HOMEWORK 3

CEE 530: Continuum Mechanics and Thermodynamics

Due: April 16, 2018

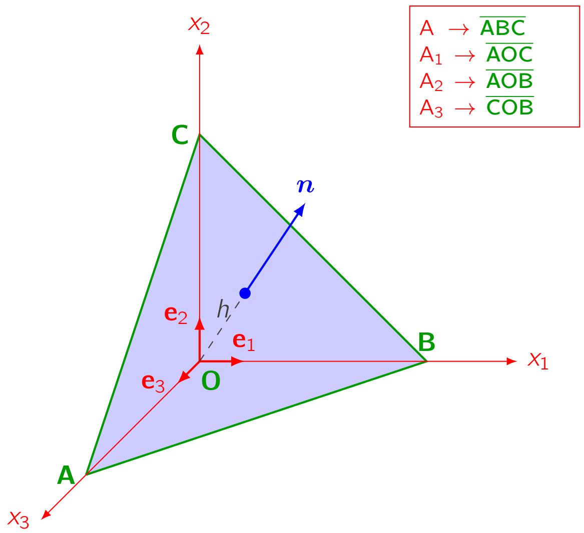
PROBLEM 1

Consider a continuum, such as a crystalline solid, in which a concentration of contaminants is diffusing. Assume that the flux per unit area j of contaminants in direction $\boldsymbol{n}$ at a point $\boldsymbol{x}$ is only a function of these two variables and time, i.e., $j = j(\boldsymbol{x}, \boldsymbol{n}, t)$. Assume furthermore that there exists a source that releases a concentration $r(\boldsymbol{x}, t)$ of contaminants per unit time. Denote with $\rho_c(\boldsymbol{x}, t)$ the concentration (or mass density) of contaminants at a point $\boldsymbol{x}$ at time t.

1. (15) Write down an integral balance statement of contaminants.
2. (15) Following the proof of Cauchy's tetrahedron theorem, and using the integral balance statement, obtain that the dependence of j on $\boldsymbol{n}$ at a point can only be linear, i.e., there exists a vector field $\boldsymbol{J}(\boldsymbol{x}, t)$ such that
$$j(\boldsymbol{x}, \boldsymbol{n}, t) = \boldsymbol{J}(\boldsymbol{x}, t) \cdot \boldsymbol{n}$$
3. (10) Obtain the local statement.

Consider a crystalline solid body $\mathcal{B}_0$ which is deforming with a motion $\boldsymbol{\varphi}$ containing a contaminant whose flux per unit area at a point $\boldsymbol{x}$ in the direction $\boldsymbol{n}$ is given by $\mathrm{j}(\boldsymbol{x}, \boldsymbol{n}, t)$. Also the contaminant is introduced at a rate $\mathrm{r}(\boldsymbol{x}, t)$.

1. The rate of change of the contaminant M is given by
$$\frac{DM}{Dt} = \int_{\boldsymbol{\varphi}(V_0,t)} \mathrm{r}(\boldsymbol{x}, t) dv$$
and also
$$\frac{DM}{Dt} = \int_{\boldsymbol{\varphi}(V_0,t)} \frac{\partial \rho_c(\boldsymbol{x}, t)}{\partial t} dv + \int_{\boldsymbol{\varphi}(\partial V_0,t)} \mathrm{j}(\boldsymbol{x}, \boldsymbol{n}, t) ds$$
which yields the integral balance statement as
$$\boxed{\int_{\boldsymbol{\varphi}(V_0,t)} \frac{\partial \rho_c(\boldsymbol{x}, t)}{\partial t} dv + \int_{\boldsymbol{\varphi}(\partial V_0,t)} \mathrm{j}(\boldsymbol{x}, \boldsymbol{n}, t) ds = \int_{\boldsymbol{\varphi}(V_0,t)} \mathrm{r}(\boldsymbol{x}, t) dv} \tag{1}$$
2. Consider the tetrahedron drawn below at a point $\boldsymbol{x}$ in our domain



the total flux across the surface is given by

$$f = \mathrm{j}(\boldsymbol{x}, -\mathbf{e}_1, t)A_1 + \mathrm{j}(\boldsymbol{x}, -\mathbf{e}_2, t)A_2 + \mathrm{j}(\boldsymbol{x}, -\mathbf{e}_3, t)A_3 + \mathrm{j}(\boldsymbol{x}, \boldsymbol{n}, t)A$$

and since $\mathrm{j}(-\boldsymbol{v}, t) = -\mathrm{j}(\boldsymbol{v}, t)$ we may re-write the above statement as

$$f = -\mathrm{j}(\boldsymbol{x}, \mathbf{e}_1, t)A_1 - \mathrm{j}(\boldsymbol{x}, \mathbf{e}_2, t)A_2 - \mathrm{j}(\boldsymbol{x}, \mathbf{e}_3, t)A_3 + \mathrm{j}(\boldsymbol{x}, \boldsymbol{n}, t)A$$

and from the balance statement of Equation 1 we can state

$$\dot{\rho}_c(\boldsymbol{x}, t)V + f - \mathrm{r}(\boldsymbol{x}, t)V = 0$$

since we know that $A_i = A\boldsymbol{n} \cdot \mathbf{e}_i$ we may rewrite the balance statement as

$$\dot{\rho}_c(\boldsymbol{x}, t)\frac{V}{A} - \mathrm{j}(\boldsymbol{x}, \mathbf{e}_1, t)n_1 - \mathrm{j}(\boldsymbol{x}, \mathbf{e}_2, t)n_2 - \mathrm{j}(\boldsymbol{x}, \mathbf{e}_3, t)n_3 + \mathrm{j}(\boldsymbol{x}, \boldsymbol{n}, t) - \mathrm{r}(\boldsymbol{x}, t)\frac{V}{A} = 0$$

and if we let $h \to 0$ we are left with

$$\mathrm{j}(\boldsymbol{x}, \boldsymbol{n}, t) = \mathrm{j}(\boldsymbol{x}, \mathbf{e}_1, t)n_1 + \mathrm{j}(\boldsymbol{x}, \mathbf{e}_2, t)n_2 + \mathrm{j}(\boldsymbol{x}, \mathbf{e}_3, t)n_3 = \mathrm{j}(\boldsymbol{x}, \mathbf{e}_i, t)n_i$$

thus

$$\boxed{\mathrm{j}(\boldsymbol{x}, \boldsymbol{n}, t) = \mathrm{J}(\boldsymbol{x}, t) \cdot \boldsymbol{n}} \tag{2}$$

3. Since we showed the linear relation $\mathrm{j}(\boldsymbol{x}, \boldsymbol{n}, t) = \mathrm{J}(\boldsymbol{x}, t) \cdot \boldsymbol{n}$ we may now re-write the balance statement of Equation 1 as

$$\int_{\varphi(V_0,t)} \frac{\partial \rho_c(\boldsymbol{x}, t)}{\partial t} dv + \int_{\varphi(\partial V_0,t)} \mathrm{J}(\boldsymbol{x}, t) \cdot \boldsymbol{n} ds = \int_{\varphi(V_0,t)} \mathrm{r}(\boldsymbol{x}, t) dv$$

using Gauss' theorem we may state

$$\int_{\varphi(V_0,t)} \left[\frac{\partial \rho_c(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot \mathrm{J}(\boldsymbol{x}, t)\right] ds = \int_{\varphi(V_0,t)} \mathrm{r}(\boldsymbol{x}, t) dv$$

and since it must hold for any arbitrary $V_0 \subseteq \mathcal{B}_0$ we may state

$$\boxed{\frac{\partial \rho_c(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot \mathrm{J}(\boldsymbol{x}, t) = \mathrm{r}(\boldsymbol{x}, t)} \tag{3}$$

PROBLEM 2

A body made out of a radioactive material is undergoing motion.

1. (10) The radioactive decay makes the body lose mass at a mass fraction rate of γ/s (This mass is converted into energy). Modify the equation of conservation of mass to take this into account, and obtain the expressions in the local and integral forms, both for the spatial and material configurations.

2. (10) The radioactive decay of the material makes it transform from its original form, material 0, into a different one, material 1. So, in addition to the mass loss, a mass fraction λ of material 0 transforms into material 1 per unit time. Write down the equations governing the evolution of the mass densities of materials 0 and 1, respectively. Provide integral and local, spatial and material forms.

1. For a body $\mathcal{B} = \varphi(\mathcal{B}_0, t)$ which loses mass at a mass fraction rate γs^{-1} ($\gamma \geq 0$) we have that the change in total mass $M^{\mathcal{B}}$ will be given by

$$\frac{DM^{\mathcal{B}}}{Dt} = -\gamma s^{-1} \int_{\mathcal{B}} \rho(\boldsymbol{x}, t) dV \equiv - \int_{\mathcal{B}} \gamma s^{-1} \rho(\boldsymbol{x}, t) dv$$

where we also know that

$$\frac{DM^{\mathcal{B}}}{Dt} = \int_{\mathcal{B}} \left[\frac{\partial \rho(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot (\rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t)) \right] dv$$

thus

$$\int_{\mathcal{B}} \left[\frac{\partial \rho(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot (\rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t)) \right] dv = - \int_{\mathcal{B}} \gamma s^{-1} \rho(\boldsymbol{x}, t) dv$$

or

$$\int_{\mathcal{B}} \left[\frac{\partial \rho(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot (\rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t)) + \gamma s^{-1} \rho(\boldsymbol{x}, t) \right] dv = 0$$

and since the above equation must hold for any arbitrary $\mathcal{B}_0$ we can localize it such that in spatial form we have the following conservation of mass statement

$$\boxed{\frac{\partial \rho(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot (\rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t)) = -\gamma s^{-1} \rho(\boldsymbol{x}, t)} \tag{4}$$

similarly in the Lagrangian description we may state

$$\frac{DM^{\mathcal{B}}}{Dt} = -\gamma s^{-1} \int_{\mathcal{B}_0} \rho_0(\boldsymbol{X}, t) dV \equiv - \int_{\mathcal{B}_0} \gamma s^{-1} \rho_0(\boldsymbol{X}, t) dV$$

and

$$\frac{DM^{\mathcal{B}}}{Dt} = \int_{\mathcal{B}_0} \frac{\partial \rho_0(\boldsymbol{X}, t)}{\partial t} dV$$

thus we obtain

$$\int_{\mathcal{B}_0} \frac{\partial \rho_0(\boldsymbol{X}, t)}{\partial t} dV = - \int_{\mathcal{B}_0} \gamma s^{-1} \rho_0(\boldsymbol{X}, t) dV$$

which, in analogous manner as before, yields

$$\boxed{\frac{\partial \rho_0(\boldsymbol{X}, t)}{\partial t} = -\gamma s^{-1} \rho_0(\boldsymbol{X}, t)} \tag{5}$$

2. Consider two materials denoted by the superscript ^{1} and ^{2}. The mass fraction exchange rate of material 1 to material 2 is denoted by λs^{-1} ($\lambda \geq 0$) such that

$$\begin{aligned}\frac{DM^1}{Dt} &= -\int_{\varphi^1(V_0,t)} (\gamma s^{-1} + \lambda s^{-1})\rho^1(\boldsymbol{x},t)dv \\ \frac{DM^2}{Dt} &= \int_{\varphi^1(V_0,t)} \lambda s^{-1}\rho^1(\boldsymbol{x},t)dv\end{aligned}$$

and

$$\begin{aligned}\frac{DM^1}{Dt} &= \int_{\varphi^1(V_0,t)} \left[\frac{\partial \rho^1(\boldsymbol{x},t)}{\partial t} + \nabla\cdot(\rho^1(\boldsymbol{x},t)\boldsymbol{v}^1(\boldsymbol{x},t))\right] dv \\ \frac{DM^2}{Dt} &= \int_{\varphi^2(V_0,t)} \left[\frac{\partial \rho^2(\boldsymbol{x},t)}{\partial t} + \nabla\cdot(\rho^2(\boldsymbol{x},t)\boldsymbol{v}^2(\boldsymbol{x},t))\right] dv\end{aligned}$$

thus gives us the relations

$$\begin{aligned}\int_{\varphi^1(V_0,t)} \left[\frac{\partial \rho^1(\boldsymbol{x},t)}{\partial t} + \nabla\cdot(\rho^1(\boldsymbol{x},t)\boldsymbol{v}^1(\boldsymbol{x},t))\right] dv &= -\int_{\varphi^1(V_0,t)} (\gamma s^{-1} + \lambda s^{-1})\rho^1(\boldsymbol{x},t)dv \\ \int_{\varphi^2(V_0,t)} \left[\frac{\partial \rho^2(\boldsymbol{x},t)}{\partial t} + \nabla\cdot(\rho^2(\boldsymbol{x},t)\boldsymbol{v}^2(\boldsymbol{x},t))\right] dv &= \int_{\varphi^1(V_0,t)} \lambda s^{-1}\rho^1(\boldsymbol{x},t)dv\end{aligned}$$

if we assume that the motion of the two materials is the same, i.e $\boldsymbol{\varphi}^1 = \boldsymbol{\varphi}^2 = \boldsymbol{\varphi}$ we can say

$$\int_{\varphi(V_0,t)} \left[\frac{\partial \rho^1(\boldsymbol{x},t)}{\partial t} + \nabla\cdot(\rho^1(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)) + (\gamma s^{-1} + \lambda s^{-1})\rho^1(\boldsymbol{x},t)\right] dv = 0$$

$$\int_{\varphi(V_0,t)} \left[\frac{\partial \rho^2(\boldsymbol{x},t)}{\partial t} + \nabla\cdot(\rho^2(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)) - \lambda s^{-1}\rho^1(\boldsymbol{x},t)\right] dv = 0$$

and thus since the above must hold for any arbitrary volume $V_0 \subseteq \mathcal{B}_0$ we must have

$$\boxed{\begin{aligned}\frac{\partial \rho^1(\boldsymbol{x},t)}{\partial t} + \nabla\cdot(\rho^1(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)) &= -(\gamma s^{-1} + \lambda s^{-1})\rho^1(\boldsymbol{x},t) \qquad (6) \\ \frac{\partial \rho^2(\boldsymbol{x},t)}{\partial t} + \nabla\cdot(\rho^2(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)) &= \lambda s^{-1}\rho^1(\boldsymbol{x},t) \qquad (7)\end{aligned}}$$

similarly from the Lagrangian perspective we may state

$$\begin{aligned}\frac{DM^1}{Dt} &= -\int_{V_0} (\gamma s^{-1} + \lambda s^{-1})\rho_0^1(\boldsymbol{X},t)dV \\ \frac{DM^2}{Dt} &= \int_{V_0} \lambda s^{-1}\rho_0^1(\boldsymbol{X},t)dV\end{aligned}$$

and

$$\begin{aligned}\frac{DM^1}{Dt} &= \int_{V_0} \frac{\partial \rho_0^1(\boldsymbol{X},t)}{\partial t}\, dV \\ \frac{DM^2}{Dt} &= \int_{V_0} \frac{\partial \rho_0^2(\boldsymbol{X},t)}{\partial t}\, dV\end{aligned}$$

which yields

$$\int_{V_0} \left[\frac{\partial \rho_0^1(\boldsymbol{X},t)}{\partial t} + (\gamma s^{-1} + \lambda s^{-1})\rho_0^1(\boldsymbol{X},t)\right] dV = 0$$

$$\int_{V_0} \left[\frac{\partial \rho_0^2(\boldsymbol{X},t)}{\partial t} - \lambda s^{-1}\rho_0^1(\boldsymbol{X},t)\right] dV = 0$$

which localized yields

$$\boxed{\begin{aligned} \frac{\partial \rho_0^1(\boldsymbol{X},t)}{\partial t} &= -(\gamma s^{-1} + \lambda s^{-1})\rho_0^1(\boldsymbol{X},t) && (8) \\ \frac{\partial \rho_0^2(\boldsymbol{X},t)}{\partial t} &= \lambda s^{-1}\rho_0^1(\boldsymbol{X},t) && (9) \end{aligned}}$$

PROBLEM 3

A fluid of density ρ is in hydrostatic equilibrium under the action of gravity in a container. The vertical direction is taken to coincide with the x_3-axis.

1. (10) The state of stress of the fluid is everywhere of the form $\boldsymbol{\sigma}(\boldsymbol{x}) = p(\boldsymbol{x})\mathbf{1}$. Find $p(\boldsymbol{x})$. Does it depend on the shape of the container?
2. (10) Prove mathematically Archimedes' principle: The buoyant force exerted by a fluid on an object is equal to the weight of the fluid displaced by the object.
3. (10) Assume the container is a cylinder of height L and radius R with the base of the cylinder against the ground and open at the top. The fluid is in contact with the atmosphere there, and the atmospheric pressure is p_{atm}. If we make an imaginary cut of the cylinder into two identical parts with a plane containing the cylinder axis, what is the total force exerted by the fluid on each half of the container?
4. (10) What is then the resultant of the traction on the walls of the container cut by the imaginary plane? The outer lateral walls of the cylinder are exposed to the nearly constant atmospheric pressure p_{atm}.

1. The integral form of the balance of linear momentum of a group of particles in hydrostatic equilibrium contained in volume v of body $\mathcal{B}$ at its current configuration can be written as

$$\int_v [\rho \boldsymbol{g} + \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x})]\, dv = \mathbf{0}$$

where $\boldsymbol{g} = -g\mathbf{e}_3$. If we take $\boldsymbol{\sigma}(\boldsymbol{x}) = p(\boldsymbol{x})\boldsymbol{I}$, with $\boldsymbol{I}$ being the second order identity tensor, we may write

$$\int_v [\rho \boldsymbol{g} + \nabla p(\boldsymbol{x})]\, dv = \mathbf{0}$$

thus for any arbitrary volume we have

$$\rho \boldsymbol{g} + \nabla p(\boldsymbol{x}, t) = \mathbf{0}$$

and if we dot the equation above with $\mathbf{e}_3$ we obtain

$$\frac{\partial p(\boldsymbol{x})}{\partial x_3} = \rho g$$

which yields

$$\boxed{p(\boldsymbol{x}) = \rho g x_3 + c_1 \qquad (10)}$$

and assuming atmospheric pressure is negligible, we can use our boundary condition $p(h) = 0$, with h being the elevation of the surface of the fluid, which gives us ($x_3 \leq h$)

$$\boxed{p(x_3) = \rho g (x_3 - h) \qquad (11)}$$

2. Consider a body of arbitrary volume v^b. This body is immersed in a fluid in hydrostatic equilibrium. We know that the resultant force acting on the body will be equal and opposite to the tractions on the water, hence

$$\boldsymbol{f}^b = -\int_{\partial v^b} \boldsymbol{t}(\boldsymbol{x}, -\boldsymbol{n}) ds = \int_{\partial v^b} \boldsymbol{\sigma}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}, \partial v^b) ds = \int_{v^b} \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}) dv$$

from above we know

$$\int_{v^b} \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}) dv = \int_{v^b} \nabla p(\boldsymbol{x}) dv = \int_{v^b} \rho g \mathbf{e}_3 dv$$

hence

$$\boxed{\boldsymbol{f}^b = \int_{v^b} \rho g \mathbf{e}_3 \, dv} \tag{12}$$

3. From the fact that the cylinder is of height L and the atmospheric pressure is p_{atm} ($p_{atm} < 0$) we can deduce that

$$p(L) = p_{atm} = \rho g L + c_1 \Rightarrow c_1 = p_{atm} - \rho g L$$

therefore

$$p(L) = \rho g(x_3 - L) + p_{atm}$$

the total force exerted by the fluids on the half of the container is given by

$$\begin{aligned}
\boldsymbol{f}^1 &= -\int_0^L \int_0^\pi p(x_3) \boldsymbol{I} \, \mathbf{e}_r R dx_3 d\vartheta - \int_0^\pi \int_0^R p(0) \boldsymbol{I} \mathbf{e}_3 r dr d\vartheta = \\
&= -\int_0^L \int_0^\pi p(x_3) \underbrace{(\cos(\vartheta)\mathbf{e}_1 + \sin(\vartheta)\mathbf{e}_2)}_{\mathbf{e}_r} R dx_3 d\vartheta - \int_0^\pi \int_0^R p(0) \mathbf{e}_3 r dr d\vartheta = \\
&= (L^2 \rho g - 2L p_{atm}) R \mathbf{e}_2 + \frac{\pi R^2}{2}(p_{atm} - \rho g L)\mathbf{e}_3
\end{aligned}$$

and thus

$$\boxed{\boldsymbol{f}^1 = (L^2 \rho g - 2L p_{atm}) R \mathbf{e}_2 + \frac{\pi R^2}{2}(p_{atm} - \rho g L)\mathbf{e}_3} \tag{13}$$

and for the opposite half we will simply have

$$\begin{aligned}
\boldsymbol{f}^2 &= -\int_0^L \int_\pi^{2\pi} p(x_3)(\cos(\vartheta)\mathbf{e}_1 + \sin(\vartheta)\mathbf{e}_2) R dx_3 d\vartheta - \int_0^\pi \int_0^R p(0) \mathbf{e}_3 r dr d\vartheta = \\
&= (2L p_{atm} - L^2 \rho g) R \mathbf{e}_2 + \frac{\pi R^2}{2}(p_{atm} - \rho g L)\mathbf{e}_3
\end{aligned}$$

or

$$\boxed{\boldsymbol{f}^2 = -(L^2 \rho g - 2L p_{atm}) R \mathbf{e}_2 + \frac{\pi R^2}{2}(p_{atm} - \rho g L)\mathbf{e}_3} \tag{14}$$

4. The resultant of the tractions on the walls of the half container, generated from the splitting of the cylinder by the imaginary plane will be given by

$$\boldsymbol{f}^1_{net} = \boldsymbol{f}^1 + \int_0^L \int_\pi^{2\pi} p_{atm}(\cos(\vartheta)\mathbf{e}_1 + \sin(\vartheta)\mathbf{e}_2) R dx_3 d\vartheta + \int_0^\pi \int_0^R p_{atm} \mathbf{e}_3 r dr d\vartheta$$

which gives

$$\boxed{\boldsymbol{f}^1_{tot} = \rho g L \left[L R \mathbf{e}_2 - \frac{\pi R^2}{2} \mathbf{e}_3 \right]} \qquad (15)$$

and similarly

$$\boxed{\boldsymbol{f}^2_{tot} = -\rho g L \left[L R \mathbf{e}_2 + \frac{\pi R^2}{2} \mathbf{e}_3 \right]} \qquad (16)$$

PROBLEM 4

Obtain the Navier?Stokes equations, which in Cartesian coordinates read

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \nabla \boldsymbol{v}\, \boldsymbol{v} = -\nabla p + \mu \Delta \boldsymbol{v}$$

which states the linear momentum balance for a viscous, incompressible fluid. To this end, use the fact that for a viscous fluid the Cauchy stress $\boldsymbol{\sigma}$ is related to the rate of deformation tensor $\boldsymbol{d} = (\nabla \boldsymbol{v} + \nabla \boldsymbol{v})/2$ as $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu \boldsymbol{d}$.

Take a control volume $v \subseteq \mathcal{B}$ with $\mathcal{B}$ being our body in its current configuration. The linear momentum of the particles contained in the volume v can be written as

$$\boldsymbol{L}(t) = \int_v \rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t) dv$$

furthermore the net external (ignoring long-range) forces actin on our volume v can be written as

$$\boldsymbol{f}(t) = \int_{\partial v} \boldsymbol{t}(\boldsymbol{x}, t) ds$$

where ∂v denotes the boundary of our volume v. From the linear momentum balance principle we can state that the change of linear moment of out particles in v is equal to the forces applied to them, hence

$$\frac{D\boldsymbol{L}(\boldsymbol{x}, t)}{Dt} = \boldsymbol{f}(t)$$

or expanded gives

$$\frac{D}{Dt} \int_v \rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t) dv = \int_{\partial v} \boldsymbol{t}(\boldsymbol{x}, t) ds$$

which gives (with conservation of mass)

$$\int_v \rho(\boldsymbol{x}, t) \frac{D}{Dt} \boldsymbol{v}(\boldsymbol{x}, t) dv = \int_{\partial v} \boldsymbol{t}(\boldsymbol{x}, t) ds$$

and the first integrand can be expanded such that (the function argument remain $(\boldsymbol{x}, t)$ for $\rho, \boldsymbol{a}, \boldsymbol{v}$ they are just dropped here for convenience)

$$\begin{aligned}
\rho \boldsymbol{a} &= \rho \frac{D\boldsymbol{v}}{Dt} = \rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \nabla \boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{v} \overbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) \right]}^{= 0 \text{ from mass conservation}} + \rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \nabla(\boldsymbol{v}) \cdot \boldsymbol{v} = \\
&= \boldsymbol{v} \frac{\partial \rho}{\partial t} + \boldsymbol{v} \nabla \cdot (\rho \boldsymbol{v}) + \rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \nabla(\boldsymbol{v}) \cdot \boldsymbol{v} = \frac{\partial (\boldsymbol{v} \rho)}{\partial t} + \boldsymbol{v} \nabla \cdot (\rho \boldsymbol{v}) + \nabla(\boldsymbol{v}) \cdot \rho \boldsymbol{v} = \\
&= \frac{\partial (\boldsymbol{v} \rho)}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v})
\end{aligned}$$

therefore we are left with

$$\int_v \left[\frac{\partial(\boldsymbol{v}(\boldsymbol{x},t)\rho(\boldsymbol{x},t))}{\partial t} + \nabla\cdot(\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\otimes\boldsymbol{v}(\boldsymbol{x},t))\right]dv = \int_{\partial v}\boldsymbol{t}(\boldsymbol{x},t)ds$$

and using Gauss' theorem we may write

$$\int_v \left[\frac{\partial(\boldsymbol{v}(\boldsymbol{x},t)\rho(\boldsymbol{x},t))}{\partial t} + \nabla\cdot(\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\otimes\boldsymbol{v}(\boldsymbol{x},t))\right]dv = \int_{v}\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{x},t)dv$$

since v is an arbitrary portion of $\mathcal{B}$ we may localize the above statement as

$$\frac{\partial(\boldsymbol{v}(\boldsymbol{x},t)\rho(\boldsymbol{x},t))}{\partial t} + \nabla\cdot(\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\otimes\boldsymbol{v}(\boldsymbol{x},t)) = \nabla\cdot\boldsymbol{\sigma}(\boldsymbol{x},t)$$

furthermore if the fluid is incompressible and $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{d}$ with $\boldsymbol{d} = (\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^\top)/2$, then we can say

$$\begin{aligned}
&\rho\frac{\partial\boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + \rho\nabla\cdot(\boldsymbol{v}(\boldsymbol{x},t)\otimes\boldsymbol{v}(\boldsymbol{x},t)) = \nabla\cdot\boldsymbol{\sigma}(\boldsymbol{x},t)\\
&\rho\frac{\partial\boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + \rho[\nabla\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},t) + \boldsymbol{v}(\boldsymbol{x},t)\nabla\cdot\boldsymbol{v}(\boldsymbol{x},t)] = \nabla\cdot[-p(\boldsymbol{x},t)\boldsymbol{I} + 2\mu\boldsymbol{d}(\boldsymbol{x},t)]\\
&\rho\frac{\partial\boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + \rho[\nabla\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},t) + \boldsymbol{v}(\boldsymbol{x},t)\nabla\cdot\boldsymbol{v}(\boldsymbol{x},t)] = -\nabla p(\boldsymbol{x},t) + \mu\nabla\cdot(\nabla\boldsymbol{v}(\boldsymbol{x},t) + \nabla\boldsymbol{v}(\boldsymbol{x},t)^\top)
\end{aligned}$$

since the body is incompressible, $J = 1$ then $\dot{J} = J\nabla\cdot\boldsymbol{v} = 0$ which implies $\nabla\cdot\boldsymbol{v} = 0$ therefore we can continue in our manipulation such that

$$\begin{aligned}
&\rho\frac{\partial\boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + \rho[\nabla\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},t) + \boldsymbol{v}(\boldsymbol{x},t)\cancelto{0}{\nabla\cdot\boldsymbol{v}(\boldsymbol{x},t)}] = -\nabla p(\boldsymbol{x},t) + \mu\nabla\cdot(\nabla\boldsymbol{v}(\boldsymbol{x},t) + \nabla\boldsymbol{v}(\boldsymbol{x},t)^\top)\\
&\rho\frac{\partial\boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + \rho\nabla\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},t) = -\nabla p(\boldsymbol{x},t) + \mu\nabla\cdot\nabla\boldsymbol{v}(\boldsymbol{x},t) + \mu\nabla\cdot\nabla\boldsymbol{v}(\boldsymbol{x},t)^\top
\end{aligned}$$

where

$$\nabla\cdot\nabla\boldsymbol{v}(\boldsymbol{x},t)^\top = \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}v_i\mathbf{e}_j\otimes\mathbf{e}_i\cdot\mathbf{e}_k = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}v_i\mathbf{e}_k\cdot\mathbf{e}_i\otimes\mathbf{e}_j = \nabla\underbrace{\nabla\cdot\boldsymbol{v}(\boldsymbol{x},t)}_{0}$$

therefore the Navier-Stokes equation may be written as

$$\boxed{\rho\frac{\partial\boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + \rho\nabla\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},t) = -\nabla p(\boldsymbol{x},t) + \mu\nabla\cdot\nabla\boldsymbol{v}(\boldsymbol{x},t) \qquad (17)}$$

or in indicial notation dropping the function argument

$$\boxed{\rho\frac{\partial v_i}{\partial t} + \rho v_{i,j}v_j = -p_{,i} + \mu v_{i,jj} \qquad (18)}$$