LECTURE 9

TOPICS

- REVIEW
- CONSTRAINTS CONT'D
- 2D TRUSSES

LOGISTICS:

- HW # 2 DUE FRIDAY
- HW # 3 OUT FRIDAY

REVIEW

LAST TIME WE SAW THAT TO APPLY DISPLACEMENTS WE EFFECTIVELY REPLACE THE EQUILIBRIUM EQUATION CORRESPONDING TO THE DOF WITH THE DISPLACEMENT CONSTRAINT.

TO DO SO:

1) WE ZERO OUT THE ROW OF [K] CORRESPONDING TO FIXED DOF

2) WE PLACE 1 ON THE DIAGONAL

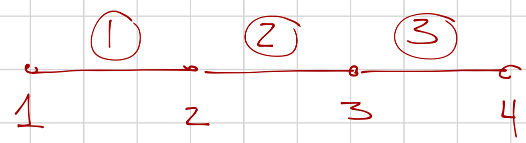
3) WE PLACE THE DISPLACEMENT VALUE IN THE LOAD VECTOR

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \xrightarrow{u_2=\alpha} \begin{Bmatrix} P_1 \\ \alpha \\ P_2 \end{Bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ 0 & 1 & 0 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

ANOTHER APPROACH IS TO GROUP THE DOFs IN FREE DOFs $\{u_f\}$ AND FIXED $\{u_s\}$ SUCH THAT

$$\begin{Bmatrix} \{P_s\} \\ \{P_f\} \end{Bmatrix} = \begin{bmatrix} [K_{ss}] & [K_{sf}] \\ [K_{fs}] & [K_{ff}] \end{bmatrix} \begin{Bmatrix} \{u_s\} \\ \{u_f\} \end{Bmatrix} \Rightarrow \begin{aligned} \{P_f\} &= [K_{fs}]\{u_s\} + [K_{ff}]\{u_f\} \\ \{P_s\} &= [K_{ss}]\{u_s\} + [K_{sf}]\{u_f\} \end{aligned}$$

FOR EXAMPLE:



$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 & 0 \\ -K^1 & K^1+K^2 & -K^2 & 0 \\ 0 & -K^2 & K^2+K^3 & -K^3 \\ 0 & 0 & -K^3 & K^3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

SUPPOSE WE FIX $u_1 = \alpha$, $u_4 = \beta$

IF WE TOOK THE FIRST APPROACH,

$$\begin{Bmatrix} \alpha \\ P_2 \\ P_3 \\ \beta \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -K^1 & K^1+K^2 & -K^2 & 0 \\ 0 & -K^2 & K^2+K^3 & -K^3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

IF WE TOOK THE SECOND APPROACH

1) REORGANIZE ROWS & COLUMNS

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 & 0 \\ -K^1 & K^1+K^2 & -K^2 & 0 \\ 0 & -K^2 & K^2+K^3 & -K^3 \\ 0 & 0 & -K^3 & K^3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

FREE $\{u_f\}$

SUPPORTED $\{u_f\}$

a) MOVE u_4 UP TOP $\Longrightarrow$ CHANGE COLUMNS

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{bmatrix} K^1 & 0 & -K^1 & 0 \\ -K^1 & 0 & K^1+K^2 & -K^2 \\ 0 & -K^3 & -K^2 & K^2+K^3 \\ 0 & K^3 & 0 & -K^3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_4 \\ u_2 \\ u_3 \end{Bmatrix}$$

b) MOVE P_4 UP TOP $\Longrightarrow$ CHANGE ROWS

$$\begin{Bmatrix} P_1 \\ P_4 \\ \hline P_2 \\ P_3 \end{Bmatrix} = \left[\begin{array}{cc|cc} K^1 & 0 & -K^1 & 0 \\ 0 & K^3 & 0 & -K^3 \\ \hline -K^1 & 0 & K^1+K^2 & -K^2 \\ 0 & -K^3 & -K^2 & K^2+K^3 \end{array}\right] \begin{Bmatrix} u_1 \\ u_4 \\ \hline u_2 \\ u_3 \end{Bmatrix}$$

$$\{P_f\} = \begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix} \quad \{P_s\} = \begin{Bmatrix} P_1 \\ P_4 \end{Bmatrix} \quad \{u_f\} = \begin{Bmatrix} u_4 \\ u_2 \end{Bmatrix} \quad \{u_s\} = \begin{Bmatrix} u_1 \\ u_4 \end{Bmatrix}$$

$$[K_{ss}] = \begin{bmatrix} k^1 & 0 \\ 0 & k^3 \end{bmatrix}, \quad [K_{ff}] = \begin{bmatrix} k^1+k^2 & -k^2 \\ -k^2 & k^2+k^3 \end{bmatrix}$$

$$[K_{fs}] = [K_{sf}] = \begin{bmatrix} -k^1 & 0 \\ 0 & -k^3 \end{bmatrix}$$

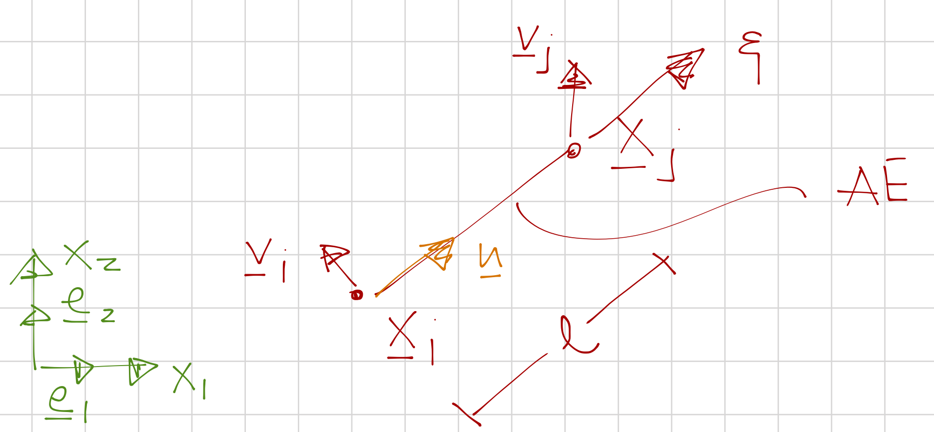
$$\underbrace{\{P_f\}}_{\text{APPLIED LOADS}} = [K_{ff}]\underbrace{\{u_f\}}_{\text{UNKNOWN DISPL}} + [K_{fs}]\{u_s\} \Rightarrow [K_{ff}]\{u_f\} = -[K_{fs}]\{u_s\} + \{P_f\}$$

$$\underbrace{\{P_s\}}_{\text{REACTION FORCES}} = [K_{ss}]\underbrace{\{u_s\}}_{\text{APPLIED DISPL}} + [K_{sf}]\underbrace{\{u_f\}}_{\text{UNKNOWN DISPL}}$$

FINISH TALKING ABOUT EIGENVALUES IN JUPYTER NOTEBOOK

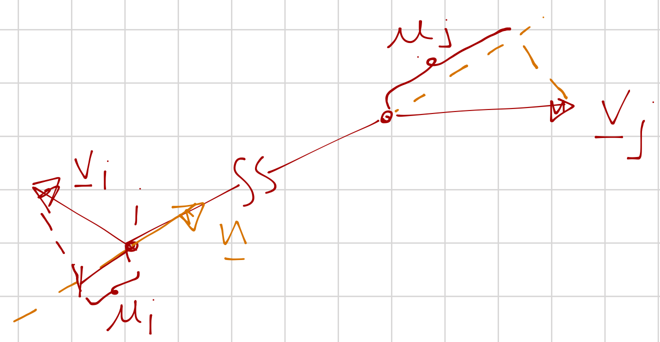
2D TRUSSES

NOW CONSIDER A TRUSS IN 2D



- $\underline{x}_i$ & $\underline{x}_j$ ARE THE COORDINATES OF THE END NODES
- $\ell = \|\underline{x}_j - \underline{x}_i\|$ IS THE LENGTH OF THE TRUSS
- $\underline{n} = \dfrac{\underline{x}_j - \underline{x}_i}{\ell}$ IS THE UNIT VECTOR ALONG TRUSS POINTING FROM NODE i TO NODE j
- $\underline{v}_i$ & $\underline{v}_j$ ARE THE VECTORS OF THE DISPLACEMENTS OF NODE i & j RESPECTIVELY
- ξ A COORDINATE ALONG THE TRUSS THAT IS 0 AT $\underline{x}_i$ AND ℓ AT $\underline{x}_j$

ONLY THE COMPONENT OF THE DISPLACEMENTS ALONG $\underline{n}$ CONTRIBUTE TO STRETCHING (FOR SMALL DEFORMATIONS)



WHERE

$$u_i = \underline{v}_i \cdot \underline{n}$$

$$u_j = \underline{v}_j \cdot \underline{n}$$

SUCH THAT THE DEFORMATION INSIDE THE TRUSS SIMILARLY TO BEFORE IS

$$u(\xi) = (u_j - u_i)\frac{\xi}{e} + u_i$$

$$u(\xi) = \underline{n} \cdot (\underline{v}_j - \underline{v}_i)\frac{\xi}{e} + \underline{n} \cdot \underline{v}_i$$

AND

$$\underline{u}(\xi) = u(\xi)\,\underline{n} = \left[\underline{n} \cdot \left((\underline{v}_j - \underline{v}_i)\frac{\xi}{e} + \underline{v}_i\right)\right]\underline{n} = (\underline{n} \otimes \underline{n})\left((\underline{v}_j - \underline{v}_i)\frac{\xi}{e} + \underline{v}_i\right)$$

NOW RECALL THAT THE INTERNAL FORCE IN THE TRUSS IS GIVEN BY

$$f = AE\frac{du}{d\xi} = \frac{AE}{e}(u_j - u_i)$$

MULTIPLY BOTH SIDES BY $\underline{n}$ TO GET THE INTERNAL FORCE VECTOR

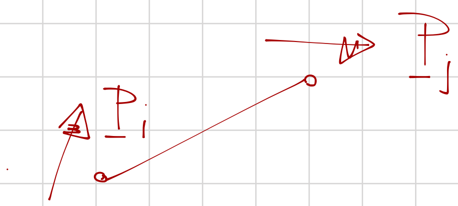
$$\underline{f} = f\underline{n} = \frac{AE}{e}(u_j - u_i)\underline{n} = \frac{AE}{e}(\underline{n} \cdot (\underline{v}_j - \underline{v}_i))\underline{n} =$$

$$= \underbrace{\left(\frac{AE}{e}\underline{n} \otimes \underline{n}\right)}_{\underline{K}}(\underline{v}_j - \underline{v}_i)$$

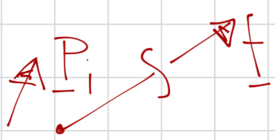
$$= \underline{\underline{k}}\,(\underline{v}_j - \underline{v}_i)$$

$$= \underline{\underline{k}}\ \{-1 \quad 1\} \begin{Bmatrix} \underline{v}_i \\ \underline{v}_j \end{Bmatrix}$$

SIMILARLY TO BEFORE CONSIDER EXTERNAL LOADS APPLIED TO NODE i, j



THEN AT i:

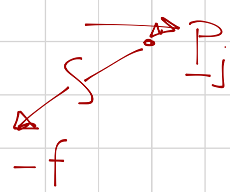


$$\underline{P}_i + \underline{f} = 0 \Rightarrow \underline{P}_i = -\underline{f} = -\underline{\underline{k}}\,(\underline{v}_j - \underline{v}_i) =$$

$$= \underline{\underline{k}}\,\underline{v}_i - \underline{\underline{k}}\,\underline{v}_j$$

$$= \{\underline{\underline{k}} \quad -\underline{\underline{k}}\} \begin{Bmatrix} \underline{v}_i \\ \underline{v}_j \end{Bmatrix}$$

AND AT j



$$\underline{P}_j - \underline{f} = 0 \Rightarrow \underline{P}_j = \{-\underline{\underline{k}} \quad \underline{\underline{k}}\} \begin{Bmatrix} \underline{v}_i \\ \underline{v}_j \end{Bmatrix}$$

SUCH THAT

$$\begin{Bmatrix} \underline{P}_i \\ \underline{P}_j \end{Bmatrix} = \begin{bmatrix} \underline{\underline{k}} & -\underline{\underline{k}} \\ -\underline{\underline{k}} & \underline{\underline{k}} \end{bmatrix} \begin{Bmatrix} \underline{v}_i \\ \underline{v}_j \end{Bmatrix}$$

WHEN WE IMPLEMENT IT IN CODE WE EXPAND THE ABOVE IN THE BASIS

$$\begin{Bmatrix} \{\underline{P}_i\}_e \\ \{\underline{P}_j\}_{\underline{e}} \end{Bmatrix} = \begin{bmatrix} [\underline{\underline{k}}]_e & [-\underline{\underline{k}}]_e \\ [-\underline{\underline{k}}]_{\underline{e}} & [\underline{\underline{k}}]_e \end{bmatrix} \begin{Bmatrix} \{\underline{v}_i\}_e \\ \{\underline{v}_j\}_{\underline{e}} \end{Bmatrix}$$

EXAMPLE

LET $\underline{x}_i = \underline{e}_1 + \underline{e}_2$, $\underline{x}_j = 4\underline{e}_1 + 5\underline{e}_2$, $AE = 5^3$

$$\ell = |\underline{x}_j - \underline{x}_i| = |3\underline{e}_1 + 4\underline{e}_2| = 5$$

$$\underline{n} = \frac{1}{5}(\underline{x}_j - \underline{x}_i) = \frac{3\underline{e}_1 + 4\underline{e}_2}{5}$$

$$\underline{\underline{k}} = \frac{AE}{\ell}\, \underline{n} \otimes \underline{n} = \frac{AE}{\ell}\frac{1}{5}(3\underline{e}_1 + 4\underline{e}_2) \otimes \frac{1}{5}(3\underline{e}_1 + 4\underline{e}_2) =$$

$$= \frac{\cancel{5^3}}{\cancel{5^3}}\left[9\underline{e}_1 \otimes \underline{e}_1 + 12\underline{e}_1 \otimes \underline{e}_2 + 12\underline{e}_2 \otimes \underline{e}_1 + 16\underline{e}_2 \otimes \underline{e}_2\right]$$

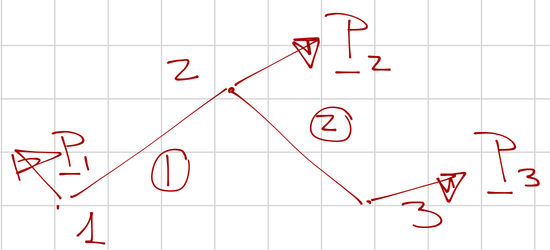
$$[\underline{\underline{K}}]_e = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

$$\begin{Bmatrix} \underline{P}_i \\ \underline{P}_j \end{Bmatrix} = \begin{bmatrix} \underline{\underline{K}} & -\underline{\underline{K}} \\ -\underline{\underline{K}} & \underline{\underline{K}} \end{bmatrix} \begin{Bmatrix} \underline{v}_i \\ \underline{v}_j \end{Bmatrix} \Rightarrow \begin{Bmatrix} P_{i1} \\ P_{i2} \\ P_{j1} \\ P_{j2} \end{Bmatrix} = \begin{bmatrix} 9 & 12 & -9 & -12 \\ 12 & 16 & -12 & -16 \\ -9 & -12 & 9 & 12 \\ -12 & -16 & 12 & 16 \end{bmatrix} \begin{Bmatrix} v_{i1} \\ v_{i2} \\ v_{j1} \\ v_{j2} \end{Bmatrix}$$

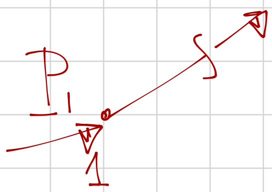
$$\underbrace{\{P\}}_{\text{FORCE VECTOR}} = \underbrace{[K]}_{\text{STIFFNESS MAT}} \overbrace{\{v\}}^{\text{DOF}}$$

SHOW JUPYTER

WHAT IF

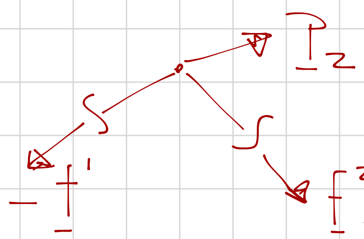


SIMILARLY TO 1-D WRITE BALANCE OF FORCES AT EACH NODE



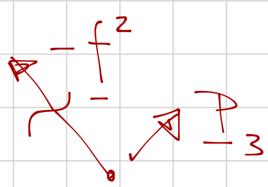
$\underline{f}^1 = \underline{\underline{K}}^1(\underline{v}_2 - \underline{v}_1)$

$$\underline{P}_1 + \underline{f}^1 = 0$$



$\underline{f}^2 = \underline{\underline{K}}^2(\underline{v}_3 - \underline{v}_2)$

$$\underline{P}_2 - \underline{f}^1 + \underline{f}^2 = 0$$



$$\underline{P}_3 - \underline{f}^2 = 0$$

$$\begin{Bmatrix} \underline{P}_1 \\ \underline{P}_2 \\ \underline{P}_3 \end{Bmatrix} = \begin{bmatrix} \underline{\underline{K}}^1 & -\underline{\underline{K}}^1 & \underline{\underline{0}} \\ -\underline{\underline{K}}^1 & \underline{\underline{K}}^1 + \underline{\underline{K}}^2 & -\underline{\underline{K}}^2 \\ \underline{\underline{0}} & -\underline{\underline{K}}^2 & \underline{\underline{K}}^2 \end{bmatrix} \begin{Bmatrix} \underline{v}_1 \\ \underline{v}_2 \\ \underline{v}_3 \end{Bmatrix}$$