LECTURE 1.

TOPICS

- VECTOR & VECTOR ALGEBRA
  - BASIS
  - DEFINITION OF VECTOR
  - DOT PRODUCT $\rightarrow$ KRONECKER DELTA
  - CROSS PRODUCT $\rightarrow$ LEVI CIVITA
  - INDICIAL NOTATION

- TENSORS & TENSOR ALGEBRA
  - DYADIC PRODUCT
  - DOUBLE DOT PRODUCT
  - TRANSPOSE
  - INVERSE
  - SINGULAR
  - EIGENVALUES & VECS
  - POSITIVE (SEMI)-DEFINITE
  - NORM OF TENSOR
  - PROJECTION

BEFORE GETTING INTO CM WE NEED TO DEFINE A FEW QUANTITIE

SCALARS & SCALAR FIELDS

$\alpha(\underline{x}) \in \mathbb{R}$ TEMPERATURE, PRESS.

VECTORS POSSES A MAGNITUDE & DIRECTION

$\underline{v}(\underline{x}) \in \mathbb{R}^d$ VELOCITY, FORCE ETC

DENOTED BY ONE UNDERLINE

REAL VECTOR SPACE

A SET V & OPERATION $+ : V \times V \to V$

SUCH THAT

$\underline{a} + \underline{b} = \underline{b} + \underline{a}$

$(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$

$\exists\, \underline{0} \in V$ S.T. $\underline{a} + \underline{0} = \underline{a}$

$\exists\, -\underline{a}$ S.T. $(-\underline{a}) + \underline{a} = \underline{0}$

FURTHER FOR ANY $\alpha \in \mathbb{R}$ & $\underline{a} \in V$

$1\,\underline{a} = \underline{a}$

$\alpha(\beta\,\underline{a}) = (\alpha\beta)\,\underline{a}$

$\alpha(\underline{a} + \underline{b}) = \alpha\underline{a} + \alpha\underline{b}$

$(\alpha + \beta)\,\underline{a} = \alpha\underline{a} + \beta\underline{a}$

NORM VECTOR SPACE

A NORM VECTOR SPACE IS A REAL VECTOR SPACE V ENDOWED WITH $\|\cdot\| : V \times V \to \mathbb{R}$

$\|\underline{a}\| \geq 0 \quad \forall\, \underline{a} \in V \quad \|\underline{a}\| = 0$ IFF $\underline{a} = \underline{0}$

$\|\alpha\,\underline{a}\| = |\alpha|\,\|\underline{a}\|$

$\|\underline{a} + \underline{b}\| \leq \|\underline{a}\| + \|\underline{b}\|$

INNER PRODUCT SPACE

IT'S A NORM VECTOR SPACE ENDOWED WITH $\cdot : V \times V \to \mathbb{R}$

$(\underline{a} + \underline{b}) \cdot \underline{c} = \underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{c} \quad \forall\, \underline{a}, \underline{b}, \underline{c} \in V$

$(\alpha\underline{a}) \cdot \underline{b} = \alpha(\underline{a} \cdot \underline{b})$

$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$

$\underline{a} \cdot \underline{a} \geq 0 \quad \forall\, \underline{a} \in V \quad \underline{a} \cdot \underline{a} = 0$ IFF $\underline{a} = 0$

A NATURAL NORM FOR THE SPACE IS

$\|\underline{a}\| = \sqrt{\underline{a} \cdot \underline{a}}$

EUCLEDIAN VECTOR SPACE

IT'S A NORMED INNER PRODUCT SPACE ENDOWED WITH A VECTOR PRODUCT $\wedge : V \times V \to V$ SUCH THAT

$$\underline{a} \wedge \underline{b} = -\underline{b} \wedge \underline{a}$$

$$(\alpha \underline{a} + \beta \underline{b}) \wedge \underline{c} = \alpha \underline{a} \wedge \underline{c} + \beta \underline{b} \wedge \underline{c}$$

$$\underline{a} \cdot (\underline{a} \wedge \underline{b}) = 0$$

$$(\underline{a} \wedge \underline{b}) \cdot (\underline{a} \wedge \underline{b}) = (\underline{a} \cdot \underline{a})(\underline{b} \cdot \underline{b}) - (\underline{a} \cdot \underline{b})^2$$

THE MOST COMMON FORM IS $\times$

BASIS

A SET OF BASIS $\{\underline{q}_i\}_i$ OF V IS A SET OF VECTORS SUCH THAT EACH $\underline{v} \in V$ CAN BE WRITTEN AS

$$\underline{v} = \sum_{i=1}^{d} v^i \underline{q}_i$$

COMPONENTS OF $\underline{v}$ IN $\underline{q}_i$ BASIS

INDEX NOTATION INTERLUDE

WE USE EINSTEIN SUMMATION CONVENTION THAT INDECES REPEATED EXACTLY TWICE IMPLY SUMMATION

$$v^i \underline{q}_i \equiv \sum_{i=1}^{d} v^i \underline{q}_i$$

REPETED INDECES ARE REFERRED AS DUMMY INDECES

NON-REPEATE ARE REFERRED AS FREE INDICES

EG CONSIDER A VECTOR $\underline{V}_K$

$$\underline{V}_K = V^i_k \, \underline{g}_i$$

i IS DUMMY K IS FREE & MUST APPEAR ON BOTH SIDES

END INTERLUDE

IF $\{\underline{g}_i\}_i$ ARE ORTHONORMAL

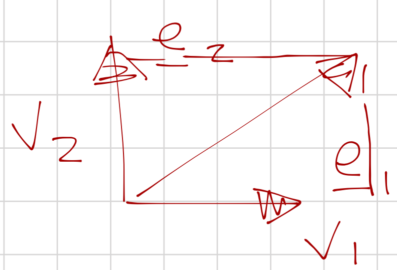
$$\underline{g}_i \cdot \underline{g}_j \overset{?}{=} \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{ELSE} \end{cases}$$

KRONECKER DELTA

THEN THE BASIS ARE OFTEN DENOTED AS $\{\underline{e}_i\}_i$ &

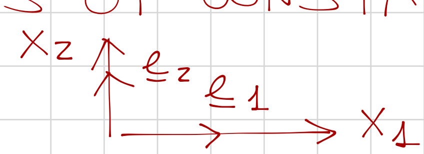
$$\underline{V} = V_i \, \underline{e}_i$$

UNIQUILEY DEFINED



$$V_i = \underline{V} \cdot \underline{e}_i$$

OFTEN BASIS ARE ASSOCIATED WITH COORDINATE NAMELY THEY ARE TANGENT VECTORS TO LINES OF CONSTANT CRDS



LATER WE WILL DISCUSS GEN BASIS & COORDS

SOME REMARKS ON PRODUCTS

DOT PRODUCT REPRESENTS A PROJECTION ON

$$\underline{a} \cdot \underline{b} = (a_i \underline{e}_i) \cdot (b_j \underline{e}_j)$$

$$= a_i b_j \, \underline{e}_i \cdot \underline{e}_j = a_i b_i$$

$$= \|a\| \|b\| \cos(\theta)$$

CROSS PRODUCT

$$\underline{a} \times \underline{b} = (a_i \underline{e}_i) \times (b_j \underline{e}_j)$$

$$= a_i b_j \, \underline{e}_i \times \underline{e}_j = a_i b_j \, \varepsilon_{ijk} \, \underline{e}_k$$

$\varepsilon_{ijk} \leftarrow$ LEVI-CIVITA PERMUTATION

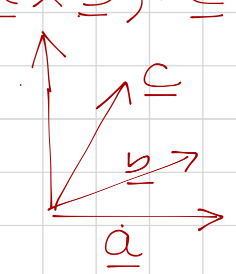
$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if EVEN PERM} \\ -1 & \text{IF ODD PERM} \\ 0 & \text{if REPEATED} \end{cases}$$

$\|\underline{a} \times \underline{b}\| \leftarrow$ REPRESENTS THE AREA OF PARALLELOGRAM

$$\|\underline{a} \times \underline{b}\| = \|\underline{a}\| \, \|\underline{b}\| \sin\theta$$

TRIPLE SCALAR PRODUCT

$(\underline{a} \times \underline{b}) \cdot \underline{c} \leftarrow$ REPRESENTS THE VOLUME



DUAL VECTOR SPACE

THE DUAL SPACE TO V IS DENOTED BY V^* AND IT'S THE SPACE OF ALL LINEAR FUNCTIONALS ON V.

NAMELY if $\underline{v}^* \in V$, $\underline{v}^* : V \to \mathbb{R}$
SUCH THAT $\forall\, \alpha \in \mathbb{R}$, $\underline{a}, \underline{b} \in V$

$$\underline{v}^*(\alpha \underline{a} + \underline{b}) = \alpha\, \underline{v}^*(\underline{a}) + \underline{v}^*(\underline{b})$$

OFTEN $\underline{v}^*(\underline{a})$ IS DENOTE, ABUSING NOTATION AS

$$\underline{v}^* \cdot \underline{a}$$

V^* HAS THE SAME DIMENSIONS OF V.

THE VECTOR SPACE V^* IS SPANNED BY THE DUAL BASIS $\{\underline{g}^i\}_i$.

THE DUAL BASIS SATISFY

$$\underline{g}^i \cdot \underline{g}_j = \delta^i_j$$

NOTE IF V IS AN EUCLEDIAN SPACE & $\underline{g}_j$ ARE ORTHONORMAL

NOTE THAT BOTH V & V^* ARE INNER PRODUCT SPACES IN THEIR OWN

ANY VECTOR $\underline{a}$ CAN BE REPRESENTED IN EITHER ONE OF THE BASIS

$$\underline{a} = a^i \underline{g}_i = a_i \underline{g}^i$$

a^i ARE OFTEN TERME CONTRAVARIANT
a_i " " " COVARIANT

&

$\underline{g}^i$ COVARIANT / $\underline{g}_i$ CONTRAVARIANT

WHY ARE THIS IMPORTANT ?

99 % OF THE TIME WE WORK IN EUCLEDIAN SPACES BUT THERE ARE INSTANCES IN WHICH WE DON'T SUCH AS STRUCTURAL MECHANICS, RELATIVITY ETC...

IN THESE INSTANCES DIFFERENTIAL GEOMETRY PLAYS A CENTRAL ROLE & THIS FORMALISM IS EXTREMELY IMPORTANT

<u>ON THE REPRESENTATION OF VECTORS</u>

$$\underline{V} = V_i \underline{g}^i = V^j \underline{g}_j$$

$$[\underline{V}]_{\underline{g}_i} = \begin{bmatrix} V^1 \\ V^2 \\ V^3 \\ \vdots \end{bmatrix}$$