LECTURE 19:

TOPICS:

- QUADRILATERAL ELEMENTS
  - PARENT DOMAIN
  - BASIS FUNCTIONS
  - ISOPARAMETRIC MAPPING
  - QUADRATURE

LOGISTICS:

- HW # 7 DUE MONDAY
- HW # 8 OUT FRI/MON DUE MONDA AFTER
- FINAL PROJECT : DUE DEAN'S DATE
  FINAL EXAM

LAST TIME WE ARRIVED AT THE WEAK FORM FOR

$$? u: \quad \Delta u = f \quad \forall \underline{x} \in \Omega$$
$$u = g \quad \forall \underline{x} \in \Gamma_D$$
$$\alpha \underline{\nabla} u \cdot \underline{n} = h \quad \forall \underline{x} \in \Gamma_N$$

WHERE THE WEAK FORM IS

$$? u \in S: \quad a(u,v) = F(v) \qquad \forall v \in V$$

WHERE

$$S = \{ u \in H^1(\Omega) \mid u|_{\Gamma_D} = g \}$$
$$V = \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \}$$

AND

$$a(u,v) = \int_\Omega \alpha \underline{\nabla} u \cdot \underline{\nabla} v \, d\Omega \ , \quad F(v) = \int_{\Gamma_N} h v \, d\Gamma - \int_\Omega f v \, d\Omega$$

AND LASTLY THE GALERKIN APPROXIMATION

$$? \, u^h \in S^h : \quad a(u^h, v^h) = F(v^h) \quad \forall \, v^h \in V^h$$

WHERE

$$S^h \subseteq S, \quad V^h \subseteq V$$

THE WAY WE CONSTRUCT S^h IS BY SUBDIVIDING THE DOMAIN

$$\Omega = \bigcup_{e=1}^{n_e} \Omega^e$$

AND BY REQUIRING THAT $u^h \in S$ BUT ALSO

$u^h|_{\Omega^e}$ TO BE A POLYNOMIAL OF ORDER p

TWO GEOMETRIES FOR Ω^e

- HEXAHEDRAL
  - QUADS
  - BRICKS
- SIMPLICIAL
  - TRIANGLES
  - TETRAHEDRA

IF u^h IS A POLYNOMIAL IN Ω^e THEN WE WANT TO CONSTRUCT BASIS FUNCTIONS φ_i^e SUCH THAT ANY POLYNOMIAL OF ORDER p IN Ω^e CAN BE WRITTEN AS

$$u^h\big|_{\Omega^e} = u_a^e \varphi_a^e$$

THE WAY WE DO THIS IS BY CREATING

1. A PARENT DOMAIN THAT IS FIXED $\hat{\Omega}$
2. A SET OF BASIS FUNCTIONS FOR POLYNOMIALS OF ORDER p ON $\hat{\Omega}$, $\hat{\varphi}_a$
3. A MAP FROM $\hat{\Omega}$ TO Ω^e DENOTED AS $\underline{x}^e$

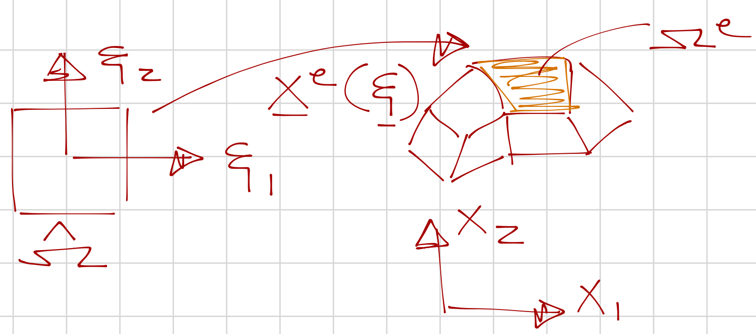
WITH $x^e(\underline{\xi})$ & $\hat{\varphi}_a(\underline{\xi})$ AND $\underline{\xi}^e(\underline{x}) = \underline{x}^e(\underline{\xi})^{-1}$

PARAMETRIC COORDINATES

WE CAN GET $\varphi_a^e(\underline{x}) = \hat{\varphi}_a(\underline{\xi}^e(\underline{x}))$

WE ACTUALLY NEVER CONSTRUCT $\varphi_a^e(\underline{x})$ AS WE DO ALL OF OUR COMPUTATIONS OVER THE PARAMETRIC DOMAIN.

QUADRILATERAL ELEMENTS



$\hat{\Omega} = [-1, 1] \times [1, 1]$

IF $\ell_a^p(\xi)$ IS THE ath LAGRANGE BASE OF ORDER p DEFINED OVER $[-1, 1]$

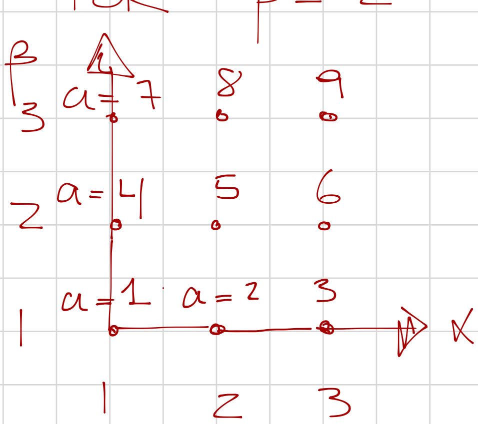
EG.

THEN WE CONSTRUCT THE BASIS FUNCTIONS OVER $\hat{\Omega}$ AS

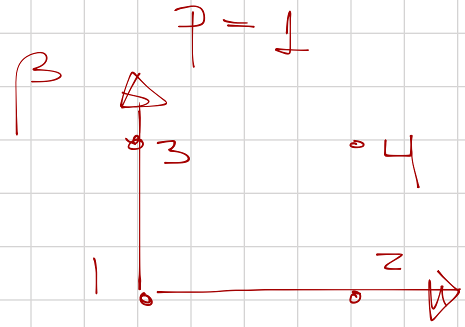
$\hat{\varphi}_a(\underline{\xi}) = \ell_\alpha^p(\xi_1)\, \ell_\beta^p(\xi_2)$

THE NUMBERING OF a AS A FUNCTION OF α & β IS SOMEWHAT ARBITRARY. ONE EXAMPLE

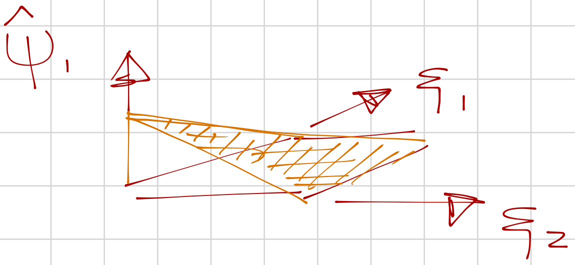
FOR $p = 2$



$p = 1$



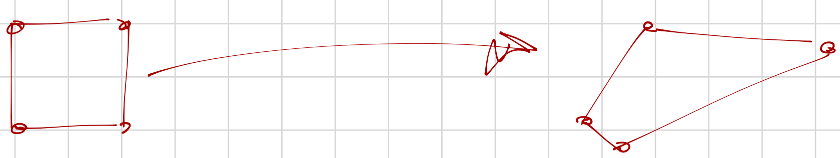
THIS ARE THE BASIS SHOWED IN CLASS LAST TIME



IF WE HAVE $\hat{\varphi}_a(\underline{\xi})$ AND THE COORDINATES OF THE NODES OVER THE PHYSICAL DOMAIN THEN WE CAN CONSTRUCT $x^e(\underline{\xi})$ BY INTERPOLATING WITH ORDER p THOSE COORDINATES.

$$\underline{x}^e(\underline{\xi}) = \hat{\varphi}_a(\underline{\xi})\, \underline{x}^e_a$$

NOTE IF $\hat{\varphi}_a(\underline{\xi})$ ARE LINEAR THEN THE EDGES OF Ω^e WILL BE LINEAR



IF $\hat{\varphi}_a(\underline{\xi})$ ARE HIGHER ORDER THE EDGES WILL BE POLYNOMIALS OF ORDER p

COMPUTING ELEMENT STIFFNESS

SIMILARLY AS BEFORE NOW WITH THE BASIS FUNCTIONS WE CAN CONSTRUCT THE ELEMENT ARRAYS. THEN WITH THE LOCAL TO GLOBAL MAP WE PIECE EVERYTHING TOGETHER

$$K^e_{ab} = \int_{\Omega^e} \alpha\, \underline{\nabla} \varphi^e_a\, \underline{\nabla} \varphi^e_b \, d\Omega$$

NOTE:

$$\underline{\nabla} \varphi^e_a = \frac{d\varphi^e_a}{dX_i}\, \underline{e}_i = \frac{d\varphi^e_a}{d\xi_\alpha}\frac{d\xi^e_\alpha}{dX_i}\, \underline{e}_i = \frac{d\varphi^e_a}{d\xi_\beta}\, \overbrace{\underline{e}_\beta \cdot \underline{e}_\alpha}^{\delta_{\beta\alpha}}\, \frac{d\xi^e_\alpha}{dX_i}\, \underline{e}_i$$

$$= \left(\underline{\nabla}_x\, \underline{\xi}^e\right)^T \left(\underline{\nabla}_{\xi}\, \varphi^e_a\right)$$

$$\underline{\xi}^e(\underline{x}^e(\underline{\xi})) = \underline{\xi} \Rightarrow \nabla_{\underline{\xi}}(\underline{\xi}^e(\underline{x}^e(\underline{\xi}))) = \nabla_{\underline{x}}\underline{\xi}^e \, \nabla_{\underline{\xi}}\underline{x}^e = \underline{\underline{1}}$$

$$\Rightarrow \nabla_{\underline{x}}\underline{\xi}^e = (\nabla_{\underline{\xi}}x^e)^{-1}$$

$$d\Omega = J^e \, d\hat{\Omega} \, , \qquad J^e = \mathrm{DET}(\nabla_{\underline{\xi}}x^e)$$

$$\Rightarrow K^e_{ab} = \int_{\hat{\Omega}} \alpha (\nabla_{\underline{\xi}}x^e)^{-T}(\nabla_{\underline{\xi}}\hat{\psi}_a)(\nabla_{\underline{\xi}}x^e)^{-T}(\nabla_{\underline{\xi}}\hat{\psi}_b) \, J^e \, d\hat{\Omega}$$

QUADRATURE

$$K^e_{ab} = \int_{\hat{\Omega}} \alpha (\nabla_{\underline{\xi}}x^e)^{-T}(\nabla_{\underline{\xi}}\hat{\psi}_a)(\nabla_{\underline{\xi}}x^e)^{-T}(\nabla_{\underline{\xi}}\hat{\psi}_b) \, J^e \, d\hat{\Omega}$$

$$= \int_{-1}^{1}\int_{-1}^{1} \underbrace{\alpha (\nabla_{\underline{\xi}}x^e)^{-T}(\nabla_{\underline{\xi}}\hat{\psi}_a)(\nabla_{\underline{\xi}}x^e)^{-T}(\nabla_{\underline{\xi}}\hat{\psi}_b) \, J^e}_{f(\xi_1, \xi_2)} \, d\xi_1 \, d\xi_2$$

$$\simeq \int_{-1}^{1} \sum_{q=1}^{n_q} f(\xi_q, \xi_2) \, w_q \, d\xi_2$$

$$\simeq \sum_{p=1}^{n_p} \sum_{q=1}^{n_q} f(\xi_q, \xi_p) \, w_q w_p$$

$\Rightarrow$ THUS THE QUAD RULE $\qquad \underline{\xi}_i = (\xi_q, \xi_p), \; w_i = w_q w_p$