Problems

MAA

June 25, 2022

Acknowledgement

Problems

1. A digital watch displays hours and minutes with AM and PM. What is the largest possible sum of the digits in the display?

   (A) 17 **(B)** 19 **(C)** 21 **(D)** 22 **(E)** 23

2. When a student multiplied the number 66 by the repeating decimal,

   $$\underline{1}.\underline{a}\ \underline{b}\ \underline{a}\ \underline{b}\ldots = \underline{1}.\overline{\underline{a}\ \underline{b}},$$

   where a and b are digits, he did not notice the notation and just multiplied 66 times $\underline{1}.\underline{a}\ \underline{b}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2-digit number $\underline{a}\ \underline{b}$?

   (A) 15 **(B)** 30 **(C)** 45 **(D)** 60 **(E)** 75

3. Sreshtha needs to estimate the quantity $\frac{a}{b} - c$, where a, b, and c are large positive integers. She rounds each of the integers so that the calculation will be easier to do mentally. In which of these situations will her answer necessarily be greater than the exact value of $\frac{a}{b} - c$?

   (A) She rounds all three numbers up.
   (B) She rounds a and b up, and she rounds $c$down.
   (C) She rounds a and c up, and she rounds $b$down.
   (D) She rounds a up, and she rounds b and $c$down.
   (E) She rounds c up, and she rounds a and $b$down.

4. What is the value of $\dfrac{11! - 10!}{9!}$?

   (A) 99 **(B)** 100 **(C)** 110 **(D)** 121 **(E)** 132

5. The sum of two positive numbers is 5 times their difference. What is the ratio of the larger number to the smaller number?

   (A) $\frac{5}{4}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{9}{5}$ **(D)** 2 **(E)** $\frac{5}{2}$

6. In a plane, four circles with radii $1, 3, 5$, and 7 are tangent to line ℓ at the same point A, but they may be on either side of ℓ. Region S consists of all the points that lie inside exactly one of the four circles. What is the maximum possible area of region S?

   (A) 24π **(B)** 32π **(C)** 64π **(D)** 65π **(E)** 84π

7. Two right circular cylinders have the same volume. The radius of the second cylinder is 10% more than the radius of the first. What is the relationship between the heights of the two cylinders?

   (A) The second height is 10% less than the first.
   (B) The first height is 10% more than the second.
   (C) The second height is 21% less than the first.
   (D) The first height is 21% more than the second.
   (E) The second height is 80% of the first.

8. Points B and C lie on $\overline{AD}$. The length of $\overline{AB}$ is 4 times the length of $\overline{BD}$, and the length of $\overline{AC}$ is 9 times the length of $\overline{CD}$. The length of $\overline{BC}$ is what fraction of the length of $\overline{AD}$?

   (A) $\frac{1}{36}$ **(B)** $\frac{1}{13}$ **(C)** $\frac{1}{10}$ **(D)** $\frac{5}{36}$ **(E)** $\frac{1}{5}$

9. Two years ago Pete was three times as old as his cousin Claire. Two years before that, Pete was four times as old as Claire. In how many years will the ratio of their ages be $2 : 1$?

   (A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

10. A three-quarter sector of a circle of radius 4 inches together with its interior can be rolled up to form the lateral surface area of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic inches?

(A) $3\pi\sqrt{5}$ **(B)** $4\pi\sqrt{3}$ **(C)** $3\pi\sqrt{7}$ **(D)** $6\pi\sqrt{3}$ **(E)** $6\pi\sqrt{7}$

11. On a 50-question multiple choice math contest, students receive 4 points for a correct answer, 0 points for an answer left blank, and -1 point for an incorrect answer. Jesse's total score on the contest was 99. What is the maximum number of questions that Jesse could have answered correctly?

    (A) 25 **(B)** 27 **(C)** 29 **(D)** 31 **(E)** 33

12. What is the minimum number of digits to the right of the decimal point needed to express the fraction $\frac{123456789}{2^{26}\cdot 5^4}$ as a decimal?

    (A) 4 **(B)** 22 **(C)** 26 **(D)** 30 **(E)** 104

13. Suppose $\cos x = 0$ and $\cos(x+z) = 1/2$. What is the smallest possible positive value of z?

    (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{5\pi}{6}$ (E) $\frac{7\pi}{6}$

14. Find the value(s) of x such that $8xy - 12y + 2x - 3 = 0$ is true for all values of y.

    (A) $\frac{2}{3}$ **(B)** $\frac{3}{2}$ or $-\frac{1}{4}$ **(C)** $-\frac{2}{3}$ or $-\frac{1}{4}$ **(D)** $\frac{3}{2}$ **(E)** $-\frac{3}{2}$ or $-\frac{1}{4}$

15. A wire is cut into two pieces, one of length a and the other of length b. The piece of length a is bent to form an equilateral triangle, and the piece of length b is bent to form a regular hexagon. The triangle and the hexagon have equal area. What is $\frac{a}{b}$?

    (A) 1 **(B)** $\frac{\sqrt{6}}{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $\frac{3\sqrt{2}}{2}$

16. The two digits in Jack's age are the same as the digits in Bill's age, but in reverse order. In five years Jack will be twice as old as Bill will be then. What is the difference in their current ages?

    (A) 9 (B) 18 (C) 27 (D) 36 (E) 45

17. The function f has the property that for each real number x in its domain, $1/x$ is also in its domain and

    $f(x) + f\left(\frac{1}{x}\right) = x$

    What is the largest set of real numbers that can be in the domain of f?

    (A) $\{x|x \neq 0\}$ (B) $\{x|x < 0\}$

    (C) $\{x|x > 0\}$$(D) {x|x$\neq -1$ and x $\neq 0$ and x $\neq 1$}

    (E) $\{-1, 1\}$

18. The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

    (A) $\frac{1}{21}$ **(B)** $\frac{1}{14}$ **(C)** $\frac{5}{63}$ **(D)** $\frac{2}{21}$ **(E)** $\frac{1}{7}$

19. Circles with radii 1, 2, and 3 are mutually externally tangent. What is the area of the triangle determined by the points of tangency?

    (A) $\frac{3}{5}$ **(B)** $\frac{4}{5}$ **(C)** 1 **(D)** $\frac{6}{5}$ **(E)** $\frac{4}{3}$

20. Let $a + ar_1 + ar_1^2 + ar_1^3 + \cdots$ and $a + ar_2 + ar_2^2 + ar_2^3 + \cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is $r_1 + r_2$?

    (A) 0 **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** $\frac{1+\sqrt{5}}{2}$ **(E)** 2

21. Let $f_1(x) = \sqrt{1-x}$, and for integers $n \geq 2$, let $f_n(x) = f_{n-1}(\sqrt{n^2 - x})$. If N is the largest value of n for which the domain of f_n is nonempty, the domain of f_N is $[c]$. What is $N + c$?

    (A) -226 **(B)** -144 **(C)** -20 **(D)** 20 **(E)** 144

22. Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

(A) $\frac{7}{36}$ **(B)** $\frac{5}{24}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{17}{72}$ **(E)** $\frac{1}{4}$

23. The solutions to the equations $z^2 = 4 + 4\sqrt{15}i$ and $z^2 = 2 + 2\sqrt{3}i$, where $i = \sqrt{-1}$, form the vertices of a parallelogram in the complex plane. The area of this parallelogram can be written in the form $p\sqrt{q} - r\sqrt{s}$, where p, q, r, and s are positive integers and neither q nor s is divisible by the square of any prime number. What is $p + q + r + s$?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

24. Amy, Beth, and Jo listen to four different songs and discuss which ones they like. No song is liked by all three. Furthermore, for each of the three pairs of the girls, there is at least one song liked by those two girls but disliked by the third. In how many different ways is this possible?

(A) 108 **(B)** 132 **(C)** 671 **(D)** 846 **(E)** 1105

25. A positive integer n is nice if there is a positive integer m with exactly four positive divisors (including 1 and m) such that the sum of the four divisors is equal to n. How many numbers in the set $\{2010, 2011, 2012, \ldots, 2019\}$ are nice?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5