HOMEWORK 2

CEE 530: Continuum Mechanics and Thermodynamics

Due: March 19, 2018

PROBLEM 1

Showing step-by-step you calculations, expand the following expressions such that the gradient operator acts only on one term at a time (e.g. $\nabla(\phi \boldsymbol{a}) = \boldsymbol{a} \otimes \nabla\phi + \phi\nabla\boldsymbol{a}$) where $\phi, \psi \in \mathbb{R}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ and $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{d\times d}$

1. (5) $\nabla(\phi\boldsymbol{a}) = \boldsymbol{a} \otimes \nabla\phi + \phi\nabla\boldsymbol{a}$
2. (5) $\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$
3. (5) $\nabla \times (\phi\boldsymbol{a}) = \nabla\phi \times \boldsymbol{a} + \phi\nabla \times \boldsymbol{a}$
4. (5) $\nabla \cdot (\boldsymbol{A}\boldsymbol{b}) = \boldsymbol{A}^\top : \nabla\boldsymbol{b} + \nabla \cdot \boldsymbol{A}^\top \cdot \boldsymbol{b}$
5. (5) $\nabla \cdot (\boldsymbol{A}\boldsymbol{B}) = \nabla\boldsymbol{A} : \boldsymbol{B} + \boldsymbol{A}\nabla \cdot \boldsymbol{B}$

PROBLEM 2

(10) Let $\mathbf{1}$ be the identity tensor and $\boldsymbol{B} \in \mathbb{R}^{d\times d}$ such that

$$\boldsymbol{A}(\boldsymbol{B}) = \text{tr}(\boldsymbol{B})\ \mathbf{1} + \frac{1}{2}(\boldsymbol{B} + \boldsymbol{B}^\top)$$

what is $\dfrac{\partial\boldsymbol{A}}{\partial\boldsymbol{B}}$? $= \mathbf{1} \otimes \mathbf{1} + \mathbb{I}^{sym}$ where $\mathbb{I}^{sym} = 1/2(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$.

PROBLEM 3

Consider a cube of side a made of a soft, deformable material. We adopt an orthonormal basis $\{\mathbf{e}_i\}_i$ parallel to the sides of the undeformed cube to describe the motion, and one of the vertices of the cube as the origin. In this way, all points in the cube have position vectors of the form $\boldsymbol{X} = X_i\mathbf{e}_i$, with $0 \leq X_i \leq a, i = 1, 2, 3$. The motion of the cube is given by

$$\varphi(\boldsymbol{X}, t) = X_1(1 + t)\ \mathbf{e}_1 + X_2(1 + t^2)\ \mathbf{e}_2 + X_3(1 + t^3)\ \mathbf{e}_3$$

so that at t = 0 the cube is at the reference configuration

1. (10) Find the material velocity field.
2. (10) Find the spatial velocity field.
3. (10) Find the spatial acceleration field.
4. (10) Compute the right Cauchy-Green deformation tensor. Does it depend on $\boldsymbol{X}$?
5. (5) Is the motion admissible at each t?
6. (10) What is the stretch ratio of vectors in the direction parallel to $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ as a function of time.
7. (10) What is the value of the cosine of the angle between directions $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1$ as a function of time?
8. (10) Can you construct an affine deformation mapping in which the directions defined by the orthonormal vectors $\boldsymbol{f}_1 = (\mathbf{e}_1 + \mathbf{e}_2)/2$, $\boldsymbol{f}_2 = (\mathbf{e}_2 - \mathbf{e}_1)/2$ and $\boldsymbol{f}_3 = \mathbf{e}_3$ are mapped into themselves (each one to itself), but stretched with stretch ratios $\lambda_1(t) = 1 + t$, $\lambda_2(t) = 1 + t^2$ and $\lambda_3(t) = 1 + t^3$, respectively? Of course, we request it to have a positive Jacobian. Express it in the $\mathbf{e}_i$ basis.

For the map

$$\varphi(\boldsymbol{X}, t) = X_1(1+t)\mathbf{e}_1 + X_2(1+t^2)\mathbf{e}_2 + X_3(1+t^3)\mathbf{e}_3$$

1. The material velocity field is given by

$$\boxed{\boldsymbol{V}(\boldsymbol{X}, t) = X_1\mathbf{e}_1 + 2X_2t\mathbf{e}_2 + 3X_3t^2\mathbf{e}_3} \tag{1}$$

2. The spatial velocity field is given by

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{V}(\varphi^{-1}(\boldsymbol{x}, t), t)$$

where

$$\varphi^{-1}(\boldsymbol{x}, t) = \frac{x_1}{(1+t)}\mathbf{e}_1 + \frac{x_2}{(1+t^2)}\mathbf{e}_2 + \frac{x_3}{(1+t^3)}\mathbf{e}_3$$

therefore

$$\boxed{\boldsymbol{v}(\boldsymbol{x}, t) = \frac{x_1}{(1+t)}\mathbf{e}_1 + \frac{2x_2}{(1+t^2)}t\mathbf{e}_2 + \frac{3x_3}{(1+t^3)}t^2\mathbf{e}_3} \tag{2}$$

3. The spatial acceleration field is given by

$$\boldsymbol{a}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{v}(\boldsymbol{x}, t)}{\partial t} + \nabla_{\boldsymbol{x}}\boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t)$$

with

$$\frac{\partial \boldsymbol{v}(\boldsymbol{x}, t)}{\partial t} = -\frac{x_1}{(1+t)^2}\mathbf{e}_1 - \frac{2tx_2}{(1+t^2)^2}\mathbf{e}_2 - \frac{3t^2x_3}{(1+t^3)^2}\mathbf{e}_3$$

and

$$\nabla_{\boldsymbol{x}}\boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t) = \frac{x_1}{(1+t)^2}\mathbf{e}_1 + \frac{x_2}{(1+t^2)^2}\mathbf{e}_2 + \frac{x_3}{(1+t^3)^2}\mathbf{e}_3$$

which gives

$$\boxed{\boldsymbol{a}(\boldsymbol{x}, t) = 0\mathbf{e}_1 + \frac{x_2 - 2tx_2}{(1+t^2)^2}\mathbf{e}_2 + \frac{x_3 - 3t^2x_3}{(1+t^3)^2}\mathbf{e}_3} \tag{3}$$

4. The deformation gradient is given by

$$[\boldsymbol{F}(\boldsymbol{X}, t)] = \begin{bmatrix} 1+t & 0 & 0 \\ 0 & 1+t^2 & 0 \\ 0 & 0 & 1+t^3 \end{bmatrix}$$

and therefore the right Cauchy-Green deformation tensor will be given by

$$\boldsymbol{C}(\boldsymbol{X}, t) = \boldsymbol{F}^T(\boldsymbol{X}, t) \cdot \boldsymbol{F}(\boldsymbol{X}, t)$$

therefore

$$\boxed{[\boldsymbol{C}(\boldsymbol{X}, t)] = \begin{bmatrix} (1+t)^2 & 0 & 0 \\ 0 & (1+t^2)^2 & 0 \\ 0 & 0 & (1+t^3)^2 \end{bmatrix}} \tag{4}$$

5. The motion is admissible at each $t > 0$. In fact the map is defined for all points $\boldsymbol{X}$, is injective and continuosly differentiable.

6. The stretch ratio in the direction of $\boldsymbol{n}^* = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ is given by

$$\lambda(\boldsymbol{X}, \boldsymbol{n}, t) = \sqrt{\boldsymbol{n} \cdot \boldsymbol{C}(\boldsymbol{X}, t) \cdot \boldsymbol{n}}$$

where

$$\boldsymbol{n} = \frac{\boldsymbol{n}^*}{||\boldsymbol{n}^*||} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$$

thus

$$\boldsymbol{n} \cdot \boldsymbol{C}(\boldsymbol{X}, t) \cdot \boldsymbol{n} = \frac{1}{3}((1+t)^2 + (1+t^2)^2 + (1+t^3)^2)$$

and therefore

$$\boxed{\lambda(\boldsymbol{X}, \boldsymbol{n}, t) = \sqrt{\frac{1}{3}((1+t)^2 + (1+t^2)^2 + (1+t^3)^2)} \qquad (5)}$$

7. Take the vectors $\boldsymbol{n}^1 = 1/\sqrt{2}(\mathbf{e}_1 + \mathbf{e}_2)$ and $\boldsymbol{n}^2 = \mathbf{e}_1$ in the reference configuration, the cosine between the two vectored after deformation will be given by

$$cos(\theta_{12}) = \frac{\boldsymbol{n}^1 \cdot \boldsymbol{C}(\boldsymbol{X}, t) \cdot \boldsymbol{n}^2}{\sqrt{\boldsymbol{n}^1 \cdot \boldsymbol{C}(\boldsymbol{X}, t) \cdot \boldsymbol{n}^1}\sqrt{\boldsymbol{n}^2 \cdot \boldsymbol{C}(\boldsymbol{X}, t) \cdot \boldsymbol{n}^2}}$$

where

$$\boldsymbol{n}^1 \cdot \boldsymbol{C}(\boldsymbol{X}, t) \cdot \boldsymbol{n}^2 = \frac{1}{\sqrt{2}}(1+t)^2$$

and

$$\sqrt{\boldsymbol{n}^2 \cdot \boldsymbol{C}(\boldsymbol{X}, t) \cdot \boldsymbol{n}^2} = \frac{1}{\sqrt{2}}\sqrt{(1+t)^2 + (1+t^2)^2}; \qquad \sqrt{\boldsymbol{n}^1 \cdot \boldsymbol{C}(\boldsymbol{X}, t) \cdot \boldsymbol{n}^1} = (1+t);$$

which gives

$$\boxed{cos(\theta_{12}) = \frac{(1+t)}{\sqrt{(1+t)^2 + (1+t^2)^2}} \qquad (6)}$$

8. The affine deformation map would be of the sort

$$\varphi(\boldsymbol{X}, t) = \boldsymbol{F} \cdot \boldsymbol{X} + \boldsymbol{x}_0$$

and it would have the following properties

$$\begin{aligned}
\boldsymbol{F} \cdot \boldsymbol{f}_1 &= \lambda_1(t)\boldsymbol{f}_1 = (1+t)\boldsymbol{f}_1 \\
\boldsymbol{F} \cdot \boldsymbol{f}_2 &= \lambda_2(t)\boldsymbol{f}_2 = (1+t^2)\boldsymbol{f}_2 \\
\boldsymbol{F} \cdot \boldsymbol{f}_3 &= \lambda_3(t)\boldsymbol{f}_3 = (1+t^3)\boldsymbol{f}_3
\end{aligned}$$

since the three vectors are orthonormal the tensor $\boldsymbol{F}$ can be build as

$$\boldsymbol{F} = (1+t)\boldsymbol{f}_1 \otimes \boldsymbol{f}_1 + (1+t^2)\boldsymbol{f}_2 \otimes \boldsymbol{f}_2 + (1+t^3)\boldsymbol{f}_3 \otimes \boldsymbol{f}_3$$

its representation in the $\boldsymbol{f}_{1\ldots3}$ set of basis will be

$$[\boldsymbol{F}]_{\boldsymbol{f}} = \begin{bmatrix} (1+t) & 0 & 0 \\ 0 & (1+t^2) & 0 \\ 0 & 0 & (1+t^3) \end{bmatrix}$$

and its representation in the $\mathbf{e}_{1\ldots3}$ set of basis will be

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{t^2}{2} + \frac{t}{2} + 1 & \frac{t}{2} - \frac{t^2}{2} & 0 \\ \frac{t}{2} - \frac{t^2}{2} & \frac{t^2}{2} + \frac{t}{2} + 1 & 0 \\ 0 & 0 & t^3 + 1 \end{bmatrix} \tag{7}$$

PROBLEM 4

A cylindrical tube of inner radius a, outer radius b, and length L is turned inside out and subsequently constrained to take the shape of a straight cylinder of the same dimensions.

1. (15) Assuming that the radial and axial fibers remain unstretched and still radial and axial, respectively, after deformation, determine the deformation mapping $\boldsymbol{\varphi}$.
2. (5) Show that $\boldsymbol{\varphi} \circ \boldsymbol{\varphi}$ is the identity mapping, i.e., turning the cylinder inside out twice returns it to its initial configuration.
3. (10) Compute the deformation gradient. Name a direction with stretched fibers.

1. The map can be given by

$$\boldsymbol{\varphi}(\boldsymbol{X}, t) = (b + a - X_r)\mathbf{e}_r(\boldsymbol{X}) + (L - X_z)\,\mathbf{e}_z \tag{8}$$

   where $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_3$ are a set of cylindrical basis.

2. If we compose $\boldsymbol{\varphi} \circ \boldsymbol{\varphi}$ we get

$$\begin{aligned} \boldsymbol{\varphi}(\boldsymbol{\varphi}(\boldsymbol{X}, t), t) &= (b + a - (b + a - X_r))\mathbf{e}_r(\boldsymbol{X}) + (L - (L - X_z))\,\mathbf{e}_z = \\ &= X_r\mathbf{e}_r + X_z\mathbf{e}_z = \boldsymbol{X} \qquad \checkmark \end{aligned} \tag{9}$$

3. The deformation gradient is given by

$$\boldsymbol{F}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{\varphi}(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}$$

   recalling that

$$\frac{\partial \mathbf{e}_r(\boldsymbol{X})}{\partial X_\theta} = \frac{1}{X_r}\mathbf{e}_\theta(\boldsymbol{X})$$

   yields the deformation gradient in the cylindrical basis as

$$[\boldsymbol{F}(\boldsymbol{X}, t)]_{\mathbf{e}_{r,\theta,z}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{(b + a - X_r)}{X_r} & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{10}$$

and if we calculate the Cauchy-Green deformation tensor we obtain

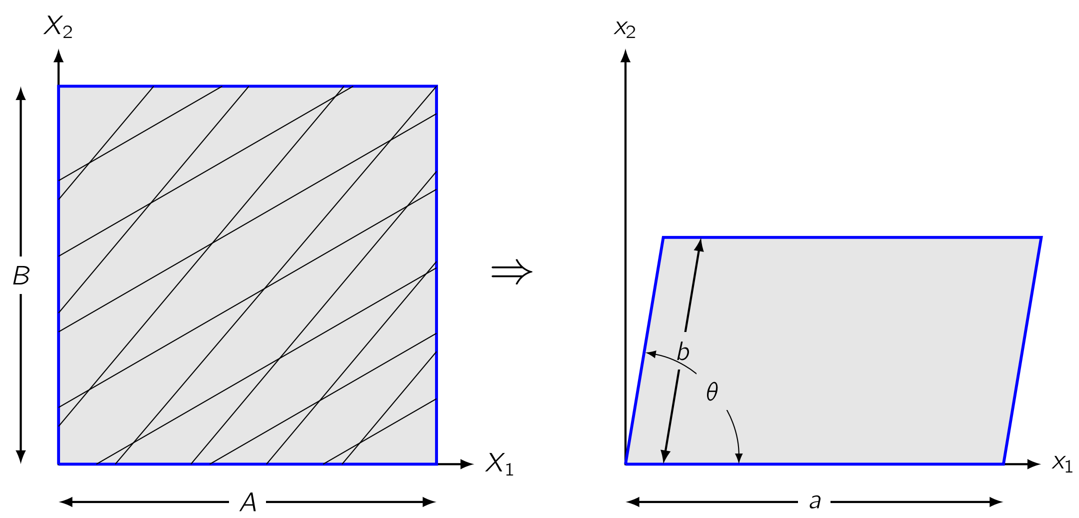
$$\boldsymbol{C}(\boldsymbol{X},t) = \boldsymbol{F}^T(\boldsymbol{X},t)\boldsymbol{F}(\boldsymbol{X},t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{(b+a-X_r)^2}{X_r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

therefore the fibers in the $\mathbf{e}_r$ (i.e. the radial) direction remain un-stretched similarly to the fibers in the $\mathbf{e}_z$ (i.e. the axial) direction. The fibers in the $\mathbf{e}_\theta$ (i.e. tangential) direction are stretched by $\lambda(\boldsymbol{X},\mathbf{e}_\theta) = (b+a-X_r)/X_r$.

PROBLEM 5

A block of rubber has a rectangular cross-section of dimensions A and B in its undeformed configuration. A set of axes is chosen such that the cross-section of the block coincides with the $X_1 - X_2$ plane. The deformation of the block is constrained in the remaining direction X_3. The cross-section of the block is reinforced by means of two sets of uniformly spaced straight wires. The two families of wires subtend an angle 2α to each other, and their bisector is at an angle β to the X_1-axis. The wires are strongly bonded to the matrix and follow its deformation. Since the metallic wires are much stiffer than the rubbery matrix, they are taken to be inextensible to a first approximation. After deformation the cross−section of the block takes a rhomboidal shape of dimensions a and b. The angle between the sides of the rhomboid is θ.

1. (15) Write down the right Cauchy-Green deformation tensor $\boldsymbol{C}$ as a function of the coordinate stretches a/A, b/B and the angle θ.
   Hint: Notice that the deformation gradient will be uniform throughout the sample, therefore, you can try to deduce $\boldsymbol{C}$ by using geometric considerations of changes of lengths and angles.
2. (10) For what value of β the sides of the block remain at an angle of $\pi/2$ after the deformation?
3. (10) Compute the coordinate stretch ratios a/A and b/B as a function of α, β, θ.
4. (10) Let $\beta = 0$ and imagine that the block is stretched in the X_1 direction and compressed in the X_2. Define a nonlinear Poisson's ratio by the relation $\nu = -E_{22}/E_{11}$, where E_{11} and E_{22} are components of the Lagrangian strain tensor $\boldsymbol{E} = (\boldsymbol{C} - \boldsymbol{I})/2$. Compute ν as a function of α. For what value of α does it follow that $\nu = 0.5$?



Take

1. We know the following

$$\begin{aligned}\mathbf{E}_1 \cdot \boldsymbol{C} \cdot \mathbf{E}_1 &= \left(\frac{a}{A}\right)^2 \\ \mathbf{E}_2 \cdot \boldsymbol{C} \cdot \mathbf{E}_2 &= \left(\frac{b}{B}\right)^2 \\ \frac{\mathbf{E}_1 \cdot \boldsymbol{C} \cdot \mathbf{E}_2}{\sqrt{\mathbf{E}_1 \cdot \boldsymbol{C} \cdot \mathbf{E}_1}\sqrt{\mathbf{E}_2 \cdot \boldsymbol{C} \cdot \mathbf{E}_2}} &= \cos(\theta)\end{aligned}$$

where the matrix representation of the right Cauchy-Green strain tensor for the case of in plane deformation is given by

$$[\boldsymbol{C}] = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $C_{12} = C_{21}$. Therefore with the three equations we can solve for the three unknowns $C_{11}, C_{22}, C_{12} = C_{21}$ such that

$$\begin{cases} C_{11} = (a/A)^2 \\ C_{22} = (b/B)^2 \\ C_{12} = C_{21} = \cos(\theta)(a/A)(b/B) \end{cases}$$

which therefore gives

$$\boxed{[\boldsymbol{C}] = \begin{bmatrix} (a/A)^2 & \cos(\theta)(a/A)(b/B) & 0 \\ \cos(\theta)(a/A)(b/B) & (b/B)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \tag{11}$$

2. In order for the sides of the block to maintain an angle of $\pi/2$ we must have

$$C_{12} = 0$$

Since we know that the wires are inextensible we must have

$$\boldsymbol{G}_1 \cdot \boldsymbol{C} \cdot \boldsymbol{G}_1 = 1$$

where $\boldsymbol{G}_1, \boldsymbol{G}_2$ are given by

$$\boldsymbol{G}_1 = \cos(\beta - \alpha)\mathbf{E}_1 + \sin(\beta - \alpha)\mathbf{E}_2$$

$$\boldsymbol{G}_2 = \cos(\beta + \alpha)\mathbf{E}_1 + \sin(\beta + \alpha)\mathbf{E}_2$$

hence

$$\begin{aligned}\boldsymbol{G}_1 \cdot \boldsymbol{C} \cdot \boldsymbol{G}_1 &= G_1^i\mathbf{E}_i \cdot C_{jk}\mathbf{E}_j \otimes \mathbf{E}_k \cdot G_1^l\mathbf{E}_l = G_1^i G_1^l C_{il} = \\ &= \cos^2(\beta - \alpha)C_{11} + \sin^2(\beta - \alpha)C_{22} + 2\cos(\beta - \alpha)\sin(\beta - \alpha)C_{12} = 1\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{G}_2 \cdot \boldsymbol{C} \cdot \boldsymbol{G}_2 &= G_2^i\mathbf{E}_i \cdot C_{jk}\mathbf{E}_j \otimes \mathbf{E}_k \cdot G_2^l\mathbf{E}_l = G_2^i G_2^l C_{il} = \\ &= \cos^2(\beta + \alpha)C_{11} + \sin^2(\beta + \alpha)C_{22} + 2\cos(\beta + \alpha)\sin(\beta + \alpha)C_{12} = 1\end{aligned}$$

and to maintain an angle of $\pi/2$ we obtain two equations such that

$$\begin{cases} \cos^2(\beta - \alpha)C_{11} + \sin^2(\beta - \alpha)C_{22} = 1 \\ \cos^2(\beta + \alpha)C_{11} + \sin^2(\beta + \alpha)C_{22} = 1 \end{cases}$$

and subtracting the second from the first gives

$$
\begin{aligned}
&(\cos^2(\beta-\alpha)-\cos^2(\beta+\alpha))C_{11}+(\sin^2(\beta-\alpha)-\sin^2(\beta+\alpha))C_{22}= \\
&=4\cos(\beta)\sin(\beta)\cos(\alpha)\sin(\alpha)C_{11}-4\cos(\beta)\sin(\beta)\cos(\alpha)\sin(\alpha)C_{22}= \\
&=\cos(\beta)\sin(\beta)[4\cos(\alpha)\sin(\alpha)C_{11}-4\cos(\alpha)\sin(\alpha)C_{22}]=0
\end{aligned}
$$

hence the sides of the block will maintain an angle of $\pi/2$ when

$$\boxed{\beta=n\frac{\pi}{2}} \tag{12}$$

with n being any integer value.

3. Since we know that the wires are inextensible we must have

$$\boldsymbol{G}_1\cdot\boldsymbol{C}\cdot\boldsymbol{G}_1=1$$

$$\boldsymbol{G}_2\cdot\boldsymbol{C}\cdot\boldsymbol{G}_2=1$$

where $\boldsymbol{G}_1,\boldsymbol{G}_2$ are given by

$$\boldsymbol{G}_1=\cos(\beta-\alpha)\mathbf{E}_1+\sin(\beta-\alpha)\mathbf{E}_2$$

$$\boldsymbol{G}_2=\cos(\beta+\alpha)\mathbf{E}_1+\sin(\beta+\alpha)\mathbf{E}_2$$

hence

$$
\begin{aligned}
\boldsymbol{G}_1\cdot\boldsymbol{C}\cdot\boldsymbol{G}_1 &= G_1^i\mathbf{E}_i\cdot C_{jk}\mathbf{E}_j\otimes\mathbf{E}_k\cdot G_1^l\mathbf{E}_l=G_1^iG_1^lC_{il}= \\
&= \cos^2(\beta-\alpha)C_{11}+\sin^2(\beta-\alpha)C_{22}+2\cos(\beta-\alpha)\sin(\beta-\alpha)C_{12}= \\
&= \cos^2(\beta-\alpha)(a/A)^2+\sin^2(\beta-\alpha)(b/B)^2+\ldots \\
&\qquad\qquad \ldots+2\cos(\beta-\alpha)\sin(\beta-\alpha)\cos(\theta)(a/A)(b/B)=1
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\boldsymbol{G}_2\cdot\boldsymbol{C}\cdot\boldsymbol{G}_2 &= G_2^i\mathbf{E}_i\cdot C_{jk}\mathbf{E}_j\otimes\mathbf{E}_k\cdot G_2^l\mathbf{E}_l=G_2^iG_2^lC_{il}= \\
&= \cos^2(\beta+\alpha)C_{11}+\sin^2(\beta+\alpha)C_{22}+2\cos(\beta+\alpha)\sin(\beta+\alpha)C_{12}= \\
&= \cos^2(\beta+\alpha)(a/A)^2+\sin^2(\beta+\alpha)(b/B)^2+\ldots \\
&\qquad\qquad \ldots+2\cos(\beta+\alpha)\sin(\beta+\alpha)\cos(\theta)(a/A)(b/B)=1
\end{aligned}
$$

once more we are left with two equations and the two unknowns $(a/A),(b/B)$ which can be solved for, such that

$$\boxed{\left\{\begin{array}{l}\cos^2(\beta-\alpha)(a/A)^2+\sin^2(\beta-\alpha)(b/B)^2+\sin(2(\beta-\alpha))\cos(\theta)(a/A)(b/B)=1\\ \cos^2(\beta+\alpha)(a/A)^2+\sin^2(\beta+\alpha)(b/B)^2+\sin(2(\beta+\alpha))\cos(\theta)(a/A)(b/B)=1\end{array}\right.} \tag{13}$$

4. Take $\beta=0,\theta=0$ which gives

$$\left\{\begin{array}{l}\cos^2(\alpha)(a/A)^2+\sin^2(\alpha)(b/B)^2=1\\ \cos^2(\alpha)(a/A)^2+\sin^2(\alpha)(b/B)^2=1\end{array}\right.$$

therefore

$$(a/A)^2=\frac{1-\sin^2(\alpha)(b/B)^2}{\cos^2(\alpha)}$$

the right Cauchy-Green strain tensor will be simplified to

$$[\boldsymbol{C}] = \begin{bmatrix} (a/A)^2 & 0 & 0 \\ 0 & (b/B)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the Euler-Lagrange strain tensor will look as such

$$[\boldsymbol{E}] = \frac{1}{2}[\boldsymbol{C} - \boldsymbol{I}] = \begin{bmatrix} (a/A)^2 - 1 & 0 & 0 \\ 0 & (b/B)^2 - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and thus

$$\nu \quad = \quad -\frac{E_{11}}{E_{22}} = -\frac{(a/A)^2 - 1}{(b/B)^2 - 1} = -\frac{\dfrac{1 - \sin^2(\alpha)(b/B)^2}{\cos^2(\alpha)} - 1}{(b/B)^2 - 1} = \tan(\beta - \alpha)^{-2}$$

therefore for $\beta = 0$

$$\boxed{\nu_{12} = \cot^2(\alpha)} \tag{14}$$