LECTURE 6:

- DEFORMATION GRADIENT
- RIGHT CAUCHY GREEN TENSOR
- METRIC CHANGES

WE DEFINED $\underline{\underline{F}}(\underline{X}) = \underline{\nabla}_{\underline{X}} \underline{\varphi}(\underline{X})$ THE DEFORMATION GRADIENT THAT MAPS ELEMENTS OF THE TANGENT SPACE AT $\underline{X}$ OF THE REFERENCE BODY (MANIFOLD) ELEMENTS OF THE TANGEN SPACE AT $\underline{\varphi}(\underline{X})$ OF THE DEFORMED BODY (MANIFOLD)

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OFTEN IT IS MORE COMMON TO LOOK AT THE DISPLACEMENT GRADIENT

$$\underline{\nabla}_{\underline{X}} \underline{U}(\underline{X}) = \underline{\nabla}_{\underline{X}} \left(\underline{\varphi}(\underline{X}) - \underline{X} \right) = \underline{\underline{F}}(\underline{X}) - \underline{\underline{I}}$$

OR

$$\underline{\nabla}_{\underline{x}} \underline{u}(\underline{x}) = \underline{\nabla}_{\underline{x}} \left(\underline{x} - \underline{\varphi}^{-1}(\underline{x}) \right) = \underline{\underline{I}} - \underline{\underline{F}}^{-1}(\underline{x})$$

SOME USEFUL RELATIONS FOLLOW DIRECTLY FROM SIMPLE CHAIN RULES

$$\underline{\nabla}_{\underline{x}} \psi = \underline{\underline{F}}^{-T} \underline{\nabla}_{\underline{X}} \psi$$

$$\underline{\nabla}_{\underline{x}} \underline{a} = \underline{\nabla}_{\underline{X}} \underline{a} \underline{\underline{F}}^{-1}$$

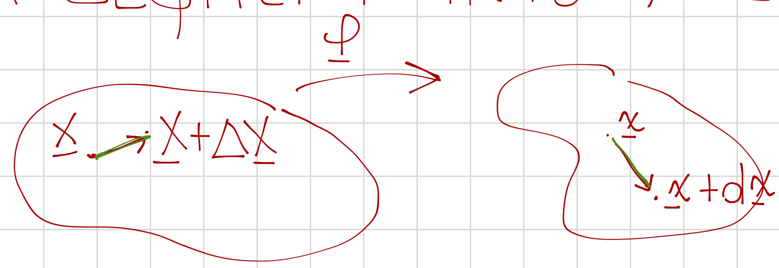
$$\underline{\nabla}_{\underline{x}} \underline{\underline{A}} = \underline{\nabla}_{\underline{X}} \underline{\underline{A}} \underline{\underline{F}}^{-1}$$

$$\underline{\nabla}_{\underline{x}} \cdot \underline{\underline{A}} = \underline{\nabla}_{\underline{X}} \underline{\underline{A}} : \underline{\underline{F}}^{-T}$$

EFFECTIVELY SINCE WE SAW THAT AS $\Delta \underline{X} \searrow 0$

$$\Delta \underline{x} = \underline{\varphi}(\underline{X}+\Delta\underline{X}) - \underline{\varphi}(\underline{X}) = \underline{\underline{F}}(\underline{X})\,\Delta\underline{X}$$

THE MAP $\underline{\varphi}$ BEHAVE LOCALLY AS AN AFFINE MAP & WE KNOW AFFINE MAPS MAP A SEGMENT INTO A SEGMENT



EG: $\underline{\varphi}(\underline{X}) = (X_1 + \alpha X_2^2)\,\underline{E}_1 + X_2\,\underline{E}_2$

$$[\nabla_{\underline{X}}\underline{\varphi}] = \begin{bmatrix} 1 & 2X_2 \\ 0 & 1 \end{bmatrix} \leftarrow \underline{\underline{F}} \text{ CAN DEPEND ON SPACE}$$

CONSIDER $\alpha = 1$

$$\underline{X}_A = 0.5(\underline{E}_1+\underline{E}_2) \Rightarrow \underline{x}_A = 0.75\,\underline{E}_1 + 0.5\,\underline{E}_2$$

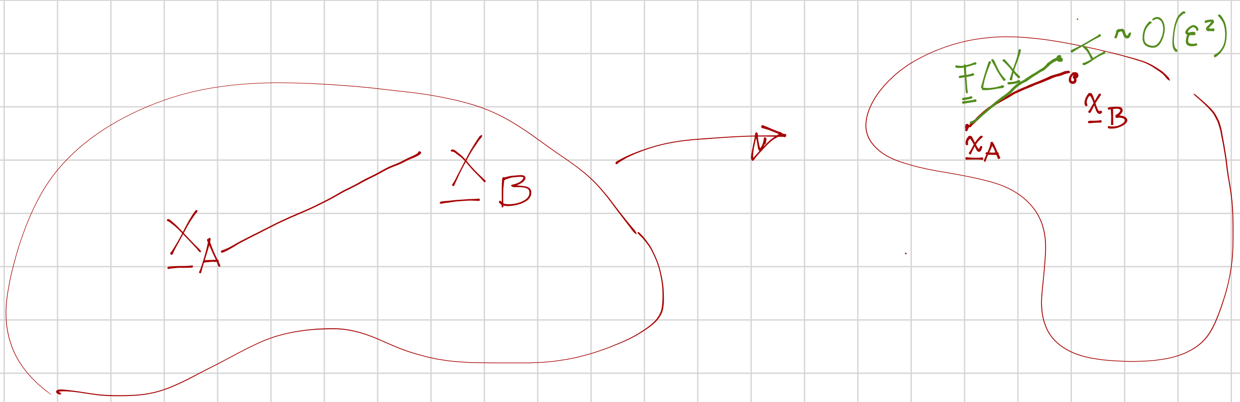
$$\underline{X}_B = (0.5+\varepsilon)(\underline{E}_1+\underline{E}_2) \qquad \underline{x}_B = (0.75 + 1.5\varepsilon + \varepsilon^2)\,\underline{E}_1 + (0.5+\varepsilon)\,\underline{E}_2$$

$$\Delta\underline{X} = \varepsilon(\underline{E}_1+\underline{E}_2) \qquad \Delta\underline{x} = (1.5\varepsilon+\varepsilon^2)\,\underline{E}_1 + \varepsilon\,\underline{E}_2$$

$$\Delta\underline{x} \approx \underline{\underline{F}}(\underline{X}_A)\,\Delta\underline{X} = \underline{X}_A + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \varepsilon \\ \varepsilon \end{Bmatrix} = \underline{X}_A + 2\varepsilon\,\underline{E}_1 + \varepsilon\,\underline{E}_2$$

$$\underline{x}_B \approx (0.75+2\varepsilon)\,\underline{E}_1 + (0.5+\varepsilon)\,\underline{E}_1 \overset{\varepsilon = 0.1}{=} 0.95\,\underline{E}_1 + 0.6\,\underline{E}_1$$

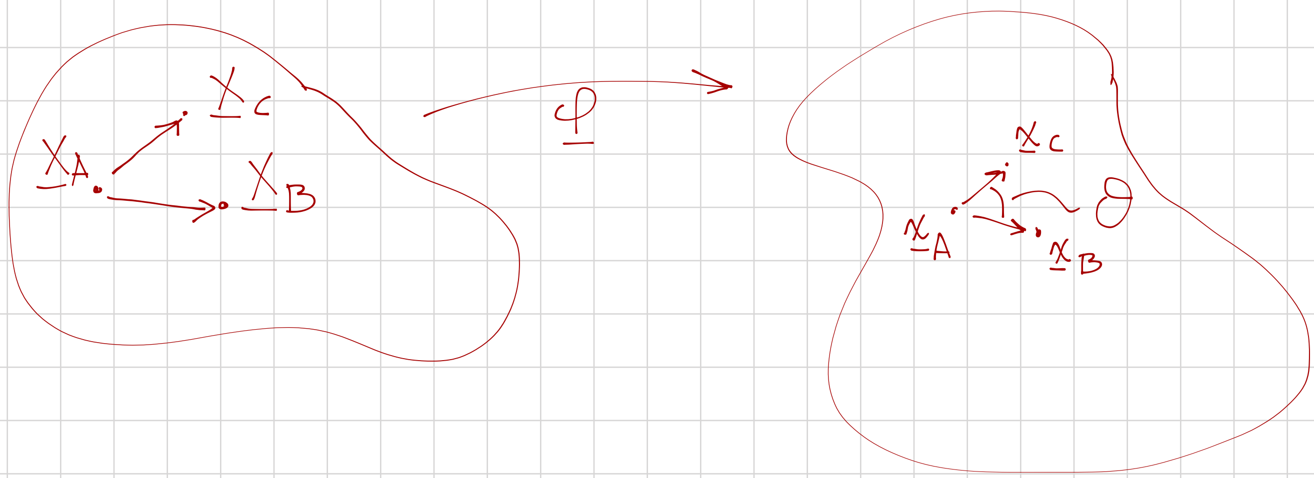
$$\text{VS } 0.801\,\underline{E}_1 + 0.6\,\underline{E}_1$$



RIGHT CAUCHY GREEN TENSOR

NOW THINK ABOUT THREE POINTS

$\underline{X}_A, \underline{X}_B, \underline{X}_C$ & $\underline{x}_i = \underline{\varphi}(\underline{X}_i)$, $i = A, B, C$



IF OUR MAP IS AFFINE

$$\underline{\varphi}(\underline{X}) = \underline{a} + \underline{\underline{F}}\,\underline{X} \qquad (\nabla \underline{\underline{F}} = 0)$$

THEN WE KNOW WE MAP SEGMENTS INTO SEG WHICH MEANS

① THE LENGHT OF $\underline{X}_B - \underline{X}_A$ IN THE DEFORMED CONFIG IS

$\|\underline{x}_B - \underline{x}_A\|$ & THE CHANGE IN LENGH

$$\lambda = \frac{\|\underline{x}_B - \underline{x}_A\|}{\|\underline{X}_B - \underline{X}_A\|} = \frac{\|\Delta \underline{x}\|}{\|\Delta \underline{X}\|}$$

② THE ANGLE BETWEEN $\underline{X}_B - \underline{X}_A$ & $\underline{X}_C - \underline{X}_A$ IN THE DEFORMED CONFIG IS GIVEN BY

$$\theta = \cos^{-1}\left[\frac{(\underline{x}_B - \underline{x}_A)\cdot(\underline{x}_C - \underline{x}_A)}{\|\underline{x}_B - \underline{x}_A\|\,\|\underline{x}_C - \underline{x}_A\|}\right]$$

WHAT IF THE MAPPING WAS NO LONGER AFFINE BUT WE ARE INTERESTED IN INFINITESIMAL VECTORS

$d\underline{x} \cdot d\underline{y}$ ⟵ INFINITESIMAL VECTOR SEGMENTS EMANATING FROM $\underline{x}_A$

$$d\underline{x} = \underline{\underline{F}}(\underline{X}_A)\, d\underline{X} \;, \quad d\underline{y} = \underline{\underline{F}}(\underline{X}_A)\, d\underline{Y}$$

$$d\underline{x} \cdot d\underline{y} = \underline{\underline{F}}\, d\underline{X} \cdot \underline{\underline{F}}\, d\underline{Y} = d\underline{X} \cdot \underline{\underline{F}}^T \underline{\underline{F}}\, d\underline{Y} = d\underline{X} \cdot \underline{\underline{C}}\, d\underline{Y}$$

$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$ IS THE RIGHT CAUCHY GREEN TENSOR

$C_{IJ} = F_{pI} F_{pJ}$ ⟵ NOTE BOTH INDECES IN REF

$\underline{\underline{C}}$ IS SYMMETRIC & POSITIVE DEFINITE

$$\mathrm{DET}(\underline{\underline{C}}) = \mathrm{DET}(\underline{\underline{F}})\, \mathrm{DET}(\underline{\underline{F}}) > 0$$

$\underline{\underline{C}}$ IS A MAP FROM $T_{\underline{X}}\Omega_0$ TO $T^*_{\underline{X}}\Omega_0$

NAMELY IT TAKES A VECTOR & RETURNS A 1-FORM

CHANGES IN LENGHT

WE ARE INTERESTED IN CHANGES IN LENGHT OF INFINITESIMAL VECTORS

$$d\underline{x} \,,\; d\underline{X} \,, \quad d\underline{x} = \underline{\varphi}(\underline{X} + d\underline{X}) - \underline{\varphi}(\underline{X})$$

$$\lambda = \frac{\|d\underline{x}\|}{\|d\underline{X}\|} = \frac{\sqrt{d\underline{X} \cdot \underline{\underline{C}}\, d\underline{X}}}{\|d\underline{X}\|} = \sqrt{\frac{d\underline{X} \cdot C\, d\underline{X}}{\|d\underline{X}\|\, \|d\underline{X}\|}} = \sqrt{\underline{N} \cdot \underline{\underline{C}}\, \underline{N}}$$

NOW, LET $\underline{N}$ DENOTE AN ARBITRARY DIRECTION

EG. $\underline{N} = \dfrac{d\underline{X}}{\|d\underline{X}\|}$ ⟵ UNIT MAGNITUDE

THEN WE DEFINE THE STRETCH RATIO AT $\underline{X}$ IN THE DIRECTION $\underline{N}$ AS

$$\lambda(\underline{X}, \underline{N}) = \sqrt{\underline{N} \cdot \underline{\underline{C}}(\underline{X})\, \underline{N}}$$

EG.

$$[\underline{\underline{F}}] = \begin{bmatrix} 1 & 2\alpha X_2 \\ 0 & 1 \end{bmatrix} \Rightarrow [\underline{\underline{C}}] = \begin{bmatrix} 1 & 0 \\ \alpha X_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha X_2 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 2\alpha X_2 \\ 2\alpha X_2 & 1+4\alpha X_2^2 \end{bmatrix}$$

THE STRETCH A $\underline{X}_p = 0.5(\underline{E}_1 + \underline{E}_2)$ IN $\underline{E}_1$ & $\underline{E}_2$ & $\underline{E}_1 + \underline{E}_2$

$$\lambda(\underline{E}_1, \underline{X}_p) = \sqrt{\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}^T \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}} = 1 \quad \leftarrow \text{NO STRETCH} \checkmark$$

$$\lambda(\underline{E}_2, \underline{X}_p) = \sqrt{\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}^T \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}} = 1+\alpha$$

$$\lambda\left(\frac{\underline{E}_1 + \underline{E}_2}{\sqrt{2}}, \underline{X}_p\right) = \frac{1}{\sqrt{2}} \sqrt{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^T \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}} = \sqrt{\frac{2+3\alpha}{2}}$$

$-\underline{N}$

CHANGES IN ANGLES

SIMILARLY AS BEFORE WE CAN SAY WE ARE INTERESTED IN CHANGES IN ANGLE BETWEEN INFINITESIMAL VECTORS

LET $d\underline{X}_1, d\underline{X}_2$ BE INFINITESIMAL FIBERS

$$d\underline{x}_1 = \underline{\underline{F}}(\underline{X}_p) d\underline{X}_1 \quad d\underline{x}_2 = \underline{\underline{F}}(\underline{X}_p) d\underline{X}_2$$

$$\cos\theta = \frac{d\underline{x}_1 \cdot d\underline{x}_2}{\|d\underline{x}_1\| \cdot \|d\underline{x}_2\|} = \frac{d\underline{X}_1 \underline{\underline{C}} d\underline{X}_2}{\lambda(\underline{N}_1, \underline{X}_p)\lambda(\underline{N}_2, \underline{X}_p)\|d\underline{X}_1\|\|d\underline{X}_2\|}$$

$$= \frac{\underline{N}_1 \cdot \underline{\underline{C}}\, \underline{N}_2}{\lambda(\underline{N}_1, \underline{X}_P)\, \lambda(\underline{N}_2, \underline{X}_P)}$$

WHERE

$$\underline{N}_1 = \frac{d\underline{X}_1}{\|d\underline{X}_1\|} \quad , \quad \underline{N}_2 = \frac{d\underline{X}_2}{\|d\underline{X}\|_2}$$

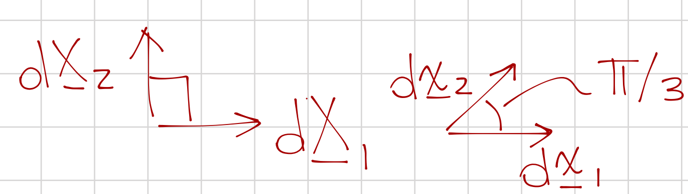
EXAMPLE:

$$\underline{N}_1 = \underline{E}_1 \;,\; \underline{N}_2 = \underline{E}_2 \;,\; \underline{X}_P = 0.5\,(\underline{E}_1 + \underline{E}_2)$$

$$\cos\theta = \frac{\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}^T \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}}{1+\alpha} = \frac{\alpha}{1+\alpha}$$

if $\alpha = 1$

$$\theta = \frac{1}{3}\pi = 60^\circ$$



WHY IS THIS IMPORTANT?

EFFECTIVELY WITHOUT EXPLICIT KNOWLEDGE OF SPATIAL QUANTITIES WE CAN COMPUTE CHANGES IN METRIC

INTERPRETING THE COMPONENTS

$$[\underline{\underline{C}}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

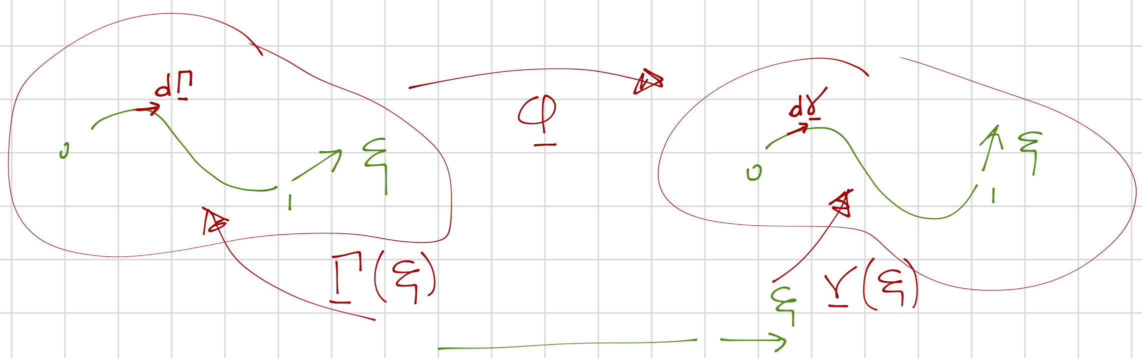
$$\lambda(\underline{E}_I, \underline{X}) = \sqrt{C_{(I)(I)}}$$

$$\cos\theta_{IJ} = \frac{C_{(I)(J)}}{\sqrt{C_{(I)(I)}}\sqrt{C_{(J)(J)}}}$$

DEFORMED CURVES

LET $\underline{\Gamma}(\xi) : [0,1] \to \Omega_0$, $\quad \underline{\gamma}(\xi) = \underline{\varphi}(\underline{\Gamma}(\xi))$

WE ARE INTERESTED IN THE LENGH OF $\underline{\gamma}(\xi)$



$$\ell(\underline{\gamma}) = \int \|d\underline{\gamma}\| = \int_0^1 \|\underline{\gamma}'(\xi)\| \, d\xi$$

$$\underline{\gamma}'(\xi) = \underline{\underline{F}}(\underline{\Gamma}(\xi))\,\underline{\Gamma}'(\xi)$$

$$\|\underline{\gamma}'(\xi)\|^2 = \underline{\Gamma}'(\xi) \cdot \underline{\underline{C}}(\underline{\Gamma}(\xi))\,\underline{\Gamma}'(\xi)$$

$$\Rightarrow \quad \ell(\underline{\gamma}) = \int_0^1 \left[\underline{\Gamma}'(\xi) \cdot \underline{\underline{C}}(\underline{\Gamma}(\xi))\,\underline{\Gamma}'(\xi)\right]^{1/2} d\xi$$

EXAMPLE

CONSIDER $\quad \underline{\Gamma}(\xi) = \xi\,(\underline{E}_1 + \underline{E}_2)$

$$[\underline{\underline{C}}] = \begin{bmatrix} 1 & 2\alpha X_2 \\ 2\alpha X_2 & 1+4\alpha X_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 2\sqrt{2}\alpha\xi \\ 2\sqrt{2}\alpha\xi & 1+8\xi^2\alpha \end{bmatrix}$$

$$\underline{\Gamma}'(\xi) = (\underline{E}_1 + \underline{E}_2)$$

$$\ell(\underline{\gamma}) = \int_0^1 \left[2 + 4\sqrt{2}\alpha\xi + 8\xi^2\alpha\right] d\xi$$

$$= \left[2\xi + 2\sqrt{2}\alpha\xi^2 + 8/3\,\xi^3\alpha\right]^{1/2}\Big|_0^1 = \left[2 + \alpha\,(2\sqrt{2} + 8/3)\right]^{1/2}$$

NOTE if $\alpha = 0$ $\sqrt{2}$ ✓

DEFORMED VOLUMES

RECALL THAT

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = a_i b_j c_k \, \varepsilon_{ijk}$$

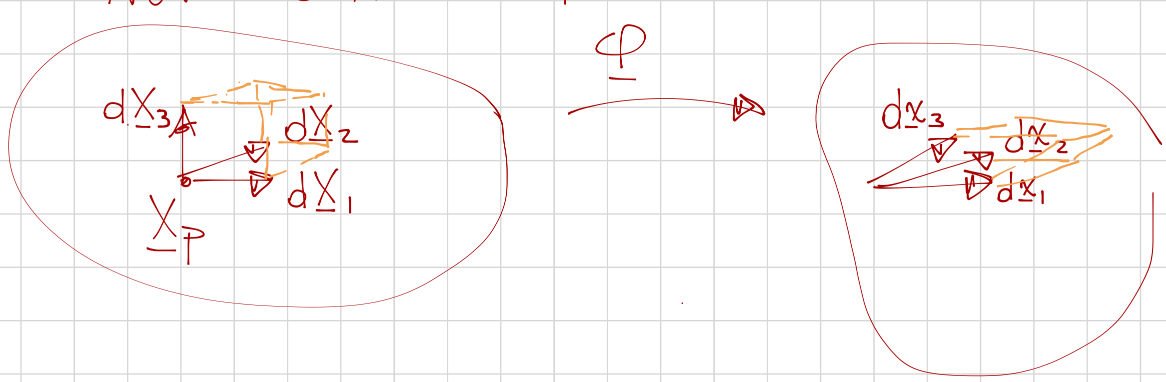
& THAT

$$\text{DET } \underline{\underline{A}} = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$$

OR EQUIVALENTLY

$$\varepsilon_{PQR} \text{ DET } \underline{\underline{A}} = \varepsilon_{ijk} A_{iP} A_{jQ} A_{kR}$$

NOW CONSIDER



$$dV_0 = |d\underline{X}_3 \cdot (d\underline{X}_1 \times d\underline{X}_2)| = |\varepsilon_{IJK} \, dX_{3I} \, dX_{1J} \, dX_{2K}|$$

$$dV = |d\underline{x}_3 \cdot (d\underline{x}_1 \times d\underline{x}_2)| = |\varepsilon_{ijk} \, dx_{3i} \, dx_{1j} \, dx_{2k}|$$

NOW WITH $\quad d\underline{x}_i = \underline{\underline{F}}(\underline{X}_P) d\underline{X}_i$

$$dV = |\underline{\underline{F}} d\underline{X}_3 \cdot (\underline{\underline{F}} d\underline{X}_1 \times \underline{\underline{F}} d\underline{X}_2)| =$$

$$= |\varepsilon_{ijk} \, F_{iP} \, dX_{3P} \, F_{jQ} \, dX_{1Q} \, F_{kR} \, dX_{2R}| =$$

$$= |\underbrace{\varepsilon_{ijk} F_{iP} F_{jQ} F_{kR}}_{\varepsilon_{PQR} \text{ DET } \underline{\underline{F}}} \, dX_{3P} \, dX_{1Q} \, dX_{2R}| =$$

$$= |\text{DET } \underline{\underline{F}} \; \varepsilon_{PQR} \, dX_{3P} \, dX_{1Q} \, dX_{2R}| =$$

$$= \underbrace{\text{DET } \underline{\underline{F}}}_{>0} \, dV_0$$

DET $\underline{\underline{F}}$ IS OFTEN DENOTED AS J & TERMED THE JACOBIAN.

<u>EXAMPLE</u>: $\underline{\varphi}(\underline{X}) = \alpha \underline{X}$, $\Omega_0 = [-1,1]^3$

$$\underline{\underline{F}} = \alpha \underline{\underline{1}} \Rightarrow \det(\alpha \underline{\underline{1}}) = \alpha^3$$

$$V = \int_{\Omega} dV = \int_{\Omega_0} J \, dV_0 = \alpha^3 8$$