MID-TERM EXAM

CEE 361: Introduction to Finite Element Methods

Thurday Oct. 19

1. The exam is closed book and closed notes.
2. The exam will start at 11:00am and end sharply at 12:20pm. If you continue writing past that, we will not grade your exam and you will not receive any credit.
3. Make sure you justify all your answers.
4. Should you have any questions please come outside of the classroom where we will be standing. Once you ask your question we will come back into the classroom and the question will be answered to the entire class.

PROBLEM 1: Some Tensor Algebra and Calculus (20 pts)

For the following assume that $\{\mathbf{e}_i\}_{i=1}^d$ is a set of orthonormal basis associated with cartesian coordinates $\{x_i\}_{i=1}^d$. For each question provide a detailed justification (i.e. do not simply answer yes or no; if you simply say yes or no you will not receive any credit).

1. (4 pts) Let $d = 2$. $u = \exp(x_1)\sin(x_2)$. Find ∇u and $\nabla \cdot (\nabla u)$.
2. (4 pts) Let $\boldsymbol{x}_a = -1\mathbf{e}_1 - 1\mathbf{e}_2$ and $\boldsymbol{x}_b = 2\mathbf{e}_1 + 1\mathbf{e}_2$. Find the projection tensor that projects vectors along the direction $\boldsymbol{a} = \boldsymbol{x}_b - \boldsymbol{x}_a$.
3. (4 pts) Are $\boldsymbol{a} = 3\mathbf{e}_1 + 5\mathbf{e}_2, \boldsymbol{b} = 6\mathbf{e}_1 + 1\mathbf{e}_3$ linearly independent? (i.e. does α exists such that $\alpha\boldsymbol{a} = \boldsymbol{b}$?)
4. (4 pts) Let $d = 2$. Construct a tensor $\boldsymbol{T}$ that rotates a vector by $\pi/2$.
5. (4 pts) Is $\boldsymbol{T}(\boldsymbol{u}) = 2(\boldsymbol{u}\cdot\mathbf{e}_1)\mathbf{e}_2 + 4(\boldsymbol{u}\cdot\mathbf{e}_2)\mathbf{e}_1$ a tensor?

PROBLEM 2:

1. Solution 1:

$$
\begin{aligned}
u &= exp(x_1)sin(x_2) \\
\nabla u &= \frac{du}{dx_i}\mathbf{e}_i \\
&= exp(x_1)sin(x_2)\mathbf{e}_1 + exp(x_1)cos(x_2)\mathbf{e}_2 \\
\\
\nabla\cdot(\nabla u) &= \frac{d(\nabla u)}{dx_i}\cdot\mathbf{e}_i \\
&= exp(x_1)sin(x_2) - exp(x_1)sin(x_2) \\
&= 0
\end{aligned}
$$

2. Solution 2:

$$
\begin{aligned}
\boldsymbol{a} &= \boldsymbol{x}_b - \boldsymbol{x}_a \\
&= 3\mathbf{e}_1 + 2\mathbf{e}_2 \\
\boldsymbol{n} &= \frac{\boldsymbol{a}}{||\boldsymbol{a}||} \\
&= \frac{3\mathbf{e}_1 + 2\mathbf{e}_2}{\sqrt{13}} \\
\boldsymbol{T} &= \boldsymbol{n} \otimes \boldsymbol{n} \\
&= \frac{9\mathbf{e}_1 \otimes \mathbf{e}_1 + 6\mathbf{e}_1 \otimes \mathbf{e}_2 + 6\mathbf{e}_2 \otimes \mathbf{e}_1 + 4\mathbf{e}_2 \otimes \mathbf{e}_2}{13}
\end{aligned}
$$

3. Solution 3:

$$
\begin{aligned}
\boldsymbol{a} &= 3\mathbf{e}_1 + 5\mathbf{e}_2 \\
\boldsymbol{b} &= 6\mathbf{e}_1 + 1\mathbf{e}_3
\end{aligned}
$$

Let α be a non-zero real number such that:

$$\alpha \boldsymbol{a} = \boldsymbol{b}$$

Therefore:

$$(3\alpha - 6)\mathbf{e}_1 + 5\alpha\mathbf{e}_2 - 1\mathbf{e}_3 = 0$$

This is not true for any α hence the vectors are lineary independent

4. Solution 4:

$$\boldsymbol{T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

5. Solution 5:

$$
\begin{aligned}
\boldsymbol{T}(\boldsymbol{u}) &= 2(\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_2 + 4(\boldsymbol{u} \cdot \mathbf{e}_2)\mathbf{e}_1 \\
\boldsymbol{T}(\alpha\boldsymbol{u}) &= 2(\alpha\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_2 + 4(\alpha\boldsymbol{u} \cdot \mathbf{e}_2)\mathbf{e}_1 \\
&= \alpha\boldsymbol{T}(\boldsymbol{u}) \\
\boldsymbol{T}(\boldsymbol{u} + \boldsymbol{v}) &= 2((\boldsymbol{u} + \boldsymbol{v}) \cdot \mathbf{e}_1)\mathbf{e}_2 + 4((\boldsymbol{u} + \boldsymbol{v}) \cdot \mathbf{e}_2)\mathbf{e}_1 \\
&= 2(\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_2 + 4(\boldsymbol{u} \cdot \mathbf{e}_2)\mathbf{e}_1 + 2(\boldsymbol{v} \cdot \mathbf{e}_1)\mathbf{e}_2 + 4(\boldsymbol{v} \cdot \mathbf{e}_2)\mathbf{e}_1 \\
&= \boldsymbol{T}(\boldsymbol{u}) + \boldsymbol{T}(\boldsymbol{v})
\end{aligned}
$$

Hence $\boldsymbol{T}(\boldsymbol{u})$ is a tensor.

PROBLEM 3:

1. Solution 1:

element	i node	j node
1	1	2
2	2	3
3	3	4
4	4	1
5	1	3

Table 1: Connectivity Array

2. Solution 2:

$$\begin{aligned} \boldsymbol{q}_1 &= [0, 0] \\ \boldsymbol{q}_3 &= [w, h] \\ &= [3, 5] \\ \boldsymbol{n} &= \frac{[3, 5]}{\sqrt{34}} \\ \boldsymbol{k}^e &= \frac{A^e E^e}{\ell^e} \boldsymbol{n} \otimes \boldsymbol{n} \end{aligned}$$

`Substituting the values we obtain:`

$$\boldsymbol{k}^e = \frac{AE}{34\sqrt{34}} \begin{bmatrix} 9 & 15 \\ 15 & 25 \end{bmatrix}$$

3. Solution 3:

`Solution :`

$$\begin{bmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \boldsymbol{P}_3 \\ \boldsymbol{P}_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{k}^1 + \boldsymbol{k}^4 + \boldsymbol{k}^5 & -\boldsymbol{k}^1 & -\boldsymbol{k}^5 & -\boldsymbol{k}^4 \\ -\boldsymbol{k}^1 & \boldsymbol{k}^1 + \boldsymbol{k}^2 & -\boldsymbol{k}^2 & \boldsymbol{O} \\ -\boldsymbol{k}^5 & -\boldsymbol{k}^2 & \boldsymbol{k}^2 + \boldsymbol{k}^3 + \boldsymbol{k}^5 & -\boldsymbol{k}^3 \\ -\boldsymbol{k}^4 & \boldsymbol{O} & -\boldsymbol{k}^3 & \boldsymbol{k}^3 + \boldsymbol{k}^4 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \end{bmatrix}$$

4. Solution 4:

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Solution :
Now,
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$$\boldsymbol{k}^1 = \frac{AE}{3}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

$$\boldsymbol{k}^2 = \frac{AE}{5}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}\boldsymbol{P}_1\\ \begin{bmatrix}P_{2x}\\ 0\end{bmatrix}\\ \boldsymbol{0}\\ \boldsymbol{P}_4\\ 0\end{bmatrix} = \begin{bmatrix}\boldsymbol{k}^1+\boldsymbol{k}^4+\boldsymbol{k}^5 & -\boldsymbol{k}^1 & -\boldsymbol{k}^5 & -\boldsymbol{k}^4 & -\boldsymbol{m}_s\\ -\frac{AE}{3}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}\frac{AE}{3} & 0\\ 0 & 1\end{bmatrix} & -\frac{AE}{5}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \boldsymbol{O} & \boldsymbol{0}\\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{0}\\ -\boldsymbol{k}^4 & \boldsymbol{O} & -\boldsymbol{k}^3 & \boldsymbol{k}^3+\boldsymbol{k}^4 & \boldsymbol{0}\\ \boldsymbol{m}_s^T & \boldsymbol{0}^T & \boldsymbol{0}^T & \boldsymbol{0}^T & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{u}_1\\ \begin{bmatrix}u_{2x}\\ u_{2y}\end{bmatrix}\\ \boldsymbol{u}_3\\ \boldsymbol{u}_4\\ \lambda\end{bmatrix}$$

5. Solution 5:

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Solution :
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$$[\boldsymbol{R}_1] = \begin{bmatrix}\boldsymbol{k}^1+\boldsymbol{k}^4+\boldsymbol{k}^5 & -\boldsymbol{k}^1 & -\boldsymbol{k}^5 & -\boldsymbol{k}^4 & -\boldsymbol{m}_s\end{bmatrix}\begin{bmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\\ \boldsymbol{u}_3\\ \boldsymbol{u}_4\\ \lambda\end{bmatrix}$$

6. Solution 6:

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Solution :
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$$[R_{2y}] = \begin{bmatrix}-\frac{AE}{3}\begin{bmatrix}0 & 0\end{bmatrix} & \begin{bmatrix}0 & \frac{AE}{5}\end{bmatrix} & -\frac{AE}{5}\begin{bmatrix}0 & 1\end{bmatrix} & \begin{bmatrix}0 & 0\end{bmatrix} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{u}_1\\ \begin{matrix}u_2x\\ u_2y\end{matrix}\\ \boldsymbol{u}_3\\ \boldsymbol{u}_4\\ \lambda\end{bmatrix}$$

PROBLEM 4: Finite Element Methods (40 pts)

Consider the following strong form: with $f, k : [0, \ell] \to \mathbb{R}$ being given functions, h, g given constants, find $u : [0, \ell] \to \mathbb{R}$ such that

$$\frac{d^2u}{dx^2} + uk = f, \quad \forall x \in (0, \ell)$$

with

$$u(0) = g, \quad \frac{du}{dx}(\ell) = h.$$

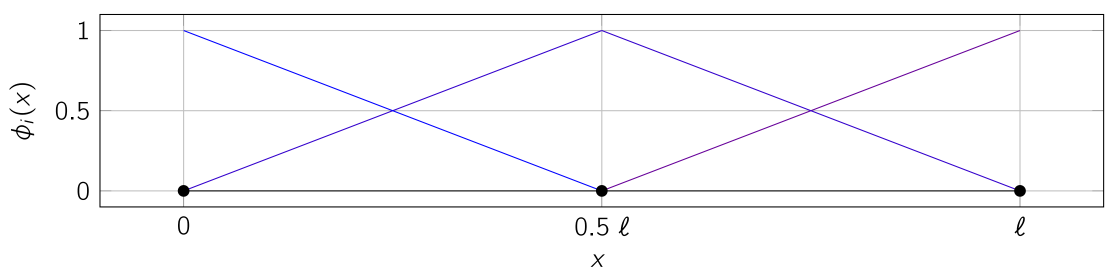
1. (5 pts) Identify the different types of boundary conditions (i.e. Neumann vs Dirichlet).
2. (5 pts) What is the set of trial functions $\mathcal{S}$?
3. (5 pts) What is the set of test functions $\mathcal{V}$?
4. (10 pts) Derive the weak formulation of the problem. You should arrive at an expression of the form $a(w, u) = F(w) \quad \forall w \in \mathcal{V}$ where a and F are to be defined accordingly.
5. (5 pts) Derive the matrix form of the problem.
6. (10 pts) Let us discretize the domain into two elements using linear basis functions (as we did in class) for a total of three degrees of freedom and three basis functions (see plot below) given by

$$\phi_1 = \begin{cases} (1 - 2x) & \text{if } x < 0.5\,\ell \\ 0 & \text{if } x \geq 0.5\,\ell \end{cases}$$

$$\phi_2 = \begin{cases} 2x & \text{if } x < 0.5\,\ell \\ 2(1 - x) & \text{if } x \geq 0.5\,\ell \end{cases}$$

$$\phi_3 = \begin{cases} 0 & \text{if } x < 0.5\,\ell \\ 2x - 1 & \text{if } x \geq 0.5\,\ell \end{cases}$$

   If $k(x) = x^2$ and $f(x) = x$, find the values of $F(\phi_1)$ and $a(\phi_2, \phi_3)$ in terms of g, h, ℓ.



PROBLEM 5:

1. Solution 1:

   i `Dirichlet :` $u(0) = g$

   ii `Neumann :` $\frac{du}{dx}(\ell) = h$

2. Solution 2:

   `The set of trial functions` $\mathcal{S}$:

$$\mathcal{S} = \{u | u \in \text{Smooth}, u(0) = g\}$$

3. Solution 3:

The set of test functions $\mathcal{V}$:

$$\mathcal{V} = \{w | w \in \text{Smooth}, w(0) = 0\}$$

4. Solution 4:

$$\frac{d^2 u}{dx^2} + uk = f$$

Multiplying both the sides by the weight w and integrating:

$$\begin{aligned}
\int_0^\ell \frac{d^2 u}{dx^2} w \ dx + \int_0^\ell uwk \ dx &= \int_0^\ell fw \ dx \\
\frac{du}{dx} w \bigg|_0^\ell - \int_0^\ell \frac{du}{dx}\frac{dw}{dx} \ dx + \int_0^\ell uwk \ dx &= \int_0^\ell fw \ dx \\
-\int_0^\ell \frac{du}{dx}\frac{dw}{dx} \ dx + \int_0^\ell uwk \ dx &= \int_0^\ell fw \ dx - hw(\ell)
\end{aligned}$$

5. Solution 5:

Galerking form:

$$\begin{aligned}
u^h &= v^h + g^h \\
v^h &= \sum_j v_j \phi_j \\
w^h &= \sum_i w_i \phi_i \\
g^h &= g\phi_1
\end{aligned}$$

Substituting in the weak form to obtain the matrix form:

$$-\sum_j \int_0^\ell \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} v_j \ dx + \sum_j \int_0^\ell \phi_i \phi_j k v_j \ dx = \int_0^\ell g \frac{d\phi_1}{dx}\frac{d\phi_i}{dx} - \int_0^\ell gk\phi_1\phi_i \ dx + \int_0^\ell f\phi_i \ dx - h\phi_i(\ell)$$

This gives us the matrix form:

$$K_{ij} v_j = F_i$$

where:

$$\begin{aligned}
K_{ij} &= -\int_0^\ell \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} \ dx + \int_0^\ell \phi_i \phi_j k \ dx \\
&= a(\phi_i, \phi_j) \\
F_i &= \int_0^\ell g \frac{d\phi_1}{dx}\frac{d\phi_i}{dx} - \int_0^\ell gk\phi_1\phi_i \ dx + \int_0^\ell f\phi_i \ dx - h\phi_i(\ell)
\end{aligned}$$

6. Solution 6:

Computing $F(\phi_1)$:

$$F(\phi_1) = \int_0^\ell g\frac{d\phi_1}{dx}\frac{d\phi_1}{dx} - \int_0^\ell gk\phi_1\phi_1 \ dx + \int_0^\ell f\phi_1 \ dx - h\phi_1(\ell)$$
$$= \int_0^{\ell/2} g\frac{d\phi_1}{dx}\frac{d\phi_1}{dx} - \int_0^{\ell/2} gk\phi_1\phi_1 \ dx + \int_0^{\ell/2} f\phi_1 \ dx$$

As,

$$\phi_1 = \begin{cases} (1-2x/\ell) & \text{if } \ x < 0.5\ell \\ 0 & \text{if } \ x \geq 0.5\ell \end{cases}$$

Substituing we obtain:

$$F(\phi_1) = \frac{2g}{\ell} - \frac{g\ell^3}{240} + \frac{\ell^2}{24}$$

Computing $a(\phi_2, \phi_3)$:

$$a(\phi_2, \phi_3) = -\int_0^\ell \frac{d\phi_2}{dx}\frac{d\phi_3}{dx} \ dx + \int_0^\ell \phi_2\phi_3 k \ dx$$

As,

$$\phi_2 = \begin{cases} 2x/\ell & \text{if } \ x < 0.5\ell \\ 2(1-x/\ell) & \text{if } \ x \geq 0.5\ell \end{cases}$$
$$\phi_3 = \begin{cases} 0 & \text{if } \ x < 0.5\ell \\ 2x/\ell - 1 & \text{if } \ x \geq 0.5\ell \end{cases}$$

Substituting we obtain:

$$a(\phi_2, \phi_3) = \frac{2}{\ell} + \frac{23\ell^3}{960}$$