HOMEWORK 1

CEE 361-513: Introduction to Finite Element Methods

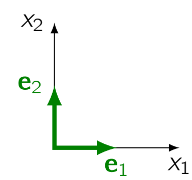
Due: Friday Sept. 29

NB: Students taking CEE 513 must complete all problems. All other students will not be graded for problems marked with $\star$, but are encourage to attempt them anyhow.

PROBLEM 1

Unless otherwise specified, you may assume that $\{\mathbf{e}_i\}_{i=1}^d$ is a set of orthonormal basis associated with a set of cartesian coordinates $\{x_i\}_{i=1}^d$ (cf. the figure on the right). Use indicial notation when appropriate.

1. Show that for two vectors $\boldsymbol{a}, \boldsymbol{b}$ the following holds $\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = 0$.
2. Let $d = 3$ and $\boldsymbol{u}(\boldsymbol{x}) = x_1x_2x_3\mathbf{e}_1 + x_1\mathbf{e}_2 + x_1\mathbf{e}_3$ compute $\nabla\boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u}$.
3. Let $d = 2$, $\boldsymbol{u}(\boldsymbol{x}) = x_1x_2\mathbf{e}_1 + x_1\mathbf{e}_2$, and $\boldsymbol{v}(\boldsymbol{x}) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. If $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \boldsymbol{u} \otimes \boldsymbol{v}$, what are the values of T_{ij}.
4. What is the value of $\boldsymbol{I} : \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity tensor.
5. Let $\boldsymbol{u}$ be a vector. Is $\boldsymbol{T}(\boldsymbol{u}) = \exp(\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_1$ a tensor? Show why or why not.
6. Let $\boldsymbol{u}$ be a vector. Is $\boldsymbol{T}(\boldsymbol{u}) = 10\,(\boldsymbol{u} \cdot \mathbf{e}_2)\mathbf{e}_1 + (\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_2$? Show why or why not.
7. $\star$ Show that $(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot (\boldsymbol{A}) = \boldsymbol{u} \otimes \boldsymbol{A}^\top\boldsymbol{v}$.
8. $\star$ Show that $\nabla \cdot (\psi\boldsymbol{u}) = \nabla\psi \cdot \boldsymbol{u} + \psi\nabla \cdot \boldsymbol{u}$ for $\boldsymbol{u} \in \mathbb{R}^d, \psi \in \mathbb{R}$.
9. $\star$ Show that $\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = \nabla\boldsymbol{u}\,\boldsymbol{v} + \boldsymbol{u}\nabla \cdot \boldsymbol{v}$.



PROBLEM 2

To practice with `Python` do the following operations

1. Let $\boldsymbol{u} = 1\mathbf{e}_1 + 2\mathbf{e}_2$. Construct a *unit* vector $\boldsymbol{n}$ such that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$. (Hint: create any vector $\boldsymbol{v}$ that is not linearly dependent with $\boldsymbol{u}$, then let $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{u}/\|\boldsymbol{u}\|\,\boldsymbol{u}$ and then let $\boldsymbol{n} = \boldsymbol{w}/\|\boldsymbol{w}\|$).
2. Let $\boldsymbol{u} = 3\mathbf{e}_1 + 2\mathbf{e}_2 + 4\mathbf{e}_3$, $\boldsymbol{v} = 5\mathbf{e}_1 + 1\mathbf{e}_2 + 4\mathbf{e}_3$. Construct a *unit* vector $\boldsymbol{n}$ that is orthogonal to $\boldsymbol{u}, \boldsymbol{v}$. (Hint: $\times$)
3. Given two points $\boldsymbol{x}_a = 1\mathbf{e}_1 + 2\mathbf{e}_2$, $\boldsymbol{x}_b = 5\mathbf{e}_1 + 7\mathbf{e}_2$, construct a tensor $\boldsymbol{T}$ that projects vectors along the direction of $\boldsymbol{a} = \boldsymbol{x}_b - \boldsymbol{x}_a$. Remember that a projection must satisfy $\boldsymbol{T}(\boldsymbol{T}(\boldsymbol{b}))$ for all vectors $\boldsymbol{b}$.
4. $\star$ Given a function $f(\boldsymbol{x}) = sin(x_1)e_2^x$ derive ∇f and plot the vector field

PROBLEM 3

1. Let u, v be sufficiently smooth functions of x. Show step-by-step that

$$\int_0^\ell \left[\frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right)\right] v\,dx = \int_0^\ell EI\frac{d^2u}{dx^2}\frac{d^2v}{dx^2}\,dx + \left[\frac{d}{dx}\left(EI\frac{d^2u}{dx^2}\right)v\right]\Bigg|_0^\ell - \left[EI\frac{d^2u}{dx^2}\frac{dv}{dx}\right]\Bigg|_0^\ell$$

where E, I are constants.

2. $\star$ Let $\boldsymbol{\sigma}(\boldsymbol{x}) \in \mathbb{R}^d \times \mathbb{R}^d, \boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$, and $\boldsymbol{\eta}(\boldsymbol{x}) \in \mathbb{R}^d$ (with both $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$ being integrable and sufficiently smooth), show that

$$\int_\Omega (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\eta} dV = \int_{\partial\Omega} \boldsymbol{\eta} \cdot \boldsymbol{\sigma} \boldsymbol{n} dS - \int_\Omega \boldsymbol{\sigma} : \nabla \boldsymbol{\eta} dV.$$