LECTURE 4

TOPICS

- REVIEW
- TENSORS CONT'D
- DIFFERENTIAL PROBLEMS
- TRUSS EQUATION

LOGISTICS

- HOMEWORK #1 POSTED
- PRECEPT 2 IN FRIEND 008 W
- POSTED JUPYTER NOTEBOOK

REVIEW

WE WERE TALKING ABOUT TENSORS THAT ARE LINEAR OPERATORS ON VECTORS SPACES.

NAMELY A TENSOR TAKES A VECTOR, PERFORMS SOME LINEAR OPERATIONS ON THE VECTOR & IT THEN RETURNS THE (LINEARLY) MODIFIED VECTOR.

WE SAW LAST CLASS THAT EFFECTIVLY, ANY LINEAR OPERATION ON A VECTOR $\underline{a} \in \mathbb{R}^d$ CAN BE EXPRESSED AS

$$(\underline{\beta} \cdot \underline{a})\, \underline{\alpha}$$

WHERE $\underline{\alpha}, \underline{\beta} \in \mathbb{R}^d$ ARE DEFINED BY THE LINEAR OPERATION WE ARE TRYING TO PERFORM.

THE ABOVE IMPLIES THAT FOR EVERY TENSOR $\underline{\underline{A}} \in \mathbb{R}^{d \times d}$ $\exists\, \underline{\alpha}, \underline{\beta} \in \mathbb{R}^d$ S.T. $\forall\, \underline{a} \in \mathbb{R}^d$

$$\underline{\underline{A}}(\underline{a}) = (\underline{\beta} \cdot \underline{a})\, \underline{\alpha}$$

TO CONSTRUCT TENSORS AS STAND ALONE OBJECTS IN TERMS OF THE VECTORS $\underline{\beta}, \underline{\alpha}$ WE INTRODUCED THE DYADIC PRODUCT.

THE DYADIC (OR TENSOR OR OUTER) PRODUCT IS DENOTED BY $\otimes$.

USING THE DYADIC PRODUCT WE CAN WRITE

$$\underline{\underline{A}} = \underline{\alpha} \otimes \underline{\beta}$$

SUCH THAT

$$\underline{\underline{A}}(\underline{a}) = (\underline{\alpha} \otimes \underline{\beta})(\underline{a}) = \underline{\alpha}\,(\underline{\beta} \cdot \underline{a})$$

DOT

THEN WE SAW THAT, LETTING $\underline{\alpha} = \alpha_i \underline{e}_i$, $\underline{\beta} = \beta_j \underline{e}_j$

$$\underline{\underline{A}} = \underline{\alpha} \otimes \underline{\beta} = (\alpha_i \underline{e}_i) \otimes (\beta_j \underline{e}_j) = \alpha_i \beta_j \, \underline{e}_i \otimes \underline{e}_j$$

AND LETTING $A_{ij} = \alpha_i \beta_j$ WE HAVE

$$\underline{\underline{A}} = A_{ij} \, \underline{e}_i \otimes \underline{e}_j$$

THUS, SIMILARLY TO VECTORS $\underline{\underline{A}}$ CAN BE EXPRESSED AS A LINEAR COMBINATION OF TENSOR BASIS $\underline{e}_i \otimes \underline{e}_j$ OR CAN BE THOUGHT OF BEING DEFINED BY ITS COEFFICIENTS A_{ij} ALONG WITH THE BASIS

THE MATRIX REPRESENTATION OF $\underline{\underline{A}}$ IN THE $\underline{e}_i$ BASIS IS

$$[\underline{\underline{A}}]_{\underline{e}} = \begin{bmatrix} A_{11} & \cdots & A_{1d} \\ \vdots & \ddots & \vdots \\ A_{d1} & \cdots & A_{dd} \end{bmatrix}$$

AND

$$[\underline{\underline{A}}(\underline{a})]_{\underline{e}} = [\underline{\underline{A}}]_{\underline{e}} [\underline{a}]_{\underline{e}}$$

EG: ROTATION IN 2D

$$\underline{\underline{A}} = 1\,\underline{e}_1 \otimes \underline{e}_2 - 1\,\underline{e}_2 \otimes \underline{e}_1 \quad , \quad \underline{a} = 2\underline{e}_1 + \underline{e}_2$$

$$\underline{\underline{A}}(\underline{a}) = (\underline{e}_1 \otimes \underline{e}_2 - \underline{e}_2 \otimes \underline{e}_1)(2\underline{e}_1 + \underline{e}_2) =$$

$$= (\underline{e}_1 \otimes \underline{e}_2)(2\underline{e}_1 + \underline{e}_2) - (\underline{e}_2 \otimes \underline{e}_1)(2\underline{e}_1 + \underline{e}_2)$$

$$= \underline{e}_1 \left(\cancelto{0}{2\underline{e}_2 \cdot \underline{e}_1} + \cancelto{1}{\underline{e}_2 \cdot \underline{e}_2} \right) - \underline{e}_2 \left(\cancelto{2}{\underline{e}_1 \cdot 2\underline{e}_1} + \cancelto{0}{\underline{e}_1 \cdot \underline{e}_2} \right)$$

$$= \underline{e}_1 - 2\underline{e}_2$$

AND

$$[\underline{\underline{A}}\,\underline{a}]_{\underline{e}} = [\underline{e}_1 - 2\underline{e}_2]_{\underline{e}} = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

SIMILARLY

$$[\underline{\underline{A}}]_{\underline{e}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad [\underline{a}]_{\underline{e}} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

$$[\underline{\underline{A}}\,\underline{a}]_{\underline{e}} = [\underline{\underline{A}}]_{\underline{e}} [\underline{a}]_{\underline{e}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} \checkmark$$

TENSOR ALGEBRA

$$\underline{\underline{A}} + \underline{\underline{B}} = (A_{ij} + B_{ij})\, \underline{e}_i \otimes \underline{e}_j \quad ; \quad [\underline{\underline{A}} + \underline{\underline{B}}]_{\underline{e}} = [\underline{\underline{A}}]_{\underline{e}} + [\underline{\underline{B}}]_{\underline{e}}$$

$$\underline{\underline{A}}\,\underline{\underline{B}} = A_{ij} B_{k\ell}\, (\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_\ell) = A_{ij} B_{k\ell}\, \underline{e}_j \cdot \underline{e}_k\, \underline{e}_i \otimes \underline{e}_\ell$$

DOT

$$= A_{ij} B_{k\ell}\, \delta_{jk}\, \underline{e}_i \otimes \underline{e}_\ell = A_{ij} B_{j\ell}\, \underline{e}_i \otimes \underline{e}_\ell$$

$$[\underline{\underline{A}}\,\underline{\underline{B}}]_{\underline{e}} = [\underline{\underline{A}}]_{\underline{e}} [\underline{\underline{B}}]_{\underline{e}}$$

$$\underline{\underline{A}} : \underline{\underline{B}} = A_{ij} B_{k\ell} (\underline{e}_i \otimes \underline{e}_j) : (\underline{e}_k \otimes \underline{e}_\ell) = A_{ij} B_{ij}$$

DOT

DOT

$$\mathrm{TRACE}(\underline{\underline{A}}) = \underline{\underline{A}} : \underline{\underline{1}} = A_{ij}\, \delta_{k\ell}\, (\underline{e}_i \otimes \underline{e}_j) : (\underline{e}_k \otimes \underline{e}_\ell) =$$

$$= A_{ij}\, \delta_{k\ell}\, \delta_{ik}\, \delta_{j\ell} = A_{ij}\, \delta_{ij} = A_{ii}$$

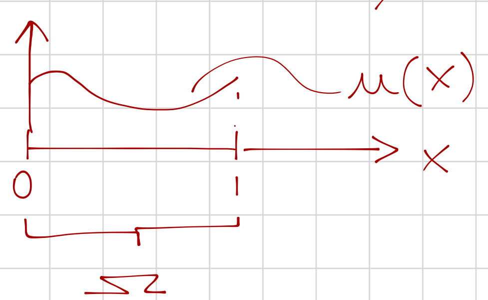
CHECK OUT PYTHON NOTEBOOK

SO FAR WE HAVE BEEN INTERESTED IN THE NOTION OF VECTORS & TENSOR BUT WHAT WE ARE REALLY AFTER ARE FUNCTIONS

FUNCTIONS CAN BE SCALAR-, VECTOR-, OR TENSOR-VALUED FUNCTIONS.

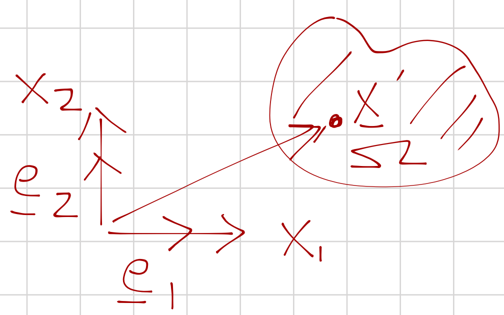
SCALAR FUNCTION

1-D $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$ (EG $\Omega = [0, 1]$



$u(x) = \cos(x) e^{x} \ldots$

2-D $\theta : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$



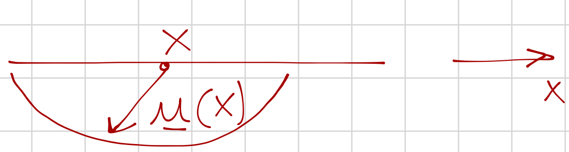
$\underline{x} = x_i \underline{e}_i$

$\theta(\underline{x}) = \cos(x_1) e^{x_2} + x_2^2 \ldots$

VECTOR FUNCTION

1-D $\underline{u} : \Omega \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}$

EG
$\underline{u}(x) = \cos(x) \underline{e}_1 + \sin(x) \underline{e}_2$



EG DEFORMATION OF A BEAM-COLUMN

2-D $\underline{v} : \Omega \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}$

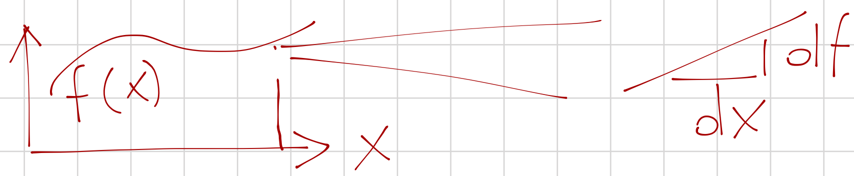
$\underline{v}(\underline{x}) = v_i(\underline{x}) \underline{e}_i = v_i(x_1, x_2 \ldots x_d) \underline{e}_i$

TESOR FUNCTION

$\underline{\underline{\sigma}} : \Omega \rightarrow \mathbb{R}^{d \times d}$ EG STRESS TENSOR

$\underline{\underline{\sigma}}(\underline{x}) = \sigma_{ij}(\underline{x})\, \underline{e}_i \otimes \underline{e}_j$

REVIEW OF CALCULUS



$\leftarrow$ SLOPE OF A FUNCTION

$$\frac{df}{dx}(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

SIMILARLY IF WE HAVE A FUNCTION OF MULTIPLE VARIABLE

$$\frac{df}{dx_i}(x_1, \ldots x_i \ldots x_d) = \lim_{\varepsilon \to 0} \frac{f(x_1, \ldots, x_i + \varepsilon, \ldots x_d) - f(x_1, \ldots, x_i, \ldots x_d)}{\varepsilon}$$

$\frac{df}{dx_i}(\underline{x})$ TELLS YOU THE CHANGE IN THE FUNCTION f WRT THE COORDINATE x_i

THE GRADIENT OF A FUNCTION

$$\underline{\nabla} f = \frac{df}{dx_i}(\underline{x})\, \underline{e}_i$$

SUMMATION

IS A VECTOR POINTING IN THE DIRECTION OF GROWTH

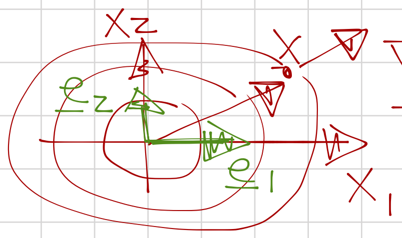
THE GRADIENT OF A FUNCTION

$$\underline{\nabla} f = \frac{df}{dx_i}(\underline{x})\,\underline{e}_i$$

SUMMATION

IS A VECTOR POINTING IN THE DIRECTION OF MAX GROWTH

EG: CONSIDER $f(\underline{x}) = x_1^2 + x_2^2$



$$\underline{\nabla} f(\underline{x}) = \frac{df}{dx_i}(\underline{x})\,\underline{e}_i$$

$$= (2x_1 + 2x_2)\,\underline{e}_1$$