HOMEWORK 2

CEE 361-513: Introduction to Finite Element Methods

Due: Friday Oct. 6

NB: Students taking CEE 513 must complete all problems. All other students will not be graded for problems marked with $\star$, but are encourage to attempt them anyhow.

PROBLEM 1

Write down the expression for the following famous PDEs (for two-dimensional problems) and provide a short description of what they represent

1. Heat equation
2. Euler-Bernoulli equations
3. Navier-Stokes equations
4. Cauchy momentum equations
5. Shallow water equations
6. Kirchoff-Love plates' equation
7. Transport equation

PROBLEM 2

1. Classify the the PDEs of Problem 1, namely specify the order and the linearity.
2. $\star$ Second order quasilinear[1] partial differential equations can be categorized as elliptic, parabolic, and hyperbolic. If we write the second order differential equation as

$$\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}\frac{\partial^2 u}{\partial x_i x_j} + (\text{lower order derivatives}) = 0$$

then it can be categorized as

Elliptic:	if all the eigenvalues of the matrix a_{ij} have the same sign
Parabolic:	if one eigenvalue of the matrix a_{ij} is equal to zero and the rest have the same sign
Hyperbolic:	if all but one eigenvalues of the matrix a_{ij} have the same sign

   (a) Write down an explanation, in terms of characteristics, of what are the main differences between elliptic, parabolic, and hyperbolic PDEs.

   (b) For the PDEs of Problem 1, if they are second order and quasilinear, state whether they are elliptic, parabolic or hyperbolic.

[1] A quasilinear PDE is any PDE whose higher order derivatives are linear.

PROBLEM 3

Consider the system of uniaxial rods as shown below. We have three rods with Youngs' modulus and cross sectional areas E_e, A_e and lengths ℓ_e for $e = 1 \dots 3$. Each element is labeled by $\boxed{e}$, $e = 1 \dots 3$. Element $\boxed{2}$ and $\boxed{3}$ are connected via a rigid link at node ③, hence the j node of element $\boxed{2}$ and $\boxed{3}$ share the same global degree of freedom (see below).

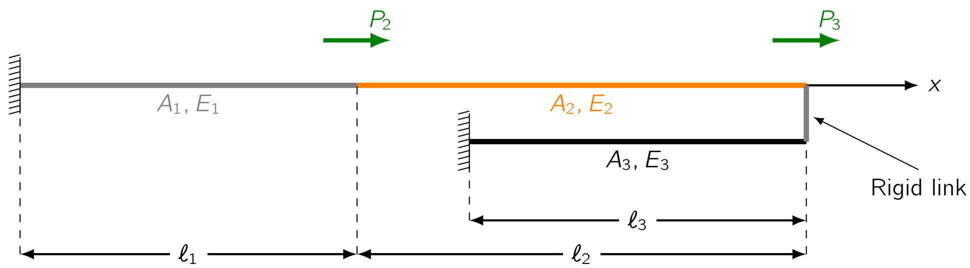

Figure 1: The system of uniaxial rods

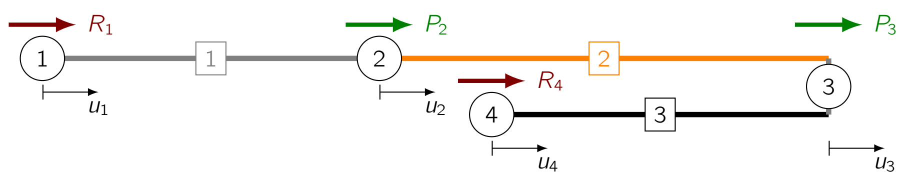

Figure 2: The system of uniaxial rods with nodes ⓝ and elements $\boxed{e}$ labeled.

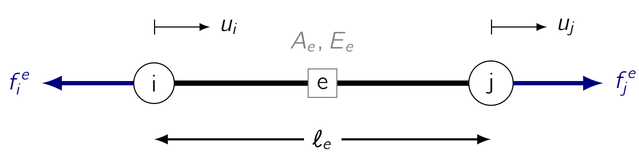

Figure 3: The local element view. Each element $\boxed{e}$ is composed of two nodes, node i and j. Their positions are denoted by x_i and x_j, $x_i < x_j$. The internal forces at node i and j for element e are denoted by $f^e_{i,j}$.

1. Write down the connectivity matrix. Namely, if we let each element be composed of two nodes (cf. Fig 3), node i and j with position x_i, x_j such that $x_i < x_j$, then identify the global node number corresponding to the each one of the local nodes. In short, fill in the table below.

Table 1: Connectivity array

element	i node	j node

2. For each element write the internal forces as the matrix vector operation of the *local element stiffness* and the *local degrees of freedom*.

3. For each element write the internal forces as the matrix vector operation of the *local element stiffness* and the *GLOBAL degrees of freedom* using the connectivity array.

4. For each node write the equilibrium equations in terms of the external forces P_k, $k = 2, 3$, the reactions R_k, $k = 1, 4$, and the internal forces $f^e_{i,j}$.
5. Let $k_i = A_i E_i / \ell_i$ for $i = 1 \dots 3$. Write down the equilibrium equations in matrix form. Namely, as we did in class, write the equilibrium equations with a load vector containing reactions and external forces, denoted it by {P}, the stiffness matrix denoted by $[K]$, and the vector of displacements $\{U\}$ such that
$$[K]\{U\} = \{P\}.$$
6. Let $u_1 = a$, $u_4 = b$ with a, b being given quantities. Apply the boundary conditions to $[K], \{P\}$.

PROBLEM 4

Consider the system of rods of Problem 3. Let $E_1 A_1 = 2$, $E_2 A_2 = 5$, $E_3 A_3 = 7$ and $\ell_1 = 5$, $\ell_2 = 10$, $\ell_3 = 7$ such that $x_1 = 0$, $x_2 = 5$, $x_3 = 15$, $x_4 = 8$.

1. Write in `python` a function `local_to_global_dof` that takes as arguments (1) the connectivity array, (2) the element number, and (3) the local degree of freedom (i or j) and returns the corresponding global degree of freedom. Namely:

```
local_to_global_dof( connectivity_array, element_number, local_dof )
```

2. Write in `python` a function `element_stiffness` that takes as arguments (1) the element Young's modulus, (2) the element cross sectional area, (3) the x_i coordinate, (4) the x_j coordinate, and returns the element stiffness. Namely:

```
element_stiffness( youngs_modulus, area , x_i, x_j )
```

3. Write a loop that assembles the global element stiffness matrix. As aid you can (don't have to) follow the following pseudocode.

```
# Create arrays of material properties
youngs_moduli ← [ E_1, ...]

areas ← [A_1, ...]

# Create the connectivity array
connectivity_array ← [[global_i, global_j], ... ]

# Create the coordinates array
coordinates_array ← [x_1,x_2, ... ]

# Initialize the global stiffness matrix
K ← a zero matrix of size num_dof × num_dof

# Loop over all elements
for each element e

    x_{i,j} ← get the crds of the i and j nodes of element e

    youngs_e, area_e ← get properties of element e

    ke ← element_stiffness( youngs_modulus_e, area_e , x_i, x_j )

    for each local dof p

        global_p ← local_to_global_dof( connectivity_array, e, p )

        for each local dof q

            global_q ← local_to_global_dof( connectivity_array, e, q )

            K[ global_p, global_q ] += ke[p,q]
```

4. As before assume u_1 and u_4 are given, namely $u_1 = 0, u_4 = 1$. Apply this boundary conditions to $[K]$ and $\{P\}$ where $P_2 = 10,\ P_3 = 2$.
5. What are the displacements of the nodes ?
6. What are the reactions R_1, R_4 ?