LECTURE 13:

- SHORT REVIEW
- BALANCE OF MOMENTUM (LINEAR & ANGULAR)
- BALANCE OF MECHANICAL ENERGY

WE SAW THE MASTER BALANCE LAW

$$I(t) = \int_{\Omega} \phi(\underline{x}, t)\, dv$$

$$\frac{D}{Dt} I(t) = \frac{D}{Dt} \int_{\Omega} \phi(\underline{x}, t)\, dv = \int_{\Omega} \frac{D}{Dt} \phi(\underline{x}, t) + \phi(\underline{x}, t) \nabla \cdot \underline{v}\, dv$$

FOR A FIXED SET OF MATERIAL POINTS CONTAINED IN Ω @ TIME t

IF $\phi = \rho(\underline{x}, t)$ WE THEN KNOW

$$\frac{D}{Dt} M = \frac{D}{Dt} \int_{\Omega} \rho(\underline{x}, t)\, dv = 0 \Rightarrow \frac{D\rho}{Dt} + \rho \nabla \cdot \underline{v} = 0$$

$$\frac{d\rho}{dt} + \nabla \cdot (\rho \underline{v}) = 0 \qquad \forall \underline{x} \in \Omega$$

CONTINUITY EQUATION OR BALANCE OF MASS

BALANCE OF MOMENTUM

LET $\underline{L}(t)$ DENOTE THE LINEAR MOMENTUM

IF WE HAVE A DISCRETE SYSTEM OF PARTICLES

$$\underline{L}(t) = \sum_i m_i \underline{v}_i(t)$$

THEN, FOLLOWING THE SECON LAW

$$\frac{d\underline{L}}{dt} = \underline{f}$$

WHERE $\underline{f}$ REPRESENTS EXTERNAL FORCES APPLIED

IN THE CONTINUUM CASE FOR $\Theta_0 \subseteq \Omega_0$, $\underline{\varphi}(\Theta_0, t) = \Theta$

$$\underline{L}(t) = \int_\Theta \rho(\underline{x},t)\, \underline{v}(\underline{x},t)\, dv$$

$$\underline{f}(t) = \int_\Theta \underline{b}(\underline{x},t)\, dv + \int_{\partial\Theta} \underline{t}(\underline{x},t)\, ds$$

EG. GRAVITY

THE SECOND LAW THEN REQUIRES

$$\frac{D}{Dt}\underline{L}(t) = \underline{f}(t)$$

$$\frac{D}{Dt}\underline{L}(t) = \int_{\Theta_0} (\dot{\rho}\underline{v} + \rho\dot{\underline{v}}) J + \rho \underline{v}\, \nabla \cdot \underline{v} J\, dv_0 =$$

$$= \int_\Theta (\dot{\rho}\underline{v} + \rho\dot{\underline{v}} + \rho\underline{v}\, \nabla\cdot\underline{v})\, dv = \int_\Theta (\rho\dot{\underline{v}} + \underline{v}(\dot{\rho} + \rho\nabla\cdot\underline{v}))\, dv$$

$$= \int_\Theta \rho\dot{\underline{v}}\, dv$$

$$\Rightarrow \int_\Theta \rho\dot{\underline{v}} = \int_\Theta \underline{b}(\underline{x},t)\, dv + \int_{\partial\Theta} \underline{t}(\underline{x},t)\, ds$$

$\underline{\underline{T}}\underline{n}$

$$\int_\Theta \rho\dot{\underline{v}} = \int_\Theta \underline{b}\, dv + \int_{\partial\Theta} \underline{\underline{T}}\underline{n}\, ds = \int_\Theta \underline{b}\, dv + \int_\Theta \nabla\cdot\underline{\underline{T}}\, dv$$

BALANCE OF LINEAR MOMENTUM

$$\Rightarrow \rho\dot{\underline{v}} = \nabla\cdot\underline{\underline{T}} + \underline{b} \qquad \forall\, \underline{x} \in \Omega \qquad (*)$$

FLUX OF INTERNAL FORCES

RECALL

$$\frac{D}{Dt}\,\underline{L}(t) = \frac{D}{Dt}\int_{\theta} \rho \underline{v}\, dv = \frac{D}{Dt}\int_{\theta_0} \rho \underline{v} J\, dV_0 = \frac{D}{Dt}\int_{\theta_0} \rho_0 \underline{V}\, dV_0$$

$$\rho_0 = \rho J, \quad \frac{D}{Dt}\rho_0 = \frac{d\rho_0}{dt} = 0$$

$$= \int_{\theta_0} \frac{D}{Dt}(\rho_0 \underline{V}) = \int_{\theta_0} \rho_0 \underline{A}\, dV_0$$

SIMILARLY

$$\underline{f} = \int_{\theta} \underline{b}\, dv + \int_{\partial\theta} \underline{\underline{T}}\,\underline{n}\, ds = \int_{\theta_0} \underline{b} J\, dV_0 + \int_{\partial\theta_0} \underline{\underline{T}}\, J \underline{\underline{F}}^{-T} \underline{N}\, dS_0 =$$

$$= \int_{\theta_0} \underline{B}\, dV_0 + \int_{\theta_0} \underline{\underline{P}}\,\underline{N}\, dS_0 = \int_{\theta_0} \underline{B}\, dV_0 + \int_{\theta_0} \underline{\nabla}_X \cdot \underline{\underline{P}}\, dS_0$$

BALANCE OF MOMENTUM IN REF CONFIG

$$\Rightarrow \quad \rho_0 \underline{A} = \underline{B} + \underline{\nabla}_X \cdot \underline{\underline{P}} \qquad \forall\ \underline{X} \in \Omega_0 \qquad (**)$$

NOTE THAT WHILE (*) HOLDS TRUE FOR EVERY POINT IN SPACE (**) IS TRUE FOR EVERY MATERIAL POINT

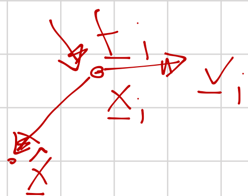
SINCE FOR FLUIDS $\underline{\underline{T}}$ OFTEN DEPENDS ONLY ON RATES OF DEF (*) IS THE EQ OF CHOICE

FOR SOLIDS ON THE OTHER HAND THE STRESS DEPENDS ON $\underline{\underline{F}}$ SO WE MUST HAVE A NOTION OF THE REF CONFIG AND (**) IS THE EQ OF CHOICE

BALANCE OF ANGULAR MOMENTUM

ANGULAR MOM FOR DISCRETE PARTICLES

$$\underline{J} = \sum_i m_i (\underline{x}_i - \hat{\underline{x}}) \times \underline{v}_i = \sum_i \underline{r}_i \times m_i \underline{v}_i$$



AND THE TORQUE

$$\underline{T} = \sum_i \underline{r}_i \times \underline{f}_i$$

BALANCE OF ANGULAR MOMENTUM REQUIRES

$$\frac{d}{dt}\underline{J} = \underline{T}$$

IN CONTINUUM MECH WE GENERALIZE THE ABOVE

$$\underline{J}(t) = \int_\Theta \underline{r} \times \rho \underline{v} \, dv$$

$$\underline{T} = \int_\Theta \underline{r} \times \underline{b} \, dv + \int_{\partial\Theta} \underline{r} \times \underline{t} \, ds$$ (WE IGNORE DISTRIBUTED COUPLES - EG POLAR MATERIALS)

$$\frac{D}{Dt}\underline{J} = \underline{T}$$

$$\Rightarrow \int_{\Theta_0} \frac{D}{Dt}\left(\underline{r} \times \rho \underline{v} \, J\right) dv_0$$

$$= \int_{\Theta_0} \left(\cancel{\overset{=\underline{v}}{\dot{\underline{r}}} \times \rho \underline{v}}^{\,0} + \underline{r} \times \rho \dot{\underline{v}}\right) J + \underline{r} \times \underline{v} \, \dot{\rho} \, J + \underline{r} \times \rho \underline{v} \, \dot{J} \quad dV$$

$$= \int_\Theta \underline{r} \times \rho \dot{\underline{v}} + \underline{r} \times \underline{v} \underbrace{\cancel{\left(\dot{\rho} + \rho \nabla \cdot \underline{v}\right)}}^{\text{BOM}} dv$$

$$= \int_\Theta \underline{r} \times \rho \dot{\underline{v}} \, dv$$

NOW CONSIDER

$$\int_{\partial\Theta} \underline{t}\times\underline{t}\, ds = \int_{\partial\Theta} \underline{t}\times \underline{\underline{T}}\,\underline{n}\, ds = \int_{\Theta} \underline{\nabla}\cdot\left(\underline{t}\times\underline{\underline{T}}\right) dv$$

$$= \int_{\Theta} \underline{t}\times\underline{\nabla}\cdot\underline{\underline{T}} + \underline{\underline{T}}^{T} : \underline{\underline{\underline{\varepsilon}}} \quad dV$$

$$\underline{\nabla}\cdot\left(\underline{t}\times\underline{\underline{T}}\right) = \left(\varepsilon_{ijk}\, t_i\, T_{jq}\right)_{,q}\, \underline{e}_k = \left(\varepsilon_{ijk}\, t_{i,q}\, T_{jq}\right)\underline{e}_k + \left(\varepsilon_{ijk}\, t_i\, T_{jq,q}\right)\underline{e}_k$$

$$= \underbrace{\varepsilon_{ijk}\,\delta_{iq}\, T_{jq}\,\underline{e}_k}_{\varepsilon_{ijk} T_{ji}\, \underline{e}_k} + \varepsilon_{ijk}\, t_i\, T_{iq,q}\,\underline{e}_k$$

LET $\underline{\underline{\underline{\varepsilon}}} = \varepsilon_{ijk}\, \underline{e}_i\otimes\underline{e}_j\otimes\underline{e}_k$

$$\int_{\Theta} \underline{t}\times\rho\dot{\underline{v}} = \int_{\Theta} \underline{t}\times\underline{b} + \underline{t}\times\underline{\nabla}\cdot\underline{\underline{T}} + \int_{\Theta} \underline{\underline{T}}^{T} : \underline{\underline{\underline{\varepsilon}}}$$

$$\int_{\Theta} \underline{t}\times\left(\rho\dot{\underline{v}} - \underline{b} - \underline{\nabla}\cdot\underline{\underline{T}}\right) dv = 0 = \int_{\Theta} \underline{\underline{T}}^{T} : \underline{\underline{\underline{\varepsilon}}}$$

$$\Rightarrow \underline{\underline{T}}^{T} : \underline{\underline{\underline{\varepsilon}}} = 0 \qquad \varepsilon_{ijk} T_{ji} = 0 \qquad \forall k$$

$$\varepsilon_{kij} T_{ji} = \frac{1}{2}\left(\varepsilon_{kij} T_{ji} - \varepsilon_{kji} T_{ji}\right)$$

$$= \frac{1}{2}\left(\varepsilon_{kij} T_{ji} - \varepsilon_{kij} T_{ij}\right)$$

$$= \frac{1}{2}\varepsilon_{kij}\left(T_{ji} - T_{ij}\right) = 0$$

$$\Rightarrow T_{ji} - T_{ij} = 0 \Rightarrow \underline{\underline{T}} = \underline{\underline{T}}^{T}$$

NOTE THAT WHILE LINEAR MOM GIVES US A PDE
ANGULAR MOMENTUM TELLS US HOW TO CONSTRUCT
AN APPROPRIATE CONSTITUTIVE RELATION

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}' \Rightarrow \underline{\underline{P}} = J\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T} \Rightarrow \underline{\underline{\sigma}} = \frac{\underline{\underline{P}}\,\underline{\underline{F}}^{T}}{J} \Rightarrow \underline{\underline{P}}\,\underline{\underline{F}}^{T} = \underline{\underline{F}}\,\underline{\underline{P}}^{T}$$

$$\underline{\underline{\tau}} = J\underline{\underline{\sigma}} \Rightarrow \underline{\underline{\tau}} = \underline{\underline{\tau}}^{T}$$

$$\underline{\underline{S}} = J\underline{\underline{F}}^{-1}\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T} \Rightarrow \underline{\underline{\sigma}} = \frac{\underline{\underline{F}}\,\underline{\underline{S}}\,\underline{\underline{F}}^{T}}{J} \Rightarrow \underline{\underline{S}} = \underline{\underline{S}}^{T}$$

BALANCE OF MECHANICAL ENERGY

NOTE THAT BALANCE OF MECHANICAL ENERGY, WHEN OTHER SOURCES OF ENERGY (THERMAL, CHEMICAL...) ARE NEGLECTED, IS NOT AN ADITIONAL CONDITION BUT A CONSEQUENCE OF BALANCE OF LINEAR MOMENTUM.

CONSIDER THE EXTERNAL POWER

$$P^{EXT} = \int_{\theta} \underline{b}\cdot\underline{v}\,dv + \int_{\partial\theta} \underline{t}\cdot\underline{v}\,ds =$$

$$= \int_{\theta} \underline{b}\cdot\underline{v} + \int_{\theta} \underline{v}\cdot\underline{\underline{\sigma}}\,\underline{n}\,ds$$

$$= \int_{\theta} \underline{b}\cdot\underline{v} + \int_{\theta} \underline{\nabla}\cdot(\underline{\underline{\sigma}}\,\underline{v})\,dv = \int_{\theta} \underline{b}\cdot\underline{v} + \underline{\nabla}\cdot\underline{\underline{\sigma}}\cdot\underline{v} + \int_{\theta} \underline{\underline{\sigma}} : \underline{\nabla}\underline{v}$$

$$= \int_{\theta} \underline{v}\cdot(\underline{b} + \underline{\nabla}\cdot\underline{\underline{\sigma}}) + \int_{\theta} \underline{\underline{\sigma}} : \underline{\nabla}\underline{v}$$

$$= \int_{\theta} \underline{v}\cdot(\rho\dot{\underline{v}}) + \int_{\theta} \underline{\underline{\sigma}} : \underline{\nabla}\underline{v} = \frac{D}{Dt}\int_{\theta} \frac{1}{2}\rho\,\underline{v}\cdot\underline{v} + \int_{\theta} \underline{\underline{\sigma}} : \underline{\nabla}\underline{v}$$

$$\Rightarrow \underbrace{\int_{\theta} \underline{b}\cdot\underline{v}\,dv + \int_{\partial\theta} \underline{t}\cdot\underline{v}\,ds}_{P^{EXT}} = \underbrace{\frac{D}{Dt}\int_{\theta} \frac{1}{2}\rho\,\underline{v}\cdot\underline{v}\,dv}_{\frac{D}{Dt}\,K} + \underbrace{\int_{\theta} \underline{\underline{\sigma}} : \underline{\nabla}\underline{v}\,dv}_{P^{INT}}$$

P^{EXT} ↑ EXTERNAL POWER

$\frac{D}{Dt}\,K$ ↑ KINETIC ENERGY

P^{INT} ↑ STRESS WORK

$$\frac{D}{Dt} K + P^{INT} = P^{EXT}$$

NOTE THAT KINETIC ENERGY IS NOT CONSERVED

IF $P^{EXT} = 0$ WE HAVE FREE VIBRATIONS

IF $\frac{D}{Dt} K = 0 \Rightarrow \rho \underline{v} = 0$ WE HAVE A QUASI-STATIC PROB.

NOTE THAT IN GENERAL WE CANNOT WRITE

$$\int_\Theta \underline{\underline{T}} : \underline{\nabla} \underline{v} \quad \text{AS} \quad \frac{D}{Dt} \int_\Theta e(\underline{x}, t)\, dv$$

(AS WE WILL SEE WE COULD HAVE DISSIPATION)

IF THE MATERIAL IS ELASTIC

$$\int_\Theta \underline{\underline{T}} : \underline{\nabla} \underline{v} = \frac{D}{Dt} \int_\Theta e\, dV$$

$$\Rightarrow \frac{D}{Dt} \underbrace{\left[\int_\Theta \frac{1}{2} \rho \underline{v} \cdot \underline{v}\, dv + \int_\Theta e\, dV \right]}_{\text{TOTAL INTERNAL ENERGY}} = P^{ext}$$

ALSO NOTE THAT THE STRESS WORK IS GIVEN

$$\underline{\underline{T}} : \underline{\nabla} \underline{v} = \underline{\underline{T}} : \underline{\nabla}^S \underline{v} = \underline{\underline{T}} : \underline{\underline{L}}$$

$\underline{\underline{T}}$ IS SAID TO BE POWER CONJUGATE TO $\underline{\underline{L}}$

SIMILARLY

$$P^{EXT} = \int_{\theta} \underline{b}\cdot\underline{v}\,dv + \int_{\partial\theta} \underline{t}\cdot\underline{v}\,ds =$$

$$= \int_{\theta_0} \underline{B}\cdot\underline{V}\,dV + \int_{\partial\theta_0} \underline{V}\cdot\underline{\underline{P}}\,\underline{N}\,dS_0 = - \text{———} + \int_{\theta} \underline{V}\cdot\nabla_{\underline{X}}\underline{\underline{P}} + \int_{\theta_0} \underline{\underline{P}} : \nabla_{\underline{X}}\underline{V}$$

$$= \int_{\theta_0} \underline{V}\,\rho_0\,\dot{\underline{V}} + \int_{\theta_0} \underline{\underline{P}} : \dot{\underline{\underline{F}}}$$

$$\Rightarrow \int_{\theta_0} \underline{B}\cdot\underline{V} + \int_{\partial\theta_0} \underline{V}\cdot\underline{T}\,dS_0 = \frac{D}{Dt}\int \frac{1}{2}\rho_0\,\underline{V}\cdot\underline{V} + \int_{\theta_0} \underline{\underline{P}} : \dot{\underline{\underline{F}}}$$

$\underline{\underline{P}}$ IS POWER CONJUGATE TO $\dot{\underline{\underline{F}}}$