LECTURE 3

- REVIEW
- TENSORS CONT'D
- REVIEW OF DIFF EQ

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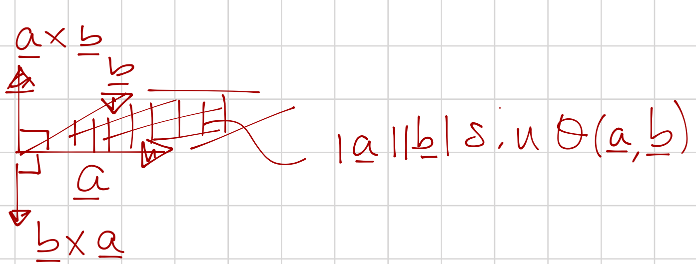
REVIEW

LAST TIME WE CONTINUED WITH VECTORS & INTRODUCED AN ADDITIONAL OPERATION AMONGST VECTORS, THE CROSS PRODUCT.

THE CROSS PRODUCT HAS ONLY MEANING WHEN WE WORK IN $\mathbb{R}^3$

FOR TWO VECTORS $\underline{a}, \underline{b} \in \mathbb{R}^3$ THE CROSS PRODUCT IS A VECTOR WITH MAGNITUDE GIVEN BY THE AREA OF THE PARALLELOGRAM CREATED BY THE TWO VECTORS & POINTIN ALONG THE NORMAL TO THE SURFACE. THE SIGN OF THE VECTOR IS DETERMINED BY THE INFAMOUS RIGHT HAND RULE.



AND WE SAW THAT

$$\underline{a} \times \underline{b} = \varepsilon_{ijk} a_i b_j \underline{e}_k$$

WHERE ε_{ijk} IS THE LEVI CIVITA SYMBOL

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{IF REPEATED INDECES} \\ 1 & \text{IF EVEN PERM} \\ -1 & \text{IF ODD PERM} \end{cases}$$

WE THEN INTRODUCED THE NOTION OF SECOND ORDER TENSOR $\underline{\underline{T}}$ AS A LINEAR OPERATOR ON THE SPACE OF VECTORS. NAMELY

$$\underline{\underline{T}} \in \mathbb{R}^{d \times d}, \quad \underline{\underline{T}} : \mathbb{R}^d \to \mathbb{R}^d$$

$$\underline{\underline{T}}(\underline{a}+\underline{b}) = \underline{\underline{T}}(\underline{a}) + \underline{\underline{T}}(\underline{b}) \quad \forall\, \underline{a}, \underline{b} \in \mathbb{R}$$

$$\underline{\underline{T}}(\alpha \underline{a}) = \alpha \underline{\underline{T}}(\underline{a}) \qquad \forall\, \alpha \in \mathbb{R},\ \underline{a} \in \mathbb{R}^d$$

TENSOR ALGEBRA

$(\underline{\underline{A}} + \underline{\underline{B}})\underline{u} = \underline{\underline{A}}\underline{u} + \underline{\underline{B}}\underline{u} = \underline{\underline{C}}\underline{u}\,, \quad \underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}}$

$\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{u} = \underline{\underline{C}}\underline{u} = \underline{\underline{A}}(\underline{\underline{B}}\underline{u})$

TENSOR PRODUCT

SINCE TENSORS ARE LINEAR OPERATORS WE CAN SHOW THAT FOR ANY TENSOR $\underline{\underline{T}}$ THERE EXIST TWO VECTORS $\underline{a}, \underline{b} \in \mathbb{R}^d$ SUCH THAT

$$\underline{\underline{T}}(\underline{c}) = (\underline{b} \cdot \underline{c})\underline{a}$$

TO CONSTRUCT $\underline{\underline{T}}$ THEN WE INTRODUCE THE DYADIC OR TENSOR OR OUTER PRODUCT $\otimes$ SUCH THAT

$$\otimes : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$$

EG: $\underline{\underline{T}} = \underline{a} \otimes \underline{b}$

THEN

$$\underline{\underline{T}}(\underline{c}) = (\underline{a} \otimes \underline{b})(\underline{c}) = \underline{a}\,(\underline{b} \cdot \underline{c})$$

NOTE THAT THE ABOVE SATISFY THE LINEARITY CONDITION

$$(\underline{a} \otimes \underline{b})(\underline{c} + \underline{d}) = \underline{a}\,(\underline{b} \cdot (\underline{c} + \underline{d})) = \underline{a}(\underline{b} \cdot \underline{c}) + \underline{a}\,(\underline{b} \cdot \underline{d}) \quad \checkmark$$

$$(\underline{a} \otimes \underline{b})(\alpha \underline{c}) = \alpha\,(\underline{a} \otimes \underline{b})\underline{c} \quad \checkmark$$

NOTE: $\underline{a} \otimes \underline{b} \neq \underline{b} \otimes \underline{a}$!!

IN FACT IF $\underline{\underline{T}} = \underline{a} \otimes \underline{b}$ WE DENOTE BY $\underline{\underline{T}}^T = \underline{b} \otimes \underline{a}$ THE TRANSPOSE OF $\underline{\underline{T}}$

LASTLY NOTE THAT IF $\underline{a} = a_i \underline{e}_i$, $\underline{b} = b_j \underline{e}_j$ THEN

$$\underline{\underline{T}} = (a_i \underline{e}_i) \otimes (b_j \underline{e}_j) = a_i b_j\, \underline{e}_i \otimes \underline{e}_j = T_{ij}\, \underline{e}_i \otimes \underline{e}_j$$

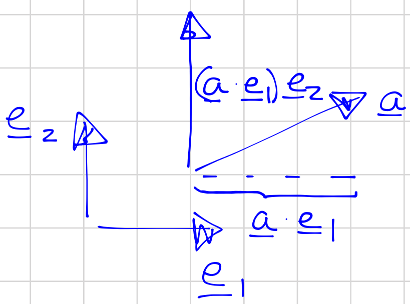
& T_{ij} ARE SAID TO BE THE COMPONENTS OF $\underline{\underline{T}}$ IN THE $\underline{e}_i$ BASIS WHICH CAN BE WRITTEN AS

$$[\underline{\underline{T}}]_{\underline{e}} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ \vdots & & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots \\ & \ddots & \end{bmatrix}$$

$$= [\underline{a}]_{\underline{e}} [\underline{b}]_{\underline{e}}^T = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \end{Bmatrix} \{ b_1 \; b_2 \; \cdots \; b_n \}$$

EG:

CONSIDER THE OPERATION OF TAKING A VECTOR GETTING ITS PROJECTION ALONG $\underline{e}_1$ & ROTATING IT IN THE $\underline{e}_2$ DIRECTION



$$\underline{\underline{T}}(\underline{a}) = (\underline{a}\cdot\underline{e}_1)\underline{e}_2 = (\underline{e}_2 \otimes \underline{e}_1)(\underline{a}) = (T_{ij}\, \underline{e}_i \otimes \underline{e}_j)(\underline{a})$$

WHERE $T_{ij} = 1$ IF $i=2$ & $j=1$ ELSE 0.

$$[\underline{\underline{T}}]_{\underline{e}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \{1 \; 0\} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$[\underline{\underline{T}}(\underline{a})]_{\underline{e}} = [\underline{\underline{T}}]_{\underline{e}} [\underline{a}]_{\underline{e}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ a_1 \end{Bmatrix} = [a_1 \underline{e}_2]_{\underline{e}}$$

NOW WHAT IF WE WANT TO CONSTRUCT A AN OPERATION THAT ROTATES 90° COUNTERCLOCKWISE ?

$\underline{b} = \underline{\underline{T}}(\underline{a})$?

, $|\underline{a}| = |\underline{b}|$

$b_1 = -a_2$, $b_2 = a_1$ $\underline{b} = -a_2 \underline{e}_1 + a_1 \underline{e}_2$

WORK IN GROUPS TO DO IT!

$$\underline{\underline{T}}(\underline{a}) = (-a_2\underline{e}_1) + a_1\underline{e}_2 = -(\underline{a}\cdot\underline{e}_2)\underline{e}_1 + (\underline{a}\cdot\underline{e}_1)\underline{e}_2$$

$$= (-\underline{e}_1\otimes\underline{e}_2)(\underline{a}) + (\underline{e}_2\otimes\underline{e}_1)(\underline{a}) =$$

$$= (-\underline{e}_1\otimes\underline{e}_2 + \underline{e}_2\otimes\underline{e}_1)(\underline{a})$$

THUS

$$[\underline{\underline{T}}]_{\underline{e}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

LASTLY A VERY SPECIAL TENSOR IS THE IDENTITY TENSOR

$$\underline{\underline{I}}(\underline{a}) = \underline{a} = a_i\underline{e}_i = a_j\underline{e}_i\delta_{ij} = (\underline{a}\cdot\underline{e}_j)\underline{e}_i\delta_{ij} =$$

$$= (\delta_{ij}\,\underline{e}_i\otimes\underline{e}_j)(\underline{a})$$

$$[\underline{\underline{I}}]_{\underline{e}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

TO CONCLUDE WE REVISIT SOME OPERATIONS ON TENSORS

$$\underline{\underline{A}} + \underline{\underline{B}} = (A_{ij} + B_{ij})\,\underline{e}_i\otimes\underline{e}_j$$

$$\underline{\underline{A}}\,\underline{\underline{B}} = A_{ik}B_{kj}\,\underline{e}_i\otimes\underline{e}_j$$

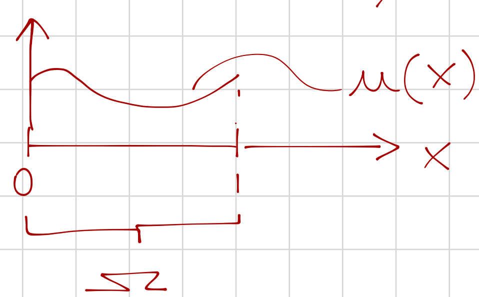
$$\underline{\underline{A}} : \underline{\underline{B}} = A_{ij}B_{ij}$$

SO FAR WE HAVE BEEN INTERESTED IN THE NOTION OF VECTORS & TENSOR BUT WHAT WE ARE REALLY AFTER ARE FUNCTIONS

FUNCTIONS CAN BE SCALAR-, VECTOR-, OR TENSOR-VALUED FUNCTIONS.

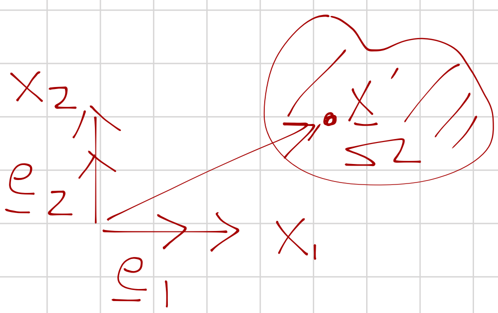
SCALAR FUNCTION

1-D $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$ (EG. $\Omega = [0, 1]$



$u(x) = \cos(x) e^{x} \dots$

2-D $\theta : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$



$\underline{x} = x_i \underline{e}_i$

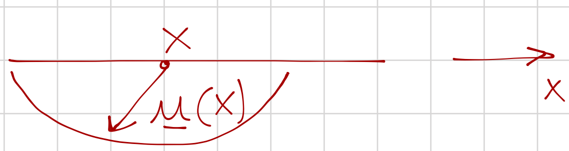
$\theta(\underline{x}) = \cos(x_1) e^{x_2} + x_2^2 \dots$

VECTOR FUNCTION

1-D $\underline{u} : \Omega \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}$

EG

$\underline{u}(x) = \cos(x) \underline{e}_1 + \sin(x) \underline{e}_2$



EG DEFORMATION OF A BEAM-COLUMN

2-D $\underline{v} : \Omega \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}$

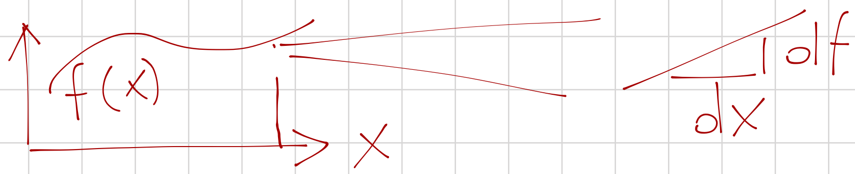
$\underline{v}(\underline{x}) = v_i(\underline{x}) \underline{e}_i = v_i(x_1, x_2 \dots x_d) \underline{e}_i$

TESOR FUNCTION

$\underline{\underline{T}} : \Omega \to \mathbb{R}^{d\times d}$ EG STRESS TENSOR

$$\underline{\underline{T}}(\underline{x}) = T_{ij}(\underline{x})\, \underline{e}_i \otimes \underline{e}_j$$

REVIEW OF CALCULUS



← SLOPE OF A FUNCTION

$$\frac{df}{dx}(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

SIMILARLY IF WE HAVE A FUNCTION OF MULTIPLE VARIABLE

$$\frac{df}{dx_i}(x_1, \dots x_i \dots x_d) = \lim_{\varepsilon \to 0} \frac{f(x_1, \dots, x_i + \varepsilon, \dots x_d) - f(x_1, \dots, x_i, \dots x_d)}{\varepsilon}$$

$\frac{df}{dx_i}(\underline{x})$ TELLS YOU THE CHANGE IN THE FUNCTION f WRT THE COORDINATE x_i

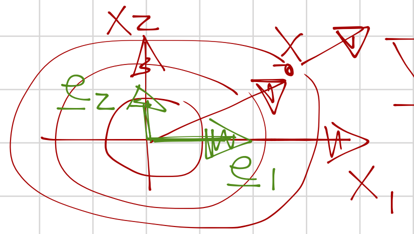
THE GRADIENT OF A FUNCTION

$$\nabla f = \frac{df}{dx_i}(\underline{x})\, \underline{e}_i$$

SUMMATION

IS A VECTOR POINTING IN THE DIRECTION OF MAX GROWTH

EG: CONSIDER $f(\underline{x}) = x_1^2 + x_2^2$



$$\underline{\nabla} f(\underline{x}) = \frac{df}{dx_i}(\underline{x})\,\underline{e}_i$$

$$= (2x_1 + 2x_2)\,\underline{e}_1$$