LECTURE 2

- TENSORS & TENSOR ALGEBRA
  - DYADIC PRODUCT
  - DOUBLE DOT PRODUCT
  - TRANSPOSE
  - INVERSE
  - SINGULAR
  - EIGENVALUES & VECS
  - POSITIVE (SEMI)-DEFINITE
  - NORM OF TENSOR
  - PROJECTION

A SECOND ORDER TENSOR IS A LINEAR MAP FROM $\mathbb{R}^N \to \mathbb{R}^N$

NAMELY IF $\underline{\underline{A}}$ IS A TENSOR, $\forall \underline{a}, \underline{b} \in \mathbb{R}^N$

$$\underline{\underline{A}}(\alpha \underline{a} + \underline{b}) = \alpha \underline{\underline{A}}(\underline{a}) + \underline{\underline{A}}(\underline{b})$$

AS $\underline{A}$ IS A LINEAR MAP ANY TENSOR $\underline{\underline{A}}$ CAN BE WRITTEN A

$$\underline{\underline{A}} = \underline{a} \otimes \underline{b}$$

WHERE $\otimes$ DENOTES THE DYADIC PRODUCT

THEN

$$\underline{\underline{A}}(\underline{c}) = \underline{a}(\underline{b} \cdot \underline{c})$$

THE DYADIC PRODUCT IS NOT COMMUTATIVE

OFTEN WE DENOTE

$\underline{\underline{A}}(\underline{c})$ BY SIMPLY OMITTING THE PARENTHESIS

$\underline{\underline{A}}\underline{c}$

THE TRANSPOSE OF A TENSOR IS SIMPLY

$$\underline{\underline{A}}^T = (\underline{a} \otimes \underline{b})^T = \underline{b} \otimes \underline{a}$$

SUCH THAT

$$\underline{\underline{A}}^T(\underline{c}) = (\underline{a} \otimes \underline{b})^T \underline{c} = \underline{b}\,(\underline{a} \cdot \underline{c})$$

SINCE $\exists\ \underline{a}, \underline{b} \in \mathbb{R}^N$ SUCH THAT $\underline{\underline{A}} = \underline{a} \otimes \underline{b}$

$$\underline{\underline{A}} = \underline{a} \otimes \underline{b} = (a_i \underline{e}_i \otimes b_j \underline{e}_j) = a_i b_j\ \underline{e}_i \otimes \underline{e}_j = A_{ij}\, \underline{e}_i \otimes \underline{e}_j$$

& $\quad \underline{\underline{A}}^T \overset{?}{=} A_{ij}\, \underline{e}_j \otimes \underline{e}_i$

IF WE USE GENERALIZED BASIS THEN

$$\underline{\underline{A}} = A_{ij}\, \underline{g}^i \otimes \underline{g}^j = A_i{}^j\, \underline{g}^i \otimes \underline{g}_j = A^i{}_j\, \underline{g}_i \otimes \underline{g}^j =$$

$$= A^{ij}\, \underline{g}_i \otimes \underline{g}_j$$

ALGEBRA OF TENSORS

$$+ : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}^{n \times n}$$

$$\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{B}} + \underline{\underline{A}} \qquad \forall\ \underline{\underline{A}}, \underline{\underline{B}} \in \mathbb{R}^{n \times n}$$

$\exists\ \underline{\underline{O}}$ S.T. $\underline{\underline{A}} + \underline{\underline{O}} = \underline{\underline{A}}$

$\exists\ (-\underline{\underline{A}})$ S.T. $\underline{\underline{A}} + (-\underline{\underline{A}}) = \underline{\underline{O}}$

$$\underline{\underline{A}}\,\underline{\underline{B}} \Rightarrow (\underline{\underline{A}}\,\underline{\underline{B}})(\underline{a}) = \underline{\underline{A}}(\underline{\underline{B}}\,\underline{a})$$

WHERE

$$\underline{\underline{A}}\,\underline{\underline{B}} = (A_{ij}\, \underline{e}_i \otimes \underline{e}_j)(B_{kl}\, \underline{e}_k \otimes \underline{e}_l) =$$

$$= A_{ij} B_{kl}\ \underline{e}_j \cdot \underline{e}_k\ \ \underline{e}_i \otimes \underline{e}_l$$

$$= A_{ij} B_{jl}\ \ \underline{e}_i \otimes \underline{e}_l$$

REMEMBER $\underline{e}_j \cdot \underline{e}_k = \delta_{jk}$ & $\quad \underline{a}_i\, \delta_{il} = \underline{a}_l$

NOTE

$$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T = A_{ij} B_{jl}\, \underline{e}_l \otimes \underline{e}_i$$

WE ALSO DEFINE THE DOUBLE DOT PRODUCT

$$\underline{\underline{A}} : \underline{\underline{B}} = A_{ij} B_{kl} (\underline{e}_i \otimes \underline{e}_j) : (\underline{e}_k \otimes \underline{e}_l)$$

WE DEFINE THE IDENTY TENSOR

$\underline{\underline{I}}$ S.T. $\underline{\underline{I}}\,\underline{u} = \underline{u}$

HENCE $\underline{\underline{I}} = \delta_{ij}\, \underline{e}_i \otimes \underline{e}_j$

$$\underline{\underline{I}}\,\underline{u} = (\delta_{ij}\, \underline{e}_i \otimes \underline{e}_j)(u_k \underline{e}_k) = \delta_{ij}\, \delta_{jk}\, u_k\, \underline{e}_i = \delta_{ik} u_k \underline{e}_i = u_i \underline{e}_i$$

TRACE OF A TENSOR

$$tr(\underline{\underline{A}}) := \underline{\underline{A}} : \underline{\underline{I}} = A_{ii}$$

DETERMINANT

$$\det(\underline{\underline{A}}) = \det [A_{ij}]_{\underline{e}_i}$$

IN A SET OF ORTHONORMAL BASIS

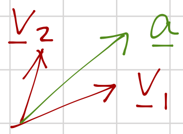
EIGENVALUES & EIGENVECTORS

FOR A TENSOR $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ THE EIGENVALUES λ_i & EIGENVECTORS $\underline{v}_i$ $i = 1 \ldots n$ SATISFY

$$\underline{\underline{A}}\, \underline{v}_i = \lambda_{(i)} \underline{v}_{(i)}$$

NO SUMMATION

INTUITIVE INTERPRETATION



$\underline{a} = a_1 \underline{v}_1 + a_2 \underline{v}_2$

$\underline{\underline{A}}\,\underline{a} = \lambda_1 a_1 \underline{v}_1 + \lambda_2 a_2 \underline{v}_2$

IF $\underline{\underline{A}}$ IS SYMMETRIC THEN THE EIGENVECTORS ARE ORTHOGONAL

IF $\lambda_i > 0 \Rightarrow \underline{\underline{A}}$ POSITIVE DEFINITE

$\lambda_i \geq 0 \Rightarrow \underline{\underline{A}}$ " SEMI-DEF

IF $\underline{\underline{A}}$ IS SYMMETRIC $\Rightarrow$ HAS ORTHOGONAL EIGENVECTORS

$$\underline{\underline{A}} = \sum_{i=1}^{n} \lambda_i \underline{v}_i \otimes \underline{v}_i \Leftarrow \text{SPECTRAL DECOMPOSITION}$$

$$[\underline{\underline{A}}]_{\underline{v}_i} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$$

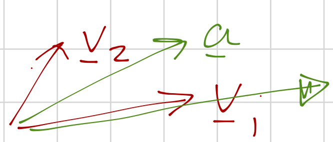
NOTE $\text{DET}(\underline{\underline{A}}) = \lambda_1 \lambda_2 \lambda_3$

INVERSE OF A TENSOR

IF $|\lambda_i| > 0$ THEN $\exists\, \underline{\underline{A}}^{-1}$ SUCH THAT

$$\underbrace{\underline{\underline{A}}^{-1}\underline{\underline{A}}}_{\circ} = \underline{\underline{I}}$$

INTUITIVELY if $\lambda_i = 0$



$\lambda_1 \underline{a} \Leftarrow$ A PROJECTION

$(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$ $\qquad (\alpha \underline{\underline{A}})^{-1} = \frac{1}{\alpha}\underline{\underline{A}}^{-1}$

$\text{DET}(\underline{\underline{A}}^{-1}) = \text{DET}(\underline{\underline{A}})^{-1}$

SYMMETRIC & SKEW SYMMETRIC PART

A IS SYMMETRIC IF $\underline{\underline{A}}^T = \underline{\underline{A}}$

$$\underline{\underline{A}}^{SYM} = \frac{1}{2}(\underline{\underline{A}} + \underline{\underline{A}}^T)$$

$$\underline{\underline{A}}^{SKEW} = \underline{\underline{A}} - \underline{\underline{A}}^{SYM} = \frac{1}{2}(\underline{\underline{A}} - \underline{\underline{A}}^T)$$

ORTHOGONAL TENSOR

$$\underline{\underline{Q}}\,\underline{u} \cdot \underline{\underline{Q}}\,\underline{v} = \underline{u} \cdot \underline{v}$$

$$\underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{I}} \Rightarrow \underline{\underline{Q}}^{-1} = \underline{\underline{Q}}^T$$

NOTE $\det(\underline{\underline{Q}}\,\underline{\underline{Q}}^T) = \det(\underline{\underline{Q}})\det(\underline{\underline{Q}}^T)$

$= \det(\underline{\underline{Q}})^2 = \det(\underline{\underline{I}}) = 1 \Rightarrow \det(\underline{\underline{Q}}) = 1$

SPHERICAL & DEVIATORIC

$$\underline{\underline{A}}^{SPH} = \alpha(\underline{\underline{A}})\,\underline{\underline{I}}, \quad \alpha = tr(\underline{\underline{A}})/n$$

$$\underline{\underline{A}}^{DEV} = \underline{\underline{A}} - \underline{\underline{A}}^{SPH}$$

AVERAGE STRETCH

$$tr(\underline{\underline{A}}^{DEV}) = tr(\underline{\underline{A}} - \alpha\underline{\underline{I}}) = tr(\underline{\underline{A}}) - n\,tr\left(\frac{\underline{\underline{A}}}{n}\right) = 0$$

TENSOR INVARIANTS

TENSOR INVARIANTS ARE PROPERTIES OF TENSORS THAT REMAIN INVARIANT UNDER CHANGES OF OBSERVER

$$I(\underline{\underline{A}}) = I(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T)$$

THREE IMPORTANT INVARIANTS ARE

$$I_1(\underline{\underline{A}}) = tr(\underline{\underline{A}}) = tr(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T : \underline{\underline{I}} = \underline{\underline{A}}\,\underline{\underline{Q}} : \underline{\underline{Q}}^T$$

$$= \underline{\underline{A}} : \underline{\underline{Q}}^T\underline{\underline{Q}} = \underline{\underline{A}} : \underline{\underline{I}} \quad (Q_{ij} A_{jk} Q_{ek} \delta_{ie} = A_{jk} Q_{ik} Q_{ij})$$

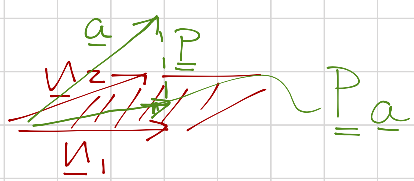
$$= \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2(\underline{\underline{A}}) = \frac{1}{2}\left[\mathrm{tr}(\underline{\underline{A}})^2 - \mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{A}})\right] = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

$$I_3(\underline{\underline{A}}) = \det(\underline{\underline{A}}) = \lambda_1 + \lambda_2 + \lambda_3$$

PROJECTION TENSOR

A PROJECTION TENSOR TAKES A VECTOR AND PROJECTS IT ONTO A HYPEPLANE (OR A LINE)



W $\underline{n}_1$ & $\underline{n}_2$ ORTHONORMAL

$$\underline{\underline{P}}\,\underline{a} = (\underline{a}\cdot\underline{n}_1)\,\underline{n}_1 + (\underline{a}\cdot\underline{n}_2)\,\underline{n}_2$$

$$\Rightarrow \underline{\underline{P}} = \underline{n}_1 \otimes \underline{n}_1 (+\ \underline{n}_2 \otimes \underline{n}_2)$$

A PROJECTION IS ALWAYS SYMMETRIC

$$\underline{\underline{P}}\,\underline{\underline{P}} = \underline{\underline{P}}$$

AN IT'S SINGULAR, WHY ?

WHAT ARE THE EIGENVAWES OF $\underline{\underline{P}}$?

HIGHER ORDER TENSORS

HIGHER ORDER TENSORS CAN BE EXPRESSED AS

$$\underline{\underline{\underline{C}}} = C_{ijk\ldots q}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \ldots \underline{e}$$

HIGHER ORDER TENSORS FOLLOW VIRTUALLY ALL THE SAME RULES OF 2nd ORDER

$$\underline{\underline{\underline{\mathbb{A}}}} = - \_\_$$

$$\mathbb{B} = - \_\_$$

$$\mathbb{C} = C_{ijkl}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l$$

$$\mathbb{A}\mathbb{B} = A_{ijkl} B_{mnop} (\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l)(\underline{e}_m \otimes \underline{e}_n \otimes \underline{e}_o \otimes \underline{e}_p)$$

$$= A_{ijkl} B_{lnop}$$

$$\mathbb{A} : \mathbb{B} = A_{ijkl} B_{klop}$$

$$\mathbb{A}^T = C_{ijkl}\, \underline{e}_l \otimes \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k$$

NOTE: DIFFERENT DEF IN HOLZAPFEL

$$\mathbb{I} : \underline{A} = \underline{A} \Rightarrow \mathbb{I} = \delta_{ik}\delta_{jl}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l$$

$$\mathbb{I}^{SYM} : \underline{A} = \underline{A}^{SYM} \Rightarrow \mathbb{I}^{SYM} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

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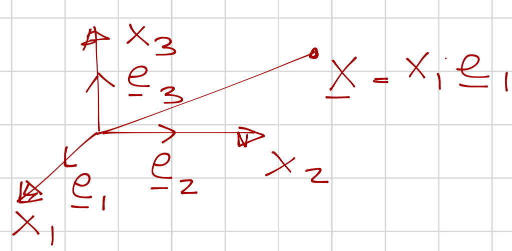
VECTOR & TENSOR CALCULUS

IF WE HAVE A SCALAR FIELD AS A FUNCTION OF SPACE

$$\phi : \mathbb{R}^n \to \mathbb{R}$$

$$\phi(\underline{x}), \quad x$$



THEN WE DEFINE THE GRADIENT OF A SCALAR

$$\underline{\nabla} \phi(\underline{x}) = \frac{d\phi}{dx_i} \underline{e}_i$$ ← NOTE THIS IS A VECTOR

IT'S A VECTOR THAT POINT TOWARD THE DIRECTION OF MAXIMUM INCREASE OF ϕ WHOSE MAGNITUDE IS THE

FOR VECTOR & TENSOR FIELDS WE MORE BROADLY DEFINE

$$\underline{\nabla}(\circ) = \frac{d(\circ)}{dx_i} \otimes \underline{e}_i \quad \Leftarrow \text{THE GRADIENT}$$

$$\underline{a}(\underline{x}) : \mathbb{R}^n \to \mathbb{R}^n$$

$$\underline{\nabla}\,\underline{a} = \frac{d(a_i \underline{e}_i)}{dx_j} \otimes \underline{e}_j = \frac{d a_i}{dx_j}\, \underline{e}_i \otimes \underline{e}_j$$

$$\underline{\nabla}\,\underline{\underline{A}} = \frac{d A_{ij}}{dx_k}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k$$

NOTE: FLUID MECH REVERSED

$$\underline{\nabla} \cdot (\circ) = \frac{d(\cdot)}{dx_i} \cdot \underline{e}_i \quad \Leftarrow \text{DIVERGENCE}$$

THE DIVERGE IS A MEASURE OF FLUX

$$\underline{\nabla} \cdot \underline{a} = \frac{d(a_i \underline{e}_i)}{dx_j} \cdot \underline{e}_j = \frac{d a_i}{dx_i} = \underline{\nabla}\,\underline{a} : \underline{\underline{1}}$$

$$\underline{\nabla} \cdot \underline{\underline{A}} = \frac{d A_{ij}}{dx_j}\, \underline{e}_i$$

IF $\underline{\nabla} \cdot \underline{a} = 0$ $\quad \underline{a}$ IS SAID TO BE SOLEINOIDAL (VOLUME PRESERVING VECTOR FIELD

$$\underline{\nabla} \times (\circ) = \underline{e}_j \times \frac{d(\circ)}{dx_j} \quad \Leftarrow \text{CURL}$$

MEASURE OF ROTATION

RETURNS A VECTOR NORMAL TO THE PLANE OF MAX ROTATION

$$\underline{\nabla} \times \underline{a} = \frac{d a_i}{dx_j}\, \varepsilon_{ijk}\, \underline{e}_k$$

$$\underline{\nabla} \times \underline{\underline{A}}$$

IF $\underline{\nabla} \times \underline{a} \Rightarrow \underline{a}$ IRROTATIONAL

$$\underline{\nabla} \times \underline{\nabla} \phi = 0$$

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{u} = 0$$