Finite Elements in 1-D

The Local Element View

Maurizio M. Chiaramonte

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

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$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find u s.t.

$$\frac{d^2u}{dx^2} = f \quad \forall x \in (a, b)$$

with

$$u = g \quad \forall x \in \Gamma_D$$

$$\frac{du}{dx} = h \quad \forall x \in \Gamma_N.$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find $u \in \mathcal{S}$ s.t.

$$a(u,v) = F(v) \quad \forall v \in \mathcal{V}$$

where

$$\mathcal{S} = \{u | u \in H^1(\Omega), u = g \, \forall x \in \Gamma_D\},$$
$$\mathcal{V} = \{v | v \in H^1(\Omega), v = 0 \, \forall x \in \Gamma_D\}$$

and

$$a(u,v) = \int_\Omega \frac{du}{dx}\frac{dv}{dx} dx, \quad F(v) = hv|_{\Gamma_N} - \int_\Omega fv dx$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Choose a finite number of (sufficiently smooth) basis functions

$$\Phi^h = \{\phi_i\}_{i=1}^N, \quad \mathcal{Z}^h = \{z^h | z^h = \sum_{i=1}^N z_i \phi_i(x)\}$$

Define a $\mathcal{S}^h \subseteq \mathcal{S}$ and $\mathcal{V}^h \subseteq \mathcal{V}$ such that

$$\mathcal{S}^h = \mathcal{S} \cap \mathcal{Z}^h,$$
$$\mathcal{V}^h = \mathcal{V} \cap \mathcal{Z}^h.$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find $u^h \in \mathcal{S}^h$ s.t.

$$a(u^h, v^h) = F(v^h) \quad \forall v^h \in \mathcal{V}^h$$

where

$$u^h(x) = u_i \phi_i(x), \quad v^h(x) = v_i \phi_i(x).$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find u_j s.t.

$$\sum_{j=1}^{N} a(\phi_i, \phi_j) u_j = F(\phi_i) \quad \forall i = 1 \dots N$$

where

$$a(\phi_i, \phi_j) = \int_\Omega \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx, \quad F(\phi_i) = h\phi_i|_{\Gamma_N} - \int_\Omega f\phi_i dx$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find u_j s.t.

$$[K]\{U\} = \{F\}$$

where

$$[K]_{ij} = a(\phi_i, \phi_j), \quad \{U\}_j = u_j, \quad \{F\}_i = F(\phi_i)$$

where

$$a(\phi_i, \phi_j) = \int_\Omega \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} dx, \quad F(\phi_i) = h\phi_i|_{\Gamma_N} - \int_\Omega f\phi_i dx$$

Constructing the Basis Functions

To construct $\mathscr{Z}^h$ we:

1. Divide the domain $\Omega = \bigcup_{e=1}^{N} \Omega^e$
2. Require that $z^h\big|_{\Omega^e}$ be a polynomial of order p
3. Ensure smoothness across boundaries of Ω^e by constraining dofs

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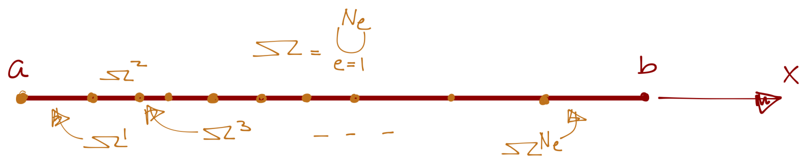

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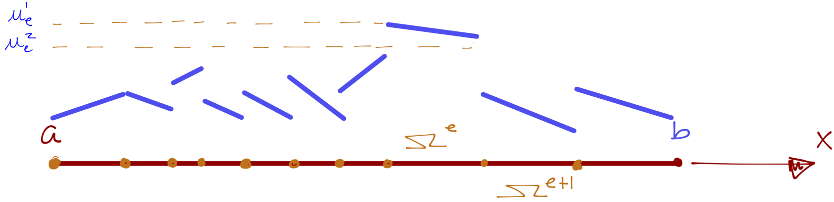

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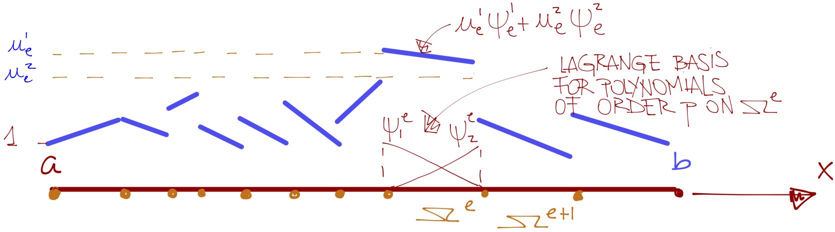

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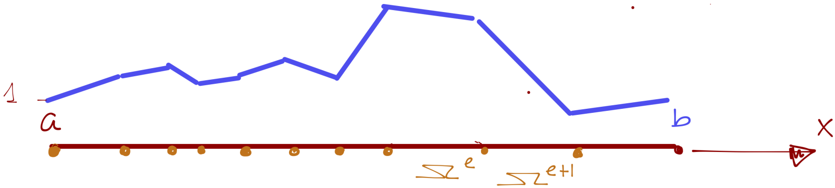
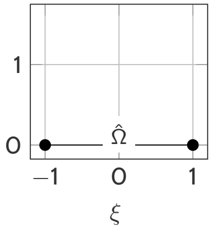


The Element View

$$[K]_{ij} = a(\phi_i, \phi_j) = \int_\Omega \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \sum_{e=1}^{N_e} \int_{\Omega^e} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx$$

The Element View

Parametric Element Domain $\hat{\Omega}$



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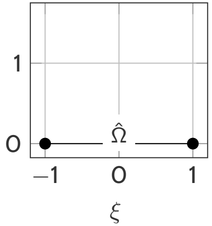
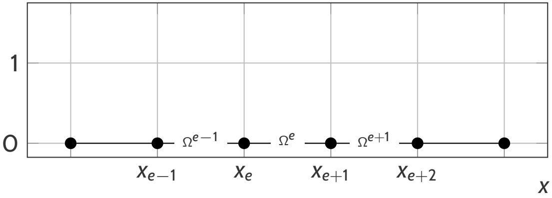


Physical/Real Domain $\Omega = \bigcup_{e=1}^{n_{el}} \Omega^e$



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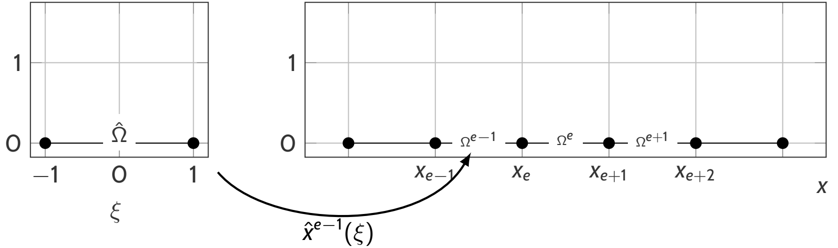

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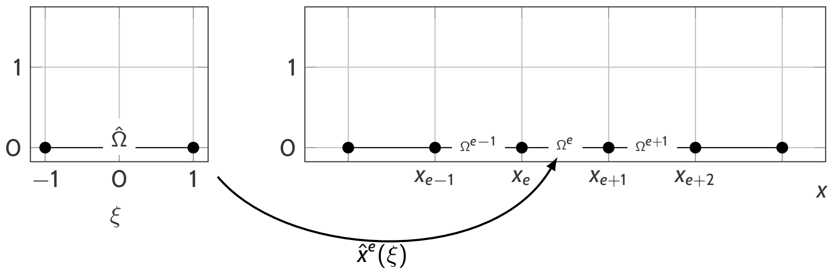

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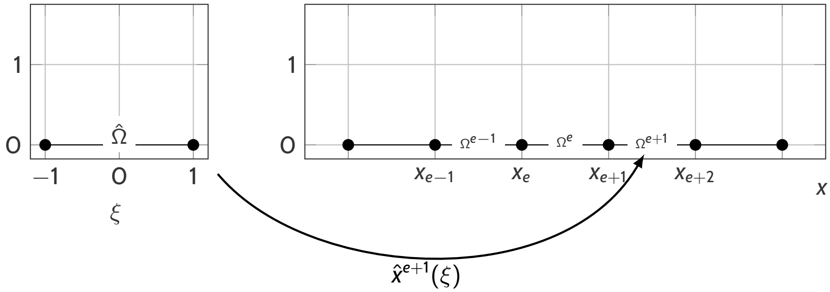

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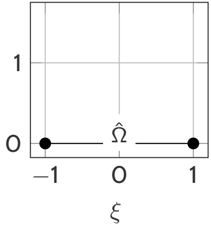

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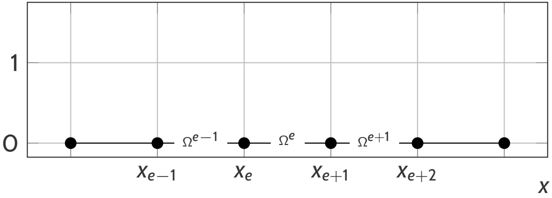

$$\hat{x}^e : \hat{\Omega} \rightarrow \Omega^e$$

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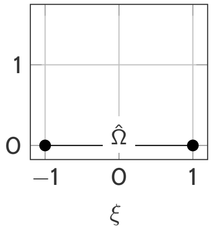
Parametric Element Domain $\hat{\Omega}$



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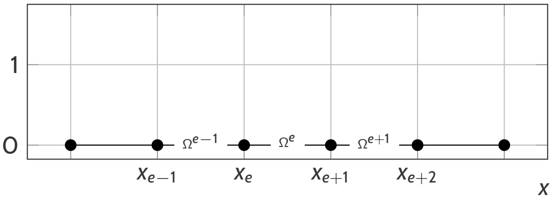
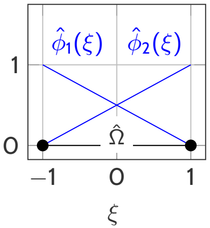


$$\hat{x}^e : [-1, 1] \rightarrow [x_i, x_{i+1}]$$

The Element View

Parametric Element Domain $\hat{\Omega}$



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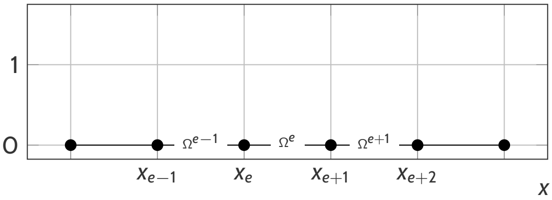

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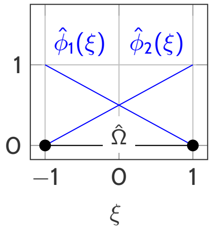

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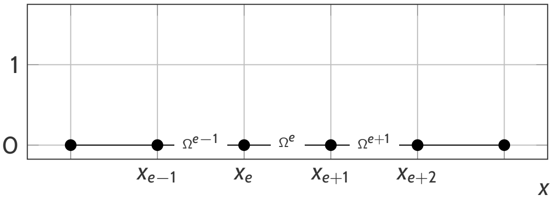

$$\phi_a^e(x) \quad \text{such that} \quad \phi_a^e(\,\hat{x}^e(\xi)\,) = \hat{\phi}_a(\xi)$$

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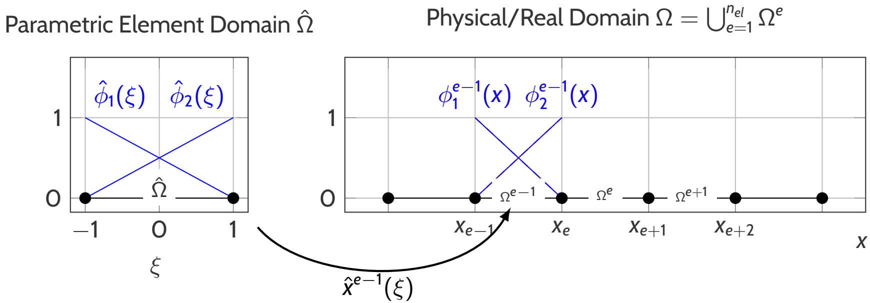

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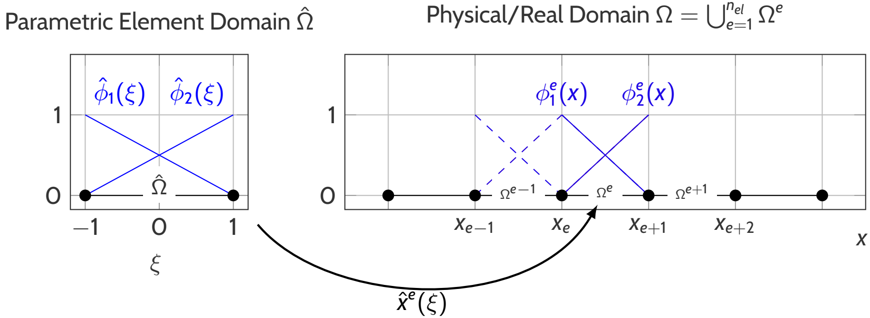

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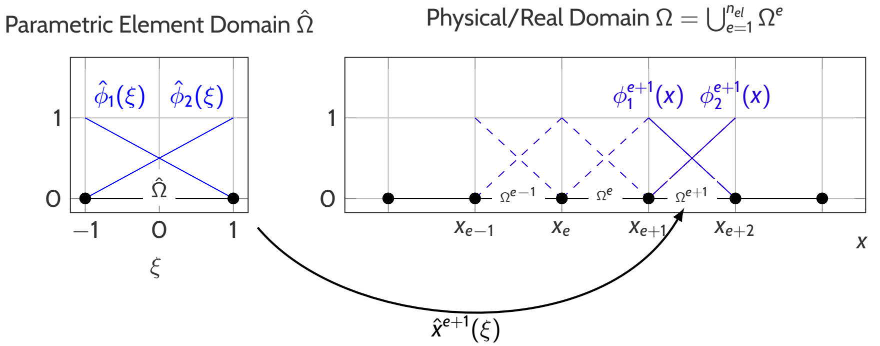

$$\phi_a^e(x) \quad \text{such that} \quad \phi_a^e(\,\hat{x}^e(\xi)\,) = \hat{\phi}_a(\xi)$$

The Element View

$$[k^e]_{ab} = a^e(\phi_a^e, \phi_b^e) = \int_{\Omega^e} \frac{d\phi_a^e}{dx}\frac{d\phi_b^e}{dx}dx = \tag{1}$$

$$= \int_{\hat{\Omega}} \frac{d\hat{\phi}_a}{d\xi}\left(\frac{dx^e}{d\xi}\right)^{-1}\frac{d\phi_b}{d\xi}\left(\frac{dx^e}{d\xi}\right)^{-1}\frac{dx^e}{d\xi}d\xi \tag{2}$$

then *assemble* into global stiffness

$$[K] = \mathbb{A}_{e=1}^{N_e}[k^e]$$

The Element View

$$[k^e]_{ab} = a^e(\phi_a^e, \phi_b^e) = \int_{\Omega^e} \frac{d\phi_a^e}{dx} \frac{d\phi_b^e}{dx} dx = \tag{1}$$

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then *assemble* into global stiffness

$$[K] = \mathbb{A}_{e=1}^{N_e}[k^e]$$

The Element View

$$\int_{\mathcal{B}} f(x)dx \approx \sum_{q=1}^{N_q} f(x_q)\omega_q$$

The Element View

$$[k^e]_{ab} = a^e(\phi_a^e, \phi_b^e) = \int_{\Omega^e} \frac{d\phi_a^e}{dx}\frac{d\phi_b^e}{dx} dx = \tag{3}$$

$$= \int_{\hat{\Omega}} \frac{d\hat{\phi}_a}{d\xi}\left(\frac{dx^e}{d\xi}\right)^{-1}\frac{d\hat{\phi}_b}{d\xi}\left(\frac{dx^e}{d\xi}\right)^{-1}\frac{dx^e}{d\xi} d\xi \tag{4}$$

then *assemble* into global stiffness

$$[K] = \mathbb{A}_{e=1}^{N_e}[k^e]$$

The Element View

$$[k^e]_{ab} = a^e(\phi^e_a, \phi^e_b) = \int_{\Omega^e} \frac{d\phi^e_a}{dx}\frac{d\phi^e_b}{dx} dx = \tag{5}$$

$$\approx \sum_{q=1}^{N_q} \left[\frac{d\hat{\phi}_a}{d\xi} \left(\frac{dx^e}{d\xi} \right)^{-1} \frac{d\hat{\phi}_b}{d\xi} \left(\frac{dx^e}{d\xi} \right)^{-1} \frac{dx^e}{d\xi} \right] \bigg|_{x_q} \omega_q \tag{6}$$

then *assemble* into global stiffness

$$[K] = \mathbb{A}_{e=1}^{N_e} [k^e]$$