# LECTURE 8:

- POLAR DECOMPOSITION
- STRAIN MEASURES

## LOGISTICS:

- HW 1 DUE TODAY
- HW 2 OUT TODAY DUE MONDAY MARCH 19
- MIDTERM EXAM NEW DATE MARCH 26

COVERS EVERYTHING WE DISCUSS TO END OF LECTURE ON 03/14

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CH 1, 2, 3.1 - 3.14, 3.18 - 3.28
(4.1 - 4.6, 4.10)

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CH 1, 2, 4, (3)

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## POLAR DECOMPOSITION

THE POLAR DECOMPOSITION PROVIDES A GREAT DEAL OF INSIGHT ON THE MOTION

CONSIDER A SOLID THAT UNDERGOES A DEFORMATION $\underline{\underline{F}}_1$ AND A ROTATION $\underline{\underline{R}}$

THE TOTAL DEFORMATION GRADIENT IS

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{F}}_1$$

NOTE THAT THE ENERGY OF SOLIDS DEPEND SOLELY ON STRETCHING & DISTORTING ATOMIC BONDS. THUS THE ENERGY OF

$$\underline{\underline{F}}_1(\Omega_0) \;\&\; \underline{\underline{R}}\,\underline{\underline{F}}_1(\Omega_0)$$

IS THE SAME.

FURTHERMORE

$$\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{F}}_1^T\underbrace{\underline{\underline{R}}^T\underline{\underline{R}}}\,\underline{\underline{F}}_1 = \underline{\underline{F}}_1^T\underline{\underline{F}}_1$$

ROTATIONS ARE ORTHONORMAL TENSORS

SO THE METRIC TENSOR IS INDEPENDENT OF RIGID BODY ROTATIONS

ALSO NOTE THAT $\underline{\underline{C}}$ IS FULLY CHARACTERIZED BY SIX COMPONENTS WHILE $\underline{\underline{F}}$ NEED NINE COMPONENTS

$\underline{\underline{F}}$ CONTAINS MORE INFORMATION THAN $\underline{\underline{C}}$

BUT CONSIDERING THAT THE ELASTIC STRAIN ENERGY DEPENDS SOLELY ON STRETCHES & ANGLE CHANGES, NO CONTINUUM THEORY OF ELASTICITY DEPENDS ON THE FULL $\underline{\underline{F}}$ BUT ONLY ON PARTS OF IT (E.G. $\underline{\underline{C}}$).

POLAR DECOMPOSITION: LET $\underline{\underline{F}}$ BE A SECOND ORDER TENSOR W/ $\det \underline{\underline{F}} \neq 0$, IT ADMITS THE DECOMPOSITION

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$$

WHERE $\underline{\underline{R}} \in O(3)$ ← ORTHOGONAL GROUP (ROTATIONS)

$\underline{\underline{U}}, \underline{\underline{V}}$ ARE SYMMETRIC SECOND ORDER TENSORS

THIS DECOMPOSITION IS UNIQUE !!!

$\underline{\underline{R}}\,\underline{\underline{U}} \leftarrow$ RIGHT POLAR DECOMPOSITION

$\underline{\underline{V}}\,\underline{\underline{R}} \leftarrow$ LEFT POLAR DECOMPOSITION

NOTE: $\det(\underline{\underline{F}}) = \det(\underline{\underline{R}}\,\underline{\underline{U}}) = \det(\underline{\underline{U}}) = \det(\underline{\underline{V}})$

NOTE THAT

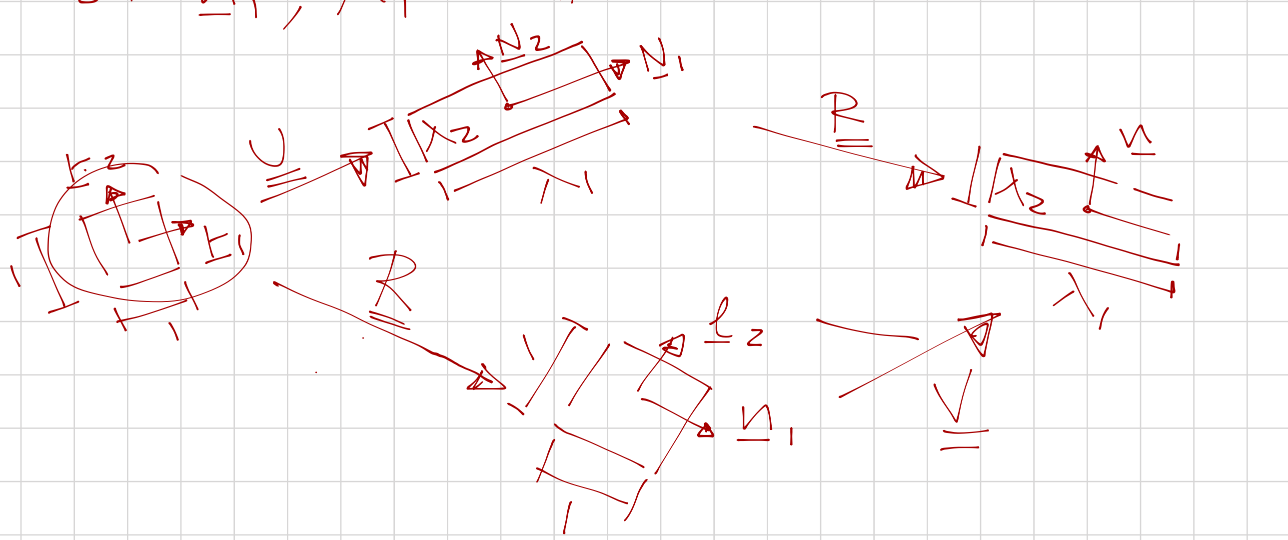
$$\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{U}}^T\underline{\underline{R}}^T\underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{U}}^T\underline{\underline{U}} = \underline{\underline{U}}\,\underline{\underline{U}}$$

THUS $\underline{\underline{C}}$ HAS THE SAME EIGENVECTOR OF $\underline{\underline{U}}$ AND ITS EIGENVALUES ARE THE SQUARED OF $\underline{\underline{U}}$

INTERPRETATION:

LET $\underline{N}_i, \lambda_i$ BE THE EIGENVALUES $\Rightarrow$ VECS OF $\underline{\underline{U}}$

LET $\underline{n}_i, \lambda_i$ " " " $\underline{\underline{V}}$




LEFT CAUCHY GREEN TENSOR

SIMILARLY TO THE RIGHT C-G THE LEFT C-G IS A MEASURE OF METRIC CHANGES MEASURED ON SPATIAL VECTORS

$\underline{\underline{b}} = \underline{\underline{F}}\,\underline{\underline{F}}^T$ (AKA THE FINGER TENSOR)

WITH $\underline{\underline{F}} = \underline{\underline{V}}\,\underline{\underline{R}}$

$$\underline{\underline{b}} = \underline{\underline{V}}\,\underline{\underline{R}}\,\underline{\underline{R}}^T\underline{\underline{V}}^T = \underline{\underline{V}}\,\underline{\underline{V}}^T = \underline{\underline{V}}\,\underline{\underline{V}}$$

NOTE THAT

$$d\underline{X}\cdot d\underline{Y} = \underline{\underline{F}}^{-1}d\underline{x}\cdot\underline{\underline{F}}^{-1}d\underline{y} = d\underline{x}\cdot\underline{\underline{F}}^{-T}\underline{\underline{F}}^{-1}d\underline{y} = d\underline{x}\cdot\underline{\underline{b}}^{-1}d\underline{y}$$

THE FINGER TENSOR ENABLE THE COMPUTATION OF SCALAR PRODUCTS IN THE REFERENCE CONFIG WITH THE ONLY KNOWLEDGE OF SPATIAL QUANTITIES

THE FINGER STRAIN TENSOR IS CONVENIENT WHE TRACKING DEFORMATION MAP IS MORE CHALLENGING THAN INVERSE $\underline{\underline{F}}$.

THEN CONSTITUTIVE RELATIONS CAN BE DEVELOPED BASED ON $\underline{\underline{b}}$

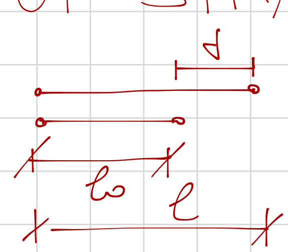
## STRAIN TENSORS

NOTE WHILE THE DISPLACEMENT IS A FUNDAMENTAL MEASURABLE QUANTITY, STRAINS ARE CONCEPTUAL MEASURES OF DEFORMATIONS.

HENCE A LARGE NUMBER OF DEFINITIONS $\exists$

RECALL A DEFINITION OF STRAIN

$$\varepsilon = \frac{\delta}{l_0} = \frac{l - l_0}{l_0}$$




ANOTHER (AKA GREEN-LAGRANGE)

$$\varepsilon_{GL} = \frac{1}{2}\left(\frac{l^2 - l_0^2}{l_0^2}\right) \leftarrow \text{STRAIN WRT REFERENCE}$$

ANOTHER EULER-ALMANSI

$$\varepsilon_{EA} = \frac{1}{2}\left(\frac{l^2 - l_0^2}{l^2}\right) \leftarrow \text{STRAIN WRT MATERIAL}$$

ANOTHER (TRUE) HECKY STRAIN

$$d\varepsilon_H = \frac{d\ell}{\ell}$$

$$\Rightarrow \int d\varepsilon_H = \int_{\ell_0}^{\ell} \frac{d\ell}{\ell} \Rightarrow \quad \varepsilon_H = \ln\left(\frac{\ell}{\ell_0}\right)$$

GREEN-LAGRANGE

$$\varepsilon_G = \frac{1}{2}\left(\frac{d\underline{x}\cdot d\underline{x} - d\underline{X}\cdot d\underline{X}}{d\underline{X}\cdot d\underline{X}}\right) = \underline{N}\cdot\left(\frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})\right)\underline{N}$$

$$= \underline{N}\cdot \underline{\underline{E}}\,\underline{N}$$

$$\Rightarrow \underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$$

NOTE $\underline{\underline{E}}$ IS DEFINED IN THE REFERENCE CONFIGURATION (REFERENTIAL STRAIN TENSOR)

$\underline{\underline{E}}$ DOES NOT DEPEND ON $\underline{\underline{R}}$

$\underline{\underline{E}}$ IS SYMMETRIC

NOTE: $\quad \underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}} = (\underline{\nabla}\underline{u} + \underline{\underline{I}})^T(\underline{\nabla}\underline{u} + \underline{\underline{I}})$

$$= \underline{\nabla}\underline{u} + \underline{\nabla}^T\underline{u} + \underline{\nabla}^T\underline{u}\,\underline{\nabla}\underline{u} + \underline{\underline{I}}$$

$$\Rightarrow \underline{\underline{E}} = \frac{1}{2}\left(\underline{\nabla}\underline{u} + \underline{\nabla}^T\underline{u} + \underline{\nabla}^T\underline{u}\,\underline{\nabla}\underline{u}\right)$$

IF WE ARE DEALING WITH INFINITESIMAL DEFORMATIONS $\|\underline{\nabla}\underline{u}\| \ll 1$

$$\underline{\nabla}^T\underline{u}\,\underline{\nabla}\underline{u} \ll \underline{\nabla}\underline{u}$$

THUS WE APPROXIMAT $\underline{\underline{E}}$ WITH

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\nabla}\underline{u} + \underline{\nabla}^T\underline{u}) = \mathrm{SYM}(\underline{\nabla}\underline{u})$$

## EULER-ALMANSI

$$e_{EA} = \frac{1}{2}\left(\frac{\ell^2-\ell_0^2}{\ell^2}\right) = \frac{1}{2}\frac{d\underline{x}}{\|d\underline{x}\|}\cdot(\underline{\underline{I}} - \underline{\underline{b}}^{-1})\frac{d\underline{x}}{\|d\underline{x}\|} = \underline{n}\cdot\underline{\underline{e}}\,\underline{n}$$

$$\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{b}}^{-1})$$

$\underline{\underline{e}}$ IS DEFINED OVER THE SPATIAL DOMAIN

REFERENTIAL STRAIN TENSOR

$- \sim$

NOTE

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) = \underline{\underline{F}}^T\left(\frac{1}{2}\left(\underline{\underline{I}} - \underline{\underline{F}}^{-T}\underline{\underline{F}}^{-1}\right)\right)\underline{\underline{F}} = \underline{\underline{F}}^T\underline{\underline{e}}\,\underline{\underline{F}}$$

## HECKY STRAIN

$$\frac{\ell}{\ell_0} = \sqrt{\frac{d\underline{X}\cdot\underline{\underline{C}}\,d\underline{X}}{d\underline{X}\cdot d\underline{X}}} = \frac{d\underline{X}}{\|d\underline{X}\|}\cdot\sqrt{\underline{\underline{C}}}\cdot\frac{d\underline{X}}{d\underline{X}} = \underline{N}\cdot\underline{\underline{U}}\,\underline{N}$$

$$\underline{\underline{H}} = \ln(\underline{\underline{U}}) \qquad \ln(\underline{\underline{U}}) = \sum_i \ln(\lambda_i)\,\underline{E}_i \otimes \underline{E}_j$$

$$\left(\underline{\underline{h}} = \ln(\underline{\underline{V}})\right)$$

NOTE IF YOU HAVE $\underline{\underline{F}} = \underline{\underline{F}}_1\underline{\underline{F}}_2$

$$\underline{\underline{C}} = (\underline{\underline{U}}_1\underline{\underline{U}}_1)(\underline{\underline{U}}_2\underline{\underline{U}}_2) \Rightarrow \sqrt{\underline{\underline{C}}} = \underline{\underline{U}}_1\underline{\underline{U}}_2$$

$$\underline{\underline{H}} = \ln(\underline{\underline{U}}_1\underline{\underline{U}}_2) = \ln(\underline{\underline{U}}_1) + \ln(\underline{\underline{U}}_2)$$

ADDITIVE DECOMPOSITION