# MID-TERM EXAM

CEE 513: Introduction to Finite Element Methods

Thurday Oct. 19

1. The exam is closed book and closed notes.
2. The exam will start at 11:00am and end sharply at 12:20pm. If you continue writing past that, we will not grade your exam and you will not receive any credit.
3. Make sure you justify all your answers.
4. Should you have any questions please come outside of the classroom where we will be standing. Once you ask your question we will come back into the classroom and the question will be answered to the entire class.

## PROBLEM 1: Some Tensor Algebra and Calculus ( 20 pts )

For the following assume that $\{\mathbf{e}_i\}_{i=1}^d$ is a set of orthonormal basis associated with cartesian coordinates $\{x_i\}_{i=1}^d$. For each question provide a detailed justification (i.e. do not simply answer yes or no; if you simply say yes or no you will not receive any credit).

1. (4 pts) Let $d = 2$. $u = \exp(x_1)\sin(x_2)$. Find $\nabla u$ and $\nabla \cdot (\nabla u)$.
2. (4 pts) Let $\boldsymbol{x}_a = -1\mathbf{e}_1 - 1\mathbf{e}_2$ and $\boldsymbol{x}_b = 2\mathbf{e}_1 + 1\mathbf{e}_2$. Find the projection tensor that projects vectors along the direction $\boldsymbol{a} = \boldsymbol{x}_b - \boldsymbol{x}_a$.
3. (4 pts) Are $\boldsymbol{a} = 3\mathbf{e}_1 + 5\mathbf{e}_2 + 2\mathbf{e}_3, \boldsymbol{b} = 2\mathbf{e}_1 + 1\mathbf{e}_3, \boldsymbol{c} = 7\mathbf{e}_1 + 5\mathbf{e}_2 + 4\mathbf{e}_3$ linearly independent?
4. (4 pts) Let $d = 2$. Construct a tensor $\boldsymbol{T}$ that rotates a vector by $\pi/2$.
5. (4 pts) A map $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is such that

$$
\begin{aligned}
\boldsymbol{T}(2\mathbf{e}_1 + 3\mathbf{e}_2) &= 5\mathbf{e}_1 - \mathbf{e}_2, \\
\boldsymbol{T}(-2\mathbf{e}_1 + 4\mathbf{e}_2) &= 3\mathbf{e}_1 + 2\mathbf{e}_2, \\
\boldsymbol{T}(\mathbf{e}_1 + 4\mathbf{e}_2) &= \mathbf{e}_1 - \mathbf{e}_2.
\end{aligned}
$$

   Is $\boldsymbol{T}$ a tensor?

## PROBLEM 2:

1. Solution 1:

$$
\begin{aligned}
u &= exp(x_1)sin(x_2) \\
\nabla u &= \frac{du}{dx_i}\mathbf{e}_i \\
&= exp(x_1)sin(x_2)\mathbf{e}_1 + exp(x_1)cos(x_2)\mathbf{e}_2
\end{aligned}
$$

$$
\begin{aligned}
\nabla \cdot (\nabla u) &= \frac{d(\nabla u)}{dx_i} \cdot \mathbf{e}_i \\
&= exp(x_1)sin(x_2) - exp(x_1)sin(x_2) \\
&= 0
\end{aligned}
$$

2. Solution 2:

$$\begin{aligned}\boldsymbol{a} &= \boldsymbol{x}_b - \boldsymbol{x}_a \\ &= 3\mathbf{e}_1 + 2\mathbf{e}_2 \\ \boldsymbol{n} &= \frac{\boldsymbol{a}}{||\boldsymbol{a}||} \\ &= \frac{3\mathbf{e}_1 + 2\mathbf{e}_2}{\sqrt{13}} \\ \boldsymbol{T} &= \boldsymbol{n} \otimes \boldsymbol{n} \\ &= \frac{9\mathbf{e}_1 \otimes \mathbf{e}_1 + 6\mathbf{e}_1 \otimes \mathbf{e}_2 + 6\mathbf{e}_2 \otimes \mathbf{e}_1 + 4\mathbf{e}_2 \otimes \mathbf{e}_2}{13}\end{aligned}$$

3. Solution 3:

$$\begin{aligned}\boldsymbol{a} &= 3\mathbf{e}_1 + 5\mathbf{e}_2 + 2\mathbf{e}_3 \\ \boldsymbol{b} &= 2\mathbf{e}_1 + 1\mathbf{e}_3 \\ \boldsymbol{c} &= 7\mathbf{e}_1 + 5\mathbf{e}_2 + 4\mathbf{e}_3\end{aligned}$$

Let $\alpha_1, \alpha_2, \alpha_3$ be a non-zero real numbers such that:

$$\alpha_1 \boldsymbol{a} + \alpha_2 \boldsymbol{b} + \alpha_3 \boldsymbol{c} = 0$$

Solving gives us:

$$\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = 0$$

Hence the vectors are linearly dependent.

4. Solution 4:

$$\boldsymbol{T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

5. Solution 5:

Let us assume that $\boldsymbol{T}(\boldsymbol{u})$ is a tensor, hence it should satisfy the properties of a tensor. Adding equation 1 and 2 and dividing by 7 gives:

$$\boldsymbol{T}(\mathbf{e}_2) = \frac{8\mathbf{e}_1 + \mathbf{e}_2}{7}$$

Multiplying equation 3 by 2 and adding to equation 2 gives; then dividing by 12:

$$\boldsymbol{T}(\mathbf{e}_2) = \frac{5\mathbf{e}_1}{12}$$

This leads to a contradiction, hence $\boldsymbol{T}(\boldsymbol{u})$ is not a tensor.

# PROBLEM 3:

# PROBLEM 4: Frame System (40 pts)

Consider a thermoelastic frame element (both bending and axial forces are relevant). We are interested in finding the deformation of a beam of length $\ell$. We let the displacement of the beam be defined by a vector valued function $\boldsymbol{w}(x) = u(x)\mathbf{e}_x + v(x)\mathbf{e}_y$. The beam is susceptible to thermal expansion and contraction.

The boundary value problem becomes: find $u(x)$ and $v(x)$ such that

$$\frac{d}{dx}(A\sigma(u(x))) = 0,$$
$$EI\frac{d^4v(x)}{dx^4} = 0,$$

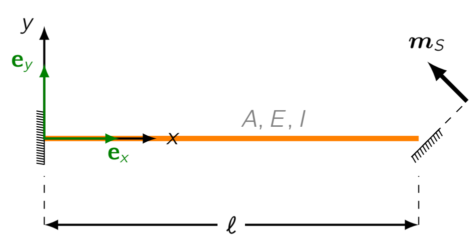
subject to boundary conditions

$$u(0) = v(0) = \theta(0) = 0 \quad (1)$$
$$\boldsymbol{w}(\ell) \cdot \boldsymbol{m}_s = 0 \quad (2)$$

where $\theta(x)$, as always, is $dv(x)/dx$ and

$$\sigma(u(x)) = E\left(\frac{du(x)}{dx} - \alpha T(x)\right).$$

Additionally, $\alpha$ and $T(x) = x$ represent the coefficient of thermal expansion and the temperature (given quantities), respectively, and $\boldsymbol{m}_S$ is a given unit normal vector defining the orientation of the plane along which the frame is allowed to slide at $x = \ell$.




Ignoring boundary conditions above follow the following steps to derive a general matrix vector equation for a thermoelastic truss:

1. (5 pts) Find a general solution for $u$ satisfying
$$\frac{d}{dx}\left[AE\left(\frac{du}{dx} - \alpha T(x)\right)\right] = 0.$$

2. (5 pts) Similarly as we did for standard truss elements, assume the displacements at $x = 0$ and at $x = \ell$ are given by the unknown constant $u_i, u_j$. Solve for your integration constants from part 1. Namely, you should arrive at an expression
$$u(x) = u_i\phi_1(x) + u_j\phi_2(x) + \phi_3(x)$$
where you need to determine $\phi_{1,2,3}$ in terms of $\alpha, \Delta T, \ldots$.

3. (10 pts) Write the equilibrium equations at the left support (node $i$) and at the right support (node $j$). Note that the internal forces are given by $A\sigma(u(x))$.

4. (5 pts) Write the matrix vector equation for the balance of axial forces. You should arrive at something of the sort
$$\begin{Bmatrix} P_i \\ P_j \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}.$$
Provide the values for $P_i, P_j$ and $k_{ab}$.

5. (5 pts) Recalling that for beams we have that the nodal displacements and rotations ($v_{i,j}$ and $\theta_{i,j}$) satisfy the following system of equations, with some given $V_{i,j}$, $M_{i,j}$,

$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} k_{fv} & k_{f\theta} & -k_{fv} & k_{f\theta} \\ k_{mv} & k_{m\theta} & -k_{mv} & \hat{k}_{m\theta} \\ -k_{fv} & -k_{f\theta} & k_{fv} & -k_{f\theta} \\ k_{mv} & \hat{k}_{m\theta} & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}$$

Merge your results from 4. into the above.

6. (10 pts) Now, apply the boundary conditions of (**??**), (**??**) to the derived system of equations from 5.

1. Solution 1:

The general solution is obtained as:

$$\frac{d}{dx}\left[AE\left(\frac{du}{dx} - \alpha T(x)\right)\right] = 0$$
$$AE\left(\frac{du}{dx} - \alpha T(x)\right) = C_1$$
$$\frac{du}{dx} - \alpha T(x) = \frac{C_1}{AE}$$
$$u = \frac{C_1 x}{AE} + \alpha\frac{x^2}{2} + C_2$$

2. Solution 2:

Setting $x = 0$

$$u_i = C_2$$

Setting $x = \ell$

$$u_j = \frac{C_1 \ell}{AE} + \alpha\frac{\ell^2}{2} + u_i$$
$$C_1 = \left(\frac{u_j - u_i}{\ell} - \frac{\alpha\ell}{2}\right) AE$$

Therefore:

$$u = u_j\left(\frac{x}{\ell}\right) + u_i\left(1 - \frac{x}{\ell}\right) - \frac{\alpha x}{2}(\ell - x)$$

3. Solution 3:

At the left support:

$$f_i = AE\left(\frac{u_j - u_i}{\ell} - \frac{\alpha\ell}{2}\right)$$

At the right support:

$$f_j = AE\left(\frac{u_j - u_i}{\ell} - \frac{\alpha\ell}{2}\right)$$

4. Solution 4:

Writing in matrix form and taking into account that there is no external loading applied.

$$\begin{bmatrix} \frac{-AE\alpha\ell}{2} \\ \frac{AE\alpha\ell}{2} \end{bmatrix} = \frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

5. Solution 5:

Merging the results:

$$\begin{bmatrix} \frac{-AE\alpha\ell}{2} \\ V_i \\ M_i \\ \frac{AE\alpha\ell}{2} \\ V_j \\ M_j \end{bmatrix} = \begin{bmatrix} \frac{AE}{\ell} & 0 & 0 & -\frac{AE}{\ell} & 0 & 0 \\ 0 & k_{fv} & k_{f\theta} & 0 & -k_{fv} & k_{f\theta} \\ 0 & k_{mv} & k_{m\theta} & 0 & -k_{mv} & \hat{k}_{m\theta} \\ -\frac{AE}{\ell} & 0 & 0 & \frac{AE}{\ell} & 0 & 0 \\ 0 & -k_{fv} & -k_{f\theta} & 0 & k_{fv} & -k_{f\theta} \\ 0 & k_{mv} & \hat{k}_{m\theta} & 0 & -k_{mv} & k_{m\theta} \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{bmatrix}$$

6. Solution 6:

Applying the boundary conditions:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{AE\alpha\ell}{2} \\ V_j \\ M_j \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{AE}{\ell} & 0 & 0 & \frac{AE}{\ell} & 0 & 0 & -m_s^1 \\ 0 & -k_{fv} & -k_{f\theta} & 0 & k_{fv} & -k_{f\theta} & -m_s^2 \\ 0 & k_{mv} & \hat{k}_{m\theta} & 0 & -k_{mv} & k_{m\theta} & 0 \\ 0 & 0 & 0 & m_s^1 & m_s^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \\ \lambda \end{bmatrix}$$

# PROBLEM 5: Finite Element Methods (40 pts)

Consider the following strong form: with $f, k : [0, \ell] \to \mathbb{R}$ being given functions, $g$ given constant, find $u : [0, \ell]$ such that

$$\frac{d^2u}{dx^2} + uk = f, \quad \forall x \in (0, 1)$$

with

$$u(0) = g, \quad \frac{du}{dx}(\ell) = \alpha u(\ell).$$

Note that the latter boundary condition is named *Robin boundary condition* (a linear combination of Dirichlet and Neumann) and for the purpose of this exercise can be treated as a Neumann boundary condition.

1. (5 pts) What is the set of trial functions $\mathcal{S}$?
2. (5 pts) What is the set of test functions $\mathcal{V}$?
3. (10 pts) Derive the weak formulation of the problem. You should arrive at an expression of the form $a(w,u) = F(w) \quad \forall w \in \mathcal{V}$ where $a$ and $F$ are to be defined accordingly.
4. (10 pts) What is the corresponding variational problem? Namely find the potential functional $\Pi$ and show that the statement: find $u \in \mathcal{S}$ s.t.

$$\langle \delta\Pi, \delta u \rangle = 0 \quad \forall \delta u \in \mathcal{V}$$

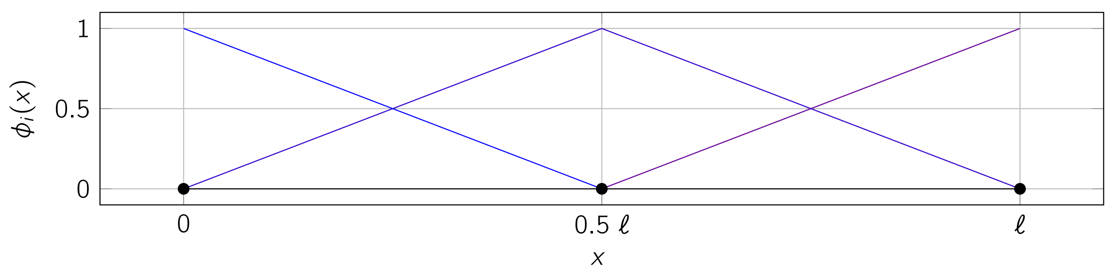
is equivalent to the weak form.
5. (5 pts) Derive the matrix form of the problem.
6. (5 pts) Let us discretize the domain into two elements using linear basis functions (as we did in class) for a total of three degrees of freedom and three basis functions (see plot below) given by

$$\phi_1 = \begin{cases} (1-2x) & \text{if } x < 0.5\,\ell \\ 0 & \text{if } x \geq 0.5\,\ell \end{cases}$$

$$\phi_2 = \begin{cases} 2x & \text{if } x < 0.5\,\ell \\ 2(1-x) & \text{if } x \geq 0.5\,\ell \end{cases}$$

$$\phi_3 = \begin{cases} 0 & \text{if } x < 0.5\,\ell \\ 2x-1 & \text{if } x \geq 0.5\,\ell \end{cases}$$

If $k(x) = x^2$ and $f(x) = x$, find the values of $F(\phi_1)$ and $a(\phi_2, \phi_3)$ in terms of $g$, $\ell$.




# PROBLEM 6:

1. Solution 1:

The set of trial functions $\mathcal{S}$:

$$\mathcal{S} = \{u | u \in \text{Smooth}, u(0) = g\}$$

2. Solution 2:

The set of test functions $\mathcal{V}$:

$$\mathcal{V} = \{w | w \in \text{Smooth}, w(0) = 0\}$$

3. Solution 3:

$$\frac{d^2u}{dx^2} + uk = f$$

`Multiplying both the sides by the weight` $w$ `and integrating:`

$$\int_0^\ell \frac{d^2u}{dx^2} w \ \ dx + \int_0^\ell uwk \ \ dx = \int_0^\ell fw \ \ dx$$
$$\frac{du}{dx} w \bigg|_0^\ell - \int_0^\ell \frac{du}{dx}\frac{dw}{dx} \ \ dx + \int_0^\ell uwk \ \ dx = \int_0^\ell fw \ \ dx$$
$$-\int_0^\ell \frac{du}{dx}\frac{dw}{dx} \ \ dx + \int_0^\ell uwk \ \ dx + \alpha u(\ell)w(\ell) = \int_0^\ell fw \ \ dx$$

4. Solution 4:

`The corresponding variational form:`

$$\int_0^\ell \frac{1}{2}\left(\frac{du}{dx}\right)^2 \ \ dx - \int_0^\ell \frac{1}{2}u^2k \ \ dx + \int_0^\ell fu \ \ dx - \frac{1}{2}\alpha(u(\ell))^2 = 0$$

5. Solution 5:

`Galerking form:`

$$u^h = v^h + g^h$$
$$v^h = \sum_j v_j\phi_j$$
$$w^h = \sum_i w_i\phi_i$$
$$g^h = g\phi_1$$

`Substituting in the weak form to obtain the matrix form:`

$$-\sum_j \int_0^\ell \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} v_j \ \ dx + \sum_j \int_0^\ell \phi_i\phi_j k v_j \ \ dx + \alpha \sum_j \phi_i(\ell)\phi_j(\ell)v_j = \int_0^\ell g\frac{d\phi_1}{dx}\frac{d\phi_i}{dx} - \int_0^\ell gk\phi_1\phi_i \ \ dx + \int_0^\ell f\phi_i \ \ dx$$

`This gives us the matrix form:`

$$K_{ij}v_j = F_i$$

`where:`

$$K_{ij} = -\int_0^\ell \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} \ \ dx + \int_0^\ell \phi_i\phi_j k \ \ dx + \alpha\phi_i(\ell)\phi_j(\ell)$$
$$= a(\phi_i, \phi_j)$$
$$F_i = \int_0^\ell g\frac{d\phi_1}{dx}\frac{d\phi_i}{dx} - \int_0^\ell gk\phi_1\phi_i \ \ dx + \int_0^\ell f\phi_i \ \ dx$$

6. Solution 6:

Computing $F(\phi_1)$ :

$$F(\phi_1) = \int_0^\ell g\frac{d\phi_1}{dx}\frac{d\phi_1}{dx} - \int_0^\ell gk\phi_1\phi_1 \ \ dx + \int_0^\ell f\phi_1 \ \ dx$$
$$= \int_0^{\ell/2} g\frac{d\phi_1}{dx}\frac{d\phi_1}{dx} - \int_0^{\ell/2} gk\phi_1\phi_1 \ \ dx + \int_0^{\ell/2} f\phi_1 \ \ dx$$

As,

$$\phi_1 = \begin{cases} (1-2x/\ell) & \text{if } \ x < 0.5\ell \\ 0 & \text{if } \ x \geq 0.5\ell \end{cases}$$

Substituing we obtain:

$$F(\phi_1) = \frac{2g}{\ell} - \frac{g\ell^3}{240} + \frac{\ell^2}{24}$$

Computing $a(\phi_2, \phi_3)$ :

$$a(\phi_2, \phi_3) = -\int_0^\ell \frac{d\phi_2}{dx}\frac{d\phi_3}{dx} \ \ dx + \int_0^\ell \phi_2\phi_3 k \ \ dx$$

As,

$$\phi_2 = \begin{cases} 2x/\ell & \text{if } \ x < 0.5\ell \\ 2(1-x/\ell) & \text{if } \ x \geq 0.5\ell \end{cases}$$
$$\phi_3 = \begin{cases} 0 & \text{if } \ x < 0.5\ell \\ 2x/\ell - 1 & \text{if } \ x \geq 0.5\ell \end{cases}$$

Substituting we obtain:

$$a(\phi_2, \phi_3) = \frac{2}{\ell} + \frac{23\ell^3}{960}$$