LECTURE 19:

- CONSTITUTIVE THEORY

RESTRICTIONS:
- SECOND LAW
- MATERIAL SYMMETRY
- MATERIAL FRAME INDIFF.
- LOCAL ACTION
- ANGULAR MOM BALANCE
- POLYCONVEXITY

A CONSTITUTIVE EQUATION (AKA EQUATION OF STATE) RELATES TWO THERMODYNAMIC STATE VARIABLES EG.

$$\underline{\underline{P}}(\underline{X},t) = \hat{\underline{\underline{P}}}(\underline{\underline{F}}(\underline{X},t),\underline{X},t)$$

IN WHAT FOLLOWS WE WILL FOCUS SOLELY ON ISOTHERMIC PROCESSES AS WELL AS REVERSIBLE (DISSIPATION = 0)

THEREFORE WE POSTULATE THE EXISTANCE OF A FREE ENERGY FUNCTION ψ S.T. $\frac{d\psi}{d\theta} = 0$

SECOND LAW

ALL CONSTITUTIVE PROCESSES MUST BE CONSISTENT WITH THE FREE ENERGY IMBALANCE

THIS IS KNOWN AS THE COLEMAN-NOLL PROCEDURE OR EXPLOITATION THAT REQUIRES THAT THE FREE ENERGY IMBALANCE

$$-\rho_0\dot{\psi} + \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \rho_0\eta\dot{\theta} - \frac{1}{\theta}\underline{Q}\cdot\nabla\theta \geq 0$$

BE SATISFIED FOR ALL CONCEIVABLE PROCESS $(\underline{\underline{F}}, \dot{\underline{\underline{F}}}, \theta, \dot{\theta}, \nabla\theta)$.

IF $\psi(\underline{\underline{F}},\theta)$ THEN

$$\left[\underline{\underline{P}} - \rho_0\frac{d\psi}{d\underline{\underline{F}}}\right] : \dot{\underline{\underline{F}}} - \rho_0\left[\frac{d\psi}{d\theta} + \eta\right]\dot{\theta} - \frac{1}{\theta}\underline{Q}\cdot\nabla\theta \geq 0$$

FOR A REVERSIBLE PROCESS THE STRICT EQUALITY HOLDS & , AS WE IGNORE TEMP.

$$\left[\underline{\underline{P}} - \rho_0 \frac{d\psi}{d\underline{\underline{F}}}\right] : \dot{\underline{\underline{F}}} = 0$$

$$\Rightarrow \underline{\underline{P}} = \rho_0 \frac{d\psi}{d\underline{\underline{F}}} = \frac{d\Psi}{d\underline{\underline{F}}}$$

IF THE REFERENCE CONFIGURATION IS STRESS FREE THEN

$\frac{d\Psi}{d\underline{\underline{F}}}(\underline{\underline{1}}) = 0$ & FURTHER WE NORMALIZE

Ψ S.T. $\Psi(\underline{\underline{1}}) = 0$

FURTHERMORE WE REQUIRE THAT AS $J \to +\infty$ OR $J \to 0^+$

$\Psi \to +\infty$

AND IN ADD. $\Psi \geq 0$

MATERIAL FRAME INDEFFERENCE

A SCALAR FUNCTION SHOULD BE INVARIANT UNDER CHANGES OF OBSERVER NAMELY $\forall \ \underline{\underline{Q}} \in SO(3)$

$$\Psi(\underline{\underline{Q}}\underline{\underline{F}}) = \Psi(\underline{\underline{F}})$$

NOTE: WE SAW IN HW THAT IN GENERAL COMPOSITION DOES NOT COMMUTE

IF FOR A GIVEN $\underline{\underline{F}}$ $\Psi(\underline{\underline{F}}) = \Psi(\underline{\underline{F}}\underline{\underline{Q}})$ THEN THE MATERIAL IS ISOTROPIC. NAMELY IT'S RESPONSE IS INVARIANT OF DIRECTION

NOTE THAT IF $\Psi = \hat{\Psi}(\underline{C})$ MFI IS SATISFIED A PRIORI

POLYCONVEXITY OF STRAIN ENERGY FUNC

WE SAW LAST TIME THAT WE CAN RECAST THE PROBLEM OF BOM AS

$$\inf_{\underline{\varphi} \in V} \left\{ \Pi[\underline{\varphi}] = \int_{\Omega_0} \psi(\underline{F}) \, dV_0 \right\}$$

WHICH WE EQUIVALENTLY SAID LET'S FIND A STAT. POINT $\underline{\varphi}$ S.T.

$$\langle \delta\Pi, \delta\underline{\varphi} \rangle = 0 \quad \forall \ \delta\underline{\varphi} \in TV$$

(AND WE SHOULD ALSO CHECK FOR SECOND VARIATIONS)

THE EXISTANCE OF A MINIMIZER IS CLOSELY TIED TO THE CONVEXITY OF ψ.

CONVEX SET:

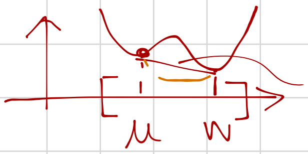
A CONVEX SET Z IS SUCH THAT $\forall u, v \in Z$ & $\forall \xi \in [0,1]$

$$w = \xi u + (1-\xi) v \quad \in Z$$

CONVEX FUNCTION

A CONVEX FUNCTION OVER A CONVEX SET IS S.T.

$$f(\xi u + (1-\xi) w) \leq \xi f(u) + (1-\xi) f(w)$$



$\xi f(u) + (1-\xi) f(w)$

A SUFFICIENT CONDITION FOR THE EXISTANCE OF A MINIMIZER IS FOR ψ TO BE CONVEX.

NAMELY

$$\psi(\xi \underline{F}_1 + (1-\xi) \underline{F}_2) \leq \xi \psi(\underline{F}_1) + (1-\xi) \psi(\underline{F}_2)$$

WHILE STRICT CONVEXITY IMPLIES THE EXISTANCE OF A MINIMIZER IT IS TOO STRONG OF A CONDITION BECAUSE

- STRICT CONVEXITY IMPLIES UNIQUENESS OF SOLUTIONS (PREVENT POSSIBLE BIFURCATIONS EG. BUCKLING)

- STRICT CONVEXITY IS INCOMPATIBLE WITH MATERIAL FRAME INDIFFERENCE

REMEMBER, FRAME INDEFFERENCE REQUIRED $\forall\ \underline{\underline{Q}} \in SO(3)$

$$\psi(\underline{\underline{F}}) = \psi(\underline{\underline{Q}}\,\underline{\underline{F}})$$

NOW CONSIDER

$$\underline{\underline{F}} = \underline{\underline{1}}, \quad [\underline{\underline{Q}}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{CLOCKWISE } 90^\circ \text{ ROT}$$

THEN $\psi(\underline{\underline{F}}) = 0$ AND $\psi(\underline{\underline{Q}}\,\underline{\underline{F}}) \overset{\text{MFI}}{=} \psi(\underline{\underline{F}}) = 0$

NOW CHOOSE $\xi = 1/2$ AND NOTE

$$\frac{1}{2}\left([\underline{\underline{Q}}] + [\underline{\underline{F}}]\right) = \begin{bmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\leftarrow$ VOLUMETRIC CONTRACTION BY 1/2 IN PLANE & 45° ROT

$$\underbrace{\underbrace{\psi\left(\tfrac{1}{2}(\underline{\underline{Q}} + \underline{\underline{F}})\right)}_{>0} \leq \tfrac{1}{2}\left(\psi(\underline{\underline{Q}}) + \psi(\underline{\underline{F}})\right) = 0}_{\text{CONTRADICTION}}$$

A MORE RELAXED CONDITION IS REQUIRING POLY-CONVEXITY

NAMELY $\psi(\underline{\underline{F}}) = \hat{\psi}(\underline{\underline{F}}, J\underline{\underline{F}}^{-T}, J)$ MUST BE CONVEX WRT $\underline{\underline{F}}, J\underline{\underline{F}}^{-T}, J$.

IN ADDITION IT MUST SATISFY CERTAIN GROWTH REQ.

$$\hat{\psi} \geq c_0\left(\|\underline{\underline{F}}\|^2 + \|J\underline{\underline{F}}^{-T}\|^{3/2}\right) - c_1$$

W/ $c_0 \in \mathbb{R}^+, c_1 \in \mathbb{R}$

EG. NEO-HOOKEAN

$$\hat{\psi}(\underline{\underline{F}}, J\underline{\underline{F}}^{-T}, J) = \frac{\mu}{2}\left[\|\underline{\underline{F}}\|^2 - 3\right] + \frac{\mu^2}{\lambda}\left[J^{-\lambda/\mu} - 1\right]$$

$$\frac{d^2\psi}{dJ} = (\lambda + \mu)\, J^{-\frac{\lambda + 2\mu}{\mu}} \geq 0$$

SINCE $J > 0 \Rightarrow \lambda, \mu > 0$ ✓

NOTE ST VENANT KIRCHOFF & OTHER MODELS BASED ON HECKY TENSOR ARE NON-POLYCONVEX

BALANCE OF ANGULAR MOMENTUM

RECALL THAT

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T, \qquad \underline{\underline{P}} = J\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T}$$

$$\Rightarrow \quad \underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T$$

PRINCIPLE OF LOCALITY

THE CONSTITUTIVE RESPONSE SHOULD ONLY DEPEND ON PRIMARY VARIABLES AT $\underline{X}$