LECTURE 31

- HIGHER ORDER TENSORS
- VECTOR & TENSOR CALCULUS
- CURVILINEAR COORDINATES

HIGHER ORDER TENSORS

HIGHER ORDER TENSORS CAN BE EXPRESSED AS

$$\underline{\underline{\underline{\mathbb{C}}}} = C_{ijk\cdots q}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \cdots \underline{e}$$

HIGHER ORDER TENSORS FOLLOW VIRTUALLY ALL THE SAME RULES OF 2nd ORDER

$$\underline{\underline{\underline{\mathbb{A}}}} = - \,\_\_$$

$$\underline{\underline{\underline{\mathbb{B}}}} = - \,\_\_$$

$$\underline{\underline{\underline{\mathbb{C}}}} = C_{ijkl}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l$$

$$\underline{\underline{\underline{\mathbb{A}}}}\,\underline{\underline{\underline{\mathbb{B}}}} = A_{ijkl} B_{mnop} (\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l)(\underline{e}_m \otimes \underline{e}_n \otimes \underline{e}_o \otimes \underline{e}_p)$$
$$= A_{ijkl} B_{lnop}$$

$$\underline{\underline{\underline{\mathbb{A}}}} : \underline{\underline{\underline{\mathbb{B}}}} = A_{ijkl} B_{klop}$$

$$\underline{\underline{\underline{\mathbb{A}}}}^T = C_{ijkl}\, \underline{e}_l \otimes \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k$$

NOTE: DIFFERENT DEF IN HOLZAPFEL

$$\underline{\underline{\underline{\mathbb{I}}}} : \underline{\underline{A}} = \underline{\underline{A}} \Rightarrow \underline{\underline{\underline{\mathbb{I}}}} = \delta_{ik}\delta_{jl}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l$$

$$\underline{\underline{\underline{\mathbb{I}}}}^{SYM} : \underline{\underline{A}} = \underline{\underline{A}}^{SYM} \Rightarrow \underline{\underline{\underline{\mathbb{I}}}}^{SYM} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

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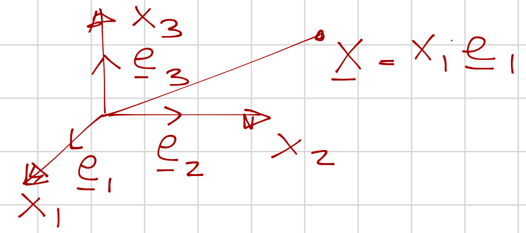
VECTOR & TENSOR CALCULUS

IF WE HAVE A SCALAR FIELD AS A FUNCTION OF SPACE

$\phi : \mathbb{R}^n \rightarrow \mathbb{R}$

$\phi(\underline{x})$, x



THEN WE DEFINE THE GRADIENT OF A SCALAR

$\underline{\nabla} \phi(\underline{x}) = \frac{d \phi}{d x_i} \underline{e}_i$ ← NOTE THIS IS A VECTOR

IT'S A VECTOR THAT POINT TOWARD THE DIRECTION OF MAXIMUM INCREASE OF ϕ WHOSE MAGNITUDE IS THE CHANGE IN ϕ

THE DIRECTIONAL DERIVATIVE IS ANOTHER EXTREMELY IMPORTANT QUANTITY & MEASURES THE RATE OF CHANGE IN A SPECIFIC DIRECTION

$$\underline{\nabla}_{\underline{n}} \underline{\Phi}(\underline{x}) := \frac{d}{d\varepsilon} \phi(\underline{x} + \varepsilon \underline{n})\Big|_{\varepsilon = 0} = \underline{\nabla} \underline{\Phi} \cdot \underline{n}$$

FOR VECTOR & TENSOR FIELDS WE MORE BROADLY DEFINE

$$\underline{\nabla}(\circ) = \frac{d(\circ)}{dx_i} \otimes \underline{e}_i$$ ← THE GRADIENT

$\underline{a}(\underline{x})\ \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\underline{\nabla}\,\underline{a} = \frac{d(a_i \underline{e}_i)}{dx_j} \otimes \underline{e}_j = \frac{d a_i}{dx_j}\, \underline{e}_i \otimes \underline{e}_j$$

$\underline{\underline{A}}(\underline{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$

$$\underline{\nabla}\,\underline{\underline{A}} = \frac{d A_{ij}}{dx_k}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k$$

NOTE: FLUID MECH REVERSED

$$\underline{\nabla}\cdot(\circ) = \frac{d(\cdot)}{dx_i} \cdot \underline{e}_i$$ ← DIVERGENCE

THE DIVERGE IS A MEASURE OF FLUX

$$\underline{\nabla}\,\underline{a} = \frac{d(a_i \underline{e}_i)}{dx_j} \cdot \underline{e}_j = \frac{d a_i}{dx_i} = \underline{\nabla}\underline{a} : \underline{\underline{1}}$$

$$\nabla \cdot \underline{\underline{A}} = \frac{dA_{ij}}{dx_j} \underline{e}_i$$

IF $\underline{\nabla} \cdot \underline{a} = 0$ $\quad \underline{a}$ IS SAID TO BE SOLEINOIDAL (VOLUME PRESERVING VECTOR FIELD

$$\underline{\nabla} \times (\circ) = \underline{e}_j \times \frac{d(\circ)}{dx_j} \quad \Leftarrow \text{CURL}$$

MEASURE OF ROTATION

RETURNS A VECTOR NORMAL TO THE PLANE OF MAX ROTATION

$$\underline{\nabla} \times \underline{a} = \frac{da_i}{dx_j} \varepsilon_{ijk} \underline{e}_k$$

$$\underline{\nabla} \times \underline{\underline{A}}$$

IF $\underline{\nabla} \times \underline{a} \Rightarrow \underline{a}$ IRROTATIONAL

$$\underline{\nabla} \times \underline{\nabla} \phi = 0$$

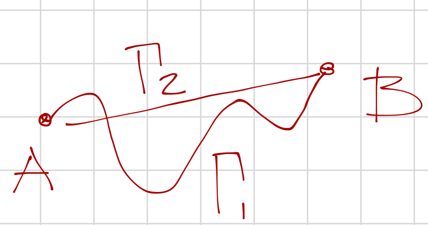
$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{u} = 0$$

NOTE THAT SINCE $\nabla \times \nabla \phi = 0$ THAT MEANS THAT ALL IRROTATIONAL VECTOR FIELD CAN BE DERIVED

FROM A POTENTIAL ϕ, HENCE ARE CONSERVATIVE

SCALAR POTENTIAL

CONSERVATIVE MEANS THAT INTEGRALS BETWEEN TWO POINTS ARE PATH INDEPENDENT



$$\int_{\Gamma_1} \underline{a} \cdot d\underline{x} = \int_{\Gamma_2} \underline{a} \cdot d\underline{x}$$

(WE WILL SEE WHY IN A SEC)

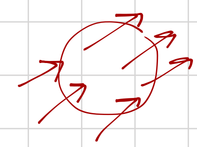
INTEGRAL THEOREMS

DIVERGENCE THEOREM

LET Ω BE A OPEN DOMAIN BOUNDED BY Γ

THEN FOR $\underline{a} \in \mathbb{R}^n$, $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$

$$\int_\Omega \underline{\nabla} \cdot \underline{a} \, d\Omega = \int_\Gamma \underline{a} \cdot \underline{n} \, d\Gamma$$ ⟵ FLUX



$$\int_\Omega \underline{\nabla} \cdot \underline{\underline{A}} \, d\Omega = \int_\Gamma \underline{\underline{A}} \, \underline{n} \, d\Gamma$$

GREEN GAUSS OSTROGRADSKII

LET $\underline{\underline{A}} = \phi \underline{\underline{I}}$

$$\int_\Omega \underline{\nabla} \phi \, d\Omega = \int_\Omega \underline{\nabla} \cdot (\phi \underline{\underline{I}}) \, d\Omega = \int_\Gamma \phi \underline{n} \, d\Gamma$$

$$\underline{\nabla} \cdot (\phi \underline{\underline{I}}) = \frac{d}{dx_i} (\phi \delta_{jk} \underline{e}_j \otimes \underline{e}_k) \cdot \underline{e}_i = \frac{d\phi_i}{dx_i} \underline{e}_i$$

STOKE'S THEOREM

LET Γ BE A CLOSED CURVE

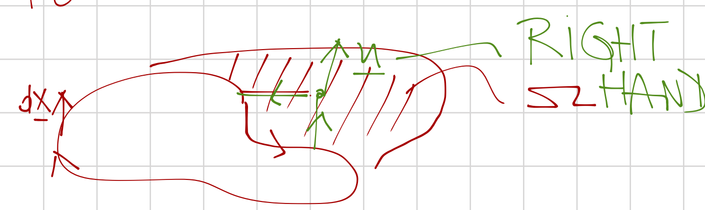
LET Ω BE AN OPEN SURFACE BOUNDED BY Γ

(THINK ABOUT BUBBLE BLOWER)

LET $d\underline{x}$ BE THE TANGENT TO Γ

LET $\underline{n}$ BE THE NORMAL TO Ω

$$\oint_\Gamma \underline{a} \cdot d\underline{x} = \int_\Omega \underline{\nabla} \times \underline{a} \cdot \underline{n} \, d\Omega$$



GRADIENT OF TENSOR VALUED FUNCTIONS

$$\phi(\underline{\underline{A}})$$

$$\frac{d\phi}{d\underline{\underline{A}}} = \frac{d\phi}{dA_{ij}} \underline{e}_i \otimes \underline{e}_j$$

$$\frac{d\underline{a}}{d\underline{\underline{A}}} = \frac{da_i}{dA_{jk}} \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k$$

$$\frac{d\underline{\underline{B}}}{d\underline{\underline{A}}} = \frac{dB_{ij}}{dA_{ke}} \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_e$$

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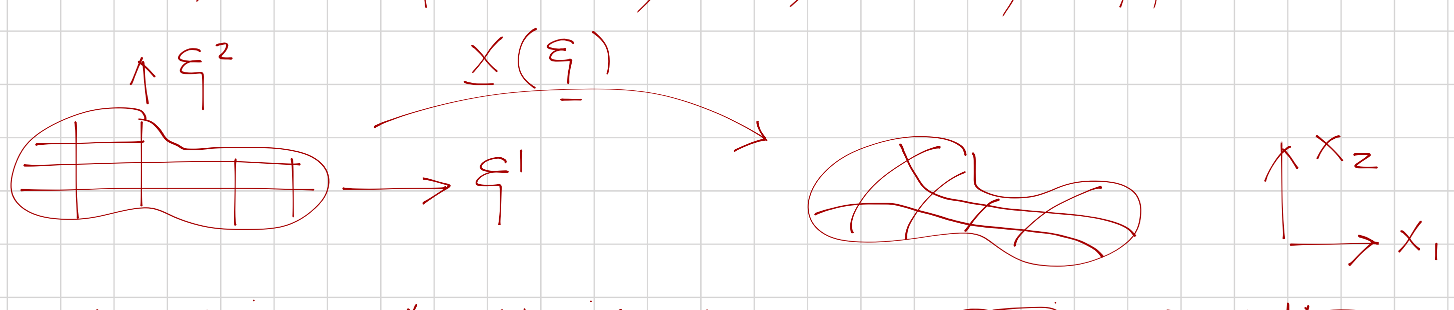
CURVILINEAR COORDINATES

OFTEN THE CASE THAT EXPRESSIN QUANTITIES IN DIFFERE COORDS CAN BE ADVAN.

EG. POLAR COORD, GENERALIZED COORD ...

$$\phi(x_1, x_2) = \hat{\phi}(r(x_1, x_2), \theta(x_1, x_2))$$



THERE EXIST A UNIQUE INVERTIBLE MAP BETWEEN THE TWO COORDINATES

$$x_i = \hat{x}_i(\xi^j)$$

$$\xi^i = \hat{\xi}^i(x_j)$$

LET $\underline{e}_i$ BE THE STANDARD CARTESIAN BASIS

THEN WE DEFINE

$$\underline{g}_i = \frac{dx_i}{d\xi^i} \underline{e}_i$$

AS THE NATURAL OR INDUCED BASIS.

NOTE THAT ξ^i IS A BIJECTION (ONE-TO-ONE ONTO - AKE INJECTION + SURJECTION) HENCE $\underline{g}_i$ WILL BE A BASIS

A VECTOR CAN THEN BE WRITTEN AS

$$\underline{a} = a^i \underline{g}_i$$

<u>EG</u>:

LET ξ^i BE THE POLAR COORDINATES r, θ S.T. $\xi^1 = r$, $\xi^2 = \theta$

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta$$

$$\underline{g}_1 = \frac{d}{d\xi^1}(x_i \underline{e}_i) = \frac{d}{dr}(x_i \underline{e}_i) = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2$$

$$\underline{g}_2 = \frac{d}{d\xi^2}(x_i \underline{e}_2) = \frac{d}{d\theta}(x_i \underline{e}_i) = -r\sin\theta\, \underline{e}_1 + r\cos\theta\, \underline{e}_2$$

<u>DUAL BASIS</u>

$$\underline{g}^i \cdot \underline{g}_j = \delta^i{}_j$$

$$\underline{g}^i = \frac{d\xi^i}{dx_j} \underline{e}_j \qquad \left(\underline{g}^i \cdot \underline{g}_j = \frac{d\xi^i}{dx_k} \underline{e}_k \cdot \frac{dx_p}{d\xi^j} \underline{e}_p = \right.$$

$$= \frac{d\xi^i}{dx_k} \frac{dx_k}{d\xi^j} = \frac{d\xi^i}{d\xi^j} = \delta^i{}_j$$

EG

$$\underline{g}^1 = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2$$

$$\underline{g}^2 = \frac{1}{r}\left(-\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2\right)$$

GRADIENT

$$\phi(\xi^i)$$

$$\underline{\nabla}\phi = \frac{d\phi}{dx_i}\underline{e}_i = \frac{d\phi}{d\xi^i}\frac{d\xi^i}{dx_j}\underline{e}_j = \frac{d\phi}{d\xi^i}\underline{g}^i$$

IF r, θ

$$\frac{d\phi}{dr}\underline{g}^1 + \frac{d\phi}{d\theta}\underline{g}^2$$

LET $\underline{e}_r = \frac{\underline{g}^1}{\|\underline{g}_1\|}$, $\underline{e}_\theta = \frac{\underline{g}^2}{\|\underline{g}_2\|} = r\,\underline{g}^2$

$$\underline{\nabla}\phi = \frac{d\phi}{dr}\underline{e}_r + \frac{1}{r}\frac{d\phi}{d\theta}\underline{e}_\theta$$