HOMEWORK 3

CEE 530: Continuum Mechanics and Thermodynamics

Due: April 16, 2018

PROBLEM 1

Consider a continuum, such as a crystalline solid, in which a concentration of contaminants is diffusing. Assume that the flux per unit area j of contaminants in direction $\boldsymbol{n}$ at a point $\boldsymbol{x}$ is only a function of these two variables and time, i.e., $j = j(\boldsymbol{x}, \boldsymbol{n}, t)$. Assume furthermore that there exists a source that releases a concentration $r(\boldsymbol{x}, t)$ of contaminants per unit time. Denote with $\rho_c(\boldsymbol{x}, t)$ the concentration (or mass density) of contaminants at a point $\boldsymbol{x}$ at time t.

1. (15) Write down an integral balance statement of contaminants.
2. (15) Following the proof of Cauchy's tetrahedron theorem, and using the integral balance statement, obtain that the dependence of j on $\boldsymbol{n}$ at a point can only be linear, i.e., there exists a vector field $\boldsymbol{J}(\boldsymbol{x}, t)$ such that
$$j(\boldsymbol{x}, \boldsymbol{n}, t) = \boldsymbol{J}(\boldsymbol{x}, t) \cdot \boldsymbol{n}$$
3. (10) Obtain the local statement.

PROBLEM 2

A body made out of a radioactive material is undergoing motion.

1. (10) The radioactive decay makes the body lose mass at a mass fraction rate of γ/s (This mass is converted into energy). Modify the equation of conservation of mass to take this into account, and obtain the expressions in the local and integral forms, both for the spatial and material configurations.
2. (10) The radioactive decay of the material makes it transform from its original form, material 0, into a different one, material 1. So, in addition to the mass loss, a mass fraction λ of material 0 transforms into material 1 per unit time. Write down the equations governing the evolution of the mass densities of materials 0 and 1, respectively. Provide integral and local, spatial and material forms.

PROBLEM 3

A fluid of density ρ is in hydrostatic equilibrium under the action of gravity in a container. The vertical direction is taken to coincide with the x_3-axis.

1. (10) The state of stress of the fluid is everywhere of the form $\boldsymbol{\sigma}(\boldsymbol{x}) = p(\boldsymbol{x})\mathbf{1}$. Find $p(\boldsymbol{x})$. Does it depend on the shape of the container?
2. (10) Prove mathematically Archimedes' principle: The buoyant force exerted by a fluid on an object is equal to the weight of the fluid displaced by the object.
3. (10) Assume the container is a cylinder of height L and radius R with the base of the cylinder against the ground and open at the top. The fluid is in contact with the atmosphere there, and the atmospheric pressure is p_{atm}. If we make an imaginary cut of the cylinder into two identical parts with a plane containing the cylinder axis, what is the total force exerted by the fluid on each half of the container?
4. (10) What is then the resultant of the traction on the walls of the container cut by the imaginary plane? The outer lateral walls of the cylinder are exposed to the nearly constant atmospheric pressure p_{atm}.

PROBLEM 4

Obtain the Navier?Stokes equations, which in Cartesian coordinates read

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \nabla \boldsymbol{v}\, \boldsymbol{v} = -\nabla p + \mu \Delta \boldsymbol{v}$$

which states the linear momentum balance for a viscous, incompressible fluid. To this end, use the fact that for a viscous fluid the Cauchy stress $\boldsymbol{\sigma}$ is related to the rate of deformation tensor $\boldsymbol{d} = (\nabla \boldsymbol{v} + \nabla \boldsymbol{v})/2$ as $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu \boldsymbol{d}$.