LECTURE 11:

TOPICS:

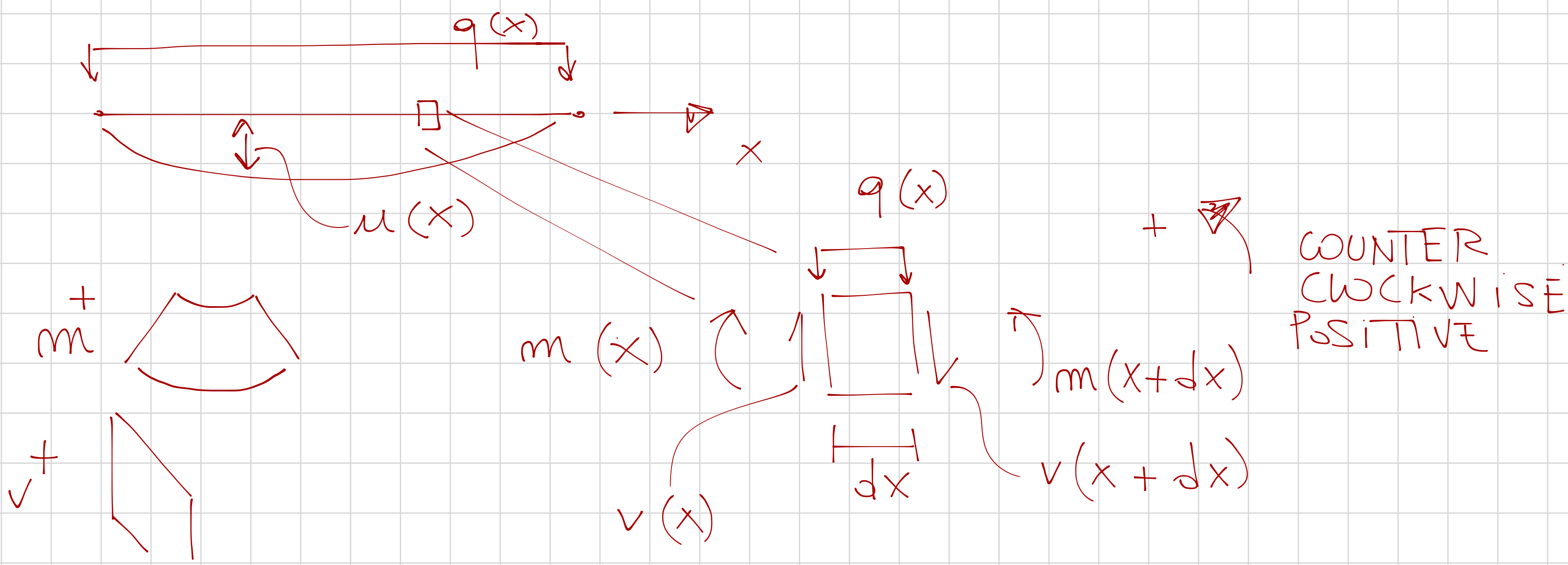
- REVIEW
- BEAM EQ IN 1D

LOGISTICS

- HW 3 DUE TOM

BEAM EQUATIONS

IN EULER BERNOULLI WE ASSUME PLANE SECTIONS REMAIN PLANE



SUM OF FORCES IN THE VERTICAL DIRECTION GIVES

$$v(x) - v(x+dx) - q\,dx = 0 \implies \frac{dV}{dx} = -q$$

BALANCE OF MOMENT ABOUT X

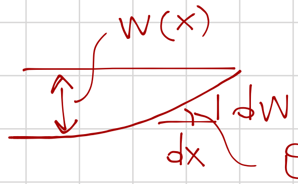
$$m(x+dx) - m(x) - v(x+dx)\,dx - (q\,dx)\,dx = 0$$

$$q\,dx = v(x) - v(x+dx)$$

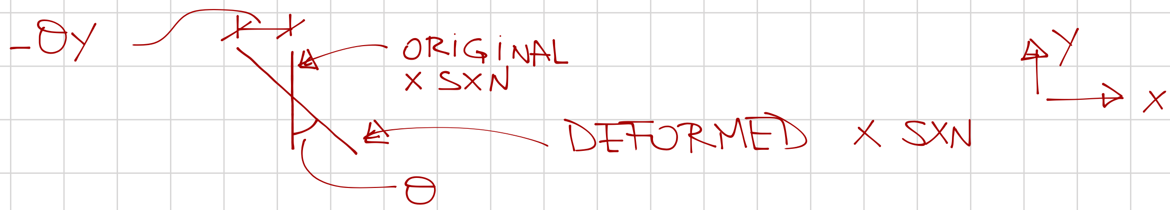
$$m(x+dx) - m(x) - \cancel{v(x+dx)\,dx}^{\,0} - v(x)\,dx + \cancel{v(x+dx)\,dx}^{\,0} = 0$$

$$m(x+dx) - m(x) = v(x)\,dx \implies \frac{dm}{dx} = v$$

USING THE KINEMATIC ASSUMPTIO OF PLANESECTIONS REMAIN PLANE AND NORMAL TO THE NEUTRAL AXIS



$$\theta = \frac{dw}{dx}$$



$$\varepsilon(x,y) = \frac{d}{dx}(-\theta y) = -\frac{d^2w}{dx^2}y \,, \quad M(x) = -\int_A \sigma y = -\int_A E\varepsilon y =$$

$$= \int_A E\frac{d^2w}{dx^2} y \, dA = E\frac{d^2w}{dx^2}\int_A y \, dA \doteq EI\frac{d^2w}{dx^2}$$

WE HAVE

$$\frac{dV}{dx} = \frac{d}{dx}\left(\frac{dM}{dx}\right) = \frac{d}{dx}\left(\frac{d}{dx} EI \frac{d^2w}{dx^2}\right) = EI\frac{d^4w}{dx^4} = -q$$

$EI\dfrac{d^4w}{dx^4} = -q$ ← BEAM EQUATION

THE DIFF PROBLEM THAN READS

$? \, w : [x_i, x_j] \to \mathbb{R}$ $\quad EI\dfrac{d^4w}{dx^4} = \cancel{-q}$ ASSUME $= 0$ FOR NOW $\quad \forall \, x \in (x_i, x_j)$

$w(x_i) = w_i \qquad dw/dx\,(x_i) = \theta(x_i) = \theta_i$

$w(x_j) = w_j \qquad dw/dx\,(x_j) = \theta(x_j) = \theta_j$

IF WE INTEGRATE THE ABOVE THE SOLUTION TAKES THE FORM

$$w(x) = c_1x^3 + c_2x^2 + c_3x + c_4$$

WE HAVE FOUR UNKNOWNS AND FOUR BOUNDARY CONDITIONS SUCH THAT WE CAN SOLVE FOR c_i, $i = 1 \ldots 4$.

IF WE DO SO WE END UP WITH A FORM OF $w(x)$ AS

$$w(x) = N_1(x)\mu_i + N_2(x)\theta_i + N_3(x)\mu_j + N_4(x)\theta_j$$

WHERE, $\boxed{\text{ASSUMING } x_i = 0,\ x_j = \ell}$

$$N_1(x) = 1 - 3\left(\frac{x}{\ell}\right)^2 + 2\left(\frac{x}{\ell}\right)^3$$

$$N_2(x) = \ell\left[\left(\frac{x}{\ell}\right) - 2\left(\frac{x}{\ell}\right)^2 + \left(\frac{x}{\ell}\right)^3\right]$$

$$N_3(x) = 3\left(\frac{x}{\ell}\right)^2 - 2\left(\frac{x}{\ell}\right)^3$$

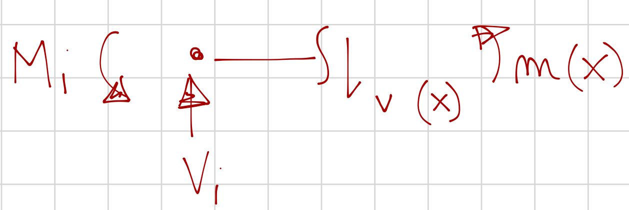
$$N_4(x) = \ell\left[\left(\frac{x}{\ell}\right)^3 - \left(\frac{x}{\ell}\right)^2\right]$$

LET LOWERCASE $m(x)$ & $v(x)$ DENOTE THE INTERNAL MOMENT AND SHEAR RESP.

RECALL

$$m(x) = EI\frac{d^2w}{dx^2}(x) \qquad v(x) = \frac{dm}{dx}(x) = EI\frac{d^3w}{dx^3}(x)$$

AT NODE i WE APPLY EXTERNAL MOMENT & FORCES



SUM MOMENTS

$$M_i + m(0) = 0$$

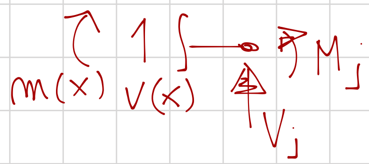
$$M_i = -m(0) = -EI\frac{d^2w(0)}{dx^2} = EI\left\{-N_1''(0) \quad -N_2''(0) \quad -N_3''(0) \quad -N_4''(0)\right\}\begin{Bmatrix}\mu_i\\ \theta_i\\ \mu_j\\ \theta_j\end{Bmatrix}$$

SUM FORCES

$$V_i - v(x_i = 0) = 0$$

$$V_i = v(0) = EI\frac{d^3w(0)}{dx^3} = EI\left\{N_1'''(0) \quad N_2'''(0) \quad N_3'''(0) \quad N_4'''(0)\right\}\begin{Bmatrix}\mu_i\\ \theta_i\\ \mu_j\\ \theta_j\end{Bmatrix}$$

SIMILARLY AT NODE j



$$M_j - m(x_j) = 0$$

$$M_j = EI \frac{d^2 w(e)}{dx^2} = EI \{ N_1''(e) \quad N_2''(e) \quad N_3''(e) \quad N_4''(e) \} \begin{Bmatrix} u_i \\ \theta_i \\ u_j \\ \theta_j \end{Bmatrix}$$

$$V_j + v(x_j = e) = 0$$

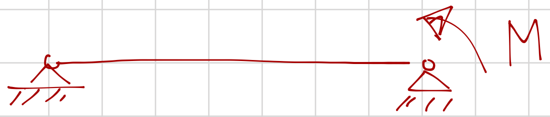
$$V_j = -v(x_j = e) = -EI \{ N'''(e) \quad N'''(e) \quad N'''(e) \quad N'''(e) \} \begin{Bmatrix} u_i \\ \theta_i \\ u_j \\ \theta_j \end{Bmatrix}$$

THUS GIVING

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = EI \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ -N_1''(0) & -N_2''(0) & -N_3''(0) & -N_4''(0) \\ -N_1'''(e) & -N_2'''(e) & -N_3'''(e) & -N_4'''(e) \\ N_1''(e) & N_2''(e) & N_3''(e) & N_4''(e) \end{bmatrix} \begin{Bmatrix} u_i \\ \theta_i \\ u_j \\ \theta_j \end{Bmatrix}$$

SEE ATTACHED SLIDES FOR CONSTRAINTS

A SIMPLE EXAMPLE



CONSIDER A BEAM OF LENGTH $\ell = 10$ WITH AN APPLIED LOAD $M = 10$

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = EI \begin{bmatrix} 12/\ell^3 & 6/\ell^2 & -12/\ell^3 & 6/\ell^2 \\ 6/\ell^2 & 4/\ell & -6/\ell^2 & 2/\ell \\ 12/\ell^3 & -6/\ell^2 & 12/\ell^3 & -6/\ell^2 \\ 6/\ell^2 & 2/\ell & -6/\ell^2 & 4/\ell \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = EI \begin{bmatrix} 12/\ell^3 & 6/\ell^2 & -12/\ell^3 & 6/\ell^2 \\ 6/\ell^2 & 4/\ell & -6/\ell^2 & 2/\ell \\ 12/\ell^3 & -6/\ell^2 & 12/\ell^3 & -6/\ell^2 \\ 6/\ell^2 & 2/\ell & -6/\ell^2 & 4/\ell^2 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$K_{ff} = \begin{bmatrix} 4/\ell & 2/\ell \\ 2/\ell & 4/\ell \end{bmatrix}$$

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} = \begin{bmatrix} 4/\ell & 2/\ell \\ 2/\ell & 4/\ell \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \frac{\ell^2}{16-4} \begin{bmatrix} 4/\ell & -2/\ell \\ -2/\ell & 4/\ell \end{bmatrix} \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} = \frac{\ell^2}{12} \begin{Bmatrix} -20/\ell \\ 40/\ell \end{Bmatrix}$$

$$\begin{Bmatrix} \{Q_s\} \\ \{Q_f\} \end{Bmatrix} = \begin{bmatrix} [K_{ss}] & [K_{fs}] \\ [K_{sf}] & [K_{ff}] \end{bmatrix} \begin{Bmatrix} \{w_s\} \\ \{w_f\} \end{Bmatrix}$$

$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \begin{bmatrix} 6/\ell^2 & 6/\ell^2 \\ -6/\ell^2 & -6/\ell^2 \end{bmatrix} \begin{Bmatrix} -20/12\,\ell \\ 40/12\,\ell \end{Bmatrix}$$