Frame Elements

Frame Elements

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} k_{fv} & k_{f\theta} & -k_{fv} & k_{f\theta} \\ k_{mv} & k_{m\theta} & -k_{mv} & \hat{k}_{m\theta} \\ -k_{fv} & -k_{f\theta} & k_{fv} & -k_{f\theta} \\ k_{mv} & \hat{k}_{m\theta} & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}$$

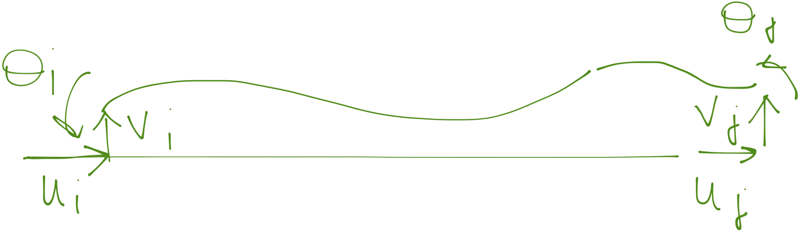
$$\begin{Bmatrix} P_i \\ P_j \end{Bmatrix} = \begin{bmatrix} k_p & -k_p \\ -k_p & k_p \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

where

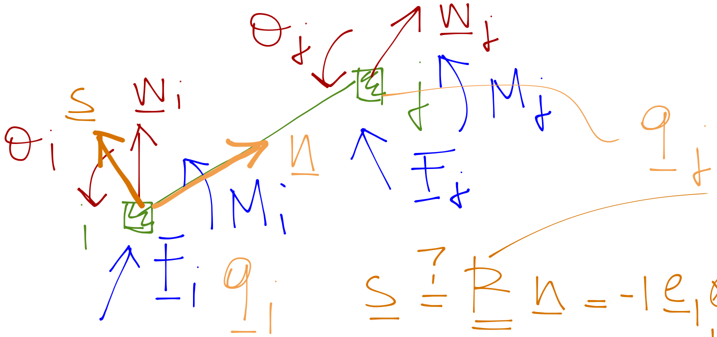
$$k_{fv} = \frac{12EI}{\ell^3}, \quad k_{mv} = k_{f\theta} = \frac{6EI}{\ell^2}, \quad k_{m\theta} = \frac{4EI}{\ell}, \quad \hat{k}_{m\theta} \frac{2EI}{\ell}, \quad k_p = \frac{AE}{\ell}.$$

Frame Elements

$$
\begin{Bmatrix} P_i \\ V_i \\ M_i \\ P_j \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} k_p & 0 & 0 & -k_p & 0 & 0 \\ 0 & k_{fv} & k_{f\theta} & 0 & -k_{fv} & k_{f\theta} \\ 0 & k_{mv} & k_{m\theta} & 0 & -k_{mv} & \hat{k}_{m\theta} \\ -k_p & 0 & 0 & k_p & 0 & 0 \\ 0 & -k_{fv} & -k_{f\theta} & 0 & k_{fv} & -k_{f\theta} \\ 0 & k_{mv} & \hat{k}_{m\theta} & 0 & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{Bmatrix}
$$



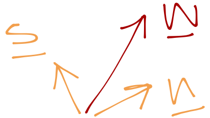
Frame Elements



$$[\underline{\underline{R}}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\underline{s} \overset{?}{=} \underline{\underline{R}}\,\underline{n} = -1\,\underline{e}_1 \otimes \underline{e}_2 + 1\,\underline{e}_2 \otimes \underline{e}_1$$

$$\ell = |\underline{q}_j - \underline{q}_i|, \quad \underline{n} = \frac{\underline{q}_j - \underline{q}_i}{\ell}$$



$$\underline{w} = \underline{w} \cdot \underline{n}\,\underline{n} + \underline{w} \cdot \underline{s}\,\underline{s}$$

Frame Elements

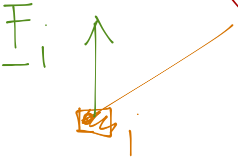
$$u_{i,j} = \underline{w}_{i,j} \cdot \underline{n}$$

$$v_{i,j} = \underline{w}_{i,j} \cdot \underline{s}$$

$$\underline{w}_i = u_i \underline{n} + v_i \underline{s}$$

$$\underline{w}_j = \underbrace{u_j \underline{n}}_{\underline{u}_j = (\underline{n} \otimes \underline{n}) \underline{w}_j} + \underbrace{v_j \underline{s}}_{\underline{v}_j = (\underline{s} \otimes \underline{s}) \underline{w}_j}$$

Frame Elements



$$f^n(x_i)\,\underline{n}$$

$$-f^s(x_i)\,\underline{s}$$

$$\underline{F}_i - f^s(x_i)\,\underline{s} + f^n(x_i)\,\underline{n} = 0$$

$$\underline{F}_i = -f^n(x_i)\,\underline{n} + f^s(x_i)\,\underline{s} =$$

$$= \frac{AE}{\ell}(u_i - u_j)\,\underline{n} + \left[\frac{12EI}{\ell^3}(v_i - v_j) + \right.$$

$$\left. + \frac{6EI}{\ell^2}(\theta_i + \theta_j)\right]\underline{s}$$

Frame Elements

$$\begin{Bmatrix} \boldsymbol{F}_i \\ M_i \\ \boldsymbol{F}_j \\ M_j \end{Bmatrix} = \begin{bmatrix} \boldsymbol{K}_{fw} & \boldsymbol{k}_{f\theta} & -\boldsymbol{K}_{fw} & \boldsymbol{k}_{f\theta} \\ \boldsymbol{k}_{mw} & k_{m\theta} & -\boldsymbol{k}_{mw} & \hat{k}_{m\theta} \\ -\boldsymbol{K}_{fw} & -\boldsymbol{k}_{f\theta} & \boldsymbol{K}_{fw} & -\boldsymbol{k}_{f\theta} \\ \boldsymbol{k}_{mw} & \hat{k}_{m\theta} & -\boldsymbol{k}_{mw} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} \boldsymbol{w}_i \\ \theta_i \\ \boldsymbol{w}_j \\ \theta_j \end{Bmatrix}$$

Where

$\boldsymbol{F}_{i,j} \in \mathbb{R}^2$ is a vector

$\boldsymbol{w}_{i,j} \in \mathbb{R}^2$ is a vector

$M_{i,j} \in \mathbb{R}$ is a scalar

$\theta_{i,j} \in \mathbb{R}$ is a scalar

$\boldsymbol{K}_{fw} \in \mathbb{R}^2 \times \mathbb{R}^2$ is a tensor

$\boldsymbol{k}_{f\theta} \in \mathbb{R}^2$ is a vector

$\boldsymbol{k}_{Mw} \in \mathbb{R}^2$ is a vector

$k_{m\theta}, \hat{k}_{m\theta} \in \mathbb{R}$ is a scalar

$$\boldsymbol{K}_{fw} = \frac{AE}{\ell}\boldsymbol{n} \otimes \boldsymbol{n} + \frac{12EI}{\ell^3}\boldsymbol{s} \otimes \boldsymbol{s}, \quad k_{m\theta} = \frac{4EI}{\ell}, \quad \hat{k}_{m\theta} = \frac{2EI}{\ell}, \quad \boldsymbol{k}_{mw} = \boldsymbol{k}_{f\theta} = \frac{6EI}{\ell^2}\boldsymbol{s}$$

Frame Elements

$$\boldsymbol{F}_i = \boldsymbol{K}_{\boldsymbol{fw}}\boldsymbol{w}_i + \boldsymbol{k}_{\boldsymbol{f}\theta}\,\theta_i - \boldsymbol{K}_{\boldsymbol{fw}}\boldsymbol{w}_i + \boldsymbol{k}_{\boldsymbol{f}\theta}\theta_j$$

Vector equation!

$$M_i = \boldsymbol{k}_{mw}\cdot\boldsymbol{w}_i + k_{m\theta}\,\theta_i - \boldsymbol{k}_{mw}\cdot\boldsymbol{w}_i + \hat{k}_{m\theta}\theta_j$$

Scalar equation!

$$\boldsymbol{F}_j = -\boldsymbol{K}_{\boldsymbol{fw}}\boldsymbol{w}_i - \boldsymbol{k}_{\boldsymbol{f}\theta}\,\theta_i + \boldsymbol{K}_{\boldsymbol{fw}}\boldsymbol{w}_i - \boldsymbol{k}_{\boldsymbol{f}\theta}\theta_j$$

Vector equation!

$$M_j = \boldsymbol{k}_{mw}\cdot\boldsymbol{w}_i + \hat{k}_{m\theta}\,\theta_i + -\boldsymbol{k}_{mw}\cdot\boldsymbol{w}_i + k_{m\theta}\theta_j$$

Scalar equation!

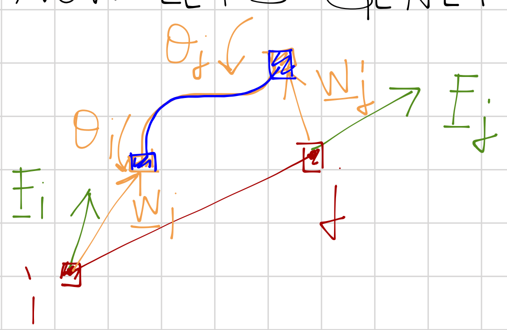
Frame Elements

$$\begin{Bmatrix} [\boldsymbol{F}_i] \\ M_i \\ [\boldsymbol{F}_j] \\ M_j \end{Bmatrix} = \begin{bmatrix} [\boldsymbol{K}_{\boldsymbol{fw}}] & [\boldsymbol{k}_{\boldsymbol{f}\theta}] & -[\boldsymbol{K}_{\boldsymbol{fw}}] & [\boldsymbol{k}_{\boldsymbol{f}\theta}] \\ [\boldsymbol{k}_{mw}]^\top & k_{m\theta} & -[\boldsymbol{k}_{mw}]^\top & \hat{k}_{m\theta} \\ -[\boldsymbol{K}_{\boldsymbol{fw}}] & -[\boldsymbol{k}_{\boldsymbol{f}\theta}] & [\boldsymbol{K}_{\boldsymbol{fw}}] & -[\boldsymbol{k}_{\boldsymbol{f}\theta}] \\ [\boldsymbol{k}_{mw}]^\top & \hat{k}_{m\theta} & -[\boldsymbol{k}_{mw}]^\top & k_{m\theta} \end{bmatrix} \begin{Bmatrix} [\boldsymbol{w}_i] \\ \theta_i \\ [\boldsymbol{w}_j] \\ \theta_j \end{Bmatrix}$$

$$\begin{Bmatrix} [\boldsymbol{F}_i]_{2\times 1} \\ M_i \\ [\boldsymbol{F}_j]_{2\times 1} \\ M_j \end{Bmatrix} = \begin{bmatrix} [\boldsymbol{K}_{fw}]_{2\times 2} & [\boldsymbol{k}_{f\theta}]_{2\times 1} & -[\boldsymbol{K}_{fw}]_{2\times 2} & [\boldsymbol{k}_{f\theta}]_{2\times 1} \\ [\boldsymbol{k}_{mw}]^{\top}_{1\times 2} & k_{m\theta} & -[\boldsymbol{k}_{mw}]^{\top}_{1\times 2} & \hat{k}_{m\theta} \\ -[\boldsymbol{K}_{fw}]_{2\times 2} & -[\boldsymbol{k}_{f\theta}]_{2\times 1} & [\boldsymbol{K}_{fw}]_{2\times 2} & -[\boldsymbol{k}_{f\theta}]_{2\times 1} \\ [\boldsymbol{k}_{mw}]^{\top}_{1\times 2} & \hat{k}_{m\theta} & -[\boldsymbol{k}_{mw}]^{\top}_{1\times 2} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} [\boldsymbol{w}_i]_{2\times 1} \\ \theta_i \\ [\boldsymbol{w}_j]_{2\times 1} \\ \theta_j \end{Bmatrix}$$

$$\frac{6EI}{\ell}[\underline{S}]^T$$

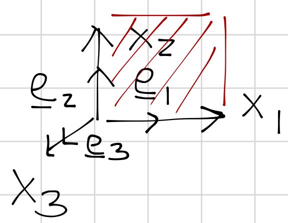
NOW LET'S GENERALIZE IT TO MULTIPLE DIMENSIONS



WE FIRST CONSTRUCT A DIRECTOR VECTOR

$\underline{n} = \dfrac{\underline{q}_j - \underline{q}_i}{\ell}$, $\ell = |\underline{q}_j - \underline{q}_i|$

THEN WE CONSTRUCT A NORMAL VECTOR



$\underline{n} = \alpha \underline{e}_1 + \beta \underline{e}_2$

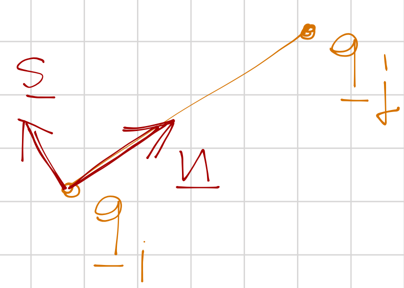
$\underline{n} \times \underline{e}_3$

OR ROTATE

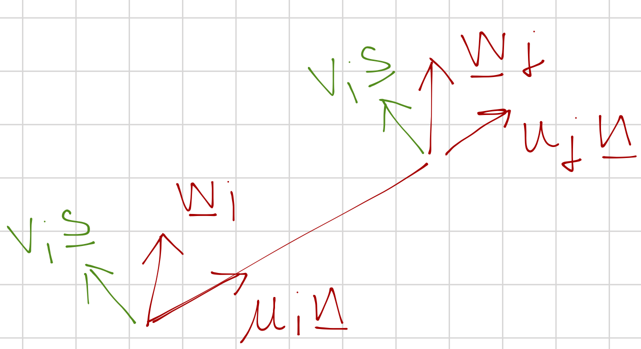
$\underline{\underline{R}} = -1\, \underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1$

S.T.

$\underline{s} = \underline{\underline{R}}\,\underline{n}$



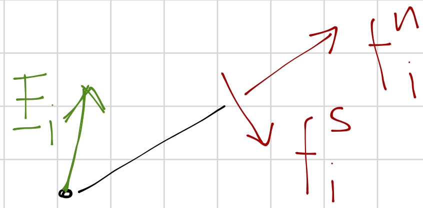
NOW, YOU RECALL THAT FOR TRUSSES



$u\underline{n} = (\underline{n}\otimes\underline{n})\,\underline{w}$

SIMILARLY NOW WE CAN WRITE

$v\underline{s} = (\underline{s}\otimes\underline{s})\,\underline{w}$



$$\underline{F}_i + f_i^n \underline{n} - f_i^s \underline{s} = 0$$

$$\Rightarrow \quad \underline{F}_i = -f_i^n \underline{n} + f_i^s \underline{s} = \frac{AE}{\ell}(u_i - u_j)\underline{n} + \left[\frac{12EI}{\ell^3}(v_i - v_j) + \frac{6EI}{\ell^2}(\theta_i + \theta_j) \right] \underline{s}$$

$$= \left[\frac{AE}{\ell} \underline{n} \otimes \underline{n} + \frac{12EI}{\ell^3} \underline{s} \otimes \underline{s} \right] (\underline{w}_i - \underline{w}_j) + \frac{6EI}{\ell^2} \underline{s} \, (\theta_i + \theta_j)$$

$$\underline{F}_i = \left[\underline{\underline{K}}_{fw} \quad \underline{K}_{f\theta} \quad -\underline{\underline{K}}_{fw} \quad \underline{K}_{f\theta} \right] \begin{Bmatrix} \underline{w}_i \\ \theta_i \\ \underline{w}_j \\ \theta_j \end{Bmatrix}$$