# Direct Stiffness Methods

## Beams and Beam-Columns

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# Review of Beams

# Beams

We arrived at the beam governing equation as

$$\frac{d}{dx^2}\left(EI\frac{dv}{dx^2}\right) = -q(x), \quad \forall x$$

where $v$ is our beam displacement. Where

$$m(x) = \left(EI\frac{dv}{dx^2}\right) \leftarrow \text{Internal moment}$$

$$f(x) = \frac{d}{dx}\left(EI\frac{dv}{dx^2}\right) \leftarrow \text{Internal shear}$$

# Beams

The analytical solution of the beam equation becomes

$$v(x) = N_1(x)\, v_i + N_2(x)\theta_i + N_3(x)v_j + N_4(x)\theta_j$$

where

$$\begin{aligned}
N_1(x) &= 1 - 3\left(\frac{x}{\ell}\right)^2 + 2\left(\frac{x}{\ell}\right)^3 \\
N_2(x) &= \ell\left[\left(\frac{x}{\ell}\right) - 2\left(\frac{x}{\ell}\right)^2 + \left(\frac{x}{\ell}\right)^3\right] \\
N_3(x) &= 3\left(\frac{x}{\ell}\right)^2 - 2\left(\frac{x}{\ell}\right)^3 \\
N_4(x) &= \ell\left[\left(\frac{x}{\ell}\right)^3 - \left(\frac{x}{\ell}\right)^2\right]
\end{aligned}$$

# Beams

The nodal equilibrium equations then reduce

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ -N_1''(0) & -N_2''(0) & -N_3''(0) & -N_4''(0) \\ -N_1'''(L) & -N_2'''(L) & -N_3'''(L) & -N_4'''(L) \\ N_1''(L) & N_2''(L) & N_3''(L) & N_4''(L) \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}.$$

# Beams

The nodal equilibrium equations then reduce

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ -N_1''(0) & -N_2''(0) & -N_3''(0) & -N_4''(0) \\ -N_1'''(L) & -N_2'''(L) & -N_3'''(L) & -N_4'''(L) \\ N_1''(L) & N_2''(L) & N_3''(L) & N_4''(L) \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}.$$

# Beams

The nodal equilibrium equations then reduce

$$\left\{\begin{matrix} V_i \\ M_i \\ V_j \\ M_j \end{matrix}\right\} = \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ -N_1''(0) & -N_2''(0) & -N_3''(0) & -N_4''(0) \\ -N_1'''(L) & -N_2'''(L) & -N_3'''(L) & -N_4'''(L) \\ N_1''(L) & N_2''(L) & N_3''(L) & N_4''(L) \end{bmatrix} \left\{\begin{matrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{matrix}\right\}.$$

# Beams

The nodal equilibrium equations then reduce

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ -N_1''(0) & -N_2''(0) & -N_3''(0) & -N_4''(0) \\ -N_1'''(L) & -N_2'''(L) & -N_3'''(L) & -N_4'''(L) \\ N_1''(L) & N_2''(L) & N_3''(L) & N_4''(L) \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}.$$

# Beams

The nodal equilibrium equations then reduce

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} k_{fv} & k_{f\theta} & -k_{fv} & k_{f\theta} \\ k_{mv} & k_{m\theta} & -k_{m\theta} & \hat{k}_{m\theta} \\ k_{fv} & -k_{f\theta} & -k_{fv} & -k_{f\theta} \\ k_{mv} & \hat{k}_{m\theta} & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}$$

where

$$k_{fv} = \frac{12EI}{\ell^3}, \quad k_{mv} = k_{f\theta} = \frac{6EI}{\ell^2}, \quad k_{m\theta} = \frac{4EI}{\ell}, \quad \hat{k}_{m\theta} \frac{2EI}{\ell}$$

# Global Assembly

# Beams

# Frame Elements

# Frame Elements

# Frame Elements