Finite Elements in 1-D

The Local Element View

Maurizio M. Chiaramonte

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

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$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find u s.t.

$$\frac{d^2u}{dx^2} = f \quad \forall x \in (a, b)$$

with

$$u = g \quad \forall x \in \Gamma_D$$

$$\frac{du}{dx} = h \quad \forall x \in \Gamma_N.$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find $u \in \mathcal{S}$ s.t.

$$a(u, v) = F(v) \quad \forall v \in \mathcal{V}$$

where

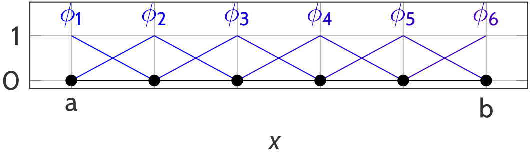
$$\mathcal{S} = \{u | u \in H^1(\Omega), u = g \, \forall x \in \Gamma_D\},$$
$$\mathcal{V} = \{v | v \in H^1(\Omega), v = 0 \, \forall x \in \Gamma_D\}$$

and

$$a(u, v) = \int_\Omega \frac{du}{dx}\frac{dv}{dx} dx, \quad F(v) = hv|_{\Gamma_N} - \int_\Omega fv dx$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$



Choose a finite number of basis functions

$$\Phi^h = \{\phi_1, \dots, \phi_N\}$$

such that

$$u(x) \approx u^h(x) = \sum_{i=1}^{N} u_i \phi_i(x)$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find $u^h \in \mathcal{S}^h$ s.t.

$$a(u^h, v^h) = F(v^h) \quad \forall v^h \in \mathcal{V}^h$$

where

$$\mathcal{S} \supseteq \mathcal{S}^h = \{u^h | u^h \in H^1(\Omega), u^h(x) = \sum_{i=1}^{N} u_i \phi_i(x), u^h = g \, \forall x \in \Gamma_D\},$$

$$\mathcal{V} \supseteq \mathcal{V}^h = \{v | v \in H^1(\Omega),\ v^h(x) = \sum_{i=1}^{N} v_i \phi_i(x),\ v^h = 0 \, \forall x \in \Gamma_D\}.$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find u_j s.t.

$$\sum_{j=1}^{N} a(\phi_i, \phi_j) u_j = F(\phi_i) \quad \forall i = 1 \dots N$$

where

$$a(\phi_i, \phi_j) = \int_\Omega \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx, \quad F(\phi_i) = h\phi_i|_{\Gamma_N} - \int_\Omega f\phi_i dx$$

FEM Roadmap

$$(S) \Leftrightarrow (W) \Rightarrow (G) \Leftrightarrow (M)$$

Find u_j s.t.

$$[K]\{U\} = \{F\}$$

where

$$[K]_{ij} = a(\phi_i, \phi_j), \quad \{U\}_j = u_j, \quad \{F\}_i = F(\phi_i)$$

where

$$a(\phi_i, \phi_j) = \int_\Omega \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx, \quad F(\phi_i) = h\phi_i|_{\Gamma_N} - \int_\Omega f\phi_i dx$$

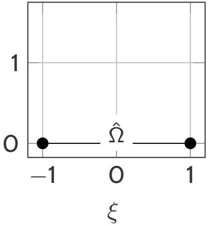
Goal of Lecture

From the global view to the local view

- How do we construct Φ^h?
- How do we efficiently compute $[K]_{ij}$, $\{F\}_j$?

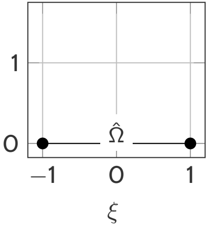
The Element View

Parametric Element Domain $\hat{\Omega}$

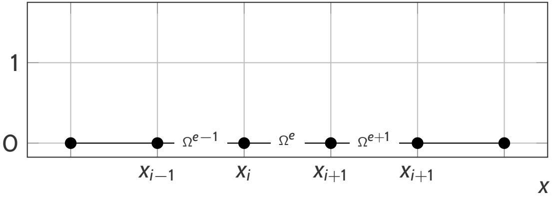


The Element View

Parametric Element Domain $\hat{\Omega}$



Physical/Real Domain $\Omega = \bigcup_{e=1}^{n_{el}} \Omega^e$



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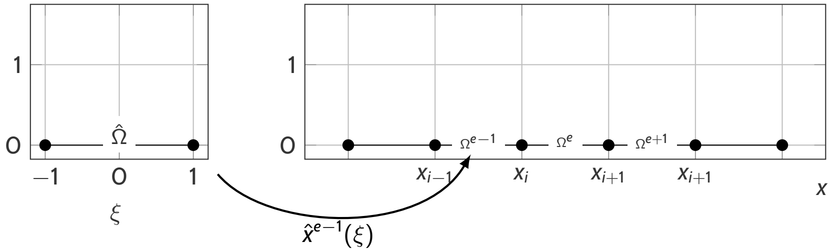

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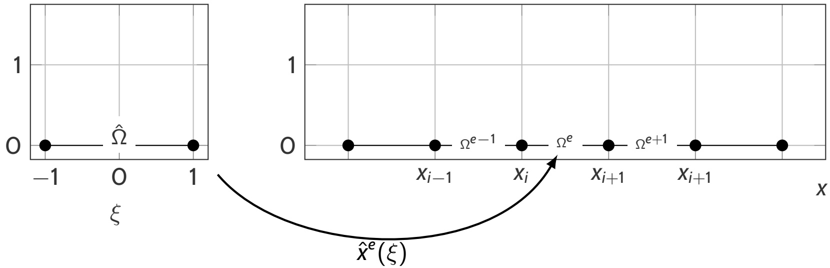

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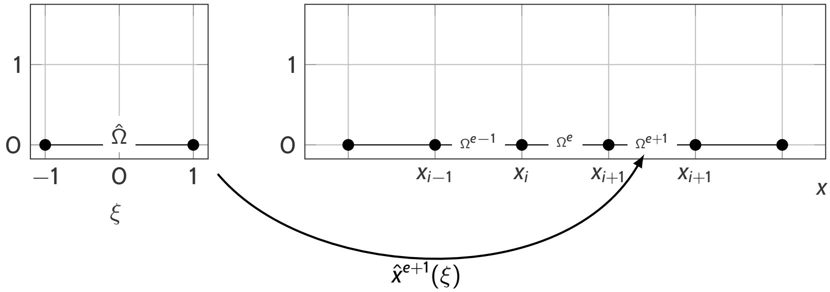

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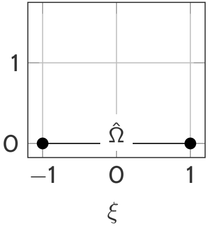
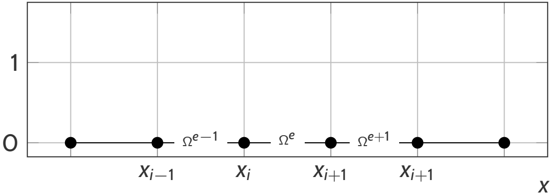
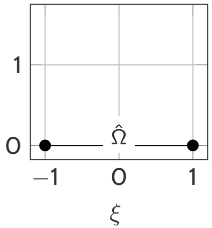




$$\hat{x}^e : \hat{\Omega} \to \Omega^e$$

The Element View

Parametric Element Domain $\hat{\Omega}$



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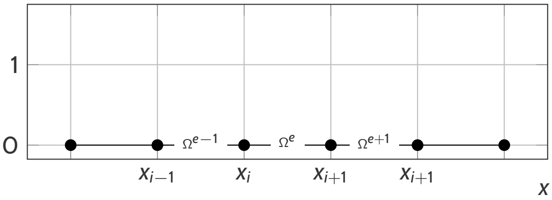
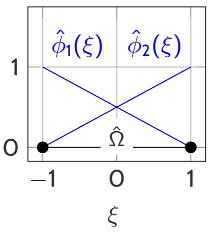


$$\hat{x}^e : [-1, 1] \to [x_i, x_{i+1}]$$

The Element View

Parametric Element Domain $\hat{\Omega}$



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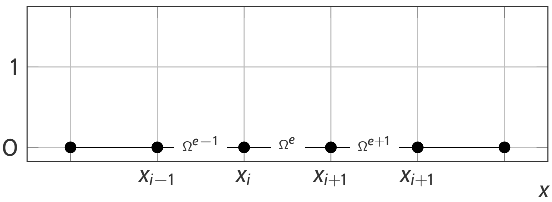

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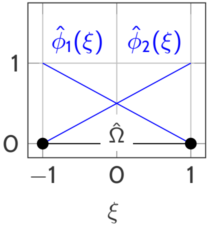

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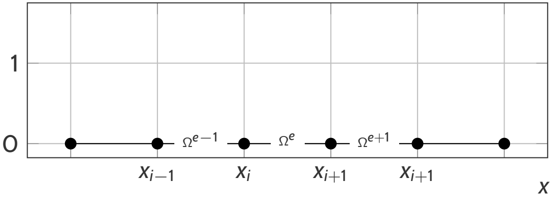
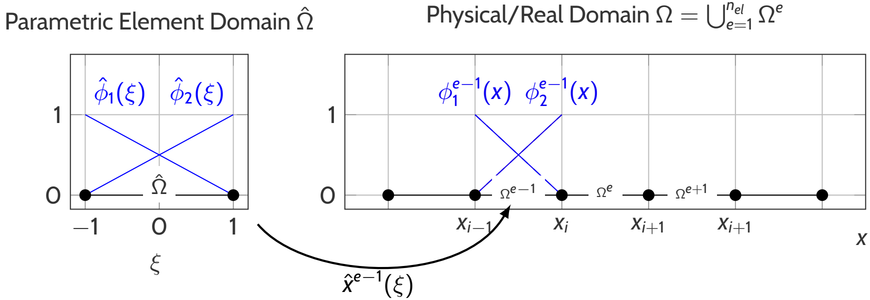


$$\phi_i^e(x) \quad \text{such that} \quad \phi_i^e(\,\hat{x}^e(\xi)\,) = \hat{\phi}_i(\xi)$$

The Element View



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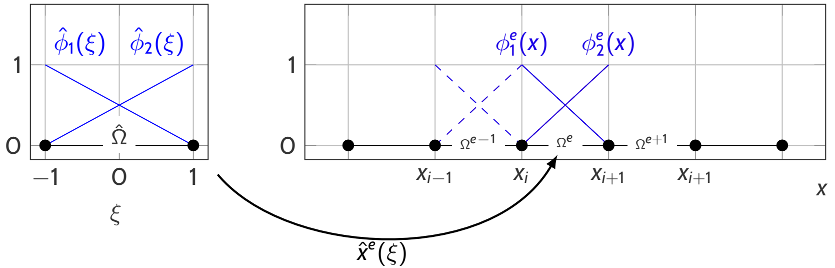
The Element View

Parametric Element Domain $\hat{\Omega}$ Physical/Real Domain $\Omega = \bigcup_{e=1}^{n_{el}} \Omega^e$



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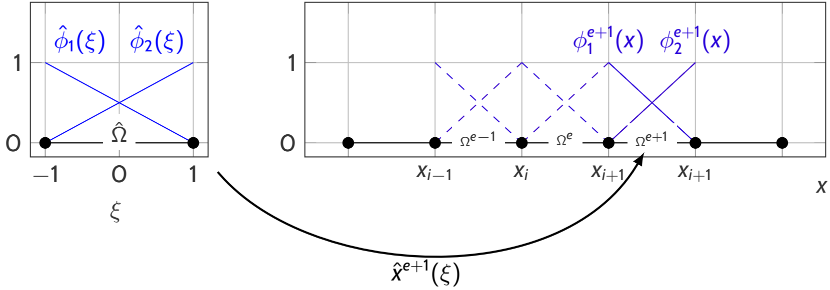

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The Element View

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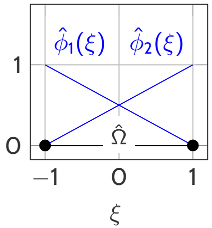

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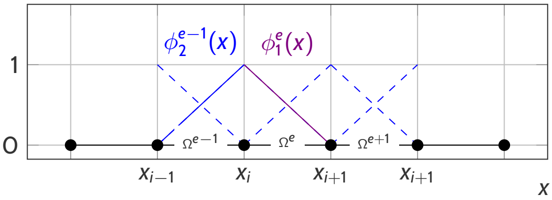
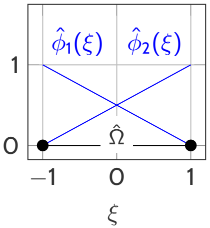


Let $\mathcal{M}$ be the local dof to global dof map

The Element View

Parametric Element Domain $\hat{\Omega}$



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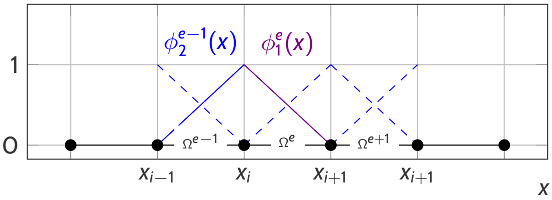
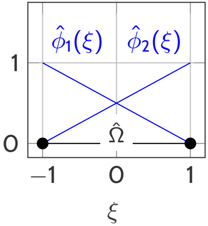


Let $\mathcal{M}$ be the local dof to global dof map : $\quad \mathcal{M}(e, 1) = i$

The Element View

Parametric Element Domain $\hat{\Omega}$



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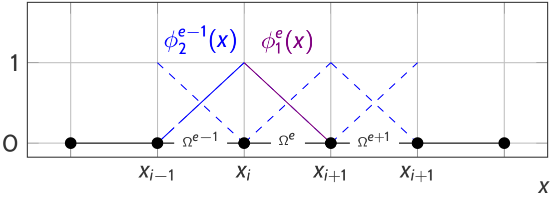
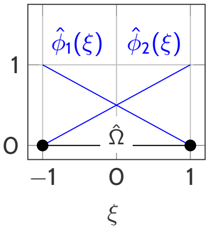


Let $\mathcal{M}$ be the local dof to global dof map : $\quad \mathcal{M}(e, 2) = i + 1$

The Element View

Parametric Element Domain $\hat{\Omega}$



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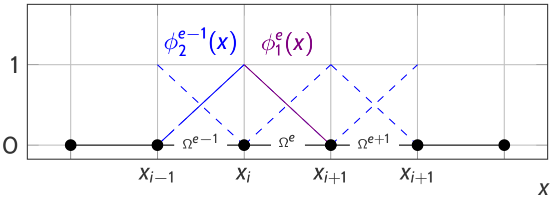

Let $\mathcal{M}$ be the local dof to global dof map

$$\phi_i(x) = \sum_{e=1}^{n_{el}} \sum_{a=1}^{n_{dof}} \begin{cases} \phi_a^e(x) & \text{if } \mathcal{M}(e,a) = i \\ 0 & \text{else} \end{cases}$$

The Element View

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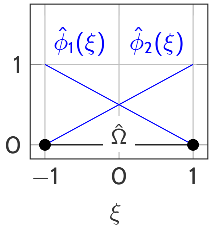

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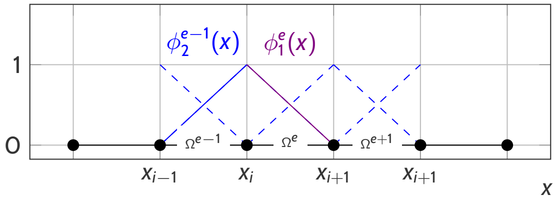
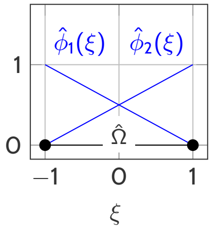


Let $\mathcal{M}$ be the local dof to global dof map

$$\phi_i(x) = \phi_2^{e-1} + \phi_1^{e}$$

The Element View

Parametric Element Domain $\hat{\Omega}$



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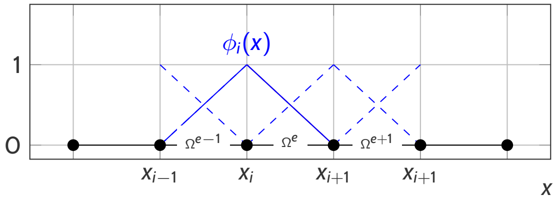

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Isoparametric Mapping

Parametric Element Domain $\hat{\Omega}$

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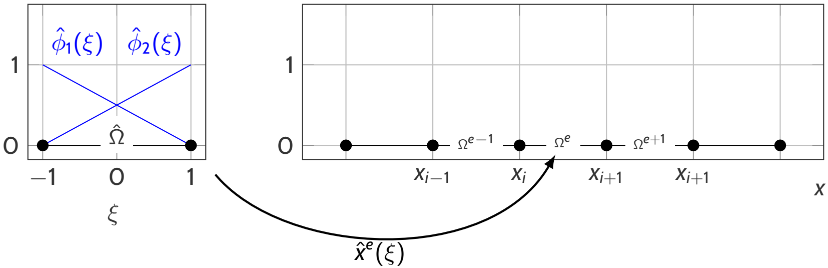
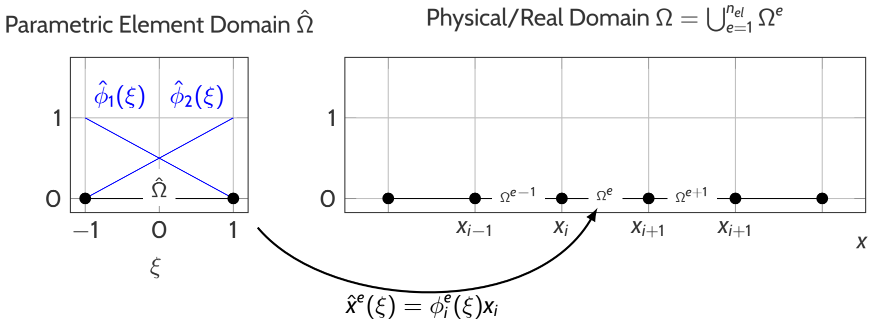


Isoparametric Mapping



Isoparametric Mapping

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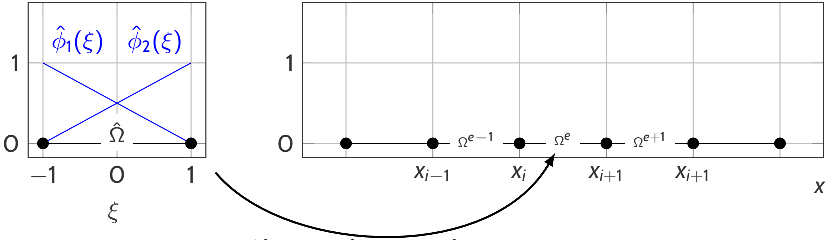

$$\hat{x}^e(\xi) = \phi_1^e(\xi)x_1 + \phi_2^e(\xi)x_2$$

The Element View

$$K_{ij} = a(\phi_i, \phi_j)$$

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$$\phi_i(x) = \sum_{e=1}^{n_{el}} \sum_{a=1}^{n_{dof}} \begin{cases} \phi_a^e(x) & \text{if } \mathcal{M}(e, a) = i \\ 0 & \text{else} \end{cases}$$

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$$K_{ij} = \sum_{e=1}^{n_{el}} a^e(\phi_i, \phi_j) = \sum_{e=1}^{n_{el}} \sum_{a=1}^{n_{dof}} \sum_{b=1}^{n_{dof}} \begin{cases} a^e(\phi_a^e, \phi_b^e) & \text{if } \mathcal{M}(e,a) = i \text{ and if } \mathcal{M}(e,b) = j \\ 0 & \text{else} \end{cases}$$

where [1]

---

[1] $\hat{\xi}(x)$ is such that $\hat{x}(\hat{\xi}(x)) = x$.

The Element View

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where [1]

$$a^e(\phi_a^e, \phi_b^e) = \int_{\Omega^e} \frac{d\phi_a^e}{dx} \frac{d\phi_b^e}{dx} dx = \int_{\hat{\Omega}} \frac{d\hat{\phi}_a}{d\xi} \frac{d\hat{\xi}}{dx} \frac{d\hat{\phi}_b}{d\xi} \frac{d\hat{\xi}}{dx} \frac{d\hat{x}}{d\xi} d\xi = k_{ab}^e$$

[1] $\hat{\xi}(x)$ is such that $\hat{x}(\hat{\xi}(x)) = x$.

The Element View

With

1. Local basis functions $\hat{\phi}(\xi)$
2. Element nodes $\{x_i\}$
3. Local to global map $\mathcal{M}(e, i)$

The Structure of a Finite Element Code

1. Loop over each element
2. Loop over each dof of the element
3. Local to global map $\mathcal{M}(e, i)$