MID-TERM PRACTICE QUESTIONS

CEE 361-513: Introduction to Finite Element Methods

Thurday Oct. 19

This are some example questions to sharpen your skills for the mid-term. In addition you should review the homework, precepts, and lecture notes, as well as Chapter 1.1 - 1.9 and 1.12 - 1.14 of the Hughes book.

PROBLEM 1

1. Let $d = 2$. $u = x_1 x_2 + c$ be a scalar where c is any arbitrary constant. Find ∇u and $\nabla \cdot (\nabla u)$.
2. Let $d = 3$. $\boldsymbol{u} = x_1 x_3 \mathbf{e}_1 + x_2 x_3 \mathbf{e}_2$. Find the gradient of $\boldsymbol{u}$.
3. Is $\boldsymbol{T}(\boldsymbol{u}) = \sin(\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_2 + \cos(\boldsymbol{u} \cdot \mathbf{e}_2)\mathbf{e}_1$ a tensor?
4. Let $\boldsymbol{x}_a = 2\mathbf{e}_1 + 5\mathbf{e}_2$ and $\boldsymbol{x}_b = 7\mathbf{e}_1 + 8\mathbf{e}_2$. Find the projection tensor that projects vectors along the direction $\boldsymbol{a} = \boldsymbol{x}_b - \boldsymbol{x}_a$.
5. Let $\{\mathbf{e}_i\}_{i=1}^3$ be a set of orthonormal basis. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{v}$ be three vectors such that $\boldsymbol{b} = \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{a}\, \boldsymbol{a} / \|\boldsymbol{a}\|^2$. Show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent (i.e. $\alpha \boldsymbol{b} + \boldsymbol{a} = 0 \Rightarrow \alpha = 0$).
6. Let $\mathrm{tr}(\boldsymbol{A}) := \boldsymbol{A} : \mathbf{1}$ be the trace of a tensor $\boldsymbol{A}$. If $f = x_2 x_3 + x_1 x_3 + x_1 x_2$, in which $\{\mathbf{e}_i\}_{i=1}^3$ is a set of orthonormal basis associated with the cartesian coordinates $\{x_i\}_{i=1}^3$. Show that $\nabla \cdot (\nabla f) = \mathrm{tr}(\nabla(\nabla(f)))$.

PROBLEM 2

Consider the truss shown below. Foreach node we have associated coordinates $\boldsymbol{q}_z$ and associated global degrees of freedom $\boldsymbol{u}_z$, where both $\boldsymbol{q}$ and $\boldsymbol{u}$ are vectors. All elements have the same E, A.

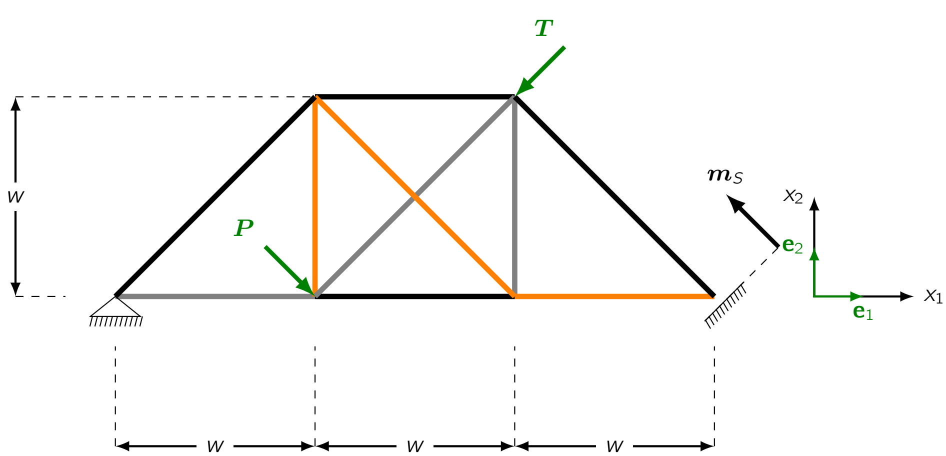


Figure 1: The system of uniaxial rods

1. Label each node and element and create a connectivity array.
2. For each element write the internal forces as the matrix vector operation of the *local element stiffness* and the *local degrees of freedom*.
3. For each element write the internal forces as the matrix vector operation of the *local element stiffness* and the *GLOBAL degrees of freedom* using the connectivity array.
4. For each node write the equilibrium equations in terms of the external forces, the reactions, and the internal forces.

5. Write down the equilibrium equations in matrix form. Namely, as we did in class, write the equilibrium equations with a load vector containing reactions and external forces, denoted it by {P}, the stiffness matrix denoted by $[K]$, and the vector of displacements $\{U\}$ such that

$$[K]\{U\} = \{P\}.$$

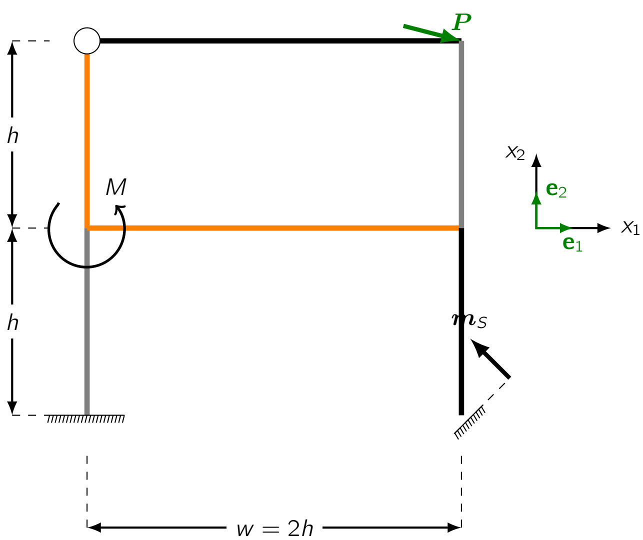
6. At the leftmost node we prevent the truss from moving. At the rightmost node we allow the truss to move along a plane whose unit normal is $\boldsymbol{m}_2$. Apply the aforementioned conditions to $[K], \{P\}$.

7. What is the reaction force at the leftmost node?

8. What is the reaction force at the rightmost nodes?

PROBLEM 3

Consider the frame shown below. At the lower- and left-most node we constrain the frame from moving in all directions and we prevent it from rotating. At the upper- and left-most node we have a hinge (hence no moment can be transferred). At the lowest- and right-most support the frame is allowed to move along a plane define by the normal $\boldsymbol{m}_S$. All elements have the same E, I, A.



1. Label each element and node and write the connectivity array.

2. For each node write the equilibrium equations in terms of the external force $\boldsymbol{P}$ and moment M, and the internal forces $\boldsymbol{f}^e_{i,j}$ and moments $m^e_{i,j}$.

3. Write the general expression of internal forces (and moments) as the matrix vector operation of the *local element stiffness* and the *local degrees of freedom*.

4. For each element write the internal forces (and moments) as the matrix vector operation of the *local element stiffness* and the *GLOBAL degrees of freedom* using the connectivity array.

5. Using $\boldsymbol{K}^e_{\boldsymbol{fw}}, \boldsymbol{k}^e_{\boldsymbol{f}\theta}, \ldots$, write down the equilibrium equations in matrix form.

6. At the lower- and left-most node we constrain the frame from moving in all directions and we prevent it from rotating. At the upper- and left-most node we have a hinge (hence no moment can be transferred). At the lowest- and right-most support the frame is allowed to move along a plane define by the normal $\boldsymbol{m}_S$. Apply the aforementioned conditions to the matrix form of the previous step.

7. How would you determine the reactions?

PROBLEM 4

Consider the following strong form: find $u : (0,1) \to \mathbb{R}$ such that

$$-\frac{d^2u}{dx^2} + u + x^3 = 0, \quad \forall x \in (0,1)$$

For each of the following boundary conditions, state the set of trial and test functions and derive the weak form.

i. $u(0) = g_0, \quad u(1) = g_1$

ii. $\frac{du}{dx}(0) = h_0, \quad u(1) = g_1$

iii. $u(0) = g_0, \quad \frac{du}{dx}(1) = h_1$

PROBLEM 5

For the above BVP derive the matrix form and, assuming linear shape functions as shown in class,

i. Derive the *element* stiffness matrix

ii. Assuming we discretize $(0,1)$ into two elements, with the element stiffness matrix derived above, assemble the global stiffness matrix.

PROBLEM 6

Consider the potential given by

$$\Pi[u] = \int_0^1 \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx + \int_0^1 \frac{u^2}{2} dx + \int_0^1 x^3 u dx.$$

Find $\langle \delta\Pi, \delta u \rangle$.