Finite Elements in n-D

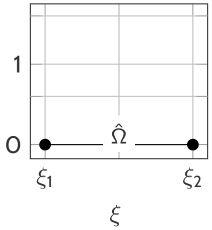
Quadrilateral Elements

Maurizio M. Chiaramonte

The Local Element View

1. Parametric element domain $\hat{\Omega}$
2. Physical element domain Ω^e
3. Map $\hat{\boldsymbol{x}}^e(\xi)$ from parametric to physical domain
4. Shape Functions $\{\hat{\phi}_a(\xi)\}_{a=1}^{n_{dof}^e}$
5. Degrees of freedom $\{u_a^e\}_{a=1}^{n_{dof}^e}$, $u^h(\xi) = \sum_{a=1}^{n_{dof}^e} u_a^e \hat{\phi}_a(\xi)$
6. Local to global dof map

Parametric Element Domain $\hat{\Omega}$



Physical/Real Domain $\Omega = \bigcup_{e=1}^{n_{el}} \Omega^e$

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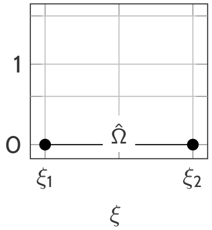

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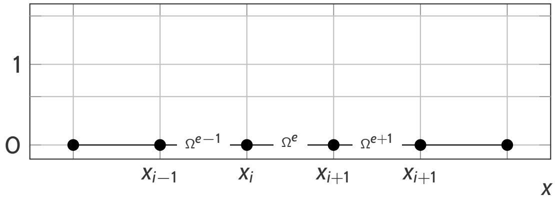

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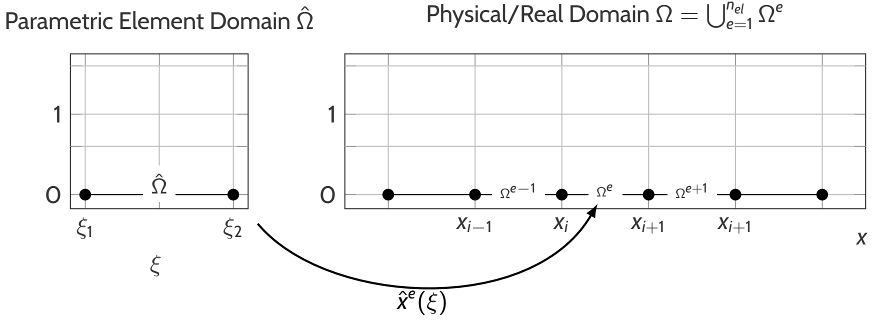

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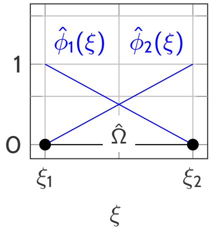

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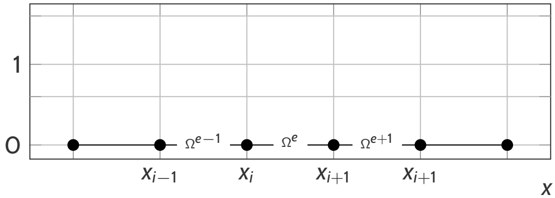

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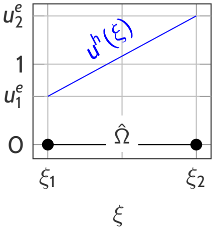

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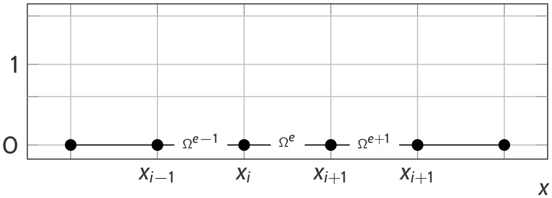

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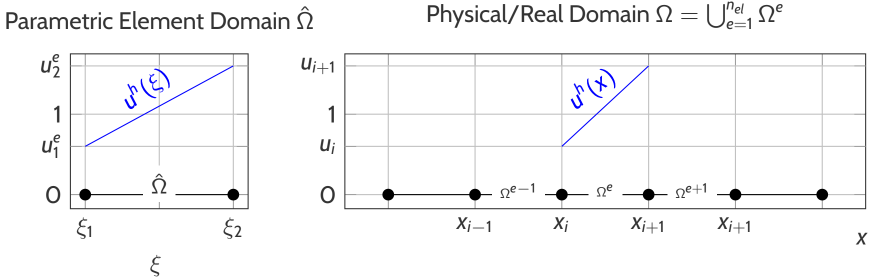
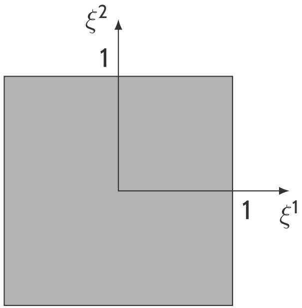
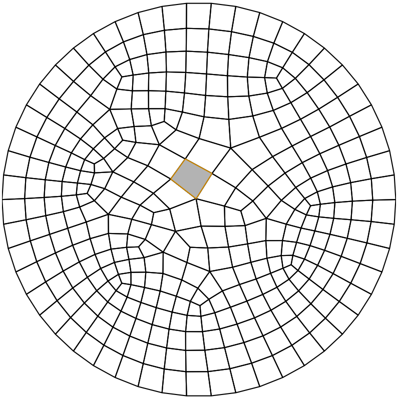


The Local Element View

Quadrilateral parametric element $\hat{\Omega}$



Quadrilateral (hexahedral) mesh



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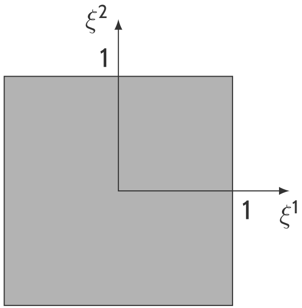

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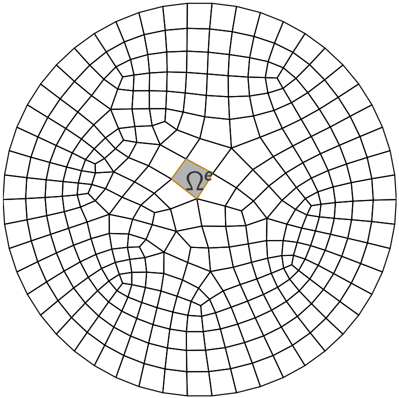

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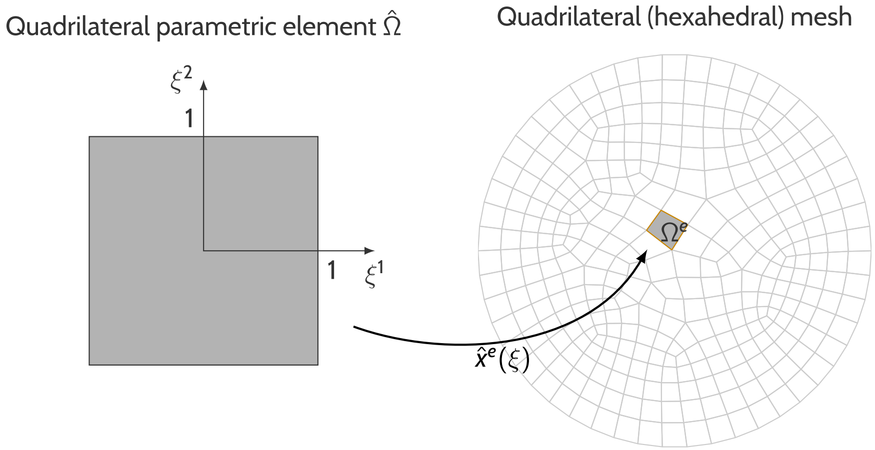
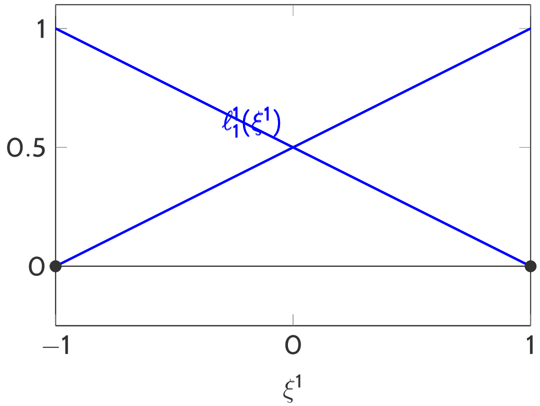


Constructing Basis Functions

The Local Element View

Quadrilateral (hexahedral) basis



$$\hat{\phi}_1(\boldsymbol{\xi}) = \ell_1^1(\xi^1)\ell_1^1(\xi^2)$$

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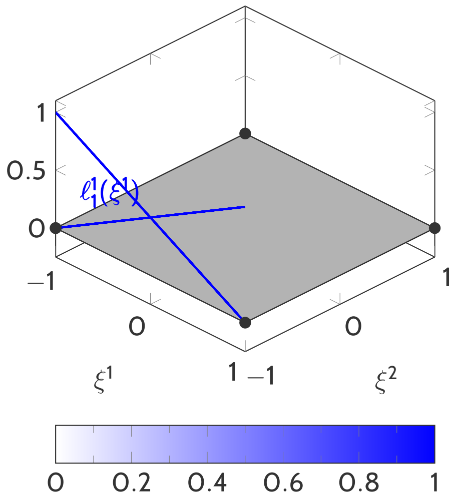

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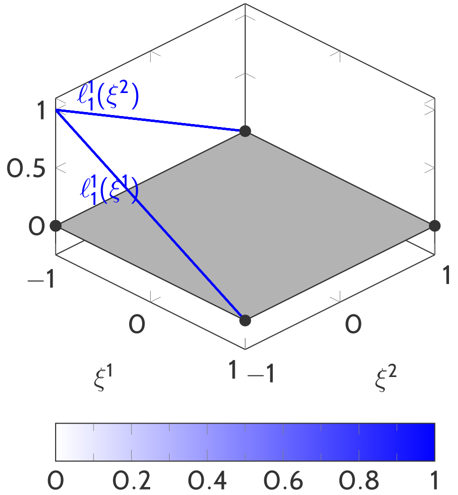

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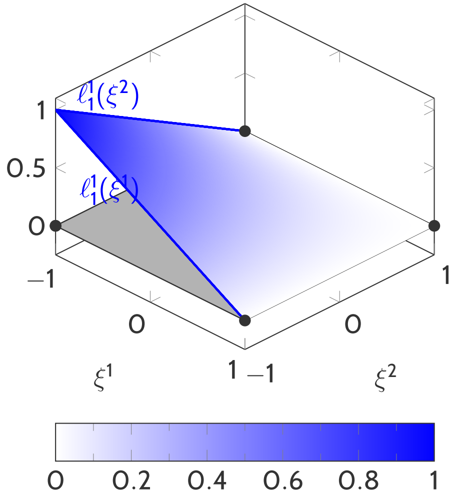

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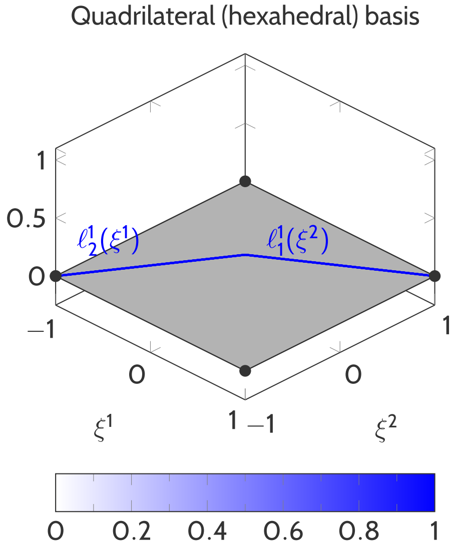

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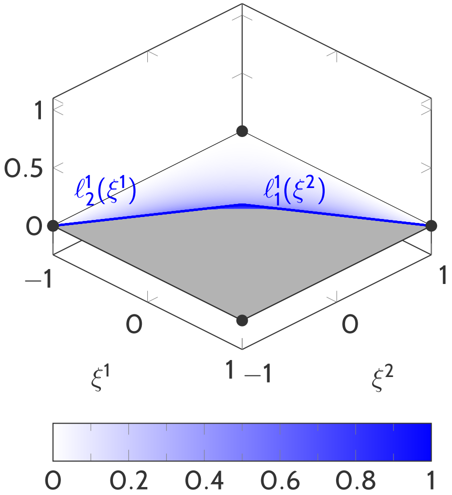

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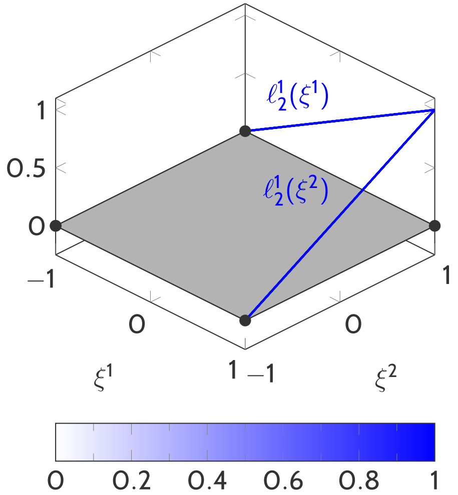

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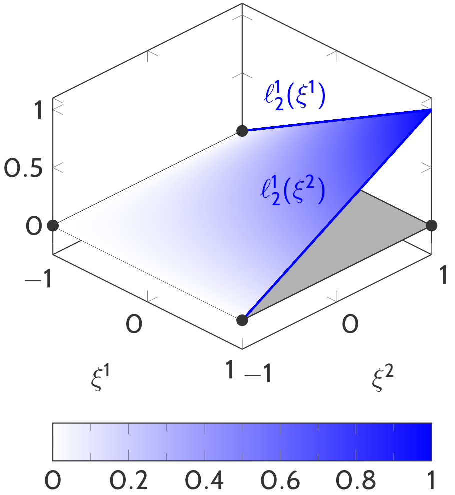

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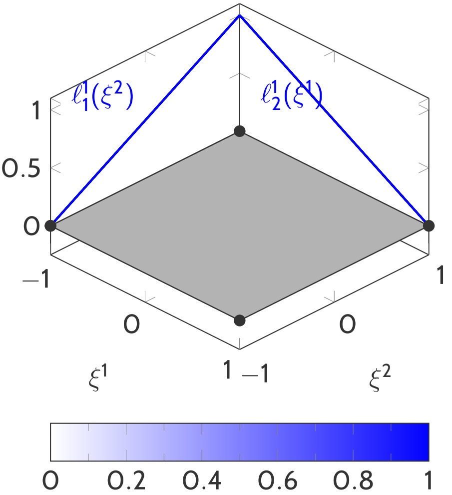

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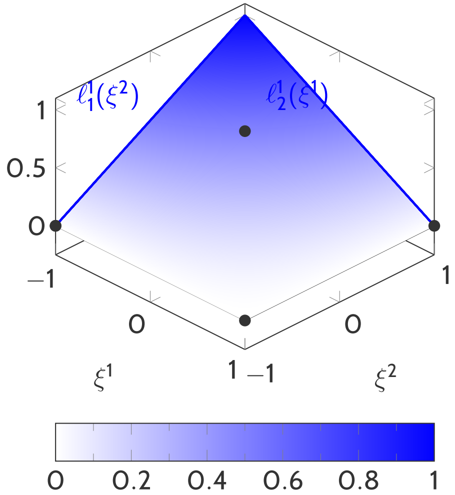

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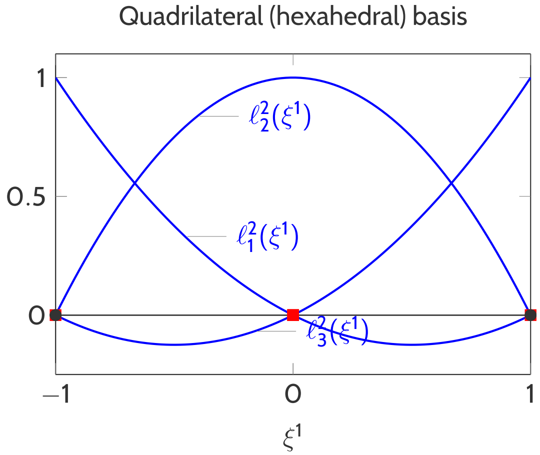

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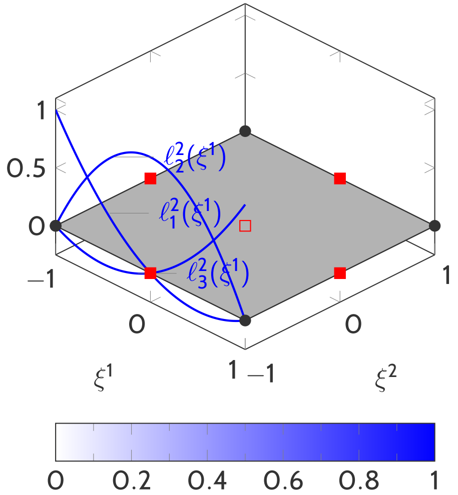

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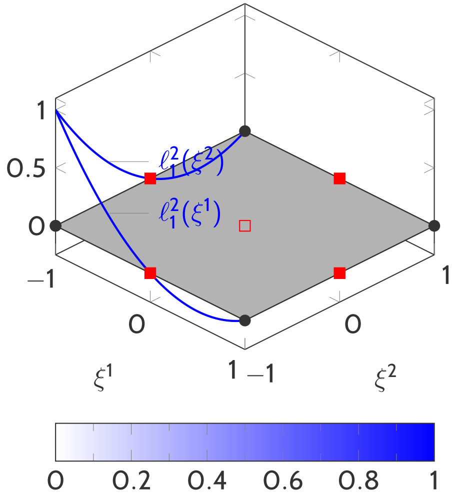

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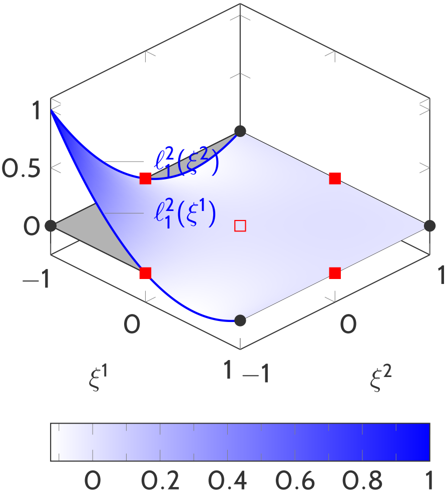
Quadrilateral (hexahedral) basis



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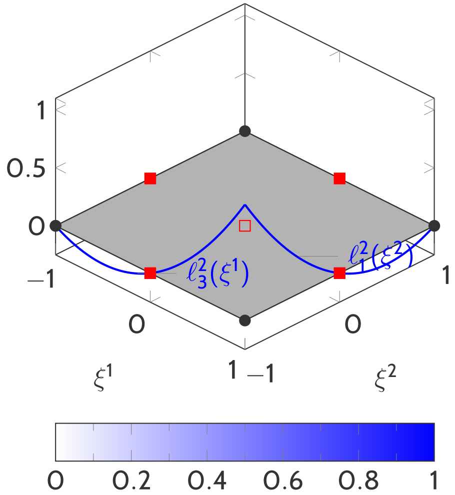

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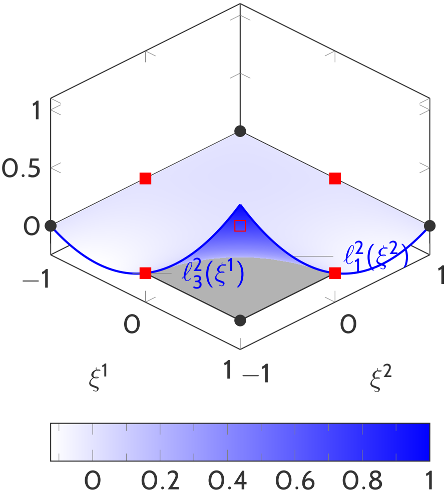

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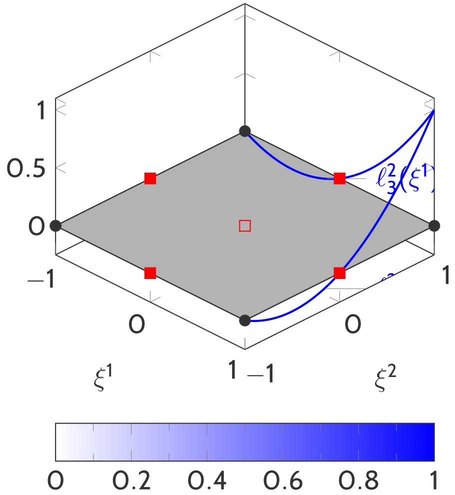

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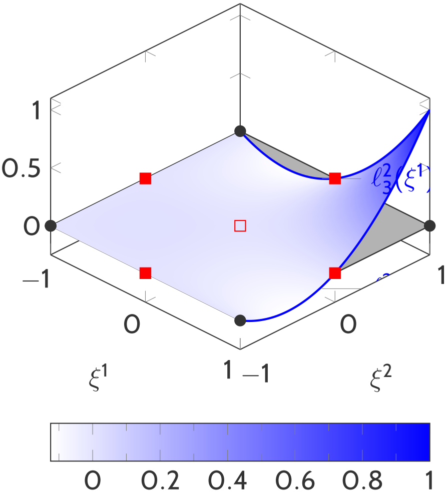
$\hat{\phi}_7(\boldsymbol{\xi}) = \ell_2^2(\xi^1)\ell_3^2(\xi^2)$

$\hat{\phi}_8(\boldsymbol{\xi}) = \ell_1^2(\xi^1)\ell_2^2(\xi^2)$

$\hat{\phi}_9(\boldsymbol{\xi}) = \ell_2^2(\xi^1)\ell_2^2(\xi^2)$

The Local Element View

Quadrilateral (hexahedral) basis



$$\hat{\phi}_1(\boldsymbol{\xi}) = \ell_1^2(\xi^1)\ell_1^2(\xi^2)$$

$$\hat{\phi}_2(\boldsymbol{\xi}) = \ell_3^2(\xi^1)\ell_1^2(\xi^2)$$

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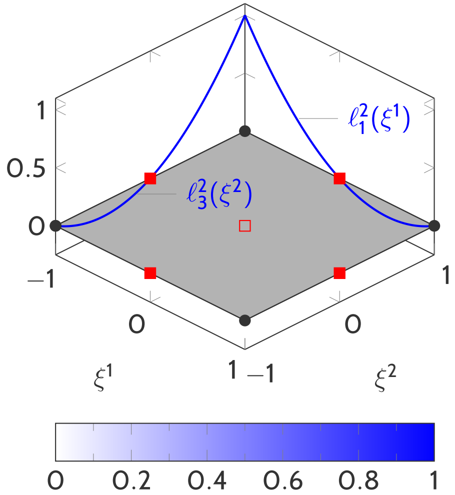
The Local Element View

Quadrilateral (hexahedral) basis



$$\hat{\phi}_1(\boldsymbol{\xi}) = \ell_1^2(\xi^1)\ell_1^2(\xi^2)$$

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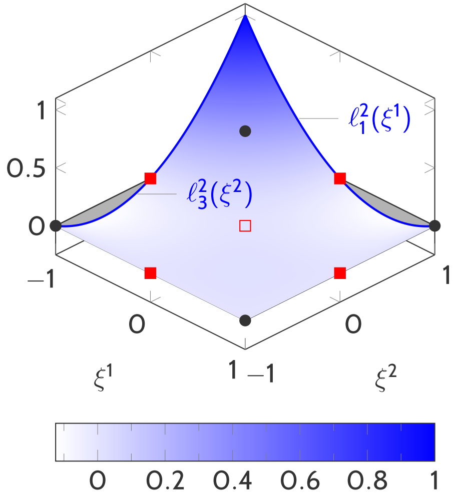
The Local Element View

Quadrilateral (hexahedral) basis



$$\hat{\phi}_1(\boldsymbol{\xi}) = \ell_1^2(\xi^1)\ell_1^2(\xi^2)$$

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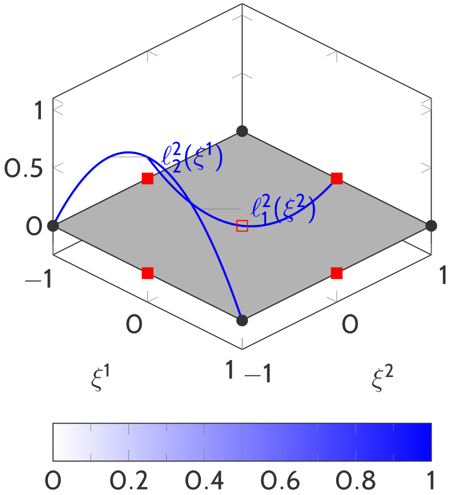
The Local Element View

Quadrilateral (hexahedral) basis



$$\hat{\phi}_1(\boldsymbol{\xi}) = \ell_1^2(\xi^1)\ell_1^2(\xi^2)$$

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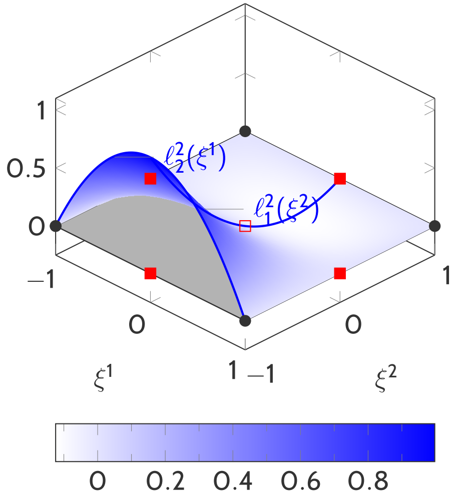
The Local Element View

Quadrilateral (hexahedral) basis



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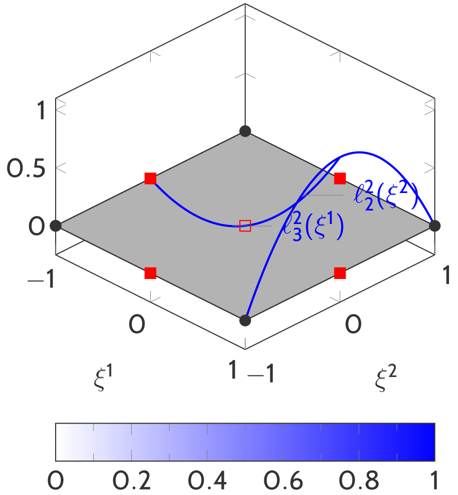
The Local Element View

Quadrilateral (hexahedral) basis



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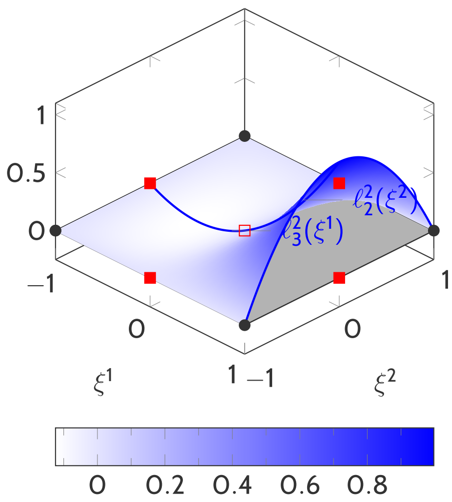
The Local Element View

Quadrilateral (hexahedral) basis



$$\hat{\phi}_1(\boldsymbol{\xi}) = \ell_1^2(\xi^1)\ell_1^2(\xi^2)$$

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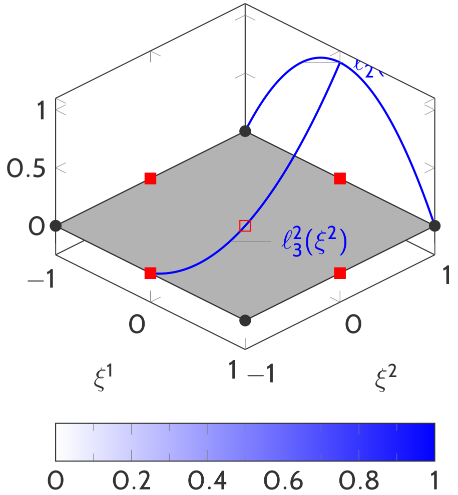
The Local Element View

Quadrilateral (hexahedral) basis



$$\hat{\phi}_1(\boldsymbol{\xi}) = \ell_1^2(\xi^1)\ell_1^2(\xi^2)$$

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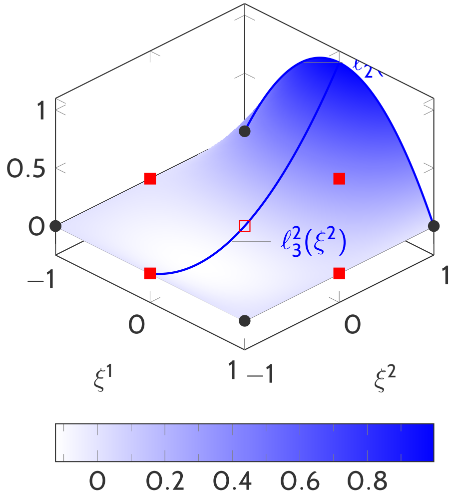
The Local Element View

Quadrilateral (hexahedral) basis



$$\hat{\phi}_1(\boldsymbol{\xi}) = \ell_1^2(\xi^1)\ell_1^2(\xi^2)$$

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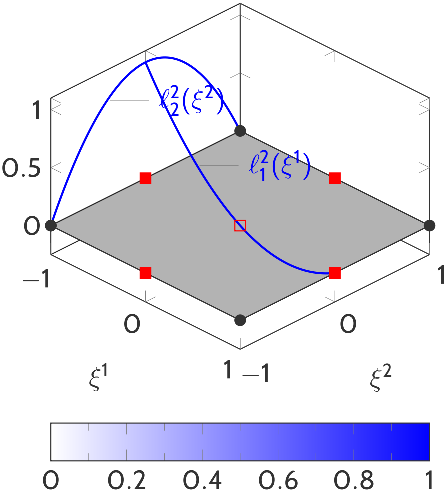
The Local Element View

Quadrilateral (hexahedral) basis



$$\hat{\phi}_1(\boldsymbol{\xi}) = \ell_1^2(\xi^1)\ell_1^2(\xi^2)$$

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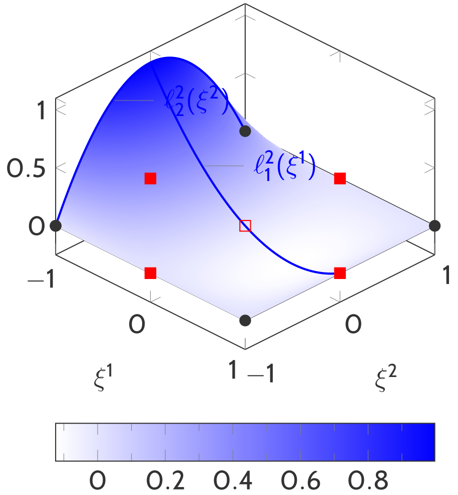
The Local Element View

Quadrilateral (hexahedral) basis



$$\hat{\phi}_1(\boldsymbol{\xi}) = \ell_1^2(\xi^1)\ell_1^2(\xi^2)$$

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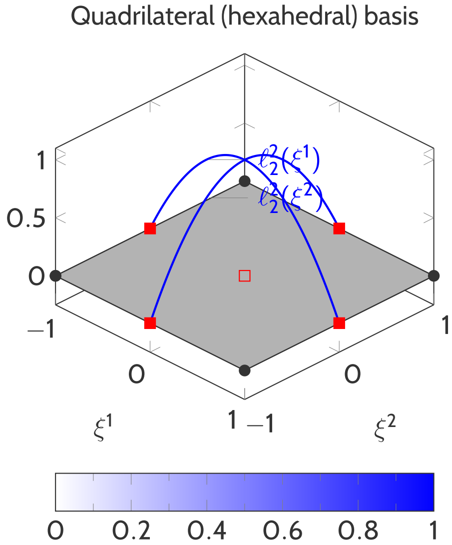
The Local Element View



$$\hat{\phi}_1(\xi) = \ell_1^2(\xi^1)\ell_1^2(\xi^2)$$

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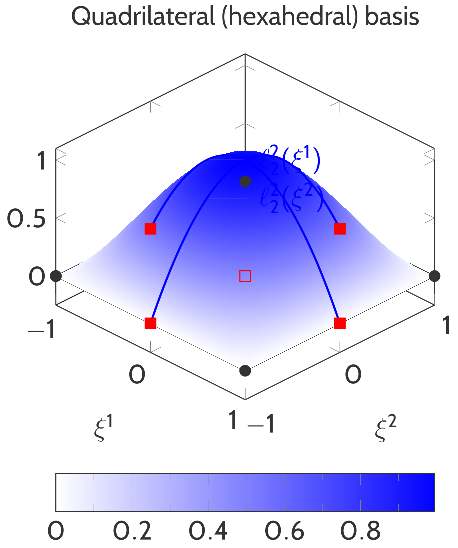
The Local Element View



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Numerical Integration

Numerical Integration

Let $\mathcal{G}(\boldsymbol{x})$ be a function defined on Ω^e and we would like to compute

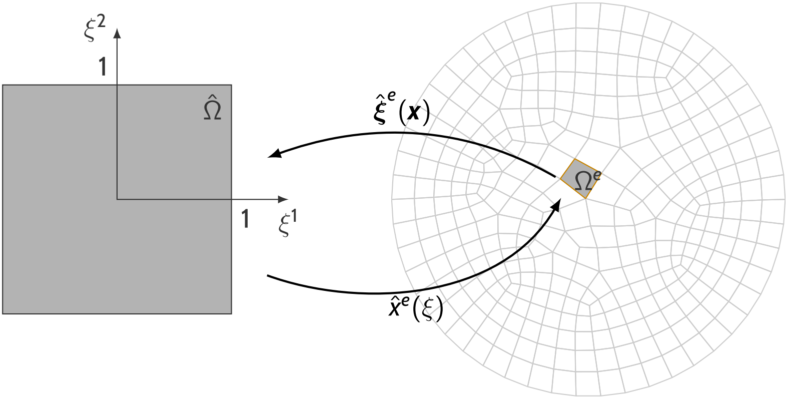
$$\int_{\Omega^e} \mathcal{G}(\boldsymbol{x}) d\Omega$$

Numerical Integration

Recall that $\hat{\boldsymbol{x}} : \hat{\Omega} \to \Omega^e$ where

$$\hat{\boldsymbol{x}}(\boldsymbol{\xi}) = \hat{\phi}_a(\boldsymbol{\xi})\boldsymbol{x}_a$$

aka *isoparametric* mapping, $\boldsymbol{x}_a$ are the coordinates (in physical space) of the DOFs.



Numerical Integration

Let $\mathcal{G}(\boldsymbol{x})$ be a function defined on Ω^e and we would like to compute

$$\int_{\Omega^e} \mathcal{G}(\boldsymbol{x}) d\Omega$$

then with $\hat{\boldsymbol{x}} : \hat{\Omega} \to \Omega^e$ with $\hat{\Omega} = [-1, 1] \times [1, 1]$ we have that

$$\int_{\Omega^e} \mathcal{G}(\boldsymbol{x}) d\Omega = \int_{\hat{\Omega}} \mathcal{G}(\hat{\boldsymbol{x}}(\boldsymbol{\xi}))\, \hat{j}(\boldsymbol{\xi}) \; d\hat{\Omega}$$

where $\hat{j}(\boldsymbol{\xi}) = \det(\nabla_{\boldsymbol{\xi}} \hat{\boldsymbol{x}}(\boldsymbol{\xi}))$.

Numerical Integration

We can re-write the integral as

$$\int_{\hat{\Omega}} \mathcal{G}(\hat{\boldsymbol{x}}(\boldsymbol{\xi}))\, \hat{j}(\boldsymbol{\xi}) \;\, d\hat{\Omega} = \int_{-1}^{1}\int_{-1}^{1} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1,\xi^2))\, \hat{j}(\xi^1,\xi^2) \;\, d\xi^1 d\xi^2$$

We can then approximate the above as

$$\int_{-1}^{1}\int_{-1}^{1} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1,\xi^2))\, \hat{j}(\xi^1,\xi^2) \;\, d\xi^1 d\xi^2$$

$$\approx \int_{-1}^{1} \sum_{q=1}^{n_Q} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1_q,\xi^2))\, \hat{j}(\xi^1_q,\xi^2)\, w_q \, d\xi^2$$

Numerical Integration

We can re-write the integral as

$$\int_{\hat{\Omega}} \mathcal{G}(\hat{\boldsymbol{x}}(\boldsymbol{\xi}))\,\hat{j}(\boldsymbol{\xi})\;d\hat{\Omega} = \int_{-1}^{1}\int_{-1}^{1} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1,\xi^2))\,\hat{j}(\xi^1,\xi^2)\;d\xi^1 d\xi^2$$

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Numerical Integration

We can re-write the integral as

$$\int_{\hat{\Omega}} \mathcal{G}(\hat{\boldsymbol{x}}(\boldsymbol{\xi}))\, \hat{j}(\boldsymbol{\xi}) \; d\hat{\Omega} = \int_{-1}^{1}\int_{-1}^{1} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1, \xi^2))\, \hat{j}(\xi^1, \xi^2) \; d\xi^1 d\xi^2$$

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Numerical Integration

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$$\begin{aligned}
&\int_{-1}^{1} \int_{-1}^{1} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1, \xi^2))\, \hat{j}(\xi^1, \xi^2) \; d\xi^1 d\xi^2 \\
\approx &\int_{-1}^{1} \sum_{q=1}^{n_Q} \mathcal{G}(\hat{\boldsymbol{x}}(\tilde{\xi}_q, \xi^2))\, \hat{j}(\tilde{\xi}_q, \xi^2)\, w_q d\xi^2 \\
\approx &\sum_{p=1}^{n_Q} \sum_{q=1}^{n_Q} \mathcal{G}(\hat{\boldsymbol{x}}(\tilde{\xi}_q, \tilde{\xi}_p))\, \hat{j}(\tilde{\xi}_q, \tilde{\xi}_p)\, w_q\, w_q \\
= &\sum_{(\tilde{\xi}_Q, \omega_Q) \in \mathcal{Q}} \mathcal{G}(\hat{\boldsymbol{x}}(\tilde{\xi}_Q))\, \hat{j}(\tilde{\xi}_Q)\, \omega_Q
\end{aligned}$$

Numerical Integration

We can re-write the integral as

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Numerical Integration

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&\int_{-1}^{1}\int_{-1}^{1} \mathcal{G}(\hat{\boldsymbol{x}}(\xi^1, \xi^2))\, \hat{j}(\xi^1, \xi^2) \;\, d\xi^1 d\xi^2 \\
&\approx \int_{-1}^{1} \sum_{q=1}^{n_Q} \mathcal{G}(\hat{\boldsymbol{x}}(\tilde{\xi}_q, \xi^2))\, \hat{j}(\tilde{\xi}_q, \xi^2)\, w_q d\xi^2 \\
&\approx \sum_{p=1}^{n_Q}\sum_{q=1}^{n_Q} \mathcal{G}(\hat{\boldsymbol{x}}(\tilde{\xi}_q, \tilde{\xi}_p))\, \hat{j}(\tilde{\xi}_q, \tilde{\xi}_p)\, w_q\, w_q \\
&= \sum_{(\tilde{\boldsymbol{\xi}}_Q, \omega_Q) \in \mathcal{Q}} \mathcal{G}(\hat{\boldsymbol{x}}(\tilde{\boldsymbol{\xi}}_Q))\, \hat{j}(\tilde{\boldsymbol{\xi}}_Q)\, \omega_Q
\end{aligned}$$