Direct Stiffness Methods

Beam-Columns

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Frame Elements

Frame Elements

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} k_{fv} & k_{f\theta} & -k_{fv} & k_{f\theta} \\ k_{mv} & k_{m\theta} & -k_{mv} & \hat{k}_{m\theta} \\ -k_{fv} & -k_{f\theta} & k_{fv} & -k_{f\theta} \\ k_{mv} & \hat{k}_{m\theta} & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

$$\begin{Bmatrix} P_i \\ P_j \end{Bmatrix} = \begin{bmatrix} k_p & -k_p \\ -k_p & k_p \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

where

$$k_{fv} = \frac{12EI}{\ell^3}, \quad k_{mv} = k_{f\theta} = \frac{6EI}{\ell^2}, \quad k_{m\theta} = \frac{4EI}{\ell}, \quad \hat{k}_{m\theta} \frac{2EI}{\ell}, \quad k_p = \frac{AE}{\ell}.$$

Frame Elements

$$\begin{Bmatrix} P_i \\ V_i \\ M_i \\ P_j \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} k_p & 0 & 0 & -k_p & 0 & 0 \\ 0 & k_{fv} & k_{f\theta} & 0 & -k_{fv} & k_{f\theta} \\ 0 & k_{mv} & k_{m\theta} & 0 & -k_{mv} & \hat{k}_{m\theta} \\ -k_p & 0 & 0 & k_p & 0 & 0 \\ 0 & -k_{fv} & -k_{f\theta} & 0 & k_{fv} & -k_{f\theta} \\ 0 & k_{mv} & \hat{k}_{m\theta} & 0 & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ \theta_i \\ u_j \\ w_j \\ \theta_j \end{Bmatrix}$$

Frame Elements

$$\begin{Bmatrix} \boldsymbol{F}_i \\ M_i \\ \boldsymbol{F}_j \\ M_j \end{Bmatrix} = \begin{bmatrix} \boldsymbol{K}_{\boldsymbol{fw}} & \boldsymbol{k}_{\boldsymbol{f}\theta} & -\boldsymbol{K}_{\boldsymbol{fw}} & \boldsymbol{k}_{\boldsymbol{f}\theta} \\ \boldsymbol{k}_{mw} & k_{m\theta} & -\boldsymbol{k}_{mw} & \hat{k}_{m\theta} \\ -\boldsymbol{K}_{\boldsymbol{fw}} & -\boldsymbol{k}_{\boldsymbol{f}\theta} & \boldsymbol{K}_{\boldsymbol{fw}} & -\boldsymbol{k}_{\boldsymbol{f}\theta} \\ \boldsymbol{k}_{mw} & \hat{k}_{m\theta} & -\boldsymbol{k}_{mw} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} \boldsymbol{w}_i \\ \theta_i \\ \boldsymbol{w}_j \\ \theta_j \end{Bmatrix}$$

Where

$\boldsymbol{F}_{i,j} \in \mathbb{R}^2$ is a vector

$\boldsymbol{w}_{i,j} \in \mathbb{R}^2$ is a vector

$M_{i,j} \in \mathbb{R}$ is a scalar

$\theta_{i,j} \in \mathbb{R}$ is a scalar

$\boldsymbol{K}_{\boldsymbol{fw}} \in \mathbb{R}^2 \times \mathbb{R}^2$ is a tensor

$\boldsymbol{k}_{\boldsymbol{f}\theta} \in \mathbb{R}^2$ is a vector

$\boldsymbol{k}_{Mw} \in \mathbb{R}^2$ is a vector

$k_{m\theta}, \hat{k}_{m\theta} \in \mathbb{R}$ is a scalar

$$\boldsymbol{K}_{\boldsymbol{fw}} = \frac{AE}{\ell}\boldsymbol{n} \otimes \boldsymbol{n} + \frac{12EI}{\ell^3}\boldsymbol{s} \otimes \boldsymbol{s}, \quad k_{m\theta} = \frac{4EI}{\ell}, \quad \hat{k}_{m\theta} = \frac{2EI}{\ell}, \quad \boldsymbol{k}_{mw} = \boldsymbol{k}_{\boldsymbol{f}\theta} = \frac{6EI}{\ell^2}\boldsymbol{s}$$

Frame Elements

$$\boldsymbol{F}_i = \boldsymbol{K}_{\boldsymbol{fw}} \boldsymbol{w}_i + \boldsymbol{k}_{\boldsymbol{f}\theta}\, \theta_i - \boldsymbol{K}_{\boldsymbol{fw}} \boldsymbol{w}_i + \boldsymbol{k}_{\boldsymbol{f}\theta} \theta_j$$ Vector equation!

$$M_i = \boldsymbol{k}_{mw} \cdot \boldsymbol{w}_i + k_{m\theta}\, \theta_i - \boldsymbol{k}_{mw} \cdot \boldsymbol{w}_i + \hat{k}_{m\theta} \theta_j$$ Scalar equation!

$$\boldsymbol{F}_j = -\boldsymbol{K}_{\boldsymbol{fw}} \boldsymbol{w}_i - \boldsymbol{k}_{\boldsymbol{f}\theta}\, \theta_i + \boldsymbol{K}_{\boldsymbol{fw}} \boldsymbol{w}_i - \boldsymbol{k}_{\boldsymbol{f}\theta} \theta_j$$ Vector equation!

$$M_j = \boldsymbol{k}_{mw} \cdot \boldsymbol{w}_i + \hat{k}_{m\theta}\, \theta_i + -\boldsymbol{k}_{mw} \cdot \boldsymbol{w}_i + k_{m\theta} \theta_j$$ Scalar equation!

Frame Elements

$$\begin{Bmatrix} [\boldsymbol{F}_i] \\ M_i \\ [\boldsymbol{F}_j] \\ M_j \end{Bmatrix} = \begin{bmatrix} [\boldsymbol{K}_{\boldsymbol{fw}}] & [\boldsymbol{k}_{\boldsymbol{f}\theta}] & -[\boldsymbol{K}_{\boldsymbol{fw}}] & [\boldsymbol{k}_{\boldsymbol{f}\theta}] \\ [\boldsymbol{k}_{mw}]^\top & k_{m\theta} & -[\boldsymbol{k}_{mw}]^\top & \hat{k}_{m\theta} \\ -[\boldsymbol{K}_{\boldsymbol{fw}}] & -[\boldsymbol{k}_{\boldsymbol{f}\theta}] & [\boldsymbol{K}_{\boldsymbol{fw}}] & -[\boldsymbol{k}_{\boldsymbol{f}\theta}] \\ [\boldsymbol{k}_{mw}]^\top & \hat{k}_{m\theta} & -[\boldsymbol{k}_{mw}]^\top & k_{m\theta} \end{bmatrix} \begin{Bmatrix} [\boldsymbol{w}_i] \\ \theta_i \\ [\boldsymbol{w}_j] \\ \theta_j \end{Bmatrix}$$

Frame Elements

$$\left\{\begin{array}{c}[\boldsymbol{F}_i]_{2\times 1}\\ M_i\\ [\boldsymbol{F}_j]_{2\times 1}\\ M_j\end{array}\right\} = \left[\begin{array}{cccc}[\boldsymbol{K}_{\boldsymbol{fw}}]_{2\times 2} & [\boldsymbol{k}_{\boldsymbol{f}\theta}]_{2\times 1} & -[\boldsymbol{K}_{\boldsymbol{fw}}]_{2\times 2} & [\boldsymbol{k}_{\boldsymbol{f}\theta}]_{2\times 1}\\ [\boldsymbol{k}_{mw}]^{\top}_{1\times 2} & k_{m\theta} & -[\boldsymbol{k}_{mw}]^{\top}_{1\times 2} & \hat{k}_{m\theta}\\ -[\boldsymbol{K}_{\boldsymbol{fw}}]_{2\times 2} & -[\boldsymbol{k}_{\boldsymbol{f}\theta}]_{2\times 1} & [\boldsymbol{K}_{\boldsymbol{fw}}]_{2\times 2} & -[\boldsymbol{k}_{\boldsymbol{f}\theta}]_{2\times 1}\\ [\boldsymbol{k}_{mw}]^{\top}_{1\times 2} & \hat{k}_{m\theta} & -[\boldsymbol{k}_{mw}]^{\top}_{1\times 2} & k_{m\theta}\end{array}\right] \left\{\begin{array}{c}[\boldsymbol{w}_i]_{2\times 1}\\ \theta_i\\ [\boldsymbol{w}_j]_{2\times 1}\\ \theta_j\end{array}\right\}$$