LECTURE 15:

- REVIEW
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SO FAR WE DERIVED ALL OF THE BALANCE LAWS THAT GOVERN THE MOTION OF CONTINUA

BALANCE OF MASS

$$\frac{d\rho}{dt} + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \qquad \forall \underline{x} \in \Omega$$

$$\frac{d\rho_0}{dt} = 0, \quad \rho_0 = \rho J \qquad \forall \underline{X} \in \Omega_0$$

BALANCE OF MOMENTUM

$$\rho \dot{\underline{v}} = \underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{b} \qquad \forall \underline{x} \in \Omega$$

$$\rho_0 \dot{\underline{V}} = \underline{\nabla} \cdot \underline{\underline{P}} + \underline{B}, \qquad \forall \underline{X} \in \Omega_0$$

$$\left(\underline{V} = \underline{v} \circ \underline{\varphi}, \ \underline{\underline{P}} = \underline{\underline{\sigma}} J \underline{\underline{F}}^{-T}, \ \underline{B} = \underline{b} J\right)$$

BALANCE OF ANGULAR MOMENTUM

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

BALANCE OF ENERGY (1st LAW)

$$\rho \dot{u} = \underline{\underline{\sigma}} : \underline{\underline{d}} + r - \underline{\nabla} \cdot \underline{q} \qquad \forall \underline{x} \in \Omega$$

$$\rho_0 \dot{u} = \underline{\underline{P}} : \underline{\underline{F}} + R - \underline{\nabla} \cdot \underline{Q} \qquad \forall \underline{X} \in \Omega_0$$

ENTROPY ENEQUALITY (2nd LAW) CLAUSIUS-DUHEM

$$\rho \gamma = \rho \dot{\eta} - \frac{1}{\theta}\left(\rho \dot{u} - \underline{\underline{\sigma}} : \underline{\underline{d}}\right) - \frac{1}{\theta^2}\, \underline{q} \cdot \underline{\nabla}_x \theta \geq 0 \qquad \forall \underline{x} \in \Omega$$

$$\rho_0 \gamma = \rho_0 \dot{\eta} - \frac{1}{\theta}\left(\rho_0 \dot{u} - \underline{\underline{P}} : \underline{\underline{\dot{F}}}\right) - \frac{1}{\theta^2}\, \underline{Q} \cdot \underline{\nabla}_X \theta \geq 0 \qquad \forall \underline{X} \in \Omega_0$$

A STRONGER FORM OF THE ABOVE

$$-\frac{1}{\theta^2}\underline{Q}\cdot\underline{\nabla}_{\underline{X}}\theta \geq 0$$ FOURIER INEQUALITY

$$\rho_0\dot{\eta}-\frac{1}{\theta}\left(\rho_0\dot{u}-\underline{\underline{P}}:\dot{\underline{\underline{F}}}\right)\geq 0$$ CLAUSIUS PLANK

UNKNOWNS

$\rho, \underline{\varphi}, \underline{v}, \underline{q}, \theta, \eta, u$ - IN PRACTICE $\underline{\varphi}$ & θ ARE PRIM

THE ABOVE BALANCE EQUATIONS ARE AUGMENTED BY EQUATIONS OF STATE (AKA CONSTITUTIVE EQUATIONS) THAT RELATE THERMODYNAMIC STATE VARIABLES.

THE BASIC ASSUMPTION OF THERMODYNAMICS IS THAT TO FULLY CHARACTERIZE OUR THERMODYNAMIC STATE (I.E. THE INTERNAL ENERGY) WE NEED n SUBSTATE VARIABLES AND A THERMAL LIKE VARIABLE.

NOTE

$$\rho_0\dot{\eta}-\frac{1}{\theta}\left(\rho_0\dot{u}-\underline{\underline{P}}:\dot{\underline{\underline{F}}}\right)\geq 0 \Rightarrow \rho_0\theta\dot{\eta}-\rho_0\dot{u}-\underline{\underline{P}}:\dot{\underline{\underline{F}}}\geq 0$$

FROM ABOVE WE CONCLUDE THAT

$$u=\hat{u}(\underline{\underline{F}}, \eta, z_i)$$

IN FACT WITH

$$\dot{u}=\frac{du}{d\eta}\dot{\eta}+\frac{du}{d\underline{\underline{F}}}:\dot{\underline{\underline{F}}}$$

$$\Rightarrow \rho_0\left[\theta-\frac{du}{d\eta}\right]\dot{\eta}+\left[\rho_0\frac{du}{d\underline{\underline{F}}}-\underline{\underline{P}}\right]:\dot{\underline{\underline{F}}}\geq 0$$

AND SINCE η, $\underline{\underline{F}}$ CAN BE THOUGHT OF ARBITRARY TRAJECTORIES WE HAVE

$$\theta=\left(\frac{du}{d\eta}\right)\bigg|_{\underline{\underline{F}}, z_i}, \quad \underline{\underline{P}}=\left(\frac{du}{d\underline{\underline{F}}}\right)\bigg|_{\eta, z_i}$$

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LEGENDRE TRANSFORM

A LEGENDRE TRANSFOR CONVERTS A CONVEX FUNCTION OF ONE SET OF VARIABLES TO ANOTHER FUNCTION OF CONJUGATE VARIABLES

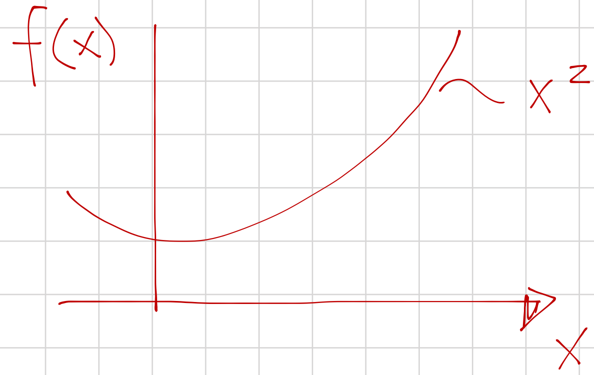
THE LEGENDRE TRANSFORM f^* OF f IS DEFINED BY THE RELATION

$$Df = (Df^*)^{-1} \quad \text{OR} \quad \equiv \quad f'(f^{*\prime}(x^*)) = x^* \ \& \ f^{*\prime}(f'(x)) = x$$

$$\begin{tikzcd} \boxed{x} \arrow[r, shift right, "Df"'] & \boxed{x^*} \arrow[l, shift right, "Df^*"'] \end{tikzcd}$$

LET x & x^* BE CONJUGATE VARIABLES THEN FOR $f(x)$ IT'S LEGENDRE TRANSFORM $f^*(x^*)$ IS GIVEN BY

$$f^*(x^*) = \inf_x f(x) - x^* x$$



$$\inf_x f(x) - x^* x \Rightarrow f'(x) - x^* = 0$$

$$\Rightarrow f'(x) = x^*$$

EX:

$f(x) = e^x$, LET x^* BE CONJUGATE TO x, $x^* = \frac{df}{dx} = e^x$

$$f^*(x^*) = \inf_x f(x) - x^* x$$

$$\inf_x e^x - x^* x \Rightarrow \quad e^x = x^* \Rightarrow \quad x = \ln x^*$$

$$f^*(x^*) = f(\ln x^*) - x^* \ln x^* = x^*(1 - \ln x^*)$$

THE INTERNAL ENERGY IS ONE OF FOUR THERMODYNAMIC POTENTIALS.

$$\psi(\theta, \underline{\underline{F}}, \ldots) = u(\eta, \underline{\underline{F}}, \ldots) - \theta\eta$$ ← HELMHOLTZ FREE EN

THE PORTION OF THE INTERNAL ENERGY ABLE TO DO WORK AT A CONSTANT TEMPERATURE.

$$h(\eta, \underline{\underline{P}}, \ldots) = u(\eta, \underline{\underline{F}}, \ldots) - \underline{\underline{P}} : \underline{\underline{F}}$$ ← ENTHALPY

PORTION OF THE INTERNAL ENERGY THAT CAN BE RELEASED AS HEAT WHEN STRESS ARE HELD CONSTANT

$$g(\theta, \underline{\underline{P}}, \ldots) = u(\eta, \underline{\underline{F}}, \ldots) - \theta\eta - \underline{\underline{P}} : \underline{\underline{F}}$$ ← GIBBS FREE ENERGY

THE PORTION OF THE FREE ENERGY AVAILABLE TO DO WORK AT A CONSTAN STRESS & TEMP.

DEPENDING WHICH THERMODYNAMIC STATE VARIABLE YOU CAN CONTROL IN AN EXPERIMENT IT MAY BE CONVENIENT TO WORK WITH ONE POTENTIAL OVER ANOTHER.

AS DEFORMATION & TEMPERATURE ARE OFTEN THE EASIEST AND MORE MEASURABLE MACROSCOPIC QUANTITIES WE OFTEN DEAL WITH $\psi(\theta, \underline{\underline{F}}, \ldots)$

WITH THE ABOVE

$$\rho_0 \dot{u} = \underline{\underline{P}} : \dot{\underline{\underline{F}}} + R - \nabla \cdot \underline{Q} \Rightarrow$$

$$\rho_0 \theta \dot{\eta} - \rho_0 \dot{u} + \underline{\underline{P}} : \dot{\underline{\underline{F}}} \geq 0 \quad \Rightarrow \quad \underline{\underline{P}} : \underline{\underline{F}} - \rho_0 (u - \theta\eta) \geq 0$$

$$\Rightarrow \quad \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \rho_0 \dot{\psi} - \rho_0 \eta \dot{\theta} \geq 0$$

ASSUME (PRINCIPLE OF LOCALITY) $\psi = \psi(\theta, \underline{\underline{F}}, \nabla\theta)$

$$\left(\underline{\underline{P}} - \rho_0 \frac{d\psi}{d\underline{\underline{F}}}\right) : \dot{\underline{\underline{F}}} - \left(\rho_0 \frac{d\psi}{d\theta} + \rho_0 \eta\right)\dot{\theta} - \rho_0 \frac{d\psi}{d\nabla\theta} \cdot \nabla\dot{\theta} \geq 0$$

SINCE THE ABOVE MUST HOLD TRUE FOR ANY ARBITRARY $\dot{\underline{\underline{F}}}, \dot{\theta}, \nabla\dot{\theta}$ (COLEMAN EXPLOITATION)

$$\Rightarrow \underline{\underline{P}} = \rho_0 \frac{d\psi}{d\underline{\underline{F}}}, \quad \eta = -\frac{d\psi}{d\theta}, \quad \frac{d\psi}{d\nabla\theta} = 0 \quad \Rightarrow \psi = \psi(\theta, \underline{\underline{F}})$$

NOTE IN GENERAL IN PROBLEMS INVOLVING INELASTICITY $\psi = \psi(\theta, \underline{\underline{F}}, \underline{z})$

INTERNAL STATE VARIABLES

WHERE AN ADDITIONAL INITIAL VALUE PROBLEM GOVERNS THE EVOLUTION OF $\underline{z}$.

GOING BACK TO CLAUSIUS - PLANK

$$-\rho_0 \frac{d\psi}{d\underline{z}} \cdot \dot{\underline{z}} \geq 0$$

OR, IF $\underline{q} = -d_{\underline{z}}\psi$, $\quad \underline{q} \cdot \dot{\underline{z}} \geq 0$

INTERNAL STATE VARIABLES ARE WIDELY USED IN THE CONSTITUTIVE FORMULATION OF DISSIPATIVE MATERIALS

INTERNAL STATE VARIABLES MAY BE OBSERVED BUT IN GENERAL NOT CONTROLLED.

THE THERMODYNAMICAL FORCES $\underline{q}$ CONJUGATE TO $\underline{z}$ ARE GENERALLY NOT EXTERNALLY DEFINED HENCE THEY DO NOT EXPLICITLY APPEAR IN BALANCE OF ENERGY

FRAME INDIFFERENCE

EFFECTIVELY BALANCE LAWS GOVERN ALL CONTINUA AND CONSTITUTIVE LAWS ARE SPECIFIC TO MATERIALS

THE STATEMENT OF MATERIAL FRAME INDIFFERENCE STATES THAT THE MATERIAL THERMOMECHANICAL POTENTIAL SHOULD BE INDEPENDENT OF OBSERVER. IE. INVARIANT UNDER TRANSLATION & ROTATION. TO FORMALIZE THE ABOVE WE NEED TO BUILD UP TO IT.

CONSIDER $\quad \underline{x} = \underline{\varphi}(\underline{X}, t)$

S.T. $\quad \underline{x}_0 = \underline{\varphi}(\underline{X}, t_0) \;,\; \underline{x}_1 = \underline{\varphi}(\underline{X}, t_1)$

NOW LET $(\underline{x}_i, t_i) \mapsto (\underline{x}_i^+, t_i^+)$ SUCH THAT DISTANCE $|\underline{x}_0 - \underline{x}_1|$ & TIME ELAPSED ARE PRESERVED.

ONE SUCH MAP IS

$$\underline{x}^+ = \underline{c}(t) + \underline{\underline{Q}}(t)\underline{x} \quad , \quad t^+ = t + \alpha$$

WHERE $\underline{\underline{Q}} \in SO(3)$.

MAPPING OF THE ABOVE FORM ARE KNOWN AS EUCLEDIAN TRANSFORMATIONS