LECTURE 15:

TOPICS:

- REVIEW
- THE ELEMENT VIEW
- FINITE ELEMENTS IN nD

LOGISTICS:

- HWKS & PROJECT

REVIEW:

THE GALERKIN FORM READ

$$? \, u^h \in S^h : \quad a(u^h, v^h) = F(v^h) \quad \forall \, v^h \in V^h$$

WHERE, GIVEN $Z^h = \{ z^h \mid z^h = z_i \phi_i \}$

$$S^h = \{ u^h \in S \cap Z^h \}, \quad V^h = \{ v^h \in V \cap Z^h \}$$

AND THE MATRIX FORM

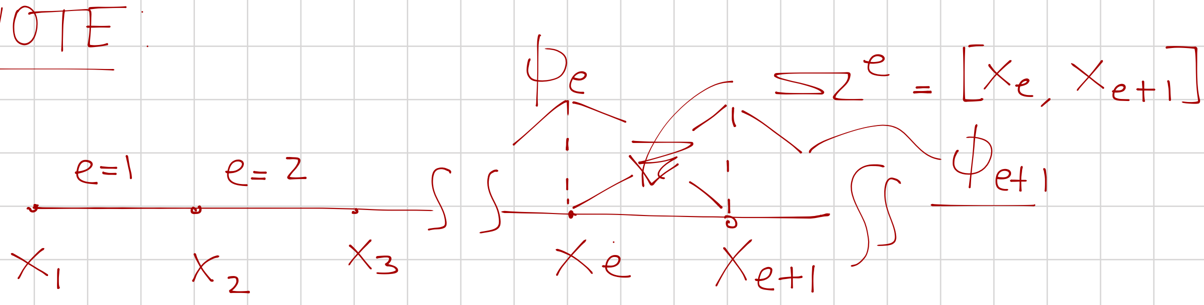
$$? \, \{u\}, \quad [K]\{u\} = \{f\}$$

WHERE

$$k_{ij} = a(\phi_i, \phi_j) = \int_\Omega AE \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \, dx$$

$$f_i = F(\phi_i) = h \, \phi_i(e) - \int_\Omega f(x) \, \phi_i(x) \, dx$$

NOTE:



$$a(\phi_i, \phi_j) = \int_\Omega AE \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \, dx = \sum_{e=1} \int_{\Omega^e} AE \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \, dx$$

$$= \sum_{e=1}^{n_{el}} a^e(\phi_i, \phi_j)$$

BUT RECALL THAT ONLY ϕ_e, ϕ_{e+1} ARE NON-ZERO IN THAT INTERVAL

THE ELEMENT VIEW

WE CAN CONSTRUCT THE BASIS FUNCTIONS BY STARTING FROM THE ELEMENTS & MOVING UP.

NOTE THAT THE FINITE ELEMENT MESH SERVES AS THE SCAFFOLD TO CONSTRUCT THE BASIS FUNCTIONS.

TO THIS EXTENT

A FINITE ELEMENT CONSIST OF

(i) A DOMAIN K^e (E.G. $K^e = [x_e, x_{e+1}]$)

(ii) NODES (EG $\{x_e, x_{e+1}\}$)

(iii) LOCAL DEGREES OF FREEDOM ($\{u^e_i\}_{i=1}^{nDOF}$)

(iv) LOCAL TO GLOBAL MAP FOR DOFs

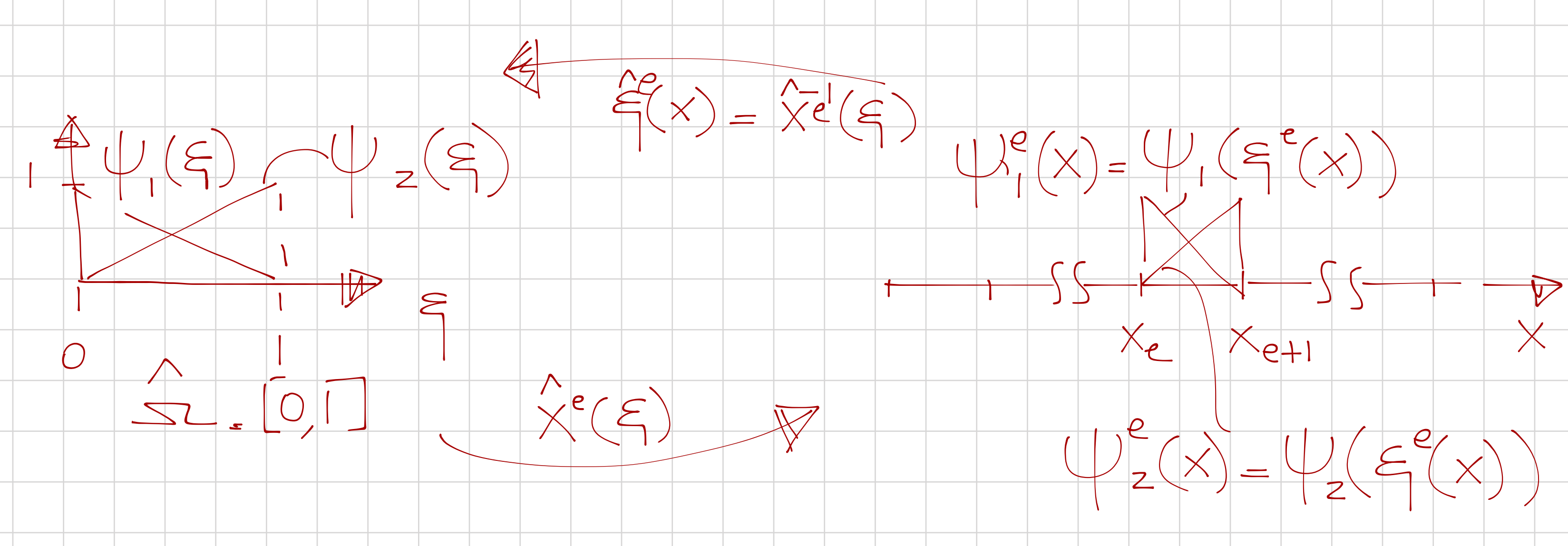
FOR EASE OF CONSTRUCT

(v) A PARENT DOMAIN $\hat{\Omega}$ (EG: $[0,1]$)

(vi) A MAP $x^e(\xi)$: $\hat{\Omega} \rightarrow \Omega^e$

BIJECTIVE

(iv) SHAPE FUNCTIONS (EG $\{\psi_1, \psi_2\}$)



$$\psi_i^e(x) = \begin{cases} \psi_i(\xi^e(x)) & \text{if } x \in K^e \\ 0 & \text{ELSE} \end{cases}$$

$$\underset{\substack{\uparrow \\ \text{GLOBAL NODE LABEL}}}{\phi_e}(x) = \begin{cases} \psi_1^e(x) & \text{if } x \in K^e \\ \psi_2^{e-1}(x) & \text{if } x \in K^{e-1} \end{cases}$$

IN REALITY WE ALWAYS WORK ON THE PARENT DOMAIN, IN FACT

$$a^e(\phi_i, \phi_j) = \int_{\Omega^e} AE \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx$$

& $$dx = \frac{dx}{d\xi}d\xi \quad , \quad \frac{d}{dx} = \frac{d}{d\xi}\frac{d\xi}{dx}$$

AND NOTE THAT, SINCE $x^e(\xi)$ IS BIJECTIVE

$$\frac{d\xi}{dx} = \left(\frac{dx}{d\xi}\right)^{-1}$$

LET a, b DENOTE LOCAL DEGREES OF FREEDOM & $M(e,a)$ THE LOCAL TO GLOBAL MAP

$$a^e(\phi_{M(e,a)}, \phi_{M(e,b)}) = \int_{\hat{\Omega}} AE \frac{d\psi_a}{d\xi}\frac{d\xi}{dx}\frac{d\psi_b}{d\xi}\frac{d\xi}{dx}\frac{dx}{d\xi}\,d\xi$$

$$= \int_{\hat{\Omega}} AE \frac{d\psi_a}{d\xi}\left(\frac{dx}{d\xi}\right)^{-1}\frac{d\psi_b}{d\xi}\left(\frac{dx}{d\xi}\right)^{-1}\frac{dx}{d\xi}\,d\xi$$

$$\underset{\substack{\uparrow \\ \text{LOCAL ELEMENT STIFFNESS}}}{K^e_{ab}} = \hat{a}^e(\psi_a, \psi_b) = \int_{\hat{\Omega}} AE \frac{d\psi_a}{d\xi}\left(\frac{dx}{d\xi}\right)^{-1}\frac{d\psi_b}{d\xi}\left(\frac{dx}{d\xi}\right)^{-1}\frac{dx}{d\xi}\,d\xi$$

$$[K] = \underset{e=1}{\overset{n_{ee}}{\mathsf{A}}} [K^e]$$

← WE COMPUTE THE LOCAL STIFFNES FOR EACH ELEMENT THEN WE ASSEMBLE

<u>NOTE</u>: $[K] \rightarrow n_{GLOBAL\ DOT} \times n_{GLOBAL\ DOT}$

$[K^e] \rightarrow n_{LOCAL\ DOT} \times n_{LOCAL\ DOT}$

THE MAP IS OFTEN CONSTRUCTED AS

$$\hat{x}^e(\xi) = x_a^e \, \varphi_a(\xi)$$

THE COORDINATE OF THE "ath" DEGREE OF FREEDOM

IN THIS CASE WE HAVE ISOPARAMETRIC MAPPING

<u>EXAMPLE</u>: (LINEAR FINITE ELEMENTS)

LET US SUBDIVIDE THE DOMAIN INTO N ELEMENTS OF SIZE $h = \ell/N$.

WE CONSTRUCT $\hat{\Omega} = [-1, 1]$

$$\varphi_1 = \frac{(\xi_2 - \xi)}{(\xi_2 - \xi_1)} = \frac{1}{2}(1 - \xi)$$

$$\varphi_2 = \frac{(\xi - \xi_1)}{(\xi_2 - \xi_1)} = \frac{1}{2}(\xi + 1)$$

<u>NOTE</u>: THESE BASIS SPAN THE SPACE OF ALL LINEAR FUNCTIONS

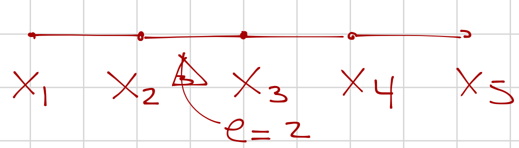
$$q(x) = a x + b$$

TWO DOFS

FOR ELEMENT e ITS NODES ARE $x_e = h \cdot (e-1)$

$x_{e+1} = h \cdot e$

EG if $e = 2$, $N = 4$, WE HAVE $N+1$ NODES



THEN THE LOCAL STIFFNESS MATRIX IS GIVEN BY $[K^e]$ AND IS OF SIZE 2×2

THE GLOBAL STIFFNESS MATRIX IS $[K]$ OF SIZE $(N+1) \times (N+1)$

$$K^e_{ab} = \hat{a}^e(\psi_a, \psi_b)$$

$$\hat{x}^e = x^e_a \psi_a = x^e_1 \frac{1}{2}(1-\xi) + \frac{1}{2} x^e_2 (1+\xi) =$$

$$= \frac{1}{2} (x^e_1 + x^e_2) + \frac{1}{2}\xi (x^e_2 - x^e_1)$$

$$= \frac{1}{2}(x_e + x_{e+1}) + \frac{1}{2}\xi(x_{e+1} - x^e)$$

$$\frac{d\hat{x}^e}{d\xi} = \frac{1}{2}(x_{e+1} - x^e) = \frac{h}{2}$$

$$K^e_{11} = \int_{\hat{\Omega}} AE \frac{d\psi_1}{d\xi}\left(\frac{d\hat{x}^e}{d\xi}\right)^{-1} \frac{d\psi_1}{d\xi}\left(\frac{d\hat{x}^e}{d\xi}\right)^{-1} \frac{d\hat{x}^e}{d\xi} d\xi =$$

$$= \int_{-1}^{1} AE \left(-\frac{1}{2}\right)\left(\frac{2}{h}\right)\left(-\frac{1}{2}\right)\left(\frac{2}{h}\right) \frac{h}{2} \, d\xi = \int_{-1}^{1} \frac{AE}{2h} d\xi = \frac{AE}{h}$$

$$[K^e] = \frac{AE}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$f^e_a = \overbrace{h\,\psi_a(\hat{\xi}^e(e))}^{\text{IF NOT ON BD } = 0} - \int_{-1}^{1} f \psi^e_a(\xi) \, d\xi$$

EXAMPLE (QUADRATIC FINITE ELEMENTS)

BEFORE WE CONSTRUCTED BASIS FUNCTIONS THAT APPROXIMATED EXACTLY LINEAR POLYNOMIALS WITHIN AN ELEMENT

SUPPOSE NOW WE WISH TO APPROXIMATE EXACTLY QUADRATIC POLYNOMIALS

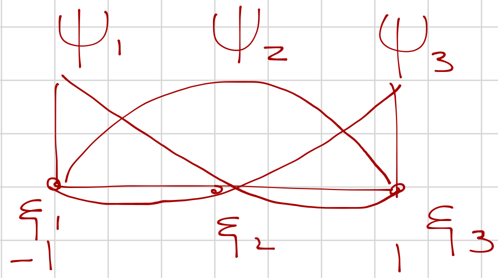
$$q(x) = ax^2 + bx + c$$

MUST HAVE 3 DOF & 3 BASIS

WE USE THE SO CALLED LAGRANGE POLYNOMIALS
LET $\hat{\Omega} = [-1, 1]$

FOR EACH BASIS FUNCTION WE HAVE A SUPPORT NODE SUCH THAT $\psi_a(\xi_b) = \delta_{ab}$

SINCE WE HAVE THREE BASIS FUNCTIONS WE MUST HAVE THREE NODES WITHIN THE ELEMENT



$$\psi_1(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{1}{2}(1-\xi)\xi$$

$$\psi_2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = (\xi+1)(1-\xi)$$

$$\psi_3(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} = \frac{1}{2}(\xi+1)\xi$$

$$x^e(\xi) = x_i^e\,\psi_i(\xi) = x_e\psi_1(\xi) + \frac{(x_{e+1}+x_e)}{2}\psi_2(\xi) + x_{e+1}\psi_3(\xi)$$

WE CHOOSE THE INTERIOR NODE TO BE THE MIDPOINT

$$= \frac{1}{2}(x_e + x_{e+1}) + \frac{1}{2}\xi(x_{e+1} - x^e)$$

$$\frac{dx^e}{d\xi} = \frac{h}{2}$$

$$k_{13}^e = \hat{a}^e(\psi_1,\psi_3) = \int_{-1}^{1} AE\,\frac{d\psi_1}{d\xi}\left(\frac{dx^e}{d\xi}\right)^{-1}\frac{d\psi_3}{d\xi}\left(\frac{dx^e}{d\xi}\right)^{-1}\left(\frac{dx^e}{d\xi}\right)d\xi$$

$$= \int_{-1}^{1} AE\left(\frac{1-2\xi}{2}\right)\left(\frac{2}{h}\right)\left(\frac{2\xi+1}{2}\right)\left(\frac{2}{h}\right)\left(\frac{h}{2}\right)d\xi$$

$$= \frac{AE}{2h}\int_{-1}^{1}(1-2\xi)(2\xi+1)\,d\xi = \frac{AE}{2h}\int_{-1}^{1}(1-4\xi^2)\,d\xi$$

$$= \frac{AE}{2h}\left[\xi - \frac{4\xi^3}{3}\right]\Big|_{-1}^{1} = \frac{AE}{2h}\left[1-\frac{4}{3} + \left(1-\frac{4}{3}\right)\right] = -\frac{1}{3}\frac{AE}{h}$$

SIMILARLY FOR OTHERS

THEN, SIMILARLY TO BEFORE WE PERFORM THE ASSEMBLY USING THE LOCAL TO GLOBAL MAP.

NOTE THAT THE INTEGRATION MAY NOT BE SO STRAIGHTFORWARD SO CANNOT BE CARRIED OUT ANALYTICALLY, THAT'S WHY WE USE NUMERICAL QUADRATURE

NUMERICAL QUADRATURE RULES ARE A SET OF POINTS $\{q_i\}_{i=1}^{nq}$ AND WEIGHTS $\{w_i\}_{i=1}^{nq}$ SUCH THAT

$$\int_{\Omega} f(x)\,dx \approx \sum_{i=1}^{nq} f(q_i)\,w_i$$

NOTE THAT THE QUADRATURE RULES ARE SPECIFIC TO DOMAINS.

THE ACCURACY IS TIED TO THE NUMBER OF INTEGRATION POINTS.

USING STANDARD GAUSS QUADRATURE WE NEED EXACTLY $(p+1)/2$ POINTS TO INTEGRATE A POLYNOMIAL OF ORDER p.