LECTURE 4:

TOPICS:

- DEFINITION OF CONTINUUM
- CONFIGURATION REFERENCE VS. CURRENT
- MOTION: DISPLACEMENT, VELOCITY, ACCELLERAT.
- MATERIAL VS SPATIAL DERIVATIVES

MATTER IS A COLLECTION OF MOLECULES FORMED BY ATOMS & SUBATOMIC PARTICLES

IF YOU THINK THAT THE SIZE OF AN ATOM IS ON THE ORDER OF 0.1 nm OR 10^{-8} cm WE HAVE ON THE ORDER OF $\sim 10^{23}$ ATOMS/GRAM

IF WE WERE INTERESTED IN THE POSITION OF ALL ATOMS IN REAL ENG APP WE WOULD HAVE ON THE ORDER $3 \cdot 10^{23}$ DEGREES OF FREEDOM AND ASSUMING INTERACTION $O(n^3)$ CALCULATIONS $\rightarrow$ PROHIBITIVE

CONTINUUM IS AN APPROXIMATION/IDEALIZATION OF LARGE COLLECTIONS OF ATOMS.

CONTINUUM IS A GAPLESS DISTRIBUTION OF MASS (THE MASS OF ATOMS) OVER A REGION (OF SPACE $\Omega \subset \mathbb{R}^3$ THAT WE CALL THE BODY.

THE TYPE OF ATOMS IS IRRELVANT FOR THE ABOVE DEF. WE COULD DEAL WITH WATER MOLECULE, JELL-O, MELTIN IRON ETC.

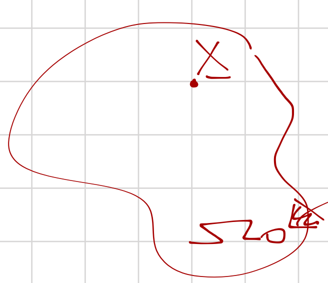
IT IS SOMEWHAT COUNTER INTUITIVE TAKING $O(10^{23})$ AND TRANSFORMING IT IN A SYSTE OF $n \rightarrow \infty$ DIM. BUT THAT IS WHAT ALLOWS US TO DESCRIBE DESCRBE THE SYSTEM AS CONTINUOS FUNCTIONS (FIELD THEORY)

THESE FIELDS REPRESENT MACROSCOPIC QUANTS AND THE EVOLUTION OF THESE MACROSCOPIC QUANTITIES CAN BE DESCRIBED THROUGH PDEs FOR WHICH WE HAVE A LOT OF TOOLS

CONTINUUM MECHANICS CAN BE SUBD.

- KINEMATICS → STUDIES DEFORMATION/MOTION
- STRESS IN A CONTINUUM
- BALANCE PRINCIPLES

IN ORDER TO TRACK THE EVOLUT DEFINE A REFERENCE CONFIG



WE DENOTE THE CONTINUUM PARTICLE WITH P WHICH HAS A REFERENTIAL POS. $\underline{X}$ ← WILL BECOME THE LABEL FOR P

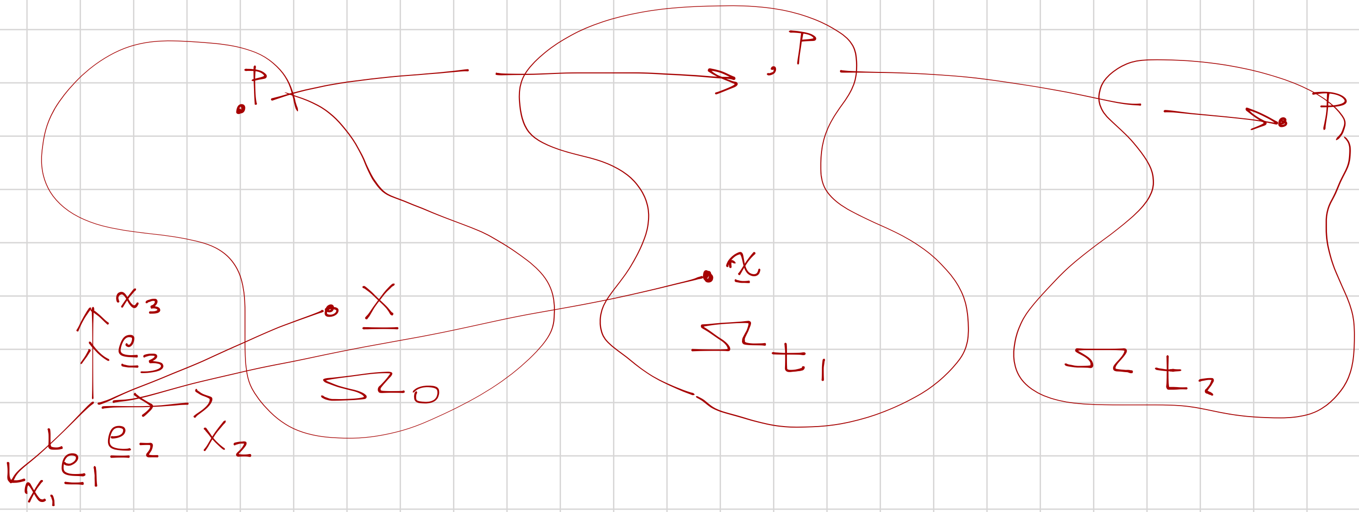
DENOTES TIME 0

NOTE THAT THE REFERENCE CONFIG. IS ARBITRARY & IT OFTEN COINCIDES WITH AN UNSTRESSED CONFIGURATION

A PARTICLE HERE IS UNDERSTOOD TO REPRESENT THE BEHAVIOR OF A COLLECTION OF ATOMS

MASS & VOLUME HERE ARE CONTINUOS

AT EACH TIME OUR CONTINUUM DEF & OCCUPIES A REGION Ω_t



WE DEFINE THE POSITION OF EACH PARTICLE VIA THE DEFORMATION MAPPING $\underline{\varphi} : \Omega_0 \to \mathbb{R}^3$

$\underline{\varphi}_t(P) = \underline{x} \quad \equiv \quad \underline{\varphi}(\underline{X}, t) = \underline{x}$

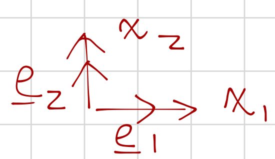
A LITTLE ABUSE OF NOT

$\underline{\varphi}(\Omega_0, t) = \Omega_t$

NOTE WE CONSTRUCT TWO COORDINATE SYSTEM

ONE THAT WE DRAW ON OUR BODY IN ITS REF CONFIG

& ONE FIXED IN SPACE



$X_i \leftarrow$ MATERIAL COORDINATES

$x_i \leftarrow$ SPATIAL COORDINATES

SO $\underline{\varphi} : \Omega_0 \to \Omega_t$ IS A VECTOR FIELD DEFINED OVER THE REFERENCE CONFIGURATION

$$\underline{x} = \underline{\varphi}(\underline{X}, t) \Rightarrow x_i = \varphi_i(X_I E_I, t)$$

IN ORDER FOR THE MAPPING TO BE ADMISSIBLE (I.E. HAVE PHYSICAL SIGNIFICANCE) IT SHOULD SATISFY:

1. $\underline{\varphi}$ MUST BE DEFINED FOR ALL POINTS IN Ω_0
2. $\underline{\varphi}$ IS INJECTIVE (EXCEPT AT THE BOUNDARY W/ CONTACT)
3. $\underline{\varphi}$ IS CONTINUOSLY DIFFERENTIABLE

2 PREVENTS INTERPENETRATION, 3 PREVENTS CRACKS.

IF 1 + 2 HOLD THAN THE MAPPING IS ONTO & ONE-TO-ONE & THUS WE HAVE THE GUARANTEE OF THE EXISTANCE OF $\underline{\varphi}^{-1}$ SUCH THAT

$$\underline{\varphi}(\underline{\varphi}^{-1}(\underline{x},t),t) = \underline{x}$$

OR EFFECTIVELY A MAP THAT ALLOWS US TO IDENTIFY, FOR EVERY POINT IN SPACE WHERE THE PARTICLE CAME FROM.

WE OFTE TALK OF $\underline{\varphi}(\underline{X},t)\big|_{\underline{X}=\underline{X}_P}$ AS THE TRAJECTORY OF A PARTICLE

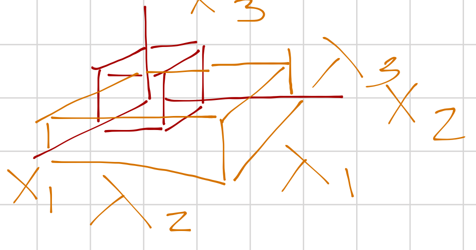
$\underline{\varphi}(\underline{X},t)\big|_{t=\hat{t}}$ THE DEFORMED CONFIGURATION

EXAMPLES:

IDENTITY MAP: $\underline{\varphi}(\underline{X},t) = \underline{X} = X_i \underline{E}_i$

STRETCHIN OF ACUBE:

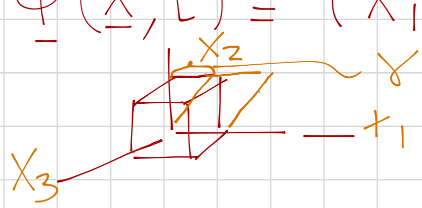
$$\underline{\varphi}(\underline{X},t) = \lambda_1 X_1 \underline{E}_1 + \lambda_2 X_2 \underline{E}_2 + \lambda_3 X_3 \underline{E}_3$$



$$\varphi_i = \lambda_{(I)} X_{(I)}$$

PURE SHEAR:

$$\underline{\varphi}(\underline{X},t) = (X_1 + \gamma X_2)\underline{E}_1 + X_2 \underline{E}_2 + X_3 \underline{E}_3$$



φ

AFFINE MAP:

$$\underline{\varphi}(\underline{X},t) = \underline{a} + \underline{\underline{F}}\,\underline{X}$$

$$\varphi_i = a_i + \underbrace{F_{iJ} X_J}$$

NOTE THE DIFFERENT INDECES!! IT CONTRACTS AN ELEMENT FROM THE REFERENCE CONFIGURATION AND IT GIVES YOU ONE IN THE DEFORMED CONFIG.

$\underline{\underline{F}}$ HERE COULD BE ANYTHI WITH THE SOLE CONDITION THAT $\det(\underline{\underline{F}}) > 0$

FOR EG

$$[\underline{\underline{F}}]_{\underline{E}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$[\underline{\underline{F}}]_{\underline{E}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

PLANE STRAIN

$$\underline{\varphi}(\underline{X},t) = \varphi_1(X_1, X_2)\,\underline{E}_1 + \varphi_2(X_1, X_2)\,\underline{E}_2 + X_3\,\underline{E}_3$$

ANTIPLANE SHEAR

$$\underline{\varphi}(\underline{X},t) = X_1\,\underline{E}_1 + X_2\,\underline{E}_2 + \varphi_3(X_1, X_2)\,\underline{E}_3$$

MATERIAL Vs SPATIAL DESCRIPTION

CONSIDER A PROPERTY OF OUR CONTINUUM
ϕ (EG MASS, TEMP)

IF $\phi = \hat{\phi}\big|_{\underline{X}}$ WE SAY WE HAVE A MATERIAL OR LAGRANGIAN DESCRIPTION

I.E FOR A FIXED PARTICLE IN THE CONTINUUM

IF $\phi = \tilde{\phi}\big|_{\underline{x}}$ WE SAY WE HAVE A SPATIAL OR EULERIAN DESCRIPTION

I.E FOR A FIXED POINT IN SPACE

DISPLACEMENT, VELOCITY & ACCELERATION

DISPLACEMENT

$$\underline{U}(\underline{X},t) = \underline{\varphi}(\underline{X},t) - \underline{X}$$

$$\underline{u}(\underline{x},t) = \underline{x} - \underline{\varphi}^{-1}(\underline{x},t)$$

VELOCITY

$$\underline{V}(\underline{X},t) = \frac{d\underline{\varphi}}{dt}(\underline{X},t)\Big|_{\underline{X}} \Leftarrow \text{FOR A FIXED PARTICLE}$$

$$\underline{v}(\underline{x},t) = \underline{V}(\underline{\varphi}^{-1}(\underline{x},t),t)$$

ACCELERATION

$$\underline{A}(\underline{X},t) = \frac{d\underline{V}}{dt}(\underline{X},t)\Big|_{\underline{X}}$$

$$\underline{a}(\underline{x},t) = \underline{A}(\underline{\varphi}^{-1}(\underline{x},t),t)$$

EG: SHRINKING OF A SPHERE

$$\underline{\varphi}(\underline{X},t) = \mathrm{EXP}(-\lambda t)\,\underline{X}$$

$$\underline{V}(\underline{X},t) = -\lambda\,\mathrm{EXP}(-\lambda t)\,\underline{X}$$

$$\underline{A}(\underline{X},t) = \lambda^2\,\mathrm{EXP}(-\lambda t)\,\underline{X}$$

$$\underline{x} = \mathrm{EXP}(-\lambda t)\,\underline{\varphi}^{-1}(\underline{x},t) \Rightarrow \underline{\varphi}^{-1}(\underline{x},t) = \mathrm{EXP}(\lambda t)\,\underline{x}$$

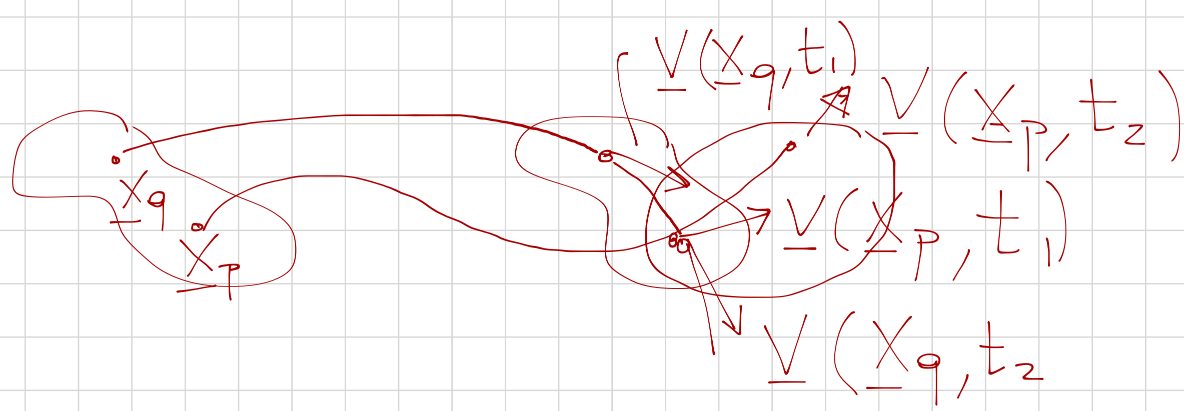
$$\underline{v}(\underline{x},t) = \underline{V}(\underline{\varphi}^{-1}(\underline{x},t),t) =$$

$$= -\lambda\underline{x}$$

$$\underline{a}(\underline{x},t) = \underline{A}(\underline{\varphi}^{-1}(\underline{x},t),t) = \lambda^2\underline{x}$$

NOW, WE KNOW $\underline{A}(\underline{X},t) = \dfrac{d\underline{V}}{dt}(\underline{X},t)$

CAN WE SAY $\underline{a}(\underline{x},t) = \dfrac{d\underline{v}}{dt}(\underline{x},t)$?



$$\frac{dv}{dt} = 0 \neq \underline{a}$$

MATERIAL GRADIENTS

if $\underline{F}(\underline{X},t)$ THEN

$$\nabla_{\underline{X}} \underline{F} = \text{GRAD}\,\underline{F} = \frac{d\underline{F}(\underline{X},t)}{d\underline{X}}\Big|_t = \frac{d\underline{F}}{dX_i}\,\underline{E}_i$$

SPATIAL GRADIENTS

IF $f(\underline{x},t)$ THEN

$$\nabla_{\underline{x}} f(\underline{x},t) = \text{GRAD}\,f = \frac{d}{d\underline{x}} f(\underline{x},t)\Big|_t = \frac{df}{dx_i}\,\underline{e}_i$$

MATERIAL TIME DERIVATIVE

IF I HAVE A SPATIAL FIELD $f(\underline{x},t)$ HOW DO I COMPUTE THE TIME DERIVATIVE OF THAT QUANTITY EXPERIENCED BY THE PARTICLE?

NAMELY

$$\frac{D(\circ)}{Dt} = \frac{d}{dt}(\cdot)\Big|_{\underline{X}}$$

if $f(\underline{x},t)$

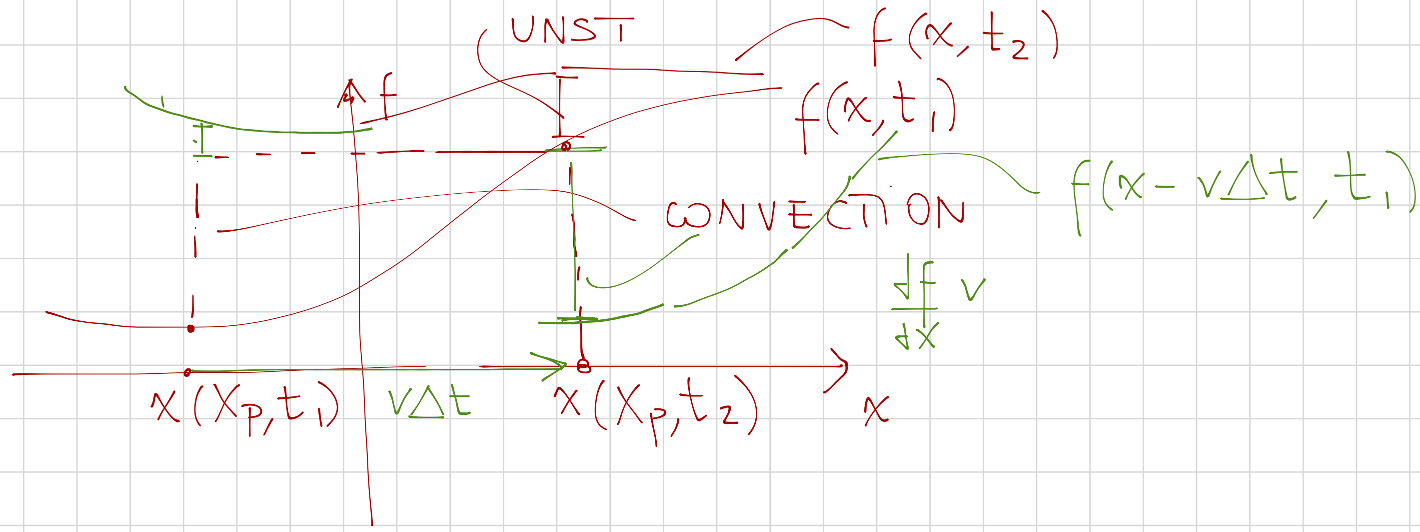
$$\frac{D}{Dt} f(\underline{x},t) = \dot{f}(\underline{x},t) = \frac{d}{dt}\left(f(\underline{x},t)\right)\Big|_{\underline{X}}$$

$$= \left(\frac{df}{dt}\right)\Big|_{\underline{x}} + \left(\frac{df}{d\underline{x}}\right)\Big|_t \left(\frac{d\underline{x}}{dt}\right)\Big|_{\underline{X}} = \frac{df}{dt}\Big|_{\underline{x}} + \frac{df}{d\underline{x}}\Big|_t \frac{d\underline{\varphi}(\underline{X},t)}{dt}\Big|_{\underline{X}}$$

$$= \frac{df}{dt}\Big|_{\underline{x}} + \frac{df}{d\underline{x}}\Big|_t \underline{v}(\underline{x},t)$$

THUS WE CAN COMPUTE THE RATE OF CHANGE OF A PARTICLE SOLELY WITH SPATIAL QUANTITIES

THINK ABOUT IT IN 1-D



IF

$\frac{d\underline{v}}{dt} = \underline{0}$ THE MOTION IS STEADY

IF

$\underline{\nabla}_x \underline{v} = 0$ ⇐ UNIFORM

IF $\underline{v} = \underline{\nabla}\phi$ ⇐ POTENTIAL FLOW

WHAT IS $\underbrace{\underline{\nabla} \times \underline{v} = 0}_{\text{NO VORTICITY!!}}$?