

# HITTING TIME AND MIXING TIME BOUNDS OF STEIN'S FACTORS

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**ABSTRACT.** For any discrete target distribution, we exploit the connection between Markov chains and Stein's method via the generator approach and express the solution of Stein's equation in terms of expected hitting time. This yields new upper bounds of Stein's factors in terms of the parameters of the Markov chain, such as mixing time and the gradient of expected hitting time. We compare the performance of these bounds with those in the literature, and in particular we improve upon an earlier estimate by [Eichelsbacher and Reinert \(2008\)](#) from  $O(e^{n \log n})$  to  $O(\log n)$  in binomial approximation, and obtain new bounds in hypergeometric approximation. As another application, the same methodology applies to bound expected hitting time via Stein's factors. This article highlights the interplay between Stein's method, modern Markov chain theory and classical fluctuation theory.

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## 1. INTRODUCTION AND MAIN RESULTS

Stein's method is well-known to be a powerful method for bounding the error rates of various distributional approximation, see e.g. [Barbour and Chen \(2005\)](#); [Diaconis and Holmes \(2004\)](#); [Ley et al. \(2017\)](#); [Ross \(2011\)](#) and the references therein. At the heart of it lies the Stein's equation

$$(1.1) \quad h(x) - \mathbb{E}h(Z) = Lf_h(x),$$

where  $Z \sim \pi$  is the target distribution with support on  $\mathcal{X}$ ,  $L$  is the Stein operator associated with  $\pi$ ,  $h$  belongs to a rich function class such as the class of indicator functions or Lipschitz continuous functions and  $f_h$  is the solution of the Stein's equation. One popular approach to identify the Stein operator  $L$  is the generator approach introduced by [Barbour \(1990\)](#); [Götze \(1991\)](#), where  $L$  is the generator of a Markov process  $X = (X_t)_{t \geq 0}$  with transition semigroup  $(P_t)_{t \geq 0}$  on the state space  $\mathcal{X}$  and stationary distribution

$\pi$ . Writing  $\pi(f) = \int f d\pi$ , the solution  $f_h$  can then be related to  $X$  via

$$(1.2) \quad f_h(x) = \int_0^\infty P_t h(x) - \pi(h) dt,$$

whenever the above integral exists. The obvious advantage of this approach is the connection with Markov processes, see e.g. [Brown and Xia \(2001\)](#); [Eichelsbacher and Reinert \(2008\)](#). In addition, the solution form of  $f_h$  naturally invites spectral techniques for Stein's method, which has been the subject of investigation in [Schoutens \(2001\)](#). We also remark that [Döbler et al. \(2015\)](#) propose an interesting iterative procedure for bounding  $f_h$  and its derivative.

Suppose now  $\mathcal{X} = \llbracket 0, N \rrbracket$  with  $N \in \mathbb{N}_0 \cup \{\infty\}$  is a countable set, where we write  $\llbracket a, b \rrbracket = \{a, a+1, \dots, b-1, b\}$  for  $a, b \in \mathbb{Z}$ . In Markov chain theory, the operator appearing on the right of (1.2)

$$D := \int_0^\infty P_t - \pi dt$$

is commonly known as deviation kernel  $D = (D(i, j))_{i, j \in \mathcal{X}}$  as in [Coolen-Schrijner and van Doorn \(2002\)](#); [Mao \(2004\)](#) (also known as fundamental matrix [Kemeny et al. \(1976\)](#), ergodic potential [Syski \(1978\)](#) or centered resolvent [Miclo \(2016\)](#)). In this paper, we further exploit this intimate connection between Markov chain theory and Stein's method, thereby allowing us to express  $f_h$  in terms of hitting time of an associated birth-death chain for any discrete target distribution  $\pi$  as in [Eichelsbacher and Reinert \(2008\)](#), and from there connects Stein's method with modern Markov chain literature and offer universal bounds of Stein's factors in terms of quantities such as mixing time and eigentime.

Before we discuss our main results, we fix our notations and revisit various parameters of countable Markov chains. We refer readers to [Aldous and Fill \(2002\)](#); [Levin et al. \(2009\)](#); [Montenegro and Tetali \(2006\)](#) for in-depth account of these topics. For any probability measure  $\mu, \nu$  with support on  $\mathcal{X}$ , the total variation distance between  $\mu$  and  $\nu$  is

$$\|\mu - \nu\|_{TV} := \sup_{A \subset \mathcal{X}} |\mu(A) - \nu(A)| = \frac{1}{2} \sum_{j \in \mathcal{X}} |\mu(j) - \nu(j)|.$$

For  $f$  on  $\mathcal{X}$ , we write  $\|f\|_\infty := \sup_x |f(x)|$ , the sup-norm of  $f$ . In this paper, we are primarily interested in the following parameters associated with an ergodic countable Markov chain  $X = (X_t)_{t \geq 0}$ :

- Worst-case mixing time: for any  $\epsilon > 0$ ,

$$t_{mix}(\epsilon) := \inf \left\{ t; \sup_i \|P_t(i, \cdot) - \pi\|_{TV} < \epsilon \right\}.$$

- Average hitting time and relaxation time:

$$t_{av} := \sum_{i, j \in \mathcal{X}} \mathbb{E}_i(\tau_j) \pi(i) \pi(j),$$

where  $\tau_A := \inf \{t; X_t \in A\}$  is the first hitting time of  $A$  by  $X$ , and as usual we write  $\tau_j = \tau_{\{j\}}$ . Note that for uniformly ergodic Markov chain, the eigentime identity [Aldous and Fill \(2002\)](#); [Cui and Mao \(2010\)](#); [Mao \(2004\)](#) is given by

$$t_{av} = \sum_{i=1}^N \frac{1}{\lambda_i} < \infty,$$

where  $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$  are eigenvalues of  $-L$ . A closely related parameter is the relaxation time

$$t_{rel} := 1/\lambda_1,$$

and for finite Markov chains we have  $t_{mix}(\epsilon) \leq t_{rel} \log(1/\epsilon\pi_{min})$  with  $\pi_{min} := \min_i \pi(i)$ , see e.g. (Levin et al., 2009, Theorem 20.6).

- Worst-case expected strong stationary time Aldous (1982):

$$t_{sst} := \sup_i \inf \{ \mathbb{E}_i(T_i) : \mathbb{P}_i(X_{T_i} = j) = \pi(j) \forall j \in \mathcal{X} \}.$$

- Worst-case expected hitting time of large set Oliveira (2012); Peres and Sousi (2015): for  $0 < \alpha < 1/2$ ,

$$t_{hit}(\alpha) := \sup_{x, A \subset \mathcal{X}, \pi(A) \geq \alpha} \mathbb{E}_x(\tau_A).$$

- Worst-case expected deviation of hitting time to a single state Aldous (1982):

$$t_{dev} := \sup_{i,k} \sum_{j \in \mathcal{X}} |\mathbb{E}_i(\tau_j) - \mathbb{E}_k(\tau_j)| \pi(j).$$

Note that Oliveira (2012); Peres and Sousi (2015) show that for finite Markov chains  $t_{mix}(1/4)$  and  $t_{hit}(\alpha)$  are equivalent up to a constant depending on  $\alpha$ , while Aldous (1982) proves the equivalence (up to some universal constants) between  $t_{mix}(1/4), t_{sst}, t_{dev}$  for reversible finite Markov chains. With the above notations in mind, we are now ready to state the main result of this paper:

**Theorem 1.1** (Hitting time as Stein's factors). *Suppose that  $\pi$  is a discrete target distribution on  $\mathcal{X} = \llbracket 0, N \rrbracket$  with  $N \in \mathbb{N}_0 \cup \{\infty\}$ . The deviation kernel  $D$  associated with  $L$  and  $\pi$  exists and is finite if and only if*

$$\sum_{i \in \mathcal{X}} \pi(i) \mathbb{E}_i(\tau_0) < \infty.$$

*In such case, if  $L$  is reversible,  $i, j \in \mathcal{X}$  and for any  $h$  such that  $\pi(|h|) < \infty$ ,*

$$\sum_{i \in \mathcal{X}} \pi(i) |h|(i) \mathbb{E}_i(\tau_j) < \infty,$$

*we can write*

$$(1.3) \quad D(j, j) = \pi(j) \sum_{i \in \mathcal{X}} \pi(i) \mathbb{E}_i(\tau_j),$$

$$(1.4) \quad D(i, j) = D(j, j) - \pi(j) \mathbb{E}_i(\tau_j),$$

$$(1.5) \quad f_h(i) = Dh(i) = \sum_{j \in \mathcal{X}} h(j) D(i, j),$$

$$(1.6) \quad \nabla f_h(i) := f_h(i+1) - f_h(i) = \sum_{j \in \mathcal{X}} h(j) \pi(j) (\mathbb{E}_i(\tau_j) - \mathbb{E}_{i+1}(\tau_j)).$$

*In particular, if  $h = \delta_j$ , the Dirac mass at  $j$ , the sup-norm of the Stein factors are*

$$\begin{aligned} \|f_{\delta_j}\|_\infty &\leq 2t_{av}, \\ \|\nabla f_{\delta_j}\|_\infty &= \pi(j) \sup_{i \in \mathcal{X}} |\mathbb{E}_i(\tau_j) - \mathbb{E}_{i+1}(\tau_j)| \leq t_{dev}. \end{aligned}$$

*Note that we can always pick  $L$  to be a birth-death process, and in such case the expected hitting time  $\mathbb{E}_i(\tau_j)$  is readily computable and is expressed solely in terms of  $\pi$ , see Remark 1.1.*

*Remark 1.1.* For any discrete distribution  $\pi$  on  $\mathcal{X}$ , it is shown in [Eichelsbacher and Reinert \(2008\)](#) that we can always pick  $L$  to be the generator of a birth-death process with birth rate  $b_i := L(i, i+1) = (i+1)\pi(i+1)/\pi(i)$  for  $i \in \llbracket 0, N-1 \rrbracket$  and death rate  $d_i := L(i, i-1) = i$  for  $i \in \llbracket 1, N \rrbracket$ . According to [Coolen-Schrijner and van Doorn \(2002\)](#); [Kijima \(1997\)](#); [Mao \(2004\)](#), for  $i, j \in \mathcal{X} = \llbracket 0, N \rrbracket$ ,

$$\mathbb{E}_i(\tau_j) = \sum_{k=i}^{j-1} \frac{1}{b_k \pi(k)} \pi(\llbracket 0, k \rrbracket) \mathbb{1}_{i < j} + \sum_{k=j}^{i-1} \frac{1}{b_k \pi(k)} \pi(\llbracket k+1, N \rrbracket) \mathbb{1}_{i \geq j},$$

and so the discrete forward gradient  $\nabla f_{\delta_j}$  can be written as

$$\nabla f_{\delta_j}(i) = \pi(j) \left( \frac{1}{b_i \pi(i)} \pi(\llbracket 0, i \rrbracket) \mathbb{1}_{i < j} + \frac{1}{b_i \pi(i)} \pi(\llbracket i+1, N \rrbracket) \mathbb{1}_{i \geq j} \right).$$

Another formula of  $\mathbb{E}_i(\tau_j)$  involves differences in summation of eigenvalues, which is given by, for example for  $i < j$ ,

$$\mathbb{E}_i(\tau_j) = \sum_{k=0}^{j-1} \frac{1}{\lambda_k^{\llbracket 0, j-1 \rrbracket}} - \sum_{k=0}^{i-1} \frac{1}{\lambda_k^{\llbracket 0, i-1 \rrbracket}},$$

where  $(\lambda_k^{\llbracket 0, i-1 \rrbracket})_{k=0}^{i-1}$  are the non-zero eigenvalues of  $-L$  restricted to  $\llbracket 0, i-1 \rrbracket$ , see e.g. [Fill \(2009\)](#); [Gong et al. \(2012\)](#) and the references therein. In any case all these expressions are expressed in terms of a given target  $\pi$ .

*Remark 1.2.* It is tempting to think that  $\|f_{\delta_j}\|_\infty$  equals to  $D(j, j)$ . Yet, while  $D(i, j) \leq D(j, j)$  for any  $i, j$ , it is unclear to the author whether  $|D(i, j)|$  is less than or equal to  $D(j, j)$ .

Theorem 1.1 reveals that hitting time and other Markov chain parameters as described are closely related to the structure and properties of  $f_h$ . In particular,  $\|\nabla f_{\delta_j}\| \leq t_{dev}$ . This upper bound allows us to bound the Stein factors using various parameters:

**Corollary 1.1** (Bounding Stein's factors via hitting and mixing time). *Suppose that  $\pi$  is a discrete distribution with finite support on  $\mathcal{X} = \llbracket 0, N \rrbracket$  and  $N < \infty$ , and  $L$  is a reversible generator. Let  $\mathcal{H} = \{h : \mathcal{X} \mapsto [0, 1]\}$  and  $i \in \mathcal{X}$ , then*

$$\begin{aligned} \sup_{h \in \mathcal{H}} |f_h(i)| &\leq \sum_{j \in \mathcal{X}} |D(i, j)| \leq 2t_{av}, \\ \sup_{h \in \mathcal{H}} |\nabla f_h(i)| &\leq \sum_{j \in \mathcal{X}} \pi(j) |\mathbb{E}_i(\tau_j) - \mathbb{E}_{i+1}(\tau_j)| \leq t_{dev} \leq \begin{cases} 10t_{sst}, \\ 5t_{mix}(1/4), \\ 5t_{rel} \log(4/\pi_{min}), \\ 5C_\alpha t_{hit}(\alpha), \end{cases} \end{aligned}$$

where  $0 < \alpha < 1/2$  and  $C_\alpha > 0$  is an universal constant depending only on  $\alpha$ .

*Remark 1.3.* In practice, we argue that the relaxation time bound is perhaps the most useful of all as exact expressions or bounds on spectral gap  $t_{rel} = 1/\lambda_1$  are readily available for many models. As a side note, we can also offer upper bounds involving log-Sobolev constant by bounding the mixing time with that, see [Diaconis and Saloff-Coste \(1996\)](#). This may perhaps yield tighter upper bound due to the double logarithm.

The rest of this paper is organized as follows. In Section 2, we present the proofs of Theorem 1.1 and Corollary 1.1. In Section 3, we illustrate our main results by detailing a few examples involving common distributions. As another application, we demonstrate a way to bound expected hitting time via Stein's factor in Section 4.

## 2. PROOFS OF THE MAIN RESULTS

**2.1. Proof of Theorem 1.1.** Coolen-Schrijner and van Doorn (2002); Mao (2004) show that  $D$  exists and is finite if and only if

$$\sum_{i \in \mathcal{X}} \pi(i) \mathbb{E}_i(\tau_j) < \infty.$$

for some  $j$  and then for all  $j$  by irreducibility. The expressions of  $D(i, j)$  and  $D(j, j)$  follow readily from (Coolen-Schrijner and van Doorn, 2002, equation 5.5, 5.7). Define the  $\alpha$ -potential kernel  $D^\alpha = (D^\alpha(i, j))_{i, j \in \mathcal{X}}$  by

$$D^\alpha(i, j) := \int_0^\infty e^{-\alpha t} (P_t(i, j) - \pi_j) dt.$$

Note that by dominated convergence theorem,

$$D^\alpha h(i) = \int_0^\infty e^{-\alpha t} (P_t h(i) - \pi(h)) dt.$$

Under the proposed assumptions on  $h$  and reversibility of  $L$ , we use (Coolen-Schrijner and van Doorn, 2002, Lemma 5.1) to arrive at

$$f_h(i) = \int_0^\infty (P_t h(i) - \pi(h)) dt = \lim_{\alpha \rightarrow 0} D^\alpha h(i) = Dh(i) = \sum_{j \in \mathcal{X}} h(j) D(i, j).$$

To prove (1.6), we see that

$$\nabla f_h(i) = \sum_{j \in \mathcal{X}} h(j) (D(i+1, j) - D(i, j)) = \sum_{j \in \mathcal{X}} h(j) \pi(j) (\mathbb{E}_i(\tau_j) - \mathbb{E}_{i+1}(\tau_j)).$$

Now, we take  $h = \delta_j$  and  $f_{\delta_j}(i) = D(i, j)$ . Using triangle inequality, we have

$$|f_{\delta_j}(i)| = |D(i, j)| \leq D(j, j) + \pi(j) \mathbb{E}_i(\tau_j) \leq \sum_{j \in \mathcal{X}} D(j, j) + \sum_{j \in \mathcal{X}} \pi(j) \mathbb{E}_i(\tau_j) = t_{av} + t_{av} = 2t_{av},$$

where we use the fact that  $\sum_j D(j, j) = t_{av}$  and the random target lemma (Levin et al., 2009, Lemma 10.1) for the second term. As for the gradient of  $f_{\delta_j}$ , it is straight forward from (1.6) that

$$\|\nabla f_{\delta_j}\|_\infty = \pi(j) \sup_{i \in \mathcal{X}} |\mathbb{E}_i(\tau_j) - \mathbb{E}_{i+1}(\tau_j)| \leq t_{dev}.$$

**2.2. Proof of Corollary 1.1.** To arrive at the first equation, we make use of (1.5),  $\|h\|_\infty \leq 1$  and triangle inequality to arrive at

$$|f_h(i)| \leq \sum_{j \in \mathcal{X}} |D(i, j)| \leq \sum_{j \in \mathcal{X}} D(j, j) + \sum_{j \in \mathcal{X}} \pi(j) \mathbb{E}_i(\tau_j) = t_{av} + \sum_{j \in \mathcal{X}} \pi(j) \mathbb{E}_i(\tau_j).$$

Note that by the random target lemma (Levin et al., 2009, Lemma 10.1), the second term is independent of  $i$  which implies  $\sum_{j \in \mathcal{X}} \pi(j) \mathbb{E}_i(\tau_j) = t_{av}$ , and desired result follows. For the second set of equation, we apply (1.6) and  $\|h\|_\infty \leq 1$  to yield

$$|\nabla f_h(i)| \leq \sum_{j \in \mathcal{X}} \pi(j) |\mathbb{E}_i(\tau_j) - \mathbb{E}_{i+1}(\tau_j)| \leq \sup_k \sum_{j \in \mathcal{X}} \pi(j) |\mathbb{E}_i(\tau_j) - \mathbb{E}_k(\tau_j)| \leq t_{dev}.$$

$t_{dev} \leq 10t_{sst}$  follows from (Aldous, 1982, Lemma 16), and making use of that together with (Aldous, 1982, Lemma 12) give

$$t_{dev} \leq 10t_{sst} \leq 5t_{mix}(1/4).$$

Utilizing the above and (Levin et al., 2009, Theorem 20.6) leads to

$$t_{dev} \leq 5t_{mix}(1/4) \leq 5t_{rel} \log(4/\pi_{min}).$$

Finally, the main result in Oliveira (2012); Peres and Sousi (2015) gives

$$t_{dev} \leq 5t_{mix}(1/4) \leq 5C_\alpha t_{hit}(\alpha).$$

### 3. BOUNDING STEIN'S FACTORS VIA HITTING AND MIXING TIME - EXAMPLES

In this section, we will discuss in detail several examples to illustrate both Theorem 1.1 and Corollary 1.1, and compare with existing bounds in the literature. Our primary comparison is the bounds in Eichelsbacher and Reinert (2008). Notable results of this section is the  $O(\log n)$  bound in Example 3.2, and new bounds in Example 3.3.

**Example 3.1** (discrete uniform on  $\llbracket 0, n-1 \rrbracket$  with  $n < \infty$ ). In our first example, we look at  $\mathcal{X} = \llbracket 0, n-1 \rrbracket$  and  $\pi(i) = 1/n$  for  $i \in \mathcal{X}$ , and we take  $L$  to be the uniform chain with  $L(i, j) = 1$  for all  $i \neq j$  as in (Aldous, 1982, Example 48), which is reversible. The nice feature about this chain is that

$$\mathbb{E}_i(\tau_j) = \begin{cases} n-1 & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Also, note that the eigenvalues of  $-L$  are 0 with multiplicity 1 and  $1/n$  with multiplicities  $n-1$ . As a result, Theorem 1.1 now reads, for any  $i, j \in \mathcal{X}$ ,

$$\begin{aligned} D(j, j) &= \left( \frac{n-1}{n} \right)^2, \\ D(i, j) &= -\frac{n-1}{n^2}, \\ t_{av} &= \frac{n-1}{n}, \\ \|f_{\delta_j}\|_\infty &= \left( \frac{n-1}{n} \right)^2 \leq 2t_{av}, \\ \|\nabla f_{\delta_j}\|_\infty &= \frac{n-1}{n}. \end{aligned}$$

As for Corollary 1.1, since  $t_{av} = O(1/n)$ ,

$$\sup_{h \in \mathcal{H}} \|f_h\|_\infty = O(1/n).$$

In addition, note that the relaxation time bound yields a crude upper bound of size  $O(n \log n)$  for the gradient of Stein's solution:

$$\sup_{h \in \mathcal{H}} |\nabla f_h(i)| \leq \frac{2(n-1)}{n} \leq \frac{5}{\lambda_1} \log(4/\pi_{\min}) = 5n \log(4n).$$

**Example 3.2** (Binomial distribution on  $\llbracket 0, n \rrbracket$  with parameters  $n$  and  $0 < p < 1$ ). In the second example, we consider  $\pi(i) = C_i^n p^i (1-p)^{n-i}$  for  $i \in \mathcal{X} = \llbracket 0, n \rrbracket$ . We take  $L$  to be a birth-death process with birth rate  $b_i = p(n-i)$  and death rate  $d_i = (1-p)i$ . The non-zero eigenvalues of  $-L$  are  $\lambda_k = k$  for  $k \in \llbracket 1, n \rrbracket$  (see e.g. [Schoutens \(2000\)](#)), and it follows from Corollary 1.1 that

$$t_{av} = \sum_{k=1}^n \frac{1}{k} = O(\log n),$$

$$\sup_{h \in \mathcal{H}} \|f_h\|_\infty = O(\log n),$$

which improves upon the previous estimate of  $O(e^{n \log n})$  in ([Eichelsbacher and Reinert, 2008](#), Example 2.11, Lemma 2.5). As for the relaxation time bound, we have

$$\sup_{h \in \mathcal{H}} \|\nabla f_h\|_\infty \leq 5n \min \left\{ \log \left( \frac{4}{1-p} \right), \log \left( \frac{4}{p} \right) \right\}.$$

This  $O(n)$  bound does not seem to be useful at all when compared with the  $O(1)$  uniform bound of  $\sup_{h \in \mathcal{H}} \|\nabla f_h\|_\infty \leq \min\{1/(1-p), 1/p\}$  as in [Eichelsbacher and Reinert \(2008\)](#).

**Example 3.3** (Hypergeometric distribution on  $\llbracket 0, r \rrbracket$  with parameters  $n, r$  and  $0 < 2r \leq n$ ). In this example, we study the hypergeometric distribution

$$\pi(i) = \frac{C_i^r C_{r-i}^{n-r}}{C_r^n}$$

for  $i \in \llbracket 0, r \rrbracket$ , and pick  $L$  to be the generator of Bernoulli Laplace model, that is, it is a birth-death chain with birth rate  $b_i$  and death rate  $d_i$  to be respectively

$$b_i = \frac{(r-i)(n-r-i)}{r(n-r)}, \quad d_i = \frac{i^2}{r(n-r)}.$$

The eigenvalues of  $-L$  are

$$\lambda_i = \frac{i(n-i+1)}{r(n-r)},$$

so Corollary 1.1 now reads

$$\sup_{h \in \mathcal{H}} |f_h(i)| \leq 2r(n-r) \sum_{i=1}^r \frac{1}{i(n-i+1)} = \frac{2r(n-r)}{n} (\log r + O(1)),$$

$$\sup_{h \in \mathcal{H}} |\nabla f_h(i)| \leq \frac{5r(n-r)}{n} \log \left( \max_{i \in \llbracket 0, r \rrbracket} \frac{4C_r^n}{C_i^r C_{r-i}^{n-r}} \right),$$

where the equality follows from ([Diaconis and Saloff-Coste, 2006](#), Page 2114). These results appear to be new.

**Example 3.4** (Geometric distribution with success probability  $0 < p < 1$ ). In this example, we use only estimates and information on hitting time to bound the Stein's factors for geometric distribution. More specifically, we look at  $\pi(i) = (1-p)^i p$  for  $i \in \mathcal{X} = \mathbb{N}_0$ , and choose  $L$  with unit per capita death rate and birth rate  $b_i = (i+1)(1-p)$ . It follows from Remark 1.1 that for any fixed  $j \in \mathcal{X}$ , if  $i < j$  then

$$\mathbb{E}_i(\tau_j) \leq \mathbb{E}_0(\tau_j) = \sum_{k=0}^{j-1} \frac{1 - (1-p)^{k+1}}{(k+1)(1-p)^{k+1}p} \leq \sum_{k=0}^{j-1} \frac{1}{(1-p)^{k+1}p}$$

For any  $i, j \in \mathcal{X}$ , and using Remark 1.1, we have

$$|\mathbb{E}_i(\tau_j) - \mathbb{E}_{i+1}(\tau_j)| \leq \frac{1}{b_i \pi(i)} \pi([0, i]) \mathbb{1}_{i < j} + \frac{1}{b_i \pi(i)} \pi([i+1, \infty]) \mathbb{1}_{i \geq j}.$$

Specializing into the case of geometric distribution leads to

$$\pi(j) |\mathbb{E}_i(\tau_j) - \mathbb{E}_{i+1}(\tau_j)| \leq (1-p)^{j-i-1} \frac{1 - (1-p)^{i+1}}{i+1} \mathbb{1}_{i < j} + \frac{(1-p)^j}{i+1} \mathbb{1}_{i \geq j}.$$

Now, using Corollary 1.1 and summing over all possible  $j$  gives

$$\sup_{h \in \mathcal{H}} |\nabla f_h(i)| \leq \frac{1 - (1-p)^{i+1}}{p(i+1)} + \frac{1 - (1-p)^{i+1}}{p(i+1)} = \frac{2(1 - (1-p)^{i+1})}{p(i+1)} \leq \frac{2}{p},$$

and so  $\sup_{h \in \mathcal{H}} \|\nabla f_h\|_\infty \leq 2/p$ . Note that in [Eichelsbacher and Reinert \(2008\)](#) they obtain

$$\begin{aligned} \sup_{h \in \mathcal{H}} |\nabla f_h(i)| &\leq \min \left\{ \frac{1}{i}, \frac{1+p}{i+1} \right\}, \\ \sup_{h \in \mathcal{H}} \|\nabla f_h\|_\infty &\leq \min\{1, 1+p\} \leq \frac{2}{p}, \end{aligned}$$

so our bound is looser than existing bound.

#### 4. BOUNDING EXPECTED HITTING TIME VIA STEIN'S FACTORS

In the previous section, we have illustrated how we can use information of hitting time and mixing time to give bounds on Stein's factors. We aim at achieving the opposite in this section and illustrate how we can obtain estimate on the expected hitting time to 0 of a Galton-Watson with immigration (GWI) process.

Recall that the generator  $L$  of GWI is a birth-death chain with  $b_i = p(r+i)$ ,  $d_i = i$  and  $\pi$  being the negative binomial distribution with parameters  $0 < p < 1$  and  $r > 0$ . Taking  $h = \delta_0$  in Theorem 1.1, for  $i \in \mathbb{N}_0$  we have

$$\nabla f_{\delta_0}(i) = \pi(0) (\mathbb{E}_i(\tau_0) - \mathbb{E}_{i+1}(\tau_0)) = (1-p)^r (\mathbb{E}_i(\tau_0) - \mathbb{E}_{i+1}(\tau_0)).$$

Next, we bound the above expression using the Stein's factor bound in [Barbour et al. \(2015\)](#) to arrive at

$$(1-p)^r |\mathbb{E}_i(\tau_0) - \mathbb{E}_{i+1}(\tau_0)| \leq \min \left\{ \frac{2}{1-p}, \frac{1+p}{(1-p)^2}, \frac{3}{2rp(1-p)^3} \right\} =: C,$$

and the above yields the following linear bound:

$$\mathbb{E}_i(\tau_0) \leq C(1-p)^{-r} i.$$

This bound can perhaps be refined by adapting non-uniform bounds of Stein's factor.



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