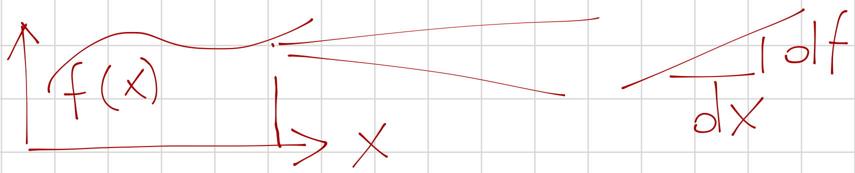
LECTURE 5:

TOPICS:

- REVIEW
- DIFFERENTIAL PROBS
- TRUSS EQ.

LAST TIME WE DISCUSSED DIFFERT FUNCTIONS SCALAR, VECTOR, TENSOR-VAWED

REVIEW OF CALCULUS



← SLOPE OF A FUNCTION

$$\frac{df}{dx}(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

SIMILARLY IF WE HAVE A FUNCTION OF MULTIPLE VARIABLE

$$\frac{df}{dx_i}(x_1, \dots x_i \dots x_d) = \lim_{\varepsilon \to 0} \frac{f(x_1, \dots, x_i + \varepsilon, \dots x_d) - f(x_1, \dots, x_i, \dots x_d)}{\varepsilon}$$

$\frac{df}{dx_i}(\underline{x})$ TELLS YOU THE CHANGE IN THE FUNCTION f WRT THE COORDINATE x_i

THE GRADIENT OF A FUNCTION

$$\underline{\nabla} f = \frac{df}{dx_i}(\underline{x})\, \underline{e}_i$$

SUMMATION

IS A VECTOR POINTING IN THE DIRECTION OF GROWTH

THE GRADIENT OF A FUNCTION

$$\underline{\nabla} f = \frac{df}{dx_i}(\underline{x})\, \underline{e}_i$$

SUMMATION

IS A VECTOR POINTING IN THE DIRECTION OF γ GROWTH

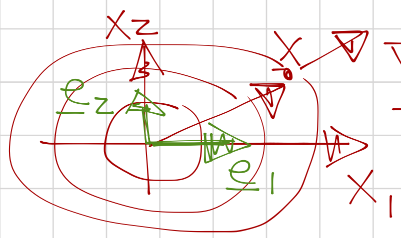
GRADIENT OF A SCALAR FUNCTION

$$\underline{\nabla} f = \frac{df}{dx_i}(\underline{x})\,\underline{e}_i$$

SUMMATION

IS A VECTOR POINTING IN THE DIRECTION OF MAX GROWTH

EG: CONSIDER $f(\underline{x}) = x_1^2 + x_2^2$



$$\underline{\nabla} f(\underline{x}) = \frac{df}{dx_i}(\underline{x})\,\underline{e}_i$$

$$= (2x_1 + 2x_2)\,\underline{e}_1$$

GRADIENT OF VECTOR FUNCTIONS

LET $\underline{u} : \Omega \rightarrow \mathbb{R}^d$ THEN

$$\underline{\nabla}\underline{u} = \frac{d\underline{u}}{dx_j} \otimes \underline{e}_j = \frac{du_i}{dx_j}\,\underline{e}_i \otimes \underline{e}_j$$

$$[\underline{\nabla}\underline{u}]_{\underline{e}} = \begin{bmatrix} du_1/dx_1 & du_2/dx_2 \\ du_2/dx_1 & du_2/dx_2 \end{bmatrix}$$

IFFECTEVILY THE GRADIENT OF A VECTOR IS A TENSOR THAT TELLS YOU HOW MUCH THE ith COMPONENT OF THE VECTOR FUNCTION $\underline{u}$ IS CHANGING IN THE jth DIRECTION

EG: LET $\underline{u}(\underline{x}) = \cos(x_1)\,\underline{e}_1 + \sin(x_2)\,\underline{e}_2$

$$\underline{\nabla}\underline{u} = \frac{d\underline{u}}{dx_j} \otimes \underline{e}_j = \frac{d\cos(x_1)}{dx_1}\,\underline{e}\otimes\underline{e}_1 + \frac{d\cos(x_1)}{dx_2}\,\underline{e}_1 \otimes \underline{e}_2$$

$$+ \frac{d\sin(x_2)}{dx_1}\,\underline{e}_1 \otimes \underline{e}_2 + \frac{d\sin(x_2)}{dx_2}\,\underline{e}_2 \otimes \underline{e}_2$$

$$= -\sin(x_1)\, \underline{e}_1 \otimes \underline{e}_1 + \cos(x_2)\, \underline{e}_2 \otimes \underline{e}_2$$

$$\Rightarrow [\underline{\nabla}\underline{u}]_{\underline{e}} = \begin{bmatrix} -\sin(x_1) & 0 \\ 0 & \cos(x_2) \end{bmatrix}$$

DIVERGENCE OPERATOR

THE DIVERGENCE OF A VECTOR $\underline{u}$ OR TENSOR $\underline{\underline{T}}$ FUNCTION IS DEFINED AS

$$\underline{\nabla} \cdot \underline{u} = \frac{\partial \underline{u}}{\partial x_i} \cdot \underline{e}_i = \frac{\partial u_j}{\partial x_i}\, \underline{e}_j \cdot \underline{e}_i = \frac{\partial u_i}{\partial x_i}$$

$$\underline{\nabla} \cdot \underline{\underline{T}} = \frac{\partial T_{ij} (\underline{e}_i \otimes \underline{e}_j)}{\partial x_k} \cdot \underline{e}_k = \frac{\partial T_{ij}}{\partial x_j}\, \underline{e}_i$$

THE DIVERGENCE IS A MEASURE OF FLUX

EX: $\underline{u} = \cos(x_1)\, \underline{e}_1 + \sin(x_2)\, \underline{e}_2$

$$\underline{\nabla} \cdot \underline{u} = -\sin(x_1) + \cos(x_2)$$

$$\underline{\underline{T}} = (3x_1 + x_2)\, \underline{e}_1 \otimes \underline{e}_1 + 5x_2\, \underline{e}_2 \otimes \underline{e}_1$$

$$\underline{\nabla} \cdot \underline{\underline{T}} = \frac{\partial T_{ij}}{\partial x_j}\, \underline{e}_i = \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2}\right) \underline{e}_1 + \left(\frac{\partial T_{21}}{\partial x_2} + \frac{\partial T_{22}}{\partial x_2}\right) \underline{e}_2$$

$$= 3\underline{e}_1 + 5\underline{e}_2$$

THERE ARE MANY OTHER DIFFERENTIAL OPERATORS

LAPLACIAN $\Delta = \underline{\nabla} \cdot \underline{\nabla} \rightarrow \Delta u = \underline{\nabla} \cdot (\underline{\nabla} u) = \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_j}\, \underline{e}_j\right) \cdot \underline{e}_i$

$$= \frac{\partial u}{\partial x_i \partial x_i}$$

CURL $\underline{\nabla} \times \underline{u} = \frac{d u_i}{d x_j} \underline{e}_i \times \underline{e}_j$

OR ANYTHING $D := \frac{d}{dt} + \Delta \longrightarrow D(u) = \underbrace{\frac{du}{dt} + \Delta u}_{= 0 \text{ HEAT EQ}}$

INTEGRAL THEOREMS

INTEGRATION BY PARTS

$$\frac{d}{dx}\left(f(x)\, g(x)\right) = \frac{df}{dx} g + \frac{dg}{dx} f$$

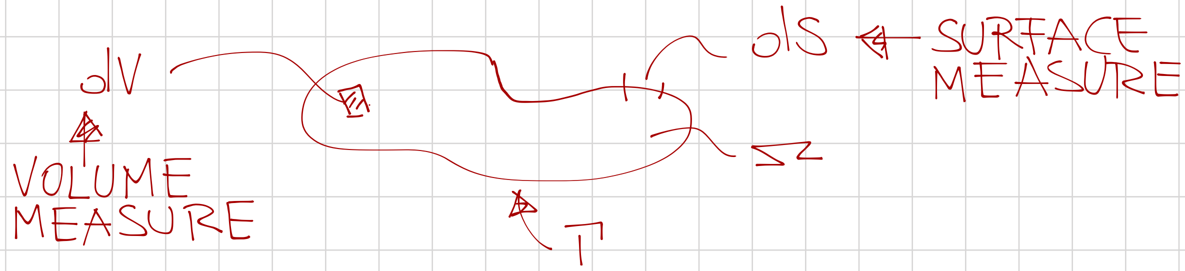
$$\int_a^b f \frac{dg}{dx} dx = \int_a^b \left[\frac{d}{dx}(f g) - \frac{df}{dx} g\right] dx = \left[f g\right]_a^b - \int_a^b \frac{df}{dx} g \, dx$$

GAUSS' (DIVERGENCE) THEOREM

$$\underline{\nabla} \cdot (\psi \underline{u}) = \frac{d \psi u}{d x_i} \cdot \underline{e}_i = \underline{u} \cdot \frac{d\psi}{dx_i} \underline{e}_i + \psi \frac{du}{dx_i} \cdot \underline{e}_i =$$

$$= \underline{u} \cdot \underline{\nabla} \psi + \psi \, \underline{\nabla} \cdot \underline{u}$$

LET $\psi : \Omega \to \mathbb{R}$, $\underline{u} : \Omega \to \mathbb{R}^d$, Γ BE THE BOUDARY OF Ω.

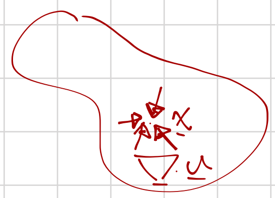


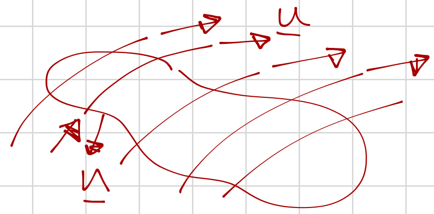
THEN

$$\int_\Omega \psi \underline{\nabla} \cdot \underline{u} \, d\Omega = \int_\Omega \underline{\nabla} \cdot (\psi \underline{u}) \, d\Omega - \int_\Omega \underline{u} \cdot \underline{\nabla} \psi \, d\Omega$$

$$= \int_\Gamma (\psi \underline{u}) \cdot \underline{n} \, d\Omega - \int_\Omega \underline{u} \cdot \underline{\nabla} \psi \, d\Omega$$

RECALL THAT $\nabla \cdot \underline{u}$ IS A MEASURE OF A NET FLUX AT ANY POINT $\underline{x}$. IF WE INTEGRATE IT THEN WE HAVE THE TOTAL FLUX IN THE DOMAIN OF INTEGRATION. BUT THE TOTAL FLUX IN THE DOMAIN OF INTEGRATION CAN BE ALSO CALCULATED AS THE THE INTEGRAL OF THE BOUNDARY FLUX





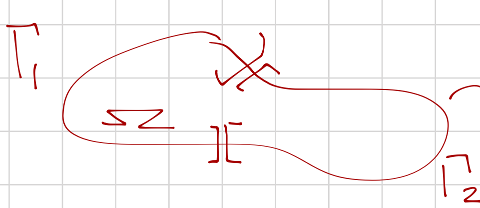
DIFFERENTIAL PROBLEMS

INGREDIENTS:

- A DOMAIN
- A DIFFERENTIAL EQ
- BOUNDARY CONDITIONS

EX: THE HEAT EQ (STEADY STATE)

FIND $u: \Omega \to \mathbb{R}$
DOMAIN



$\sim \Gamma = \Gamma_1 \cup \Gamma_2$

UNKNOWN

GIVEN

$$\Delta u = f \qquad \forall \underline{x} \in \Omega$$

S.T.

$u = \hat{u} \qquad \forall \underline{x} \in \Gamma_1$ ← BOUNDARY CONDITIONS ON THE FUNC

$\nabla u \cdot \underline{n} = h \qquad \forall \underline{x} \in \Gamma_2$ ← BOUNDARY CONDITIONS ON GRAD

A DIFFERENTIAL PROBLEM IS SAID TO BE WELL POSED IF

- A SOLUTION EXISTS
- THE SOLUTION IS UNIQUE
- THE SOLUTION'S BEHAVIOR CHANGES CONTINUOSLY

WITH THE PROBLEM DATA (EG. DIFFUSION CONST, BC ..)