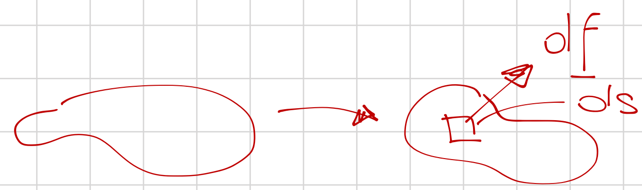
LECTURE 12:

- MOHR'S CIRCLE SEE MASE 3.8
- BALANCE LAWS

— N —

FIRST PIOLA

$d\underline{f} = \underline{\sigma}\,\underline{n}\,ds$



$d\underline{f} = \underline{P}\,\underline{N}\,dS_0$

THE 1st PIOLA KIRCHOFF RELATES A FORCE ACTING IN THE CURRENT CONFIGURATION $d\underline{f}$ TO AN AREA ELEMENT IN THE REFERENCE CONFIG $\underline{N}\,dS$.

EFFECTIVELY $\underline{T} = \underline{P}\,\underline{N} = \dfrac{d\underline{f}}{dS_0}$ ← $\underline{T}$ POINTS IN THE RIGH DIRECTION BUT IT'S MAGNITUDE IS SCALED

BALANCE LAWS

BALANCE OF MASS

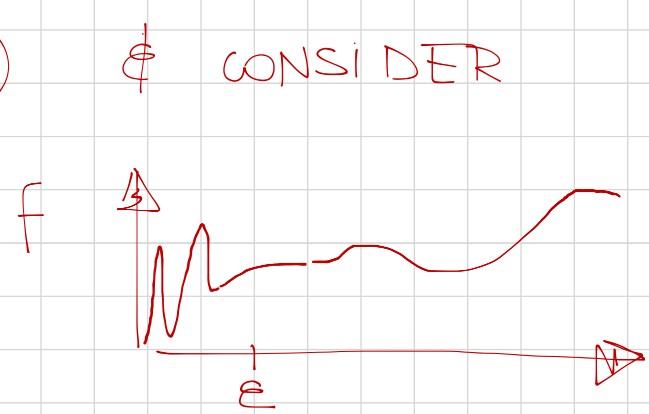
MASS IS ALWAYS CONSERVED

IF WE LET $m(\cdot)$ DENOTE THE MASS OF $(\cdot)$, $\Theta_0 \subseteq \Omega_0$ AND $\Theta = \underline{\phi}(\Theta_0, t)$ THEN WE MUST HAVE

$m(\Theta_0) = m(\Theta)$.

CONSIDER $B_\varepsilon(\underline{X})$ & CONSIDER

$f(\varepsilon) = \dfrac{m(B_\varepsilon(\underline{X}))}{|B_\varepsilon(\underline{X})|}$



WE INTRODUCE A FUNCTION $\rho_0(\underline{X})$, INDEPENDENT OF TIME, KNOWN A THE REFERENCE DENSITY

$$\rho_0(\underline{X}) = \lim_{r \searrow 0^+} \frac{m(B_r(\underline{X}))}{|B_r(\underline{X})|}$$

THIS SHOULD BE ε

THE ABOVE IS SUCH THAT

$$m(\theta_0) = \int_{\theta_0} \rho_0(\underline{X}) \, dV_0$$

SIMILARLY WE CAN DEFINE THE SPATIAL DENSITY AS

$$\rho(\underline{x}, t) = \lim_{r \searrow 0^+} \frac{m(B_r(\underline{x}))}{|B_r(\underline{x})|}$$

SUCH THAT

$$m(\theta_0) = m(\theta) = \int_{\theta} \rho(\underline{x}, t) \, dv$$

NOTE THAT

$$\int_{\theta_0} \rho_0(\underline{X}) \, dV_0 = \int_{\theta} \rho(\underline{x}, t) \, dv = \int_{\theta_0} \rho(\underline{\varphi}(\underline{X}, t), t) \, J \, dV_0$$

RECALL $J \, dV_0 = dv$

THE ABOVE IMPLIES

$$\int_{\theta_0} \left(\rho_0(\underline{X}) - \rho(\underline{\varphi}(\underline{X}, t), t) J \right) dV_0 = 0$$

SINCE θ_0 IS ARBITRARY, & AS LONG AS $|\theta_0| > 0$ $m(\theta_0) > 0$ IT FOLLOWS

$$\rho_0(\underline{X}) - \rho(\underline{\varphi}(\underline{X}, t), t) J = 0 \Rightarrow \rho_0 = \rho J$$

NOW NOTE THAT

$$\overline{m(\Theta)}^{\bullet} = \frac{D}{Dt} m(\Theta) = \frac{d}{dt} m(\Theta)\Big|_{\Theta_0} = \frac{D}{Dt}\int_\Theta \rho(\underline{x},t)\,dv = 0$$

$$= \frac{D}{Dt}\int_{\Theta_0} \rho(\underline{x},t)\, J\, dV_0 = \int_{\Theta_0} \frac{D}{Dt}\left(\rho(\underline{x},t)\, J\right) dV_0$$

$$= \int_{\Theta_0} (\dot\rho J + \rho \dot J)\, dV_0$$

RECALL

$$\dot J = \frac{dJ}{d\underline{\underline{F}}} : \dot{\underline{\underline{F}}} = \underbrace{J\underline{\underline{F}}^{-T} : \dot{\underline{\underline{F}}}} = J\underline{\underline{1}} : \underbrace{\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}} = J\underline{\underline{1}} : \nabla_x \underline{v} = J\,\nabla\cdot\underline{v}$$

$$= \int_{\Theta_0} (\dot\rho + \rho\nabla\cdot\underline{v})\, J\, dV_0 = 0 \qquad *$$

$$= \int_\Theta(\dot\rho + \rho\nabla\cdot\underline v)\,dV$$

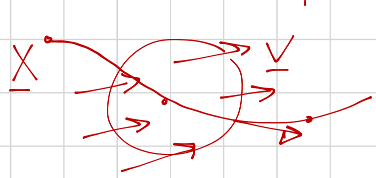
NOW NOTE

$$\dot\rho = \frac{d\rho}{dt}\Big|_x + \nabla\rho\cdot\underline v$$

$$= \int_\Theta \frac{d\rho}{dt}\,dv + \int_\Theta(\nabla\rho\cdot\underline v + \rho\nabla\cdot\underline v)\,dv = \int_\Theta \frac{d\rho}{dt}\,dv + \int_\Theta \nabla\cdot(\rho\underline v)\,dv = 0$$

$$\Rightarrow \int_\Theta \frac{d}{dt}\rho\, dv = -\int_{\partial\Theta}\rho\underline v\cdot\underline n\, dv \qquad \Leftarrow \text{FREEZIN } \Theta!$$

FOR A FIXED SPATIAL VOLUME Θ WE HAVE THAT THE RATE OF CHANGE OF MASS INSIDE THE VOLUME IS EQUAL TO THE NEGATIVE OUTWARD FLUX



GOING BACK TO (*) SINCE $\Theta = \underline{\varphi}(\Theta_0)$ IS ARB

$$\Rightarrow \boxed{\dot{\rho} + \rho \underline{\nabla} \cdot \underline{v} = 0 \quad \forall \ \underline{x} \in \Omega}$$

THIS IS KNOWN AS THE CONTINUITY EQUATION

$$\frac{d\rho}{dt} + \underline{\nabla}\rho \cdot \underline{v} + \rho \underline{\nabla} \cdot \underline{v} = \frac{d\rho}{dt} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

RATE OF CHANGE OF DENSITY → EQUAL TO NEGATIVE OUTWARD FLUX

NOTE THE REFERENTIAL CONSERVATION OF DENSITY IS TRIVIAL

$$\frac{D}{Dt}(m(\Theta_0)) = \int_{\Theta_0} \rho_0 \, dV_0 = \frac{D}{Dt}\rho_0 = 0$$

<u>NOTE</u>:

VOLUME CHANGE

$$\frac{d\rho}{dt} + \underline{\nabla}\rho \cdot \underline{v} + \rho \underline{\nabla} \cdot \underline{v} = 0$$

TRANSPORT

IF $J = 1$ ← ISOCHORIC OR VOLUME PRESERVING MOTION

$\dot{J} = 0 = J \underline{\nabla} \cdot \underline{v}$, SINCE $J > 0 \Rightarrow \underline{\nabla} \cdot \underline{v} = 0$

THUS FOR ISOCHORIC MOTIONS

$$\frac{d\rho}{dt} + \underline{\nabla}\rho \cdot \underline{v} = 0 \quad \forall \ \underline{x} \in \Omega$$

$$\underline{\nabla} \cdot \underline{v} = 0 \quad \forall \ \underline{x} \in \Omega$$

REYNOLD'S TRANSPORT THEOREM

LET $\phi = \phi(\underline{x}, t)$ DESCRIBE SOME SCALAR QUANTITY OF A PARTICLE IN SPACE PER UNIT SPATIAL VOLUME

THE TOTAL QUANTITY WITH, $\Omega(t) = \underline{\varphi}(\Omega_0, t)$

$$I(t) = \int_{\Omega(t)} \phi(\underline{x}, t)\, dv$$

WE ARE INTERESTED IN HOW ϕ EVOLVES WITH TIME.

$$\frac{D}{Dt} I(t) = \frac{D}{Dt} \int_{\Omega(t)} \phi(\underline{x}, t)\, dv = \frac{D}{Dt} \int_{\Omega_0} \phi(\underline{\varphi}(\underline{X}, t), t) J \, dV_0$$

$$= \int_{\Omega_0} (\dot{\phi} J + \phi \dot{J})\, dV = \int_{\Omega_0} (\dot{\phi} + \phi \nabla \cdot \underline{v}) J \, dV_0 =$$

$$= \int_{\Omega} (\dot{\phi} + \nabla \cdot \underline{v}\, \phi)\, dV = \int_{\Omega} \frac{\partial \phi}{\partial t} + \int_{\partial \Omega} \phi \underline{v} \cdot \underline{n}$$

$$\frac{D}{Dt} \int_{\Omega} \phi \, dv = \underbrace{\int_{\Omega} \frac{\partial \phi}{\partial t}}_{\text{LOCAL RATE OF CHANGE}} + \underbrace{\int_{\partial \Omega} \phi \underline{v} \cdot \underline{n}}_{\text{RATE OF TRANSPORT (OUTWARD FLUX)}}$$

ANOTHER IMPORTANT CASE IS WHEN WE HAVE A QUANTITY PER UNIT MASS (EG CONCENTRATION OR MASS FRACTION)

EG $\quad c(\underline{x}, t) = \phi(\underline{x}, t) / \rho(\underline{x}, t)$

$$I(t) = \int_{\Omega(t)} \phi(\underline{x}, t)\, dV = \int_{\Omega(t)} c(\underline{x}, t)\, \rho(\underline{x}, t)\, dv$$

NOTE NOW

$$\dot{I}(t) = \int_{\Omega} \left(\dot{\phi} + \phi \nabla \cdot \underline{v} \right) dv = \int_{\Omega} \left(\dot{\overline{c\rho}} + c\rho \nabla \cdot \underline{v} \right) dv$$

$$= \int_{\Omega} \left(\dot{c}\rho + c\dot{\rho} + c\rho \nabla \cdot \underline{v} \right) dv =$$

$$= \int_{\Omega} \left(\dot{c}\rho + c \underbrace{\left(\dot{\rho} + \rho \nabla \cdot \underline{v} \right)}_{=0} \right) dv =$$

$$= \int_{\Omega} \left(\dot{c}\rho \right) dv$$

BALANCE OF MOMENTUM

LET $\underline{L}(t)$ DENOTE THE LINEAR MOMENTUM

IF WE HAVE A DISCRETE SYSTEM OF P

$$\underline{L}(t) = \sum_i m_i \underline{v}_i(t)$$

THEN, FOLLOWING THE SECOND LAW

$$\frac{d\underline{L}}{dt} = \underline{f}$$

WHERE $\underline{f}$ REPRESENTS EXTERNAL FORCES

IN THE CONTINUUM CASE FOR $\Theta_0 \subseteq \Omega_0$, $\underline{\varphi}(\Theta_0, t) = \Theta$

$$\underline{L}(t) = \int_{\Theta} \rho(\underline{x}, t) \, \underline{v}(\underline{x}, t) \, dv$$

$$\underline{f}(t) = \int_{\Theta} \underline{b}(\underline{x}, t) \, dv + \int_{\partial\Theta} \underline{t}(\underline{x}, t) \, ds$$

E.G. GRAVITY

THE SECOND LAW THEN REQUIRES

$$\frac{D}{Dt}\underline{L}(t) = \underline{f}(t)$$

$$\frac{D}{Dt}\underline{L}(t) = \int_{\Theta_0} \left[(\dot{\rho}\,\underline{v} + \rho\,\dot{\underline{v}})J + \rho\,\underline{v}\;\underline{\nabla}\cdot\underline{v}\,J\right] dv_0 =$$

$$= \int_{\Theta} (\dot{\rho}\,\underline{v} + \rho\,\dot{\underline{v}} + \rho\,\underline{v}\;\underline{\nabla}\cdot\underline{v})\,dv = \int_{\Theta} \left(\rho\,\dot{\underline{v}} + \underline{v}(\dot{\rho} + \rho\,\underline{\nabla}\cdot\underline{v})\right) dv$$

$$= \int_{\Theta} \rho\,\dot{\underline{v}}\,dv$$

$$\Rightarrow \int_{\Theta} \rho\,\dot{\underline{v}} = \int_{\Theta} \underline{b}(\underline{x}, t)\,dv + \int_{\partial\Theta} \underbrace{\underline{t}(\underline{x}, t)}_{\underline{\underline{T}}\,\underline{n}}\,ds$$

$$\int_{\Theta} \rho\,\dot{\underline{v}} = \int_{\Theta} \underline{b}\,dv + \int_{\Theta} \underline{\underline{T}}\,\underline{n}\,ds = \int_{\Theta} \underline{b}\,dv + \int_{\Theta} \underline{\nabla}\cdot\underline{\underline{T}}\,dV$$

$$\Rightarrow \rho\,\dot{\underline{v}} = \underbrace{\underline{\nabla}\cdot\underline{\underline{T}}}_{\text{FLUX OF INTERNAL FORCES}} + \underline{b} \qquad \forall\, \underline{x} \in \Omega$$