Finite Elements in n-D

Triangular Elements

Maurizio M. Chiaramonte

The Local Element View

1. Parametric element domain $\hat{\Omega}$
2. Physical element domain Ω^e
3. Map $\hat{\boldsymbol{x}}^e(\xi)$ from parametric to physical domain
4. Shape Functions $\{\hat{\phi}_a(\xi)\}_{a=1}^{n^e_{dof}}$
5. Degrees of freedom $\{u^e_a\}_{a=1}^{n^e_{dof}}$, $u^h(\xi) = \sum_{a=1}^{n^e_{dof}} u^e_a \hat{\phi}_a(\xi)$
6. Local to global dof map

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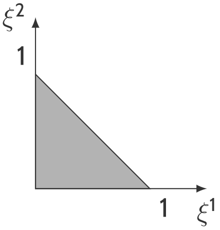
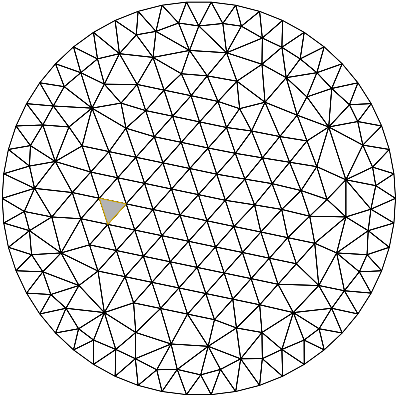
The Local Element View

Triangular parametric element $\hat{\Omega}$



Triangular (simplicial) mesh



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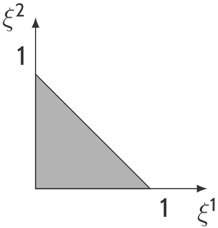

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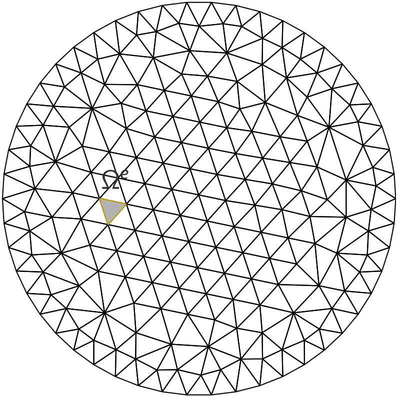

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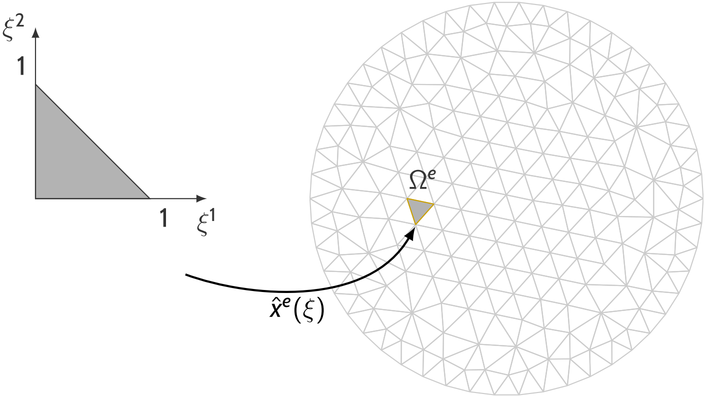
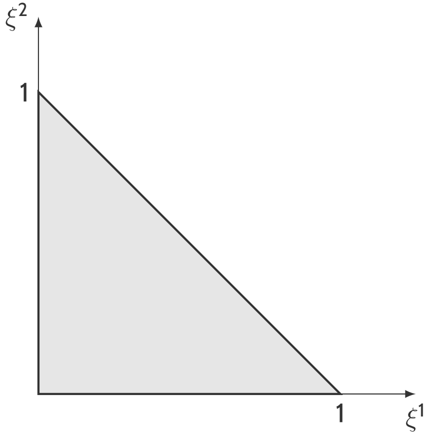


Constructing Basis Functions

Barycentric Coordinates



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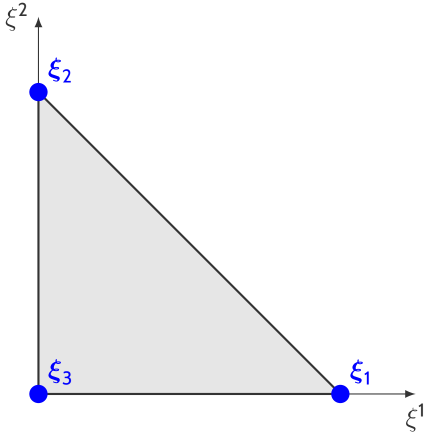

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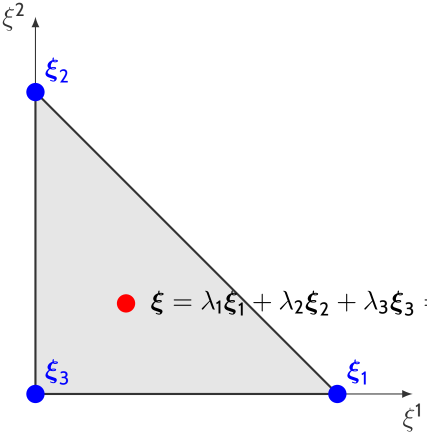
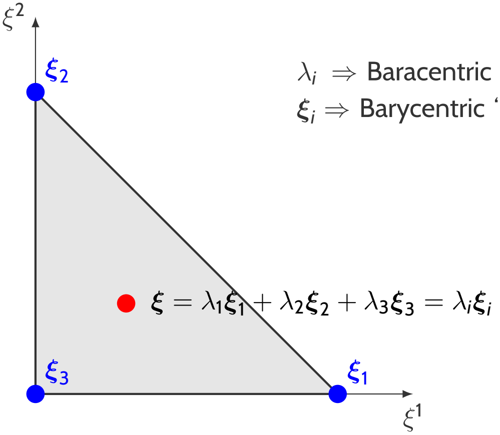


$$\boldsymbol{\xi} = \lambda_1 \boldsymbol{\xi}_1 + \lambda_2 \boldsymbol{\xi}_2 + \lambda_3 \boldsymbol{\xi}_3 = \lambda_i \boldsymbol{\xi}_i$$

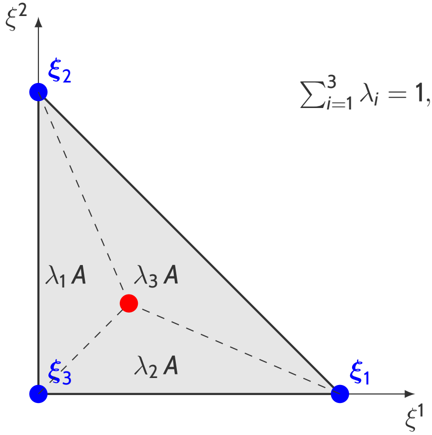
Barycentric Coordinates



$\lambda_i \Rightarrow$ Baracentric coordinates,

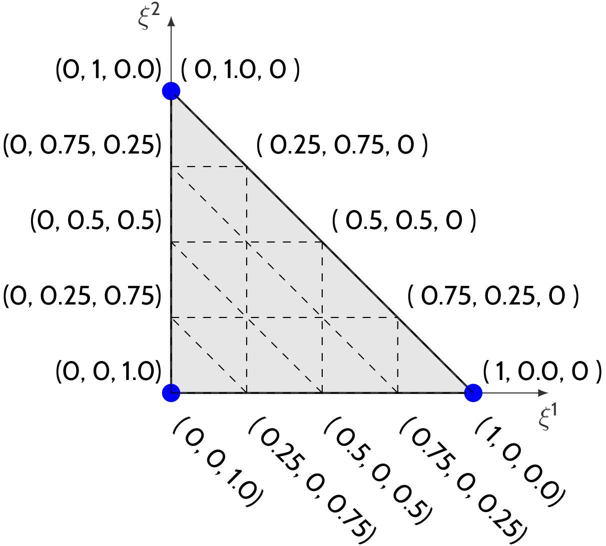
$\boldsymbol{\xi}_i \Rightarrow$ Barycentric "basis"

Barycentric Coordinates



$\sum_{i=1}^{3} \lambda_i = 1,\ A = 1/2$

Barycentric Coordinates



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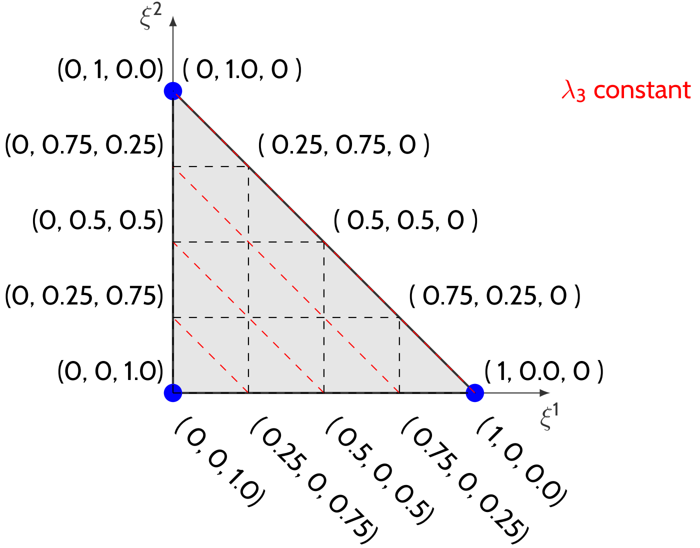

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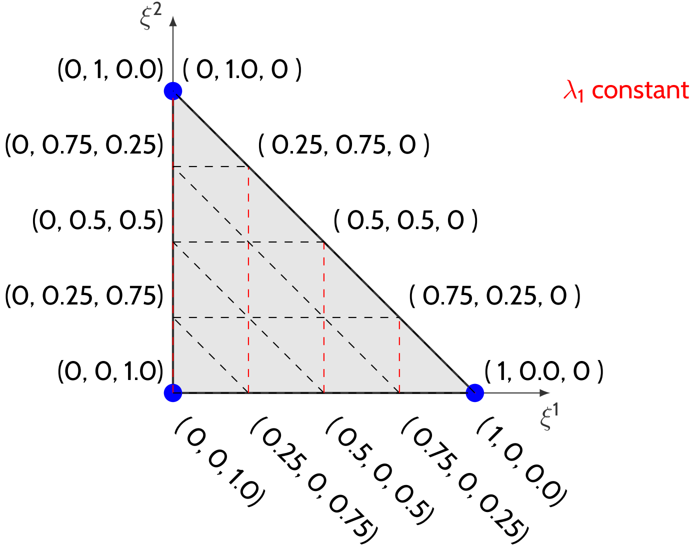

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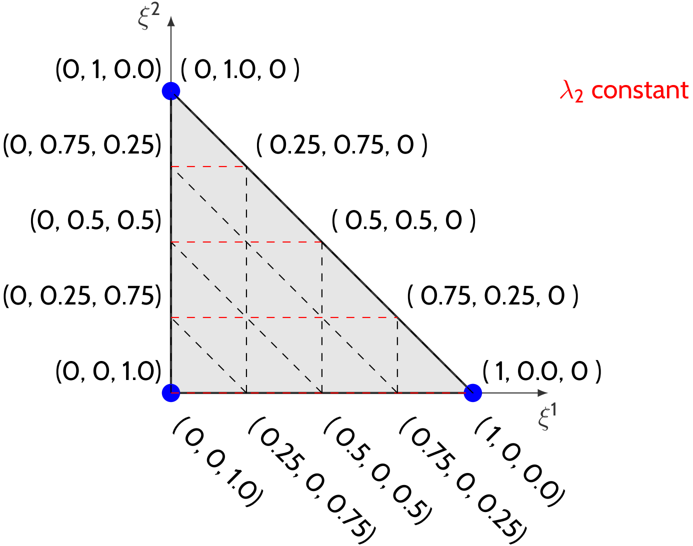

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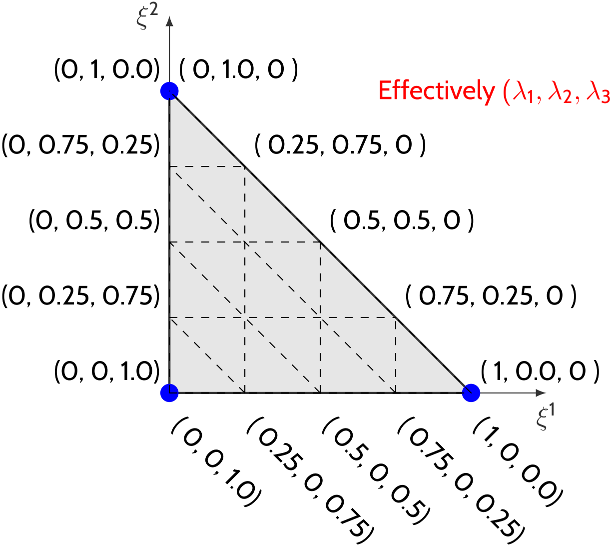


Effectively $(\lambda_1, \lambda_2, \lambda_3) = (\xi^1, \xi^2, 1 - \xi^1 - \xi^2)$

The Local Element View

For elements of order p the shape functions are

$$\hat{\phi}_a(\boldsymbol{\lambda}) = T_i(\lambda_1)T_j(\lambda_2)T_k(\lambda_3)$$

for $i, j, k = 0 \ldots p$, where

$$T_i(z) = \begin{cases} \ell_i^i(z/z_i) & \text{if } i \neq 0 \\ 1 & \text{if } i = 0 \end{cases}$$

and

$$a = \hat{a}(i, j, k).$$

and, for example, for $p = 2$

$$z_i = 0, 1/2, 1$$

are the uniformly spaced nodes in one dimensions for the Lagrange basis construction.

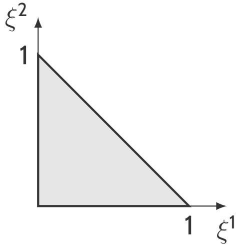
The Local Element View

The elements of order p have

$$n_{dof} = \frac{1}{2}(p+1)(p+2)$$

DOFs which correspond to the function evaluation at the points $\{x_i\}_{i=0}^{n_{dof}-1}$ where

$$\boldsymbol{x} = (i/p, j/p), \quad 0 \leq i + j \leq p = 1$$



The Local Element View

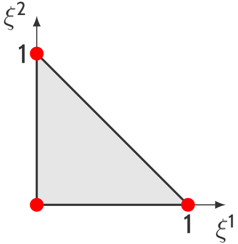
The elements of order $p = 1$ have

$$n_{dof} = \frac{1}{2}(p+1)(p+2) = 3$$

DOFs which correspond to the function evaluation at the points $\{x_i\}_{i=0}^{2}$ where

$$\boldsymbol{x} = (i/p, j/p), \quad 0 \leq i + j \leq p = 1$$

$$\hat{\phi}_a(\boldsymbol{\lambda}) = T_i(\lambda_1) T_j(\lambda_2) T_k(\lambda_3), \quad i, j, k = 0 \dots p$$



$$\hat{\phi}_1(\boldsymbol{\lambda}) = T_1(\lambda_1) T_0(\lambda_2) T_0(\lambda_2) = \ell_1^1(\lambda_1) = \ell_1^1(\xi^1) = \xi^1$$

$$\hat{\phi}_2(\boldsymbol{\lambda}) = T_0(\lambda_1) T_1(\lambda_2) T_0(\lambda_2) = \ell_1^1(\lambda_2) = \ell_1^1(\xi^2) = \xi^2$$

$$\hat{\phi}_3(\boldsymbol{\lambda}) = T_0(\lambda_1) T_0(\lambda_2) T_1(\lambda_3)$$

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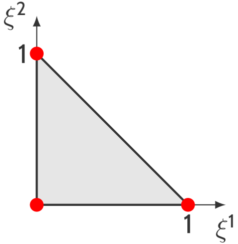

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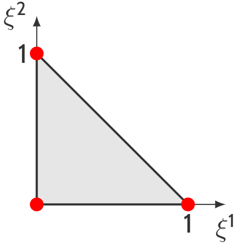

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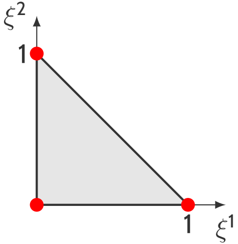

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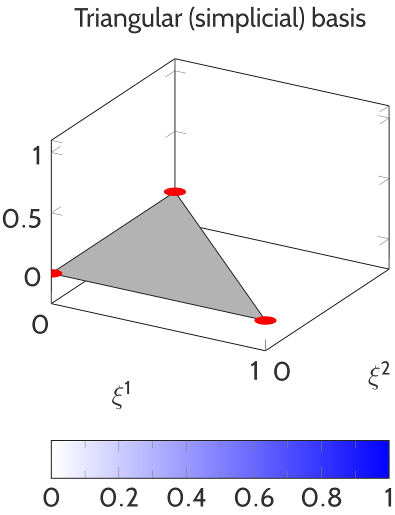

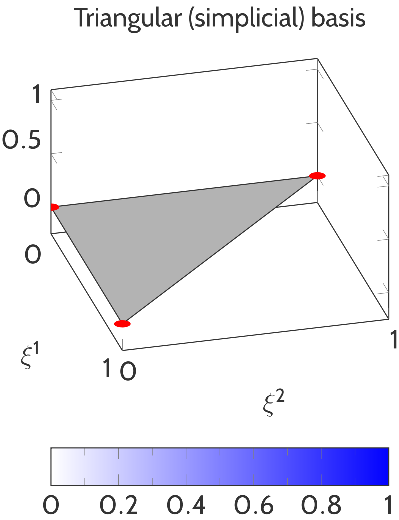

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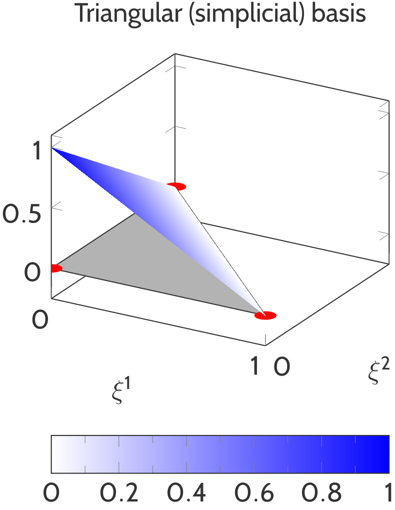

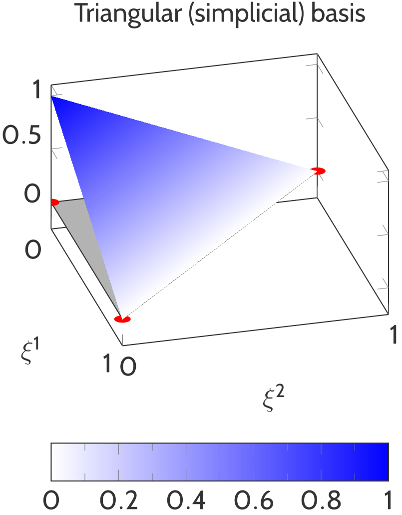

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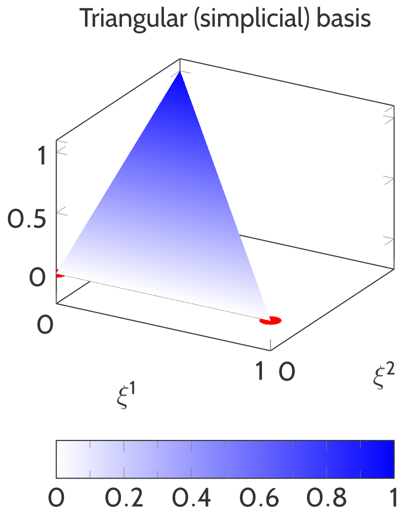

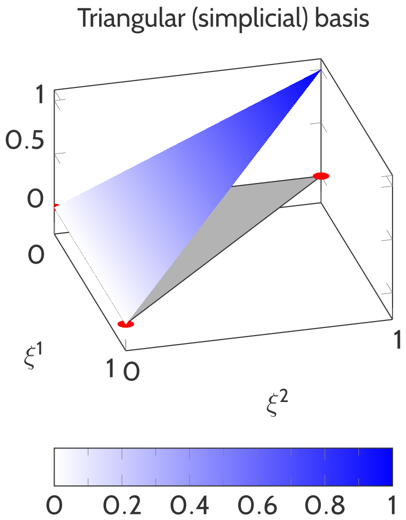

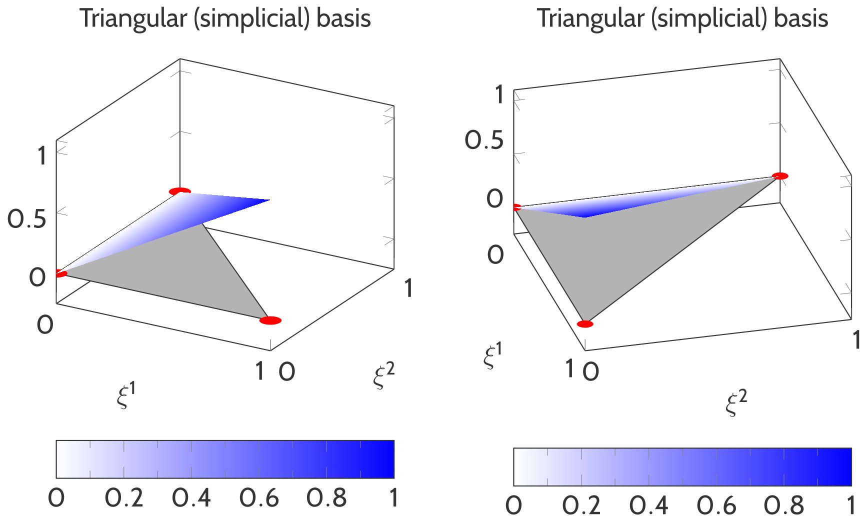
Triangular (simplicial) basis
1
0.5
0
0
1 0
1
ξ^1
ξ^2
0
0.2
0.4
0.6
0.8
1
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1

The Local Element View

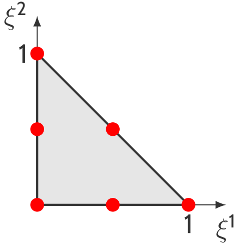
The elements of order $p = 2$ have

$$n_{dof} = \frac{1}{2}(p+1)(p+2) = 6$$

DOFs which correspond to the function evaluation at the points $\{x_i\}_{i=0}^{5}$ where

$$\boldsymbol{x} = (i/p, j/p), \quad 0 \leq i + j \leq p = 2$$

$$\hat{\phi}_a(\boldsymbol{\lambda}) = T_i(\lambda_1) T_j(\lambda_2) T_k(\lambda_3), \quad i, j, k = 0 \ldots p$$



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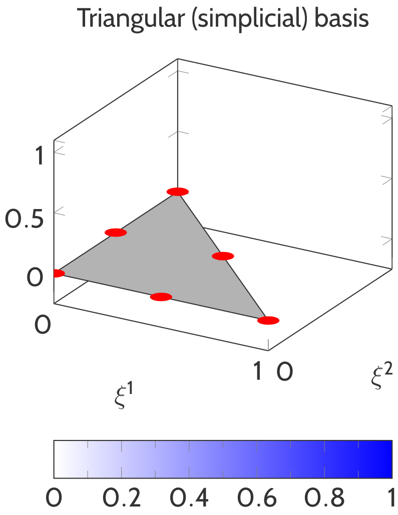

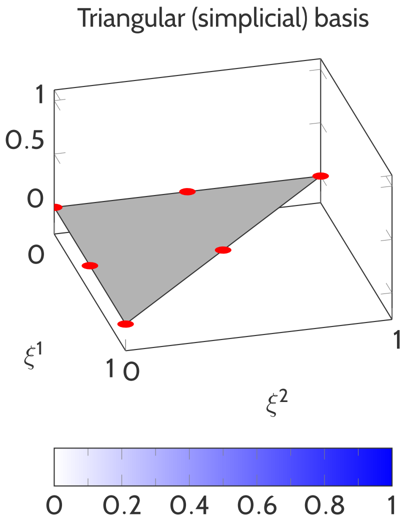

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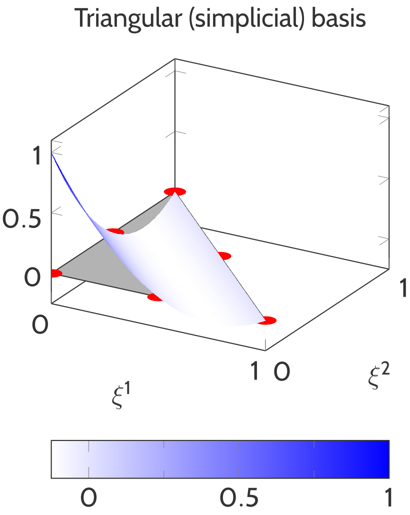

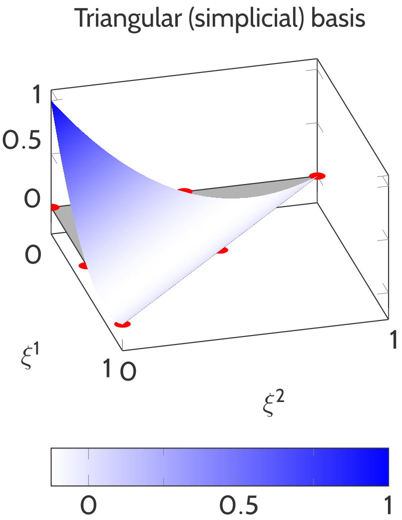

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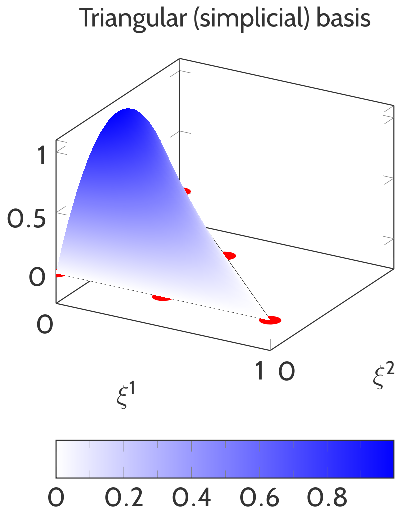

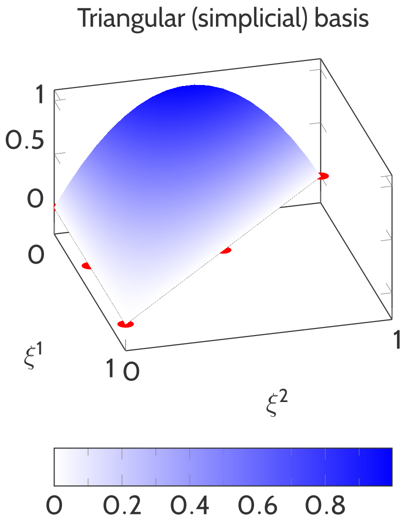

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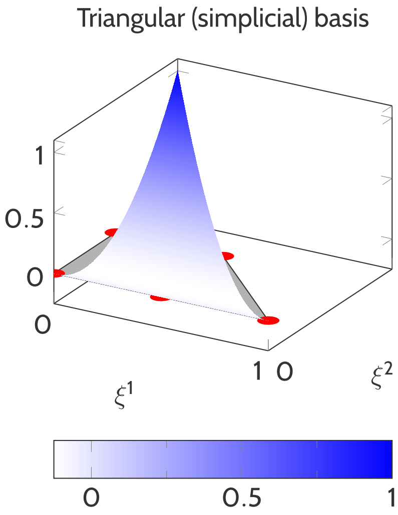

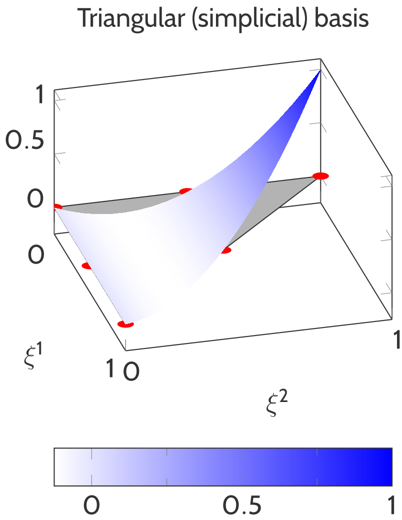

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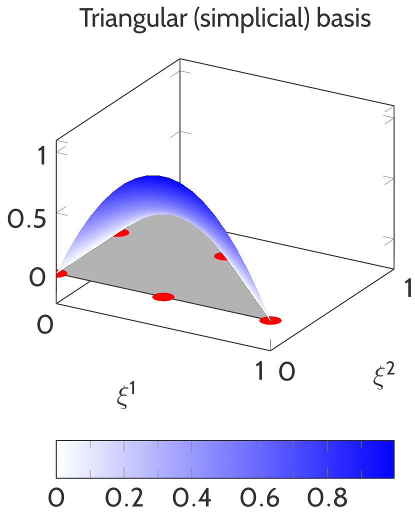
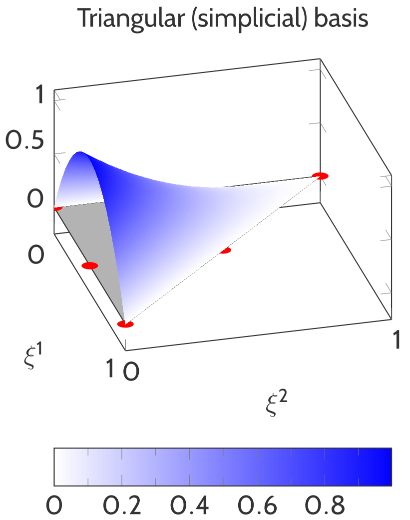
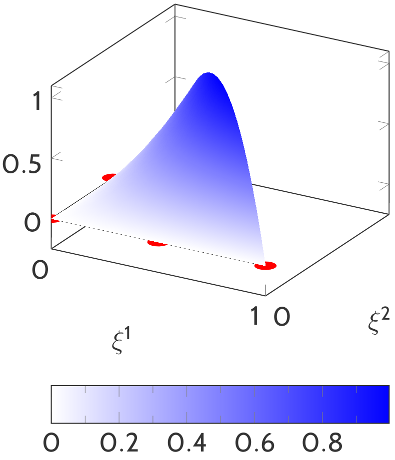




The Local Element View

Triangular (simplicial) basis



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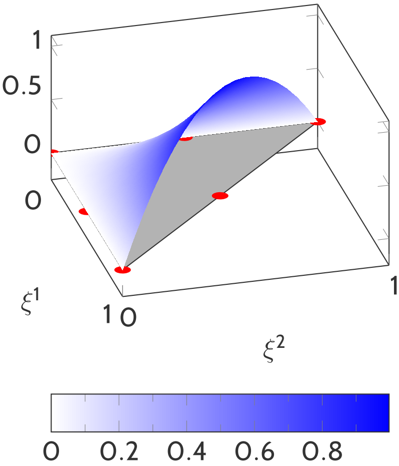

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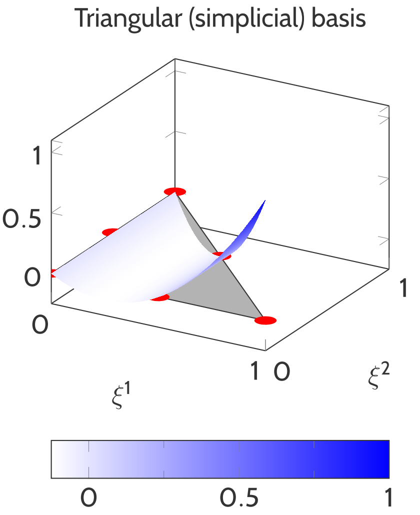
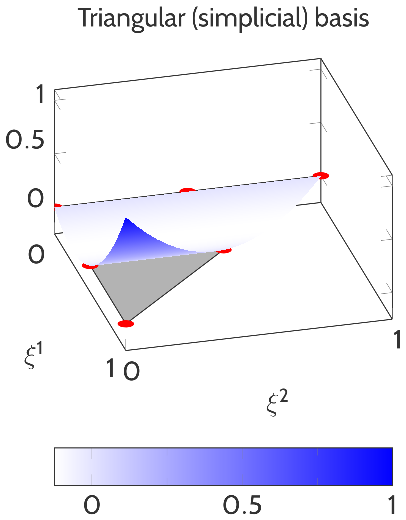




The Local Element View

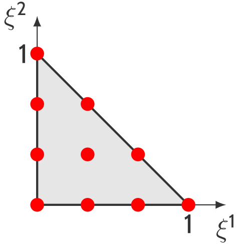
The elements of order $p = 3$ have

$$n_{dof} = \frac{1}{2}(p+1)(p+2) = 10$$

DOFs which correspond to the function evaluation at the points $\{x_i\}_{i=0}^{9}$ where

$$\boldsymbol{x} = (i/p, j/p), \quad 0 \leq i + j \leq p = 3$$

$$\hat{\phi}_a(\boldsymbol{\lambda}) = T_i(\lambda_1)T_j(\lambda_2)T_k(\lambda_3), \quad i, j, k = 0 \ldots p$$



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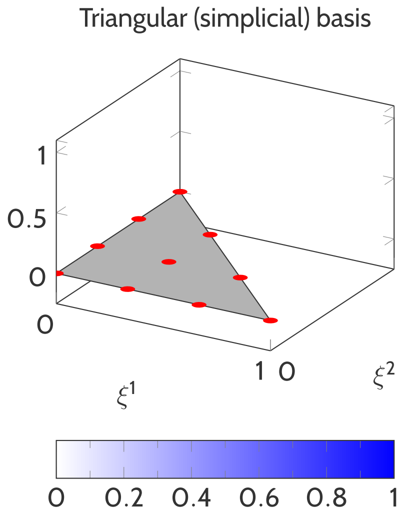
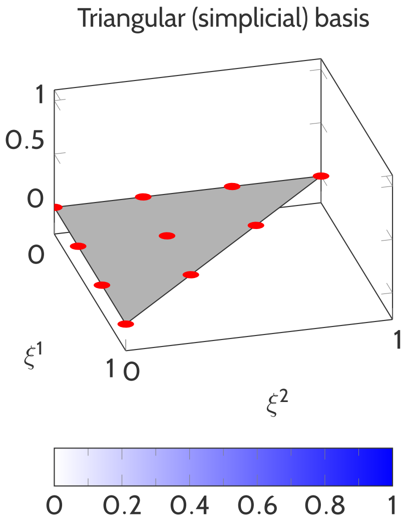
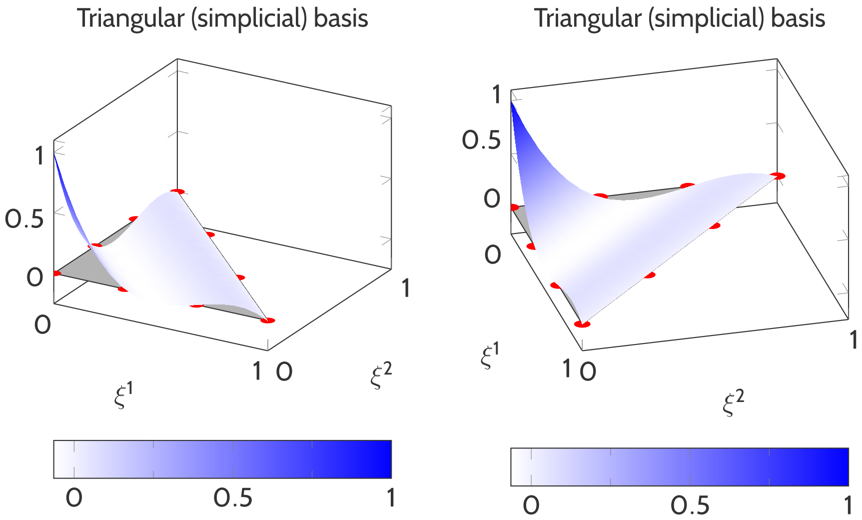


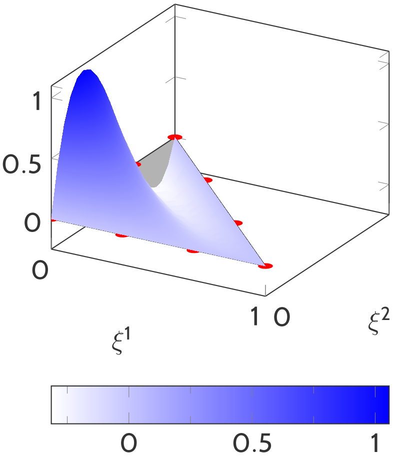


The Local Element View

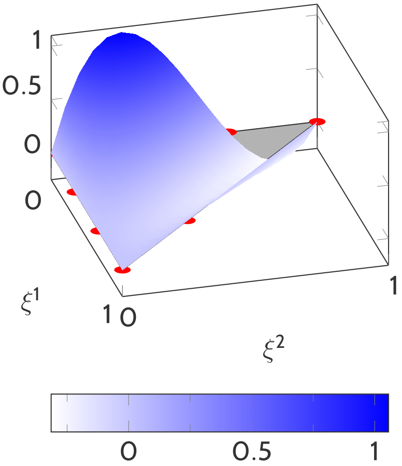


The Local Element View

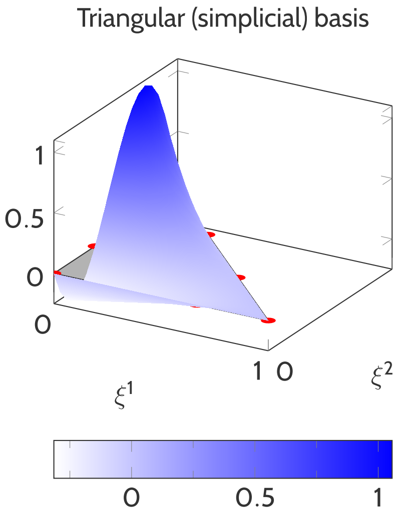
Triangular (simplicial) basis

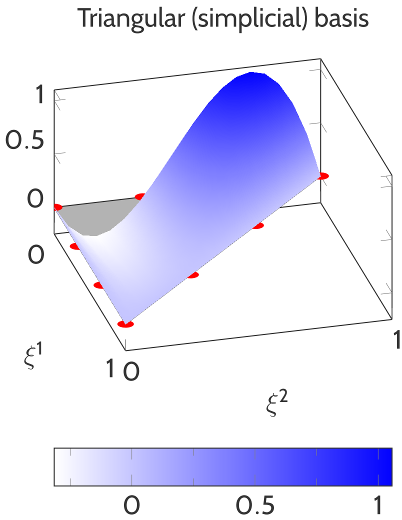


Triangular (simplicial) basis

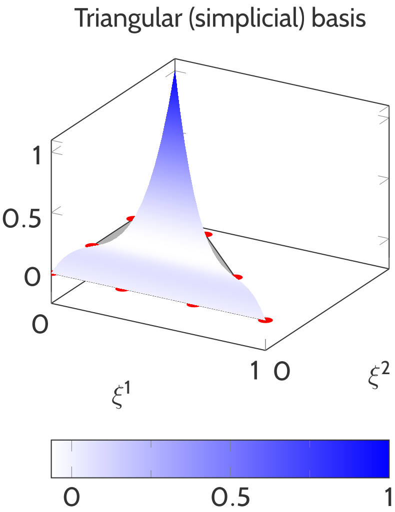


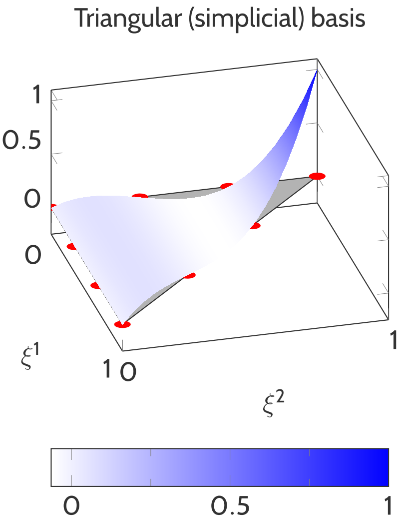
The Local Element View



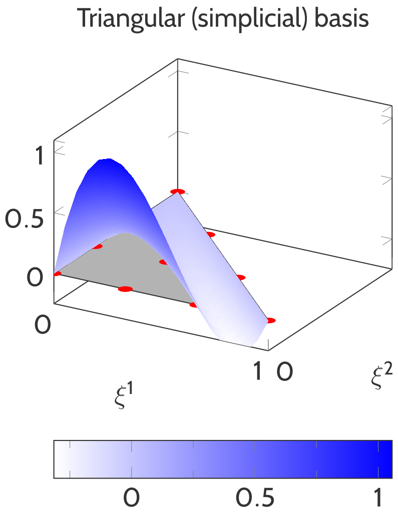


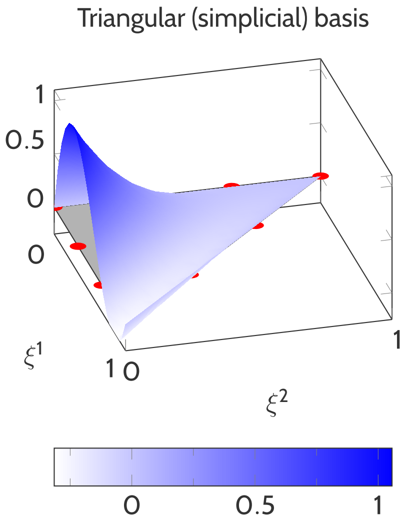
The Local Element View



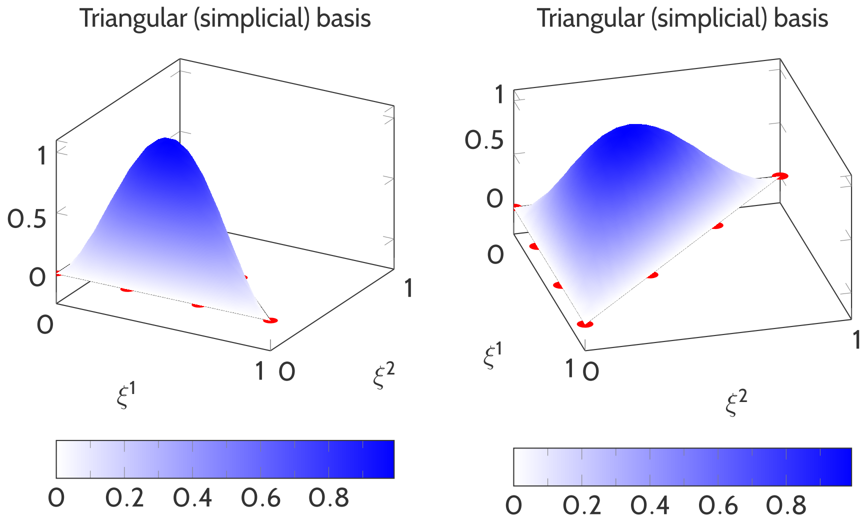


The Local Element View

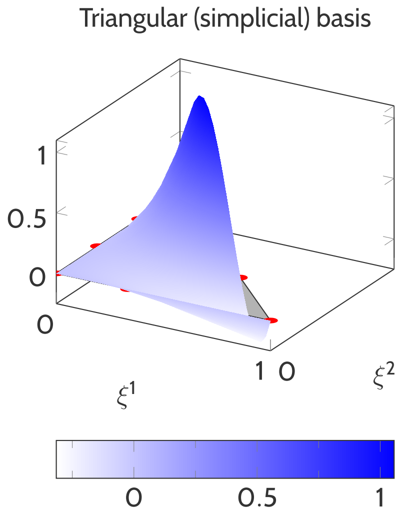


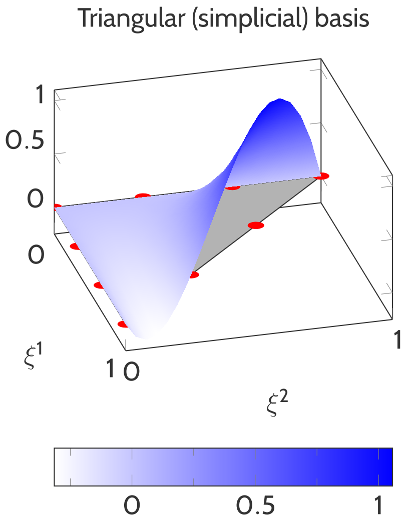


The Local Element View

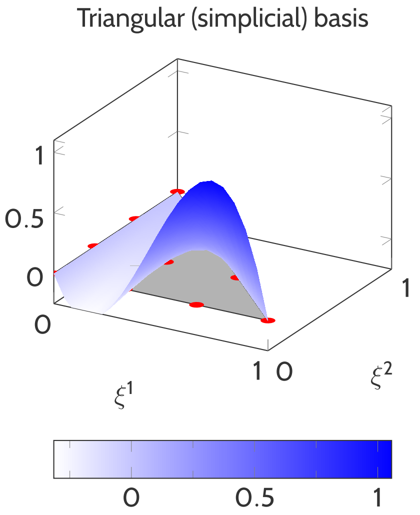


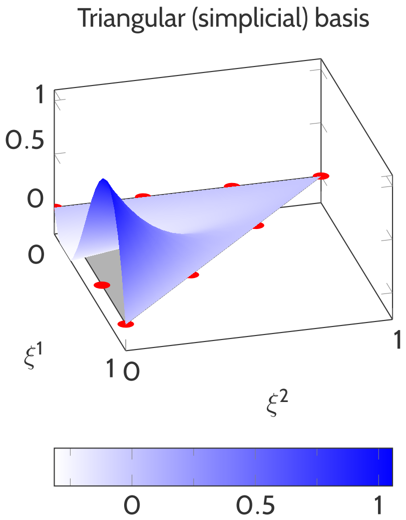
The Local Element View



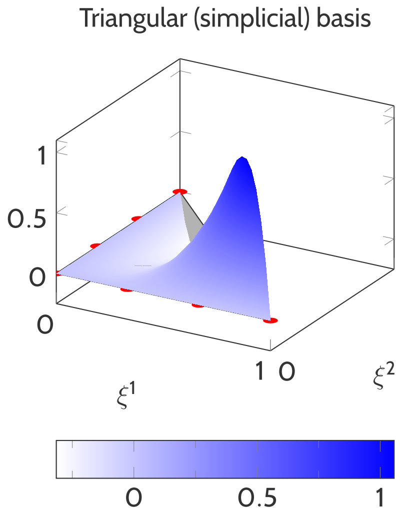


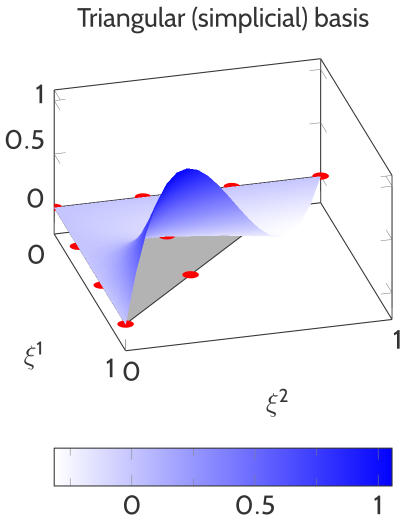
The Local Element View



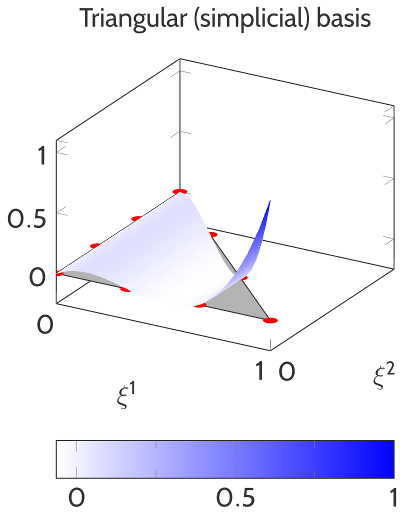


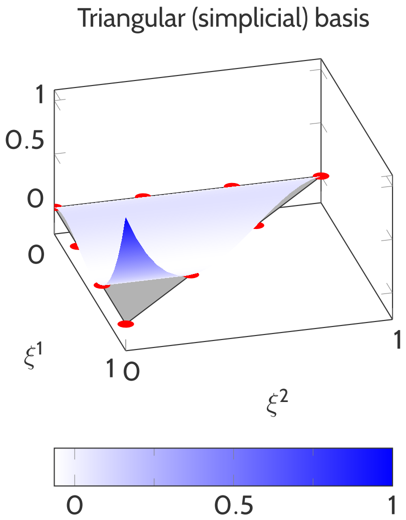
The Local Element View





The Local Element View





Numerical Integration

Numerical Integration

Let $\mathcal{G}(\boldsymbol{x})$ be a function defined on Ω^e and we would like to compute

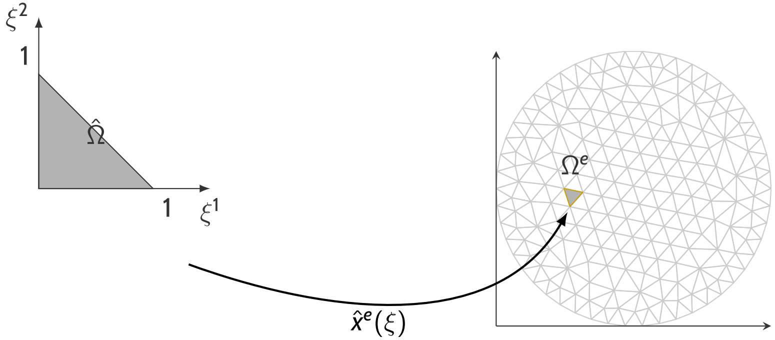
$$\int_{\Omega^e} \mathcal{G}(\boldsymbol{x}) d\Omega$$

Numerical Integration

Recall that $\hat{\boldsymbol{x}} : \hat{\Omega} \to \Omega^e$ where

$$\hat{\boldsymbol{x}}(\boldsymbol{\xi}) = \hat{\phi}_a(\boldsymbol{\xi})\boldsymbol{x}_a$$

aka *isoparametric* mapping, $\boldsymbol{x}_a$ are the coordinates (in physical space) of the DOFs.



Numerical Integration

Let $\mathcal{G}(\boldsymbol{x})$ be a function defined on Ω^e and we would like to compute

$$\int_{\Omega^e} \mathcal{G}(\boldsymbol{x}) d\Omega$$

then with $\hat{\boldsymbol{x}} : \hat{\Omega} \to \Omega^e$ with $\hat{\Omega}$ being the unit triangle, we have that

$$\int_{\Omega^e} \mathcal{G}(\boldsymbol{x}) d\Omega = \int_{\hat{\Omega}} \mathcal{G}(\hat{\boldsymbol{x}}(\boldsymbol{\xi}))\, \hat{j}(\boldsymbol{\xi}) \; d\hat{\Omega}$$

where $\hat{j}(\boldsymbol{\xi}) = \det(\nabla_{\boldsymbol{\xi}} \hat{\boldsymbol{x}}(\boldsymbol{\xi}))$.

Numerical Integration

We can re-write the integral as

$$\int_{\hat{\Omega}} \mathcal{G}(\hat{\boldsymbol{x}}(\boldsymbol{\xi}))\, \hat{j}(\boldsymbol{\xi}) \;\, d\hat{\Omega} = \int_0^1 \int_0^{1-\xi^2} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1, \xi^2))\, \hat{j}(\xi^1, \xi^2) \;\, d\xi^1 d\xi^2$$

We can then approximate the above as

$$\int_0^1 \int_0^{1-\xi^2} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1, \xi^2))\, \hat{j}(\xi^1, \xi^2) \;\, d\xi^1 d\xi^2 = \sum_{(\boldsymbol{\xi}_q, \omega_q) \in \mathcal{Q}} \mathcal{G}(\hat{\boldsymbol{x}}(\boldsymbol{\xi}_q))\, \hat{j}(\boldsymbol{\xi}_q)\, \omega_q$$

Numerical Integration

We can re-write the integral as

$$\int_{\hat{\Omega}} \mathcal{G}(\hat{\boldsymbol{x}}(\boldsymbol{\xi}))\,\hat{j}(\boldsymbol{\xi})\ d\hat{\Omega} = \int_0^1 \int_0^{1-\xi^2} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1,\xi^2))\,\hat{j}(\xi^1,\xi^2)\ d\xi^1 d\xi^2$$

We can then approximate the above as

$$\int_0^1 \int_0^{1-\xi^2} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1,\xi^2))\,\hat{j}(\xi^1,\xi^2)\ d\xi^1 d\xi^2 = \sum_{(\tilde{\boldsymbol{\xi}}_q,\omega_q)\in\mathcal{Q}} \mathcal{G}(\hat{\boldsymbol{x}}(\tilde{\boldsymbol{\xi}}_q))\,\hat{j}(\tilde{\boldsymbol{\xi}}_q)\,\omega_q$$

Numerical Integration

We can re-write the integral as

$$\int_{\hat{\Omega}} \mathcal{G}(\hat{\boldsymbol{x}}(\boldsymbol{\xi}))\, \hat{j}(\boldsymbol{\xi}) \; d\hat{\Omega} = \int_0^1 \int_0^{1-\xi^2} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1, \xi^2))\, \hat{j}(\xi^1, \xi^2) \; d\xi^1 d\xi^2$$

We can then approximate the above as

$$\int_0^1 \int_0^{1-\xi^2} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1, \xi^2))\, \hat{j}(\xi^1, \xi^2) \; d\xi^1 d\xi^2$$

$$= \sum_{(\tilde{\xi}_Q, \omega_Q) \in Q} \mathcal{G}(\hat{\boldsymbol{x}}(\tilde{\xi}_Q))\, \hat{j}(\tilde{\xi}_Q)\, \omega_Q$$

Numerical Integration

We can re-write the integral as

$$\int_{\hat{\Omega}} \mathcal{G}(\hat{\boldsymbol{x}}(\boldsymbol{\xi}))\, \hat{j}(\boldsymbol{\xi}) \; d\hat{\Omega} = \int_0^1 \int_0^{1-\xi^2} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1, \xi^2))\, \hat{j}(\xi^1, \xi^2) \; d\xi^1 d\xi^2$$

We can then approximate the above as

$$\begin{aligned} &\int_0^1 \int_0^{1-\xi^2} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1, \xi^2))\, \hat{j}(\xi^1, \xi^2) \; d\xi^1 d\xi^2 \\ &\qquad = \sum_{(\tilde{\boldsymbol{\xi}}_Q, \omega_Q) \in \mathcal{Q}} \mathcal{G}(\hat{\boldsymbol{x}}(\tilde{\boldsymbol{\xi}}_Q))\, \hat{j}(\tilde{\boldsymbol{\xi}}_Q)\, \omega_Q \end{aligned}$$