Direct Stiffness Methods

Beams and Beam-Columns

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Review of Beams

Beams

We arrived at the beam governing equation as

$$\frac{d}{dx^2}\left(EI\frac{dw}{dx^2}\right) = -q(x), \quad \forall x$$

where w is our beam displacement.

Also we have

$$m(x) = EI\frac{d^2u}{dx^2} \qquad \leftarrow \text{Internal moment}$$

$$v(x) = \frac{dm}{dx}(x) = EI\frac{d^3u}{dx^3} \qquad \leftarrow \text{Internal shear}$$

Beams

The differential problem

Find $w[0, \ell] \to \mathbb{R}$ such that

$$\frac{d}{dx^2}\left(EI\frac{dw}{dx^2}\right) = 0, \quad \forall x \in (0, \ell)$$

and

$$\begin{aligned} w(0) = w_i, &\quad w'(0) = \theta(0) = \theta_i \\ w(\ell) = w_j, &\quad w'(\ell) = \theta(\ell) = \theta_j \end{aligned}$$

Beams

The analytical solution of the above problem is

$$w(x) = N_1(x)\, w_i + N_2(x)\theta_i + N_3(x)w_j + N_4(x)\theta_j$$

where

$$N_1(x) = 1 - 3\left(\frac{x}{\ell}\right)^2 + 2\left(\frac{x}{\ell}\right)^3$$

$$N_2(x) = \ell\left[\left(\frac{x}{\ell}\right) - 2\left(\frac{x}{\ell}\right)^2 + \left(\frac{x}{\ell}\right)^3\right]$$

$$N_3(x) = 3\left(\frac{x}{\ell}\right)^2 - 2\left(\frac{x}{\ell}\right)^3$$

$$N_4(x) = \ell\left[\left(\frac{x}{\ell}\right)^3 - \left(\frac{x}{\ell}\right)^2\right]$$

Beams

The nodal equilibrium equations then reduce

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = EI \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ -N_1''(0) & -N_2''(0) & -N_3''(0) & -N_4''(0) \\ -N_1'''(\ell) & -N_2'''(\ell) & -N_3'''(\ell) & -N_4'''(\ell) \\ N_1''(\ell) & N_2''(\ell) & N_3''(\ell) & N_4''(\ell) \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}.$$

Beams

The nodal equilibrium equations then reduce

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = EI \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ -N_1''(0) & -N_2''(0) & -N_3''(0) & -N_4''(0) \\ -N_1'''(\ell) & -N_2'''(\ell) & -N_3'''(\ell) & -N_4'''(\ell) \\ N_1''(\ell) & N_2''(\ell) & N_3''(\ell) & N_4''(\ell) \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}.$$

Beams

The nodal equilibrium equations then reduce

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = EI \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ -N_1''(0) & -N_2''(0) & -N_3''(0) & -N_4''(0) \\ -N_1'''(\ell) & -N_2'''(\ell) & -N_3'''(\ell) & -N_4'''(\ell) \\ N_1''(\ell) & N_2''(\ell) & N_3''(\ell) & N_4''(\ell) \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}.$$

Beams

The nodal equilibrium equations then reduce

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = EI \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ -N_1''(0) & -N_2''(0) & -N_3''(0) & -N_4''(0) \\ -N_1'''(\ell) & -N_2'''(\ell) & -N_3'''(\ell) & -N_4'''(\ell) \\ N_1''(\ell) & N_2''(\ell) & N_3''(\ell) & N_4''(\ell) \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}.$$

Beams

The nodal equilibrium equations then reduce

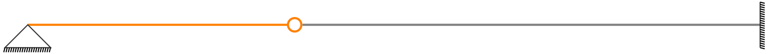
$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} k_{fv} & k_{f\theta} & -k_{fv} & k_{f\theta} \\ k_{mv} & k_{m\theta} & -k_{mv} & \hat{k}_{m\theta} \\ -k_{fv} & -k_{f\theta} & k_{fv} & -k_{f\theta} \\ k_{mv} & \hat{k}_{m\theta} & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

where

$$k_{fv} = \frac{12EI}{\ell^3}, \quad k_{mv} = k_{f\theta} = \frac{6EI}{\ell^2}, \quad k_{m\theta} = \frac{4EI}{\ell}, \quad \hat{k}_{m\theta} \frac{2EI}{\ell}$$

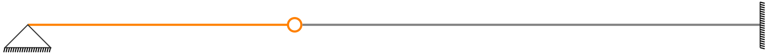
Global Assembly

Global Assembly of Beams



Element	Node i	Node j
1	1	2
2	2	3

Global Assembly of Beams



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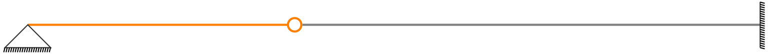
Global Assembly of Beams



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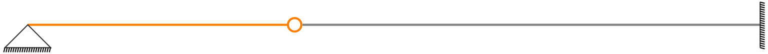

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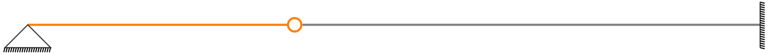

Element	Node i	Node j
1	1	2
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Global Assembly of Beams

$$\begin{Bmatrix} w_i \\ M_i \\ w_j \\ M_j \end{Bmatrix} = \begin{bmatrix} k_{fv} & k_{f\theta} & -k_{fv} & k_{f\theta} \\ k_{mv} & k_{m\theta} & -k_{mv} & \hat{k}_{m\theta} \\ -k_{fv} & -k_{f\theta} & k_{fv} & -k_{f\theta} \\ k_{mv} & \hat{k}_{m\theta} & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & \\ & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

Global Assembly of Beams

$$\begin{Bmatrix} w_i \\ M_i \\ w_j \\ M_j \end{Bmatrix} = \begin{bmatrix} k_{fv} & k_{f\theta} & -k_{fv} & k_{f\theta} \\ k_{mv} & k_{m\theta} & -k_{mv} & \hat{k}_{m\theta} \\ -k_{fv} & -k_{f\theta} & k_{fv} & -k_{f\theta} \\ k_{mv} & \hat{k}_{m\theta} & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} k^1_{fv} & k^1_{f\theta} & -k^1_{fv} & k^1_{f\theta} & 0 & 0 \\ k^1_{mv} & k^1_{m\theta} & -k^1_{mv} & \hat{k}^1_{m\theta} & 0 & 0 \\ -k^1_{fv} & -k^1_{f\theta} & k^1_{fv} & -k^1_{f\theta} & & \\ k^1_{mv} & \hat{k}^1_{m\theta} & -k^1_{mv} & k^1_{m\theta} & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

Global Assembly of Beams

$$\begin{Bmatrix} w_i \\ M_i \\ w_j \\ M_j \end{Bmatrix} = \begin{bmatrix} k_{fv} & k_{f\theta} & -k_{fv} & k_{f\theta} \\ k_{mv} & k_{m\theta} & -k_{mv} & \hat{k}_{m\theta} \\ k_{fv} & -k_{f\theta} & k_{fv} & -k_{f\theta} \\ k_{mv} & \hat{k}_{m\theta} & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} k^1_{fv} & k^1_{f\theta} & -k^1_{fv} & k^1_{f\theta} & 0 & 0 \\ k^1_{mv} & k^1_{m\theta} & -k^1_{mv} & \hat{k}^1_{m\theta} & 0 & 0 \\ -k^1_{fv} & -k^1_{f\theta} & k^1_{fv}+k^2_{fv} & -k^1_{f\theta}+k^2_{f\theta} & -k^2_{fv} & k^2_{f\theta} \\ k^1_{mv} & \hat{k}^1_{m\theta} & -k^1_{mv}+k^2_{mv} & k^1_{m\theta}+k^2_{m\theta} & -k^2_{mv} & \hat{k}^2_{m\theta} \\ 0 & 0 & -k^2_{fv} & -k^2_{f\theta} & k^2_{fv} & -k^2_{f\theta} \\ 0 & 0 & k^2_{mv} & \hat{k}^2_{m\theta} & -k^2_{mv} & k^2_{m\theta} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

Constraints

Global Assembly of Beams

We want to enforce

$$w_1 = \alpha, \quad w_3 = \beta, \quad \theta_3 = \gamma$$

in

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} k^1_{fv} & k^1_{f\theta} & -k^1_{fv} & k^1_{f\theta} & 0 & 0 \\ k^1_{mv} & k^1_{m\theta} & -k^1_{mv} & \hat{k}^1_{m\theta} & 0 & 0 \\ -k^1_{fv} & -k^1_{f\theta} & k^1_{fv}+k^2_{fv} & -k^1_{f\theta}+k^2_{f\theta} & -k^2_{fv} & k^2_{f\theta} \\ k^1_{mv} & \hat{k}^1_{m\theta} & -k^1_{mv}+k^2_{mv} & k^1_{m\theta}+k^2_{m\theta} & -k^2_{mv} & \hat{k}^2_{m\theta} \\ 0 & 0 & -k^2_{fv} & -k^2_{f\theta} & k^2_{fv} & -k^2_{f\theta} \\ 0 & 0 & k^2_{mv} & \hat{k}^2_{m\theta} & -k^2_{mv} & k^2_{m\theta} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

Global Assembly of Beams

We want to enforce

$$w_1 = \alpha, \quad w_3 = \beta, \quad \theta_3 = \gamma$$

in

$$\begin{Bmatrix} \alpha \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ k^1_{mv} & k^1_{m\theta} & -k^1_{mv} & \hat{k}^1_{m\theta} & 0 & 0 \\ -k^1_{fv} & -k^1_{f\theta} & k^1_{fv}+k^2_{fv} & -k^1_{f\theta}+k^2_{f\theta} & -k^2_{fv} & k^2_{f\theta} \\ k^1_{mv} & \hat{k}^1_{m\theta} & -k^1_{mv}+k^2_{mv} & k^1_{m\theta}+k^2_{m\theta} & -k^2_{mv} & \hat{k}^2_{m\theta} \\ 0 & 0 & -k^2_{fv} & -k^2_{f\theta} & k^2_{fv} & -k^2_{f\theta} \\ 0 & 0 & k^2_{mv} & \hat{k}^2_{m\theta} & -k^2_{mv} & k^2_{m\theta} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

Global Assembly of Beams

We want to enforce

$$w_1 = \alpha, \quad w_3 = \beta, \quad \theta_3 = \gamma$$

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$$\begin{Bmatrix} \alpha \\ M_1 \\ V_2 \\ M_2 \\ \beta \\ M_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ k^1_{mv} & k^1_{m\theta} & -k^1_{mv} & \hat{k}^1_{m\theta} & 0 & 0 \\ -k^1_{fv} & -k^1_{f\theta} & k^1_{fv}+k^2_{fv} & -k^1_{f\theta}+k^2_{f\theta} & -k^2_{fv} & k^2_{f\theta} \\ k^1_{mv} & \hat{k}^1_{m\theta} & -k^1_{mv}+k^2_{mv} & k^1_{m\theta}+k^2_{m\theta} & -k^2_{mv} & \hat{k}^2_{m\theta} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k^2_{mv} & \hat{k}^2_{m\theta} & -k^2_{mv} & k^2_{m\theta} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

Global Assembly of Beams

We want to enforce

$$w_1 = \alpha, \quad w_3 = \beta, \quad \theta_3 = \gamma$$

in

$$\begin{Bmatrix} \alpha \\ M_1 \\ V_2 \\ M_2 \\ \beta \\ \gamma \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ k^1_{mv} & k^1_{m\theta} & -k^1_{mv} & \hat{k}^1_{m\theta} & 0 & 0 \\ -k^1_{fv} & -k^1_{f\theta} & k^1_{fv}+k^2_{fv} & -k^1_{f\theta}+k^2_{f\theta} & -k^2_{fv} & k^2_{f\theta} \\ k^1_{mv} & \hat{k}^1_{m\theta} & -k^1_{mv}+k^2_{mv} & k^1_{m\theta}+k^2_{m\theta} & -k^2_{mv} & \hat{k}^2_{m\theta} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

Global Assembly of Beams

We want to enforce

$$w_1 = \alpha, \quad w_3 = \beta, \quad \theta_3 = \gamma$$

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$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} k^1_{fv} & k^1_{f\theta} & -k^1_{fv} & k^1_{f\theta} & 0 & 0 \\ k^1_{mv} & k^1_{m\theta} & -k^1_{mv} & \hat{k}^1_{m\theta} & 0 & 0 \\ -k^1_{fv} & -k^1_{f\theta} & k^1_{fv}+k^2_{fv} & -k^1_{f\theta}+k^2_{f\theta} & -k^2_{fv} & k^2_{f\theta} \\ k^1_{mv} & \hat{k}^1_{m\theta} & -k^1_{mv}+k^2_{mv} & k^1_{m\theta}+k^2_{m\theta} & -k^2_{mv} & \hat{k}^2_{m\theta} \\ 0 & 0 & -k^2_{fv} & -k^2_{f\theta} & k^2_{fv} & -k^2_{f\theta} \\ 0 & 0 & k^2_{mv} & \hat{k}^2_{m\theta} & -k^2_{mv} & k^2_{m\theta} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

Global Assembly of Beams

We want to enforce

$$w_1 = \alpha, \quad w_3 = \beta, \quad \theta_3 = \gamma$$

in

$$\{Q\} = [K]\{w\}$$

where

$$\{Q\} = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix}, \quad \{w\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

Global Assembly of Beams

We want to enforce

$$w_1 = \alpha, \quad w_3 = \beta, \quad \theta_3 = \gamma$$

in

$$\{Q\} = [K]\{w\}$$

where

$$[K] = \begin{bmatrix} k^1_{fv} & k^1_{f\theta} & -k^1_{fv} & k^1_{f\theta} & 0 & 0 \\ k^1_{mv} & k^1_{m\theta} & -k^1_{mv} & \hat{k}^1_{m\theta} & 0 & 0 \\ -k^1_{fv} & -k^1_{f\theta} & k^1_{fv}+k^2_{fv} & -k^1_{f\theta}+k^2_{f\theta} & -k^2_{fv} & k^2_{f\theta} \\ k^1_{mv} & \hat{k}^1_{m\theta} & -k^1_{mv}+k^2_{mv} & k^1_{m\theta}+k^2_{m\theta} & -k^2_{mv} & \hat{k}^2_{m\theta} \\ 0 & 0 & -k^2_{fv} & -k^2_{f\theta} & k^2_{fv} & -k^2_{f\theta} \\ 0 & 0 & k^2_{mv} & \hat{k}^2_{m\theta} & -k^2_{mv} & k^2_{m\theta} \end{bmatrix}$$

Global Assembly of Beams

We want to enforce

$$w_1 = \alpha, \quad w_3 = \beta, \quad \theta_3 = \gamma$$

in

$$\begin{Bmatrix} \{Q_s\} \\ \{Q_f\} \end{Bmatrix} = \begin{bmatrix} [K_{ss}] & [K_{sf}] \\ [K_{fs}] & [K_{ff}] \end{bmatrix} \begin{Bmatrix} \{w_s\} \\ \{w_f\} \end{Bmatrix}$$

where

$$\{Q_s\} = \begin{Bmatrix} V_1 \\ V_3 \\ M_3 \end{Bmatrix}, \quad \{w_s\} = \begin{Bmatrix} w_1 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

$$\{Q_f\} = \begin{Bmatrix} M_1 \\ V_2 \\ M_2 \end{Bmatrix}, \quad \{w_f\} = \begin{Bmatrix} \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

Global Assembly of Beams

We want to enforce

$$w_1 = \alpha, \quad w_3 = \beta, \quad \theta_3 = \gamma$$

in

$$\left\{ \begin{array}{c} \{Q_s\} \\ \{Q_f\} \end{array} \right\} = \left[\begin{array}{cc} [K_{ss}] & [K_{sf}] \\ [K_{fs}] & [K_{ff}] \end{array} \right] \left\{ \begin{array}{c} \{w_s\} \\ \{w_f\} \end{array} \right\}$$

where

$$\{Q_s\} = \left\{ \begin{array}{c} V_1 \\ V_3 \\ M_3 \end{array} \right\}, \quad \{w_s\} = \left\{ \begin{array}{c} w_1 \\ w_3 \\ \theta_3 \end{array} \right\}$$

$$\{Q_f\} = \left\{ \begin{array}{c} M_1 \\ V_2 \\ M_2 \end{array} \right\}, \quad \{w_f\} = \left\{ \begin{array}{c} \theta_1 \\ w_2 \\ \theta_2 \end{array} \right\}$$

Global Assembly of Beams

We want to enforce

$$w_1 = \alpha, \quad w_3 = \beta, \quad \theta_3 = \gamma$$

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where

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} k^1_{fv} & k^1_{f\theta} & -k^1_{fv} & k^1_{f\theta} & 0 & 0 \\ k^1_{mv} & k^1_{m\theta} & -k^1_{mv} & \hat{k}^1_{m\theta} & 0 & 0 \\ -k^1_{fv} & -k^1_{f\theta} & k^1_{fv}+k^2_{fv} & -k^1_{f\theta}+k^2_{f\theta} & -k^2_{fv} & k^2_{f\theta} \\ k^1_{mv} & \hat{k}^1_{m\theta} & -k^1_{mv}+k^2_{mv} & k^1_{m\theta}+k^2_{m\theta} & -k^2_{mv} & \hat{k}^2_{m\theta} \\ 0 & 0 & -k^2_{fv} & -k^2_{f\theta} & k^2_{fv} & -k^2_{f\theta} \\ 0 & 0 & k^2_{mv} & \hat{k}^2_{m\theta} & -k^2_{mv} & k^2_{m\theta} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$