LECTURE 10:

TOPICS:

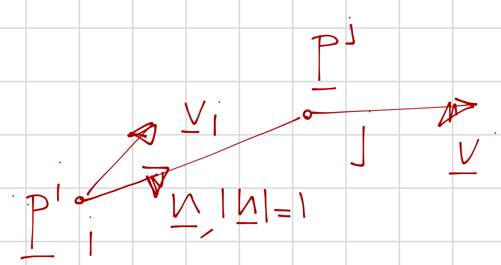
- REVIEW
- TRUSSES IN nD FINISHED
- CONSTRAINTS
- INTRO TO BEAMS

LOGISTICS

- HW# 3 DUE FRIDAY
- HW# 2 SOLUTION POSTED

REVIEW:

RECALL



WE LOOKED FOR THE LONGITUDINAL FORCES AND FORCE BALANCE & FOUND

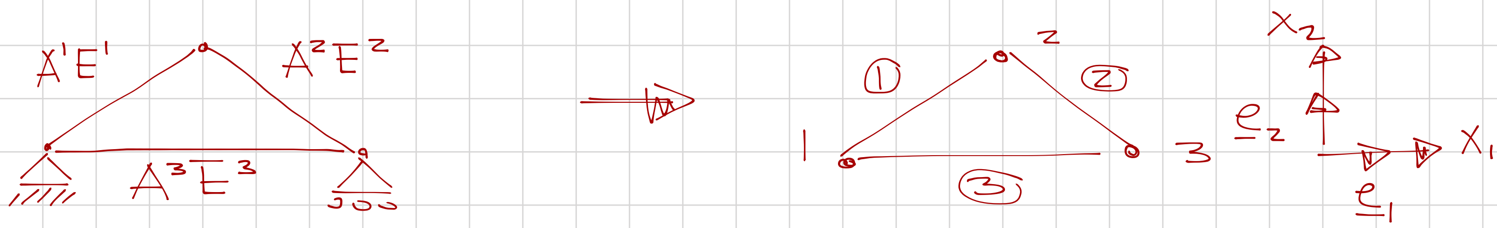
$$\underline{f} = \left(\frac{AE}{e}\underline{n}\otimes\underline{n}\right)\left(\underline{v}^j - \underline{v}^i\right)$$

SUCH THAT FORCE BALANCE GIVES

$$\begin{Bmatrix}\underline{p}^i\\ \underline{p}^j\end{Bmatrix} = \begin{Bmatrix}-\underline{f}\\ \underline{f}\end{Bmatrix} = \begin{bmatrix}\underline{\underline{k}} & -\underline{\underline{k}}\\ -\underline{\underline{k}} & \underline{\underline{k}}\end{bmatrix}\begin{Bmatrix}\underline{v}^i\\ \underline{v}^j\end{Bmatrix}$$

WHERE $\underline{\underline{k}} = \frac{AE}{e}\underline{n}\otimes\underline{n}$

SYSTEMS OF TRUSSES



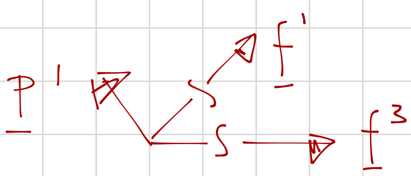
CONSIDER THE ABOVE TRUSS WERE THE POSITION OF EACH CONNECTION IS GIVEN BY $\underline{p}_i$, $i = 1, 2, 3$

SUCH THAT THE LENGTHS OF EACH ELEMENT ARE GIVEN BY

$$\ell^1 = |\underline{p}^2 - \underline{p}^1|, \quad \ell^2 = |\underline{p}^3 - \underline{p}^2|, \quad \ell^3 = |\underline{p}^1 - \underline{p}^3|$$

WE CAN SUM FORCES AT EACH NODE

AT NODE 1:



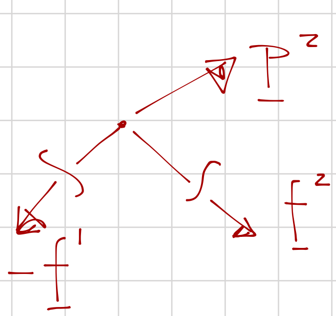
$$\underline{f}^1 = \frac{A^1E^1}{\ell^1}(\underline{n}^1 \otimes \underline{n}^1)(\underline{v}^2 - \underline{v}^1), \quad \underline{n}^1 = \frac{\underline{p}^2 - \underline{p}^1}{\ell^1}$$

$$\underline{f}^3 = \frac{A^3E^3}{\ell^3}(\underline{n}^3 \otimes \underline{n}^3)(\underline{v}^3 - \underline{v}^1), \quad \underline{n}^3 = \frac{\underline{p}^3 - \underline{p}^1}{\ell^3}$$

$$\underline{P}_1 + \underline{f}^1 + \underline{f}^3 = 0 \implies \underline{P}_1 = -\underline{f}^1 - \underline{f}^3 = -\underline{\underline{K}}^1(\underline{v}^2 - \underline{v}^1) - \underline{\underline{K}}^3(\underline{v}^3 - \underline{v}^1)$$

$$= \left\{ \underline{\underline{K}}^1 + \underline{\underline{K}}^3 \quad -\underline{\underline{K}}^1 \quad -\underline{\underline{K}}^3 \right\} \begin{bmatrix} \underline{v}^1 \\ \underline{v}^2 \\ \underline{v}^3 \end{bmatrix}$$

AT NODE 2:

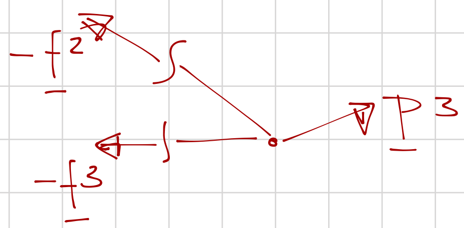


$$\underline{f}^2 = \frac{A^2E^2}{\ell^2}(\underline{n}^2 \otimes \underline{n}^2)(\underline{v}^3 - \underline{v}^2), \quad \underline{n}^2 = \frac{\underline{p}^3 - \underline{p}^2}{\ell^2}$$

$$\underline{P}^2 - \underline{f}^1 + \underline{f}^2 = 0 \longrightarrow \underline{P}_2 = \underline{f}^1 - \underline{f}^2 = \underline{\underline{K}}^1(\underline{v}^2 - \underline{v}^1) - \underline{\underline{K}}^2(\underline{v}^3 - \underline{v}^2) =$$

$$= \left\{ -\underline{\underline{K}}^1 \quad \underline{\underline{K}}^1 + \underline{\underline{K}}^2 \quad -\underline{\underline{K}}^2 \right\} \begin{Bmatrix} \underline{v}^1 \\ \underline{v}^2 \\ \underline{v}^3 \end{Bmatrix}$$

AT NODE 3:



$$\underline{P}^3 - \underline{f}^2 - \underline{f}^3 = 0 \Rightarrow \underline{P}^3 = \underline{f}^2 + \underline{f}^3 = \underline{\underline{K}}^2(\underline{V}^3 - \underline{V}^2) + \underline{\underline{K}}^3(\underline{V}^3 - \underline{V}^1) =$$

$$= \left\{ -\underline{\underline{K}}^3 \quad -\underline{\underline{K}}^2 \quad \underline{\underline{K}}^2 + \underline{\underline{K}}^3 \right\} \begin{Bmatrix} \underline{V}^1 \\ \underline{V}^2 \\ \underline{V}^3 \end{Bmatrix}$$

AND WHEN WE PUT IT TOGETHER

$$\begin{Bmatrix} \underline{P}^1 \\ \underline{P}^2 \\ \underline{P}^3 \end{Bmatrix} = \begin{bmatrix} \underline{\underline{K}}^1 + \underline{\underline{K}}^3 & -\underline{\underline{K}}^1 & -\underline{\underline{K}}^3 \\ -\underline{\underline{K}}^1 & \underline{\underline{K}}^1 + \underline{\underline{K}}^2 & -\underline{\underline{K}}^2 \\ -\underline{\underline{K}}^3 & -\underline{\underline{K}}^2 & \underline{\underline{K}}^2 + \underline{\underline{K}}^3 \end{bmatrix} \begin{Bmatrix} \underline{V}^1 \\ \underline{V}^2 \\ \underline{V}^3 \end{Bmatrix}$$

SIMILARLY TO 1D TRUSSES WE CAN CONSTRUCT THE ABOVE WITH CONNECTIVITY ARRAYS

ELEMENT	NODE i	NODE j
1	1	2
2	2	3
3	1	3

THEN WE CONSTRUCT A LOCAL TO GLOBAL MAP

$M($ ELEMENT NUMBER , LOCAL DOF (i OR j) $)$ = GLOBAL DOF

AND WITH THE ABOVE "LOOP" OVER ALL ELEMENTS

ELEMENT STIFFNESS

$$\begin{bmatrix} \underline{\underline{K}}^1 & -\underline{\underline{K}}^1 \\ -\underline{\underline{K}}^1 & \underline{\underline{K}}^1 \end{bmatrix}$$

$M(1,i)\,M(1,i)$
$(1,\ 1)$

$M(1,i)\,M(1,j)$
$(1,\ 2)$

$M(1,j)\,M(1,i)$
$(2,\ 1)$

$M(1,j)\,M(1,j)$
$(2,\ 2)$

$$\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \underline{\underline{K}}^1 & -\underline{\underline{K}}^1 & \\ -\underline{\underline{K}}^1 & \underline{\underline{K}}^1 & \\ & & \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$\begin{bmatrix} \underline{\underline{K}}^2 & -\underline{\underline{K}}^2 \\ -\underline{\underline{K}}^2 & \underline{\underline{K}}^2 \end{bmatrix} \qquad \begin{bmatrix} \underline{\underline{K}}^1 & -\underline{\underline{K}}^1 & \\ -\underline{\underline{K}}^1 & \underline{\underline{K}}^1+\underline{\underline{K}}^2 & -\underline{\underline{K}}^2 \\ & -\underline{\underline{K}}^2 & \underline{\underline{K}}^2 \end{bmatrix}$$

$$\begin{bmatrix} \underline{\underline{K}}^3 & -\underline{\underline{K}}^3 \\ -\underline{\underline{K}}^3 & \underline{\underline{K}}^3 \end{bmatrix} \qquad \begin{bmatrix} \underline{\underline{K}}^1+\underline{\underline{K}}^3 & -\underline{\underline{K}}^1 & -\underline{\underline{K}}^3 \\ -\underline{\underline{K}}^1 & \underline{\underline{K}}^1+\underline{\underline{K}}^2 & -\underline{\underline{K}}^2 \\ -\underline{\underline{K}}^3 & -\underline{\underline{K}}^2 & \underline{\underline{K}}^2+\underline{\underline{K}}^3 \end{bmatrix}$$

CONSTRAINTS

SIMILARLY TO 1D WE CAN APPLY CONSTRAINTS TO EACH INDIVIDUAL COMPONENT OF THE DISPLACEMENT BY (i) ZEROING OUT THE ROW
(ii) PLACING 1 ON THE DIAGONAL
(iii) REPLACING THE VALUE IN THE LOAD VECTOR

EG: SUPPOSE WE WANT TO PRESCRIBE $\underline{V}^3 \cdot \underline{e}_2 = V^3_2$
(i.e. THE VALUE OF THE VERTICAL DISPL AT NODE 3)

$$\begin{Bmatrix} P^1_1 \\ P^1_2 \\ P^2_1 \\ P^2_2 \\ P^3_1 \\ P^3_2 \end{Bmatrix} = \begin{bmatrix} \ddots & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ -K^3_{11} & -K^3_{12} & -K^2_{11} & -K^2_{12} & K^2_{11}+K^3_{11} & K^2_{12}+K^3_{12} \\ -K^3_{21} & -K^3_{22} & -K^2_{21} & -K^2_{22} & K^2_{21}+K^3_{21} & K^2_{22}+K^3_{22} \end{bmatrix} \begin{Bmatrix} V^1_1 \\ V^1_2 \\ V^2_1 \\ V^2_2 \\ V^3_1 \\ V^3_2 \end{Bmatrix}$$

(i) ZERO OUT ROW CORRESPONDING TO V_2^3

$$\begin{Bmatrix} P_1^1 \\ P_2^1 \\ P_1^2 \\ P_2^2 \\ P_1^3 \\ P_2^3 \end{Bmatrix} = \begin{bmatrix} \ddots & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & & & \\ -K_{11}^3 & -K_{12}^3 & -K_{11}^2 & -K_{12}^2 & K_{11}^2+K_{11}^3 & K_{12}^2+K_{12}^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} V_1^1 \\ V_2^1 \\ V_1^2 \\ V_2^2 \\ V_1^3 \\ V_2^3 \end{Bmatrix}$$

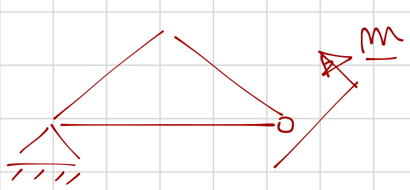
(ii) PLACE 1 ON THE DIAG

$$\begin{Bmatrix} P_1^1 \\ P_2^1 \\ P_1^2 \\ P_2^2 \\ P_1^3 \\ P_2^3 \end{Bmatrix} = \begin{bmatrix} \ddots & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & & & \\ -K_{11}^3 & -K_{12}^3 & -K_{11}^2 & -K_{12}^2 & K_{11}^2+K_{11}^3 & K_{12}^2+K_{12}^3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} V_1^1 \\ V_2^1 \\ V_1^2 \\ V_2^2 \\ V_1^3 \\ V_2^3 \end{Bmatrix}$$

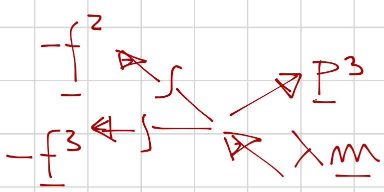
(iii) PLACE THE VALUE OF THE PRESCRIBED DISPL IN THE LOAD VECT

$$\begin{Bmatrix} P_1^1 \\ P_2^1 \\ P_1^2 \\ P_2^2 \\ P_1^3 \\ \alpha \end{Bmatrix} = \begin{bmatrix} \ddots & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & & & \\ -K_{11}^3 & -K_{12}^3 & -K_{11}^2 & -K_{12}^2 & K_{11}^2+K_{11}^3 & K_{12}^2+K_{12}^3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} V_1^1 \\ V_2^1 \\ V_1^2 \\ V_2^2 \\ V_1^3 \\ V_2^3 \end{Bmatrix}$$

NOW, WHAT IF WE DO NOT WANT TO PRESCRIBE EACH INDIVIDUAL COMPONENT BUT YET A CONSTRAIN ON AN INCLINE PLANE?



WHAT WE DO IS INTRODUCE A NEW VARIABLE λ THAT REPRESENTS THE MAGNITUDE OF THE REACTION SUCH THAT



$$\underline{P}^3 - \underline{f}^2 - \underline{f}^3 + \lambda \underline{m} = 0$$

$$\Rightarrow \underline{P}^3 = \underline{f}^2 + \underline{f}^3 - \lambda \underline{m} = \underline{\underline{K}}^2(\underline{v}^3 - \underline{v}^2) + \underline{\underline{K}}^3(\underline{v}^3 - \underline{v}^1) - \lambda \underline{m}$$

$$= \{ -\underline{\underline{K}}^3 \quad -\underline{\underline{K}}^2 \quad \underline{\underline{K}}^2 + \underline{\underline{K}}^3 \quad -\underline{m} \} \begin{Bmatrix} \underline{v}^1 \\ \underline{v}^2 \\ \underline{v}^3 \\ \lambda \end{Bmatrix}$$

NOW NOTE THAT WE HAVE INTRODUCED A NEW VARIABLE (λ) BUT NOT AN ADDITIONAL EQ.

WE MUST ADD A NEW EQ TO GET A SOLN

$$\underline{m} \cdot \underline{v}^3 = 0$$

THE PROJECTION OF $\underline{v}^3$ ALONG $\underline{m}$

SUCH THAT THE FINAL SYSTEM

$$\begin{Bmatrix} \underline{P}^1 \\ \underline{P}^2 \\ \underline{P}^3 \\ 0 \end{Bmatrix} = \begin{bmatrix} \underline{\underline{K}}^1 + \underline{\underline{K}}^3 & -\underline{\underline{K}}^1 & -\underline{\underline{K}}^3 & \underline{0} \\ -\underline{\underline{K}}^1 & \underline{\underline{K}}^1 + \underline{\underline{K}}^2 & -\underline{\underline{K}}^2 & \underline{0} \\ -\underline{\underline{K}}^3 & -\underline{\underline{K}}^2 & \underline{\underline{K}}^2 + \underline{\underline{K}}^3 & -\underline{m} \\ 0 & 0 & \underline{m} & 0 \end{bmatrix} \begin{Bmatrix} \underline{v}^1 \\ \underline{v}^2 \\ \underline{v}^3 \\ \lambda \end{Bmatrix}$$

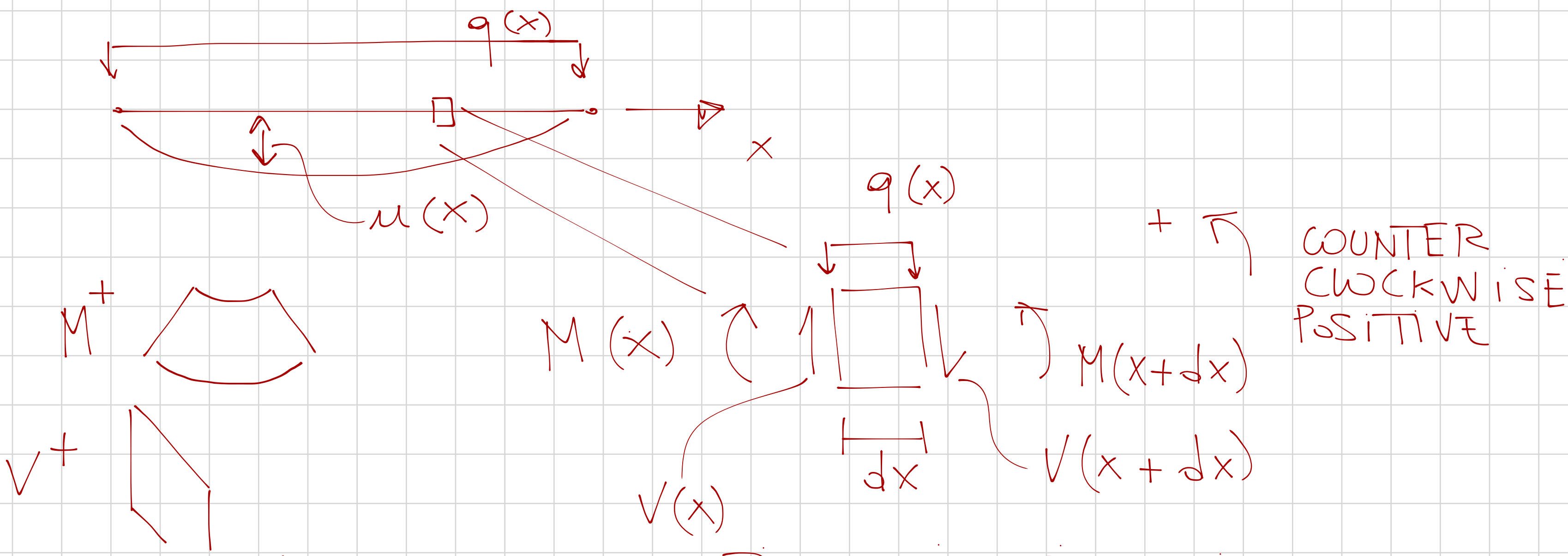
NOTE: THE LAST EQ IS A SCALAR EQ WHILE ALL OTHERS ARE VECTOR EQ.

LAST EQ

$$\{0\} = \lfloor 0 \ 0 \ 0 \ 0 \ m_1 \ m_2 \ 0 \ 0 \rfloor \begin{Bmatrix} v_1^1 \\ v_2^1 \\ v_1^2 \\ v_2^2 \\ v_1^3 \\ v_2^3 \\ \lambda \end{Bmatrix}$$

BEAM EQUATIONS

IN EULER BERNOULLI WE ASSUME PLANE SECTIONS REMAIN PLANE



SUM OF FORCES IN THE VERTICAL DIRECTION GIVES

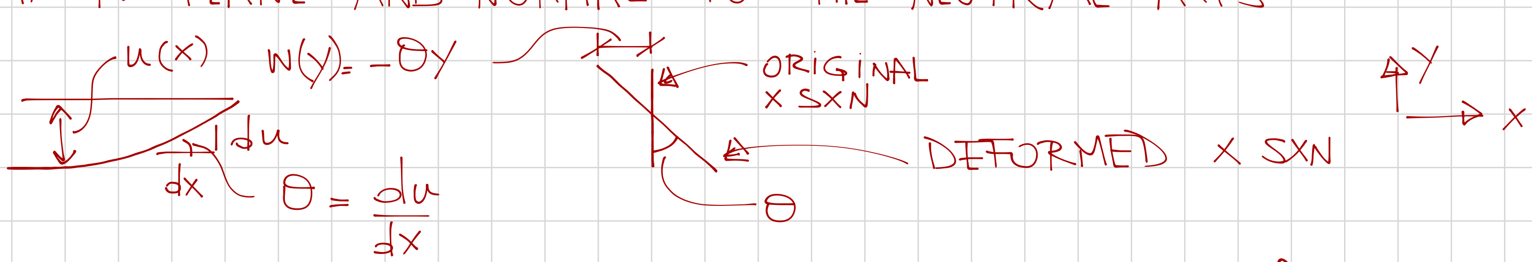
$$V(x) - V(x+dx) - q\,dx = 0 \implies \frac{dV}{dx} = -q$$

BALANCE OF MOMENT ABOUT

$$M(x+dx) - M(x) - V(x)\,dx = 0$$

$$\frac{dM}{dx} = V$$

USING THE KINEMATIC ASSUMPTIO OF PLAN SECTONS REMAIN PLANE AND NORMAL TO THE NEUTRAL AXIS



$$\varepsilon(x,y) = \frac{dw}{dx} = -\frac{d\theta}{dx}y = -\frac{d^2u}{dx^2}y, \quad M(x) = -\int_A \sigma y = -\int_A E\varepsilon y =$$

$$= \int_A E\frac{d^2u}{dx^2}y\,dA = E\frac{d^2u}{dx^2}\int_A y\,dA = EI\frac{d^2u}{dx^2}$$

WE HAVE

$$\frac{d}{dx}V = \frac{d}{dx}\left(\frac{dM}{dx}\right) = \frac{d}{dx}\left(\frac{d}{dx}EI\frac{d^2u}{dx^2}\right) = EI\frac{d^4u}{dx^4} = -q$$

$$EI\frac{d^4u}{dx^4} = -q \quad \longleftarrow \text{BEAM EQUATION}$$

THE DIFF PROBLEM THAN READS

$? \, u : [x_i, x_j] \to \mathbb{R} \quad : \quad EI \dfrac{d^4 u}{dx^4} = -\cancel{q}$ ASSUME $= 0$ FOR NOW $\quad \forall \, x \, (x_i, x_j)$

$$u(x_i) = u_i \qquad du/dx\,(x_i) = \theta(x_i) = \theta_i$$
$$u(x_j) = u_j \qquad du/dx\,(x_j) = \theta(x_j) = \theta_j$$

IF WE INTEGRATE THE ABOVE THE SOLUTION TAKES THE FORM

$$u(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4$$

WE HAVE FOUR UNKNOWNS AND FOUR BOUNDARY CONDITIONS SUCH THAT WE CAN SOLVE FOR $c_i, \, i = 1 \ldots 4$.

IF WE DO SO WE END UP WITH A FORM OF $u(x)$ AS

$$u(x) = N_1(x) u_i + N_2(x) \theta_i + N_3(x) u_j + N_4(x) \theta_j$$

WHERE, $\left| \text{ASSUMING } \quad x_i = 0, \; x_j = \ell \right|$

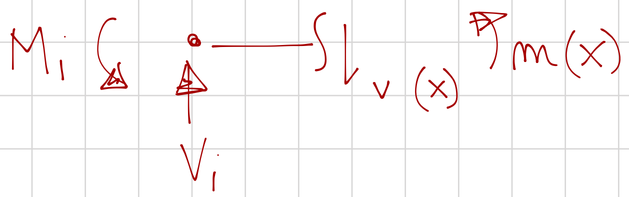
$$N_1(x) = 1 - 3\left(\frac{x}{\ell}\right)^2 + 2\left(\frac{x}{\ell}\right)^3$$
$$N_2(x) = \ell\left[\left(\frac{x}{\ell}\right) - 2\left(\frac{x}{\ell}\right)^2 + \left(\frac{x}{\ell}\right)^3\right]$$
$$N_3(x) = 3\left(\frac{x}{\ell}\right)^2 - 2\left(\frac{x}{\ell}\right)^3$$
$$N_4(x) = \ell\left[\left(\frac{x}{\ell}\right)^3 - \left(\frac{x}{\ell}\right)^2\right]$$

LET LOWERCASE $m(x)$ & $v(x)$ DENOTE THE INTERNAL MOMENT AND SHEAR RESP.

RECALL

$$m(x) = EI \frac{d^2 u}{dx^2}(x) \qquad v(x) = \frac{dm}{dx}(x) = EI \frac{d^3 u}{dx^3}(x)$$

AT NODE i WE APPLY EXTERNAL MOMENT & FORCES



SUM MOMENTS

$$M_i + m(0) = 0$$

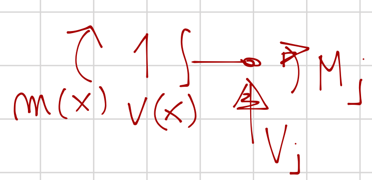
$$M_i = -m(0) = -EI\frac{d^2u}{dx^2}(0) = EI\{-N_1''(0) \quad -N_2''(0) \quad -N_3''(0) \quad -N_4''(0)\}\begin{Bmatrix} u_i \\ \theta_i \\ u_j \\ \theta_j \end{Bmatrix}$$

SUM FORCES

$$V_i - v(x_i = 0) = 0$$

$$V_i = v(0) = EI\frac{d^3u}{dx^3} = EI\{N_1'''(0) \quad N_2'''(0) \quad N_3'''(0) \quad N_4'''(0)\}\begin{Bmatrix} u_i \\ \theta_i \\ u_j \\ \theta_j \end{Bmatrix}$$

SIMILARLY AT NODE j



$$M_j - m(x_j) = 0$$

$$M_j = EI\frac{d^2u}{dx^2} = EI\{N_1''(\ell) \quad N_2''(\ell) \quad N_3''(\ell) \quad N_4''(\ell)\}\begin{Bmatrix} u_i \\ \theta_i \\ u_j \\ \theta_j \end{Bmatrix}$$

$$V_j + v(x_j = \ell) = 0$$

$$V_j = -v(x_j = \ell) = -EI\{N'''(\ell) \quad N'''(\ell) \quad N'''(\ell) \quad N'''(\ell)\}\begin{Bmatrix} u_i \\ \theta_i \\ u_j \\ \theta_j \end{Bmatrix}$$

THUS GIVING

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = EI \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ -N_1''(0) & -N_2''(0) & -N_3''(0) & -N_4''(0) \\ -N_1'''(e) & -N_2'''(e) & -N_3'''(e) & -N_4'''(e) \\ N_1''(e) & N_2''(e) & N_3''(e) & N_4''(e) \end{bmatrix} \begin{Bmatrix} u_i \\ \theta_i \\ u_j \\ \theta_j \end{Bmatrix}$$