LECTURE 10

STRESS

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SHORT RANGE FORCES → ATOMIC INTERACTIONS

LONG RANGE FORCES → GRAVITATIONAL FORCES ETC.

SHORT RANGE FORCES ARE TRANSMITTED THROUGH THE BODY

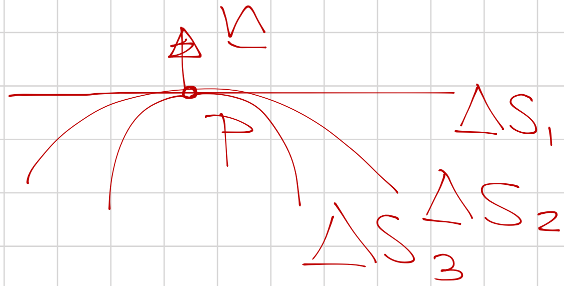
CONSIDER AN IMMAGINARY SURFACE IN OUR BODY SEPARATING THE BODY INTO TWO

WE DEFINE THE TRACTION VECTOR $\underline{t}$ AS THE FORCE PER UNIT AREA AT A POINT P

TRACTION AT A POINT P IN OUR CONTINUUM

$$\underline{t}_S(P) = \lim_{t \to 0} \frac{\Delta \underline{F}}{|S \cap B_t(P)|} = \lim_{t \to 0} \frac{\Delta \underline{F}}{\Delta S}$$

CONSIDER SURFACES THROUGH P ALL HAVING SAME NORM $\underline{n}$



IN THE LIMIT $t \to 0$ ALL $\dfrac{\Delta \underline{F}_i}{\Delta S_i}$ CONVERGE TO

IE. THE VALUE OF THE TRACTION VECTOR IS INHERENTLY LOCAL & DEPEND ON POSITION & THE SURFACE NORMAL

LASTLY NOTE THAT ON OPPOSITE FACES OF THE CUT

$$\Delta \underline{F}(\underline{n}) = -\Delta \underline{F}(-\underline{n}) \Rightarrow \underline{t}(\underline{x}, \underline{n}) = -\underline{t}(\underline{x}, -\underline{n})$$

(NOTE WE OMIT DEPENDENCE ON TIME FOR SIMPLICITY

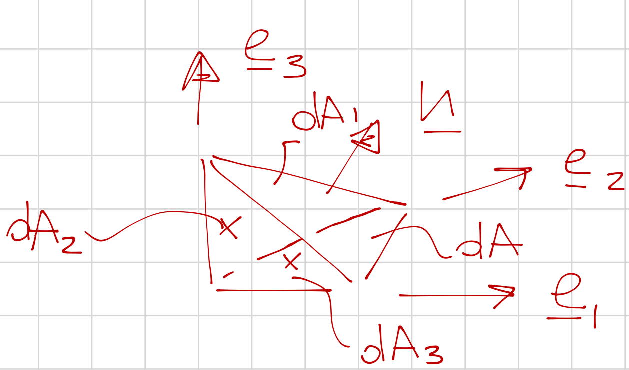
CAUCHY'S TETRAHEDRON THEOREM

$\underline{t}(\underline{x}, \underline{n})$ IS LINEAR IN $\underline{n}$.

BY DEFINITION $\underline{t}_x(\ )$ IS A TENSOR

THEREFORE CAUCHY'S TETRAHEDRON THEOREM STATES THAT $\exists\ \underline{\underline{T}} \in \mathbb{R}^{d \times d}$ SUCH THAT

$$\underline{t} = \underline{\underline{T}}\,\underline{n}$$



$$\underline{t}(-\underline{e}_3)\, dA_3 + \underline{t}(-\underline{e}_1)\, dA_1 + \underline{t}(-\underline{e}_2)\, dA_2 + \underline{t}(\underline{n})\, dA = 0$$

$$dA_i = \underline{n} \cdot \underline{e}_i\, dA$$

$$\underline{t}(-\underline{e}_3)\, \underline{n} \cdot \underline{e}_3 + \underline{t}(\underline{e}_2)\, \underline{n} \cdot \underline{e}_2 + \underline{t}(\underline{e}_1)\, \underline{e}_1 \cdot \underline{n} + \underline{t}(\underline{n}) = 0$$

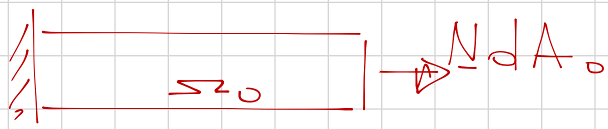
WITH $\underline{t}(-\underline{n}) = -\underline{t}(\underline{n})$

$$\underline{t}(\underline{n}) = \underline{t}(\underline{e}_1)\overbrace{\underline{e}_1 \cdot \underline{n}}^{n_1} + \underline{t}(\underline{e}_2)\overbrace{\underline{e}_2 \cdot \underline{n}}^{n_2} + \underline{t}(\underline{e}_3)\overbrace{\underline{e}_3 \cdot \underline{n}}^{n_3} = \underline{t}(n_1 \underline{e}_1 + n_2 \underline{e}_2 + n_3 \underline{e}_3)$$

$\underline{t}$ IS LINEAR IN $\underline{n}$!

THUS FAR WE HAVE DEFINED A TRACTION
WE KNOW IT DEPENDS ON THE POSITION & THE NORM ALONE.

$\underline{t}$ → FORCE PER UNIT AREA
DEPENDS ON THE NORM TO THE SURFACE



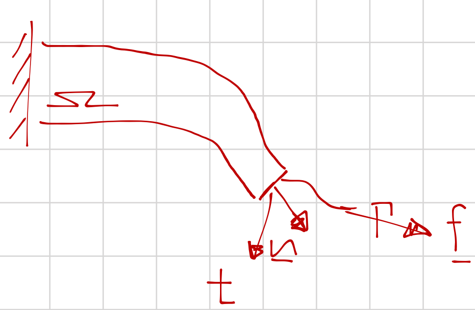
WHICH NORM & WHICH AREA?

WE HAVE MULTIPLE STRESS TENSORS:

CAUCHY STRESS TENSOR

IT MAPS NORMALS IN THE DEFORMED CONFIG INTO THE TRACTION VECTOR IN THE DEFORMED CON

$$\underline{\underline{T}}(\underline{x})\,\underline{n}(\underline{x}) = \underline{t}(\underline{x})$$



$\underline{t}$ ← FORCE PER UNIT DEFORMED AREA

$\underline{n}$ ← NORM TO DEFORMED SURFACE

$\underline{f}$ ← RESULTANT FORCE

$\underline{\underline{T}}$ ← CAUCHY'S STRESS TENSOR

$$\underline{f} = \int_\Gamma \underline{t}\, ds = \int_\Gamma \underline{\underline{T}}\,\underline{n}\, ds$$

FIRST PIOLA KIRCKOFF STRESS TENSOR

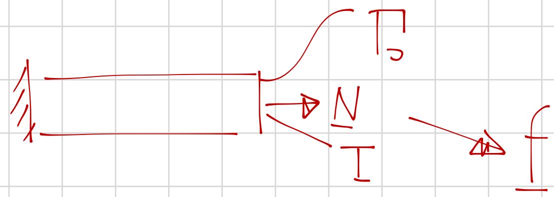
IT MAPS NORMALS IN THE REFERENCE CONFIG TO TRACTION VECTORS OVER REFERENCE

$\underline{\underline{P}}(\underline{X})\underline{N}(\underline{X}) = \underline{T}(\underline{X})$

$\underline{T}(\underline{X}) \leftarrow$ FORCE PER UNIT REFERENCE

$\underline{N} \leftarrow$ NORMAL TO REF.

$\underline{\underline{P}} \leftarrow$ FIRST PIOLA-KIRCHOFF



$$\underline{f} = \int_{\Gamma_0} \underline{\underline{P}}\,\underline{N}\, dS_0 = \int_{\Gamma} \underline{\underline{\sigma}}\,\underline{n}\, ds$$

HOW ARE $\underline{\underline{P}}$ & $\underline{\underline{\sigma}}$ RELATED

$\underline{n}\, ds = J \underline{\underline{F}}^{-T} \underline{N}\, dS_0 \leftarrow$ NANSONS FORMULA

$$\Rightarrow \int_{\Gamma} \underline{\underline{\sigma}}\,\underline{n}\, ds = \int_{\Gamma_0} \underline{\underline{\sigma}}\, J \underline{\underline{F}}^{-T} \underline{N}\, dS_0 = \int_{\Gamma_0} \underline{\underline{P}}\, N\, dS_0$$

$$\Rightarrow \quad \underline{\underline{P}} = \underline{\underline{\sigma}}\, J \underline{\underline{F}}^{-T}$$

NOTE THAT $J\underline{\underline{F}}^{-T}$ IS ALSO KNOWN AS THE PIOLA TRANSFORM

NOTE $\underline{\underline{\sigma}} \rightarrow$ HAS BOTH LEGS IN SPATIAL

$\underline{\underline{P}} \rightarrow$ IS A MIXED TENSOR JUST LIK $\underline{\underline{F}}$

WE WILL SEE LATER THAT WHILE $\underline{\underline{\sigma}}$ IS SYMMETRIC, $\underline{\underline{P}}$ IN GENERAL IT'S NOT

OTHER STRESS TENSORS

SECOND PIOLA STRESS TENSOR

$$\underline{\underline{S}} = \underline{\underline{F}}^{-1}\underline{\underline{P}} = J\,\underline{\underline{F}}^{-1}\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T}$$

- SYMMETRIC
- ARISES NATURALLY WHEN FORMULATING CONST LAWS AS A FUNCTION OF $\underline{\underline{C}}$
- TWO MATERIAL INDECES

KIRCHOFF STRESS

$$\underline{\underline{\tau}} = J\,\underline{\underline{\sigma}}$$

- ARISES NATURALLY WHEN FORMULATING CONST LAWS AS A FUNCTION OF $\underline{\underline{b}}$
- SYMMETRIC

NOTE:

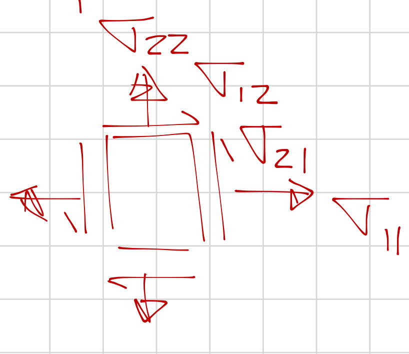
$\underline{\underline{S}}$ IS THE CONTRAVARIANT PULL BACK OF $\underline{\underline{\tau}}$

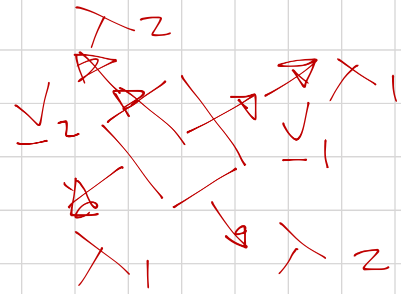
$$\underline{\underline{S}} = \underline{\varphi}^{*}(\underline{\tau}^{\#}) = \underline{\underline{F}}^{-1}\underline{\underline{\tau}}\,\underline{\underline{F}}^{-T} = J\,\underline{\underline{F}}^{-1}\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T}$$

VALUES OF STRESS

NOTE SINCE $\underline{\underline{\sigma}}$ IS SYMMETRIC THE IT HAS ORTHOGONAL EIGENVECTORS

$$\underline{\underline{\sigma}} = \sum_i \lambda_i \, \underline{v}_i \otimes \underline{v}_i$$





THE EIGENVALUES ARE PRINCIPAL STRESSES & SHEAR STRESSES VANISH

INVARIANTS

$I_1 = tr(\underline{\underline{\sigma}})$,

$\underline{\underline{\sigma}}$ CAN BE DECOMPOSED IN A MEAN HYDROSTATIC PRESSURE & A DEVIATORIC PART

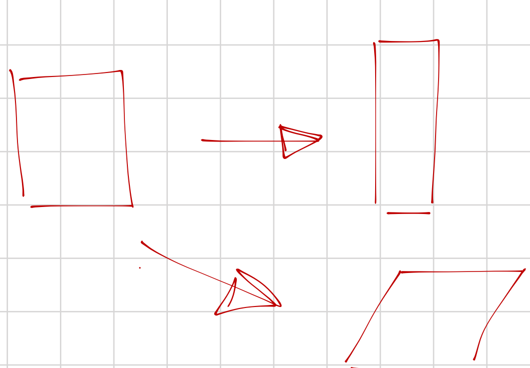
$$\underline{\underline{\sigma}} = \underline{\underline{s}} + \frac{1}{3} tr(\underline{\underline{\sigma}}) = \underline{\underline{s}} + \frac{1}{3} I_1$$

AN IMPORTANT INVARANT

$$J_2(\underline{\underline{\sigma}}) = \frac{1}{2} tr(\underline{\underline{s}}) = \frac{1}{3} I_1^2 - I_2 = \frac{1}{2}\left[tr(\underline{\underline{\sigma}}^2) - \frac{1}{3} tr(\underline{\underline{\sigma}})^2 \right]$$

J_2 IS VERY IMPORTAN FOR PLASTICITY

NOTE THAT $\underline{\underline{s}}$ IS THE STREE COMPONENTS THAT DEFORMS



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BALANCE PRINCIPLES

REYNOLDS TRANPORT THEOREM