HOMEWORK 4

CEE 530: Continuum Mechanics and Thermodynamics

Due: April 30, 2018

PROBLEM 1

Assume a free energy function having the general form $\psi = \hat{\psi}(a, b, c, d)$ and a local Dissipation function of the form $\rho\mathcal{D}_{loc} = A\dot{a} - \rho\dot{\psi} - \rho B\dot{b} \geq 0$.

1. (10) Apply Coleman's Exploitation Method to obtain the constitutive equations for the variables A and B for a non-dissipative process.
2. (10) Based on the previous result, which variables from a to d can be excluded from the constitutive equation $\hat{\psi}(a, b, c, d)$?

1. The local dissipation inequality for a non-dissipative process becomes

$$A\dot{a} - \rho\dot{\psi}(a, b, c, d) - \rho B\dot{b} = 0$$

which must hold true for all possible choices of $\dot{a}, \dot{b}, \dot{c}\ \dot{d}$. Differentiating gives

$$\left(A - \rho\frac{\partial\psi}{\partial a}\right)\dot{a} - \rho\left(B + \frac{\partial\psi}{\partial b}\right)\dot{b} + \frac{\partial\psi}{\partial c}\dot{c} + \frac{\partial\psi}{\partial d}\dot{d} = 0$$

which thus implies

$$A = \rho\frac{\partial\psi}{\partial a} \tag{1}$$

$$B = -\frac{\partial\psi}{\partial b} \tag{2}$$

$$\frac{\partial\psi}{\partial c} = 0 \tag{3}$$

$$\frac{\partial\psi}{\partial d} = 0 \tag{4}$$

2. which in turns implies $\psi = \hat{\psi}(a, b)$.

PROBLEM 2

Consider a one dimensional material whose Helmholtz free energy is given by

$$\rho\hat{\psi}(\varepsilon, \theta) = \frac{1}{2}E\varepsilon^2 - E\alpha_t\varepsilon(\theta - \theta_0) + \rho c\left[(\theta - \theta_0) - \theta\ln\frac{\theta}{\theta_0}\right]$$

with $\varepsilon = \partial_x\varphi$ and

$$q(\partial_x\theta) = -\kappa\partial_x\theta, \quad \kappa > 0.$$

1. (10) Obtain in detail the explicit expression of the entropy.
2. (10) Obtain in detail the expression for stress.
3. (10) What restrictions do we have on E, k, c, α_t?

1.

$$\eta = -\frac{\partial \psi}{\partial \theta} = \frac{E\alpha_t \varepsilon}{\rho} + c \ln \frac{\theta}{\theta_0}$$

2.

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi}{\partial \epsilon} = E\varepsilon = E\alpha_t(\theta - \theta_0).$$

3.

$$k > 0$$

PROBLEM 3

Consider the Helmholtz free energy for a Neo-Hookean material

$$\psi(\boldsymbol{F}, \theta) = \frac{\mu(\theta)}{2} \boldsymbol{F} : \boldsymbol{F} + \frac{\lambda(\theta)}{2} (\log J)^2 - \mu(\theta) \log J$$

where μ and λ are material parameters that depend on the temperature.

1. (10) Obtain in detail the explicit expression of the entropy.
2. (10) Obtain in detail the expression for stress (first Piola-Kirhoff).

1.

$$\eta = -\rho_0 \frac{\partial \psi}{\partial \theta} = \frac{\mu'(\theta)}{2} \boldsymbol{F} : \boldsymbol{F} + \frac{\lambda'(\theta)}{2} (\log J)^2 - \mu'(\theta) \log J$$

2.

$$\boldsymbol{P} = \rho_0 \frac{\partial \psi}{\partial \boldsymbol{F}} = \rho_0 \mu(\theta)(\boldsymbol{F} - \boldsymbol{F}^{-\top}) + \rho_0 \lambda(\theta) \log J \boldsymbol{F}^{-\top}$$