LECTURE 13:

TOPICS:

- OVERVIEW FINITE ELEMENTS
- STRONG & WEAK FORM
- GALERKIN APPROXIMATION
- MATRIX FORM

LOGISTICS:

- HW 4 DUE FRIDAY
- HW 5 OUT FRIDAY & DUE A WEEK AFTER

OVERVIEW OF FEM

STRONG FORM (S) ← TRADITIONAL DIFFERENTIAL PROBLEM

⇕ ← EQUIVALENT

WEAK FORM (W) ← DIFFERENTIAL PROBLEM IN INTEGRAL FORM

⇓ ← NOT EQUIVALENT

GALERKIN APPROXIMATION (G) ← APPROXIMATION OF WEAK FORM

⇕

MATRIX FORM (M) ← ALGEBRAIC SYSTEM OF EQ FOR THE GALERKIN APPROX

STRONG FORM (S)

LET'S START WITH THE CANONICAL PROBLEM

FIND $u : [0, \ell] \rightarrow \mathbb{R}$ ST

$$AE \frac{d^2 u}{dx^2} = f \qquad \forall x \in (0, \ell)$$

W/

$$u(0) = g \quad \leftarrow \text{DIRICHLET}$$

$$AE \frac{du}{dx}(\ell) = h \quad \leftarrow \text{NEUMANN}$$

NOTE: HERE THE SPACE OF ALL POSSIBLE SOLUTIONS IS GIVEN BY

u HAS TWO DERIVATIVES

$$u \in S = \{ u \in C^2([0, \ell]) \mid u(0) = g \}$$

IS $u(x) = \dfrac{1}{x - \ell/2}$ A POSSIBLE SOLUTION? NO!

OFTEN THIS IS THE STARTING POINT FOR NUM METH.

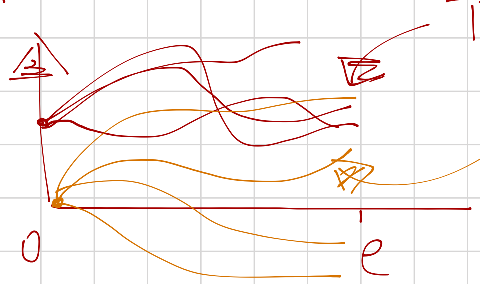
WEAK FORM (W)

TO DEFINE THE WEAK FORM

$S \leftarrow$ TRIAL FUNCTIONS: THE SET OF ALL POSSIBLE FUNCTIONS THAT ARE SMOOTH ENOUGH & RESPECT THE ESSENTIAL BOUNDARY CONDITIONS

$V \leftarrow$ TEST FUNCTIONS: THIS IS THE SET OF ALL FUNCTIONS THAT IF ADDED TO THE TRIAL FUNCTIONS YIELD A TRIAL FUNCTION

EG



FOR OUR PROBLEM

$$S = \{ u \in SMOOTH([0,\ell]) \mid u(0) = g \}$$
$$V = \{ v \in SMOOTH([0,\ell]) \mid v(0) = 0 \}$$

NOTE FOR $u \in S$, $v \in V$

$$w = u + v \in S$$

IN FACT SINCE

- $u \in SMOOTH, v \in SMOOTH \implies w \in SMOOTH$
- $w(0) = u(0) + v(0) = g + 0 = g \implies w(0) = g$

NOW DEFINE THE RESIDUAL

$$R(x) = AE \frac{d^2u}{dx^2} - f$$

IN THEORY $R(x) = 0$ FOLLOWING THE STRONG FORM.

THEREFORE

$$\int_0^\ell R(x)\, v(x)\, dx = 0 \qquad \forall v \in V$$

NOW EXPAND

$$\int_0^\ell R(x)\, v(x)\, dx = \int_0^\ell \left[AE \frac{d^2u}{dx^2} - f \right] v\, dx =$$

$$\int_0^\ell \left[AE \frac{d^2u}{dx^2} v - fv \right] dx = \left[AE \frac{du}{dx} v \right]\Big|_0^\ell - \int_0^\ell AE \frac{du}{dx} \frac{dv}{dx}\, dx - \int_0^\ell fv\, dx$$

USING THE NEUMANN BC & $v(0)=0$

$$-[h v(e)] - \int_0^e AE \frac{du}{dx}\frac{dv}{dx}\,dx - \int_0^e f v\,dx = 0 \qquad \forall v \in V$$

TAKING THE SECOND TERM TO THE RHS

$$\int_0^e AE \frac{du}{dx}\frac{dv}{dx}\,dx = h v(e) - \int_0^e f v\,dx \qquad \forall v \in V$$

OR: FIND $u \in S$

$$a(u,v) = F(v) \qquad \forall v \in V$$

WHERE

$$a(u,v) = \int_0^e AE \frac{du}{dx}\frac{dv}{dx}\,dx$$

$$F(v) = h v(e) - \int_0^e f v\,dx$$

SOME COMMENTS ON SMOOTHNESS

NOTE THAT $F(v)$ DOES NOT DEPEND ON u.

TO HAVE A WELL POSED PROBLEM & GUARANTEE THE EXISTANCE OF u WE MUST HAVE

$$a(u,u) \geq 0, \quad a(u,u) = 0 \Rightarrow u = 0 \qquad \text{(COERCIVITY)}$$

$$a(u,u) < \infty \quad \left(\text{OR } a(u,u) < \alpha \int_0^e u^2 dx\right) \qquad \text{(CONTINUITY)}$$

THE SECOND CONDITION REQUIRES

$$\int_0^e \left(\frac{du}{dx}\right)^2 < \infty$$

THE SPACE OF FUNCTIONS THAT SATISFY THE ABOVE ARE DENOTED BY

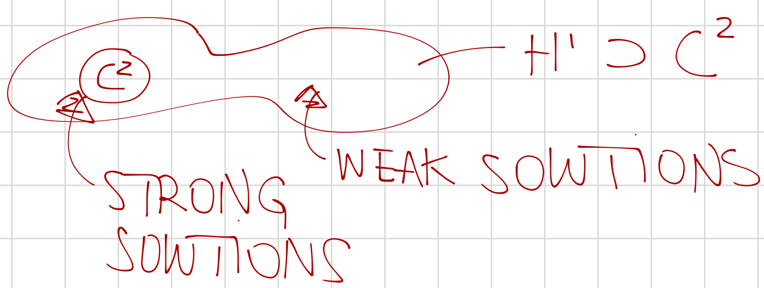
$$H^1([0,e]) = \left\{u \mid \int_0^e u^2 + \left(\frac{du}{dx}\right)^2 dx < \infty\right\}$$

THEREFORE

$$S = \{ u \in H^1([0,\ell]) \mid u(0) = g \}$$
$$V = \{ v \in H^1([0,\ell]) \mid v(0) = 0 \}$$

NOTE THAT FOR THE STRONG FORM THE SET OF ADMISSIBLE SOL $u \in C^2([0,\ell])$

MORE RESTRICTING THAN H^1



NOTE: THE WEAK FORM & STRONG FORM ARE EQUIVALENT

$$?u : AE\frac{d^2u}{dx^2} = f \; + BC \iff ?u \in S : a(u,v) = F(v) \; \forall v \in V$$

TO SHOW $(S) \Rightarrow (W)$ SEE ABOVE

TO SHOW $(W) \Rightarrow (S)$ RECALL

$$\int_0^\ell R(x)\, w(x)\, dx = 0 \xrightarrow[\text{OF CALC OF VAR}]{\text{FUNDAMENTAL LEMMA}} R(x) = 0$$

THUS, WHEN WE DEVISE A NUMERICAL METHOD WE CAN START WITH THE STRONG OR WEAK FORM EQUIVALENTLY

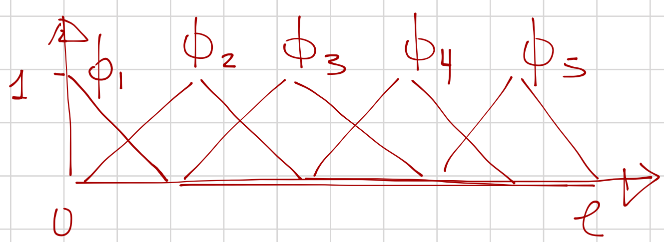
GALERKIN APPROXIMATION (G)

WHAT THE GALERKING APPROXIMATION STATES IS THAT INSTEAD OF LOOK FOR A SOLUTION TO THE WEAK FORM AMONGST ALL POSSIBLE SOLUTIONS S WE DEFINE A SUBSET OF CANDIDATE SOLUTIONS S^N SUCH THAT

$S^N \subseteq S$ (CONFORMING FINITE ELEMENTS)

S^N IS A FINITE DIMENSIONAL SET OF CANDIDATE FUNCTIONS

EG:



BASE

COEFFICIENT

$$S^5 = \{ u \mid u = \sum_{i=2}^{5} \phi_i u_i + g \phi_1 \}$$

TO ENFORCE DIRICHLET BC

NOTE THAT OFTEN INSTEAD OF USING N FOR PARAMET. $h \sim \frac{1}{N}$ IS USED

THUS INSTEAD OF LOOKIN FOR A SOLUTION IN S WE LOOK FOR THE BEST SOLUTION IN S^h

NAMELY: FIND $u^h \in S^h$

$$a(u^h, v^h) = F(v^h) \qquad \forall\ v^h \in V^h$$

WHERE V^h IS DEFINED AS BEFORE, NAMELY FOR ALL $u^h \in S^h$ & $v^h \in V^h$

$w^h = u^h + v^h \in S^h$

NOTE: THE ABOVE IS AN APPROXIMATION INFACT THE SOLUTION OF (W) MAY NOT LAY IN S^h BUT AS $h \searrow 0$ OR $N \nearrow \infty$ $u^h \to u$

ALSO NOTE ONE THING

LET $u^h \in S^h$ BE THE SOLUTION TO (G) AND $u \in S$ THE SOLUTION OF (W) AND $v^h \in V^h$.

FOR SIMPLICITY LET $g^h(x)$ SUCH THAT $g^h(0)=g$ AND

$$u^h = z^h + g^h \quad , \quad u = z + g^h$$

WHERE $z^h \in V^h$, $z \in V$

$$a(u^h, v^h) = F(v^h) \qquad (*)$$

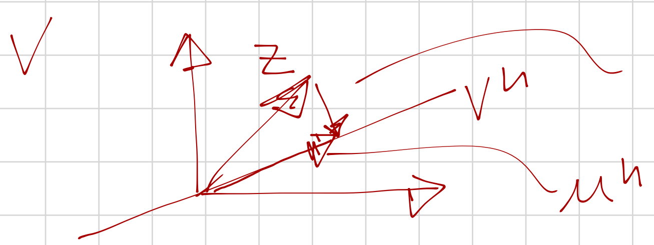
$$a(u, v^h) = F(v^h) \qquad (**)$$

NOW SUBTRACT $(**)$ FROM $(*)$

$$a(u^h - u, v^h) = a(z^h - z, v^h) = 0 \quad \Leftarrow \text{GALERKIN ORTHOGONALITY}$$

YOU MAY THINK OF $a(\cdot,\cdot)$ AS THE DOT (OR INNER) PRODUCT BETWEEN TWO FUNCTIONS.

WHEN THE DOT PRODUCT IS ZERO THE TWO FUNCTIONS ARE PERPENDICULAR



THEREFORE THE ERROR e IS PERPENDICULAR TO THE SPACE OF POSSIBLE SOLUTIONS. HENCE THE GALERKIN APPROXIMATION u^h IS THE CLOSEST FUNCTION IN S^h TO $u \in S$.

MATRIX FORM (M)

IF S^h IS FINITE DIMENSIONAL

$$S^h = \{ u^h \mid u^h = u_i \phi_i \}$$

$$V^h = \{ v^h \mid v^h = v_i \phi_i \}$$

THEN

$$a(u^h, v^h) = F(v^h) \qquad \forall\ v^h \in V^h$$

BUT

$$F(v^h) = F(\phi_i v_i) = v_i\ F(\phi_i) = \{v\}^T\{f\}$$

$$a(v^h, u^h) = a(v_i\phi_i, u_j\phi_j) = v_i\ a(\phi_i, \phi_j)\ u_j = \{v\}^T[K]\{u\}$$

WHERE $\{v\} = \begin{Bmatrix} v_1 \\ v_2 \\ \vdots \end{Bmatrix}$ $[K] = \begin{bmatrix} K_{11} & K_{12} & - & - \\ K_{21} & \ddots & & \\ \vdots & & & \end{bmatrix}$, $k_{ij} = a(\phi_i, \phi_j)$, $\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \end{Bmatrix}$

THEREFORE

$$a(u^h, v^h) = F(v^h) \iff \{v\}^T \overbrace{\left([K]\{u\} - \{f\}\right)}^{=0} \quad \forall\ \{v\}$$

$$\iff [K]\{u\} = \{f\}$$

<u>NOTE</u>: $k_{ij} = k_{ji}$ $\qquad$ $[K]$ IS SYMMETRIC

$$a(u^h, u^h) \geq 0 \Rightarrow \{u\}^T[K]\{u\} \geq 0$$ POSITIVE SEMI-DEFINITE