Compressed Sensing:

Teaching Session on Highly Accelerated fMRI

October 4th, 2013
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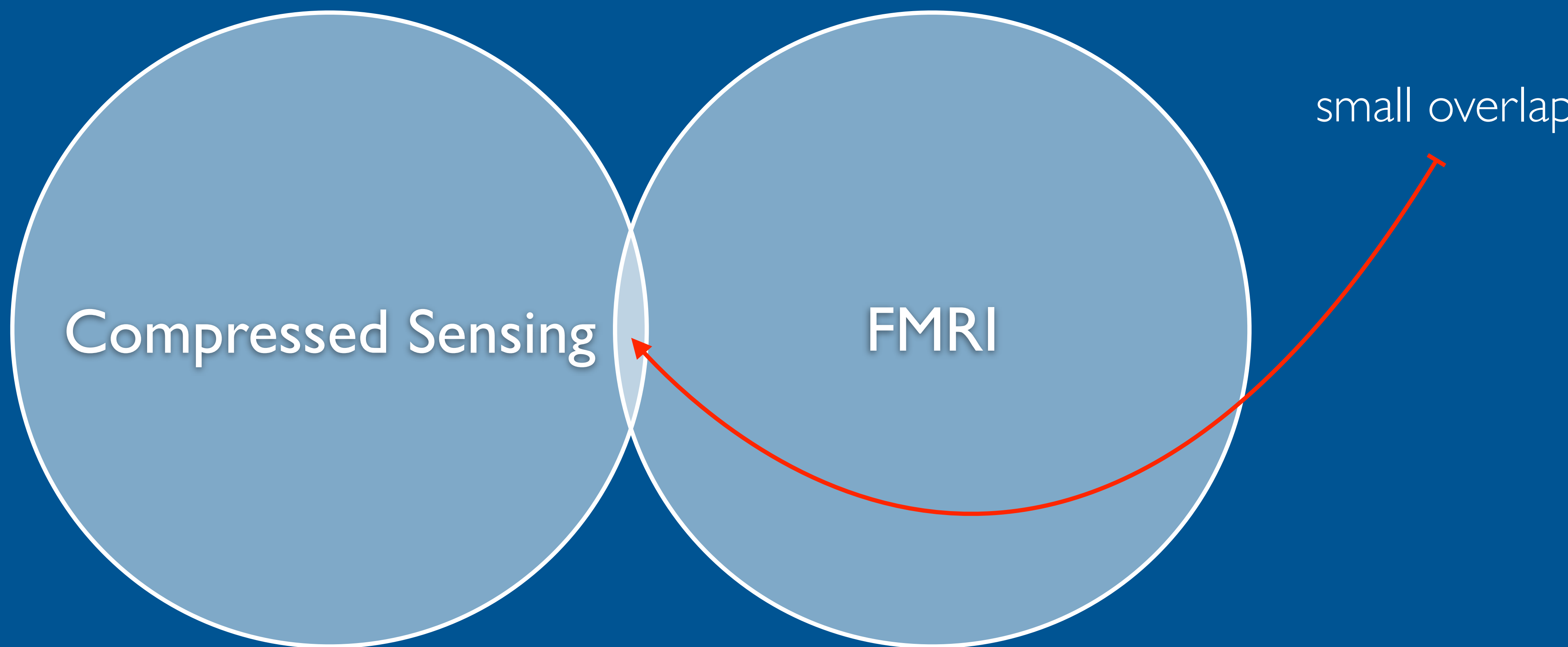
Introduction



- Compressed Sensing (CS) is an incredibly popular tool for accelerating MR image acquisition
- Unfortunately, not quite as popular for use in accelerating FMRI

Compressed Sensing Basics ⟼ **Compressed Sensing & FMRI** ⟼ **"Sparsity"-related FMRI Acceleration**

Essential CS Glossary

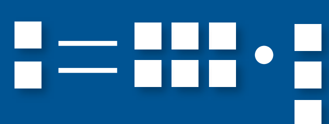
- **Sparsity**
  - → A signal is **sparse** if it has few non-zero coefficients

Sparse

Non-sparse

- **Under-determined linear systems**
  - → Under-determined: fewer measurements than unknowns

Short and Fat

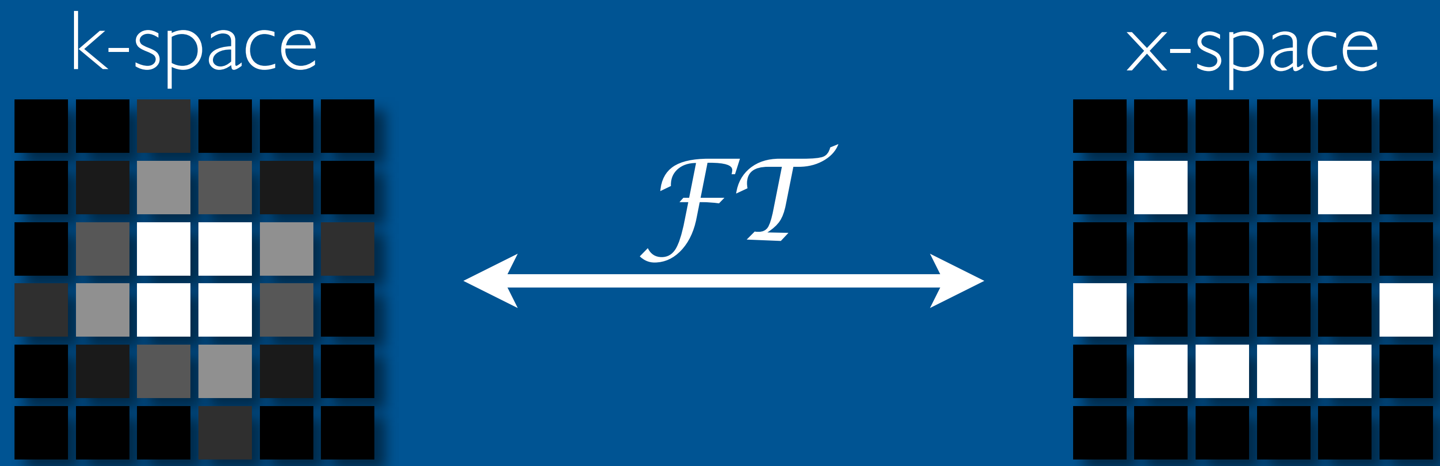


- **Basis and Transform**
  - → A basis is something that can be used to represent a signal
  - → A transform can take a signal from one basis to another
  - → e.g., the Fourier transform takes a signal into the Fourier basis

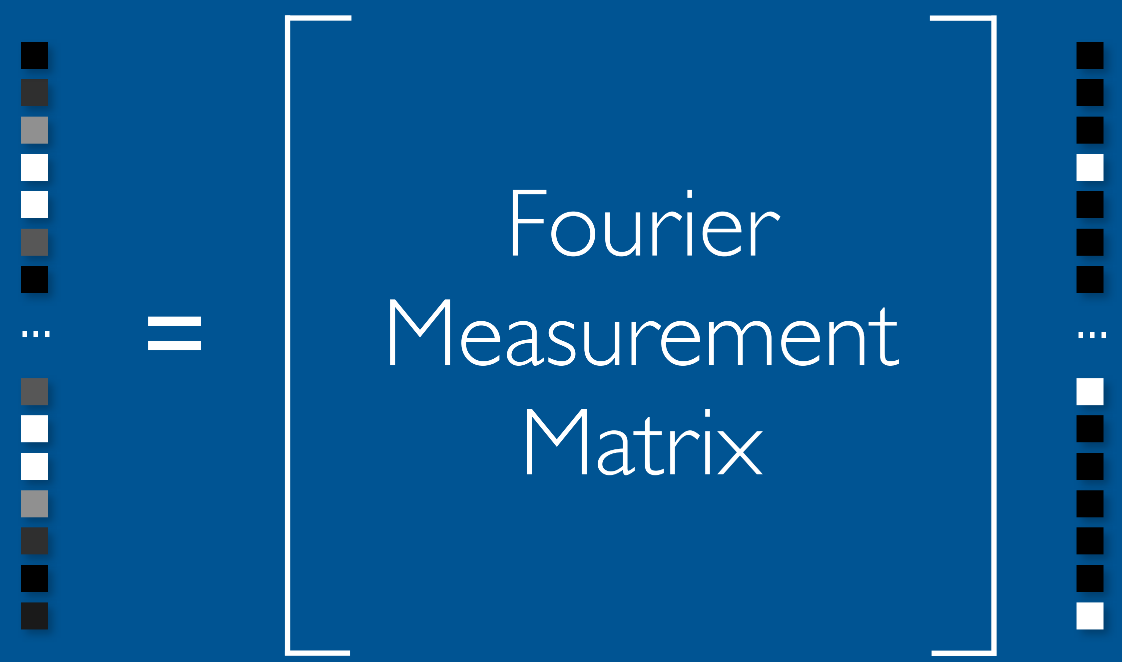
Essential CS Glossary

- L_0 **norm*** (# non-zero values)
  - → $||x||_0 = \sum (x_i \mathrel{!=} 0)$
  - → Direct measure of sparsity, very difficult to work with
- L_1 **norm** (sum of absolute coefficients)
  - → $||x||_1 = \sum |x_i|$
  - → Indirect proxy for sparsity, less difficult to work with
- L_2 **norm** (root sum of squared coefficients)
  - → $||x||_2 = (x^T x)^{0.5} = (\sum x_i^2)^{0.5}$
  - → Euclidean distance norm, easy to work with

CS Basic Intuition



– Imaging as a linear system

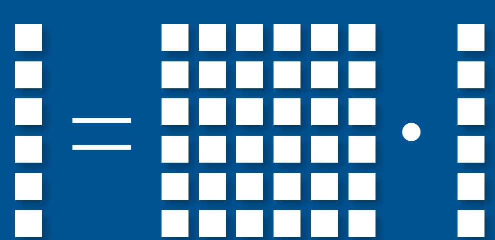


$$k = M \cdot x$$

CS Basic Intuition

- Acceleration involves reducing the number of measurements for faster encoding of the image data
  - → Acceleration factor **R** dictates how many rows of the measurement/encoding matrix are omitted
  - → With fewer rows than columns, F is under-determined

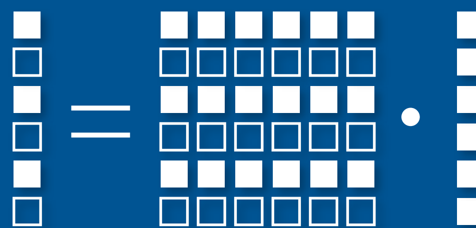
R=1



Slow but invertible
A unique solution exists

vs

R=2

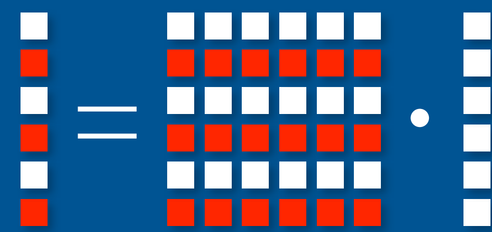


Faster, but non-invertible
Many possible solutions

To solve R > 1, need additional information or constraints

CS Basic Intuition

- Parallel Imaging:



coil #1
coil #2

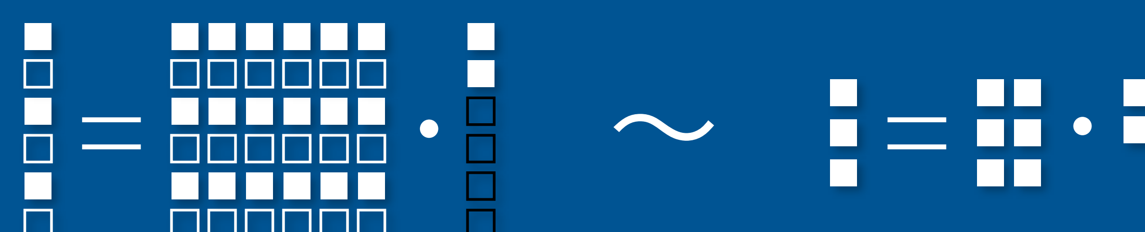
  → Multiple coils increase the number of effective measurements

  → System is no longer under-determined

- Pseudoinverse method:

  → Pick the solution with minimum signal energy (min L_2 norm)

- Compressed Sensing:



  → Pick the solution with maximum sparsity (min L_1 norm)

  → In sparse data, there are effectively fewer unknowns, which reduces sampling/measurements requirement

CS Imaging Principles

1. **Sparsity & compressibility**

   → The sparsity property ensures the image has less information than its dimensionality suggests

2. **Random & incoherent sampling**

   → The reduced sampling need to be applied with some constraints to ensure reconstruction is successful

3. **Iterative & nonlinear reconstruction**

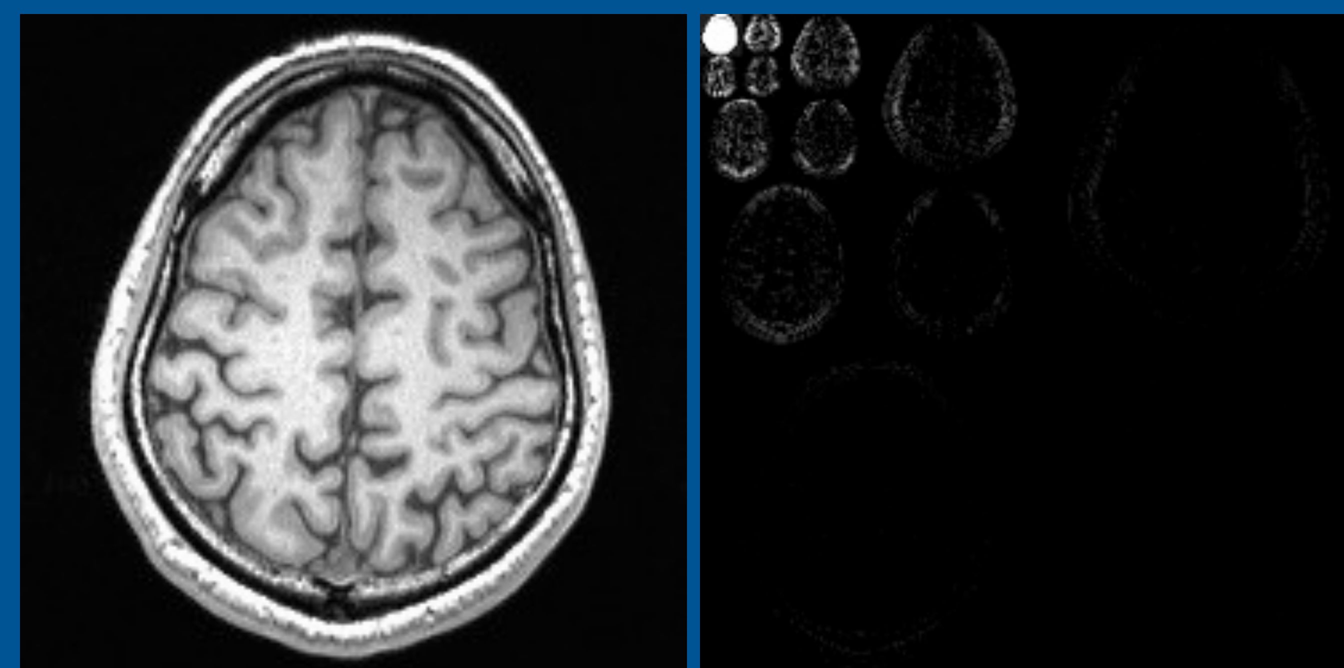
   → Image reconstruction is now an optimisation problem (namely L_1 minimization)

Sparsity & Compressibility

- MR Angiograms are sparse in native image space

- Structural brain images can be sparse using a wavelet representation

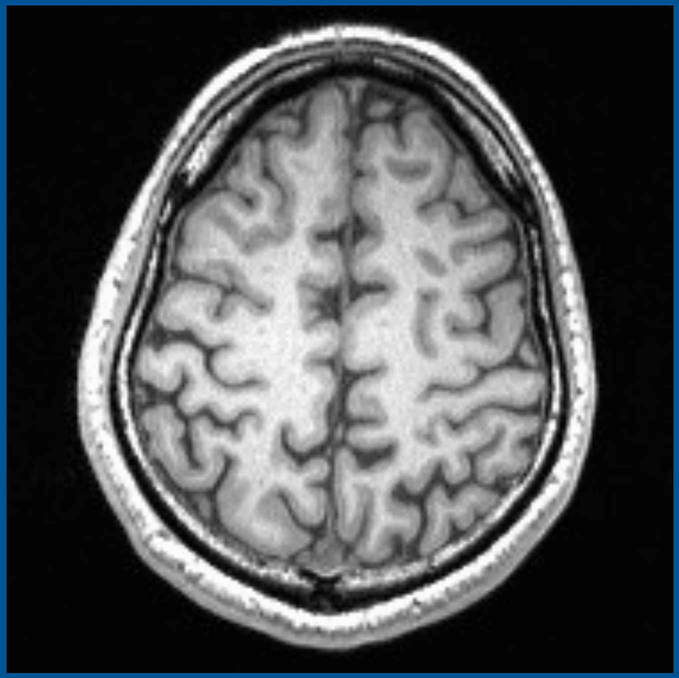
Image Domain | Wavelet Domain



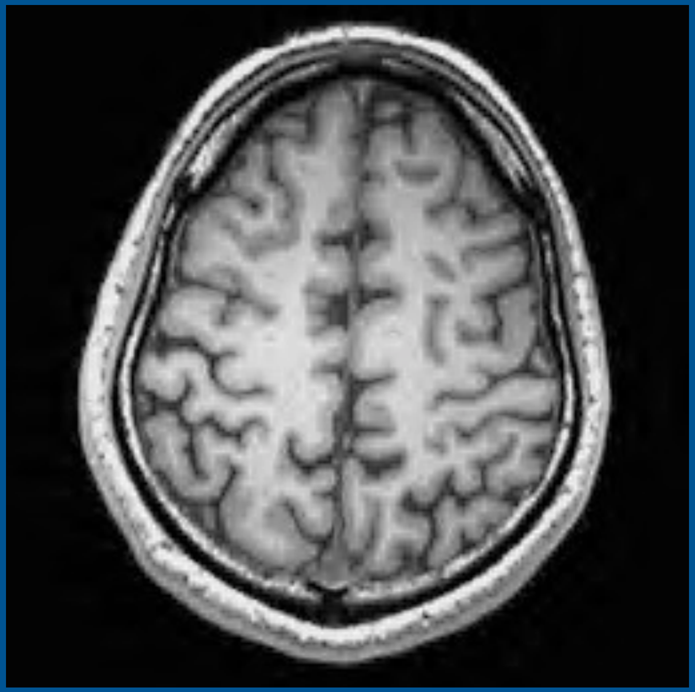
Sparse signals are highly compressible

Sparsity & Compressibility

- 128x128 brain image at varying compression factors



1x
{16384 unknowns}



10x
{1638 unknowns}

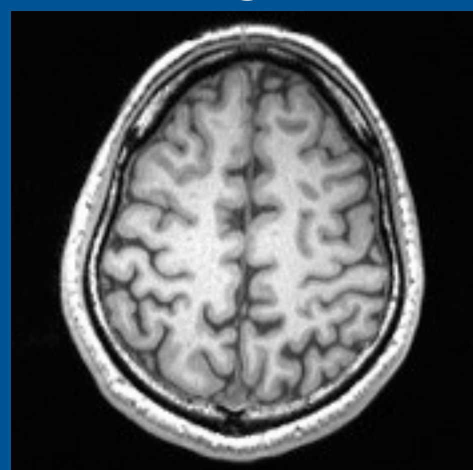
20x
{819 unknowns}

100x
{164 unknowns}

Sparsity & Compressibility

- Compression reduces the inefficiency in data representation by removing redundancy in the signal

Original



Inefficient

10x compression →

Compressed

Efficient

- Compressed sensing aims to exploit compressibility not for storage, but for speed

→ Fewer unknowns ➞ fewer measurements ➞ faster imaging

→ Rule of thumb: 2-5 measurements for every sparse point[1]

→ Acceleration factor is directly linked to sparsity

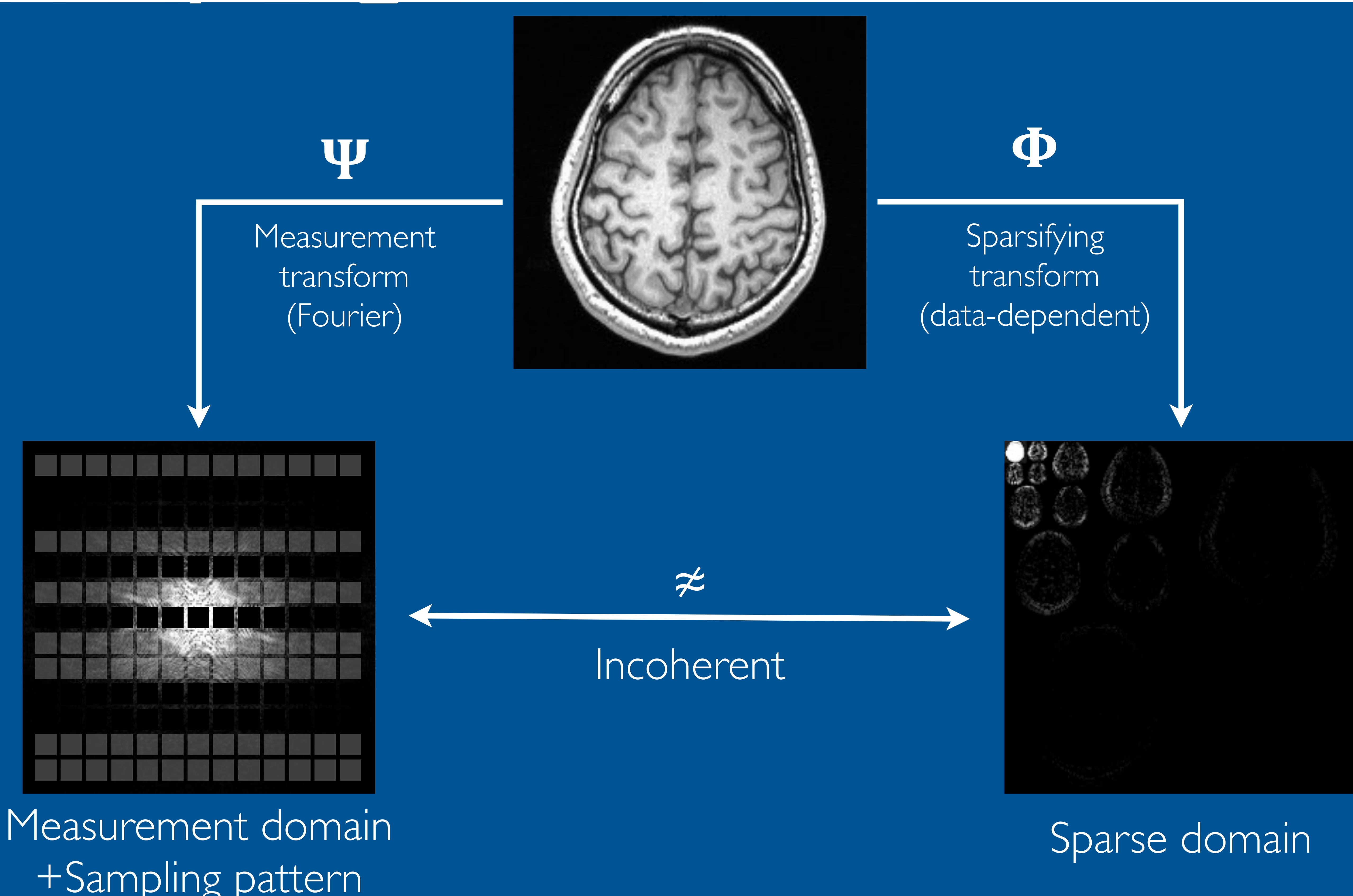
1. Lustig et al., Magn Reson Med 2007

Sampling and Incoherence

- The data cannot be sampled in the domain in which it is sparse
  - → "Needle in a haystack" problem

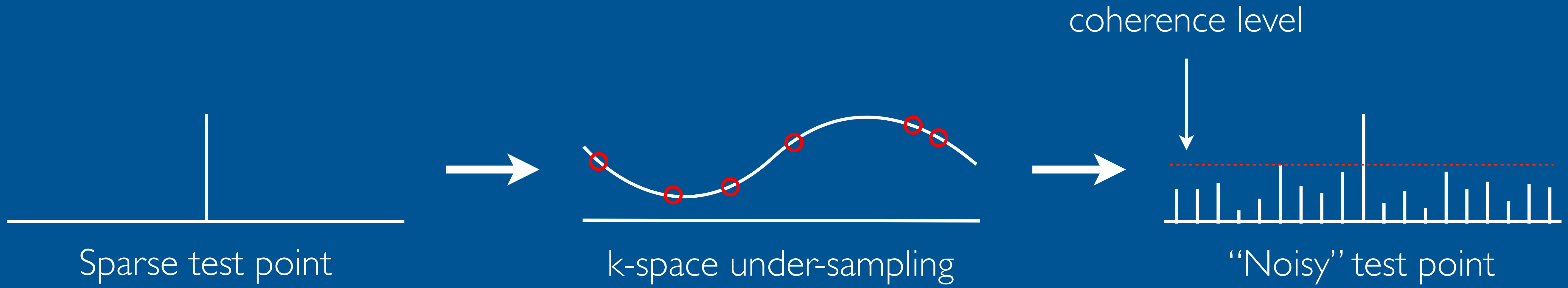
- Two distinct transforms are needed in CS
  - → Sparsifying transform $\mathbf{\Phi}$ takes data to the sparse domain
  - → Measurement transform $\mathbf{\Psi}$ takes data to the sampling domain
- These transforms need to be incoherent or uncorrelated

Sampling and Incoherence



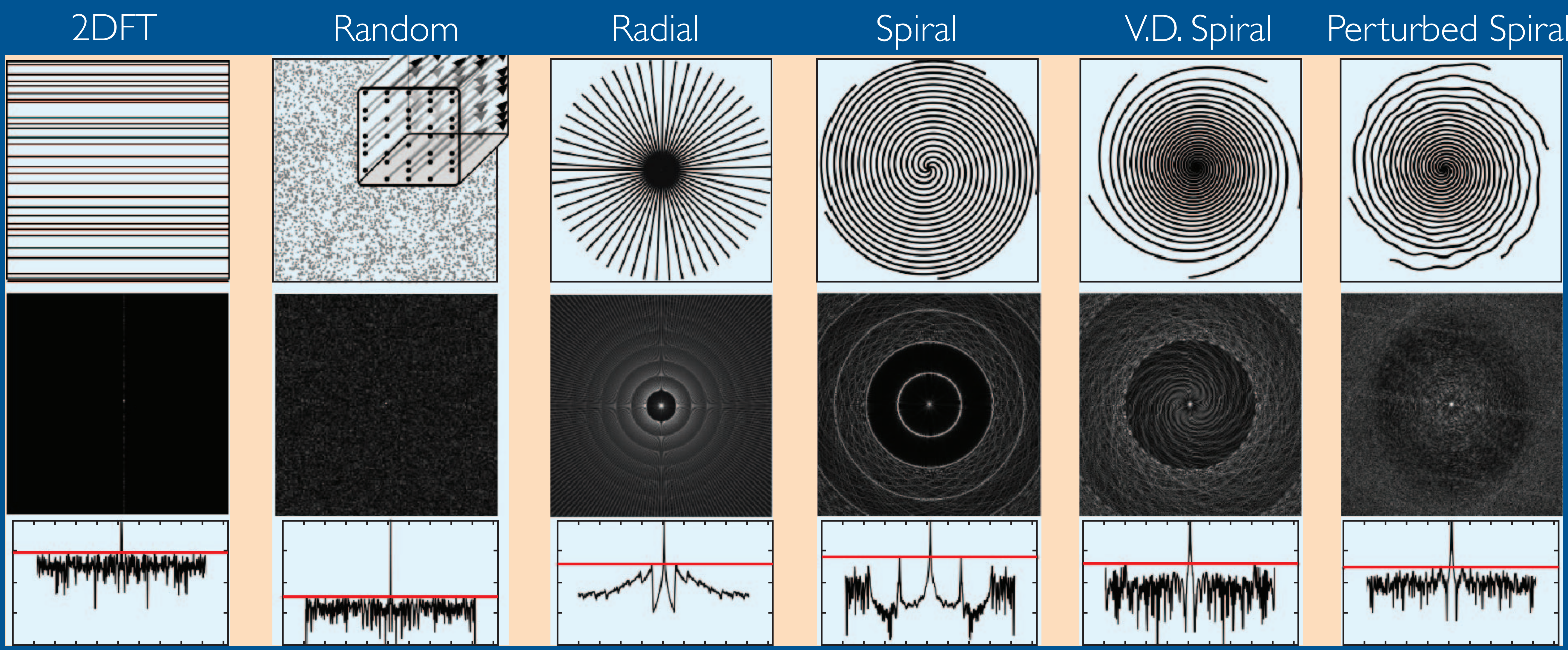
Sampling and Incoherence

- Define coherence using transform point spread function (TPSF)
  1. Consider a single point in the sparse domain
  2. Transform that point into k-space, sub-sample, and go back to the sparse representation
  3. The coherence between transforms (with sub-sampling) is the largest residual "noise" or artefact in the single point signal



- The incoherence requirement involves the measurement and sparse domains, ***not necessarily the image domain***

Sampling and Incoherence



Lustig et al., IEEE Sig Proc Mag 2008

- Sampling patterns balance incoherence and gradient efficiency
- Variable density patterns can exploit knowledge of k-space signal energy distribution

CS Reconstruction

- CS reconstruction aims to recover the true original signal using the following known information:
  1. The value of the signal at the sampled points
  2. That the signal is sparse in some known representation

	Pseudoinverse	Compressed Sensing
Solves	$\min \|\|x\|\|_2$ s.t. $y = Mx$	$\min \|\|x\|\|_1$ s.t. $y = Mx$
Solution	$\hat{x} = M^*(MM^*)^{-1}Mx$	No analytic solution

CS Reconstruction

$$\begin{array}{c} \min ||x||_1 \\ \text{s.t. } y = Mx \end{array} \xrightarrow[\text{to be noisy}]{\text{Allow measurements}} \begin{array}{c} \min ||x||_1 \\ \text{s.t. } ||y - Mx||_2 < \epsilon \end{array}$$

- You will often see it in an unconstrained form:

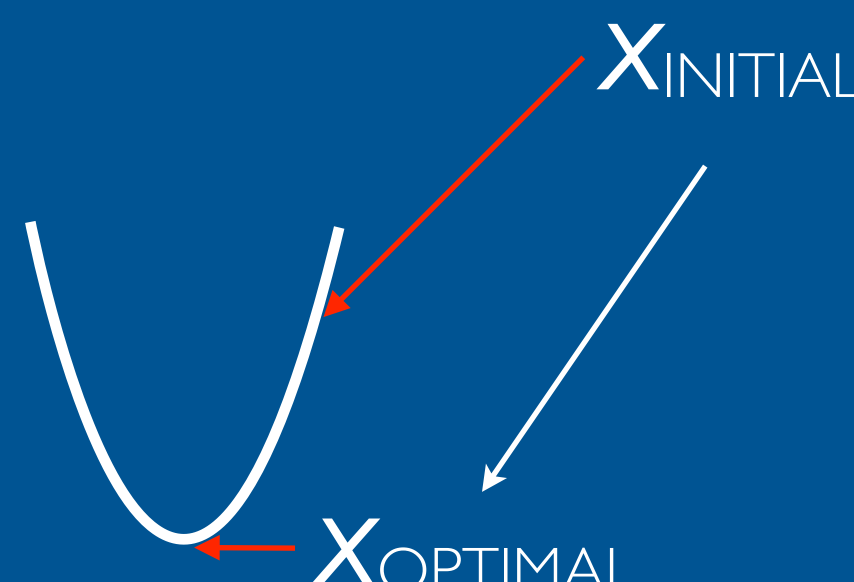
$$\min_x \frac{1}{2}||\Psi x - y||_2^2 + \lambda ||\Phi x||_1$$

CS Reconstruction

The solution is found by solving this convex optimisation problem :

"Find the signal *X* that minimizes this function of *X*"

$$\min_{x} \frac{1}{2}||\Psi x - y||_2^2 + \lambda||\Phi x||_1$$



CS Reconstruction

This is called the objective or cost function

$$\min_x \overbrace{\underbrace{\frac{1}{2}||\Psi x - y||_2^2}_{\text{Data consistency promoting term}} + \lambda \underbrace{||\Phi x||_1}_{\text{Sparsity promoting term}}}$$

Sparsity promoting term

Data consistency promoting term

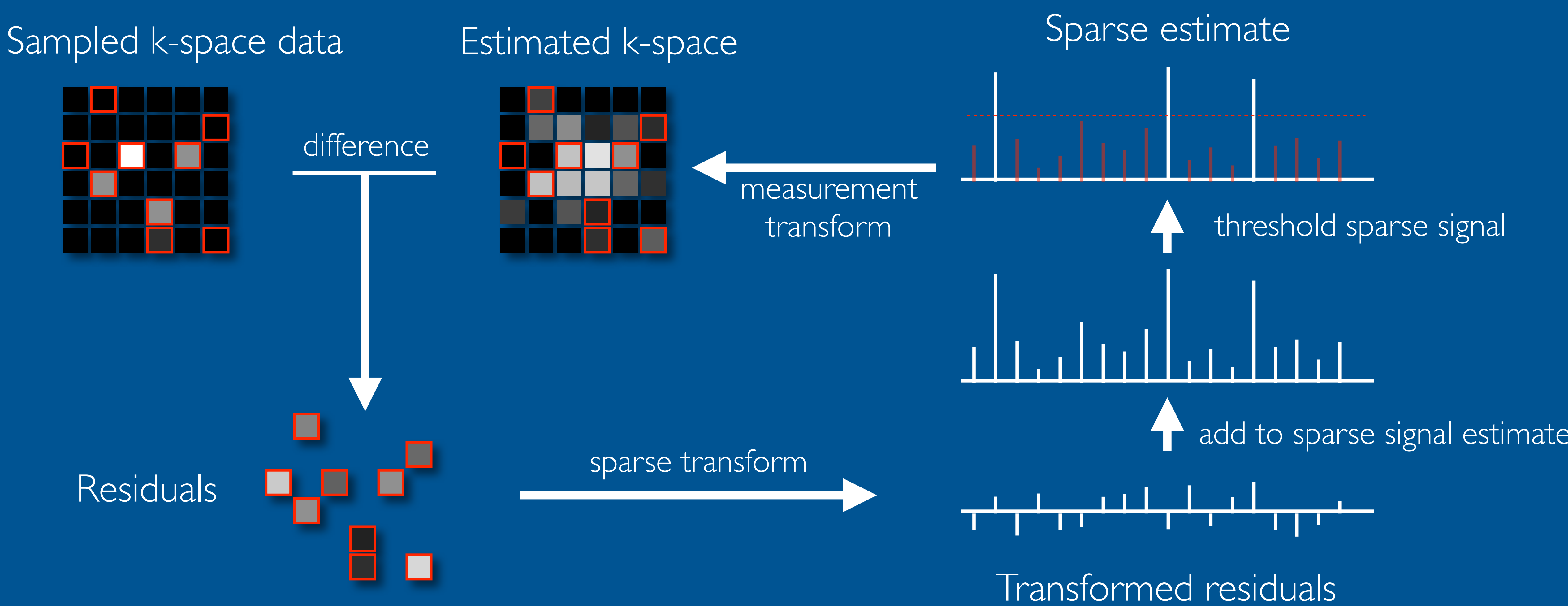
CS Reconstruction

- Lots of algorithms to solve this and related problems

  → gradient descent, linear programming, greedy algorithms...

Find the sparsest solution that is consistent with the measured data

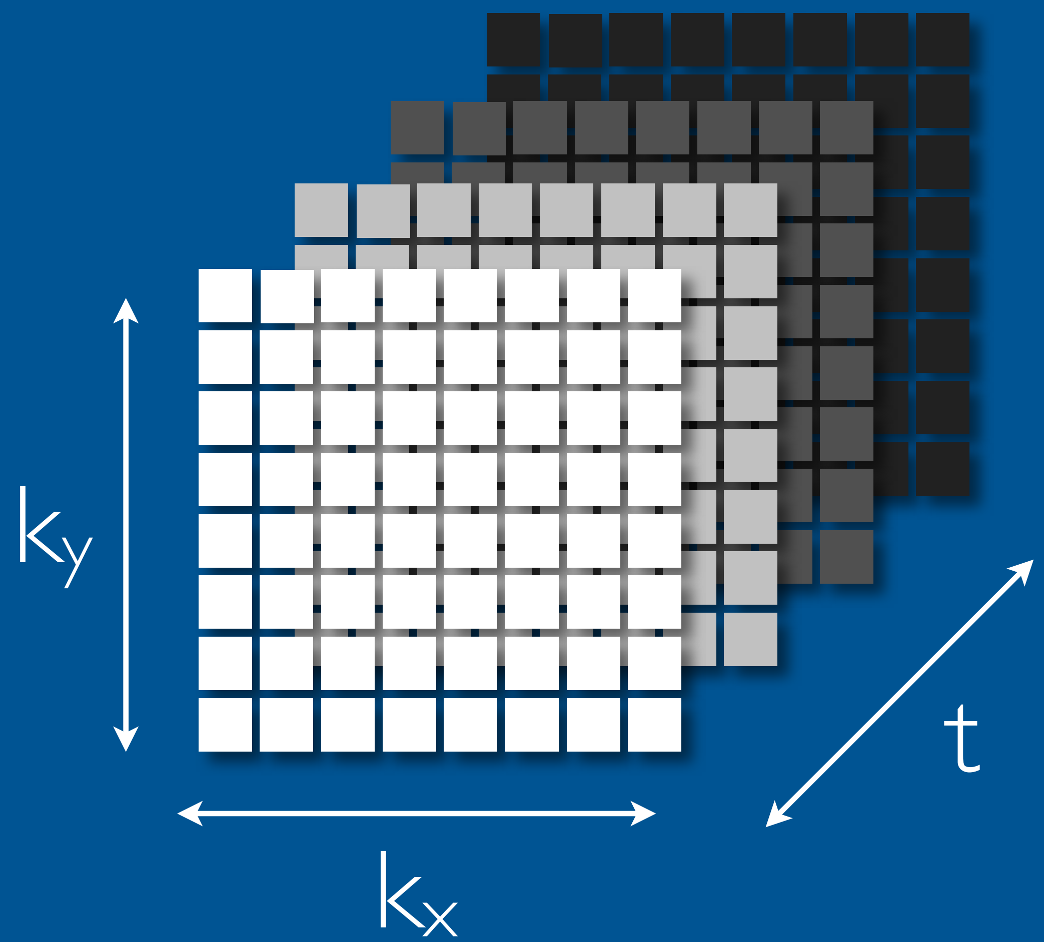
- Iterative Hard Thresholding[1]:



1. Blumensath et al., Appl Comp Harmon Anal 2009

Sparse k-t Imaging

- Extra dimensions, like time, can be integrated assuming there are appropriate sparsifying transforms for them
- k-space + time = k-t space

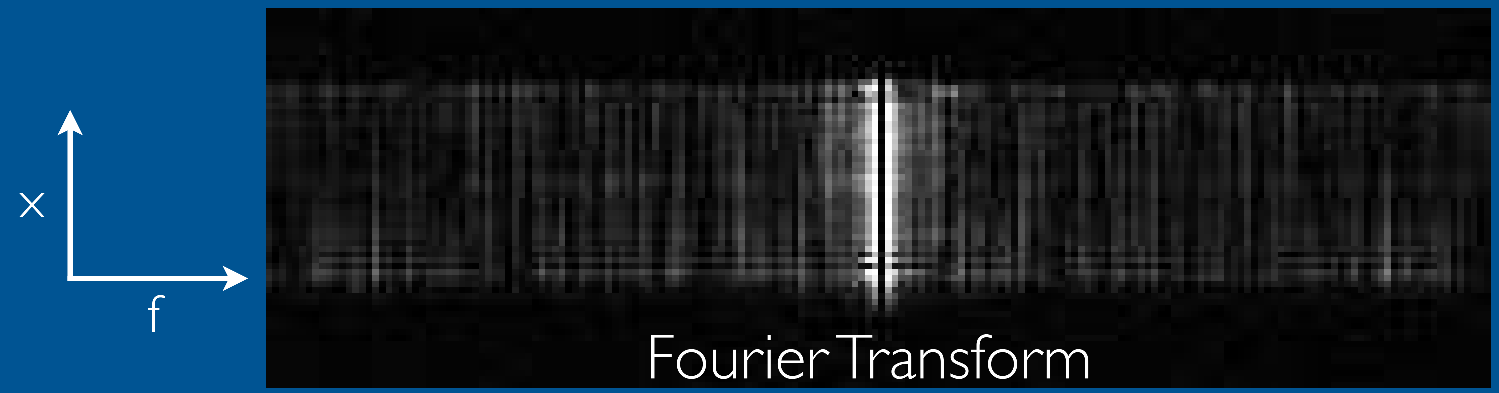


Multi-dimensional array

k-t matrix

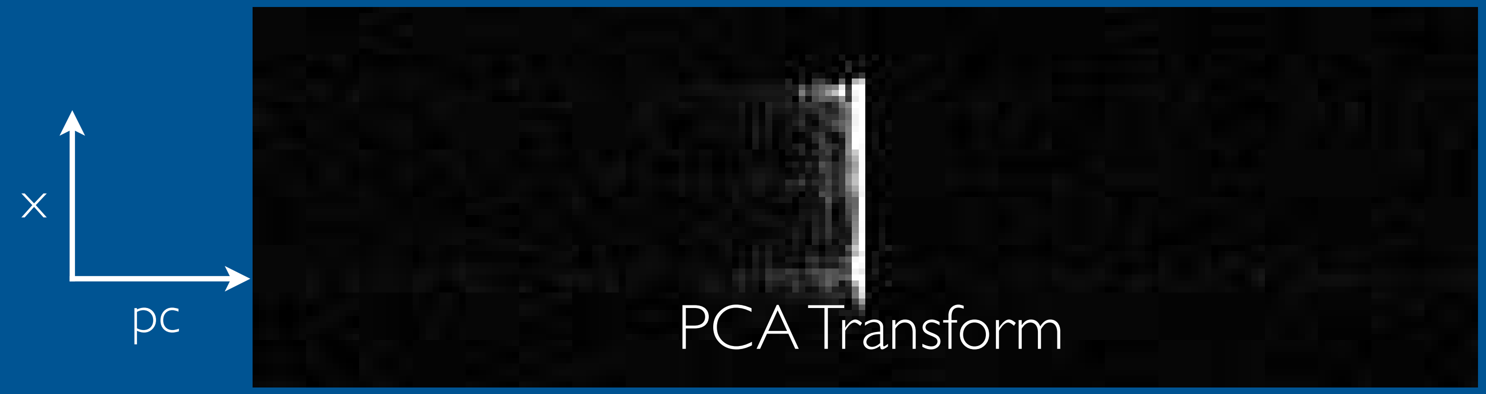
CS and FMRI

- FMRI time-series are not typically sparse in the frequency domain



Jung et al. Phys Med Biol 2007

- Apply CS time-independently on each image in the time series[1,2]
  - → Purely image-based acceleration can be limiting
- Alternative transforms can sparsify temporal dimension
  - → Principal component analysis (PCA) for sparsification

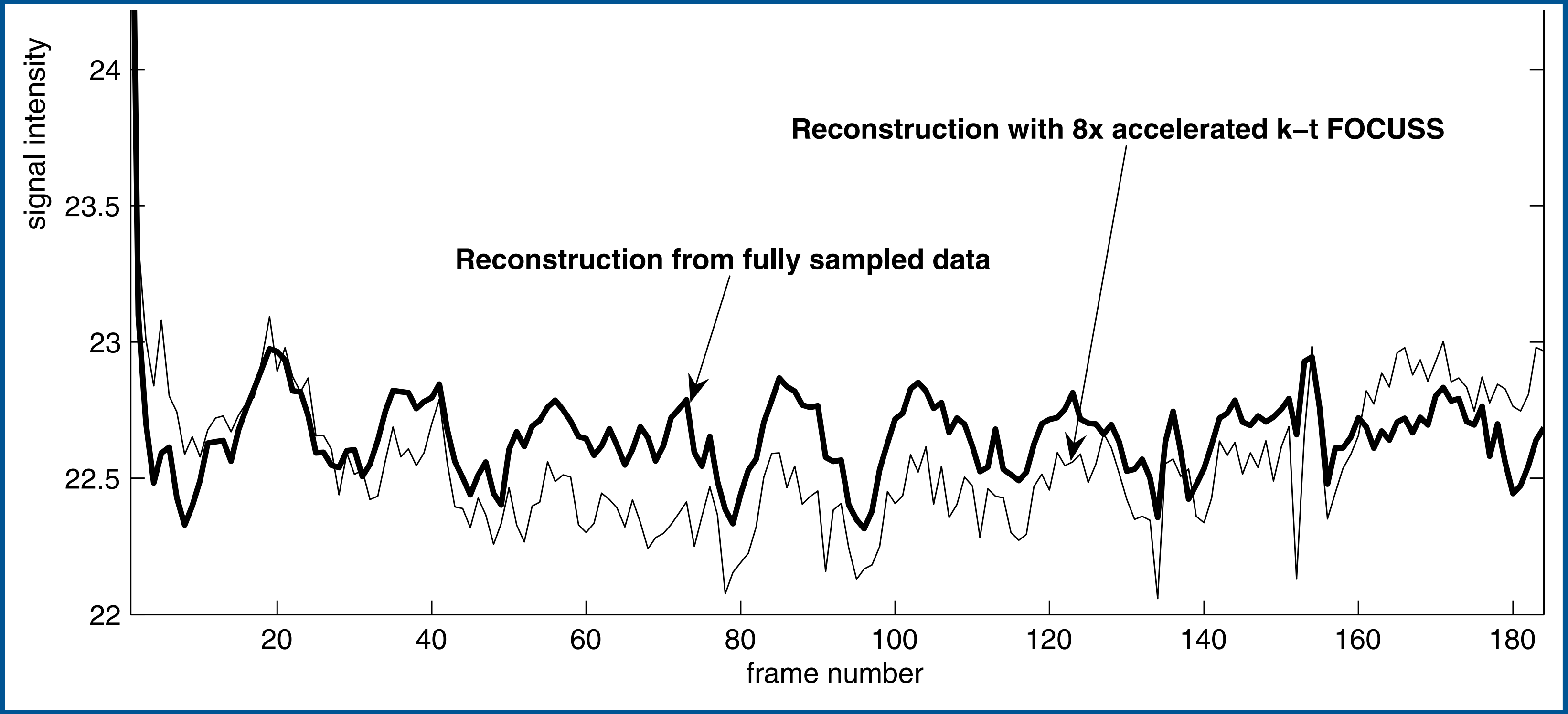


Jung et al. Phys Med Biol 2007

1. Jeromin et al. BioMedical Engineering Online. 2012
2. Holland et al. Magn Reson Med. 2013

CS and FMRI

- Fourier transform → Fourier basis functions
- PCA transform → Data derived principal component basis set



Jung et al. Phys Med Biol 2007

- If k-t data are sparse under PCA $\iff$ k-t matrix is low rank
  → Minimizing rank of a matrix is very similar to maximizing sparsity of a signal or image

Low-rank FMRI Acceleration

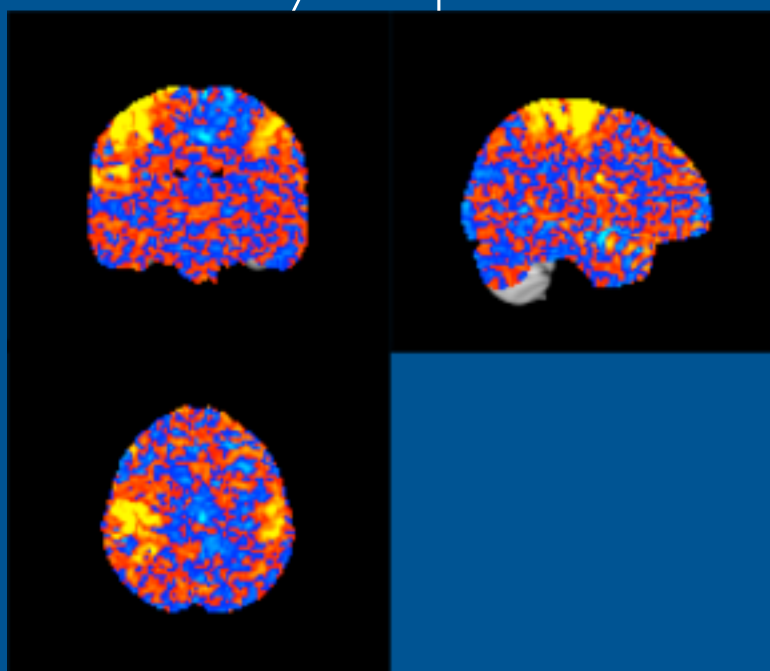
- **Matrix Completion** methods recover under-sampled k-t matrices using rank-minimisation instead of sparsity minimisation
  - → L_1 minimisation on vector of eigenvalues
- One advantage of using Matrix Completion on k-t matrices
  - → Sparsifying basis not needed *a priori*
  - → The rank-minimizing bases are estimated in matrix recovery
- Low-rank[1] or a combination of low-rank and sparsity[2] constraints are being explored now to accelerate FMRI acquisition

1. Chiew et al., ISMRM 2013, #3274
2. Lam et al., ISMRM 2013, #2620

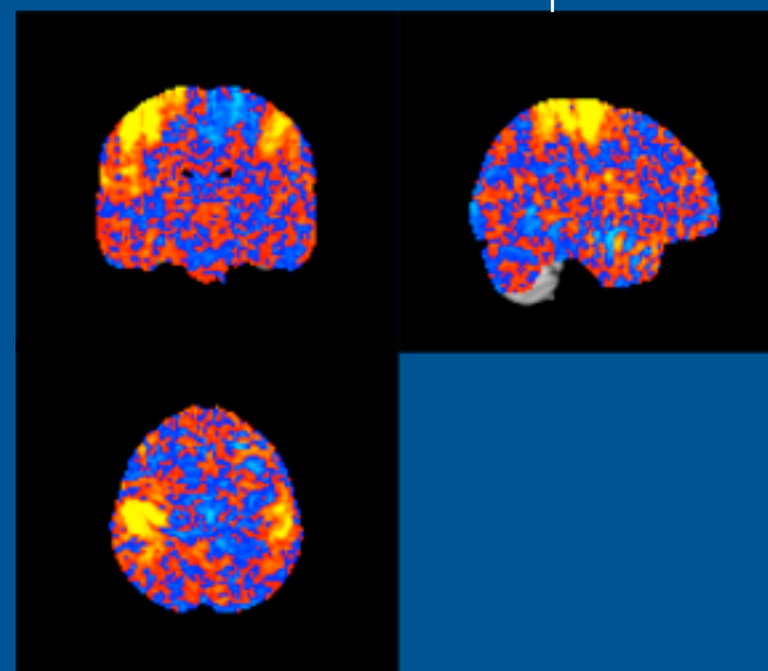
Low-rank FMRI Acceleration

Right somato-motor resting state network

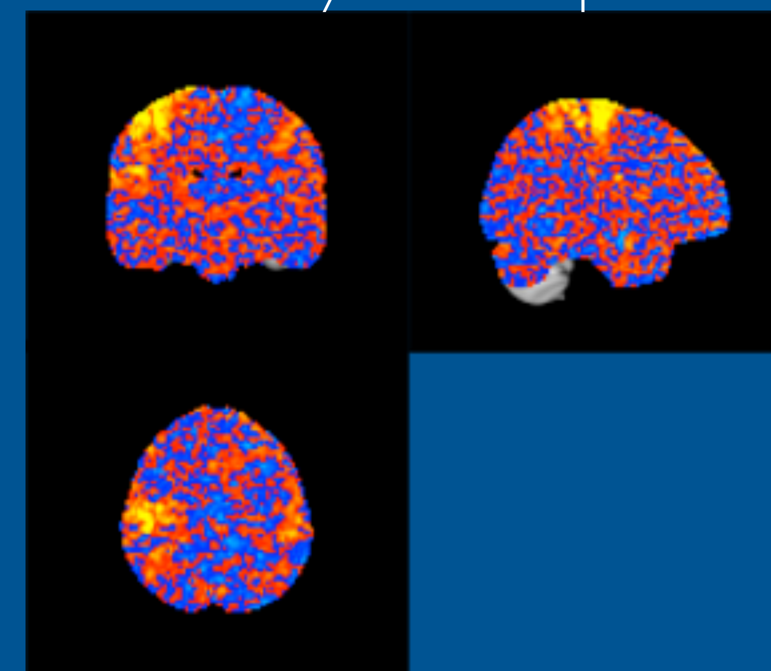
Fully sampled



4x Under-sampled

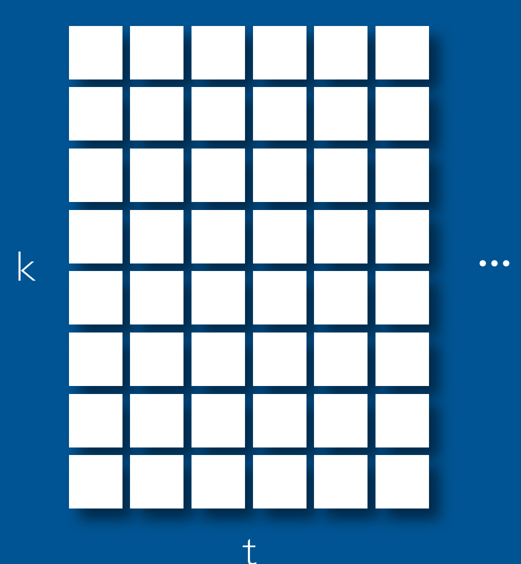


Every 4th time point

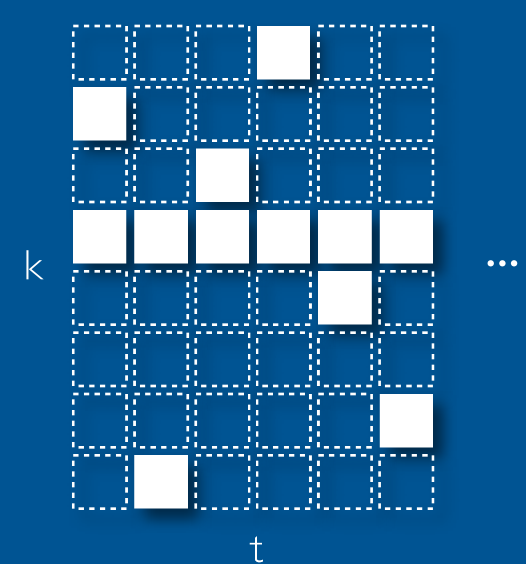


Chiew et al., unpublished data

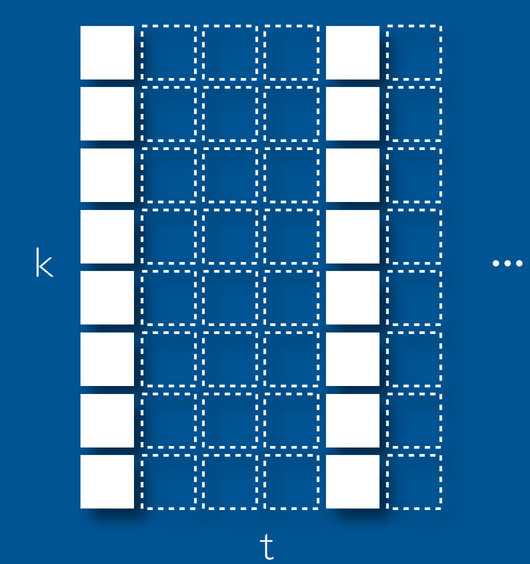
100% data



25% data



25% data



Summary

1. **Sparsity & compressibility**

   → Sparsity permits more efficient (faster) sampling

2. **Random & incoherent sampling**

   → Efficient sampling requires incoherence

3. **Iterative & nonlinear reconstruction**

   → Image reconstruction is an optimisation problem

4. **FMRI acceleration with CS and related techniques**

   → Much room for investigation

   → Sparsity and/or low-rank constraints may be useful in FMRI

Compressed Sensing

Questions?

CS Basic Intuition

- Standard Fourier imaging
  - → Matrix is the discrete Fourier transform matrix (F)
  - → Each row of F corresponds to the measurement of a point in k-space

$$k_{(n\times 1)} = F_{(n\times n)} \cdot x_{(n\times 1)}$$

- To find $x = F^{-1} \cdot k$, need length(k) == length(x)
  - → Need as many independent measurements as unknowns
  - → Otherwise F is not invertible and no unique solution exists

CS Reconstruction

λ is a parameter that controls relative importance of both terms

$$\min_X \frac{1}{2}||\Psi_\Omega X - \Psi_\Omega Y||_2^2 + \lambda||\Phi X||_1$$

Additional information about the data ***X*** can be leveraged to further constrain the solution by adding terms to the objective

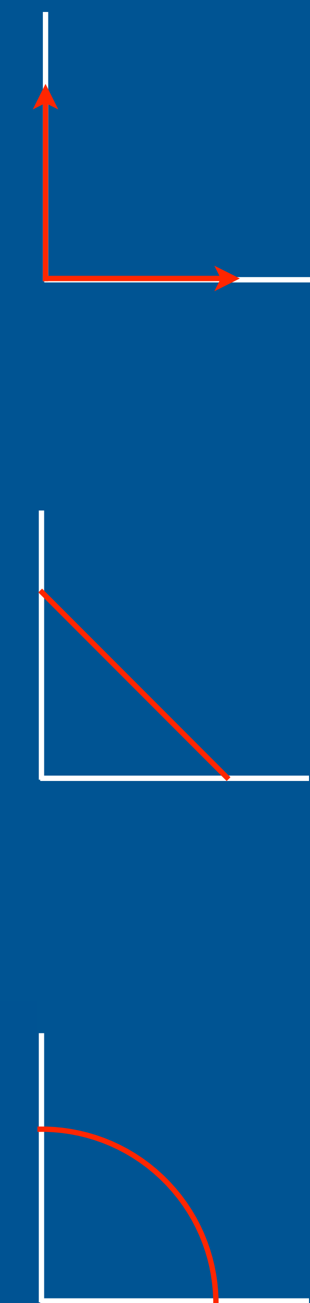
$$\min_X \ldots + \lambda_2 \cdot \mathrm{TV}\{X\}$$

Essential CS Glossary

- **L_0 norm*** (# non-zero values)
  - → $||x||_0 = \sum (x_i \mathrel{!=} 0)$
  - → Direct measure of sparsity, very difficult to work with
- **L_1 norm** (sum of absolute coefficients)
  - → $||x||_1 = \sum |x_i| = 12$
  - → Indirect proxy for sparsity, less difficult to work with
- **L_2 norm** (root sum of squared coefficients)
  - → $||x||_2 = (x^T x)^{0.5} = (\sum x_i^2)^{0.5} = 7.87$
  - → Euclidean distance norm, easy to work with

2D Level Sets

$||x||_n = 1$



CS Reconstruction

- Lots of methods to solve optimization problems
  - → gradient descent, linear programming, etc...
- Other methods
  - → FOCUSS solvers[1]
  - → Greedy algorithms[2]

Find the sparsest solution that is consistent with the measured data

- Iterative Hard Thresholding[3]:

Sets all but the largest **s** coefficients to zeros

Measured data

$$x^{n+1} = H_s(x^n + \Phi^*(y - \Phi x^n))$$

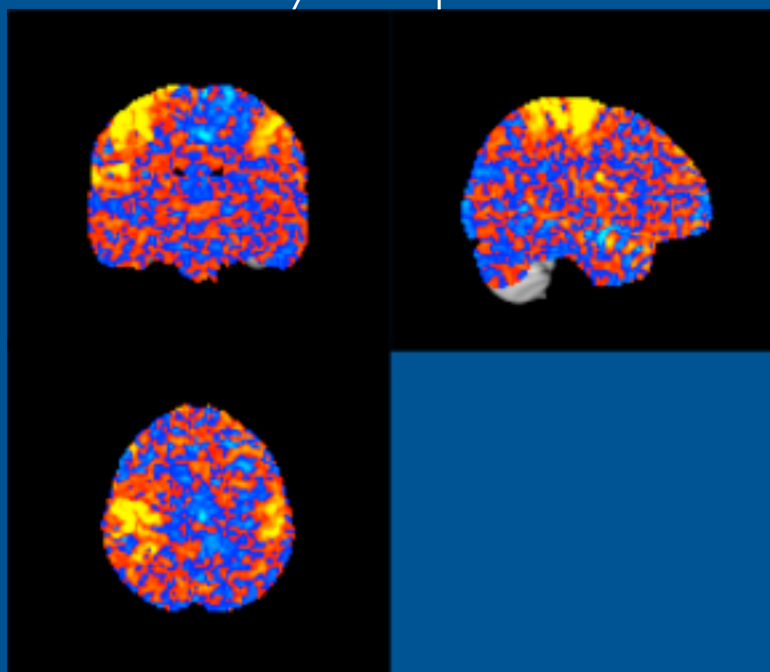
n+1th sparse signal estimate

Transforms sparse signal to and from measurement space

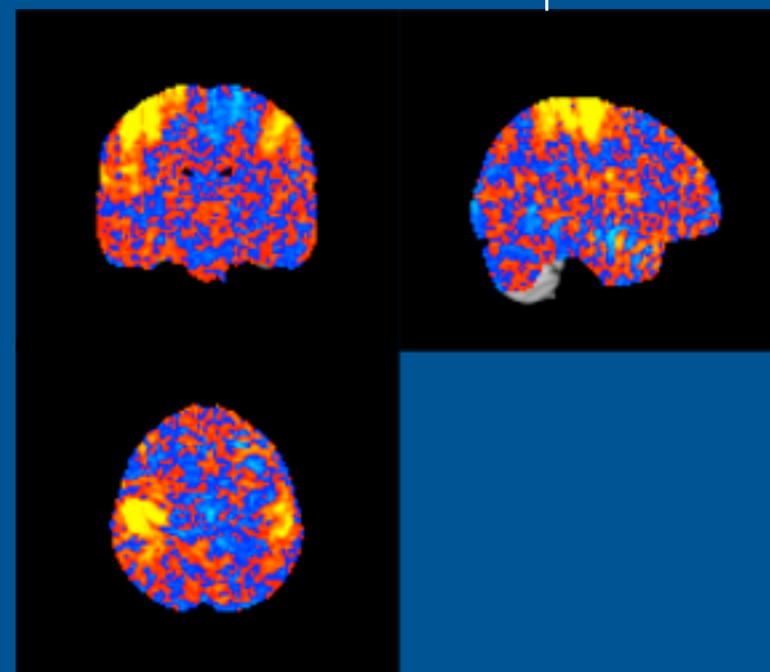
1. Jung et al., Phys Med Biol 2007
2. Usman et al., Phys Med Biol 2011
3. Blumensath et al., Appl Comp Harmon Anal 2009

Low-rank FMRI Acceleration

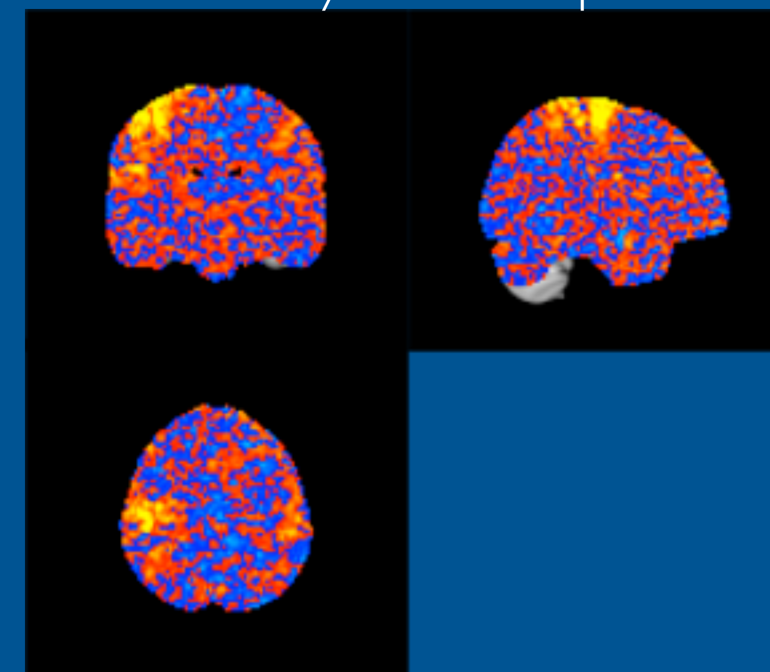
Fully sampled



4x under-sampled



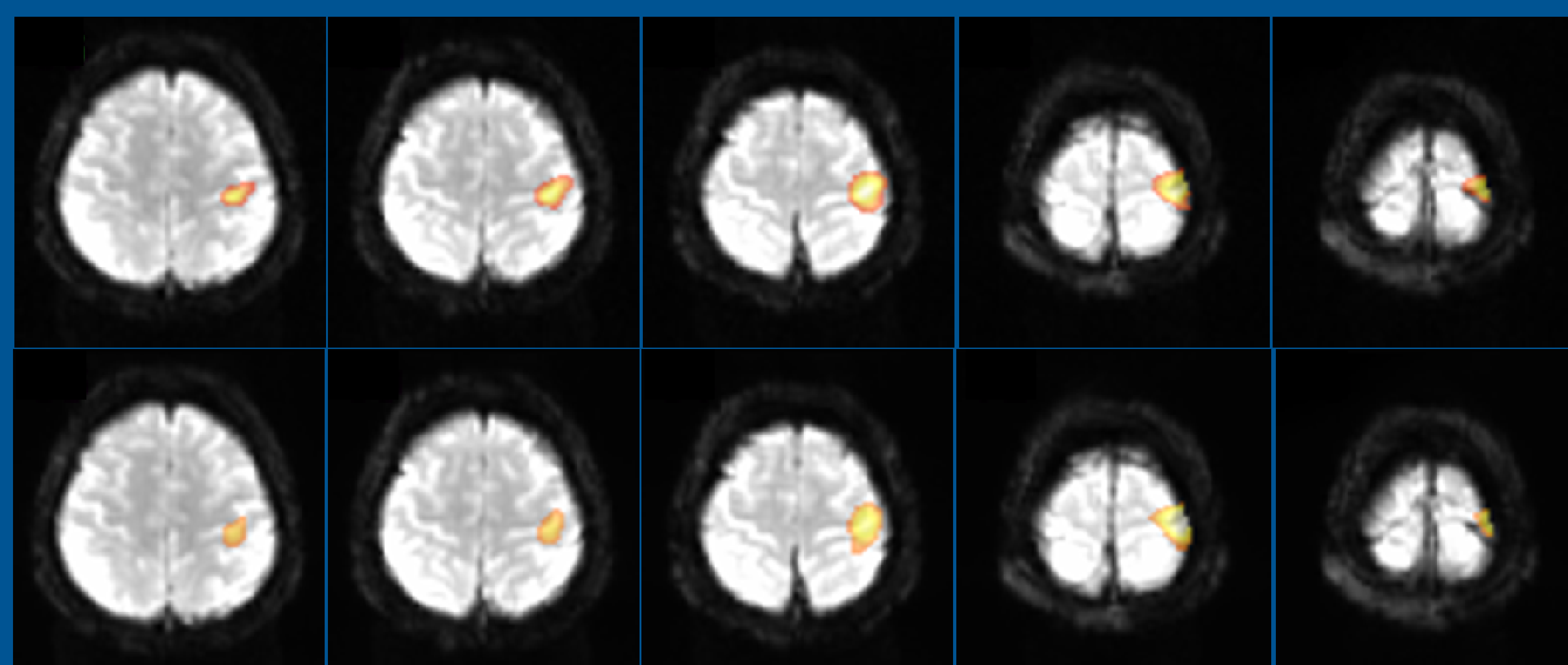
Every 4th time point



Chiew et al., unpublished data

Fully sampled

5x
Under-sampled



Lam et al., ISMRM 2013

Overview

- Essential glossary of terms and ideas
- Basic intuition behind compressed sensing
  → How can you get more "information" than you sample?
- CS imaging principles
  → Data sparsity
  → Sampling and incoherence
  → Image reconstruction
- k-t sparse and application of CS to FMRI
  → Low-rank k-t acceleration methods for FMRI