PRECEPT 2

CEE 361-513: Introduction to Finite Element Methods

Monday Sept. 25

Unless otherwise specified, you may assume that $\{\mathbf{e}_i\}_{i=1}^d$ is a set of orthonormal basis associated with a set of cartesian coordinates $\{x_i\}_{i=1}^d$.

PROBLEM 1

Let $\{\mathbf{e}_i\}_{i=1}^d$ is a set of orthonormal basis associated with a set of cartesian coordinates $\{x_i\}_{i=1}^d$ and $d = 3$
$\boldsymbol{u}(\boldsymbol{x}) = x_1x_2\mathbf{e}_1 + x_1^2\mathbf{e}_2 + x_1x_3\mathbf{e}_3$ compute $\nabla\boldsymbol{u}$ and $\nabla\cdot\boldsymbol{u}$.
Solution :

$$\begin{aligned}
\nabla\boldsymbol{u} &= \frac{\partial\boldsymbol{u}}{\partial x_j}\otimes\mathbf{e}_j \\
&= x_2\mathbf{e}_1\otimes\mathbf{e}_1 + 2x_1\mathbf{e}_2\otimes\mathbf{e}_1 + x_3\mathbf{e}_3\otimes\mathbf{e}_1 + x_1\mathbf{e}_1\otimes\mathbf{e}_2 + x_1\mathbf{e}_3\otimes\mathbf{e}_3 \\
\nabla\cdot\boldsymbol{u} &= \frac{\partial\boldsymbol{u}}{\partial x_j}\cdot\mathbf{e}_j \\
&= x_2 + x_1
\end{aligned}$$

PROBLEM 2

Let $\{\mathbf{e}_i\}_{i=1}^d$ is a set of orthonormal basis associated with a set of cartesian coordinates $\{x_i\}_{i=1}^d$ and $d = 3$
Prove the following identity:

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}$$

Solution :

$$\begin{aligned}
\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= a_i\mathbf{e}_i\times(b_j\mathbf{e}_j\times c_k\mathbf{e}_k) \\
&= a_i\mathbf{e}_i\times(b_jc_k\epsilon_{jkl}\mathbf{e}_l) \\
&= a_i\mathbf{e}_i\times(b_jc_k\epsilon_{jkl}\mathbf{e}_l) \\
&= a_ib_jc_k\epsilon_{jkl}\mathbf{e}_i\times\mathbf{e}_l \\
&= a_ib_jc_k\epsilon_{jkl}\epsilon_{ilm}\mathbf{e}_m
\end{aligned}$$

Since the indices are repeated it is basically a summation of many many terms.
Using the property of Levi-Civita symbol:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{ifeven permutation} \\ -1 & \text{ifodd permutation} \\ 0 & \text{ifrepeated} \end{cases}$$

The terms with the following conditions go to zero:

$$\begin{array}{ccc} j = k & j = l & k = l \\ i = l & i = m & l = m \end{array}$$

So the only terms that are left in the summation are with [Since we are in]:

$$\begin{array}{cc} j = i & k = m \\ k = i & j = m \end{array}$$

$$
\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= a_i b_i c_m \epsilon_{iml} \epsilon_{ilm} \mathbf{e}_m + a_i b_m c_i \epsilon_{mil} \epsilon_{ilm} \mathbf{e}_m \\
&= -a_i b_i c_m \mathbf{e}_m + a_i b_m c_i \mathbf{e}_m \\
&= (\boldsymbol{a} \cdot \boldsymbol{c}) \, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b}) \, \boldsymbol{c}
\end{aligned}
$$

PROBLEM 3

Using Python solve the following problems.

1. Let $\boldsymbol{a} = 1.0\mathbf{e}_1 + 2.0\mathbf{e}_2$ and $\boldsymbol{b} = 4.0\mathbf{e}_1 + 5.0\mathbf{e}_2$. Find the angle between the two vectors

   Solution :

```
"""
Python code
"""
import numpy as np
import numpy.linalg as LA
# Define two vectors
a = np.array([1.0,2.0])
b = np.array([4.0,5.0])
# Define the dot product
ab_dot = np.dot(a,b)
print(ab_dot)

# Find the norm
norm_a = LA.norm(a)
norm_b = LA.norm(b)
print(norm_b,norm_a)

# Compute the angle
theta = np.arccos(ab_dot/(norm_a*norm_b))
print(theta)
```

2. Find the outer product of the above two vectors

   Solution :

```
# Find the tensor
T = np.outer(a,b)
print(T)
```