionless formalism. In the above expressions, local derivative and convective (inertia) term on the left-hand side, and pressure gradient and diffusive (viscous) term on the right-hand side, are recognizable.

A.3.1 Convective Term

Equation (A.13) is the Navier-Stokes equation in conservative (or divergence or also flux) form, according to the form in which the convective term is written.

By expanding the nonlinear term of equation (A.13) and invoking mass conservation (A.6), one has:

$$\nabla \cdot (\boldsymbol{u}\boldsymbol{u}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u}(\nabla \cdot \boldsymbol{u}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \tag{A.15a}$$

$$\frac{\partial}{\partial x_j}(u_i u_j) = u_i \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} = u_j \frac{\partial u_i}{\partial x_j} \tag{A.15b}$$

and the Navier-Stokes equation in nonconservative (or convective) form is obtained:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} \tag{A.16a}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{A.16b}$$

together with the corresponding nondimensional form:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u} \tag{A.17a}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j \partial x_j}. \tag{A.17b}$$

Another form of the Navier-Stokes equation is the rotational. By considering equation (A.16) and using the vector identity:

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \frac{1}{2}\nabla|\boldsymbol{u}|^2 - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = \frac{1}{2}\nabla u^2 - \boldsymbol{u}\times\boldsymbol{\omega} \tag{A.18a}$$

$$u_j\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_i}(u_j u_j) - e_{ijk}u_j\left(e_{klm}\frac{\partial u_m}{\partial x_l}\right) = \frac{1}{2}\frac{\partial}{\partial x_i}(u^2) - e_{ijk}u_j\omega_k \tag{A.18b}$$

where e_{ijk} is the alternating unit tensor and $|\boldsymbol{u}|^2 = u_j u_j = u^2$, one has:

$$\frac{\partial \boldsymbol{u}}{\partial t} - (\boldsymbol{u}\times\boldsymbol{\omega}) = -\frac{1}{\rho}\nabla\left(p + \frac{1}{2}u^2\right) + \nu\nabla^2\boldsymbol{u} \tag{A.19a}$$

$$\frac{\partial u_i}{\partial t} - e_{ijk}u_j\omega_k = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p + \frac{1}{2}u^2\right) + \nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}, \tag{A.19b}$$

together with the corresponding nondimensional form:

$$\frac{\partial \boldsymbol{u}}{\partial t} - (\boldsymbol{u}\times\boldsymbol{\omega}) = -\nabla\left(p + \frac{1}{2}u^2\right) + \frac{1}{Re}\nabla^2\boldsymbol{u} \tag{A.20a}$$

$$\frac{\partial u_i}{\partial t} - e_{ijk}u_j\omega_k = -\frac{\partial}{\partial x_i}\left(p + \frac{1}{2}u^2\right) + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j\partial x_j}, \tag{A.20b}$$

where $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ is the vorticity and $(p + u^2/2)$ is the pressure head (total or dynamic pressure).

It is also possible to express the nonlinear term of the Navier-Stokes equation in skew-symmetric form. Using the vector identity:

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}(\nabla\cdot\boldsymbol{u})=(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=\frac{1}{2}[\nabla\cdot(\boldsymbol{uu})+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] \qquad (A.21a)$$

$$u_j\frac{\partial u_i}{\partial x_j}+\frac{1}{2}u_i\frac{\partial u_j}{\partial x_j}=u_j\frac{\partial u_i}{\partial x_j}=\frac{1}{2}\left[\frac{\partial}{\partial x_j}(u_iu_j)+u_j\frac{\partial u_i}{\partial x_j}\right], \qquad (A.21b)$$

the corresponding form of the complete equation is obtained:

$$\frac{\partial \boldsymbol{u}}{\partial t}+\frac{1}{2}[\nabla\cdot(\boldsymbol{uu})+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]=-\frac{1}{\rho}\nabla p+\nu\nabla^2\boldsymbol{u} \qquad (A.22a)$$

$$\frac{\partial u_i}{\partial t}+\frac{1}{2}\left[\frac{\partial}{\partial x_j}(u_iu_j)+u_j\frac{\partial u_i}{\partial x_j}\right]=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}, \qquad (A.22b)$$

together with the corresponding nondimensional form:

$$\frac{\partial \boldsymbol{u}}{\partial t}+\frac{1}{2}[\nabla\cdot(\boldsymbol{uu})+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]=-\nabla p+\frac{1}{Re}\nabla^2\boldsymbol{u} \qquad (A.23a)$$

$$\frac{\partial u_i}{\partial t}+\frac{1}{2}\left[\frac{\partial}{\partial x_j}(u_iu_j)+u_j\frac{\partial u_i}{\partial x_j}\right]=-\frac{\partial p}{\partial x_i}+\frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j\partial x_j}. \qquad (A.23b)$$

In computational fluid dynamics and turbulence simulation the different forms of the Navier-Stokes equations are of relevance with respect to the numerical technique that is adopted for their integration. This relevance is mainly related to the ability of the different forms of the nonlinear term in preserving momentum, vorticity, energy or other invariants in the discretized sense, in absence of both viscous dissipation and errors in the temporal derivatives.

The conservative form preserves momentum easily when discretized. The nonconservative form in general does not conserve momentum and kinetic energy, when discretized. The rotational form preserves vorticity and enstrophy besides momentum and kinetic energy. The skew-symmetric form preserves kinetic energy easily and is accurate when spectral techniques are used (Gresho 1991).

Horiuti (1987) compared conservative and rotational forms of the Navier-Stokes equations in a study of turbulent channel flow in which a Fourier-finite difference method is used. He found that the combined use of the rotational form of the convective term and second-order central finite differences along the domain's normal direction can yield to remarkable truncation errors near the walls. Zang (1991) compared rotational and skew-symmetric forms of the Navier-Stokes equation and one of his conclusions was that the rotational form produces aliasing errors more damaging with respect to those produced by the the skew-symmetric form. Kravcenko & Moin (1995) found that for divergence and convective forms of the nonlinear term of the momentum equation, spectral methods are energy conserving only if dealiased. Conservative and nonconservative forms of the convective term of the Navier-Stokes equation have been discretized using fifth-order accurate upwind-biased approximations by Tafti (1996). No remarkable differences were found in the accuracy of the calculations, with respect to the form of the nonlinear term.

A.3.2 Diffusive Term

Other forms of the Navier-Stokes equation can be written by considering different forms of the viscous term:

$$\nabla^2 \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u}) = -\nabla \times (\nabla \times \boldsymbol{u}) = -\nabla \times \boldsymbol{\omega} \,. \tag{A.24}$$

One has:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u}\boldsymbol{u}) = -\frac{1}{\rho}\nabla p - \nu \nabla \times (\nabla \times \boldsymbol{u}) = -\frac{1}{\rho}\nabla p - \nu \nabla \times \boldsymbol{\omega} \tag{A.25}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\frac{1}{\rho}\nabla p-\nu\nabla\times(\nabla\times\boldsymbol{u})=-\frac{1}{\rho}\nabla p-\nu\nabla\times\boldsymbol{\omega} \tag{A.26}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}-(\boldsymbol{u}\times\boldsymbol{\omega})=-\frac{1}{\rho}\nabla\left(p+\frac{1}{2}u^2\right)-\nu\nabla\times(\nabla\times\boldsymbol{u})=-\frac{1}{\rho}\nabla\left(p+\frac{1}{2}u^2\right)-\nu\nabla\times\boldsymbol{\omega} \tag{A.27}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}+\frac{1}{2}[\nabla\cdot(\boldsymbol{u}\boldsymbol{u})+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]=-\frac{1}{\rho}\nabla p-\nu\nabla\times(\nabla\times\boldsymbol{u})=-\frac{1}{\rho}\nabla p-\nu\nabla\times\boldsymbol{\omega} \tag{A.28}$$

that are conservative, nonconservative, rotational and skew-symmetric forms of the Navier-Stokes equation with the viscous term expressed in divergence-curl and curl forms.

Another form of the diffusive term of the Navier-Stokes equation is the stress-divergence form:

$$\nabla^2\boldsymbol{u}+\nabla(\nabla\cdot\boldsymbol{u})=\nabla^2\boldsymbol{u}=\nabla\cdot\left[(\nabla\boldsymbol{u})+(\nabla\boldsymbol{u})^T\right] \tag{A.29}$$

from which additional forms of the complete equation, commonly used in the the Finite Element method (Gresho 1991), can be written:

$$\frac{\partial \boldsymbol{u}}{\partial t}+\nabla\cdot(\boldsymbol{u}\boldsymbol{u})=-\frac{1}{\rho}\nabla p+\nu\nabla\cdot\left[(\nabla\boldsymbol{u})+(\nabla\boldsymbol{u})^T\right] \tag{A.30}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\frac{1}{\rho}\nabla p+\nu\nabla\cdot\left[(\nabla\boldsymbol{u})+(\nabla\boldsymbol{u})^T\right] \tag{A.31}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}-(\boldsymbol{u}\times\boldsymbol{\omega})=-\frac{1}{\rho}\nabla\left(p+\frac{1}{2}u^2\right)+\nu\nabla\cdot\left[(\nabla\boldsymbol{u})+(\nabla\boldsymbol{u})^T\right] \tag{A.32}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}+\frac{1}{2}[\nabla \cdot (\boldsymbol{uu})+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]=-\frac{1}{\rho}\nabla p+\nu\nabla\cdot\left[(\nabla\boldsymbol{u})+(\nabla\boldsymbol{u})^T\right]. \tag{A.33}$$

A.4 Pressure Equation

By applying the divergence operator to the Navier-Stokes equation (form A.13) and using mass conservation, an elliptic Poisson equation for the pressure is obtained:

$$\nabla\cdot[\nabla\cdot(\boldsymbol{uu})]=-\frac{1}{\rho}\nabla^2 p \tag{A.34a}$$

$$\frac{\partial^2}{\partial x_i \partial x_j}(u_j u_i)=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_i}, \tag{A.34b}$$

together with the corresponding nondimensional form:

$$\nabla\cdot[\nabla\cdot(\boldsymbol{uu})]=-\nabla^2 p \tag{A.35a}$$

$$\frac{\partial^2}{\partial x_i \partial x_j}(u_j u_i)=-\frac{\partial^2 p}{\partial x_i \partial x_i}. \tag{A.35b}$$

It can be deduced that, for incompressible fluids and with a given velocity field, the pressure is uniquely determined from the solution of equations (A.34) or (A.35), once appropriate boundary conditions for the velocity are enforced.
Similarly (considering form A.16) one has:

$$\nabla\cdot[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]=-\frac{1}{\rho}\nabla^2 p \tag{A.36a}$$

$$\frac{\partial}{\partial x_i}\left(u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_i} \tag{A.36b}$$

and the corresponding nondimensional form:

$$\nabla \cdot [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] = -\nabla^2 p \tag{A.37a}$$

$$\frac{\partial}{\partial x_i}\left(u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial^2 p}{\partial x_i \partial x_i}. \tag{A.37b}$$

Note that in the above expressions the viscous term does not appear, according to the fact that the pressure is not related to shear. In the nondimensional forms of the above equations the dipendent variable, besides the velocity, is the nondimensional pressure $p/\rho U^2$.

A.5 Vorticity Equation

By applying the curl operator to the Navier-Stokes equation (the application of the curl operator to the continuity equation leads to $\nabla \cdot \boldsymbol{\omega} = 0$), one obtains the vorticity transport equation. Using the vector identity:

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - \boldsymbol{\omega}(\nabla \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} + \boldsymbol{u}(\nabla \cdot \boldsymbol{\omega}) =$$

$$-\nabla \cdot (\boldsymbol{u}\boldsymbol{\omega}) + \nabla(\boldsymbol{\omega}\boldsymbol{u}) \tag{A.38a}$$

$$e_{ijk}\frac{\partial}{\partial x_j}(e_{klm}u_l\omega_m) = \omega_j \frac{\partial u_i}{\partial x_j} - \omega_i \frac{\partial u_j}{\partial x_j} - u_j \frac{\partial \omega_i}{\partial x_j} + u_i \frac{\partial \omega_j}{\partial x_j} =$$

$$-\frac{\partial}{\partial x_j}\left(u_j\omega_i\right)+\frac{\partial}{\partial x_j}\left(\omega_j u_i\right), \tag{A.38b}$$

one has:

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+\nabla\cdot\left(\boldsymbol{u\omega}\right)=\nabla\cdot\left(\boldsymbol{\omega u}\right)+\nu\nabla^2\boldsymbol{\omega} \tag{A.39a}$$

$$\frac{\partial \omega_i}{\partial t}+\frac{\partial}{\partial x_j}\left(u_i\omega_j\right)=\frac{\partial}{\partial x_j}\left(\omega_i u_j\right)+\nu\frac{\partial^2\omega_i}{\partial x_j\partial x_j}, \tag{A.39b}$$

together with the corresponding nondimensional form:

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+\nabla\cdot\left(\boldsymbol{u\omega}\right)=\nabla\cdot\left(\boldsymbol{\omega u}\right)+\frac{1}{Re}\nabla^2\boldsymbol{\omega} \tag{A.40a}$$

$$\frac{\partial \omega_i}{\partial t}+\frac{\partial}{\partial x_j}\left(u_i\omega_j\right)=\frac{\partial}{\partial x_j}\left(\omega_i u_j\right)+\frac{1}{Re}\frac{\partial^2\omega_i}{\partial x_j\partial x_j} \tag{A.40b}$$

or also:

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{\omega}=\left(\boldsymbol{\omega}\cdot\nabla\right)\boldsymbol{u}+\nu\nabla^2\boldsymbol{\omega} \tag{A.41a}$$

$$\frac{\partial \omega_i}{\partial t}+u_j\frac{\partial \omega_i}{\partial x_j}=\omega_j\frac{\partial u_i}{\partial x_j}+\nu\frac{\partial^2\omega_i}{\partial x_j\partial x_j} \tag{A.41b}$$

and the corresponding nondimensional form:

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{\omega}=\left(\boldsymbol{\omega}\cdot\nabla\right)\boldsymbol{u}+\frac{1}{Re}\nabla^2\boldsymbol{\omega} \tag{A.42a}$$

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} . \qquad \text{(A.42}b\text{)}$$

Note that in the above expressions the pressure term does not appear according to the fact that the vorticity is related to shear and not to pressure. The first term on the right-hand side of the vorticity transport equation is the vortex-stretching term, that has no analogous term in the Navier-Stokes equations in primitive variables. In the nondimensional forms of the above equations the dipendent variable, besides the velocity, is the nondimensional vorticity $\omega_i L/U$.

A.6 Energy Equation

By performing the dot product of $\boldsymbol{u}$ with the Navier-Stokes equation ($|\boldsymbol{u}|^2 = u_j u_j = u^2$), an equation describing the evolution of the kinetic energy $E = u^2/2$ is obtained.
One has:

$$\frac{\partial E}{\partial t} + \nabla \cdot (\boldsymbol{u}E) = -\frac{1}{\rho} \nabla(\boldsymbol{u}p) + \nu \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u} \qquad \text{(A.43}a\text{)}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}(u_j E) = -\frac{1}{\rho} \frac{\partial}{\partial x_i}(u_i p) + \nu u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad \text{(A.44}b\text{)}$$

or also:

$$\frac{\partial E}{\partial t} + (\boldsymbol{u} \cdot \nabla) E = -\frac{1}{\rho} \nabla (\boldsymbol{u} \cdot \nabla) p + \nu \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u} \qquad \text{(A.45}a\text{)}$$

$$\frac{\partial E}{\partial t} + u_j \frac{\partial E}{\partial x_j} = -\frac{1}{\rho} u_i \frac{\partial p}{\partial x_i} + \nu u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j} . \qquad \text{(A.45}b\text{)}$$

The following identities can be used:

$$u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j} = u_i \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{A.46}$$

$$\frac{\partial}{\partial x_j}\left[u_i\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] = \frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + u_i \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{A.47}$$

that give:

$$\frac{\partial E}{\partial t} + u_j \frac{\partial E}{\partial x_j} = -\frac{1}{\rho} u_i \frac{\partial p}{\partial x_i} + \nu u_i \frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{A.48}$$

$$\frac{\partial E}{\partial t} + u_j \frac{\partial E}{\partial x_j} = -\frac{1}{\rho} u_i \frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left[u_i\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] - \nu \frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{A.49}$$

A.7 Enstrophy Equation

By performing the dot product of $\boldsymbol{\omega}$ with the vorticity transport equation ($\left|\boldsymbol{\omega}^2\right| = \omega_i \omega_i = \omega^2$), an equation for the evolution of the enstrophy $\Omega = \omega^2 / 2$ is obtained. One has:

$$\frac{\partial \Omega}{\partial t} + \nabla \cdot (\boldsymbol{u}\Omega) = \boldsymbol{\omega} \cdot \nabla \cdot (\boldsymbol{\omega u}) + \nu \boldsymbol{\omega} \cdot \nabla^2 \boldsymbol{\omega} \tag{A. 50a}$$

$$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial x_j}(u_j \Omega) = \omega_i \frac{\partial}{\partial x_j}(\omega_i u_j) + \nu \omega_i \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \tag{A.50b}$$

or also:

$$\frac{\partial \Omega}{\partial t} + (\boldsymbol{u} \cdot \nabla)\Omega = \boldsymbol{\omega} \cdot (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \nu\boldsymbol{\omega} \cdot \nabla^2 \boldsymbol{\omega} \tag{A.51a}$$

$$\frac{\partial \Omega}{\partial t} + u_j \frac{\partial \Omega}{\partial x_j} = \omega_i \omega_j \frac{\partial u_i}{\partial x_j} + \nu\omega_i \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} . \tag{A.51b}$$

The following identities can be used:

$$\omega_i \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} = \omega_i \frac{\partial}{\partial x_j}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right) \tag{A.52}$$

$$\frac{\partial}{\partial x_j}\left[\omega_i\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right)\right] = \frac{\partial \omega_i}{\partial x_j}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right) + \omega_i \frac{\partial}{\partial x_j}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right) \tag{A.53}$$

that give:

$$\frac{\partial \Omega}{\partial t} + u_j \frac{\partial \Omega}{\partial x_j} = \omega_i \omega_j \frac{\partial u_i}{\partial x_j} + \nu\omega_i \frac{\partial}{\partial x_i}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right) \tag{A.54}$$

$$\frac{\partial \Omega}{\partial t} + u_j \frac{\partial \Omega}{\partial x_j} = \omega_i \omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial}{\partial x_j}\left[\omega_i\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right)\right] - \nu \frac{\partial \omega_i}{\partial x_j}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right). \tag{A.55}$$

A.8 Systems of Coordinates

In any orthogonal curvilinear coordinate system $x_i\,(x_1, x_2, x_3)$, some of the operators most frequently used in fluid dynamics have the following general form in terms of scale factors $h_i\,(h_1, h_2, h_3)$:

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial x_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial x_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial x_3}\boldsymbol{e}_3 \tag{A.56}$$

$$\nabla \cdot \boldsymbol{u} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 u_1) + \frac{\partial}{\partial x_2}(h_3 h_1 u_2) + \frac{\partial}{\partial x_3}(h_1 h_2 u_3)\right] \tag{A.57}$$

$$\nabla \times \boldsymbol{u} = \frac{1}{h_1 h_2 h_3}\left[\left(\frac{\partial}{\partial x_2}(h_3 u_3) - \frac{\partial}{\partial x_3}(h_2 u_2)\right)h_1\boldsymbol{e}_1 - \left(\frac{\partial}{\partial x_1}(h_3 u_3) - \frac{\partial}{\partial x_3}(h_1 u_1)\right)h_2\boldsymbol{e}_2\right.$$

$$\left. + \left(\frac{\partial}{\partial x_1}(h_2 u_2) - \frac{\partial}{\partial x_2}(h_1 u_1)\right)h_3\boldsymbol{e}_3\right] \tag{A.58}$$

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial x_3}\right)\right] \tag{A.59}$$

where $\boldsymbol{e}_i$ ($\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$) are the unit vectors in the directions of increasing x_i (x_1, x_2, x_3) and f is a scalar function. The values of h_1, h_2, h_3 in three frequently used orthogonal curvilinear coordinate systems, are:

i) rectangual cartesian coordinates (x,y,z):

$$h_1 = h_x = 1; \quad h_2 = h_y = 1; \quad h_3 = h_z = 1; \tag{A.60}$$

ii) cylindrical polar coordinates (r,θ,z):

$$h_1 = h_r = 1; \quad h_2 = h_\theta = r; \quad h_3 = h_z = 1; \tag{A.61}$$

iii) spherical polar coordinates (r,θ,φ):

$$h_1 = h_r = 1\,; \quad h_2 = h_\theta = r\,; \quad h_3 = h_\varphi = r\sin\theta\,. \tag{A.62}$$

A.9 Mass Conservation

The equation of mass conservation in rectangular coordinates (x,y,z) is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u_x\right) + \frac{\partial}{\partial y}\left(\rho u_y\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) = 0\,, \tag{A.63}$$

in cylindrical polar coordinates (r,θ,z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) = 0\,, \tag{A.64}$$

and in spherical polar coordinates (r,θ,φ):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\rho u_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\left(\rho u_\varphi\right) = 0\,. \tag{A.65}$$

A.10 Momentum Equation

The momentum equation for a Newtonian incompressible viscous fluid with constant properties in rectangular coordinates (x,y,z) is:

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) \tag{A.66a}$$

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) \tag{A.66b}$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right). \quad \text{(A.66}c\text{)}$$

The momentum equation for a Newtonian incompressible viscous fluid with constant properties in cylindrical polar coordinates (r,θ,z) is:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=$$

$$\rho g_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right] \quad \text{(A.67}a\text{)}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=$$

$$\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right] \quad \text{(A.67}b\text{)}$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=$$

$$\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]. \quad \text{(A.67}c\text{)}$$

The momentum equation for a Newtonian incompressible viscous fluid with constant properties in spherical polar coordinates (r,θ,φ) is:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\varphi}{r\sin\theta}\frac{\partial u_r}{\partial \varphi}-\frac{u_\theta^2+u_\varphi^2}{r}\right)=$$

$$\rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right)\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u_r}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u_r}{\partial \varphi^2}\right.$$

$$\left. - \frac{2}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right) - \frac{2}{r^2 \sin\theta}\frac{\partial u_\varphi}{\partial \varphi}\right] \qquad \text{(A.68}a\text{)}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\varphi}{r \sin\theta}\frac{\partial u_\theta}{\partial \varphi} + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cot\theta}{r}\right) =$$

$$\rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right)\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u_\theta}{\partial \varphi^2}\right.$$

$$\left. + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial u_\varphi}{\partial \varphi}\right] \qquad \text{(A.68}b\text{)}$$

$$\rho\left(\frac{\partial u_\varphi}{\partial t} + u_r \frac{\partial u_\varphi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\varphi}{\partial \theta} + \frac{u_\varphi}{r \sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\varphi u_r}{r} + \frac{u_\theta u_\varphi \cot\theta}{r}\right) =$$

$$\rho g_\varphi - \frac{1}{r \sin\theta}\frac{\partial p}{\partial \varphi} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(u_\varphi \sin\theta\right)\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u_\varphi}{\partial \varphi^2}\right.$$

$$\left. + \frac{2}{r^2 \sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial u_\theta}{\partial \varphi}\right]. \qquad \text{(A.68}c\text{)}$$

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