LECTURE 9:

- RATES OF DEFORMATION
- COVARIANT CONTRAVARIANT VECTORS
- PUSH FORWARD PULL BACK
- INTRO TO STRESS

RATES OF DEFORMATION

$$\underline{l} = \underline{\nabla}_x \underline{v}(\underline{x},t) \quad \Leftarrow \text{SPATIAL VELOCITY GRADIENT}$$

$$\dot{\underline{\underline{F}}} = \frac{d}{dt}\left(\underline{\underline{F}}\right)\Big|_{\underline{X}} = \underline{\nabla}_X \left(\frac{d\underline{\varphi}}{dt}\right) = \underline{\nabla}_X \underline{V}(\underline{X},t) \quad \Leftarrow \text{MAT VEL GRAD}$$

$$\underline{l} = \underline{\nabla}_x \underline{v} = \underline{\nabla}_x \left(\underline{V}(\underline{\varphi}^{-1}(\underline{x},t),t\right) = \underline{\nabla}_X \underline{V}\, \underline{\nabla}_x \underline{\varphi}^{-1} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$$

$\underline{l}$ CAN BE ADDITIVELY DECOMPOSED

$$\underline{l} = \underline{d} + \underline{w}$$

$$\underline{d} = \text{SYM}(\underline{l}) \quad \Leftarrow \text{RATE OF DEFORMATION TENSOR}$$

$$\underline{w} = \text{SKEW}(\underline{l}) \quad \Leftarrow \text{SPIN (VORTICITY) TENSOR}$$

BOTH $\underline{d}$ & $\underline{w}$ ARE SPATIAL QUANTITIES

IMPORTANT:

$$\dot{J} = \frac{dJ}{d\underline{\underline{F}}} : \dot{\underline{\underline{F}}} = J\underline{\underline{F}}^{-T} : \dot{\underline{\underline{F}}} = J\underline{\underline{1}} : \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1} = J\underline{\underline{1}} : \underline{l} = J\underline{\nabla}_x \cdot \underline{v}$$

THIS IMPLIES THAT FOR VOLUME PRESERVING DEF

$$J = 1 \Rightarrow \dot{J} = 0 \Rightarrow \underline{\nabla}_x \cdot \underline{v} = 0$$

COVARIANT & CONTRAVARIANT VECTORS

RECALL THAT WE SAID TANGENT VECTORS $\underline{A}$ IN THE REFERENCE CONFIGURATION CAN BE MAPPED INTO THE DEFORMED AS

$$\underline{a} = \underline{\underline{F}}\,\underline{A}$$

NOTE: $\underline{\underline{F}} = \underline{\underline{F}}(\underline{X}, t) \Rightarrow \underline{a} = \underline{a}(\underline{x}, t)$

ALSO WE SAW (NANSON'S FORMULA) THAT NORMAL VECTORS MAP AS

$$\underline{n} = \underline{\underline{F}}^{-T}\underline{N}$$

$$\left(\underline{T} \cdot \underline{N} = \underline{t} \cdot \underline{n} = 0 \Rightarrow (\underline{T} \cdot \underline{N} - \underline{t} \cdot \underline{n}) = 0 \Rightarrow (\underline{T} \cdot \underline{N} - \underline{t} \cdot \underline{n}) = 0 \right.$$

$$\Rightarrow (\underline{\underline{F}}^{-1}\underline{t} \cdot \underline{N} - \underline{t} \cdot \underline{n}) = 0 \Rightarrow \underline{t} \cdot (\underline{\underline{F}}^{-T}\underline{N} - \underline{n}) = 0 \Rightarrow \underline{\underline{F}}^{-T}\underline{N} = \underline{n}$$

WHE $\underline{n}(\underline{x}, t)$

A SPATIAL VECTOR FIELD $\underline{a}$ IS SAID TO CONVECT WITH THE BODY IF THERE EXIST A REFERENCE VECTOR FIELD $\underline{A}$ SUCH THAT

$$\underline{a}(\underline{x}, t) = \underline{\underline{F}}(\underline{X}, t)\,\underline{A}(\underline{X}) \quad \longleftarrow \text{COVARIANT}$$

OR

$$\underline{a}(\underline{x}, t) = \underline{\underline{F}}^{-T}(\underline{X}, t)\,\underline{A}(\underline{X}) \quad \longleftarrow \text{CONTRAVARIANT}$$

THUS, IF A VECTOR FIELD IS COVARIANT

$$\dot{\overline{\underline{\underline{F}}^{-1}\underline{a}}} = \dot{\underline{A}} = 0$$

$\dot{\underline{A}} = 0 \longleftarrow$ DOES NOT DEPEND ON TIME

$$\Rightarrow \dot{\overline{\underline{\underline{F}}^{-1}\underline{a}}} = \dot{\underline{\underline{F}}}^{-1}\underline{a} + \underline{\underline{F}}^{-1}\dot{\underline{a}} \Rightarrow \dot{\underline{a}} = -\underline{\underline{F}}\,\dot{\underline{\underline{F}}}^{-1}\underline{a} = \underline{\underline{l}}\,\underline{a}$$

$$\left(\underline{\underline{l}} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1} = \dot{\overline{\underline{\underline{F}}\,\underline{\underline{F}}^{-1}}} - \underline{\underline{F}}\,\dot{\underline{\underline{F}}}^{-1} = -\underline{\underline{F}}\,\dot{\underline{\underline{F}}}^{-1} \right.$$

& IF A SPATIAL VECTOR FIELD IS CONTRAVARIANT

$$\overline{\underline{\underline{F}}^T \underline{a}} = 0 \Rightarrow \dot{\underline{\underline{F}}}^T \underline{a} + \underline{\underline{F}}^T \dot{\underline{a}} = 0 \Rightarrow \dot{\underline{a}} = -\underline{\underline{F}}^{-T} \dot{\underline{\underline{F}}}^T \underline{a} = -\underline{L}^T \underline{a}$$

COVARIANT & CONTRAVARIANT BASIS

LET $\underline{g}_i$ BE A SET OF LINEARLY INDEPENDENT (BASIS) SPATIAL VECTORS THAT CONVECT WITH THE BODY TANGENTIALLY

THEN WE KNOW $\exists$ $\underline{G}_i$ SUCH THAT

$$\underline{g}_i = \underline{\underline{F}} \underline{G}_i \quad \& \quad \dot{\underline{G}}_i = 0$$

SINCE $\underline{\underline{F}}$ IS INVERTIBLE $\underline{G}_i$ IS ALSO A SET OF BASIS

$$\dot{\underline{g}}_i = \underline{L} \underline{g}_i$$

LET $\underline{g}^j$ BE THE DUAL BASIS TO $\underline{g}_i$ SUCH THAT $\underline{g}^i \cdot \underline{g}_j = \delta^i_j$

IT FOLLOWS

$$\overline{\underline{g}_i \cdot \underline{g}^j} = 0 \Rightarrow \dot{\underline{g}}_i \cdot \underline{g}^j + \underline{g}_i \cdot \dot{\underline{g}}^j = 0 \Rightarrow \underline{L}\underline{g}_i \cdot \underline{g}^j + \underline{g}_i \cdot \dot{\underline{g}}^j = 0$$

$$\Rightarrow \underline{g}_i \cdot (\underline{L}^T \underline{g}^j + \dot{\underline{g}}^j) = 0 \Rightarrow \dot{\underline{g}}^j = -\underline{L}^T \underline{g}^j$$

THUS, IF $\underline{g}^j$ IS A DUAL BASIS TO A TANGENTIALLY CONVECTIVE (COVARIANT) BASIS $\underline{g}_i$, THEN $\underline{g}^j$ IS CONTRAVARIANT (CONVECTS NORMALLY) & $\exists$

$$\underline{G}^j = \underline{\underline{F}}^T \underline{g}^j \,, \quad \dot{\underline{G}}^j = 0$$

WE CAN WRITE ANY TENSOR $\underline{\underline{A}} = A_{ij}\, \underline{g}^i \otimes \underline{g}^j = A^{ij}\, \underline{g}_i \otimes \underline{g}_j$

WHERE A_{ij} ← COVARIANT COMPONENTS

A^{ij} ← CONTRAVARIANT COMPONENTS

COVARIANT & CONTRAVARIANT TENSORS

CONSIDE TWO COVARIANT VECTOR FIELDS $\underline{a} = \underline{\underline{F}}\underline{A}$, $\underline{b} = \underline{\underline{F}}\underline{B}$
LET $\underline{\underline{G}}$ BE A SPATIAL TENSOR

$$\underline{a} \cdot \underline{\underline{G}}\underline{b} = \underline{\underline{F}}\underline{A} \cdot \underline{\underline{G}}\underline{\underline{F}}\underline{B} = \underline{A} \cdot \underline{\underline{F}}^T \underline{\underline{G}}\underline{\underline{F}}\underline{B}$$

IF

$\overline{\underline{\underline{F}}^T \underline{\underline{G}}\underline{\underline{F}}}^{\circ} = 0$ THEN $\underline{\underline{G}}$ IS TERMED A COVARIANT TENSOI

AND

$\underline{\underline{F}}^T \underline{\underline{G}}\underline{\underline{F}}$ IS THE COVARIANT PULL BACK OF $\underline{\underline{G}}$

NOTE THE ABOVE IS EQUIVALENT TO $\overline{G_{ij}}^{\circ} = 0$

EQUIVALENTLY LET $\underline{a} = \underline{\underline{F}}^{-T}\underline{A}$, $\underline{b} = \underline{\underline{F}}^{-T}\underline{B}$ &

$$\underline{a} \cdot \underline{\underline{G}}\underline{b} = \underline{A} \cdot \underline{\underline{F}}^{-1} \underline{\underline{G}}\underline{\underline{F}}^{-T}\underline{B}$$

IF

$\overline{\underline{\underline{F}}^{-1} \underline{\underline{G}}\underline{\underline{F}}^{-T}}^{\circ} = 0$ THEN $\underline{\underline{G}}$ IS TERMED A CONTRAVARIANT TENSOR

AND

$\underline{\underline{F}}^{-1} \underline{\underline{G}}\underline{\underline{F}}^{-1}$ IS THE CONTRAVARIANT PI BACK OF $\underline{\underline{G}}$

NOTE THE ABOVE IS EQUIVALENT TO $\overline{G^{ij}}^{\circ} = 0$

NOTE THAT THE COVARIANT PULL BACK (AND PUSH FOWARD) ARE OFTEN DENOTED BY $\underline{\varphi}^*(\cdot)^\flat$ ($\underline{\varphi}_*(\ )^\flat$)

NOTE THAT THE CONTRAVARIANT PULL BACK (AND PUSH FOWARD) ARE OFTEN DENOTED BY $\underline{\varphi}^*(\cdot)^\#$ ($\underline{\varphi}_*(\ )^\#$)

COROTATIONAL VECTOR FIELD

IF A SPATIAL VECTOR FIELD SPINS WITH THE BODY IS CALLED COROTATIONAL NAMELY

$$\dot{\underline{a}} = \underline{\underline{W}}\,\underline{a}$$

IF $\underline{a}$ & $\underline{b}$ ARE COROTATIONAL VECTOR FIELDS

$$\overline{\underline{a}\cdot\underline{b}}^{\circ} = \dot{\underline{a}}\cdot\underline{b} + \underline{a}\cdot\dot{\underline{b}} = \underline{\underline{W}}\,\underline{a}\cdot\underline{b} + \underline{a}\cdot\underline{\underline{W}}\,\underline{b} = \underline{b}\cdot\underline{\underline{W}}\,\underline{a} + \underline{b}\cdot\underline{\underline{W}}^T\underline{a} = \underline{b}\cdot\underline{\underline{W}}\,\underline{a} - \underline{b}\cdot\underline{\underline{W}}\,\underline{a} = 0$$

HENCE : THE ANGLE BETWEEN THE VECTORS REMAIN UNCHANGED

NOW LET $\underline{e}_i$ BE A SET OF COROTATIONAL ORTHONORMAL BASIS &

$$G_{ij} = \underline{e}_i\cdot\underline{\underline{G}}\,\underline{e}_j$$

THEN

$$\dot{\overline{G_{ij}}} = \dot{\overline{\underline{e}_i\cdot\underline{\underline{G}}\,\underline{e}_j}} = \underline{e}_i\cdot\dot{\underline{\underline{G}}}\,\underline{e}_j + \underline{e}_i\cdot\underline{\underline{W}}^T\underline{\underline{G}}\,\underline{e}_j + \underline{e}_i\cdot\underline{\underline{G}}\,\underline{\underline{W}}\,\underline{e}_j =$$

$$= \underline{e}_i\cdot\left(\dot{\underline{\underline{G}}} - \underline{\underline{W}}\,\underline{\underline{G}} + \underline{\underline{G}}\,\underline{\underline{W}}\right)\cdot\underline{e}_j$$

IF $\dot{\overline{G_{ij}}} = 0$ OR $\dot{\underline{\underline{G}}} - \underline{\underline{W}}\,\underline{\underline{G}} + \underline{\underline{G}}\,\underline{\underline{W}}$ THEN $\underline{\underline{G}}$ IS A COROTATIONAL TENSOR

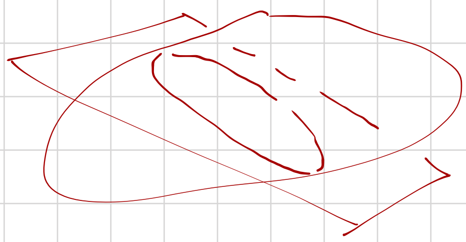
EFFECTIVELY THE TENSOR ROTATES WITH THE BODY WHICH IS IMPORTANT NOTION FOR INVARIANCE

STRESS

SHORT RANGE FORCES $\rightarrow$ ATOMIC INTERACTIONS

LONG RANGE FORCES $\rightarrow$ GRAVITATIONAL FORCES ETC.

SHORT RANGE FORCES ARE TRANSMITTED THROUGH THE BODY



THE TRACTION AT A POINT P IN OUR CONTINUUM

$$\underline{t}_S(P) = \lim_{r \to 0} \frac{\Delta \underline{F}}{|S \cap B_r(P)|} = \lim_{r \to 0} \frac{\Delta \underline{F}}{\Delta S}$$

EFFECTIVELY A TRACTION IS AN AVERAGE OF FORCES ACTING ON AN INFINITESIMA AREA ELEMENT

$$d\underline{f} = \underline{t}\, ds = \underline{T}\, dS$$

FOLLOWIN CAUCHY'S PRINCIPLE $\underline{t} = \underline{t}(\underline{x}, \underline{n})$

IE. THE VALUE OF THE TRACTION VECTOR IS INHERENTLY LOCAL & DEPEND ON POSITION & THE SURFACE NORMAL

LASTLY NOTE THAT ON OPPOSITE FACES OF THE CUT

$$\Delta \underline{F}(\underline{n}) = -\Delta \underline{F}(-\underline{n}) \implies \underline{t}(\underline{x}, \underline{n}) = -\underline{t}(\underline{x}, -\underline{n})$$

(NOTE WE OMIT DEPENDENCE ON TIME FOR SIMPLICITY

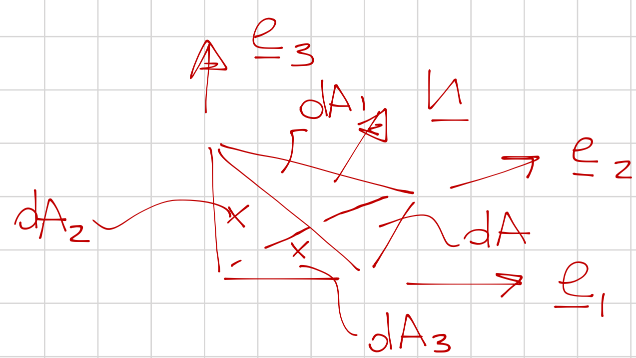
CAUCHY'S TETRAHEDRON THEOREM

$\underline{t}(\underline{x},\underline{n})$ IS LINEAR IN $\underline{n}$

BY DEFINITION $\underline{t}_{\underline{x}}(\ )$ IS A TENSOR

THEREFORE CAUCHY'S TETRAHEDRON THEOREM STATES THAT $\exists\ \underline{\underline{\sigma}} \in \mathbb{R}^{d\times d}$ SUCH THAT

$$\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$$



$$\underline{t}(-\underline{e}_3)\,dA_3 + \underline{t}(-\underline{e}_1)\,dA_1 + \underline{t}(-\underline{e}_2)\,dA_2 + \underline{t}(\underline{n})\,dA = 0$$

$$dA_i = \underline{n}\cdot\underline{e}_i\,dA$$

$$\underline{t}(-\underline{e}_3)\,\underline{n}\cdot\underline{e}_3 + \underline{t}(-\underline{e}_2)\,\underline{n}\cdot\underline{e}_2 + \underline{t}(-\underline{e}_1)\,\underline{e}_1\cdot\underline{n} + \underline{t}(\underline{n}) = 0$$

WITH $\underline{t}(-\underline{n}) = -\underline{t}(\underline{n})$

$$\underline{t}(\underline{n}) = \underline{t}(\underline{e}_1)\overbrace{\underline{e}_1\cdot\underline{n}}^{n_1} + \underline{t}(\underline{e}_2)\overbrace{\underline{e}_2\cdot\underline{n}}^{n_2} + \underline{t}(\underline{e}_3)\overbrace{\underline{e}_3\cdot\underline{n}}^{n_3} = \underline{t}(n_1\underline{e}_1 + n_2\underline{e}_2 + n_3\underline{e}_3)$$

$\underline{t}$ IS LINEAR IN $\underline{n}$!

SYMMETRY OF CAUCHY'S STRESS TENSOR