HOMEWORK 1

CEE 530: Continuum Mechanics and Thermodynamics

Due: April 30, 2018

PROBLEM 1

Show that the following identities are true for $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ being invertible second order tensors, and $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ being the identity tensor show

1. (5) Divergence:

$$\nabla \cdot (\boldsymbol{A}^\top \boldsymbol{a}) = (\nabla \cdot \boldsymbol{A}) \cdot \boldsymbol{a} + \boldsymbol{A} : \nabla \boldsymbol{a}$$

$$\nabla \cdot (\boldsymbol{A}^\top \boldsymbol{a}) = (A_{ij} a_i), j = A_{ij,j} a_i + A_{ij} a_{i,j}$$

2. (5) Trace:

$$\frac{\partial \operatorname{tr} \boldsymbol{A}}{\partial \boldsymbol{A}} = \boldsymbol{I}$$

$$\frac{\partial \boldsymbol{A} : \boldsymbol{I}}{\partial \boldsymbol{A}} = \boldsymbol{I} : \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} = \mathbb{I} : \boldsymbol{I} = \boldsymbol{I}$$

3. (5) Determinant:

$$\frac{\partial \det \boldsymbol{A}}{\partial \boldsymbol{A}} = \det \boldsymbol{A} \boldsymbol{A}^{-\top}$$

This is a lengthy one. See any continuum mechanics textbook.

PROBLEM 2

(30) Consider the surface given by the mapping

$$\begin{aligned} \boldsymbol{x}(\xi^1, \xi^2) = &(\alpha + \beta \cos(\xi^1)) \cos \xi^2 \mathbf{e}_1 \\ &+ (\alpha + \beta \cos(\xi^1)) \sin \xi^2 \mathbf{e}_2 \\ &+ \beta \sin \xi^1 \mathbf{e}_3 \end{aligned}$$

where $(\xi^1, \xi^2) \in [0, 2\pi] \times [0, 2\pi]$. Plot this surface.

Given a function

$$\phi(\xi^1, \xi^2) = \exp(\xi^1 \xi^2)$$

compute the $\nabla_{\boldsymbol{x}} \phi$ on the specified surface.

Hint: to construct the dual basis you can use this expression

$$\boldsymbol{g}^i = \frac{(\boldsymbol{g}_i \wedge \boldsymbol{g}_j) \cdot \boldsymbol{g}_j}{\boldsymbol{g}_i \cdot (\boldsymbol{g}_i \wedge \boldsymbol{g}_j) \cdot \boldsymbol{g}_j}, \quad i \neq j$$

where

$$\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a}.$$

Show why this is true!!

Also, I highly recommend using a symbolic software such as `Mathematica` or `Pythons`'s `SymPy` to compute the actual dual basis.

Recall

$$\nabla \phi = \frac{\partial \phi}{\partial \xi^i} \boldsymbol{g}^i$$

where $\boldsymbol{g}^i$ are the dual basis to

$$\boldsymbol{g}_1 = \frac{\partial \boldsymbol{x}}{\partial \xi^1} = -\beta \sin\left(\xi^1\right)\cos\left(\xi^2\right)\mathbf{e}_1 - \beta \sin\left(\xi^1\right)\sin\left(\xi^2\right)\mathbf{e}_2 \tag{1}$$

$$\boldsymbol{g}_1 = \frac{\partial \boldsymbol{x}}{\partial \xi^2} = -\left(\alpha + \beta \cos\left(\xi^1\right)\right)\sin\left(\xi^2\right)\mathbf{e}_1 + \left(\alpha + \beta \cos\left(\xi^1\right)\right)\cos\left(\xi^2\right)\mathbf{e}_2 + \beta \cos\left(\xi^2\right)\mathbf{e}_3. \tag{2}$$

And

$$\boldsymbol{g}^1 = -\frac{\cos\left(\xi^2\right)}{\beta \sin\left(\xi^1\right)}\mathbf{e}_1 - \frac{\sin\left(\xi^2\right)}{\beta \sin\left(\xi^1\right)}\mathbf{e}_2 \tag{3}$$

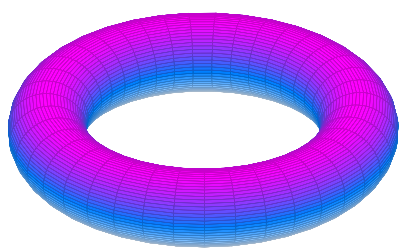
$$\boldsymbol{g}^2 = -\frac{\left(\alpha + \beta \cos\left(\xi^1\right)\right)\sin\left(\xi^2\right)}{\alpha^2 + 2\alpha\beta\cos\left(\xi^1\right) + \beta^2\cos^2\left(\xi^1\right) + \beta^2\cos^2\left(\xi^2\right)}\mathbf{e}_1 \tag{4}$$

$$+ \frac{\left(\alpha + \beta \cos\left(\xi^1\right)\right)\cos\left(\xi^2\right)}{\alpha^2 + 2\alpha\beta\cos\left(\xi^1\right) + \beta^2\cos^2\left(\xi^1\right) + \beta^2\cos^2\left(\xi^2\right)}\mathbf{e}_2 \tag{5}$$

$$+ \frac{\beta \cos\left(\xi^2\right)}{\alpha^2 + 2\alpha\beta\cos\left(\xi^1\right) + \beta^2\cos^2\left(\xi^1\right) + \beta^2\cos^2\left(\xi^2\right)}\mathbf{e}_3. \tag{6}$$

Then

$$\nabla \phi = \phi(\boldsymbol{g}^1 + \boldsymbol{g}^2).$$



PROBLEM 3

(30) In this problem we are interested in playing with the change of basis representation.

Consider a set of basis $\boldsymbol{g}_i$ that are expressed in terms of a set of some other basis $\boldsymbol{f}_i$ as $\boldsymbol{g}_i = \alpha_{ji}\mathbf{e}_j$. We can then write

$$\boldsymbol{a} = a_i \boldsymbol{g}_i = a_i \alpha_{ji} \mathbf{e}_j$$

which thus implies

$$[\boldsymbol{a}]_{\boldsymbol{g}} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}, \quad [\boldsymbol{a}]_{\mathrm{e}} = \begin{Bmatrix} a_i\alpha_{1i} \\ a_i\alpha_{2i} \\ a_i\alpha_{3i} \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = Q[\boldsymbol{a}]_{\boldsymbol{g}}.$$

Now for a tensor $\boldsymbol{T}$ we know

$$[\boldsymbol{T}]_{\mathrm{e}}[\boldsymbol{a}]_{\mathrm{e}} = [\boldsymbol{Ta}]_{\mathrm{e}} = Q[\boldsymbol{Ta}]_{\boldsymbol{g}} = Q[\boldsymbol{T}]_{\boldsymbol{g}}[\boldsymbol{a}]_{\boldsymbol{g}} = Q[\boldsymbol{T}]_{\boldsymbol{g}}Q^{-1}[\boldsymbol{a}]_{\mathrm{e}} \Rightarrow [\boldsymbol{T}]_{\mathrm{e}} = Q[\boldsymbol{T}]_{\boldsymbol{g}}Q^{-1}$$

Let $\boldsymbol{g}_i$ be a set of basis that can be written as

$$\boldsymbol{g}_1 = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 \tag{7}$$
$$\boldsymbol{g}_2 = \mathbf{e}_2 - \mathbf{e}_3 \tag{8}$$
$$\boldsymbol{g}_3 = 2\mathbf{e}_1 + \mathbf{e}_2 \tag{9}$$

where $\mathbf{e}_i$ are a set of *orthonormal* basis.

If

$$[\boldsymbol{T}]_{\boldsymbol{g}} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 4 \\ 3 & 4 & 3 \end{bmatrix}$$

is $\boldsymbol{T}$ a symmetric tensor?

$$Q = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$[\boldsymbol{T}]_{\mathrm{e}} = Q[\boldsymbol{T}]_{\boldsymbol{g}}Q^{-1} = \begin{bmatrix} 16.00 & -23.00 & -31.00 \\ 6.00 & -8.00 & -13.00 \\ 1.00 & -3.00 & -2.00 \end{bmatrix}$$

Since $[\boldsymbol{T}]_{\mathrm{e}}$ is not symmetric then $\boldsymbol{T}$ is not symmetric.

PROBLEM 4

Consider the square $[-1, 1] \times [-1, 1]$ to denote the reference configuration Ω_0 of a body. A series of equidistant lines parallel to the coordinate axes are drawn on Ω_0 , separated by a distance 0.2 and beginning at the square boundary. The body is deformed by the deformation mapping

$$\varphi(\boldsymbol{X}) = (X_1 + \alpha X_1 (X_2 - 1)(X_2 + 1))\boldsymbol{E}_1 + (X_2 - \alpha X_2(X_1 - 1)(X_1 + 1))\boldsymbol{E}_2$$

where $\{\boldsymbol{E}_i\}_{i=1,2}$ is the orthonormal basis parallel to the axes, and α is a real constant.

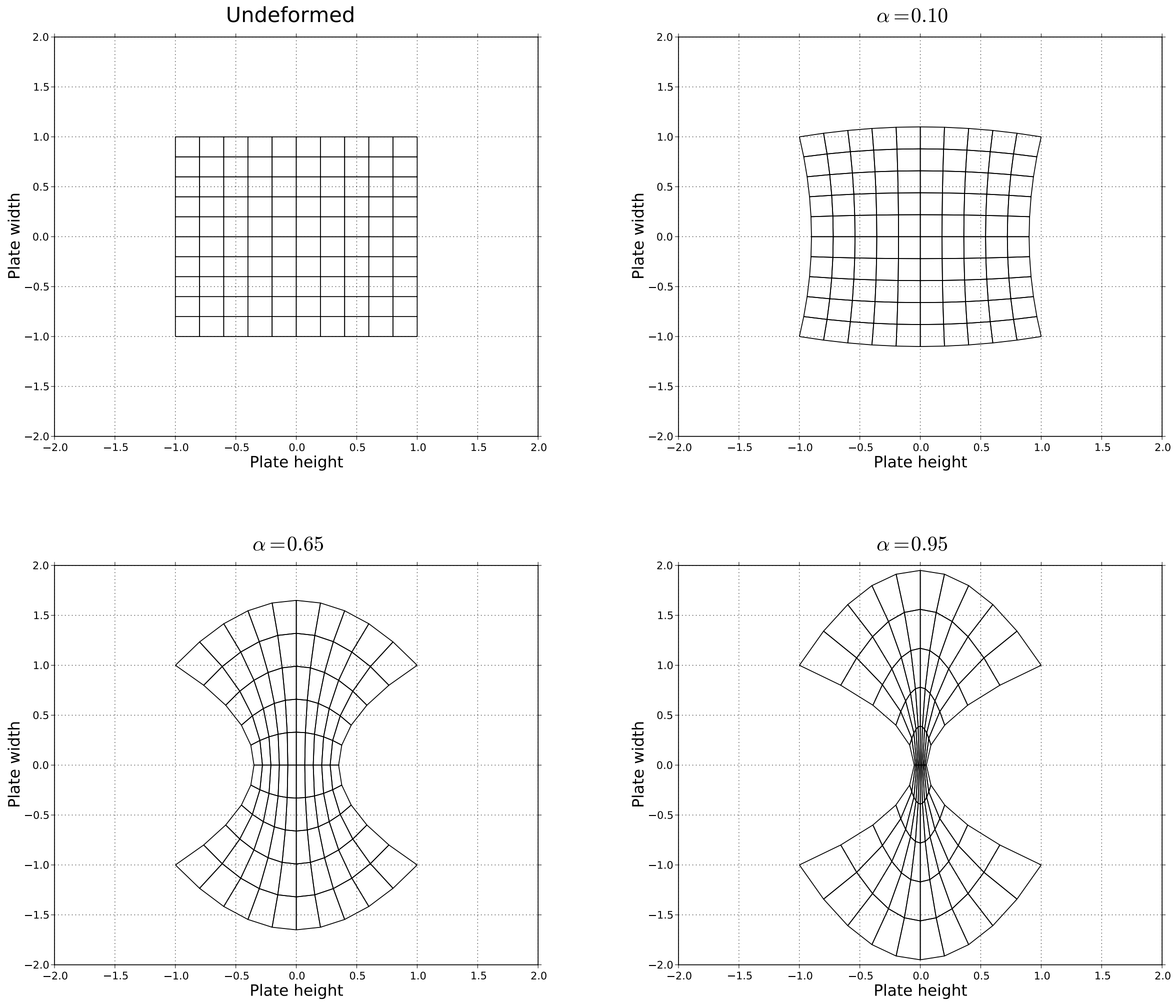
1. (10) Plot the deformed configuration of the body for $\alpha = 0.1, 0.6, 0.95$, including the deformed configuration of each one of the lines.
2. (10) What is the position in the reference configuration of the particle located at $\boldsymbol{x} = 0$ in the deformed configuration?
3. (10) For $\alpha = 0.1$, what is the position in the reference configuration of the particle located at $\boldsymbol{x} = 0.1(\boldsymbol{E}_1 + \boldsymbol{E}_2)$ in the deformed configuration?
4. (10) Is the deformation admissible for any $0 \leq \alpha < 1$ Justify (graphically). What happens at $\alpha = 1$?

Take $\mathcal{B}_0 = [-1, 1] \times [-1, 1]$ and $\varphi : \mathcal{B}_0 \to \mathbb{R}^2$ where

$$\varphi(\boldsymbol{X}) = (X_1 + \alpha X_1(X_2 - 1)(X_2 + 1))\mathbf{E}_1 + (X_2 - \alpha X_2(X_1 - 1)(X_1 + 1))\mathbf{E}_2$$

where $\mathbf{E}_i$ is a set of orthonormal basis for $\mathbb{R}^2$.

1. For $\alpha = 0.1, 0.6.095$ we get the following deformed shapes



2. For $\boldsymbol{x} = \mathbf{0}$ the reference position is $\boldsymbol{X} = \mathbf{0}$. In fact we can double check

$$\varphi(\mathbf{0}) = 0\mathbf{E}_1 + 0\mathbf{E}_2$$

Note that the above holds true only if $0 \leq |\alpha| < 1$, otherwise the motion is no longer injective and there would be no unique position in the reference configuration for the particle at $\boldsymbol{x} = \mathbf{0}$

3. The motion $\boldsymbol{\varphi}(\boldsymbol{X})$ gives us

$$\begin{cases} x_1 = X_1 + \alpha X_1(X_2 - 1)(X_2 + 1) \\ x_2 = X_2 - \alpha X_2(X_1 - 1)(X_1 + 1) \end{cases}$$

for $\alpha = 0.1$ and $\boldsymbol{x} = 0.1(\mathbf{E}_1 + \mathbf{E}_2)$ we get

$$\begin{cases} 0.1 = X_1 + 0.1X_1(X_2 - 1)(X_2 + 1) \\ 0.1 = X_2 - 0.1X_2(X_1 - 1)(X_1 + 1) \end{cases}$$

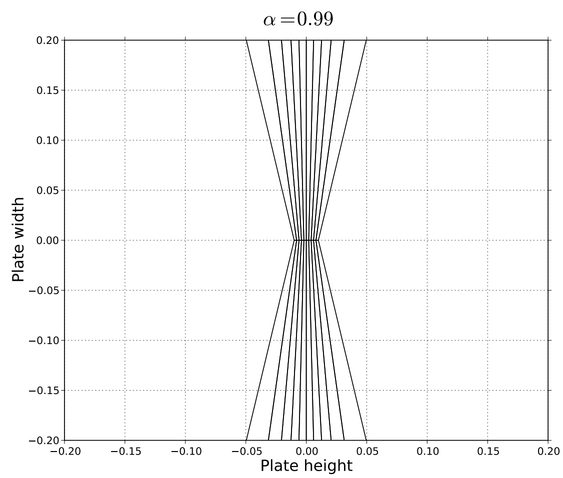
letting matlab solve for (X_1, X_2) we get

$$\boldsymbol{X} \approx 0.111\mathbf{E}_1 + 0.091\mathbf{E}_2$$

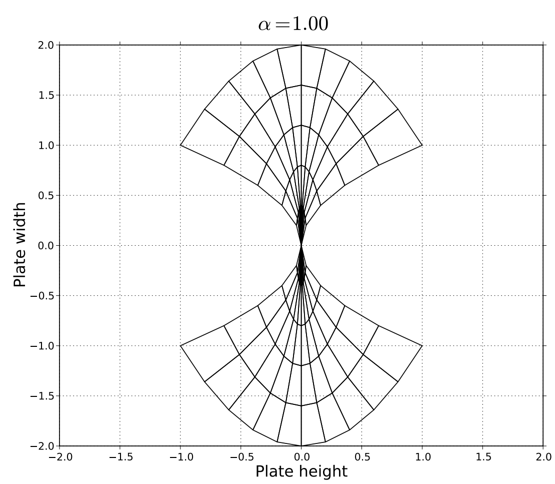
and to double check plug back into $\varphi(\boldsymbol{X})$ such that

$$\varphi(\boldsymbol{X}) = (0.111 + 0.1 \cdot 0.111(0.091^2 - 1))\mathbf{E}_1 + (0.091 - 0.1 \cdot 0.091(0.111^2 - 1))\mathbf{E}_2 = 0.1(\mathbf{E}_1 + \mathbf{E}_2) \quad \checkmark$$

4. Yes the deformation is admissible for any $0 \le \alpha < 1$ since the deformation mapping φ is defined for all points in $\mathcal{B}_0$, and the deformation mapping $\varphi(\boldsymbol{X})$ is injective and the deformation mapping $\varphi(\boldsymbol{X})$ is continuously differentiable. In fact even for $\alpha = 0.99$ all points remain distinct as shwon in the figure below which is a zoomed in on the necking at $\boldsymbol{x} = \mathbf{0}$



On the other hand, for $\alpha = 1$, all of the points of the form $\boldsymbol{X} = c\mathbf{E}_1, c \in [-1, 1]$ collapse to $\boldsymbol{x} = \mathbf{0}$, therefore the map is no longer injective and thus not admissible. See below!



PROBLEM 5

Consider the deformation mappings $\varphi_1 : \Omega_0 \to \Omega_1$ and $\varphi_2 : \Omega_1 \to \Omega_2$. The composition of the two is defined as the deformation mapping $\boldsymbol{\phi} : \Omega_0 \to \Omega_2$ such that $\boldsymbol{\phi} = \varphi_1 \circ \varphi_2$, or $\boldsymbol{\phi}(\boldsymbol{X}) := \varphi_2(\varphi_1(\boldsymbol{X}))$. Let $[\varphi_1(\boldsymbol{X})]_{\boldsymbol{E}} = (X_1, \lambda X_2, X_3)$ (uniaxial stretching) and $[\varphi_2(\boldsymbol{X})]_{\boldsymbol{E}} := (X_1 + \tan\alpha X_2, X_2, X_3)$ (simple shear), $\boldsymbol{X} \in \mathbb{R}^3$. Here, all vector components are with respect to an orthonormal basis $\boldsymbol{E}_i$ attached to a set of

Cartesian coordinate axes X_i. Consider the square of side a, $\Omega_0 = [0, a] \times [0, a] \times 0$ with respect to these axes. Hence, two basis vectors are parallel to the sides of the square, and one is orthogonal to its plane.

1. (10) Carry out the composition of mappings $\varphi_2 \circ \varphi_1$ and $\varphi_1 \circ \varphi_2$.
2. (10) Plot and compare the results. Does composition of mappings commute?

(If you are interested, you could also try an experiment, by composing two different rigid body rotations with any object within your reach ? Is the final position of the object the same after performing both compositions?)
Take the deformation

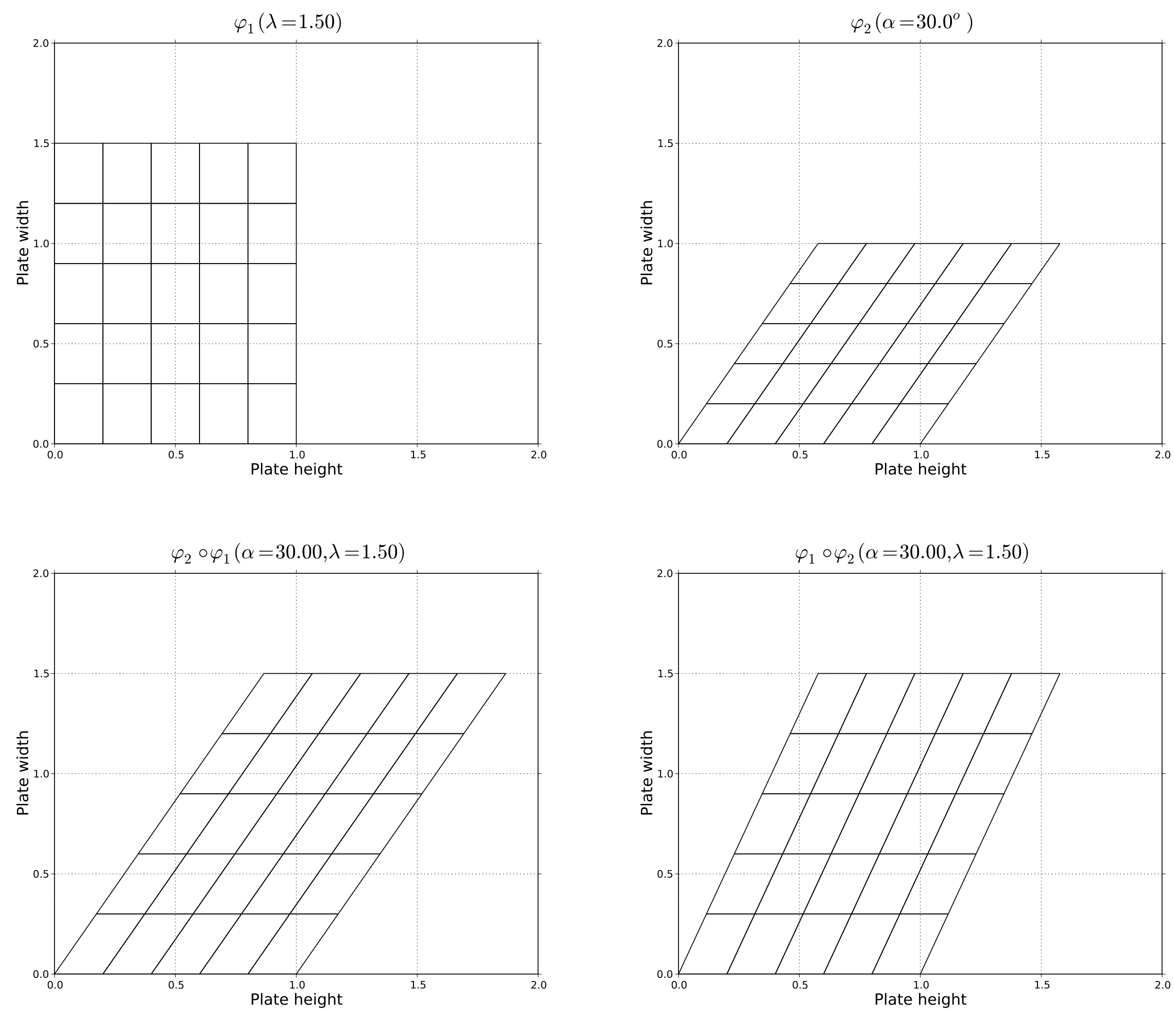
1. The composition $\varphi_2 \circ \varphi_1$ gives

$$\phi_{21}(\boldsymbol{X}) = \varphi_2(\varphi_1(\boldsymbol{X})) = \varphi_2(X_1, \lambda X_2, X_3) = (X_1 + \tan(\alpha)\lambda X_2, \lambda X_2, X_3)$$

   and the composition $\varphi_1 \circ \varphi_2$ gives

$$\phi_{12}(\boldsymbol{X}) = \varphi_1(\varphi_2(\boldsymbol{X})) = \varphi_1(X_1 + \tan(\alpha)X_2, X_2, X_3) = (X_1 + \tan(\alpha)X_2, \lambda X_2, X_3)$$

2. The plots for the deformations give us



   hence the composition of mappings does not commute when stretching take place.

PROBLEM 6

In a certain region the spatial velocity components of $\boldsymbol{v} = \boldsymbol{v}(x, t)$ in a Cartesian vector basis $\mathbf{e}_i$ are given as

$$v_1 = -\alpha\,(x_1^3 + x_1 x_2)\exp(-\beta t) \tag{10}$$
$$v_2 = \alpha\,(x_1^2 x_2 + x_2^3)\exp(-\beta t) \tag{11}$$
$$v_3 = 0 \tag{12}$$

where $\alpha, \beta > 0$ are given constants.

1. (15) Find the components of the spatial acceleration field $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, t)$ at point $[\boldsymbol{x}]_{\mathrm{e}} = (1, 0, 0)$ and time $t = 0$.
2. (15) Measurements of the pressure field throughout the continuum returned the following spatial field

$$p(\boldsymbol{x}, t) = (x_1 + x_2)^2.$$

What is the rate of change of pressure on a material particle at the same point and time?

1. Given the velocity field

$$\begin{aligned} v_1 &= -\alpha(x_1^3 + x_1 x_2^2)e^{-\beta t} \\ v_2 &= \alpha(x_1^2 x_2 + x_2^3)e^{-\beta t} \\ v_3 &= 0 \end{aligned}$$

The spatial acceleration field $\boldsymbol{a}(\boldsymbol{x}, t)$ will be given by

$$\boldsymbol{a}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{v}(\boldsymbol{x}, t)}{\partial t} + \nabla_{\boldsymbol{x}} \boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t)$$

where

$$[\nabla_{\boldsymbol{x}} \boldsymbol{v}(\boldsymbol{x}, t)] = \begin{bmatrix} \alpha(3x_1^2 + x_2^2)e^{-\beta t} & 2\,\alpha\, x_1 x_2 e^{-\beta t} & 0 \\ 2\,\alpha\, x_1 x_2 e^{-\beta t} & \alpha(x_1^2 + 3x_2^2)e^{-\beta t} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence

$$\nabla_{\boldsymbol{x}} \boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t) = (3\alpha^2 e^{-2\beta t} x_1 (x_1^2 + x_2^2)^2)\mathbf{e}_1 + (3\alpha^2 e^{-2\beta t} x_2 (x_1^2 + x_2^2)^2)\mathbf{e}_2$$

and

$$\frac{\partial \boldsymbol{v}(\boldsymbol{x}, t)}{\partial t} = \beta\alpha(x_1^3 + x_1 x_2^2)e^{-\beta t}\mathbf{e}_1 + \beta\alpha(x_1^2 x_2 + x_2^3)e^{-\beta t}\mathbf{e}_2$$

therefore

$$\begin{aligned} \boldsymbol{a}(\boldsymbol{x}, t) = \alpha e^{-2\beta t} x_1 (x_1^2 + x_2^2)(\beta e^{\beta t} + \alpha(3x_1^2 - x_2^2))\mathbf{e}_1 + \dots \\ \dots - \alpha e^{-2\beta t} x_2 (x_1^2 + x_2^2)(\beta e^{\beta t} + \alpha(x_1^2 - 3x_2^2))\mathbf{e}_2 \end{aligned}$$

and for $\boldsymbol{x} = \mathbf{e}_1$ and $t = 0$ it simplifies to

$$\boxed{\boldsymbol{a}(\boldsymbol{x}, t) = \alpha(3\alpha + \beta)} \tag{13}$$

2. Given

$$p(\boldsymbol{x}, t) = (x_1 + x_2)^2$$

the rate of change of the pressure of a give particle going through a fixed point at a given time will be given by the material time derivative hence

$$\frac{Dp(\boldsymbol{x}, t)}{Dt} = \frac{\partial p(\boldsymbol{x}, t)}{\partial t} + \nabla_{\boldsymbol{x}} p(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t)$$

where

$$\nabla_{\boldsymbol{x}} p(\boldsymbol{x}, t) = 2(x_1 + x_2)\mathbf{e}_1 + 2(x_1 + x_2)\mathbf{e}_2$$

and

$$\nabla_{\boldsymbol{x}} p(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t) = -2\alpha e^{-\beta t} \left(x_1^4 - x_2^4\right)$$

and note that

$$\frac{\partial p(\boldsymbol{x}, t)}{\partial t} = 0$$

therefore

$$\boxed{\frac{Dp(\boldsymbol{x}, t)}{Dt} = -2\alpha} \tag{14}$$