LECTURE 5

- SOLID VS FLUID GRAD & HW
- MATERIAL TIME DERIVATIVE
- LOCALLY ADMISSIBLE MAP
- DEFORMATION GRADIENT
- RIGHT CAUCHY GREEN

MATERIAL TIME DERIVATIVE

IF I HAVE A SPATIAL FIELD $f(\underline{x},t)$ HOW DO I COMPUTE THE TIME DERIVATIVE OF THAT QUANTITY EXPERIENCED BY THE PARTIC ?

NAMELY

$$\frac{D(\circ)}{Dt} = \frac{d}{dt}(\cdot)\Big|_{\underline{X}}$$

if $f(\underline{x},t)$

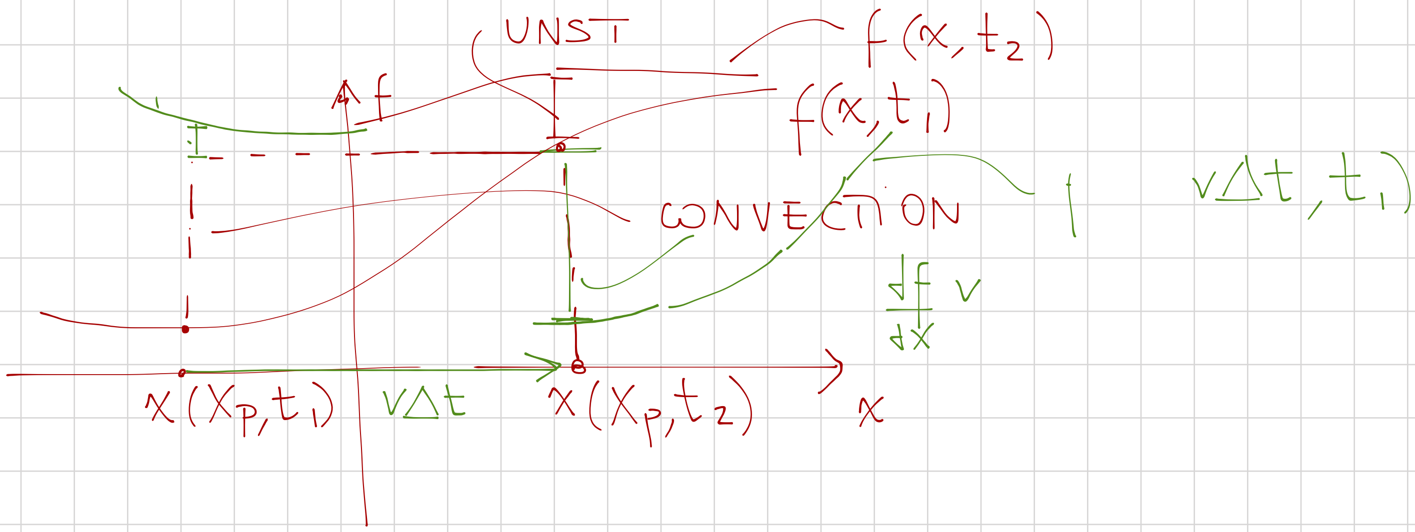
$$\frac{D}{Dt} f(\underline{x},t) = \dot{f}(\underline{x},t) = \frac{d}{dt}\left(f(\underline{x},t)\right)\Big|_{\underline{X}}$$

$$= \left(\frac{df}{dt}\right)\Big|_{\underline{x}} + \left(\frac{df}{d\underline{x}}\right)\Big|_t \left(\frac{d\underline{x}}{dt}\right)\Big|_{\underline{X}} = \frac{df}{dt}\Big|_{\underline{x}} + \frac{df}{d\underline{x}}\Big|_t \frac{d\underline{\varphi}(\underline{X},t)}{dt}\Big|_{\underline{X}}$$

$$= \frac{df}{dt}\Big|_{\underline{x}} + \frac{df}{d\underline{x}}\Big|_t \underline{v}(\underline{x},t)$$

THUS WE CAN COMPUTE THE RATE OF CHANGE OF A PARTICLE SOLELY WITH SPATIAL QUANTITIES

THINK ABOUT IT IN 1-D



IF

$\frac{d\underline{v}}{dt} = \underline{0}$ THE MOTION IS STEADY

IF

$\underline{\nabla}_{\underline{x}} \underline{v} = 0 \Leftarrow$ UNIFORM

IF $\underline{v} = \underline{\nabla}\phi \Leftarrow$ POTENTIAL FLOW

WHAT IS $\underbrace{\underline{\nabla} \times \underline{v} = 0}_{\text{NO VORTICITY!!}}$?

LOCALLY ADMISSIBLE MAP

ESSENTIALLY A MAPPING IS ADMISSIBLE IF IT IS ONE-TO-ONE.

SHOWING THAT A MAPPING IS ONE-TO-ONE GLOBALLY IS A CHALLENG TASK.

ADMISSIBILITY CAN BE CHECKED LOCALLY NAMELY BY ONLY CONSIDERING A SMALL NEIGHBORHOOD OF PARTICLES.

TO CHECK THE ABOVE CONSIDER TAYLOR SERIES EXP

$$\underline{\varphi}(\underline{X}+\Delta\underline{X}) = \underline{\varphi}(\underline{X}) + \frac{d\underline{\varphi}}{d\underline{X}}(\underline{X})\,\Delta\underline{X} + O(\|\Delta\underline{X}\|^2)$$

FOR $\Delta\underline{X} \searrow$

$$\underline{\varphi}(\underline{X}+\Delta\underline{X}) \approx \underline{\varphi}(\underline{X}) + \underline{\underline{F}}(\underline{X})\,\Delta\underline{X}$$

WHERE $\underline{\underline{F}}(\underline{X}) = \frac{d\underline{\varphi}}{d\underline{X}}(\underline{X})$

IF

$$\underline{\varphi}(\underline{X}+\Delta\underline{X}) = \underline{\varphi}(\underline{X})$$

THAN THE MAPPING IS NO LONGER 1-TO-1 & FOR THAT TO HAPPEN

$$\underline{\underline{F}}(\underline{X})\,\Delta\underline{X} = 0 \Rightarrow \text{DET}\,\underline{\underline{F}}(\underline{X}) = 0$$

LOCALLY ADMISSIBLE MAP:

A MAP $\underline{\varphi}$ IS LOCALLY ADMISSIBLE IFF

$$\text{DET}\,\underline{\nabla}_{\underline{X}}\underline{\varphi} \neq 0$$

NOTE THAT A MAPPING THAT IS NOT LOCALLY ADMISSIBLE COULD STILL BE 1-TO-1

$\Omega_0 = [-1, 1]^3$, $\underline{\varphi}(\underline{X}) = X_1^3 \underline{E}_1 + X_2 + X_3$

THIS MAP HAS AN INVERSE

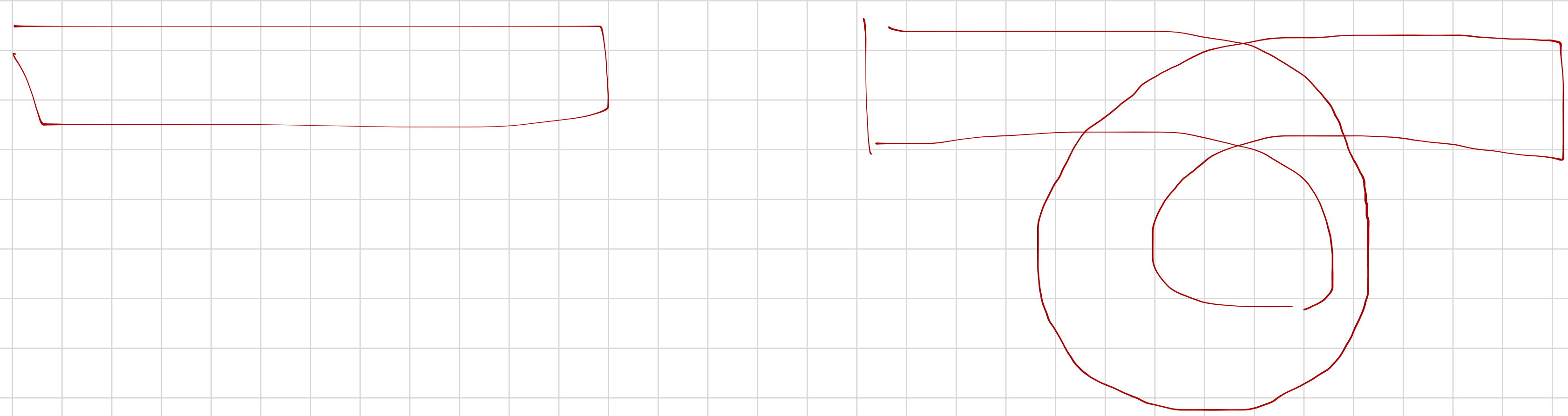
$\underline{\varphi}^{-1}(\underline{x}) = x_1^{1/3} \underline{e}_1 + x_2 \underline{e}_2 + x_3 \underline{e}_3$

BUT

DET $\underline{\nabla}\,\underline{\varphi} = 3X_1^2 \Leftarrow$ AT $X_1 = 0$ DET $\underline{\nabla}\,\underline{\varphi} = 0$

IN GENERAL IF DET $\underline{\nabla}\,\underline{\varphi} \neq 0$ $\exists$ A NEIGHBORHOOD THAT IS LOCALLY ADMISSIBLE

NOT GENERALLY TRUE GLOBALLY



$\underline{\varphi}$ LOCALLY ADMISSIBLE $\neq$ ONE-TO-ONE

<u>ON THE SIGN OF DET $\underline{\nabla}\,\underline{\varphi}$</u>

NOTE AT TIME $t=0$ DET $\underline{\nabla}\,\underline{\varphi} = 1 > 0$

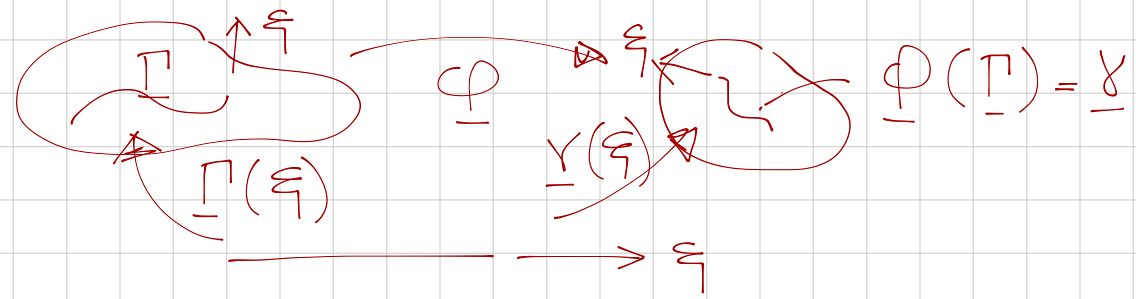
IF THE MAPPING REMAINS LOCALLY ADMISSIBLE THEN DET $\underline{\nabla}\,\underline{\varphi} > 0$ $\forall t$

THIS FOLLOWS FROM BOLZANOS THEOREM THAT SINCE $\underline{\varphi}$ IS SMOOTH IN TIME, IN ORDER FOR DET $\underline{\nabla}\,\underline{\varphi} < 0$ THERE MUST EXIST A TIME t^* SUCH THAT DET $\underline{\nabla}\,\underline{\varphi} = 0$

BOLZANO'S THEOREM - MEAN VALUE THM
- INTERMEDIATE VALUE THM

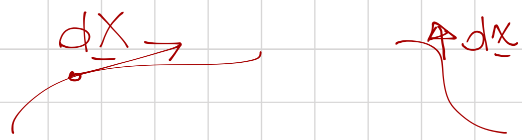
DEFORMATION GRADIENT

CONSIDER A CURVE



LET $d\underline{X} := \frac{d\underline{\Gamma}}{d\xi} d\xi$ BEING AN INFINITESIMAL TANGENT VECTOR TO $\underline{\Gamma}$

$d\underline{x} := \frac{d\underline{\gamma}}{d\xi} d\xi$ BEING AN INFINITESIMAL TANGENT VECTOR TO $\underline{\gamma}$



$$\underline{\gamma}(\xi) = \underline{\varphi}(\underline{\Gamma}(\xi)) = \underline{\varphi} \circ \underline{\Gamma}(\xi)$$

$$\Rightarrow \frac{d\underline{\gamma}}{d\xi} d\xi = \frac{d\underline{\varphi}}{d\underline{X}} \frac{d\underline{\Gamma}}{d\xi} d\xi \Rightarrow d\underline{x} = \underline{\underline{F}}(\underline{X}) d\underline{X} \quad (*)$$

WHERE $\underline{\underline{F}}(\underline{X}) = \nabla \underline{\varphi}$ IS KNOWN AS THE DEFORMATION GRADIENT.

(*) IS A LINEAR TRANSFORMATION WHICH GENERATES A SPATIAL VECTOR FROM TRANSFORMING MATERIAL VECTOR

THUS $\underline{\underline{F}}$ IS SAID TO BE A TWO POINT TENSOR

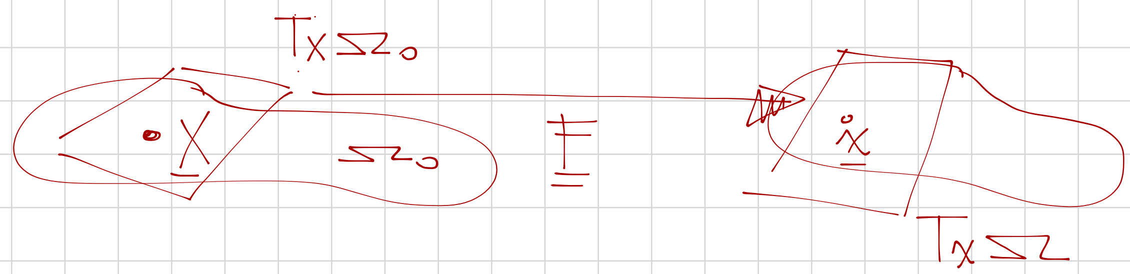
IT COMPONENS F_{iJ}

UPPER CASE REFERENCE

LOWER CASE SPATIAL

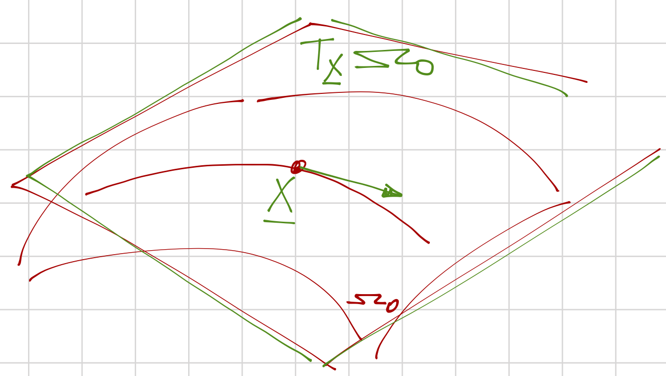
$\underline{\underline{F}}$ IS ALSO REFERRE TO AS THE TANGENT MAP

$$\underline{\underline{F}}(\underline{X}) : T_{\underline{X}}\Omega_0 \longrightarrow T_{\underline{x}}\Omega$$



$T_{\underline{X}}\Omega_0$ DENOTES THE TANGENT SPACE OF Ω_0 AT $\underline{X}$.

$T_{\underline{X}}\Omega_0$ IS A VECTOR SPACE CONTAINING ALL VECTORS THAT DEFINE ALL POSSIBLE DIRECTION OF CURVES PASSING THROUG $\underline{X}$



EXPONENTIAL MAP:

EXP: $T_{\underline{X}}\Omega_0 \longrightarrow \Omega_0$

IT IS IMPORTANT TO NOTE THAT $\underline{\underline{F}}(\underline{X})$ MAPS VECTORS ONLY ORIGINATING AT $\underline{X}$ (OR IN A INFINITESIMAL NEIGHBORHOOD).

ANOTHER WAY OF SEEING THIS IS VIA THE TAYLOR SERIES EXP.

$$\underline{\varphi}(\underline{X}) - \underline{\varphi}(\underline{Y}) = \underline{\underline{F}}(\underline{X})(\underline{Y} - \underline{X}) + o(\|\underline{Y} - \underline{X}\|)$$

THUS $\underline{\underline{F}}$ IS A GOOD APPROXIMATION ONLY

$$\|\underline{Y} - \underline{X}\| \longrightarrow 0$$

SIMILARLY

$$\underline{\Gamma}(\xi) = \underline{\varphi}^{-1}(\underline{\gamma}(\xi)) \Rightarrow d\underline{X} = \underline{\underline{F}}^{-1} d\underline{x}$$

WITH

$$\underline{\underline{F}}^{-1}(\underline{x}) = \frac{d\underline{\varphi}}{d\underline{x}}(\underline{x}) = \left(\underline{\underline{F}}(\underline{\varphi}^{-1}(\underline{x}))\right)^{-1}$$

WHERE $\underline{\underline{F}}^{-1}(\underline{x})$ MAPS SPATIAL VECTORS INTO MATERIAL VECTOR

$$\underline{\underline{F}}^{-1} : T_{\underline{x}}\Omega \rightarrow T_{\underline{X}}\Omega_0 \quad \& \quad F^{-1}_{\underline{I}j}$$

(NOTE WE ALREADY ESTABLISHED DET $\underline{\underline{F}} \neq 0$)

NOTE THAT IF $\underline{\underline{F}}$ DOES NOT DEPEND ON SPACE THE $\underline{\underline{F}}$ IS TERMED AS HOMOGENEOUS AND $\underline{\varphi}$ IS AN AFFINE MAP

$$\underline{\varphi}(\underline{X}) = \underline{a} + \underline{\underline{F}}\,\underline{X} \Rightarrow \underline{\nabla}\underline{\varphi} = \underline{\underline{F}}$$

IF $\underline{\underline{F}}$ DEPENDS ON SPACE, THEN IT IS INHOMOGENEOUS

EG: $\underline{\varphi}(\underline{X}) = (X_1 + \alpha X_2^2)\underline{E}_1 + X_2\underline{E}_2$

$$\Rightarrow [\underline{\nabla}\underline{\varphi}]_{\underline{E}} = \begin{bmatrix} 1 & 2\alpha X_2 \\ 0 & 1 \end{bmatrix}$$ DEPEND ON SPACE

OFTEN IT IS MORE COMMON TO LOOK AT THE DISPLACEMENT GRADIENT

$$\underline{\nabla}_{\underline{X}}\underline{U}(\underline{X}) = \underline{\nabla}_{\underline{X}}(\underline{\varphi}(\underline{X}) - \underline{X}) = \underline{\underline{F}}(\underline{X}) - \underline{\underline{I}}$$

OR

$$\underline{\nabla}_{\underline{x}}\underline{u}(\underline{x}) = \underline{\nabla}_{\underline{x}}(\underline{x} - \underline{\varphi}^{-1}(\underline{x})) = \underline{\underline{I}} - \underline{\underline{F}}^{-1}(\underline{x})$$

SOME USEFUL RELATIONS FOLLOW DIRECTLY FROM SIMPLE CHAIN RULES

$$\underline{\nabla}_{\underline{x}} \psi = \underline{\underline{F}}^{-T} \underline{\nabla}_{\underline{X}} \psi$$

$$\underline{\nabla}_{\underline{x}} \underline{a} = \underline{\nabla}_{\underline{X}} \underline{a} \, \underline{\underline{F}}^{-1}$$

$$\underline{\nabla}_{\underline{x}} \underline{\underline{A}} = \underline{\nabla}_{\underline{X}} \underline{\underline{A}} \, \underline{\underline{F}}^{-1}$$

$$\underline{\nabla}_{\underline{x}} \cdot \underline{\underline{A}} = \underline{\nabla}_{\underline{X}} \underline{\underline{A}} : \underline{\underline{F}}^{-T}$$