PRECEPT 3 (HW 2)

CEE 361-513: Introduction to Finite Element Methods

Wednesday Oct. 3

PROBLEM 1 DIFFERENTIAL PROBLEMS

What is the order and linearity of the following differential problems. If linear, propose a set of sufficient boundary conditions and the physical description of those boundary conditions.
Steady-state heat equation:

$$\Delta \mathbf{u} = \nabla^2 \mathbf{u} = 0 \tag{1}$$

Korteweg de Vries equation (describes stationary waves or solitons)

$$\frac{\delta \mathbf{u}}{\delta t} + \frac{\delta^3 \mathbf{u}}{\delta x^3} - 6\mathbf{u}\frac{\delta \mathbf{u}}{\delta x} = 0 \tag{2}$$

1. Linear, order 2. Boundary conditions: Dirichlet = hold temperature $\mathbf{u}$ fixed; Neumann = constant rate of cooling or heating.
2. Nonlinear, order 3.

PROBLEM 2 1D TRUSS

Consider the following 1-D truss system:

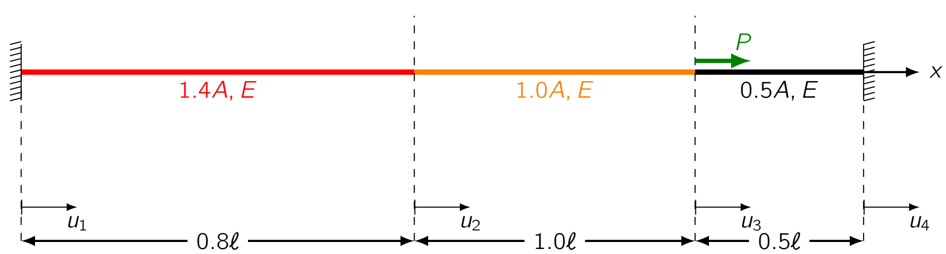

Figure 1: The 1-D Truss system

Using the information provided below solve for the displacement of each node, the unknown reactions. Draw the final free body diagram, with element forces, displacements, and reactions labeled.

$$\ell = 3.0\text{m} \qquad A = 1000\text{mm}^2$$
$$E = 200\text{GPa} \qquad P = 60\text{kN}$$

2.1 Connectivity + Free Body Diagram

The first step is to prepare the labeling conventions: write the connectivity matrix for the system (relating the local node numbers with the global node numbers) and draw a free body diagram of the system (include prescribed loads and displacements; and unknown reactions, loads, and displacements).

element	i node	j node
1	1	2
2	2	3
3	3	4

Table 1: Connectivity Matrix

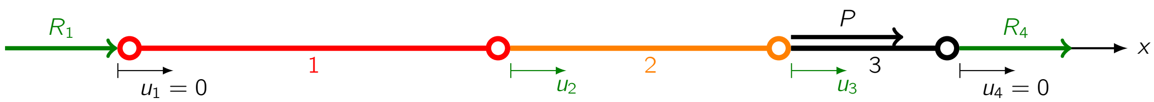

Figure 2: The Complete FBD

2.2 Element Stiffness Formulation

We then write the element stiffness matrices. The notation convention is the same as that followed in class for the local node labels (i, j).

$$\begin{bmatrix} -f_i^1 \\ f_j^1 \end{bmatrix} = \begin{bmatrix} \frac{A_1E_1}{\ell_1} & -\frac{A_1E_1}{\ell_1} \\ -\frac{A_1E_1}{\ell_1} & \frac{A_1E_1}{\ell_1} \end{bmatrix} \begin{bmatrix} u_i^1 \\ u_j^1 \end{bmatrix} \qquad \begin{bmatrix} -f_i^2 \\ f_j^2 \end{bmatrix} = \begin{bmatrix} \frac{A_2E_2}{\ell_2} & -\frac{A_2E_2}{\ell_2} \\ -\frac{A_2E_2}{\ell_2} & \frac{A_2E_2}{\ell_2} \end{bmatrix} \begin{bmatrix} u_i^2 \\ u_j^2 \end{bmatrix}$$

$$\begin{bmatrix} -f_i^3 \\ f_j^3 \end{bmatrix} = \begin{bmatrix} \frac{A_3E_3}{\ell_3} & -\frac{A_3E_3}{\ell_3} \\ -\frac{A_3E_3}{\ell_3} & \frac{A_3E_3}{\ell_3} \end{bmatrix} \begin{bmatrix} u_i^3 \\ u_j^3 \end{bmatrix}$$

2.3 Local to Global DOF

Using the connectivity matrix we replace the local node numbers (i, j) with the global node numbers in the element stiffness.

$$\begin{bmatrix} -f_i^1 \\ f_j^1 \end{bmatrix} = \begin{bmatrix} \frac{A_1E_1}{\ell_1} & -\frac{A_1E_1}{\ell_1} \\ -\frac{A_1E_1}{\ell_1} & \frac{A_1E_1}{\ell_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \begin{bmatrix} -f_i^2 \\ f_j^2 \end{bmatrix} = \begin{bmatrix} \frac{A_2E_2}{\ell_2} & -\frac{A_2E_2}{\ell_2} \\ -\frac{A_2E_2}{\ell_2} & \frac{A_2E_2}{\ell_2} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} -f_i^3 \\ f_j^3 \end{bmatrix} = \begin{bmatrix} \frac{A_3E_3}{\ell_3} & -\frac{A_3E_3}{\ell_3} \\ -\frac{A_3E_3}{\ell_3} & \frac{A_3E_3}{\ell_3} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

2.4 Nodal Equilibrium

Writing the equilibrium equations for the 4 nodes:

$$\begin{aligned} R_1 &= -f_i^1 \\ 0 &= f_j^1 - f_i^2 \\ P &= -f_j^2 + f_i^3 \\ R_4 &= f_j^3 \end{aligned}$$

2.5 Assemble global stiffness matrix from element stiffness matrices

Let $k_i = A_i E_i / \ell_i$ for $i = 1...3$. We can write down the equilibrium equations in matrix form. Namely, as we did in class, write the equilibrium equations with a load vector containing reactions and external forces, denoted it by $\{P\}$, the stiffness matrix denoted by $[K]$, and the vector of displacements $\{U\}$ such that

$$[K]\{U\} = \{P\}$$

Let us denote

$$k_1 = \frac{1.4AE}{0.8\ell} \qquad k_2 = \frac{AE}{\ell} \qquad k_3 = \frac{0.5AE}{0.5\ell}$$

$$\begin{bmatrix} R_1 \\ 0 \\ P \\ R_4 \end{bmatrix} = \begin{bmatrix} -f_i^1 \\ f_j^1 - f_i^2 \\ -f_j^2 + f_i^3 \\ f_i^3 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

2.6 Apply Boundary Conditions

For our given problem:

$$u_1 = 0 \qquad u_4 = 0 \qquad P = 60\text{kN}$$
$$R_1 = ? \qquad R_4 = ? \qquad u_2 = ? \qquad u_3 = ?$$

Modifying our stiffness and force matrices to reflect the knowns and unknowns:

$$\begin{bmatrix} R_1 \\ 0 \\ 60 \\ R_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 + k_3 & k_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

2.7 Solve for unknown displacements and reactions

All the prescribed displacements and boundary conditions are zero (fixed nodes 1 and 4), so the only unknowns are $u_2 \& u_3$ and we can solve for them using the "unknown" portions of the formulation: rows 2 and 3 and the center of the global stiffness matrix (rows and columns for global nodes 2 and 3):

$$\begin{bmatrix} 0 \\ 60 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

This leaves a 2 x 2 matrix, which i computationally inexpensive to invert, allowing us to directly solve for the unknown displacements:

$$u_2 = 0.1078mm$$
$$u_3 = 0.297mm$$

The reactions can be found by multiplying the corresponding rows of the original stiffness matrix with the displacement vector and accounting again for the boundary conditions of fixed end nodes.

$$R_1 = -k_1 u_2 = -12.58kN$$
$$R_4 = -k_3 u_3 = -47.4kN$$

In general we would not solve the matrix by hand (too tedious, especially with larger systems). The negative signs indicate that the original FBD has the wrong direction for the reaction forces. The final FBD, where (T) indicates tension and (C) indicates compression:

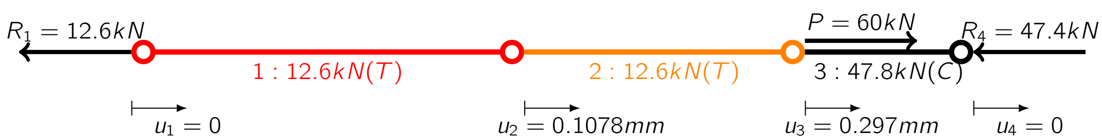

Figure 3: Final Complete FBD