Problems

MAA

December 31, 2021

Acknowledgement

Problems

1. What is $(20-(2010-201))+(2010-(201-20))$?

   (A) -4020 **(B)** 0 **(C)** 40 **(D)** 401 **(E)** 4020

2. Members of the Rockham Soccer League buy socks and T-shirts. Socks cost \$4 per pair and each T-shirt costs \$5 more than a pair of socks. Each member needs one pair of socks and a shirt for home games and another pair of socks and a shirt for away games. If the total cost is \$2366, how many members are in the League?

   (A) 77 (B) 91 (C) 143 (D) 182 (E) 286

3. In the expression $c \cdot a^b - d$, the values of a, b, c, and d are 0, 1, 2, and 3, although not necessarily in that order. What is the maximum possible value of the result?

   (A) 5 (B) 6 (C) 8 (D) 9 (E) 10

4. A football game was played between two teams, the Cougars and the Panthers. The two teams scored a total of 34 points, and the Cougars won by a margin of 14 points. How many points did the Panthers score?

   (A) 10 **(B)** 14 **(C)** 17 **(D)** 20 **(E)** 24

5. A drawer contains red, green, blue, and white socks with at least 2 of each color. What is the minimum number of socks that must be pulled from the drawer to guarantee a matching pair?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 8 **(E)** 9

6. The faces of each of 7 standard dice are labeled with the integers from 1 to 6. Let p be the probabilities that when all 7 dice are rolled, the sum of the numbers on the top faces is 10. What other sum occurs with the same probability as p?

   (A) 13 **(B)** 26 **(C)** 32 **(D)** 39 **(E)** 42

7. A quadratic equation $ax^2 - 2ax + b = 0$ has two real solutions. What is the average of these two solutions?

   (A) 1 (B) 2 (C) $\frac{b}{a}$ (D) $\frac{2b}{a}$ (E) $\sqrt{2b-a}$

8. Suppose that a and b are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. Then the pair (a, b) is

   (A) $(-2, 1)$ (B) $(-1, 2)$ (C) $(1, -2)$ (D) $(2, -1)$ (E) $(4, 4)$

9. What is the median of the following list of 4040 numbers?

   $$1, 2, 3, \ldots, 2020, 1^2, 2^2, 3^2, \ldots, 2020^2$$

   (A) 1974.5 **(B)** 1975.5 **(C)** 1976.5 **(D)** 1977.5 **(E)** 1978.5

10. There is a positive integer n such that $(n+1)! + (n+2)! = n! \cdot 440$. What is the sum of the digits of n?

    (A) 3 **(B)** 8 **(C)** 10 **(D)** 11 **(E)** 12

11. A coin is altered so that the probability that it lands on heads is less than $\frac{1}{2}$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $\frac{1}{6}$. What is the probability that the coin lands on heads?

    (A) $\frac{\sqrt{15}-3}{6}$ **(B)** $\frac{6-\sqrt{6\sqrt{6}+2}}{12}$ **(C)** $\frac{\sqrt{2}-1}{2}$ **(D)** $\frac{3-\sqrt{3}}{6}$ **(E)** $\frac{\sqrt{3}-1}{2}$

12. Driving at a constant speed, Sharon usually takes 180 minutes to drive from her house to her mother's house. One day Sharon begins the drive at her usual speed, but after driving $\frac{1}{3}$ of the way, she hits a bad snowstorm and reduces her speed by 20 miles per hour. This time the trip takes her a total of 276 minutes. How many miles is the drive from Sharon's house to her mother's house?

    (A) 132 **(B)** 135 **(C)** 138 **(D)** 141 **(E)** 144

13. For how many integers x is the point $(x, -x)$ inside or on the circle of radius 10 centered at $(5, 5)$?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

14. Let $a < b < c$ be three integers such that a, b, c is an arithmetic progression and a, c, b is a geometric progression. What is the smallest possible value of c?

(A) -2 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 6

15. For which of the following values of k does the equation $\frac{x-1}{x-2} = \frac{x-k}{x-6}$ have no solution for x?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

16. A red ball and a green ball are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k = 1, 2, 3....$ What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

(A) $\frac{1}{4}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{3}{7}$

17. Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

(A) 5 **(B)** $\sqrt{26}$ **(C)** $3\sqrt{3}$ **(D)** $2\sqrt{7}$ **(E)** $\sqrt{30}$

18. Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

(A) 0 (B) 1 (C) 4 (D) 7 (E) 8

19. Triangle ABC has $AB = 27$, $AC = 26$, and $BC = 25$. Let I be the intersection of the internal angle bisectors of $\triangle ABC$. What is BI?

(A) 15 **(B)** $5 + \sqrt{26} + 3\sqrt{3}$ **(C)** $3\sqrt{26}$ **(D)** $\frac{2}{3}\sqrt{546}$ **(E)** $9\sqrt{3}$

20. Circles with centers $(2, 4)$ and $(14, 9)$ have radii 4 and 9, respectively. The equation of a common external tangent to the circles can be written in the form $y = mx + b$ with $m > 0$. What is b?

(A) $\frac{908}{119}$ (B) $\frac{909}{119}$ (C) $\frac{130}{17}$ (D) $\frac{911}{119}$ (E) $\frac{912}{119}$

21. Andy's lawn has twice as much area as Beth's lawn and three times as much area as Carlos' lawn. Carlos' lawn mower cuts half as fast as Beth's mower and one third as fast as Andy's mower. If they all start to mow their lawns at the same time, who will finish first?

(A) Andy (B) Beth (C) Carlos (D) Andy and Carlos tie for first. (E) All three tie.

22. Debra flips a fair coin repeatedly, keeping track of how many heads and how many tails she has seen in total, until she gets either two heads in a row or two tails in a row, at which point she stops flipping. What is the probability that she gets two heads in a row but she sees a second tail before she sees a second head?

(A) $\frac{1}{36}$ **(B)** $\frac{1}{24}$ **(C)** $\frac{1}{18}$ **(D)** $\frac{1}{12}$ **(E)** $\frac{1}{6}$

23. For a particular peculiar pair of dice, the probabilities of rolling 1, 2, 3, 4, 5, and 6, on each die are in the ratio $1 : 2 : 3 : 4 : 5 : 6$. What is the probability of rolling a total of 7 on the two dice?

(A) $\frac{4}{63}$ (B) $\frac{1}{8}$ (C) $\frac{8}{63}$ (D) $\frac{1}{6}$ (E) $\frac{2}{7}$

24. Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y, and Z the midpoints of sides $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$, respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

(A) $\frac{3}{8}\sqrt{3}$ **(B)** $\frac{7}{16}\sqrt{3}$ **(C)** $\frac{15}{32}\sqrt{3}$ **(D)** $\frac{1}{2}\sqrt{3}$ **(E)** $\frac{9}{16}\sqrt{3}$

25. A plane contains points A and B with $AB = 1$. Let S be the union of all disks of radius 1 in the plane that cover $\overline{AB}$. What is the area of S?

(A) $2\pi + \sqrt{3}$ **(B)** $\frac{8\pi}{3}$ **(C)** $3\pi - \frac{\sqrt{3}}{2}$ **(D)** $\frac{10\pi}{3} - \sqrt{3}$ **(E)** $4\pi - 2\sqrt{3}$