LECTURE 2:

- REVIEW
- VECTORS CONT'D
- TENSORS

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REVIEW OF VECTORS

LAST CLASS WE SAW THAT VECTORS ARE A QUANTITY WITH MAGNITUDE & DIRECTION

WE INTRODUCED THE NOTION OF A VECTOR SPACE & THE NOTION OF BASIS $\{\underline{e}_i\}_{i=1}^{n}$ OF THE SPACE AS A SET OF VECTORS SUCH THAT ANY VECTOR $\underline{a} \in V$ CAN BE WRITTEN AS A LINEAR COMBINATION OF $\underline{e}_i$, NAMELY

$$\underline{a} = \sum_{i=1}^{n} a_i \underline{e}_i$$

WHERE a_i ARE SCALAR COEFFICIENTS.

WE SAID THAT IF

$$\underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} = \delta_{ij}$$

KRONECKER DELTA

THEN OUR BASIS ARE ORTHONORMAL, IN WHICH CASE THE a_i ARE <u>UNIQUE</u>.

WE INTRODUCED THE NOTION OF EINSTEIN'S INDICIAL NOTATION

$$a_i b_i = \sum_{i=1}^{n} a_i b_i$$

WHEREBY IF AN INDEX IS REPEATED EXACTLY TWICE IT IS SUMMED OVER & CALLED A DUMMY INDEX.

AN INDEX THAT IS REPEATE ONLY ONCE IS CALLED

A FREE INDEX

E.G.

$$\underline{q}_j = a_i b_i \underline{e}_j \quad , \quad \text{"i" IS A DUMMY INDEX, "j" IS A FREE INDEX}$$

WE INTRODUCED THE DOT PRODUCT

$$\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}|\cos\theta(\underline{a},\underline{b})$$

& IF WE LET $\underline{n} = \dfrac{\underline{b}}{|\underline{b}|}$, SUCH THAT $|\underline{n}| = \dfrac{|\underline{b}|}{|\underline{b}|} = 1$

THEN

$$\underline{a}\cdot\underline{n} = |\underline{a}|\cancelto{1}{|\underline{n}|}\cos\theta(\underline{a},\underline{n}) = |\underline{a}|\cos\theta(\underline{a},\underline{n})$$

IT'S THE PROJECTION OF $\underline{a}$ ALONG THE DIRECTION DEFINED BY $\underline{n}$.

NOTE: WE SAID THAT WE CAN WRITE $\underline{a} = a_i\underline{e}_i$ THEN

DUMMY INDEX

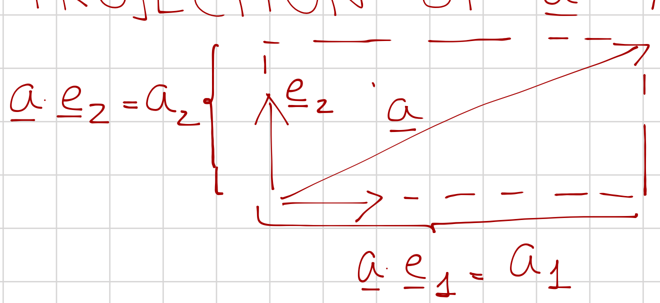
$$\underline{a}\cdot\underline{e}_j = a_i \underbrace{\underline{e}_i\cdot\underline{e}_j}_{\delta_{ij}} = a_i\delta_{ij} = (a_1\delta_{1j} + \dots + a_n\delta_{nj}) = a_j$$

FREE INDEX

EVERYTIME δ_{ij} APPEARS YOU REPLACE INDEX i WITH j IN THE EXPRESSION

$$b_k\delta_{k\ell} = b_\ell$$

NOW NOTE THAT SINCE $|\underline{e}_i| = 1$, $\underline{a}\cdot\underline{e}_j$ IS THE PROJECTION OF $\underline{a}$ ALONG THE $\underline{e}_i$ DIRECTION



EX: WHAT IS $\alpha \underline{a} \cdot \underline{b} = ?$

$\alpha \underline{a} \cdot \underline{b} = \alpha (a_i \underline{e}_i) \cdot (b_j \underline{e}_j) = \alpha a_i b_j \; \underline{e}_i \cdot \underline{e}_j = \alpha a_i b_j \delta_{ij} = \alpha a_i b_i$

WE KNOW THAT $a_i = \underline{a} \cdot \underline{e}_i$ IS UNIQUE THUS WE CAN UNIQLY REPRESENT $\underline{a}$ AS AN ARRAY

$$[\underline{a}]_{\underline{e}} = \begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix}$$

NOTE THIS SUBSCRIPT IS VERY IMPORTANT AS IT INDICATES THE COEFFICIENTS ARE FOR THE $\{\underline{e}_i\}_i$ BASIS.

NOW, WE CAN CARRY OUT $\underline{a} \cdot \underline{b}$ AS

$$[\underline{a}]_{\underline{e}}^T [\underline{b}]_{\underline{e}} = \{a_1 \; a_2 \; \ldots \; a_n\} \begin{Bmatrix} b_1 \\ \vdots \\ b_n \end{Bmatrix} = a_1 b_1 + \ldots + a_n b_n = a_i b_i \; \checkmark$$

IMPORTAN TO HAVE THESAME BASE

$- \, N$

HOW WOULD YOU CONSTRUCT A NORMAL VECTOR $\underline{n}$ TO $\underline{a}$?

1) TAKE ANY VECTOR THAT IS NOT CO-LINEAR W/ $\underline{a}$, AND DENOTE IT BY $\underline{b}$.

2) CONSTRUCT THE PROJECTION OF $\underline{b}$ ALONG $\underline{a}$

$$\underline{m} = \frac{\underline{a}}{|\underline{a}|}$$ ← $\underline{m}$ IS THE UNIT VECTOR ALONG $\underline{a}$

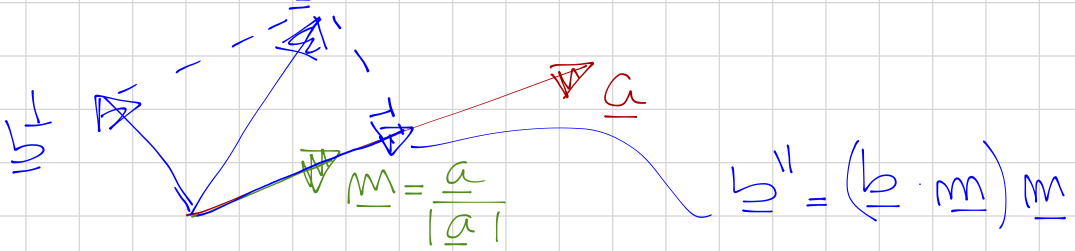
$$b^{\parallel} = \underline{b} \cdot \underline{m}$$

$$\underline{b}^{\parallel} = (\underline{b} \cdot \underline{m})\, \underline{m}$$

3) SUBTRACT THE PROJECTION FROM $\underline{b}$

$$\underline{b}^{\perp} = \underline{b} - \underline{b}^{\parallel}$$

b



$$\underline{b}^{\perp} = \underline{b} - \underline{b}^{\parallel} = \underline{b} - (\underline{b} \cdot \underline{m})\,\underline{m} = \underline{b} - (\underline{b} \cdot \underline{a})\frac{\underline{a}}{|\underline{a}|^2}$$

CROSS PRODUCT

IN $\mathbb{R}^3$ ANOTHER IMPORTANT OPERATION IS THE CROSS PRODUCT $\times$

NOTE THAT WHILE THE DOT PRODUCT IS COMMUTATIVE

I.E. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$

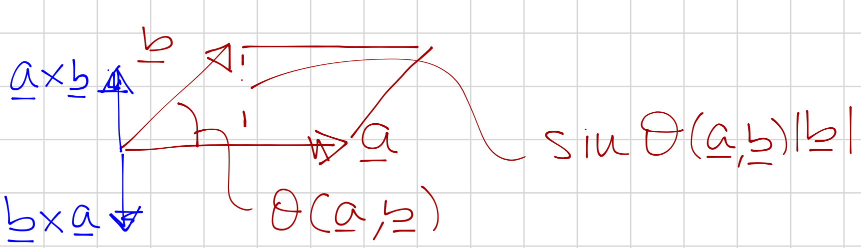
THE CROSS PRODUCT IS NOT COMMUTATIVE, NAMELY

$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$

THE CROSS PRODUCT BETWEEN TWO VECTORS GIVE ANOTHER VECTOR WHOSE MAGNITUDE IS THE AREA OF THE PARALLELOGRAM SUBDUED BY THE VECTORS & THE DIRECTION IS GIVEN BY THE RIGHT HAND RULE

E.G. LET $\underline{c} = \underline{a} \times \underline{b}$ THEN WE HAVE

$$|\underline{c}| = |\underline{a} \times \underline{b}| = |\underline{a}|\,|\underline{b}| \sin\theta(\underline{a}, \underline{b})$$



THE AREA OF THE PARALLELOGRAM IS

$$A = |\underline{a}|\,|\underline{b}| \sin\theta(\underline{a}, \underline{b}) = |\underline{c}|$$

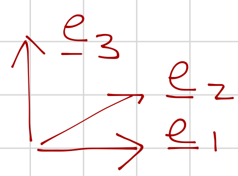
IN INDICIAL NOTATION

$$\underline{a} \times \underline{b} = (a_i \underline{e}_i) \times (b_j \underline{e}_j) = a_i b_j \; \underline{e}_i \times \underline{e}_j$$

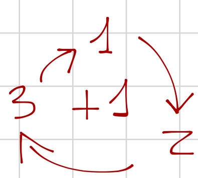
NOW LOOK @ $\underline{e}_i \times \underline{e}_j = \varepsilon_{ijk}\, \underline{e}_k$ ← $\underline{e}_i \times \underline{e}_j$ MUST BE PERP TO $\underline{e}_i$ & $\underline{e}_j$

WHAT ABOUT ε_{ijk} ?

if $i=j$ THE $\underline{e}_i \times \underline{e}_j = 0$ BECAUSE THEY ARE PARALLEL

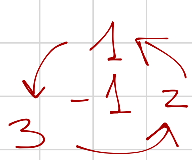


if $i=1$ & $j=2 \Rightarrow k=3$ $\quad \varepsilon_{ijk} = 1$
$i=2$ & $j=3 \Rightarrow k=1$ $\quad \varepsilon_{ijk} = 1$
$i=3$ & $j=1 \Rightarrow k=2$ $\quad \varepsilon_{ijk} = 1$



EVEN PERM

if $i=2$ & $j=1 \Rightarrow k=3$ $\quad \varepsilon_{ijk} = -1$



ODD PERM

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if } i=j \text{ OR } j=k \text{ OR } k=i \\ 1 & \text{EVEN PERMUTATION} \\ 2 & \text{ODD ''} \end{cases}$$ (REPEATED)

ε_{ijk} → LEVI-CIVITA'S SYMBOL

$$\underline{a} \times \underline{b} = \text{DET} \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = (a_2 b_3 - a_3 b_2)\underline{e}_1 - (a_1 b_3 - a_3 b_1)\underline{e}_2 + (a_1 b_2 - a_2 b_1)\underline{e}_3 =$$

$$= a_i b_j \varepsilon_{ijk} \underline{e}_k = a_i b_j \varepsilon_{ij1} \underline{e}_1$$

$$+ a_i b_j \varepsilon_{ij2} \underline{e}_2 + a_i b_j \varepsilon_{ij3} \underline{e}_3 = (a_2 b_3 \varepsilon_{231} + a_3 b_2 \varepsilon_{321})\underline{e}_1 + \ldots$$

$$- (a_2 b_3 - a_3 b_2)\underline{e}_1$$

TENSORS

SO FAR WE HAVE SEEN VECTOR AS A USEFUL MATHEMATICAL QUANTITY THAT HAS BOTH A MAGNITUDE & A DIRECTION.

WE ALSO SAW THA WE MIGHT WANT TO PERFORM OPERATIONS ON THIS VECTORS.

EG. GIVEN A VECTOR $\underline{a}$ & $\underline{b}$ WHAT IS THE PROJECTION OF $\underline{b}$ ALONG $\underline{a}$?

TO ACHIEVE LINEAR OPERATIONS ON VECTORS WE INTRODUCE THE NOTION OF TENSORS.

AS HINTED, A 2nd ORDER TENSOR IS A LINEAR OPERATOR THAT ACTS ON A VECTOR & RETURNS A VECTOR.

EG. LET $\underline{a} \in \mathbb{R}^d$ & $\underline{\underline{T}}$ A SECOND ORDER TENSO

NOTE THE TWO LINES DENOTE A 2nd ORDER TENSOR!

THEN $\underline{\underline{T}} : \mathbb{R}^d \to \mathbb{R}^d$ ($\underline{\underline{T}}$ TAKES SOMETHING IN $\mathbb{R}^d$ & RETURNS SOMETHING IN $\mathbb{R}^d$)

$$\underline{y} = \underline{\underline{T}}(\underline{a}) = \underline{\underline{T}}\underline{a}, \quad \underline{y} \in \mathbb{R}^d$$

OFTEN OMIT PARENTHESIS

SINCE $\underline{\underline{T}}$ IS LINEAR FORALL $\underline{a}, \underline{b} \in \mathbb{R}^d$ & $\alpha \in \mathbb{R}$

$$\underline{\underline{T}}(\underline{a}+\underline{b}) = \underline{\underline{T}}(\underline{a}) + \underline{\underline{T}}(\underline{b})$$

$$\underline{\underline{T}}(\alpha \underline{a}) = \alpha \underline{\underline{T}}(\underline{a})$$

EG: IS IT A TENSOR?

i) $\underline{\underline{T}}(\underline{a}) = \underline{a}/2$? $\underline{\underline{T}} : \mathbb{R}^d \to \mathbb{R}^d$ ✓

$\underline{\underline{T}}(\underline{a}+\underline{b}) = (\underline{a}+\underline{b})/2 = \underline{a}/2 + \underline{b}/2 = \underline{\underline{T}}(\underline{a}) + \underline{\underline{T}}(\underline{b})$ ✓

$\underline{\underline{T}}(\alpha \underline{a}) = \alpha \underline{a}/2 = \alpha \underline{\underline{T}}(\underline{a})$ ✓

ii) $\underline{\underline{T}}(\underline{a}) = (\underline{a} \cdot \underline{a})\underline{a}$? $\underline{\underline{T}} : \mathbb{R}^d \to \mathbb{R}^d$ ✓

$\underline{\underline{T}}(\alpha \underline{a}) = \alpha^3 \underline{\underline{T}}(\underline{a}) \neq \alpha \underline{\underline{T}}(\underline{a})$ X

TENSOR ALGEBRA

$(\underline{\underline{A}} + \underline{\underline{B}})\underline{u} = \underline{\underline{A}}\underline{u} + \underline{\underline{B}}\underline{u} = \underline{\underline{C}}\underline{u}$, $\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}}$

$\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{u} = \underline{\underline{C}}\underline{u} = \underline{\underline{A}}(\underline{\underline{B}}\underline{u})$

TENSOR PRODUCT

SINCE TENSORS ARE LINEAR OPERATORS WE CAN SHOW THAT FOR ANY TENSOR $\underline{\underline{T}}$ THERE EXIST TWO VECTORS $\underline{a}, \underline{b} \in \mathbb{R}^d$ SUCH THAT

$\underline{\underline{T}}(\underline{c}) = (\underline{b} \cdot \underline{c})\underline{a}$

TO CONSTRUCT $\underline{\underline{T}}$ THEN WE INTRODUCE THE DYADIC OR TENSOR OR OUTER PRODUCT $\otimes$ SUCH THAT

$\otimes : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$

EG: $\underline{\underline{T}} = \underline{a} \otimes \underline{b}$

THEN

$\underline{\underline{T}}(\underline{c}) = (\underline{a} \otimes \underline{b})(\underline{c}) = \underline{a}(\underline{b} \cdot \underline{c})$

NOTE THAT THE ABOVE SATISFY THE LINEARITY CONDITION

$$(\underline{a}\otimes\underline{b})(\underline{c}+\underline{d}) = \underline{a}(\underline{b}\cdot(\underline{c}+\underline{d})) = \underline{a}(\underline{b}\cdot\underline{c}) + \underline{a}(\underline{b}\cdot\underline{d}) \checkmark$$

$$(\underline{a}\otimes\underline{b})(\alpha\underline{c}) = \alpha(\underline{a}\otimes\underline{b})\underline{c} \checkmark$$

NOTE: $\underline{a}\otimes\underline{b} \neq \underline{b}\otimes\underline{a}$!!

IN FACT IF $\underline{\underline{T}} = \underline{a}\otimes\underline{b}$ WE DENOTE BY $\underline{\underline{T}}^T = \underline{b}\otimes\underline{a}$ THE TRANSPOSE OF $\underline{\underline{T}}$

LASTLY NOTE THAT IF $\underline{a} = a_i\underline{e}_i$, $\underline{b} = b_j\underline{e}_j$ THEN

$$\underline{\underline{T}} = (a_i\underline{e}_i)\otimes(b_j\underline{e}_j) = a_i b_j\, \underline{e}_i\otimes\underline{e}_j = T_{ij}\, \underline{e}_i\otimes\underline{e}_j$$

& T_{ij} ARE SAID TO BE THE COMPONENTS OF $\underline{\underline{T}}$ IN THE $\underline{e}_i$ BASIS WHICH CAN BE WRITTEN AS

$$[\underline{\underline{T}}]_{\underline{e}} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ \vdots & & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots \\ & \ddots & \end{bmatrix}$$

$$= [\underline{a}]_{\underline{e}}[\underline{b}]_{\underline{e}}^T = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \end{Bmatrix}\{b_1\ b_2\ \cdots\ b_n\}$$