Finite Elements in n-D

The Local Element View

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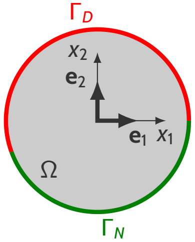
FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$



FEM Roadmap

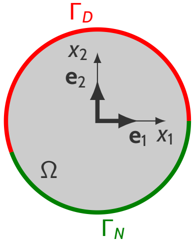
$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find u s.t.

$$\alpha \Delta u = f \quad \forall \boldsymbol{x} \in \Omega$$

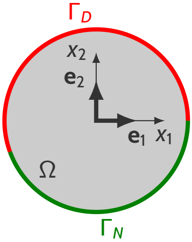
with

$$\begin{aligned} u &= g, \quad \forall \boldsymbol{x} \in \Gamma_D, \\ \alpha \nabla u \cdot \boldsymbol{n} &= h, \quad \forall \boldsymbol{x} \in \Gamma_N. \end{aligned}$$



FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$



Find $u \in \mathcal{S}$ s.t.

$$a(u, v) = F(v) \quad \forall v \in \mathcal{V}$$

where

$$\mathcal{S} = \{u | u \in H^1(\Omega), u = g \,\forall x \in \Gamma_D\},$$
$$\mathcal{V} = \{v | v \in H^1(\Omega), v = 0 \,\forall x \in \Gamma_D\},$$

and

$$a(u, v) = \int_\Omega \alpha \nabla u \cdot \nabla v \, d\Omega, \quad F(v) = \int_{\Gamma_N} hv \, d\Gamma - \int_\Omega fv d\Omega$$

FEM Roadmap

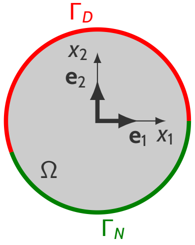
$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Choose a finite number of basis functions

$$\Phi^h = \{\phi_1, \dots, \phi_N\}$$

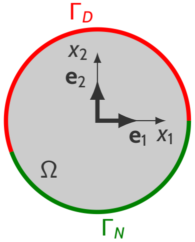
such that

$$u(x) \approx u^h(x) = \sum_{i=1}^{N} u_i \phi_i(x)$$



FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$



Find $u^h \in \mathcal{S}^h$ s.t.

$$a(u^h, v^h) = F(v^h) \quad \forall v^h \in \mathcal{V}^h$$

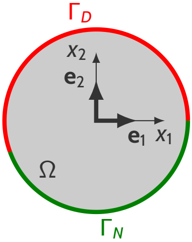
where

$$\mathcal{S} \supseteq \mathcal{S}^h = \{u^h | u^h \in H^1(\Omega), u^h(x) = \sum_{i=1}^{N} u_i \phi_i(x), u^h = g \, \forall x \in \Gamma_D\},$$

$$\mathcal{V} \supseteq \mathcal{V}^h = \{v | v \in H^1(\Omega), \; v^h(x) = \sum_{i=1}^{N} v_i \phi_i(x), \; v^h = 0 \, \forall x \in \Gamma_D\}.$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$



Find u_j s.t.

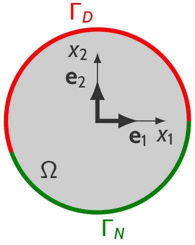
$$\sum_{j=1}^{N} a(\phi_i, \phi_j)u_j = F(\phi_i) \quad \forall i = 1 \dots N$$

where

$$a(\phi_i, \phi_j) = \int_\Omega \alpha \nabla\phi_i \nabla\phi_j d\Omega, \quad F(\phi_i) = \int_{\Gamma_N} h\phi_i d\Gamma - \int_\Omega f\phi_i d\Omega$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$



Find u_j s.t.

$$[K]\{U\} = \{F\}$$

where

$$[K]_{ij} = a(\phi_i, \phi_j), \quad \{U\}_j = u_j, \quad \{F\}_i = F(\phi_i)$$

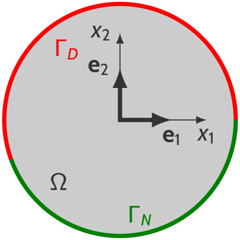
where

$$a(\phi_i, \phi_j) = \int_\Omega \alpha \nabla \phi_i \nabla \phi_j d\Omega, \quad F(\phi_i) = \int_{\Gamma_N} h\phi_i d\Gamma - \int_\Omega f\phi_i d\Omega$$

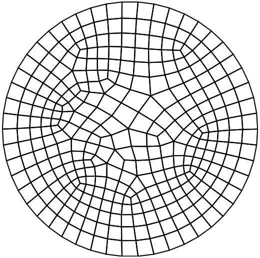
Constructing the Finite Dimensional Space

1. Subdivide the domain Ω in small element subdomains Ω^e ($\Omega = \bigcup_{e=1}^{n_{el}} \Omega^e$)
2. Over each element domain Ω^e construct basis functions w/ dof
3. Over each element domain Ω^e compute the local element arrays $[K^e], \{f^e\}$
4. With a local to global dof map assemble global arrays $[K], \{f\}$

Subdividing the Domain - The Finite Element Mesh



Quadrilateral (hexahedral) mesh



Triangular (simplicial) mesh