LECTURE 7:

- METRIC CHANGES CONTINUED

EXAMPLES OF METRIC CHANGES

$$\underline{\phi}(\underline{X}) = (X_1 + \alpha X_2^2)\underline{E}_1 + X_2\underline{E}_2$$

EG. (STRETCH RATIO)

$$[\underline{\underline{F}}] = \begin{bmatrix} 1 & 2\alpha X_2 \\ 0 & 1 \end{bmatrix} \Rightarrow [\underline{\underline{C}}] = \begin{bmatrix} 1 & 0 \\ \alpha X_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha X_2 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 2\alpha X_2 \\ 2\alpha X_2 & 1 + 4\alpha X_2^2 \end{bmatrix}$$

THE STRETCH A $\underline{X}_p = 0.5(\underline{E}_1 + \underline{E}_2)$ IN $\underline{E}_1$ & $\underline{E}_2$ & $\underline{E}_1 + \underline{E}_2$

$$\lambda(\underline{E}_1, \underline{X}_p) = \sqrt{\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}^T \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}} = 1 \quad \Leftarrow \text{NO STRETCH} \checkmark$$

$$\lambda(\underline{E}_2, \underline{X}_p) = \sqrt{\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}^T \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}} = 1 + \alpha$$

$$\lambda\left(\frac{\underline{E}_1 + \underline{E}_2}{\sqrt{2}}, \underline{X}_p\right) = \frac{1}{\sqrt{2}} \sqrt{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^T \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}} = \sqrt{\frac{2 + 3\alpha}{2}}$$

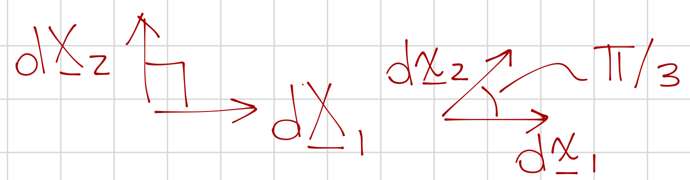
EXAMPLE: (CHANGE IN ANGLE)

$\underline{N}_1 = \underline{E}_1, \quad \underline{N}_2 = \underline{E}_2, \quad \underline{X}_P = 0.5\,(\underline{E}_1 + \underline{E}_2)$

$$\cos\theta = \frac{\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}^T \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}}{1+\alpha} = \frac{\alpha}{1+\alpha}$$

if $\alpha = 1$

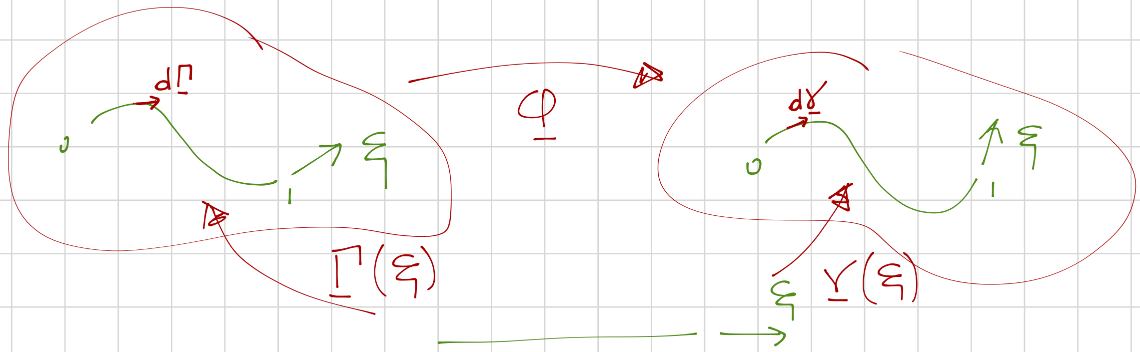
$\theta = \frac{1}{3}\pi = 60^\circ$



DEFORMED CURVES

LET $\underline{\Gamma}(\xi) : [0,1] \to \Omega_0$, $\underline{\gamma}(\xi) = \underline{\varphi}(\underline{\Gamma}(\xi))$

WE ARE INTERESTED IN THE LENGTH OF $\underline{\gamma}(\xi)$



$$\ell(\underline{\gamma}) = \int \|d\underline{\gamma}\| = \int_0^1 \|\underline{\gamma}'(\xi)\| \, d\xi$$

$$\underline{\gamma}'(\xi) = \underline{\underline{F}}(\underline{\Gamma}(\xi))\, \underline{\Gamma}'(\xi)$$

$$\|\underline{\gamma}'(\xi)\|^2 = \underline{\Gamma}'(\xi) \cdot \underline{\underline{C}}(\underline{\Gamma}(\xi))\, \underline{\Gamma}'(\xi)$$

$$\Rightarrow \quad \ell(\underline{\gamma}) = \int_0^1 \left[\underline{\Gamma}'(\xi) \cdot \underline{\underline{C}}(\underline{\Gamma}(\xi))\, \underline{\Gamma}'(\xi)\right]^{1/2} d\xi$$

EXAMPLE

CONSIDER $\underline{\Gamma}(\xi) = \xi\,(\underline{E}_1 + \underline{E}_2)$

$$[\underline{\underline{C}}] = \begin{bmatrix} 1 & 2\alpha X_2 \\ 2\alpha X_2 & 1 + 4\alpha X_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 2\sqrt{2}\alpha\xi \\ 2\sqrt{2}\alpha\xi & 1 + 8\xi^2\alpha \end{bmatrix}$$

$$\underline{\Gamma}'(\xi) = (\underline{E}_1 + \underline{E}_2)$$

$$\ell(\underline{\gamma}) = \int_0^1 \left[2 + 4\sqrt{2}\alpha\xi + 8\xi^2\alpha\right] d\xi$$

$$= \left[2\xi + 2\sqrt{2}\alpha\xi^2 + 8/3\,\xi^3\alpha\right]^{1/2}\Big|_0^1 = \left[2 + \alpha\left(2\sqrt{2} + 8/3\right)\right]^{1/2}$$

NOTE if $\alpha = 0$ $\sqrt{2}$ ✓

DEFORMED VOLUMES

RECALL THAT

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = a_i b_j c_k \varepsilon_{ijk}$$

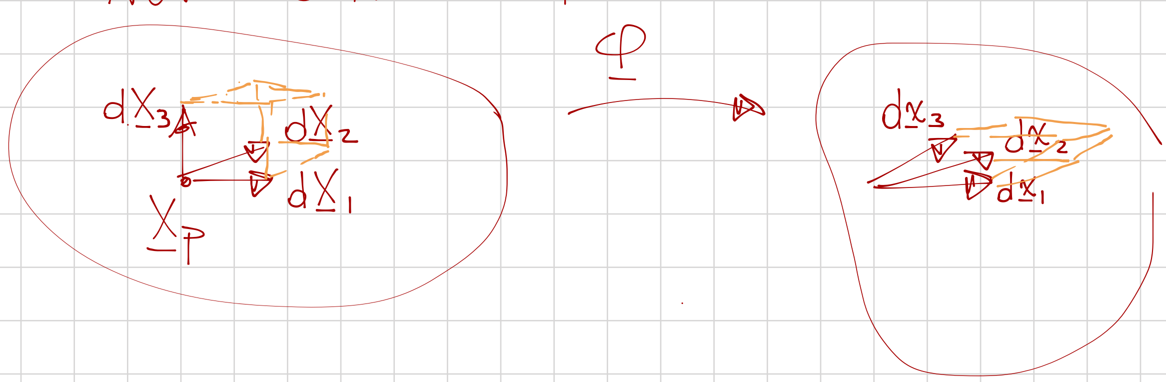
& THAT

$$\text{DET } \underline{\underline{A}} = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$$

OR EQUIVALENTLY

$$\varepsilon_{PQR} \text{ DET } \underline{\underline{A}} = \varepsilon_{ijk} A_{iP} A_{jQ} A_{kR}$$

NOW CONSIDER



$$dV_0 = \left| d\underline{X}_3 \cdot (d\underline{X}_1 \times d\underline{X}_2) \right| = \left| \varepsilon_{IJK} \, dX_{3_I} \, dX_{1_J} \, dX_{2_K} \right|$$

$$dV = \left| d\underline{x}_3 \cdot (d\underline{x}_1 \times d\underline{x}_2) \right| = \left| \varepsilon_{ijk} \, dx_{3_i} \, dx_{1_j} \, dx_{2_k} \right|$$

NOW WITH $d\underline{x}_i = \underline{\underline{F}}(\underline{X}_P) d\underline{X}_i$

$$dV = \left| \underline{\underline{F}} d\underline{X}_3 \cdot (\underline{\underline{F}} d\underline{X}_1 \times \underline{\underline{F}} d\underline{X}_2) \right| =$$

$$= \left| \varepsilon_{ijk} \, F_{iP} \, dX_{3P} \, F_{jQ} \, dX_{1Q} \, F_{kR} \, dX_{2R} \right| =$$

$$= \left| \underbrace{\varepsilon_{ijk} F_{iP} F_{jQ} F_{kR}}_{\varepsilon_{PQR} \text{ DET } \underline{\underline{F}}} \, dX_{3P} \, dX_{1Q} \, dX_{2R} \right| =$$

$$= \left| \text{DET } \underline{\underline{F}} \; \varepsilon_{PQR} \, dX_{3P} \, dX_{1Q} \, dX_{2R} \right| =$$

$$= \underbrace{\text{DET } \underline{\underline{F}}}_{>0} \, dV_0$$

DET $\underline{\underline{F}}$ IS OFTEN DENOTED AS J & TERMED THE JACOBIAN

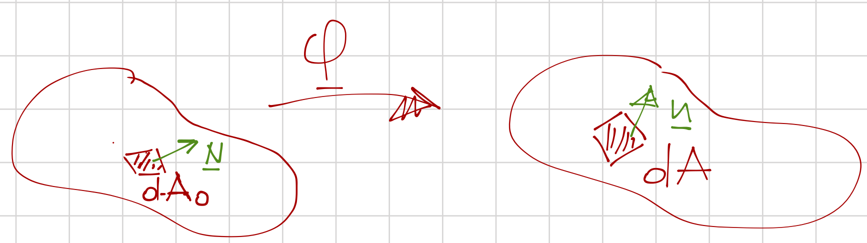
<u>EXAMPLE</u> $\quad \underline{\varphi}(\underline{X}) = \alpha \underline{X}, \quad \Omega_0 = [-1,1]^3$

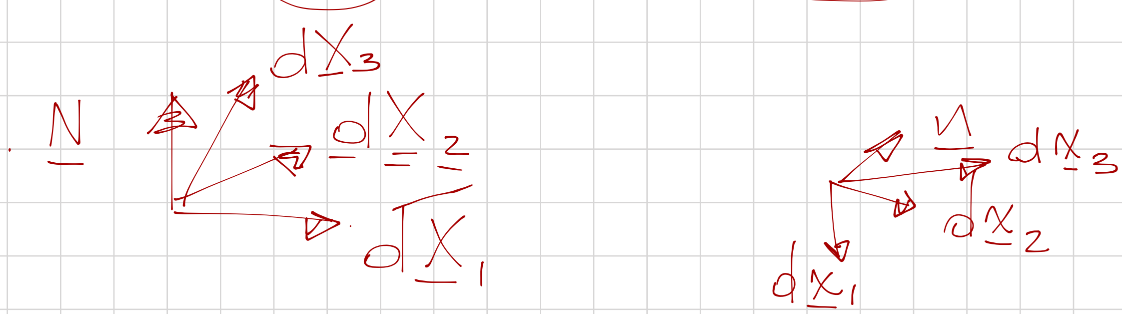
$$\underline{\underline{F}} = \alpha \underline{\underline{1}} \Rightarrow \det(\alpha \underline{\underline{1}}) = \alpha^3$$

$$V = \int_{\Omega} dV = \int_{\Omega_0} J \, dV_0 = \alpha^3 8$$

<u>DEFORMED AREAS</u>

CONSIDER AN AREA ELEMENT IN THE REFERENCE CONFIGURATION dA_0 WITH A UNIT NORMAL $\underline{N}$ AND CORRESPONDING dA & $\underline{n}$





SUCH THAT

$$dA_0 \underline{N} = d\underline{X}_1 \times d\underline{X}_2 \qquad dA \underline{n} = d\underline{x}_1 \times d\underline{x}_2$$

NOW CONSIDER THE VOLUME GIVEN BY

$$dV_0 = d\underline{X}_3 \cdot (dA_0 \underline{N}) = \frac{1}{J} dV = \frac{1}{J} d\underline{x}_3 \cdot (dA \underline{n})$$

$$\Rightarrow J \, dA_0 \, d\underline{X}_3 \cdot \underline{N} - dA \, d\underline{x}_3 \cdot \underline{n} = 0$$

$$\Rightarrow J \, dA_0 \, d\underline{X}_3 \cdot \underline{N} - dA \, \underline{\underline{F}} d\underline{X}_3 \cdot \underline{n} = 0$$

$$\Rightarrow d\underline{X}_3 \cdot (J \, dA_0 \underline{N} - dA \, \underline{\underline{F}}^T \underline{n}) = 0$$

$$\Rightarrow \quad J \, \underline{N} \, dA_0 = dA \, \underline{\underline{F}}^T \underline{n}$$

$$\Rightarrow \quad J \underline{\underline{F}}^{-T} \underline{N} \, dA_0 = dA \, \underline{n}$$

$$\Rightarrow \quad \| J \underline{\underline{F}}^{-T} \underline{N} \| \, dA_0 = dA \qquad (\| \underline{n} \| = 1)$$