LECTURE 1:

INTRO TO VECTORS & TENSORS

- SCALARS, VECTORS, TENSORS
- VECTOR 'ALGEBRA' → (THE USE & THE STUDY OF MATHEMATICAL SYMBOLS & THE RULES FOR MANIPULATING THEM)
- TENSOR ALGEBRA
- INDEX NOTATION

~ ——

A $\boxed{\text{SCALAR}}$ IS SIMPLY A QUANTITY REPRESENTED BY A REAL NUMBERS.

SOME EXAMPLES INCLUDE:

- TEMPERATURE
- MASS
- SPEED

ETC..

ALGEBRA OF SCALARS IS TRIVIAL, WHAT WE LEARN EARLY ON: ADDITION, MULT, ETC...

A $\boxed{\text{VECTOR}}$ IS A QUANTITY THAT IS REPRESENTED BY A MAGNITUDE & A DIRECTION

SOME EXAMPLES INCLUDE

- FORCE
- VELOCITY
- ETC..

WE REPRESENT A VECTOR WITH AN UNDERLINE

EG.

$\underline{a}, \underline{b}, \underline{c}$ ..

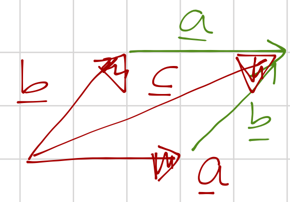
OFTE WE TALK ABOUT A VECTOR SPACE V AS A SET OF VECTORS.

SOME OPERATIONS HOLD AMONGST THE OBJECTS OF V

ADDITION

if $\underline{a}, \underline{b} \in V = \mathbb{R}^2$ ARE IN

$$\underline{c} = \underline{a} + \underline{b}$$



SCALAR MULTIPLICATION

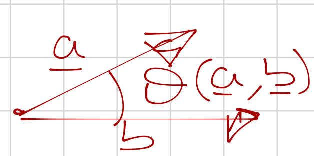
IT RESCALES THE VECTOR EG FOR $\underline{a} \in \mathbb{R}^2$
$\alpha \in \mathbb{R}, \alpha > 1$

$\alpha \underline{a}$ HAS THE SAME DIRECTION OF $\underline{a}$ BUT DIFFERENT MAGNITUDE

VECTOR MULTIPLICATIO - DOT PRODUCT

GIVE TWO VECTORS $\underline{a}, \underline{b} \in V$ THE DOT PRODUCT IS DEFINED AS

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta(\underline{a}, \underline{b})$$



WHERE θ REPRESENTS THE IN-PLANE ANGLE SUBDUED BY THE TWO VECTORS.

$|\underline{a}|$ & $|\underline{b}|$ DENOTE THE MAGNITUDE OR "LENGHT" OF $\underline{a}, \underline{b}$ RESPECTIVELY.

NOTE. LET $\underline{n} := \frac{1}{|\underline{b}|} \underline{b}$ THEN $|\underline{n}| = |\underline{b}| / |\underline{b}| = 1$

$$\underline{a} \cdot \underline{n} = |\underline{a}| \, |\underline{n}| \cos(\theta(\underline{a}, \underline{n})) = |\underline{a}| \cos \theta(\underline{a}, \underline{n})$$

$\underline{a}\cdot\underline{n}$ ← THE PROJECTION OF $\underline{a}$ ALONG THE DIRECTION OF $\underline{n}$.

OF COURSE IF $\underline{a}\perp\underline{b}$ THEN $\theta(\underline{a},\underline{b}) = \pi/2 = 90°$
AND THEREFORE

$$\underline{a}\cdot\underline{b}=0$$

BASIS

A VECTOR SPACE CAN BE DESCRIBED BY A SET OF BASIS.

A SET OF BASIS IS A SET OF VECTORS THAT "SPAN" THE ENTIRE SPACE, MEANING THAT ANY VECTOR CAN BE WRITTEN AS THE LINEAR COMBINATION OF BASIS.

FOR EXAMPLE, LET $\{\underline{q}_1, \underline{q}_2\}$ BE BASIS FOR V THEN ANY VECTOR $\underline{a}$ CAN BE WRITTEN AS

$$\underline{a} = a_1\underline{q}_1 + a_2\underline{q}_2$$

IF $\underline{q}_1$ IS PERPENDICULAR TO $\underline{q}_2$ THAN $\{\underline{q}_1, \underline{q}_2\}$ ARE A SET OF ORTHOGONAL BASIS.

IF THEY HAVE UNIT LENGHT (I.E. $|\underline{q}_i|=1$) THEY ARE CALLED ORTHONORMAL AND OFTEN DENOTED BY $\underline{e}_i$

FOR EXAMPLE, $\{\underline{e}_1, \underline{e}_2\}$ AS DRAWN BELOW ARE ORTHONORMAL BASIS OF $\mathbb{R}^2$ & SO ARE $\{\underline{e}_1^*, \underline{e}_2^*\}$

NOW NOTE THAT IF $\{\underline{e}_i\}_{i=1}^{2}$ ARE ORTHONORMAL

$\underline{e}_i \cdot \underline{e}_i = 1$: THE PROJECTION OF $\underline{e}_i$ ALONG $\underline{e}_i$ WHICH IS EXACTLY $|\underline{e}_i| = 1$

$\underline{e}_i \cdot \underline{e}_j = 0$ IF $i \neq j$ BECAUSE THEY ARE ORTHOGONAL

IN SHORT

$$\underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & \text{IF } i = j \\ 0 & \text{IF } i \neq j \end{cases} = \delta_{ij} \Leftarrow \text{KRONECKER DELTA}$$

INDICIAL NOTATION

WE KNOW THAT IF $\{\underline{e}_i\}$ IS A SET OF ORTHONORMAL BASIS THE WE CAN WRITE ANY VECTOR AS

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + \dots + a_n \underline{e}_n = \sum_{i=1}^{n} a_i \underline{e}_i$$

EINSTEIN'S SUMMATION CONVENTION

IF AN INDEX IS REPEATE EXACTLY TWICE IT IMPLIES A SUMMATION

$a_i \underline{e}_i$ REPEATED EXACTLY TWICE IMPLIES $\sum_{i=1}^{n} a_i \underline{e}_i$

INDICES THAT ARE REPEATED EXACTLY TWICE ARE CALLED DUMMY INDICES

THEY ARE CALLE DUMMY BECAUSE THEY CAN BE REPLACED WITH ANYTHING

$$a_i \underline{e}_i = a_j \underline{e}_j = a_k \underline{e}_k = \sum_{z=1}^{n} a_z \underline{e}_z$$

INDICES THAT ARE ONLY ONCE ARE FREE INDICES & CANNOT BE REPLACED

$$\underline{q}_j = a_i b_i \underline{e}_j = a_k b_k \underline{e}_j \quad (\text{THIS IMPLIES } \forall j = 1 \dots n)$$

NOTE

$$\underline{a} \cdot \underline{e}_1 = a_1 \underline{e}_1 \cdot \underline{e}_1 + a_2 \underline{e}_2 \cdot \underline{e}_1 + \dots + a_n \underline{e}_n \cdot \underline{e}_1 = a_1$$

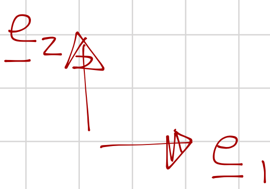
SIMILARLY

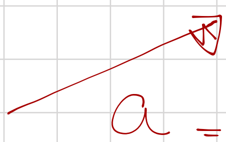
$$a_i = \underline{a} \cdot \underline{e}_i$$

WE CAN THEREFORE REPRESENT A VECTOR $\underline{a}$ IN AN ARRAY

$$[\underline{a}]_{\underline{e}} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \end{Bmatrix}$$

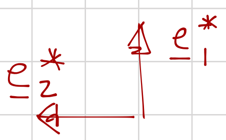
FOR EXAMPLE





$$\underline{a} = 2\underline{e}_1 + 1\underline{e}_2 \qquad [\underline{a}]_{\underline{e}} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

NOW LET



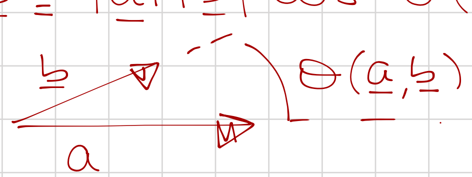
$$\underline{a} = \underline{e}_1 - 2\underline{e}_2^* \qquad [\underline{a}]_{\underline{e}^*} = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

SO A VECTOR IS MORE THAN JUST AN ARRAY !!

DOT PRODUCT REVISITED

YOU RECALL THAT FOR $\underline{a}, \underline{b} \in V$

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta(\underline{a}, \underline{b})$$



NOW, WITH $\{\underline{e}_i\}$ BEING ORTHONORMAL BASIS

$$\underline{a} = a_i \underline{e}_i, \quad \underline{b} = b_j \underline{e}_j$$

$$\underline{a} \cdot \underline{b} = (a_i \underline{e}_i) \cdot (b_j \underline{e}_j) = a_i b_j \, \underbrace{\underline{e}_i \cdot \underline{e}_j}_{} = a_i b_j \overbrace{\delta_{ij}}^{= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}} = a_i b_i = \sum_{i=1}^{n} a_i b_i$$

SO $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta(\underline{a}, \underline{b}) = a_i b_i$

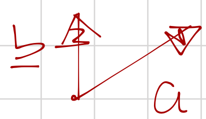
OR

$$\underline{a}\cdot\underline{b} = [\underline{a}]_{\underline{e}}^{T}\,[\underline{b}]_{\underline{e}} = \{a_1\ a_2\}\begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = a_1 b_1 + a_2 b_2$$

NOTE: $\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$!! SO THE DOT PRODUCT IS COMMUTATIVE

HOW WOULD YOU CONSTRUCT A UNIT NORMAL VECTOR TO $\underline{a}$?

LET $\underline{b}$ S.T. $\theta(\underline{a},\underline{b}) \neq 0$



$\underline{a}\cdot\underline{n} = 0 \qquad \underline{n} = \underline{b} - (\underline{b}\cdot\underline{a})\dfrac{\underline{a}}{\|\underline{a}\|^2}$ $\Leftarrow$ NORMAL BUT NOT UNIT MAGNITUDE

$\underline{m} = \dfrac{\underline{n}}{\|\underline{n}\|}$ $\Leftarrow$ UNIT NORMAL

$\underline{a}\cdot\underline{m} = 0 \quad \& \quad \|\underline{m}\| = 1$