LECTURE 14:

TOPICS:

- REVIEW
- BASIS FUNCTIONS
- ELEMENT VIEW

LOGISTICS:

- HW & PROJECT

$$(S) \iff (W) \implies (G) \iff (M)$$

STRONG FORM

$? \, u \in C^2([0,\ell])$: $\quad AE \dfrac{d^2u}{dx^2} = f \quad \forall x \in (0,\ell)$

$u(0) = g$ ← ESSENTIAL (DIRICHLET)

$AE \dfrac{du}{dx}(\ell) = h$ ← NATURAL (NEUMANN)

WEAK FORM:

$S = \{u \in H^1([0,\ell]) \mid u(0) = g\}$ ← TRIAL SOLUTIONS

$V = \{v \in H^1([0,\ell]) \mid v(0) = 0\}$ ← TEST FUNCTIONS

$? \, u \in S$: $a(v,u) = F(v) \quad \forall v \in V$

$$a(v,u) = \int_0^\ell AE \frac{du}{dx}\frac{dv}{dx}\,dx, \qquad F(v) = h\,v(\ell) - \int_0^\ell f v\,dx$$

GALERKIN APPROXIMATION

$S^h \subseteq S, \quad V^h \subseteq V$

$S^h = \{u^h \in S \mid u^h(x) = u_i \phi_i(x)\}$

$V^h = \{v^h \in V \mid v^h(x) = v_i \phi_i(x)\}$

$? \, u^n \in S^n : \quad a(u^n, v^n) = F(v^n) \quad \forall v^n \in V$

MATRIX FORM

$\{v\}\left([K]\{u\} - \{f\}\right) = 0 \quad \forall \{v\}$

$\Rightarrow \quad [K]\{u\} = \{f\}$

$K_{ij} = a(\phi_i, \phi_j)$

NOTE: $K_{ij} = K_{ji}$ $\qquad$ $[K]$ IS SYMMETRIC

$a(u^n, u^n) \geq 0 \Rightarrow \{u\}^T [K]\{u\} \geq 0$ $\qquad$ POSITIVE SEMI-DEFINITE

SHOW JUPYTER

HOW DO WE CONSTRUCT BASIS FUNCTIONS?

$u^n(x) = u_1 \phi_1(x) + u_2 \phi_2(x) + \ldots \in H^1([0, \ell])$

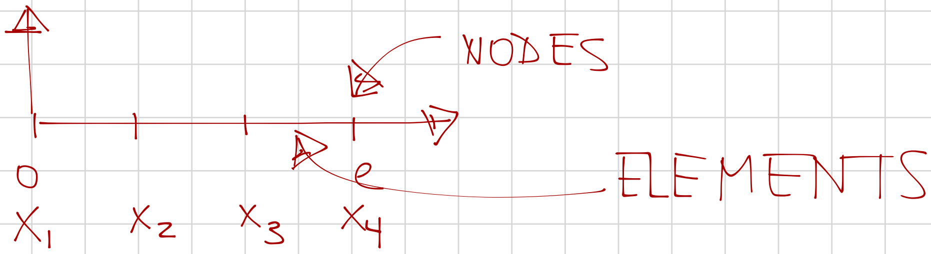
$\Rightarrow \phi_i(x) \in H^1([0,1]) \quad (\text{OR } C^0([0,\ell]))$

IDEALLY:

- BE SMOOTH ENOUGH
- HAVE LOCAL SUPPORT (I.E. COMPUTING K_{ij}, f_i IS CHEAP)
- THE LARGER THE BASIS THE BETTER THE APPROX

HOW CAN WE CONSTRUCT

- SUBDIVIDE THE DOMAIN



- HAVE THE FUNCTION SATISFY $\phi_i(x_j) = \delta_{ij}$

$\phi_i(x_j) = \delta_{ij}$

THE ABOVE GIVES $u^h(x_i) = u_i$

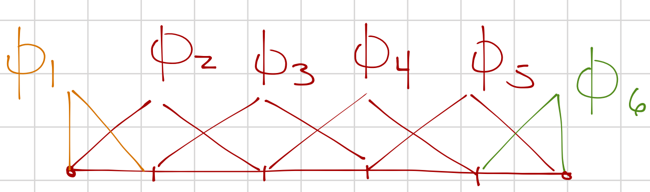
- WITHIN THE ELEMENT WE CAN CHOOSE A POLYNOMIAL (SEE WEIERSTRASS THM)

AN EXAMPLE: LINEAR PIECEWISE POLYNOMIALS

$$\phi_i(x) = \begin{cases} \dfrac{x - x_i}{x_i - x_{i-1}} & \text{if } x \in [x_{i-1}, x_i] \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x \in [x_i, x_{i+1}] \\[2ex] 0 & \text{ELSE} \end{cases} \qquad i = 2 \dots N-1$$

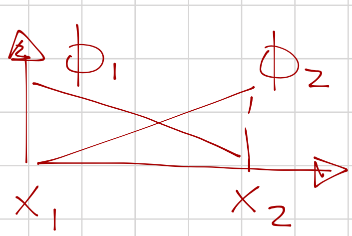
$$\phi_1(x) = \begin{cases} \dfrac{x_2 - x}{x_2 - x_1} & \text{if } x \in [x_1, x_2] \\[2ex] 0 & \end{cases}$$

$$\phi_6(x) = \begin{cases} \dfrac{x - x_6}{x_6 - x_5} & \text{if } x \in [x_5, x_6] \\[2ex] 0 & \end{cases}$$



NOTE: $\sum_i \phi_i = 1$ AND u^h IN THIS SPACE APPROXIMATE LINEAR FUNCTIONS EXACTLY

EXAMPLE: CONSIDER $N=2$



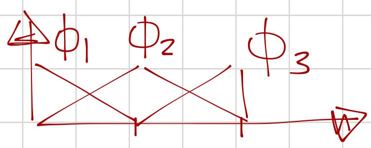
$x_1 = 0, \; x_2 = \ell$

$$K_{11} = a(\phi_1, \phi_1) = AE \int_0^{\ell} \underbrace{\frac{d\phi_1}{dx}}_{-1/h} \underbrace{\frac{d\phi_1}{dx}}_{-1/h} dx = \frac{AE}{\ell}$$

$$K_{12} = a(\phi_1, \phi_2) = AE \int_0^{\ell} \frac{d\phi_1}{dx} \frac{d\phi_2}{dx} dx = -\frac{AE}{\ell}$$

$$[\bar{K}] = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

CONSIDER $N=3$



$$K_{23} = a(\phi_2, \phi_3) = AE\int_0^{\ell} \frac{d\phi_2}{dx}\frac{d\phi_3}{dx}\,dx =$$

$$= \int_{x_1}^{x_2} \frac{d\phi_2}{dx}\overset{0}{\cancel{\frac{d\phi_3}{dx}}}\,dx + \int_{x_1}^{x_2} \frac{d\phi_2}{dx}\frac{d\phi_3}{dx} = \int_{x_1}^{x_2} \overset{-1/h}{\frac{d\phi_2}{dx}}\,\overset{1/h}{\frac{d\phi_3}{dx}}\,dx$$

$$= \frac{AE}{h}$$

$$K = \frac{AE}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

THE ELEMENT VIEW

WE CAN CONSTRUCT THE BASIS FUNCTIONS BY STARTING FROM THE ELEMENTS & MOVING UP.

NOTE THAT THE FINITE ELEMENT MESH SERVES AS THE SCAFFOLD TO CONSTRUCT THE BASIS FUNCTIONS.

TO THIS EXTENT

A FINITE ELEMENT CONSIST OF

(i) A DOMAIN K^e (E.G. $K^e = [x_e, x_{e+1}]$)

(ii) NODES (EG $\{x_e, x_{e+1}\}$)

(iii) DEGREES OF FREEDOM (EG $\{u_e, u_{e+1}\}$)

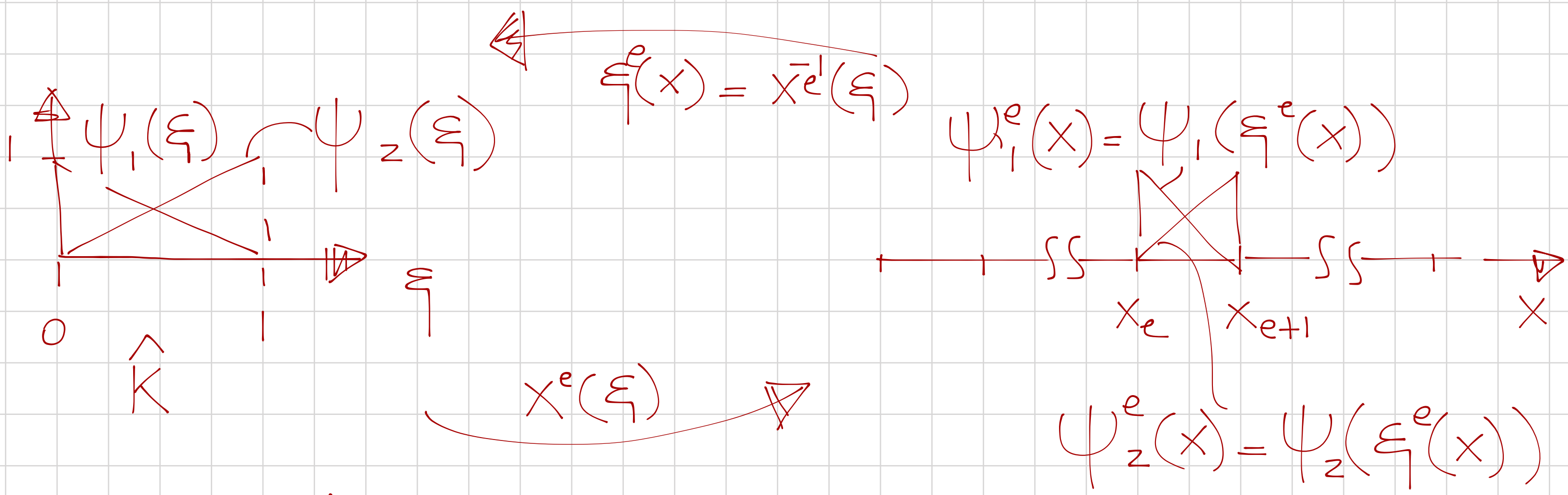
FOR EASE OF CONSTRUCT

(v) A PARENT DOMAIN $\hat{K}$ (EG $[0,1]$)

(vi) A MAP $x^e(\xi) : \hat{K} \to K^e$

BIJECTIVE

(iv) SHAPE FUNCTIONS (EG $\{\psi_1, \psi_2\}$)



$$\psi_i^e(x) = \begin{cases} \psi_i(\xi^e(x)) & \text{if } x \in K^e \\ 0 & \text{ELSE} \end{cases}$$

$$\phi_e(x) = \begin{cases} \psi_1^e(x) & \text{if } x \in K^e \\ \psi_2^{e-1}(x) & \text{if } x \in K^{e-1} \end{cases}$$

GLOBAL NODE LABEL

<u>NOTE</u>: IN PRACTICE WE NEVER CONSTRUCT ϕ