HOMEWORK 1

CEE 530: Continuum Mechanics and Thermodynamics

Due: March 5, 2018

PROBLEM 1

Show that the following identities are true for $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ being invertible second order tensors, and $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ being the identity tensor show

1. Divergence:
$$\nabla \cdot (\boldsymbol{A}^\top \boldsymbol{a}) = (\nabla \cdot \boldsymbol{A}) \cdot \boldsymbol{a} + \boldsymbol{A} : \nabla \boldsymbol{a}$$

2. Trace:
$$\frac{\partial \operatorname{tr} \boldsymbol{A}}{\partial \boldsymbol{A}} = \boldsymbol{I}$$

3. Determinant:
$$\frac{\partial \det \boldsymbol{A}}{\partial \boldsymbol{A}} = \det \boldsymbol{A} \boldsymbol{A}^{-\top}$$

PROBLEM 2

Consider the surface given by the mapping

$$\begin{aligned} \boldsymbol{x}(\xi^1, \xi^2) = & (\alpha + \beta \cos(\xi^1)) \cos \xi^2 \mathbf{e}_1 \\ & + (\alpha + \beta \cos(\xi^1)) \sin \xi^2 \mathbf{e}_2 \\ & + \beta \sin \xi^2 \mathbf{e}_3 \end{aligned}$$

where $(\xi^1, \xi^2) \in [0, 2\pi] \times [0, 2\pi]$. Plot this surface.

Given a function

$$\phi(\xi^1, \xi^2) = \exp(\xi^1 \xi^2)$$

compute the $\nabla_{\boldsymbol{x}} \phi$ on the specified surface.

Hint: to construct the dual basis you can use this expression

$$\boldsymbol{g}^i = \frac{(\boldsymbol{g}_i \wedge \boldsymbol{g}_j) \cdot \boldsymbol{g}_j}{\boldsymbol{g}_i \cdot (\boldsymbol{g}_i \wedge \boldsymbol{g}_j) \cdot \boldsymbol{g}_j}, \quad i \neq j$$

where

$$\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a}.$$

Show why this is true!!

Also, I highly recommend using a symbolic software such as `Mathematica` or `Pythons`'s `SymPy` to compute the actual dual basis.

PROBLEM 3

In this problem we are interested in playing with the change of basis representation.

Consider a set of basis $\boldsymbol{g}_i$ that are expressed in terms of a set of some other basis $\boldsymbol{f}_i$ as $\boldsymbol{g}_i = \alpha_{ji} \mathbf{e}_j$. We can then write

$$\boldsymbol{a} = a_i \boldsymbol{g}_i = a_i \alpha_{ji} \mathbf{e}_j$$

which thus implies

$$[\boldsymbol{a}]_{\boldsymbol{g}} = \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\}, \quad [\boldsymbol{a}]_{\mathrm{e}} = \left\{ \begin{array}{c} a_i\alpha_{1i} \\ a_i\alpha_{2i} \\ a_i\alpha_{3i} \end{array} \right\} = \left[\begin{array}{ccc} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{array} \right] \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\} = Q[\boldsymbol{a}]_{\boldsymbol{g}}.$$

Now for a tensor $\boldsymbol{T}$ we know

$$[\boldsymbol{T}]_{\mathrm{e}}[\boldsymbol{a}]_{\mathrm{e}} = [\boldsymbol{T}\boldsymbol{a}]_{\mathrm{e}} = Q[\boldsymbol{T}\boldsymbol{a}]_{\boldsymbol{g}} = Q[\boldsymbol{T}]_{\boldsymbol{g}}[\boldsymbol{a}]_{\boldsymbol{g}} = Q[\boldsymbol{T}]_{\boldsymbol{g}}Q^{-1}[\boldsymbol{a}]_{\mathrm{e}} \Rightarrow [\boldsymbol{T}]_{\mathrm{e}} = Q[\boldsymbol{T}]_{\boldsymbol{g}}Q^{-1}$$

Let $\boldsymbol{g}_i$ be a set of basis that can be written as

$$\boldsymbol{g}_1 = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 \tag{1}$$
$$\boldsymbol{g}_2 = \mathbf{e}_2 - \mathbf{e}_3 \tag{2}$$
$$\boldsymbol{g}_3 = 2\mathbf{e}_1 + \mathbf{e}_2 \tag{3}$$

where $\mathbf{e}_i$ are a set of *orthonormal* basis.

If

$$[\boldsymbol{T}]_{\boldsymbol{g}} = \left[\begin{array}{ccc} 2 & 0 & 3 \\ 0 & 1 & 4 \\ 3 & 4 & 3 \end{array} \right]$$

is $\boldsymbol{T}$ a symmetric tensor?

PROBLEM 4

Consider the square $[-1, 1] \times [-1, 1]$ to denote the reference configuration Ω_0 of a body. A series of equidistant lines parallel to the coordinate axes are drawn on Ω_0 , separated by a distance 0.2 and beginning at the square boundary. The body is deformed by the deformation mapping

$$\varphi(\boldsymbol{X}) = (X_1 + \alpha\, X_1\, (X_2 - 1)(X_2 + 1))\boldsymbol{E}_1 + (X_2 - \alpha X_2(X_1 - 1)(X_1 + 1))\boldsymbol{E}_2$$

where $\{\boldsymbol{E}_i\}_{i=1,2}$ is the orthonormal basis parallel to the axes, and α is a real constant.

1. Plot the deformed configuration of the body for $\alpha = 0.1, 0.6, 0.95$, including the deformed configuration of each one of the lines.

2. What is the position in the reference configuration of the particle located at $\boldsymbol{x} = 0$ in the deformed configuration?

3. For $\alpha = 0.1$, what is the position in the reference configuration of the particle located at $\boldsymbol{x} = 0.1(\boldsymbol{E}_1 + \boldsymbol{E}_2)$ in the deformed configuration?

4. Is the deformation admissible for any $0 \leq \alpha < 1$ Justify (graphically). What happens at $\alpha = 1$?

PROBLEM 5

Consider the deformation mappings $\boldsymbol{\varphi}_1 : \Omega_0 \to \Omega_1$ and $\boldsymbol{\varphi}_2 : \Omega_1 \to \Omega_2$. The composition of the two is defined as the deformation mapping $\boldsymbol{\phi} : \Omega_0 \to \Omega_2$ such that $\boldsymbol{\phi} = \boldsymbol{\varphi}_1 \circ \boldsymbol{\varphi}_2$, or $\boldsymbol{\phi}(\boldsymbol{X}) := \boldsymbol{\varphi}_2(\boldsymbol{\varphi}_1(\boldsymbol{X}))$. Let $[\boldsymbol{\varphi}_1(\boldsymbol{X})]_{\boldsymbol{E}} = (X_1, \lambda X_2, X_3)$ (uniaxial stretching) and $[\boldsymbol{\varphi}_2(\boldsymbol{X})]_{\boldsymbol{E}} := (X_1 + \tan\alpha X_2, X_2, X_3)$ (simple shear), $\boldsymbol{X} \in \mathbb{R}^3$. Here, all vector components are with respect to an orthonormal basis $\boldsymbol{E}_i$ attached to a set of Cartesian coordinate axes X_i. Consider the square of side a, $\Omega_0 = [0, a] \times [0, a] \times 0$ with respect to these axes. Hence, two basis vectors are parallel to the sides of the square, and one is orthogonal to its plane.

1. Carry out the composition of mappings $\boldsymbol{\varphi}_2 \circ \boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_1 \circ \boldsymbol{\varphi}_2$.

2. Plot and compare the results. Does composition of mappings commute?

(If you are interested, you could also try an experiment, by composing two different rigid body rotations with any object within your reach ? Is the final position of the object the same after performing both compositions?)

PROBLEM 6

In a certain region the spatial velocity components of $\boldsymbol{v} = \boldsymbol{v}(x, t)$ in a Cartesian vector basis $\mathbf{e}_i$ are given as

$$v_1 = -\alpha\,(x_1^3 + x_1 x_2)\exp(-\beta t) \tag{4}$$
$$v_2 = \alpha\,(x_1^2 x_2 + x_2^3)\exp(-\beta t) \tag{5}$$
$$v_3 = 0 \tag{6}$$

where $\alpha, \beta > 0$ are given constants.

1. Find the components of the spatial acceleration field $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, t)$ at point $[\boldsymbol{x}]_\mathrm{e} = (1, 0, 0)$ and time $t = 0$.

2. Measurements of the pressure field throughout the continuum returned the following spatial field

$$p(\boldsymbol{x}, t) = (x_1 + x_2)^2.$$

   What is the rate of change of pressure on a material particle at the same point and time?