Finite Elements in n-D

Computing Element arrays

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Computing the Element Arrays

Constructing the Element Arrays

Recall that

$$\boldsymbol{K} = \mathbb{A}_{e=1}^{n_{el}} \boldsymbol{k}^e, \quad \boldsymbol{F} = \mathbb{A}_{e=1}^{n_{el}} \boldsymbol{f}^e$$

where $\mathbb{A}$ is the assembly operator, and

$$\boldsymbol{k}_{ab}^e = a^e(\phi_a^e, \phi_b^e) = \int_{\Omega^e} \nabla_{\boldsymbol{x}} \phi_a^e(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{x}} \phi_b^e d\Omega =$$

$$= \int_{\hat{\Omega}} [(\nabla_\xi \hat{\boldsymbol{x}}^e(\xi))^{-\top} \nabla_\xi \hat{\phi}_a(\xi)] \cdot [(\nabla_\xi \hat{\boldsymbol{x}}^e(\xi))^{-\top} \nabla_\xi \hat{\phi}_b(\xi)] \, \hat{j}(\xi) d\hat{\Omega}$$

$$\approx \sum_{(\tilde{\xi}_Q, \omega_Q) \in \mathcal{Q}} [(\nabla_\xi \hat{\boldsymbol{x}}^e(\tilde{\xi}_Q))^{-\top} \nabla_\xi \hat{\phi}_a(\tilde{\xi}_Q)] \cdot [(\nabla_\xi \hat{\boldsymbol{x}}^e(\tilde{\xi}_Q))^{-\top} \nabla_\xi \hat{\phi}_b(\tilde{\xi}_Q)] \, \hat{j}^e(\tilde{\xi}_Q) \omega_Q$$

where $\hat{j}^e(\xi) = \det(\nabla_\xi \hat{\boldsymbol{x}}^e(\xi))$.

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&= \int_{\hat{\Omega}} [(\nabla_{\boldsymbol{\xi}} \hat{\boldsymbol{x}}^e(\boldsymbol{\xi}))^{-\top} \nabla_{\boldsymbol{\xi}} \hat{\phi}_a(\boldsymbol{\xi})] \cdot [(\nabla_{\boldsymbol{\xi}} \hat{\boldsymbol{x}}^e(\boldsymbol{\xi}))^{-\top} \nabla_{\boldsymbol{\xi}} \hat{\phi}_b(\boldsymbol{\xi})] \, \hat{j}(\boldsymbol{\xi}) d\hat{\Omega} \\
&\approx \sum_{(\tilde{\boldsymbol{\xi}}_Q, \omega_Q) \in \mathcal{Q}} [(\nabla_{\boldsymbol{\xi}} \hat{\boldsymbol{x}}^e(\tilde{\boldsymbol{\xi}}_Q))^{-\top} \nabla_{\boldsymbol{\xi}} \hat{\phi}_a(\tilde{\boldsymbol{\xi}}_Q)] \cdot [(\nabla_{\boldsymbol{\xi}} \hat{\boldsymbol{x}}^e(\tilde{\boldsymbol{\xi}}_Q))^{-\top} \nabla_{\boldsymbol{\xi}} \hat{\phi}_b(\tilde{\boldsymbol{\xi}}_Q)] \, \hat{j}^e(\tilde{\boldsymbol{\xi}}_Q) \omega_Q
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where $\hat{j}^e(\boldsymbol{\xi}) = \det(\nabla_{\boldsymbol{\xi}} \hat{\boldsymbol{x}}^e(\boldsymbol{\xi}))$.

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$$\begin{aligned}
\boldsymbol{f}_a^e = F^e(\phi_a^e) &= \int_{\Omega^e} \phi_a^e(\boldsymbol{x}) f(\boldsymbol{x}) d\Omega \\
&= \int_{\hat{\Omega}} \hat{\phi}_a(\boldsymbol{\xi}) f(\hat{\boldsymbol{x}}^e(\boldsymbol{\xi})) j^e(\boldsymbol{\xi}) d\hat{\Omega} = \\
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\end{aligned}$$

where $\hat{j}(\xi) = \det(\nabla_\xi \hat{\boldsymbol{x}}^e(\xi))$.

[1]for simplicity we are assuming homogeneous Neumann boundary conditions hence

$$h = 0 \Rightarrow \int_{\Gamma_N} h \phi_i d\Gamma = 0$$

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