LECTURE 16

TOPICS:

- REVIEW
- ND FINITE ELEMENTS

LOGISTICS

- HW 5 DUE MONDAY
- HW 6 OUT FRIDAY DUE FRIDAY AFTER
- NEXT TUESDAY LECTURE WILL BE RECORDED
- FINAL PROJECT WILL BE DUE ON DEAN'S DATE

REVIEW:

RECALL

$$(S) \Longleftrightarrow (W) \Longrightarrow (G) \Longleftrightarrow (M)$$

(S) WE STARTED FROM THE STRONG FORM

(W) TO GET TO THE WEAK FORM:

i) WE DEFINE THE SET OF TRIAL SOLUTIONS

TRIAL SOLUTIONS ARE ALL THE FUNCTIONS THAT COULD BE A SOLUTION. NAMELY:

a) THEY ARE SMOOTH ENOUGH
b) THEY RESPECT ESSENTIAL BOUNDARY COND

ii) WE DEFINE THE SET OF TEST FUNCTIONS

TEST FUNCTIONS ARE PERTURBATIONS OF TRIAL SOLUTIONS IN THE SENSE THAT IF YOU ADD A TEST FUNCTION TO A TRIAL SOLUTION YOU OBTAIN A TRIAL SOLUTION.

THEREFORE TRIAL SOLUTIONS NEE TO SATISFY

a) THEY ARE SMOOTH ENOUGH
b) THEY RESPECT HOMOGENEOUS ESSENTIAL BC

EG:

$?u:\quad EI\dfrac{d^4u}{dx^4}=f \quad \forall x\in(0,\ell) \qquad (*)$

$$\underbrace{u(0)=u_0,\ \frac{du}{dx}(0)=\theta_0}_{\text{DIRICHLET BC}} \qquad EI\frac{d^2u}{dx^2}(\ell)=m \quad EI\frac{d^3u}{dx^3}=t$$

$\Rightarrow \; S=\{u\in \text{SMOOTH}([0,\ell])\mid u(0)=u_0,\ \frac{du}{dx}(0)=\theta_0\}$

$V=\{v\in \text{SMOOTH}([0,\ell])\mid v(0)=0,\ \frac{dv}{dx}(0)=0\}$

iii) WE CONSTRUCT THE STATEMENT OF <u>WEIGHTED RESIDUAL</u>

THE RESIDUAL IS THE FUCTION THAT GIVEN A FUNCTION u IT TELLS US HOW MUCH IT VIOLATES THE DIFFERENTIAL EQUATION

$R=EI\dfrac{d^4u}{dx^4}-f \leftarrow$ SHOUL BE ZERO.

THE STATEMENT OF WEIGHTED RESIDUALS SAYS THAT THE WEIGHTED AVERAGE OF THE RESIDUAL SHOULD BE ZERO, FOR EVERY POSSIBLE WEIGHT.

$$\int_\Omega R(u)\,v\,dx=0 \qquad \forall x\in(0,\ell)$$

WEIGHTED SUM OF "ERROR" IN STRONG FORM

IV) LASTLY WE WANT TO BALANCE THE DERIVATIVES BETWEEN TRIAL SOLUTIONS & TEST FUNCTIONS.

THIS IS DONE BECAUSE:

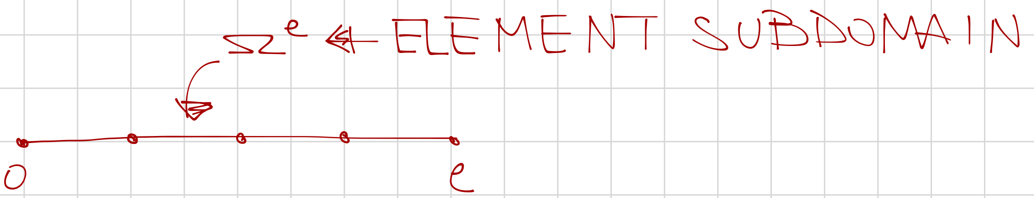
a) WE WANT TRIAL SOLUTIONS & TEST FUNCTIONS TO HAVE THE SAME SMOOTHNESS

b) WE WANT TO REDUCE THE SMOOTHNESS REQUIREMENT AS MUCH AS POSSIBLE.

(G) AFTER (W) WE SAY THAT WE APPROXIMATE THE SPACES OF TRIAL SOLUTIONS & TEST FUNCTIONS WITH SMALLER SETS OF CANDIDATE FUNCTIONS.

$$S \approx S^h \subseteq S \, , \quad V \approx V^h \subseteq V$$

PARTICULARLY WE DECIDE TO APPROXIMATE THESE FUNCTIONS WITH POLYNOMIALS OF ORDER P WITHIN SUBDOMAINS KNOWN AS OUR ELEMENTS



WHY DO WE CHOOSE POLYNOMIALS?

BECAUSE WEIRSTRASS THEOREM TELLS US THAT FOR ANY VALUE ε THERE EXIST A POLYNOMIAL $g_p(x)$ THAT APPROXIMATES A FUNCTION f WITH ε TOLERANCE, NAMELY

$$\| f(x) - g_p(x) \| = \int_{\Omega} | f(x) - g_p(x) | \, dx < \varepsilon$$

THAT MEANS THAT WITHIN EACH ELEMENT OUR SPACE S^h MUST CONTAIN POLYNOMIALS OF ORDER P

FROM HERE WE THEN CONSTRUCT OVER EACH ELEMENT A SPACE OF POLYNOMIALS OF ORDER P.

BASIS FUNCTIONS FOR THESE SPACES ARE GIVEN BY THE LAGRANGE POLYNOMIAL BASIS

$$P^p(\Omega^e) = \{ z(x) \mid z(x) = z_i \, \ell_i^p(x) \}$$

WHERE THE NUMBER OF BASIS IS $p+1$

$p=1$ $\quad z(x) = z_1 \ell_1^1(x) + z_2 \ell_2^1(x) = ax + b$

$p=2$ $\quad z(x) = z_1 \ell_1^2(x) + z_2 \ell_2^2(x) + z_3 \ell_3^2(x) = ax^2 + bx + c$

EXAMPLE (QUADRATIC FINITE ELEMENTS)

BEFORE WE CONSTRUCTED BASIS FUNCTIONS THAT APPROXIMATED EXACTLY LINEAR POLYNOMIALS WITHIN AN ELEMENT

SUPPOSE NOW WE WISH TO APPROXIMATE EXACTLY QUADRATIC POLYNOMIALS

$$q(x) = ax^2 + bx + c$$

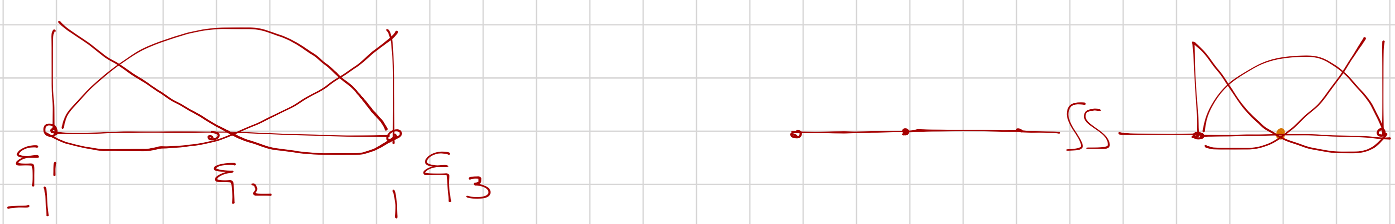
MUST HAVE 3 DOF & 3 BASIS

WE USE THE SO CALLED LAGRANGE POLYNOMIALS

LET $\hat{\Omega} = [-1, 1]$

FOR EACH BASIS FUNCTION WE HAVE A SUPPORT NODE SUCH THAT $\psi_a(\xi_b) = \delta_{ab}$

SINCE WE HAVE THREE BASIS FUNCTIONS WE MUST HAVE THREE NODES WITHIN THE ELEMENT



$$\psi_1(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{1}{2}(1-\xi)\xi$$

$$\psi_2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = (\xi+1)(1-\xi)$$

$$\psi_3(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} = \frac{1}{2}(\xi+1)\xi$$

$$x^e(\xi) = x_1^e\,\psi_1(\xi) = x_e\psi_1(\xi) + \frac{(x_{e+1}+x_e)}{2}\psi_2(\xi) + x_{e+1}\psi_3(\xi)$$

WE CHOOSE THE INTERIOR NODE TO BE THE MIDPOINT

$$= \frac{1}{2}(x_e + x_{e+1}) + \frac{1}{2}\xi(x_{e+1} - x^e)$$

$$\frac{dx^e}{d\xi} = \frac{h}{2}$$

$$k^e_{13} = \hat{a}^e(\psi_1, \psi_3) = \int_{-1}^{1} AE \frac{d\psi_1}{d\xi}\left(\frac{dx^e}{d\xi}\right)^{-1}\frac{d\psi_3}{d\xi}\left(\frac{dx^e}{d\xi}\right)^{-1}\left(\frac{dx^e}{d\xi}\right)d\xi$$

$$= \int_{-1}^{1} AE \left(\frac{1-2\xi}{2}\right)\left(\frac{2}{h}\right)\left(\frac{2\xi+1}{2}\right)\left(\frac{2}{h}\right)\left(\frac{h}{2}\right)d\xi$$

$$= \frac{AE}{2h}\int_{-1}^{1}(1-2\xi)(2\xi+1)\,d\xi = \frac{AE}{2h}\int_{-1}^{1}(1-4\xi^2)\,d\xi$$

$$= \frac{AE}{2h}\left[\xi - \frac{4\xi^3}{3}\right]\Bigg|_{-1}^{1} = \frac{AE}{2h}\left[1-\frac{4}{3} + \left(1-\frac{4}{3}\right)\right] = -\frac{1}{3}\frac{AE}{h}$$

SIMILARLY FOR OTHERS