LECTURE 7:

TOPICS:

- REVIEW
- 1-D TRUSS MATRIX S.A.

LOGISTICS:

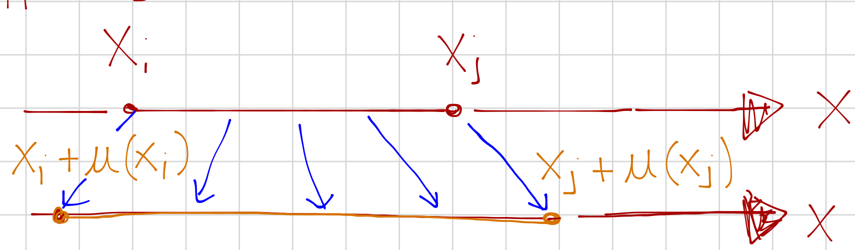
- HW # 1 DUE YESTERDAY
- HW # 2 OUT FRI DUE FRI AFTER

REVIEW:

LAST TIME WE SAW THAT THE GOVERNIN EQ FOR A 1-D TRUSS IS GIVEN BY

$$AE \frac{du}{dx}(x) = b(x)$$

WHERE $u(x)$ IS THE DISPLACEMENT OF THE TRUSS



BEFORE DEFORMATION

AFTER DEFORMATION

WITH THE ABOVE IN MIND WE FORMULATED THE PROBLEM OF INTEREST AS

FIND $u : [x_i, x_j] \rightarrow \mathbb{R}$ SUCH THAT

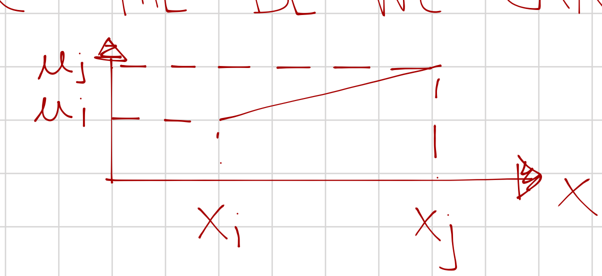
$$AE \frac{du}{dx}(x) = 0 \qquad \forall\ x \in (x_i, x_j)$$

$$u(x_i) = u_i$$

$$u(x_j) = u_j$$

GIVEN

SIMPLY BY INTEGRATING TWICE THE DE WE CAN SHOW THAT,

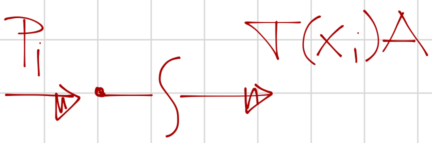


$$u(x) = (u_j - u_i)\frac{(x - x_i)}{(x_j - x_i)} + u_i$$

FROM THERE WE SAID, LET US APPLY POINT LOADS AT x_i & x_j GIVEN RESPECTIVELY AS P_i, P_j.

WHAT DO u_i & u_j NEED TO SATISFY IN ORDER FOR THE TRUSS TO BE IN BALANCE?

AT x_i:



$$P_i + \sigma(x_i)A = 0$$

$$P_i = -\sigma(x_i)A$$

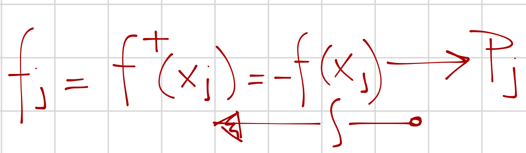
$$\sigma(x_i) = E\frac{du}{dx}(x_i), \quad \frac{du}{dx}(x) = \frac{d}{dx}\left((u_j - u_i)\frac{(x - x_i)}{(x_j - x_i)} + u_i\right) =$$

$$= \frac{u_j - u_i}{x_j - x_i} = \frac{u_j - u_i}{e}$$

$$\Rightarrow P_i = -\sigma(x_i)A = -E\frac{du}{dx}(x_i)A = -E\frac{(u_j - u_i)}{e}A =$$

$$= \frac{AE}{e}(u_i - u_j) = \frac{AE}{e}\{1 \; -1\}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

SIMILARLY AT x_j:



$$P_j - f(x_j) = 0 \Longrightarrow P_j = f(x_j) = \frac{AE}{e}\{-1 \; 1\}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

SO IN ORDER TO HAVE EQUILIBRIUM WE MUST HAVE

$$\begin{aligned} P_i &= -f_i \\ P_j &= f_j \end{aligned} \Longrightarrow \begin{Bmatrix} P_i \\ P_j \end{Bmatrix} = \begin{Bmatrix} -f_i \\ f_j \end{Bmatrix} = \frac{AE}{e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

NOW, GIVEN P_i, P_j, WE WANT TO FIND THE VALUE OF u_i, u_j SUCH THAT THE TRUSS IS IN EQUILIBRIUM

u_i, u_j ARE CALLED THE DEGREES OF FREEDOM

$K = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ IS CALLED THE STIFFNESS MATRIX

NOTE FROM ABOVE THAT

$$\begin{Bmatrix} P_i \\ P_j \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \qquad k = \frac{AE}{\ell}$$

IS A SYSTEM OF TWO EQUATION.

IF WE PRESCRIBE THE VALUES OF u_i & u_j (EFFECTIVELY OUR DIRICHLET BC) THEN $P_{i,j}$ CAN BE THOUGH OF AS REACTION FORCES RATHER THAN EXTERAL LOADS.

ON THE OTHER HAND IF WE DO NOT PRESCRIBE THE VALUE OF u_i (OR u_j), P_i (OR P_j) CAN BE APPLIED EXTERNAL FORCES.

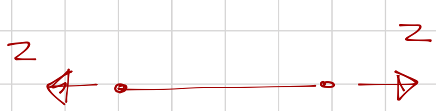
NOTE: WE MUST ALWAYS PRESCRIBE EITHER u_i OR u_j OTHERWISE WE HAVE NO DIRICHLET BC AND THE PROBLEM IS ILL POSED

EG:

1) ASSUME WE APPLY A DISPLACEMENT $u_i = -1$, $u_j = 1$
$k = 1$

WHAT ARE THE FORCES REQUIRED TO ATTIAN SUCH DISPLACEMENTS?

$$\begin{Bmatrix} R_i \\ R_j \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$$



NOTE THE ENTIRE TRUSS IS IN EQ

2) ASSUME WE HOLD x_i FIXED (I.E. $u_i = 0$) AND WE APPLY A LOAD $P_j = 10$.

HOW MUCH DOES x_j MOVES (IE WHAT IS u_j)?

$$\begin{Bmatrix} R_i \\ 10 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_j \end{Bmatrix}$$

WE CAN LOOK A THE SECOND ROW

$$\{10\} = \{-1 \quad 1\} \begin{Bmatrix} 0 \\ u_j \end{Bmatrix} = u_j \Rightarrow \quad u_j = 10$$

NOTE: THIS LAST EXAMPLE IS SOLVING A DIFFERENT PROBLEM

$? \; u : [x_i, x_j] \to \mathbb{R}$ SUCH THAT

$$AE \frac{d^2 u}{dx^2} = 0 \quad \forall x$$

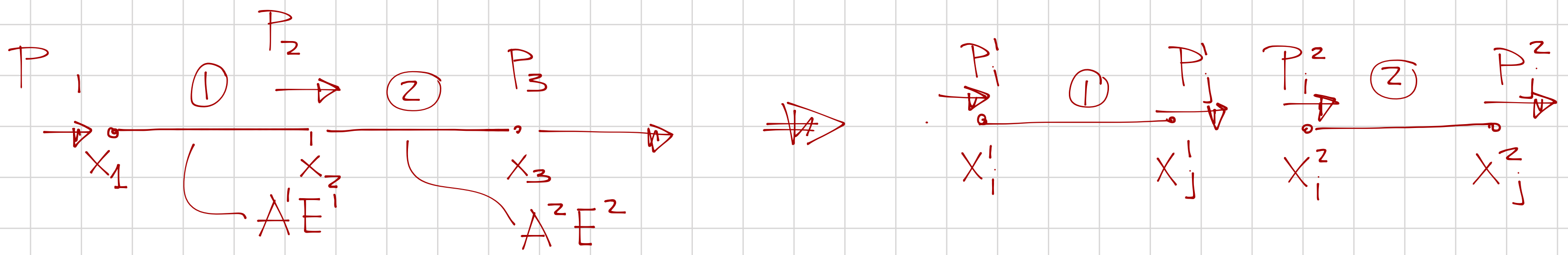
DIRICHLET ... NEUMANN

$$u(x_i) = u_i = 0, \quad AE \frac{du}{dx}(x_j) = P_j = 10$$

IMPORTANT

!!! NOTE !! YOU CANNOT HAVE ALL BCs TO BE NEUMANN !! NEED AT LEAST ONE DIRICHLET !!

NOW, WAT IF



WE CAN THINK OF THIS AS BEING AS A DIFF PROB COMPOSED OF TWO SUBPROBLEMS

$$? \; u^1 : [x_1, x_2] \to \mathbb{R} \quad A^1E^1 \frac{d^2 u}{dx^2} = 0 \quad \forall x \in (x_1, x_2)$$

$$? \; u^2 : [x_2, x_3] \to \mathbb{R} \quad A^2E^2 \frac{d^2 u}{dx^2} = 0 \quad \forall x \; (x_2, x_3)$$

AND $u^1(x_2) = u^1(x^1_j) = u^1_j = u^2_i = u^2(x^2_i) = u^2(x_2)$

FOR CONVENIENCE LET $k^1 = \frac{A^1E^1}{\ell^1}$, $k^2 = \frac{A^2E^2}{\ell^2}$

DOING A SIMILAR EXERCISE AS BEFORE

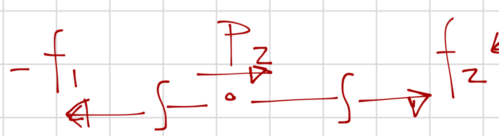
AT NODE 1:

FORCE IN ELEMENT ①

$$f_1 = A^1E^1\frac{du^1}{dx}(x_1) = \frac{A^1E^1}{\ell^1}(u_2-u_1).$$

$$P_1 + f_1 = 0 \Rightarrow P_1 = -f_1 = \{k_1 \quad -k_1\}\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \{k_1 \quad -k_1 \quad 0\}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}$$

AT NODE 2:



FORCE IN ELEMENT ②

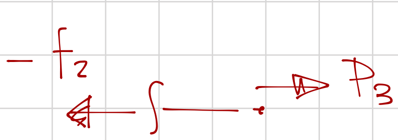
$$f_1 = A^1E^1\frac{du^1}{dx}(x_2) = \frac{A^1E^1}{\ell^1}(u_2-u_1)$$

$$f_2 = A^2E^2\frac{du^2}{dx}(x_2) = \frac{A^2E^2}{\ell^2}(u_3-u_2)$$

$$P_2 - f_1 + f_2 = 0 \Rightarrow P_2 = -k_1u_1 + (k_1+k_2)u_2 - k_2u_3$$

$$= \{-k_1 \quad k_1+k_2 \quad -k_2\}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}$$

AT NODE 3:



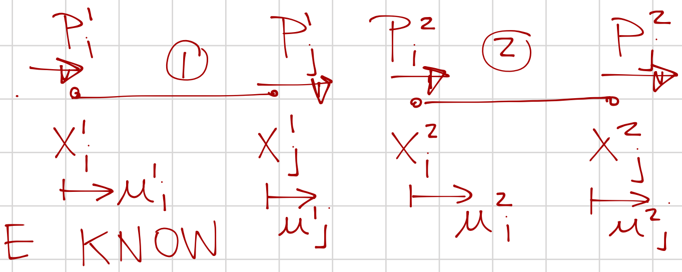
$$A^2E^2\frac{du^2}{dx}(x_3) = \frac{A^2E^2}{\ell^2}(u_3-u_2) = f_2$$

$$P_3 - f_2 = 0 \Rightarrow P_3 = k_2(u_3-u_2) = \{0 \quad -k_2 \quad k_2\}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}$$

IF WE PUT IT ALL TOGETHER

$$\underbrace{\begin{Bmatrix}P_1\\P_2\\P_3\end{Bmatrix}}_{\text{LOAD VECTOR}} = \begin{Bmatrix}-f_1\\f_1-f_2\\f_2\end{Bmatrix} = \underbrace{\begin{bmatrix}k_1 & -k_1 & 0\\-k_1 & k_1+k_2 & -k_2\\0 & -k_2 & k_2\end{bmatrix}}_{\text{GLOBAL STIFFNESS MATRIX}}\underbrace{\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}}_{\text{VECTOR OF DOFs OR UNKNOWNS}}$$

WE CAN TAKE A DIFFERENT VIEW.



WE KNOW

$$\begin{Bmatrix} P_i^1 \\ P_j^1 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_i^1 \\ u_j^1 \end{Bmatrix} \qquad \begin{Bmatrix} P_i^2 \\ P_j^2 \end{Bmatrix} = \begin{bmatrix} k^2 & -k^2 \\ -k^2 & k^2 \end{bmatrix} \begin{Bmatrix} u_i^2 \\ u_j^2 \end{Bmatrix}$$

AND ALSO $u_j^1 = u_i^2 = u_2$

$u_{i,j}^n \leftarrow$ LOCAL DEGRES OF FREEDOM

$u_n \leftarrow$ GLOBAL DEGREES OF FREEDOM

$$\begin{Bmatrix} P_i^1 \\ P_j^1 \\ P_i^2 \\ P_j^2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ k_1 & k_1 & 0 & 0 \\ 0 & 0 & k^2 & -k^2 \\ 0 & 0 & -k^2 & k^2 \end{bmatrix} \begin{Bmatrix} u_i^1 \\ u_j^1 \\ u_i^2 \\ u_j^2 \end{Bmatrix}$$

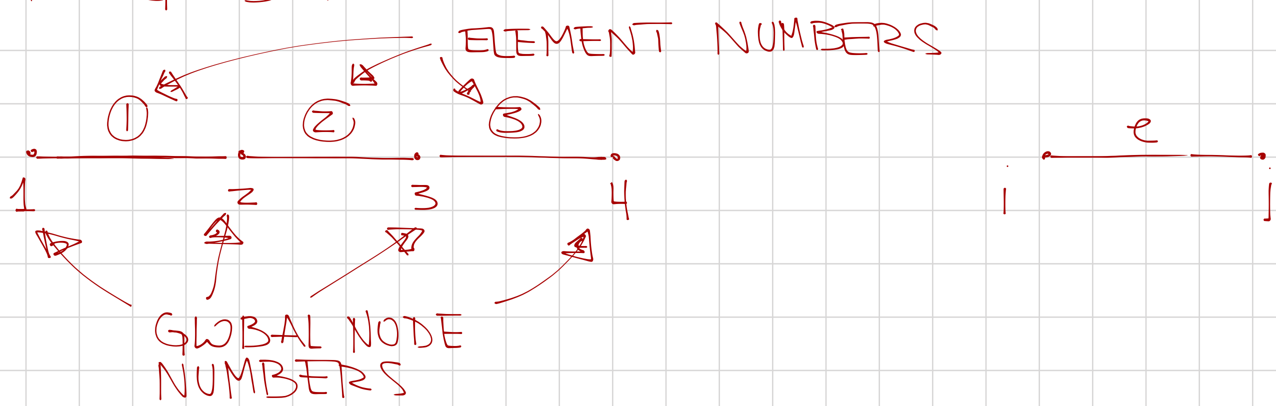
BUT WE KNOW $u_j^1 = u_i^2$ SUCH THAT

$$\begin{Bmatrix} P_i^1 \\ P_j^1 \\ P_i^2 \\ P_j^2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ k_1 & k_1 & 0 & 0 \\ 0 & 0 & k^2 & -k^2 \\ 0 & 0 & -k^2 & k^2 \end{bmatrix} \begin{Bmatrix} u_i^1 \\ u_2 \\ u_2 \\ u_j^2 \end{Bmatrix} \xrightarrow{\text{CONDENSE}} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

ELEMENT VIEW & ASSEMBLY

WE CAN TAKE A MORE METHODOLOGICAL APPROACH

START BY CONSTRUCTING A MAP FROM LOCAL TO GLOBAL DOF



CONNECTIVITY ARRAY

ELEMENT #	CONNECTIVITY i , j	
1	1	2
2	2	3
3	3	4

$$K^e = \begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix} \end{array}$$

$$K = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \text{FILL IN CLASS} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

APPLYING DISPLACEMENTS OR BC

SUPPOSE WE HAVE THE FOLLOWING STIFFNESS MATRIX

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

SUPPOSE WE HAVE DIRICHLET CONDITIONS SUCH THAT $u_1 = \hat{u}_1$.

IN THIS CASE u_1 IS NO LONGER UKNOWN.

SO WE

1) ZERO OUT THE ROW OF K THAT CORRESPONDS TO THE DEGREE OF FREEDOM

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

2) WE PLACE 1 ON THE DIAGONAL

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

3) WE REPLACE THE ROW IN LOAD VECTOR WITH THE VALUE OF THE PRESCRIBED DISPL

$$\begin{Bmatrix} \hat{u}_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

EFFECTIVELY WE ARE REPLACING THE FIRST EQUATION

$$P_1 = K_1 u_1 - K_2 u_2$$

WITH

$$\hat{u}_1 = u_1$$