LECTURE 20:

TOPICS:

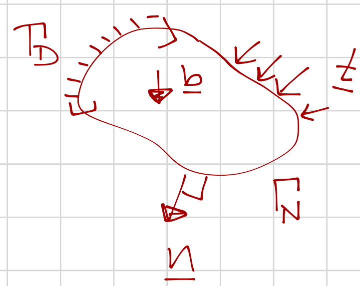
- LINEAR ELASTICITY

LOGISTICS:

- HW # 8 DUE MONDAY
- FINAL PROJECT DUE JANUARY 15th

LINEAR ELASTICITY

FIND $\underline{u} : \Omega \to \mathbb{R}^d$ S.T.



$$\underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{\underline{\nabla}}\, \underline{u}) = \underline{b} \quad \forall \underline{x} \in \Omega$$

$$\underline{u} = \underline{g} \quad \forall \underline{x} \in \Gamma_D$$

$$\underline{\underline{\sigma}}\, \underline{n} = \underline{t} \quad \forall \underline{x} \in \Gamma_N$$

AND

$$\underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}) = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}} = \lambda \, \mathrm{tr}(\underline{\underline{\varepsilon}}) \underline{\underline{1}} + 2\mu \underline{\underline{\varepsilon}}, \qquad \underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\underline{\nabla}}\, \underline{u} + \underline{\underline{\nabla}}\, \underline{u}^T\right)$$

NOTE: $\mathrm{tr}(\underline{\underline{\varepsilon}}) = \mathrm{tr}(\underline{\underline{\nabla}}\, \underline{u}) \leftarrow$ PROVE IT

— N

$\underline{\underline{\sigma}} \leftarrow$ STRESS TENSOR, $\underline{\underline{\varepsilon}} \leftarrow$ STRAIN TENSOR

$\lambda, \mu \leftarrow$ LAME'S PARAMETERS

$\underline{b} \leftarrow$ BODY FORCES (EG. GRAVITY)

$\underline{t} \leftarrow$ BOUNDARY TRACTIONS/FORCES

WEAK FORM

$$S = \{ \underline{u} \in [H^1(\Omega)]^d \mid \underline{u} = \underline{g} \quad \forall \underline{x} \in \Gamma_D \}$$

$$V = \{ \underline{v} \in [H^1(\Omega)]^d \mid \underline{v} = \underline{0} \quad \forall \underline{x} \in \Gamma_D \}$$

$$\int_\Omega (\underline{\nabla} \cdot \underline{\underline{\sigma}} - \underline{b}) \cdot \underline{v} \, d\Omega = 0 \qquad \forall \underline{v} \in V$$

$$\int_\Omega \left[(\underline{\nabla} \cdot \underline{\underline{\sigma}}) \cdot \underline{v} - \underline{b} \cdot \underline{v} \right] d\Omega = 0 \qquad \forall \underline{v} \in V$$

$$\underline{\nabla} \cdot (\underline{\underline{\sigma}}\, \underline{v}) = \underline{\underline{\sigma}} : \underline{\nabla} \underline{v} + \underline{\nabla} \cdot \underline{\underline{\sigma}} \cdot \underline{v} \Rightarrow (\underline{\nabla} \cdot \underline{\underline{\sigma}}) \cdot \underline{v} = \underline{\nabla} \cdot (\underline{\underline{\sigma}}\, \underline{v}) - \underline{\underline{\sigma}} : \underline{\nabla} \underline{v}$$

$$\int_\Omega \left[\underline{\nabla} \cdot (\underline{\underline{\sigma}}\, \underline{v}) - \underline{\underline{\sigma}} : \underline{\nabla} \underline{v} - \underline{b} \cdot \underline{v} \right] d\Omega = 0 \qquad \forall \underline{x} \in \Omega$$

USE DIVERGENCE THM AND SYMMETRY OF $\underline{\underline{\sigma}}$

$$\int_\Gamma (\underline{\underline{\sigma}}\, \underline{n}) \cdot \underline{v} \, d\Gamma - \int_\Omega \underline{\underline{\sigma}} : \underline{\nabla} \underline{v} \, d\Omega - \int_\Omega \underline{b} \cdot \underline{v} \, d\Omega = 0 \qquad \forall \underline{v} \in V$$

NOTE: $\underline{v} = 0$ ON Γ_D & $\underline{\underline{\sigma}}\, \underline{n} = \underline{t}$ ON Γ_N

$$\int_{\Gamma_N} \underline{t} \cdot \underline{v} \, d\Gamma - \int_\Omega \underline{\underline{\sigma}} : \underline{\nabla} \underline{v} \, d\Omega - \int_\Omega \underline{b} \cdot \underline{v} \, d\Omega = 0 \qquad \forall \underline{v} \in V$$

(W) ? $\underline{u} \in S$:

$$a(\underline{v}, \underline{u}) = F(\underline{v}) \qquad \forall \underline{v} \in V$$

$$a(\underline{v}, \underline{u}) = \int_\Omega \underline{\underline{\sigma}} : \underline{\nabla} \underline{v} \, d\Omega = \int_\Omega \left(\lambda \, tr(\underline{\nabla} \underline{u}) \underline{\underline{1}} + 2\mu \underline{\underline{\varepsilon}} \right) : \underline{\nabla} \underline{v} \, d\Omega$$

$$F(\underline{v}) = \int_{\Gamma_N} \underline{t} \cdot \underline{v} \, d\Gamma - \int_\Omega \underline{b} \cdot \underline{v} \, d\Omega$$

GALERKIN APPROXIMATION

? $\underline{u}^h \in S^h \subseteq S$:

$$a(\underline{v}^h, \underline{u}^h) = F(\underline{v}^h) \qquad \forall\, \underline{v}^h \in V$$

MATRIX FORM

$\underline{u}^h(\underline{x}) = \underline{u}_a \phi_a(\underline{x})$, $\quad \underline{v}^h = \underline{v}_a \phi_a(\underline{x})$

NOTE: $\underline{\nabla} \underline{u}^h = \underline{u}_i \otimes \underline{\nabla}\phi_i$, $\quad \underline{\nabla} \cdot \underline{u}^h = \underline{u}_i \cdot \underline{\nabla}\phi_i = tr(\underline{\nabla}\underline{u}^h)$

$$a(\underline{v}^h, \underline{u}^h) = \int_{\Omega} \underline{\underline{\sigma}}(\underline{\nabla}\underline{u}^h) : \underline{\nabla}\underline{v}^h \, d\Omega =$$

$$= \int_{\Omega} \left(\lambda\, tr(\underline{\nabla}\underline{u}^h)\underline{\underline{1}} + 2\mu\, \underline{\underline{\varepsilon}}\right) : \underline{\nabla}\underline{v}^h \, d\Omega =$$

$$= \int_{\Omega} \left(\lambda\, tr(\underline{\nabla}\underline{u}^h)\, tr(\underline{\nabla}\underline{v}^h) + \mu\left(\underline{\nabla}\underline{u}^h + \underline{\nabla}\underline{u}^{h\,T}\right) : \underline{\nabla}\underline{v}^h\right) d\Omega =$$

$$= \int_{\Omega} \Big[\lambda (\underline{v}_i \cdot \underline{\nabla}\phi_i)(\underline{\nabla}\phi_j \cdot \underline{u}_j) + \mu\big[(\underline{v}_i \cdot \underline{\nabla}\phi_j)(\underline{\nabla}\phi_i \cdot \underline{u}_j)$$

$$+ (\underline{v}_i \cdot \underline{u}_j)(\underline{\nabla}\phi_i \cdot \underline{\nabla}\phi_j)\big]\Big] d\Omega =$$

$$= \int_{\Omega} \underline{v}_i \cdot \left[\lambda\, \underline{\nabla}\phi_i \otimes \underline{\nabla}\phi_j + \mu\, \underline{\nabla}\phi_j \otimes \underline{\nabla}\phi_i + \mu \underline{\underline{1}}\, \underline{\nabla}\phi_i \cdot \underline{\nabla}\phi_j\right] \cdot \underline{u}_j \, d\Omega$$

$$\underline{v}_i \cdot \underline{\underline{K}}_{ij}\, \underline{u}_j$$

$$\underline{\underline{K}}_{ij} = \int_{\Omega} \left[\lambda \underline{\nabla}\phi_i \otimes \underline{\nabla}\phi_j + \mu\left[\underline{\nabla}\phi_j \otimes \underline{\nabla}\phi_i + \underline{\nabla}\phi_i \cdot \underline{\nabla}\phi_j\, \underline{\underline{1}}\right]\right] d\Omega$$

NOTE THAT HERE $\underline{\underline{K}}_{ij}$ IS A TENSOR ENTRY IN THE GLOBAL STIFFNESS

THE COMPONENTS ARE FOUND BY

$$[\underline{\underline{k}}_{ij}]_{fg} = \underline{e}_f \cdot \underline{\underline{k}}_{ij} \underline{e}_g$$

$$= \int_\Omega \Big[\lambda (\underline{\nabla}\phi_i \cdot \underline{e}_f)(\underline{\nabla}\phi_j \cdot \underline{e}_g) + \mu \big[(\underline{\nabla}\phi_j \cdot \underline{e}_f)(\underline{\nabla}\phi_i \cdot \underline{e}_g) + \underline{\nabla}\phi_i \cdot \underline{\nabla}\phi_j \, \delta_{fg} \big] \Big] d\Omega$$

SIMILARLY TO BEFORE WE COMPUTE ELEMENT STIFFNESSES $\underline{\underline{k}}^e_{ab}$ & THEN ASSEMBLE THE GLOBAL SYSTEM WHERE

$$\underline{\underline{k}}^e_{ab} = \int_{\Omega^e} \Big[\lambda \underline{\nabla}\psi^e_a \otimes \underline{\nabla}\psi^e_b + \mu \big[\underline{\nabla}\psi^e_a \otimes \underline{\nabla}\psi^e_b + \underline{\nabla}\psi^e_a \cdot \underline{\nabla}\psi^e_b \, \underline{\underline{1}} \big] \Big] d\Omega$$