HOMEWORK 2

CEE 530: Continuum Mechanics and Thermodynamics

Due: March 19, 2018

PROBLEM 1

Showing step-by-step you calculations, expand the following expressions such that the gradient operator acts only on one term at a time (e.g. $\nabla(\phi \boldsymbol{a}) = \boldsymbol{a} \otimes \nabla\phi + \phi\nabla\boldsymbol{a}$) where $\phi, \psi \in \mathbb{R}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ and $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{d\times d}$

1. (5) $\nabla(\phi \boldsymbol{a})$
2. (5) $\nabla(\phi\,\psi)$
3. (5) $\nabla \times (\phi \boldsymbol{a})$
4. (5) $\nabla \cdot (\boldsymbol{A}\boldsymbol{b})$
5. (5) $\nabla \cdot (\boldsymbol{A}\boldsymbol{B})$

PROBLEM 2

Let $\mathbf{1}$ be the identity tensor and $\boldsymbol{B} \in \mathbb{R}^{d\times d}$ such that

$$\boldsymbol{A}(\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{B})\ \mathbf{1} + \frac{1}{2}(\boldsymbol{B} + \boldsymbol{B}^\top)$$

what is $\dfrac{\partial \boldsymbol{A}}{\partial \boldsymbol{B}}$?

PROBLEM 3

Consider a cube of side a made of a soft, deformable material. We adopt an orthonormal basis $\{\mathbf{e}_i\}_i$ parallel to the sides of the undeformed cube to describe the motion, and one of the vertices of the cube as the origin. In this way, all points in the cube have position vectors of the form $\boldsymbol{X} = X_i\mathbf{e}_i$, with $0 \leq X_i \leq a$, $i = 1, 2, 3$. The motion of the cube is given by

$$\boldsymbol{\varphi}(\boldsymbol{X}, t) = X_1(1 + t)\ \mathbf{e}_1 + X_2(1 + t^2)\ \mathbf{e}_2 + X_3(1 + t^3)\ \mathbf{e}_3$$

so that at t = 0 the cube is at the reference configuration

1. (10) Find the material velocity field.
2. (10) Find the spatial velocity field.
3. (10) Find the spatial acceleration field.
4. (10) Compute the right Cauchy-Green deformation tensor. Does it depend on $\boldsymbol{X}$?
5. (5) Is the motion admissible at each t?
6. (10) What is the stretch ratio of vectors in the direction parallel to $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ as a function of time.
7. (10) What is the value of the cosine of the angle between directions $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1$ as a function of time?
8. (10) Can you construct an affine deformation mapping in which the directions defined by the orthonormal vectors $\boldsymbol{f}_1 = (\mathbf{e}_1 + \mathbf{e}_2)/2$, $\boldsymbol{f}_2 = (\mathbf{e}_2 - \mathbf{e}_1)/2$ and $\boldsymbol{f}_3 = \mathbf{e}_3$ are mapped into themselves (each one to itself), but stretched with stretch ratios $\lambda_1(t) = 1 + t$, $\lambda_2(t) = 1 + t^2$ and $\lambda_3(t) = 1 + t^3$, respectively? Of course, we request it to have a positive Jacobian. Express it in the $\mathbf{e}_i$ basis.

PROBLEM 4

A cylindrical tube of inner radius a, outer radius b, and length L is turned inside out and subsequently constrained to take the shape of a straight cylinder of the same dimensions.

1. (15) Assuming that the radial and axial fibers remain unstretched and still radial and axial, respectively, after deformation, determine the deformation mapping $\boldsymbol{\varphi}$.

2. (5) Show that $\boldsymbol{\varphi} \circ \boldsymbol{\varphi}$ is the identity mapping, i.e., turning the cylinder inside out twice returns it to its initial configuration.

3. (10) Compute the deformation gradient. Name a direction with stretched fibers.

PROBLEM 5

A block of rubber has a rectangular cross-section of dimensions A and B in its undeformed configuration. A set of axes is chosen such that the cross-section of the block coincides with the $X_1 - X_2$ plane. The deformation of the block is constrained in the remaining direction X_3. The cross-section of the block is reinforced by means of two sets of uniformly spaced straight wires. The two families of wires subtend an angle 2α to each other, and their bisector is at an angle β to the X_1-axis. The wires are strongly bonded to the matrix and follow its deformation. Since the metallic wires are much stiffer than the rubbery matrix, they are taken to be inextensible to a first approximation. After deformation the cross−section of the block takes a rhomboidal shape of dimensions a and b. The angle between the sides of the rhomboid is θ.

1. (15) Write down the right Cauchy-Green deformation tensor $\boldsymbol{C}$ as a function of the coordinate stretches a/A, b/B and the angle θ.
   Hint: Notice that the deformation gradient will be uniform throughout the sample, therefore, you can try to deduce $\boldsymbol{C}$ by using geometric considerations of changes of lengths and angles.

2. (10) For what value of β the sides of the block remain at an angle of $\pi/2$ after the deformation?

3. (10) Compute the coordinate stretch ratios a/A and b/B as a function of α, β, θ.

4. (10) Let $\beta = 0$ and imagine that the block is stretched in the X1 direction and compressed in the X_2. Define a nonlinear ?Poisson?s ratio? by the relation $\nu = -E_{22}/E_{11}$, where E_{11} and E_{22} are components of the Lagrangian strain tensor $\boldsymbol{E} = (\boldsymbol{C} - \boldsymbol{I})/2$. Compute ν as a function of α. For what value of α does it follow that $\nu = 0.5$?