LECTURE 6:

TOPICS

LOGISTICS:

- HW 1 DUE TOM
- HW 2 OUT FRI, DUE FRI

LAST TIME WE DISCUSSED SOME DIFFERENTIAL OPERATORS SOME OF WHICH ARE

- GRADIENTS: FOR SCALAR FUNCTIONS IT'S A VECTOR FIELD POINTING IN THE DIRECTION OF MAX GROWTH

  FOR VECTOR FUNCTIONS IT'S A TENSOR FIELD

- DIVERGENCE: THE DIVERGENCE OF A SCALAR FUNCTION IS NOT DEFINED

  THE DIVERGENCE OF A VECTOR FIELD IS A SCALAR FIELD THAT MEASURE FLUX

  THE DIVERGENCE OF A TENSOR FIELD IS A VECTOR FIELD THAT MEASURE FLUX

WE ALSO SAW TWO INTEGRAL THMS

$$\int_a^b f \frac{dg}{dx}\, dx = \left[f g \right]\Big|_a^b - \int_a^b \frac{df}{dx}\, g\, dx$$

UNIT OUTWAR NORMAL TO Ω

$$\int_{\Omega} \psi \nabla \cdot \underline{u}\, dv = \int_{\Gamma} \psi \underline{u} \cdot \underline{n}\, ds - \int_{\Omega} \underline{u} \cdot \nabla \psi\, dv$$

Ω DOMAIN

Γ BOUNDARY

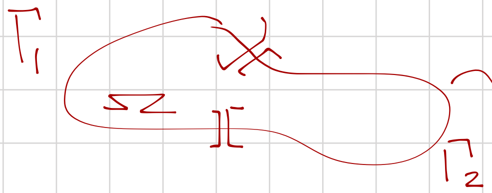
DIFFERENTIAL PROBLEMS

INGREDIENTS:

- A DOMAIN
- A DIFFERENTIAL EQ
- BOUNDARY CONDITIONS

EX: THE HEAT EQ (STEADY STATE)

FIND $u : \Omega \to \mathbb{R}$
← DOMAIN



$\Gamma = \Gamma_1 \cup \Gamma_2$

UNKNOWN GIVEN

$$\Delta u = f \qquad \forall \underline{x} \in \Omega$$

S.T.

$u = \hat{u} \qquad \forall \underline{x} \in \Gamma_1$ ← BOUNDARY CONDITIONS ON THE FUN

$\nabla u \cdot \underline{n} = h \qquad \forall \underline{x} \in \Gamma_2$ ← BOUNDARY CONDITIONS ON GRAD

A DIFFERENTIAL PROBLEM IS SAID TO BE WELL POSED IF

- A SOLUTION EXISTS
- THE SOLUTION IS UNIQUE
- THE SOLUTION'S BEHAVIOR CHANGES CONTINUOSLY WITH THE PROBLEM DATA (EG. DIFFUSION CONST, BC ..)

CLASSIFICATION OF PDEs

ORDER

THE HIGHEST PARTIAL DERIVATIVE

EG KIRCHOFF-LOVE

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0$$

$\uparrow$ FOURTH ORDER

LINEARITY

IF THE DIFF EQ IS WRITTEN AS

$$D(u) = 0$$

THEN IT'S LINEAR IF

$$D(\alpha u(x)) = \alpha D(u(x)) \qquad \forall \alpha \in \mathbb{R}$$

$$D(u(x) + w(x)) = D(u(x)) + D(w(x)) \qquad \forall u, w : \Omega \to \mathbb{R}$$

BOUNDARY CONDITIONS

LET THE ORDER OF THE PDE BE $2m$

IF WE PRESCRIBE BOUNDARY VALUES FOR THE

$(0, m-1)$ - DERIVATIVES $\to$ ESSENTIAL BOUNDARY CONDITIONS (OR DIRICHLET)

$(m, 2m)$ - DERIVATIVES $\to$ NATURAL (OR NEUMANN) BOUNDARY CONDITIONS

EG:

?u: $\Delta u = 0 \quad \forall \; \underline{x} \in \Omega$

$u = \hat{u} \quad \forall \; \underline{x} \in \Gamma_1$ ← DIRICHLET

$\nabla u \cdot \underline{n} = \hat{q} \quad \forall \; \underline{x} \in \Gamma_2$ ← NEUMANN

THE LAPLACIAN IS 2nd ORDER $2m = 2 \Rightarrow m = 1$

ALL BOUNDARY CONDITIONS ON FUNCTION ARE ESSENTIAL

ALL BOUNDARY CONDITIONS ON GRADIENTS ARE NEUMANN

BEAM EQ

?u: $EI \dfrac{d^4u}{dx^4} = 0 \quad \forall \; x \in [a,b]$

$u(a) = \hat{u}$ $\qquad \dfrac{d^2u}{dx^2} = \hat{M}$

$\dfrac{du}{dx}(a) = \hat{\theta}$ $\qquad \dfrac{d^3u}{dx^3}(b) = \hat{V}$

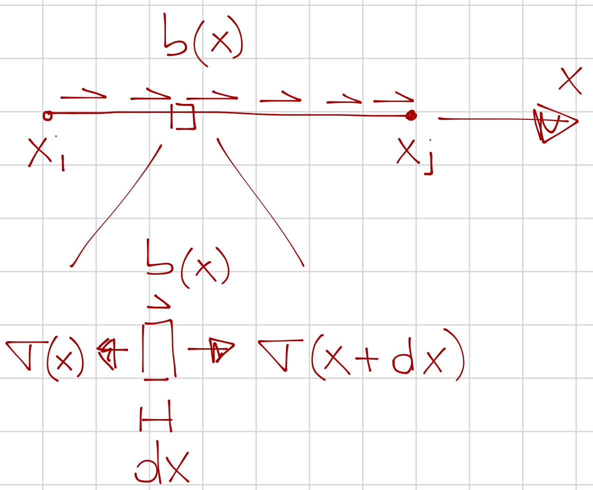
DIFF EQ IS OF ORDER 4 → $2m = 4 \Rightarrow m = 2$

0 – 1 DERIVATIVES → DIRICHLET

2 – 3 " → NEUMANN

TRUSS EQUATION

CONSIDER A TRUSS



WHERE $\sigma = P/A$ IS THE STRESS
b = SOME BODY FORCE

SUM THE FORCES

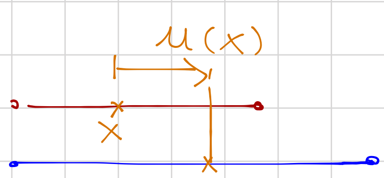
$$A\big(\sigma(x+dx) - \sigma(x)\big) + b(x)\cdot dx = 0$$

$$\Rightarrow A\frac{\big(\sigma(x+dx) - \sigma(x)\big)}{dx} = b(x)$$

$\Rightarrow$ IN THE LIMIT AS $dx \to 0$

$$A\frac{d\sigma}{dx} = b$$

LET $u(x)$ BE THE DISPLACEMENT OF THE TRUSS



THE STRAIN IS GIVEN BY $\varepsilon(x) = \frac{du}{dx}(x)$

IF WE ASSUME LINEAR ELASTICITY THE

$$\sigma(x) = E\frac{du}{dx}(x)$$

E: YOUNG'S MODULUS

IF WE REPLACE IN THE DE

$$A\frac{d\sigma}{dx} = A\frac{d}{dx}\left(E\frac{du}{dx}\right) = b \Rightarrow AE\frac{d^2u}{dx^2} = b$$

SO THE PROBLEM OF FINDING THE DISPLACEMENT OF A TRUSS BECOMES

FIND $u : [x_i, x_j] \rightarrow \mathbb{R}$ S.T.

$$AE \frac{d^2 u}{dx^2} = b \quad \forall\, x\ (x_i, x_j)$$

$$u(x_i) = u_i$$
$$u(x_j) = u_j$$

FOR NOW ASSUME $b = 0$ THEN

$$AE \frac{d^2 u}{dx^2} = 0$$

THE SOLUTION CAN THEREFORE BE A LINEAR FUNC (INTEGRATE THE ABOVE TWICE)

$$u = C_1 x + C_2$$

NOW IF WE USE THE BOUNDARY CONDITIONS

$$u(x_i) = u_i = C_1 x_i + C_2 \qquad *$$

$$u(x_2) = u_j = C_1 x_j + C_2 \qquad **$$

SUBTRACT * FROM **

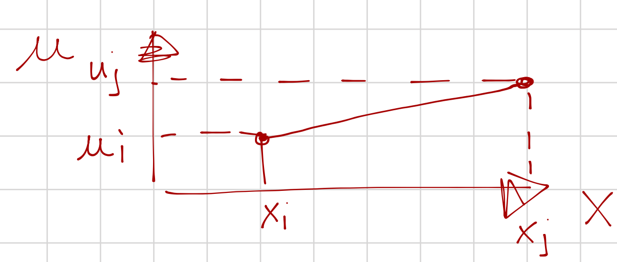
$$u_j - u_i = C_1 (x_j - x_i) \Rightarrow C_1 = \frac{u_j - u_i}{x_j - x_i}$$

FROM * SOLVE FOR C_2

$$C_2 = u_i - \frac{(u_j - u_i)\, x_i}{(x_j - x_i)}$$

$$\Rightarrow u(x) = \left(\frac{u_j - u_i}{x_j - x_i}\right) x + u_i - \left(\frac{u_j - u_i}{x_j - x_i}\right) x_i$$

$$= \frac{(u_j - u_i)(x - x_i)}{(x_j - x_i)} + u_i$$



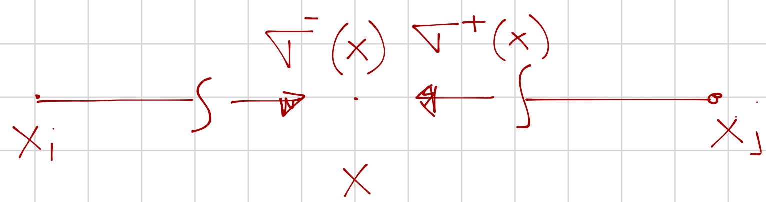
FORCE BALANCE

NOW THAT WE HAVE THE DISPLACEMENT FUNCTION $u(x) : [x_i, x_j] \to \mathbb{R}$ PARAMETERIZED BY THE END DISPLACEMENTS WE ARE INTERESTED AT LOOKING AT FORCES

RECALL THE STRESS INSIDE THE TRUSS IS

$$\sigma(x) = E \frac{du}{dx}(x)$$

EFFECTIVELY WE CUT OUR TRUSS AND LOOK AT THE STRESSES



$\bar{\sigma}(x)$ & $\sigma(x)$ MUST BE EQUAL & OPPOSITE FOR EQUILIBRIUM TO $\sigma^+(x) = -\sigma(x)$

WITH THE STRESS WE CAN RECOVER FORCE AS

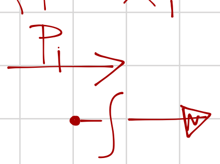
$$f(x) = A\sigma(x)$$

NOW LET'S LOOK AT FORCE EQUILIBRIUM AT x_i AND x_j.

ASSUME WE APPLY TWO FORCES P_i, P_j AT x_i, x_j.

WE WANT TO MAKE SURE THE TRUSS IS IN EQUILIBRIUM!

AT x_i :



$$\bar{f}(x_i) = f(x_i) = f_i$$

WHERE

$$f(x_i) = A\sigma(x_i) = AE \frac{du}{dx}(x_i) = AE \frac{(u_j - u_i)}{\underbrace{(x_j - x_i)}_{\ell}}$$

AND THEREFORE

$$P_i = -f_i = -\frac{AE}{\ell}\{-1 \quad 1\}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \frac{AE}{\ell}\{1 \quad -1\}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

SIMILARLY AT X_j:

$$f_j = f^+(x_j) = -f(x_j) \longrightarrow P_j$$

$$P_j - f(x_j) = 0 \Longrightarrow P_j = f(x_j) = \frac{AE}{\ell}\{-1 \quad 1\}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

SO IN ORDER TO HAVE EQUILIBRIUM WE MUST HAVE

$$\begin{aligned} P_i &= -f_i \\ P_j &= f_j \end{aligned} \Longrightarrow \begin{Bmatrix} P_i \\ P_j \end{Bmatrix} = \begin{Bmatrix} -f_i \\ f_j \end{Bmatrix} = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

NOW, GIVEN P_i, P_j, WE WANT TO FIND THE VALUE OF u_i, u_j SUCH THAT THE TRUSS IS IN EQUILIBRIUM

u_i, u_j ARE CALLED THE DEGREES OF FREEDOM

$K = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ IS CALLED THE STIFFNESS MATRIX