Compressed Sensing
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Introduction

In recent years, compressed sensing (subsequently abbreviated as "CS") strategies have enabled great advances in accelerating the acquisition of MR images. In simple terms, CS allows images to be reconstructed from far fewer measured k-space points than predicted by Nyquist sampling theory, so long as the data have a sparse representation. Since imaging time is roughly proportional to the measured k-space matrix size, this facilitates much faster imaging.

In a simplified history, CS emerged from theoretical work by Candès et al. [1], and Donoho [2] in the early-to-mid 2000s. These highly theoretical works established the mathematical basis for CS, and even discuss application to imaging. The first demonstration of CS in MRI was published by Lustig et al. in 2005 [3], with the first applications to dynamic imaging appearing in the literature in 2006 (k-t sparse [4]) and 2007 (k-t FOCUSS [5]).

Over the past few years, CS has been applied to an enormous range of applications within MR imaging, including functional MRI [5]. In addition, acceleration strategies based on similar, but different constraints such as k-t matrix rank [6] have been introduced as a potential alternative to CS for dynamic imaging, and can also employ random sampling with iterative recovery methods [7]. More recent work has explored joint sparsity- and rank-based image reconstruction, resulting in a hybrid of CS and low-rank matrix recovery techniques for dynamic imaging [8, 9].

Basic Justification for Compressed Sensing

First we briefly review the interpretation of the standard imaging procedure as a linear system. Consider an image, x, in vector form. The image x represents the unknown quantity to be measured, with its entries reflecting a particular weighting of the magnetization properties in a given position. Traditionally, to recover x, because we cannot directly probe its entries, we measure a set of linear functionals[1] on x:

$$b = Ax \tag{1}$$

where each row of the sensing or measurement matrix A corresponds to a different functional, and the results are stored in the measurement vector b. When the rows of A are Fourier basis functions, b is the familiar k-space data. To recover x, the inverse problem needs to be solved:

$$x = A^{-1}b \tag{2}$$

In the standard, deterministic, linear imaging framework, for x to be normally recoverable, A must have full rank. That is, there need to be as many linearly independent measurements of x as there are entries in x. Equivalently, the k-space matrix size must be the same as the desired image size, or that A must have as many linearly independent rows as there are unknowns in x. When this sampling requirement is satisfied, A can be trivially inverted to solve (2).

Acceleration strategies focus on reducing the number of rows in A, and thereby reducing imaging time. This however, causes A to be rank deficient, and causes the inverse problem to become ill-posed[2]. Additional information or constraints need to be imposed to find an appropriate solution.

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[1] A linear functional is something that takes a vector as input and returns a scalar, like the inner product. Every measured k-space point is the output of a linear functional, namely the inner product between the image vector and the Fourier basis vector corresponding to that k-space location.

[2] A problem is ill-posed if no unique solution exists. For example, linear equations are ill-posed if there are fewer rows than columns in the matrix, which means that there are fewer constraints than unknowns.

Where parallel imaging approaches incorporate additional information via spatially varying coil sensitivity projections (and are restricted in application to appropriate multi-coil measurement systems), CS introduces the mathematical constraint of sparsity to find a solution (and is restricted in application to data that have sparse representations). The sparsity or compressibility of a signal refers to the property of having very few non-zero coefficients (in some representation). That is, either the native signal or some transformation thereof contains mostly zeros. By definition, a k-sparse signal is one with exactly k non-zero coefficients.

(a)   (b)   (c)
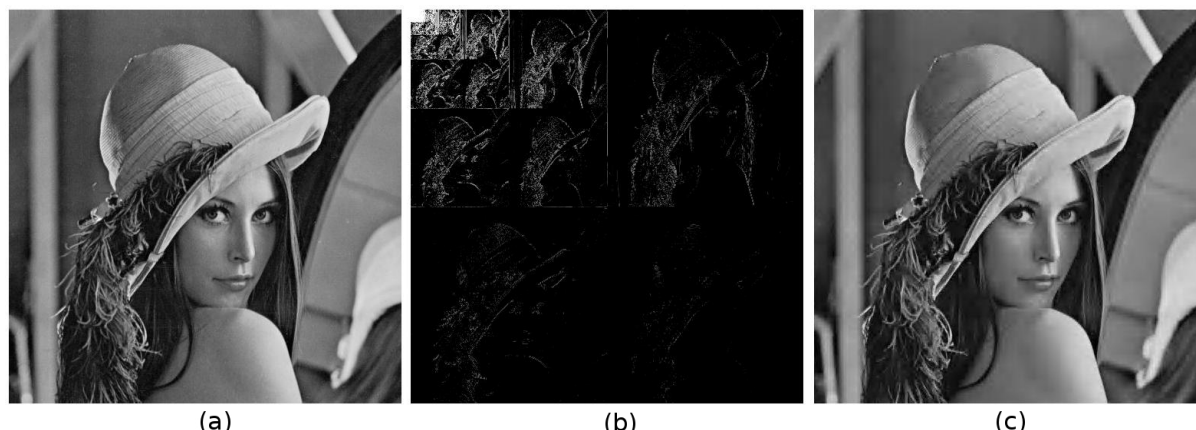
[Figure 1 – (a) Original 512x512 image. (b) Log-scaled Daubechies-4 wavelet transform of (a), windowed to emphasize the sparsity of the image in the wavelet domain. (c) Reconstructed image after retaining only the top 5% of wavelet coefficients, and setting the remaining 95% to zero. The near perfect reconstruction in (c) highlights the compressibility of the original image]

The compressibility of a sparse signal comes from the fact that it takes far fewer non-trivial (i.e., non-zero) numbers to represent the signal than its dimensionality would suggest. This provides the intuitive basis for understanding the ability of CS to reconstruct images from what seems like "too few" measurements. With *a priori* knowledge about the sparsity of the image, there is actually less information or fewer degrees of freedom that need to be captured in the acquisition.

Imaging with Compressed Sensing
There are 3 main components that are central to CS:

A. An appropriate sparsifying transform
As mentioned above, the image to be compressively sensed must be sparse for image reconstruction to be successful. Some MR image data, like MR angiograms, are natively sparse, others require linear transforms such as the finite difference (good for piecewise continuous images) and discrete wavelet transforms (often used for structural images). The sparse representation of the image is not restricted to orthonormal basis sets, and over-complete dictionaries[3] can also be used to sparsify images [10].

The existence and knowledge of a sparse representation of the image ensures its compressibility, which allows it to be efficiently reconstructed from relatively few samples. The number of samples required to guarantee reconstruction fidelity is related to its sparsity, and although theoretical bounds exist, they are not practically useful. In application, heuristics and rules of thumb are used to estimate maximally practical undersampling factors. For example,

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[3] A dictionary is a set that spans the image space, which is often over-complete and therefore non-orthogonal. Dictionaries are often constructed to maximize sparsity in an image or class of images. Contrast this with a Fourier basis set, which is a data-independent, analytically defined orthonormal basis.

sampling factors of 2-5 points for every sparse coefficient [11], and a 4-to-1 ratio [12] have been quoted in the literature.

B. Incoherent and random sampling

To guarantee robust reconstruction of the sparse images, particularly in the presence of noise, measurement matrices A must obey the restricted isometry property (RIP)[4], which essentially ensures that distances between k-sparse vectors are preserved in the measurement domain [13]. This property is the crux of reconstruction guarantees in CS.

Since the measurement domain in MRI is fixed[5], we cannot generate measurement arbitrary matrices with ideal RIP characteristics. Instead, to make sure our sensing or measurement is robust, we need to satisfy two conditions: that the measurement basis and the sparsifying basis are *incoherent*, and that the samples are drawn from a *random* subset of the measurement basis. This is sufficient to ensure that the RIP is satisfied.

Incoherence between two basis sets refers to the degree of correlation between them [4]. Let Ψ represent the sparsifying basis, and let our measurement domain be F, the discrete Fourier basis. If Ψ and F are maximally incoherent, no two basis vectors in either set are highly correlated. Incoherence is related to uncertainty principles in physics, in that a signal that is highly localized in Ψ (i.e. sparse), it will be spread out and highly diffuse in F. The standard basis e shares this property with the Fourier basis. This requirement can be understood intuitively with the following example. Imagine the sparse and measurement basis to be identical, and therefore maximally coherent. If the signal is highly sparse, then under-sampling the signal in the same domain will likely miss all of the non-zero signal elements. The incoherence requirement ensures that no matter where the samples are taken, information about the signal is being captured.

(a) (b) (c)
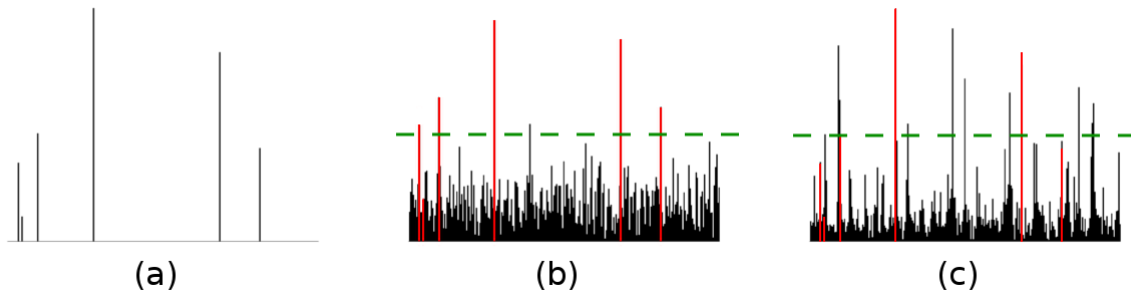
[Figure 2 – (a) Original sparse signal. (b) Random undersampling of the sparse signal in the Fourier domain, transformed back into the sparse domain. Most of the original sparse signal components (in red) are visible above a noise threshold (green dashed line). (c) Regular undersampling results in many aliased signal components, and the original signal components are not distinguishable from artefact.]

Incoherence is necessary but not sufficient to build a robust measurement matrix. Sampling in F also needs to be random. That is, measured data points must be randomly distributed in k-space. Without this condition, aliasing artefacts can confound signal estimation. Although most of the CS theoretical development deals with uniformly random sampling, in MR, practical imaging considerations such as efficient gradient trajectories and k-space signal energy concentration mean that quasi-random, density weighted sampling patterns are most often used [11].

The overall efficacy of a particular k-space sampling pattern can be evaluated using a transform point-spread-function (TPSF) analysis, which can be computed using the following relation [11]:

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[4] A matrix A obeys the RIP of order k if and only if there exists a number $\delta \ll 1$ such that:
$$(1 - \delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta)\|x\|_2^2$$
where x is in the set of all k-sparse vectors.

[5] Assuming of course that imaging occurs with linear gradients, collecting samples in Fourier or k-space.

$$TPSF(i;j) = e_j \Psi F_U^* F_U \Psi^* e_i \qquad (3)$$

which evaluates the effect of the impulse at *i* on location *j* after going through the undersampled measurement process and back. The TPSF is ideally as close to a delta function as possible, with minimal and irregular side lobes [11].

C. Image reconstruction as an optimization problem

Once the sparsity and sampling requirements are satisfied, the last important consideration is the actual image reconstruction. Ideally, the reconstruction problem is solved by identifying the sparsest solution that is consistent with the measured data [1]. Directly maximizing sparsity, however, is a computationally intractable problem. Instead, the most popular alternative has been to use L_1 minimization to approximate sparsity[6]:

$$\min_x \|x\|_1 \ such \ that \ \|y - Ax\|_2 < \epsilon \qquad (4)$$

where the L_1 norm is just the sum of the absolute value of the coefficients of *x*, and ε is a small value accommodating noise in the measurement process. Note that although L_1 minimization is the most popular approach to CS image reconstruction, it is not the only alternative. Greedy nonlinear algorithms[7], such as iterative hard thresholding can reconstruct sparse solutions without relying on the L_1 norm at all [14].

Equation (3) itself can be solved directly, or transformed into an unconstrained minimization problem through the use of Lagrange multipliers, although this comes at a cost of an additional tuning parameter λ, which weights the relative importance of the data consistency term with the sparsity promoting term:

$$\min_x \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_1 \qquad (5)$$

This is also referred to as the "basis pursuit denoising" problem, and a non-linear conjugate gradient algorithm is outlined in [11]. Any convex optimization solver can be used to solve (5), however.

Application to Dynamic Imaging and FMRI

The above discussion has been restricted to reconstruction of sparse images, without consideration for MR time series data. Very early in the development of CS MRI, Lustig et al. developed the framework for sparse dynamic MRI acting on k-t space [4]. The extension is nearly trivial:

i. Sampling must be randomized along the t-dimension as well as over k-space, so that no temporal or spatial coherence artefacts are produced
ii. As the k- and t-dimensions are independent, a separate transform must exist that properly sparsifies the temporal axis (e.g., Fourier transform for periodic data)

Many of the applications of CS to dynamic MRI have followed the same basic strategy as above. These have been most successful in applications such as cardiac cine MRI [15], which exhibit very high degrees of temporal periodicity, and therefore sparsity. Newer developments in dynamic CS include methods like k-t group sparse [16] and k-t ISD [17], which exploit additional information about the structure or support[8] of the x-f signal in the reconstruction.

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[6] The L_1 norm is the convex envelope of the "L_0" pseudo-norm, which counts non-zero coefficients.

[7] A greedy algorithm arrives at a solution though a series of locally optimal steps in the hopes of arriving at a global optimum.

[8] The support of a signal is the set of locations on which it is non-zero.

Early work in sparse k-t imaging using the FOCUSS solver [5], which extended the k-t BLAST/SENSE undersampling framework [18], explored the application to FMRI in retrospective sampling simulations. One of the primary barriers to the application of sparse k-t imaging in FMRI is that it has a broadband temporal spectrum, and is not easily sparsifed using the Fourier transform. To overcome this, an empirical sparsifying basis set was determined using a principal component analysis of a set of training data, although this requires additional data collection, either in-line (reducing the effective acceleration) or as a separate acquisition.

Some groups have explored the use of standard, time-independent CS in FMRI [19, 20], by simply performing the sparse reconstruction on an image-by-image basis. These methods do not exploit any temporal signal characteristics, which can limit their effectiveness, particularly given that image acceleration in FMRI is limited by optimal echo times for BOLD contrast. Furthermore, image-by-image reconstruction may induce spurious contrast fluctuations, which may impact FMRI statistical inference.

Related k-t Acceleration Strategies

Recently, work in k-t acceleration has incorporated rank constraints in place of or in addition to sparsity constraints, by considering k-t space as a $k{\times}t$ matrix. With this formulation, guarantees about the rank of the data can lead to accelerated imaging using outer product signal models [6, 21], because low-rank matrices have greatly reduced degrees of freedom, in the same way that sparsity ensures compressibility.

Many of the new developments in recovery of randomly undersampled low-rank matrices, called the "matrix completion" problem [22], are directly applicable to k-t acceleration, and are in many ways directly analogous to sparsity-based imaging [7]. In fact, a low-rank matrix can be thought of as having a sparse spectral decomposition. The advantage of these methods is that unlike in typical CS, the basis under which this "sparsity" occurs does not have to be known *a priori*. Instead, the basis is estimated as part of the rank-minimizing reconstruction. There is work however, demonstrating blind CS approaches for dynamic MRI [23]. Joint sparsity and rank constraints have been successful in exploiting the strengths of both constraints simultaneously [8, 9], and recent work using robust PCA methods [24] has explored k-t acceleration based on a sparse + low-rank decomposition [25].

Finally, some of these newer methods are starting to be explored in FMRI [26, 27], although this is still a relatively unexplored topic of research.

Conclusion

The introduction of compressed sensing has had an enormous impact on MRI and the development of accelerated imaging strategies. Improvements in sampling strategies and reconstruction algorithms, the incorporation of extra information, constraint and data structure into CS strategies, synergies with other acceleration methods like parallel imaging[9], and new applications are a few examples of recent developments in the literature that contribute to the exciting and fast-growing field of accelerated and dynamic MRI.

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[9] Discussion of which is omitted here for brevity.

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