Low-Rank & Structured Low-Rank Reconstruction Approaches

Image Reconstruction

ISMRM 2021

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Declaration of Financial Interests or Relationships

Speaker Name: Mark Chiew

I have no financial interests or relationships to disclose with regard to the subject matter of this presentation.

Goals

Intuition and Mechanics

A Motivating Example

The Netflix Problem

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		5		4					3		
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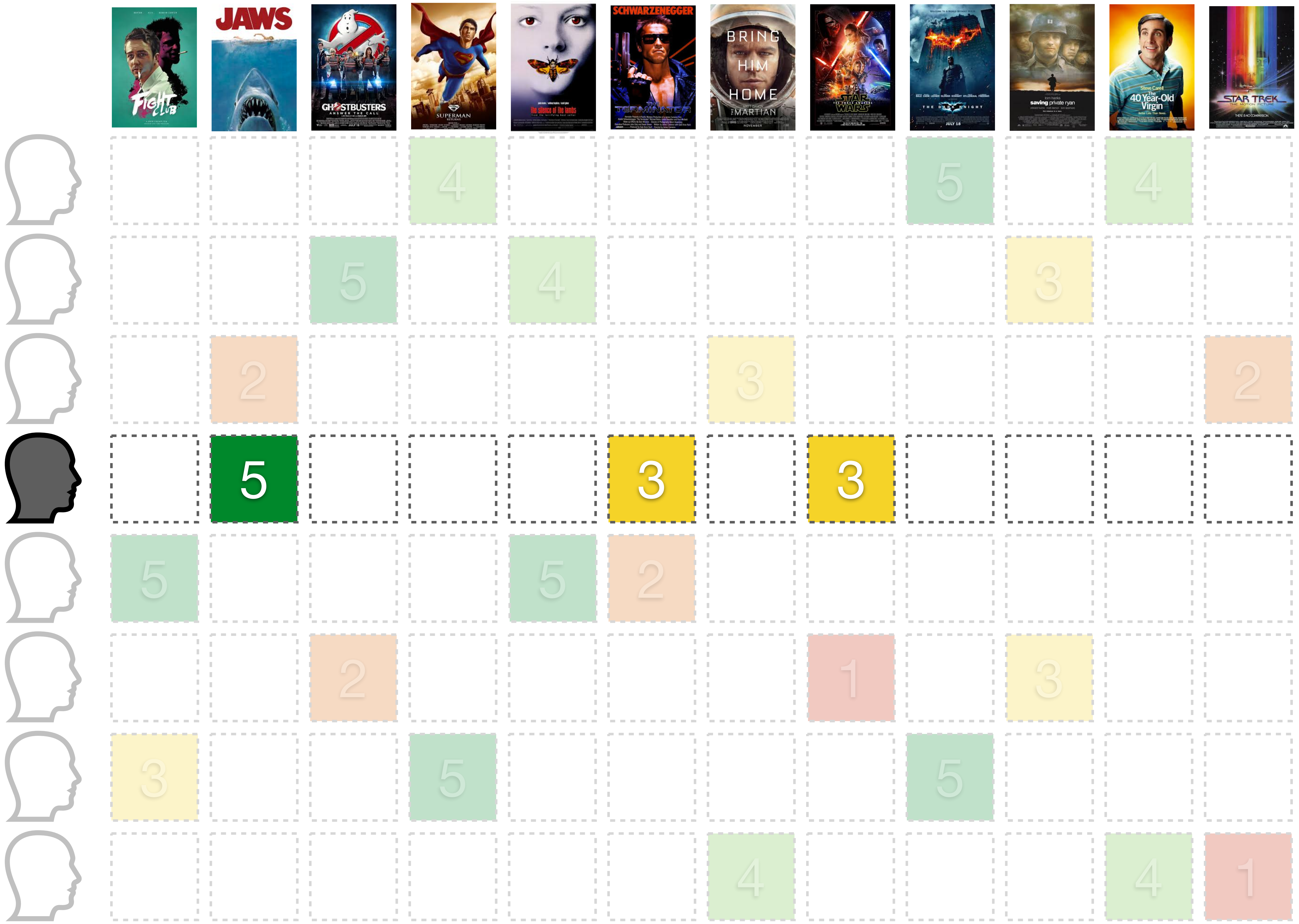
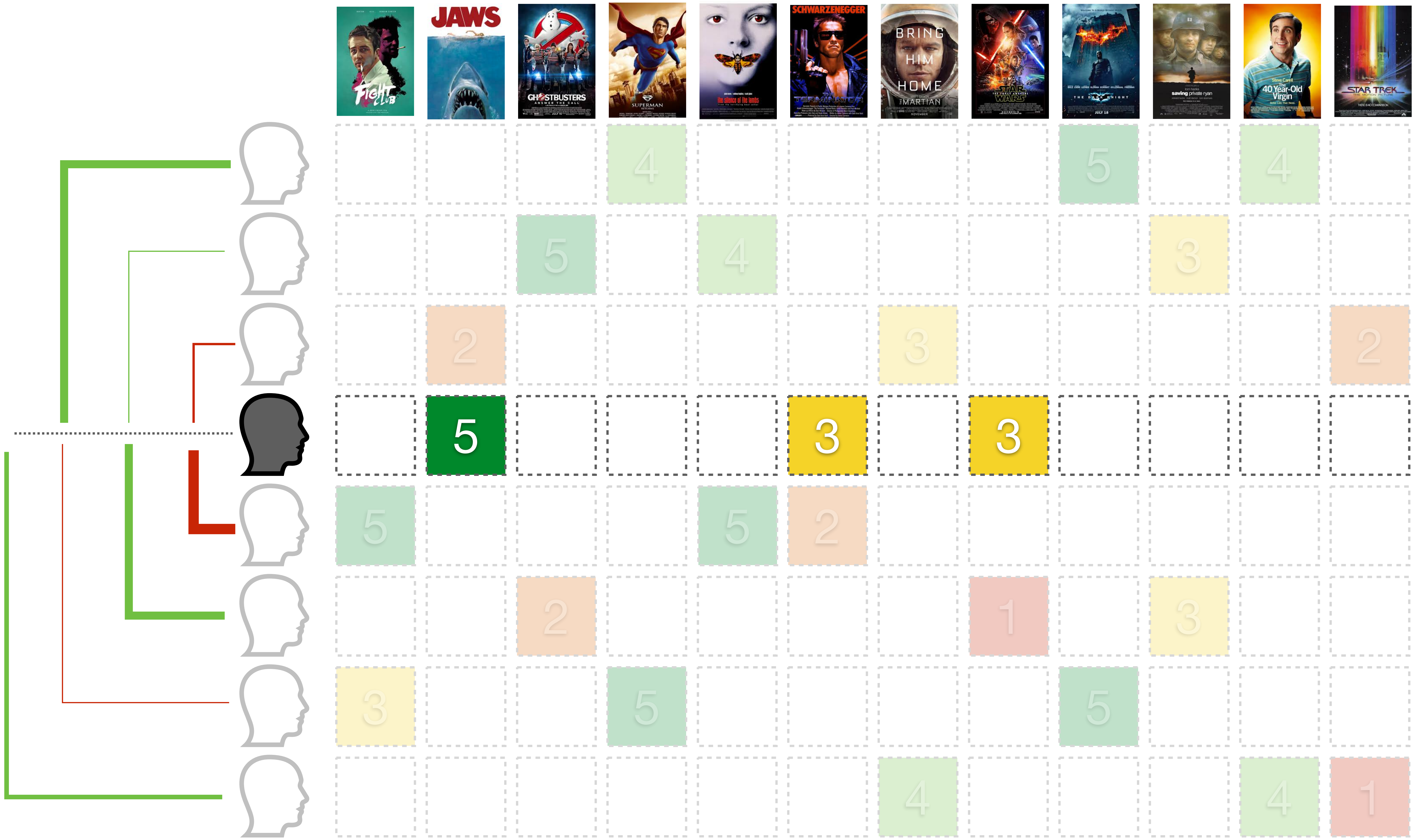
The Netflix Problem



The Netflix Problem

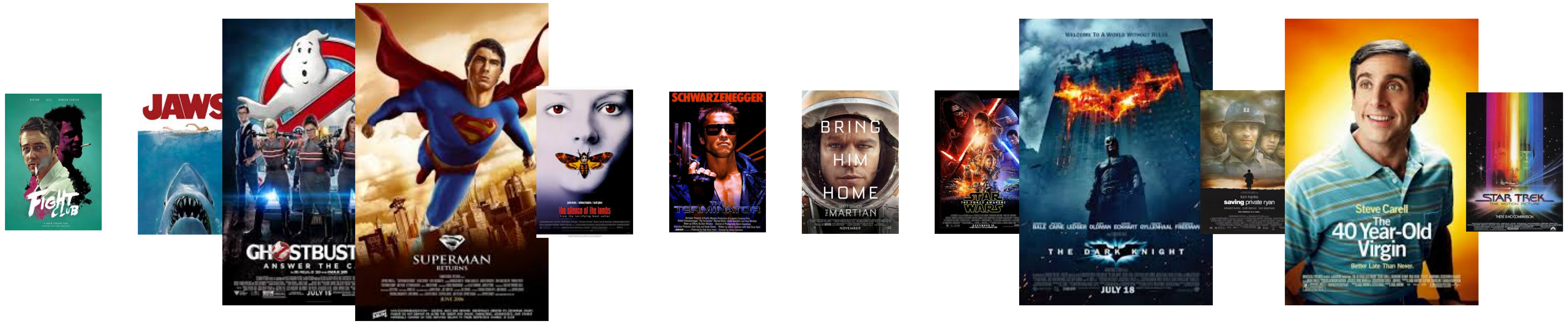


The Netflix Problem

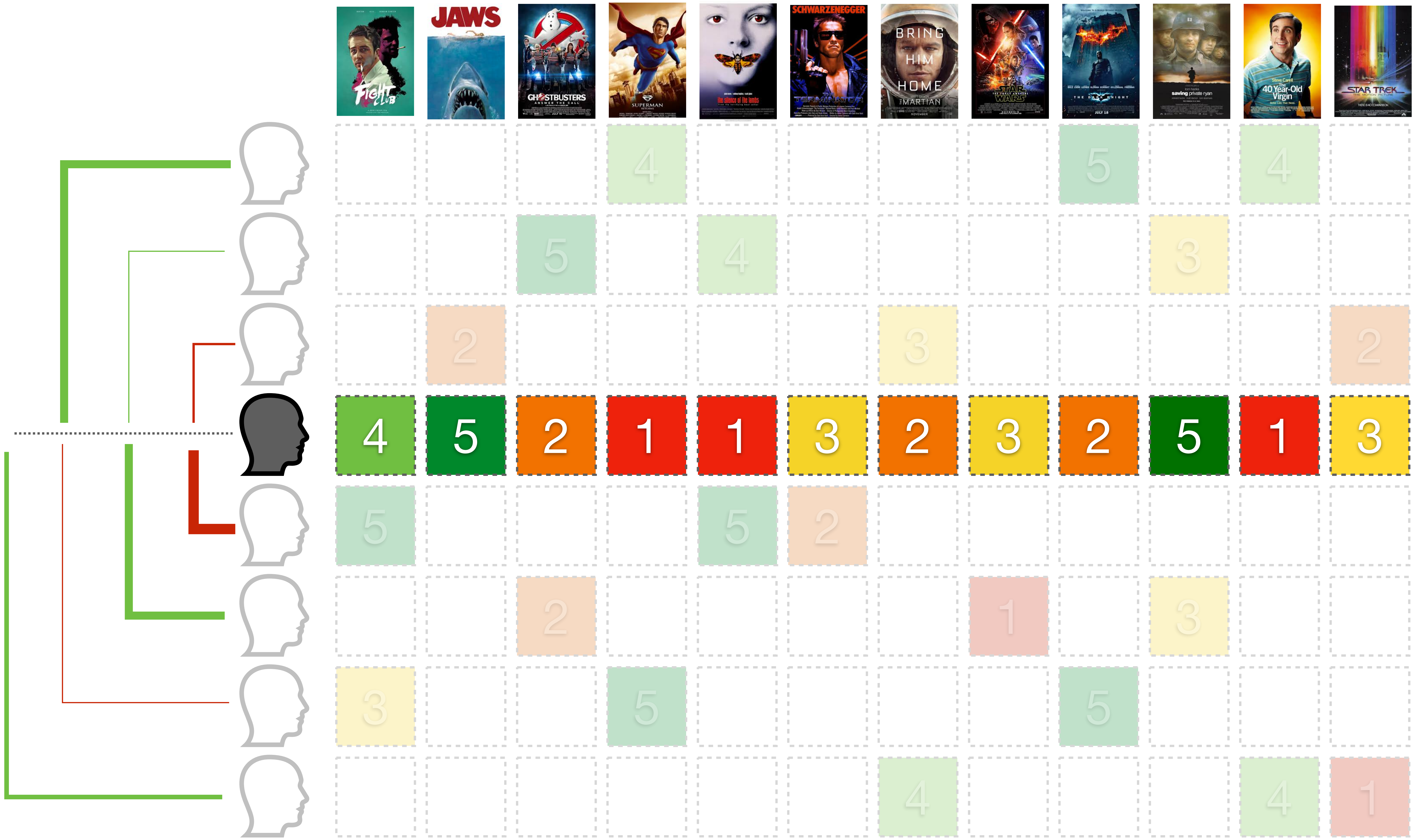
Drama/Horror

Sci-Fi

Action/Comedy



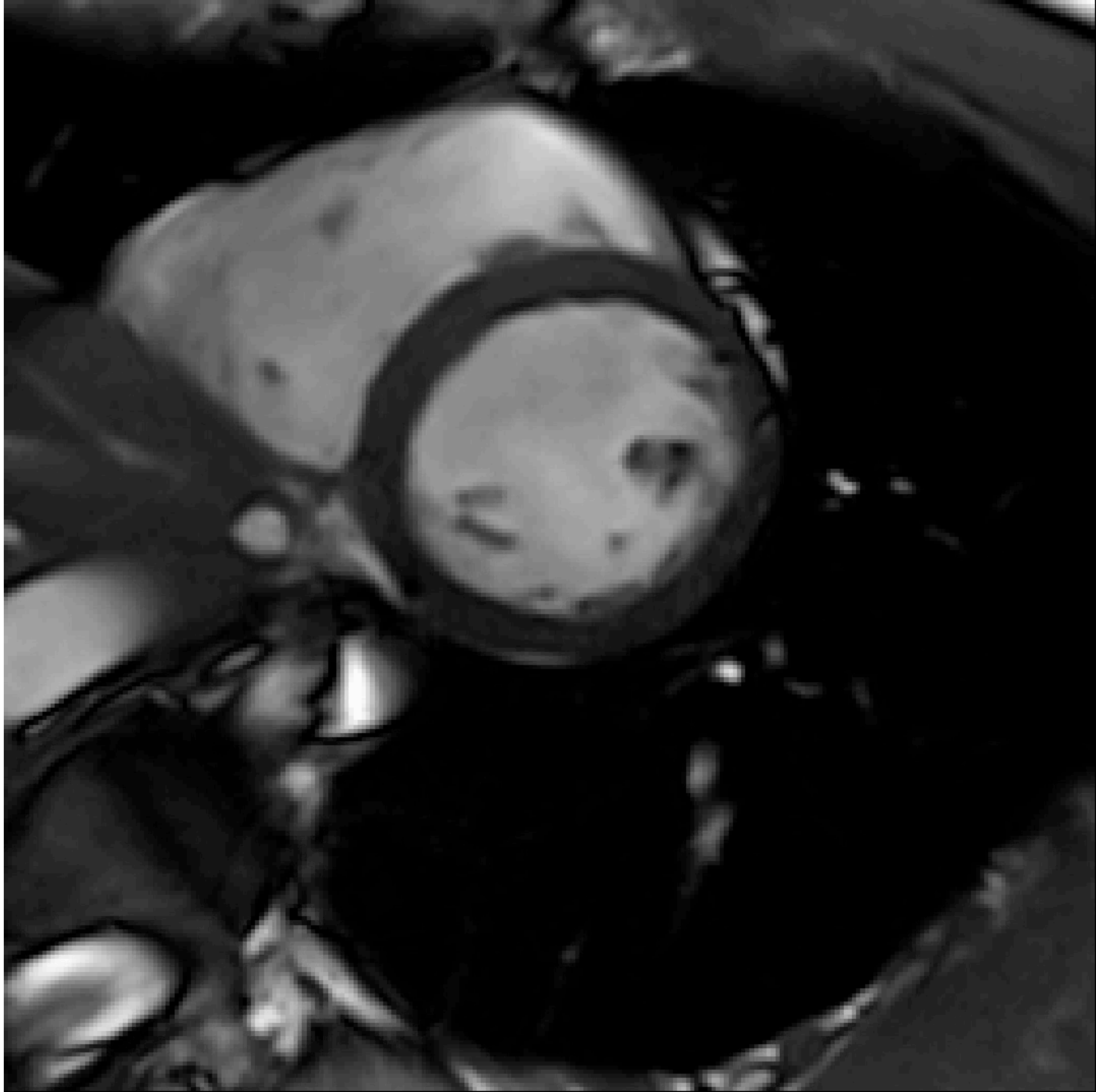
The Netflix Problem



Imaging Example

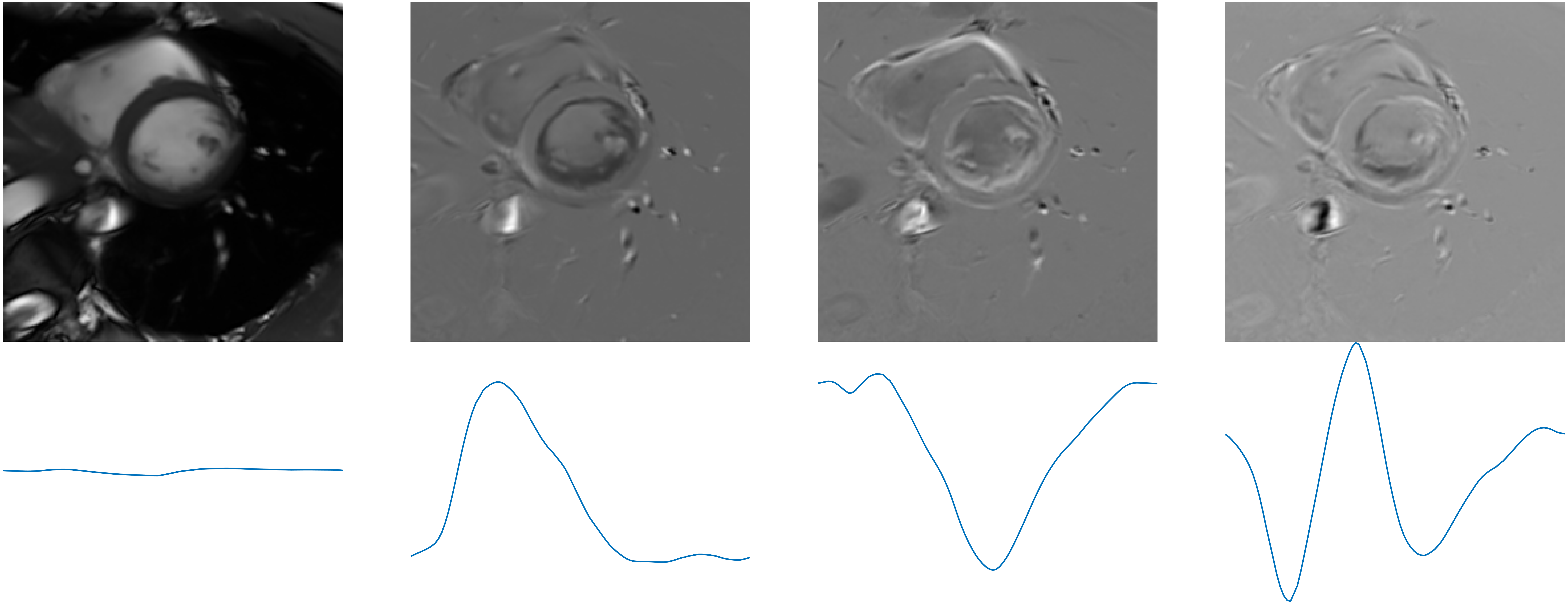
Users → Voxels

Movies → Time



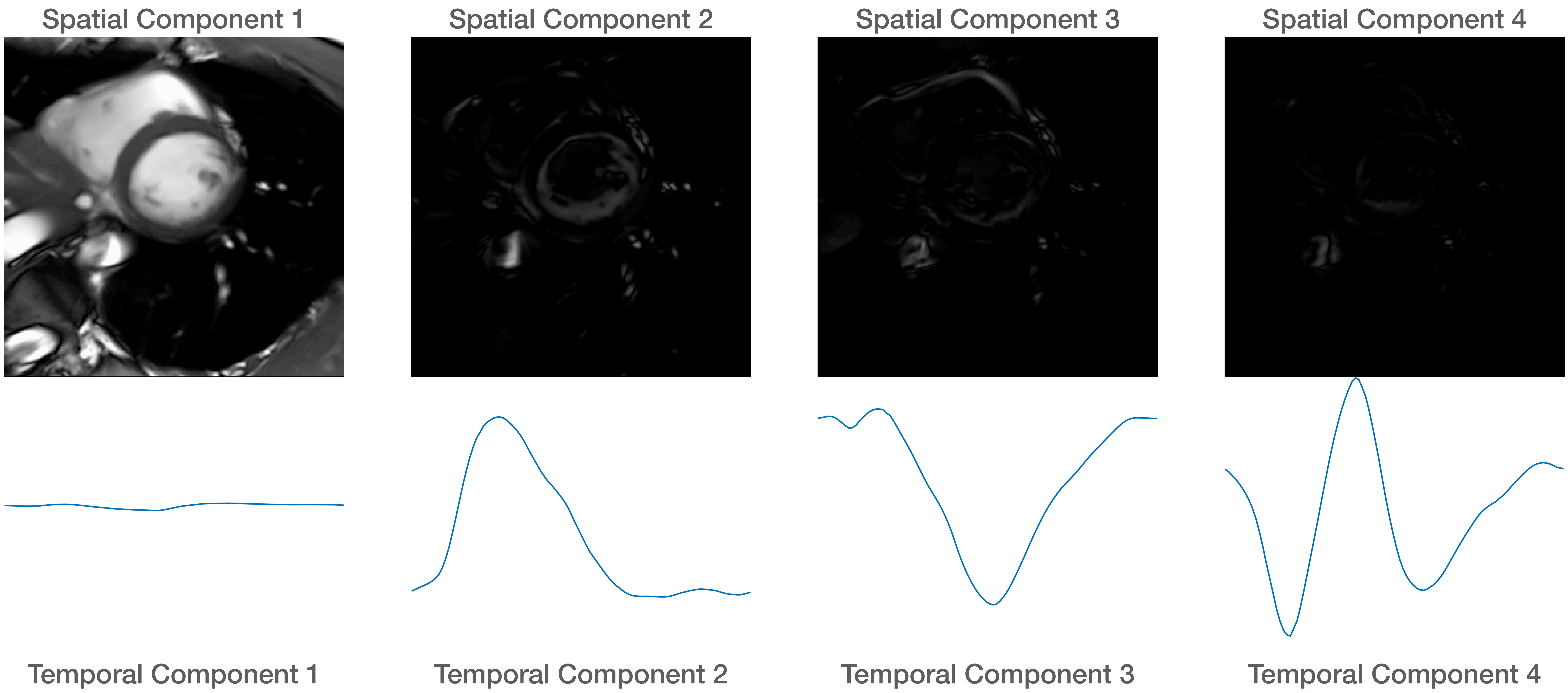
Imaging Example

Spatial Component 1 | Spatial Component 2 | Spatial Component 3 | Spatial Component 4

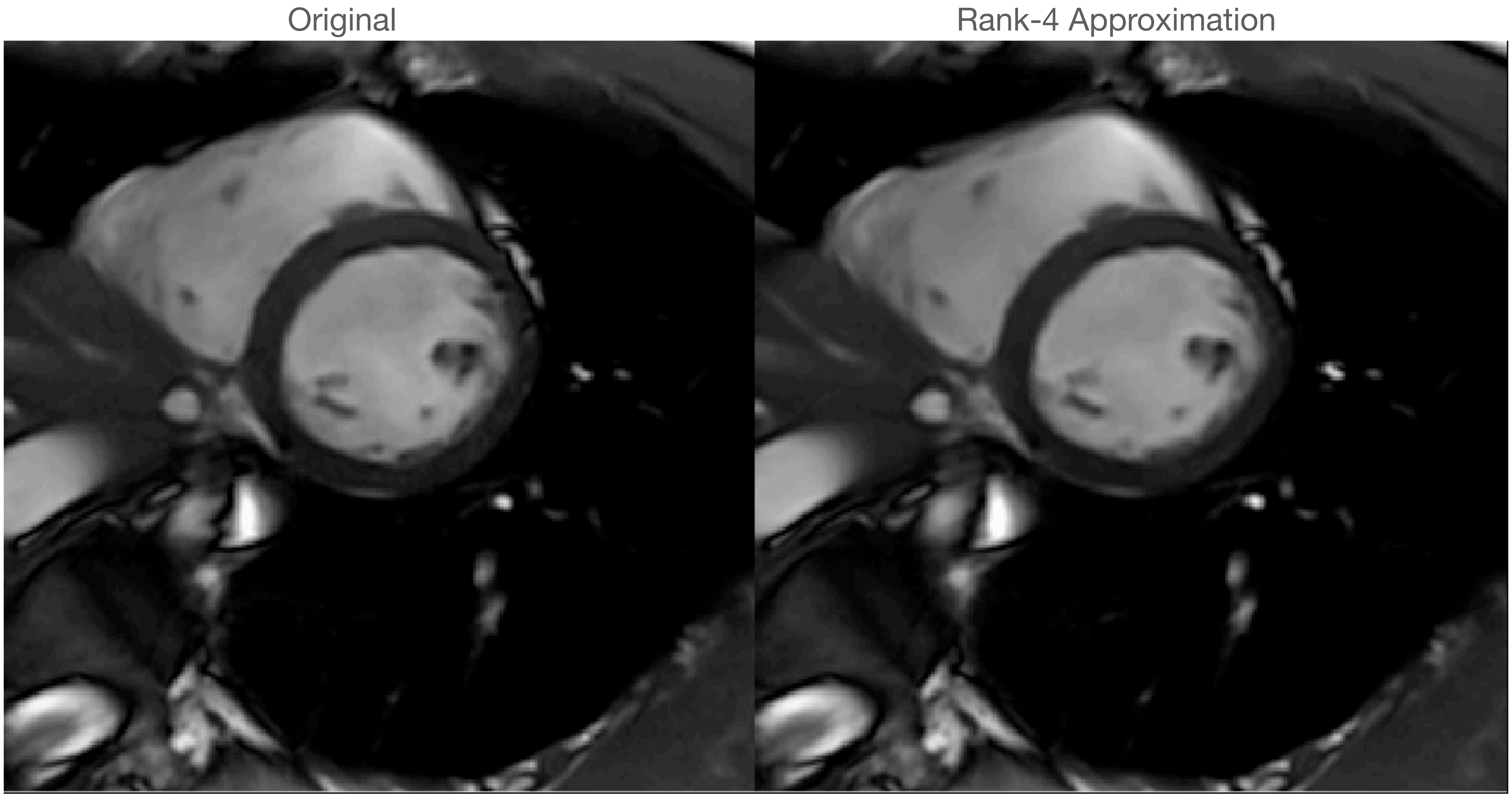


Temporal Component 1 | Temporal Component 2 | Temporal Component 3 | Temporal Component 4

Imaging Example

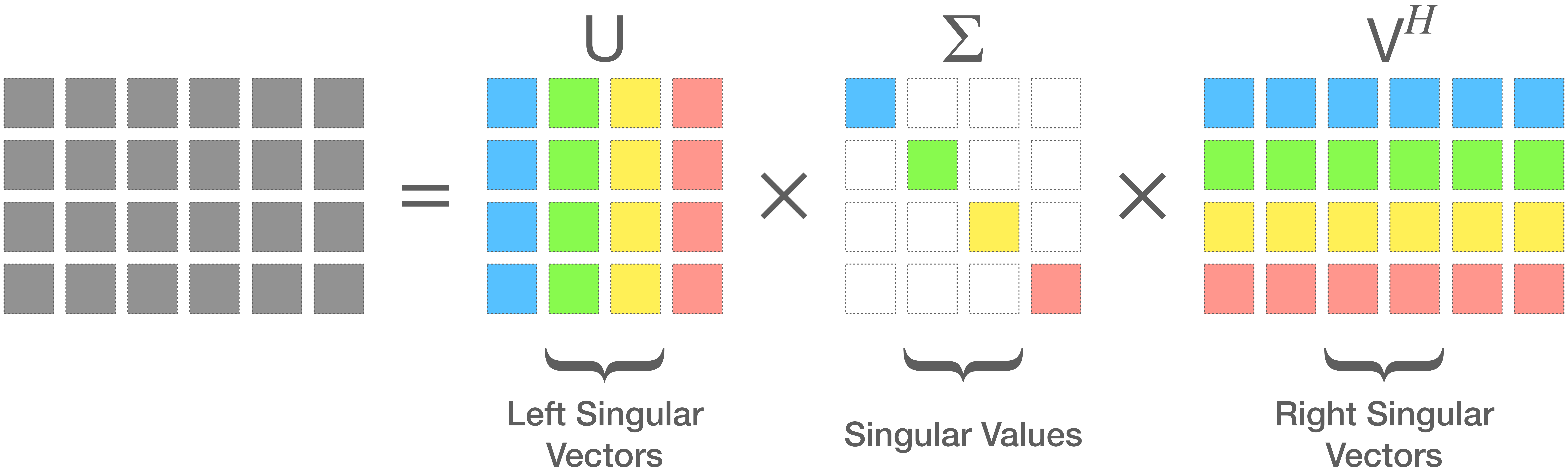


Imaging Example

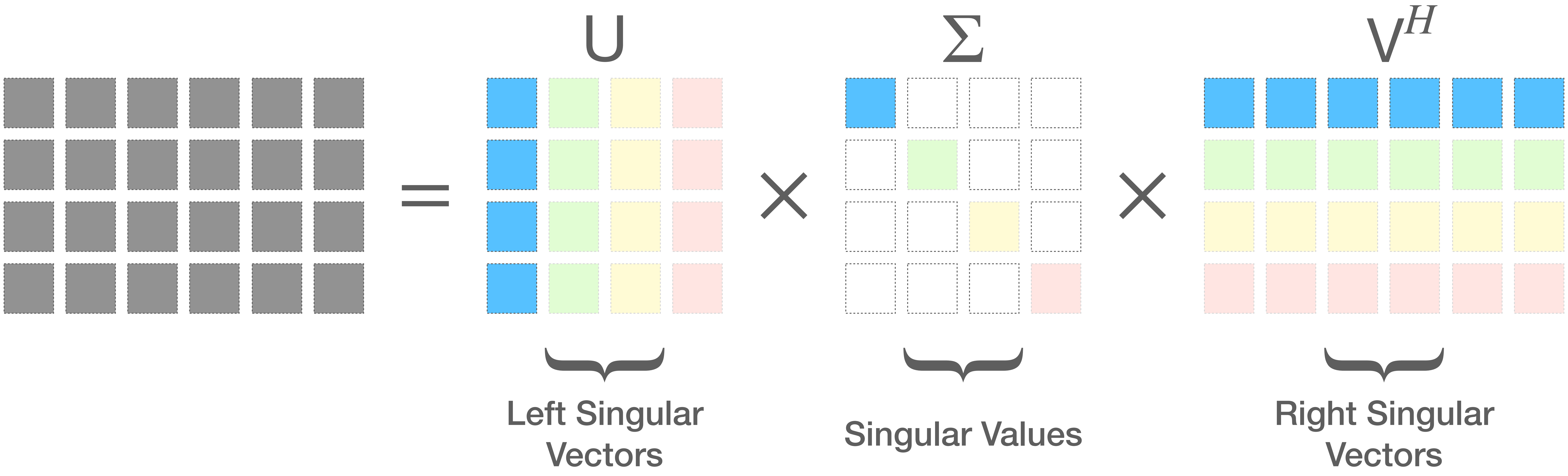


Background

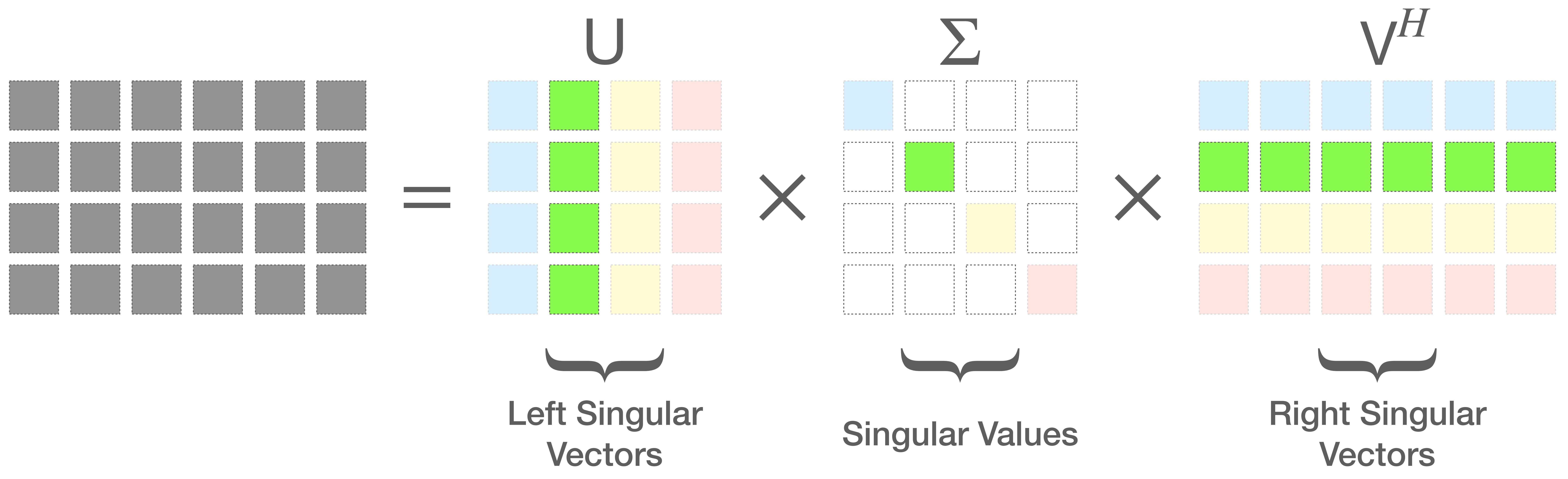
Singular Value Decomposition



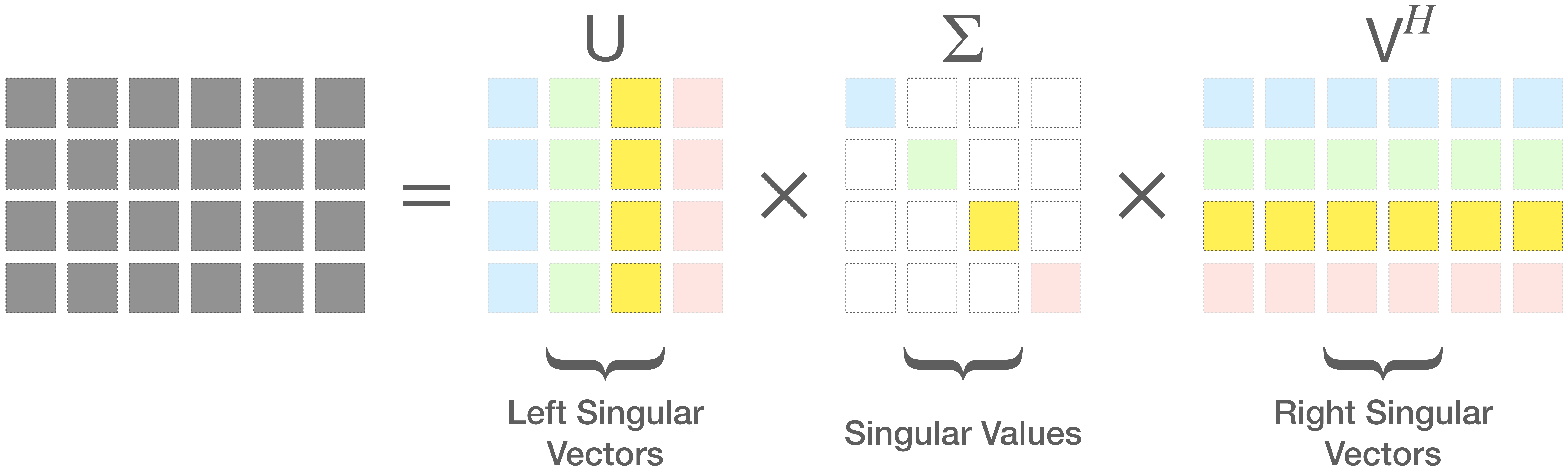
Singular Value Decomposition



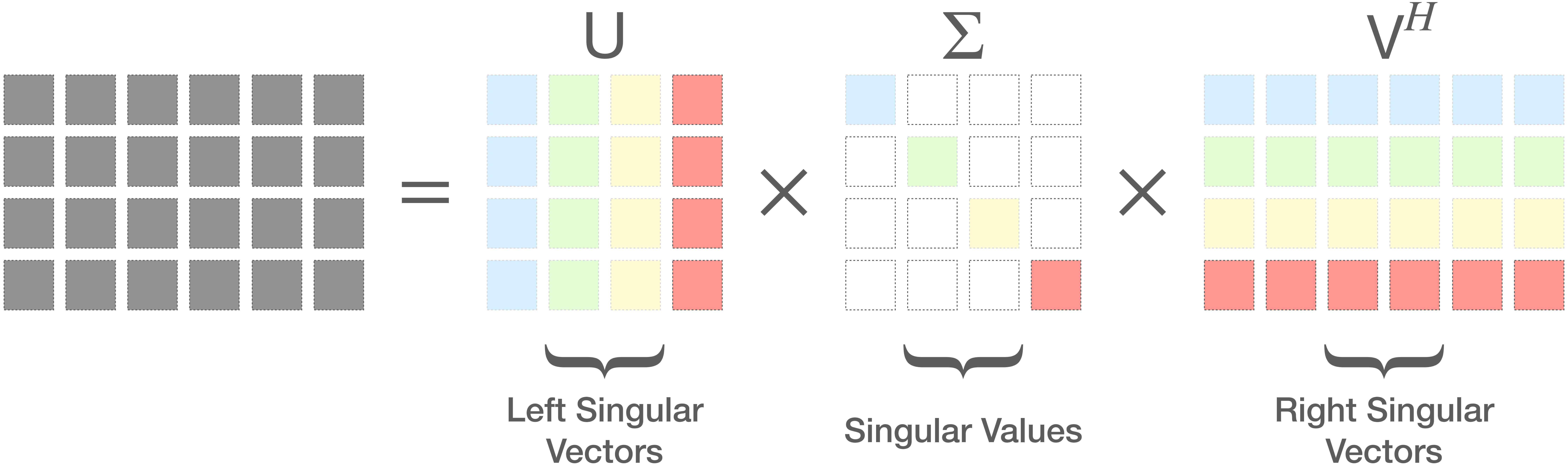
Singular Value Decomposition



Singular Value Decomposition

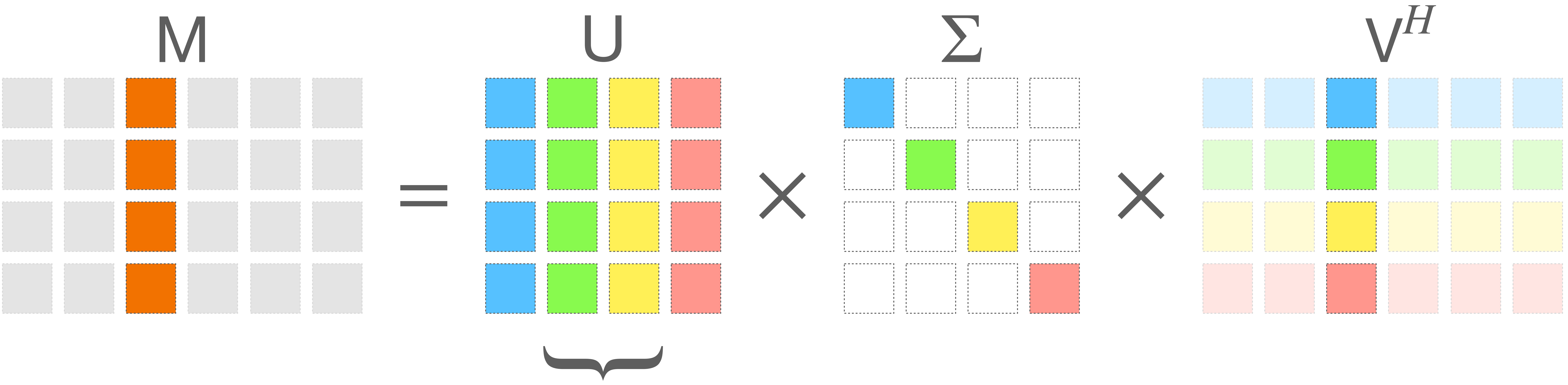


Singular Value Decomposition



Singular Value Decomposition

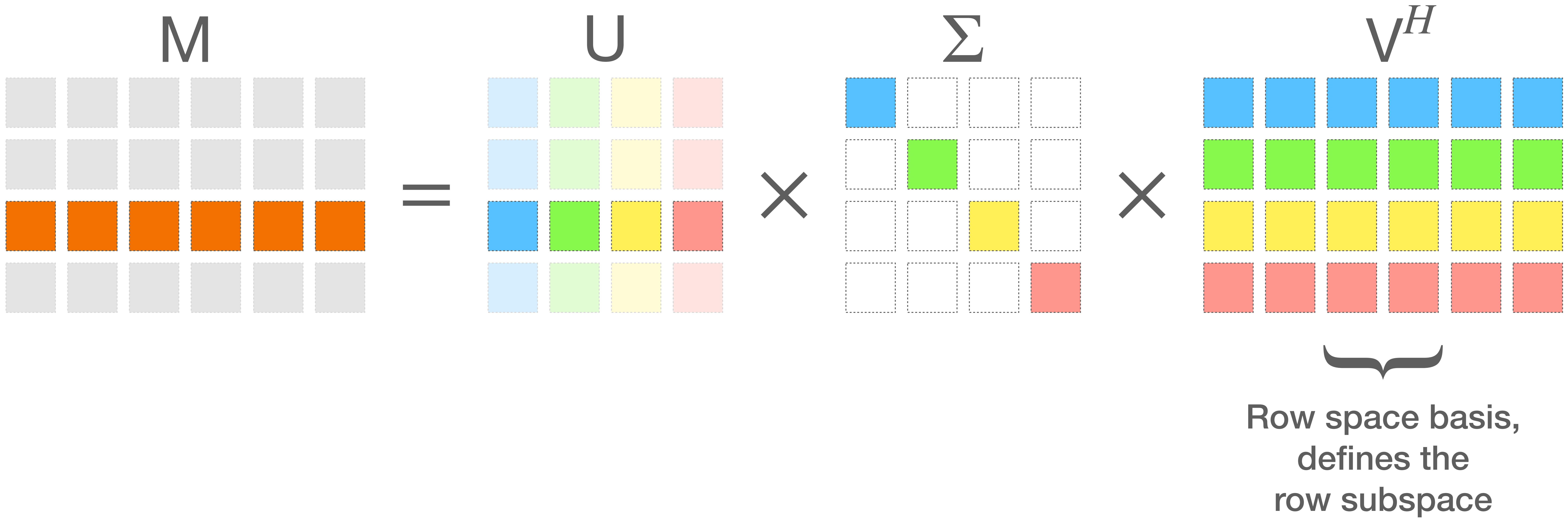
Any column of M is a linear combination of columns in U



Column space basis,
defines the
column subspace

Singular Value Decomposition

Any row of M is a linear combination of rows in V^H



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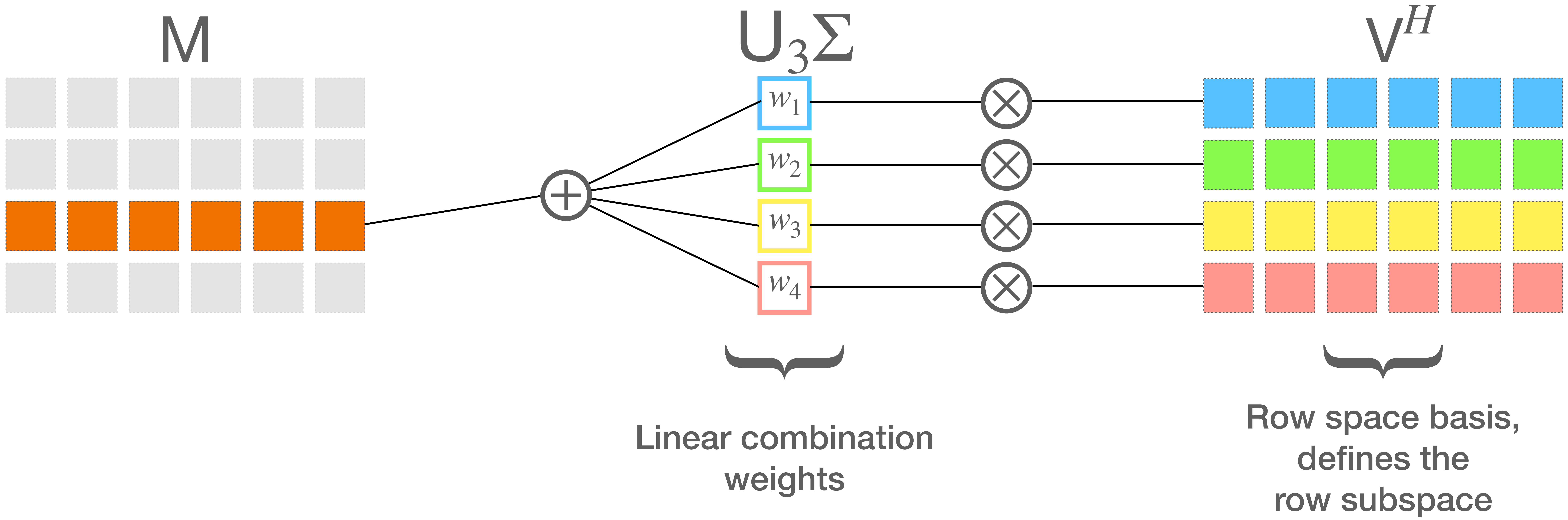
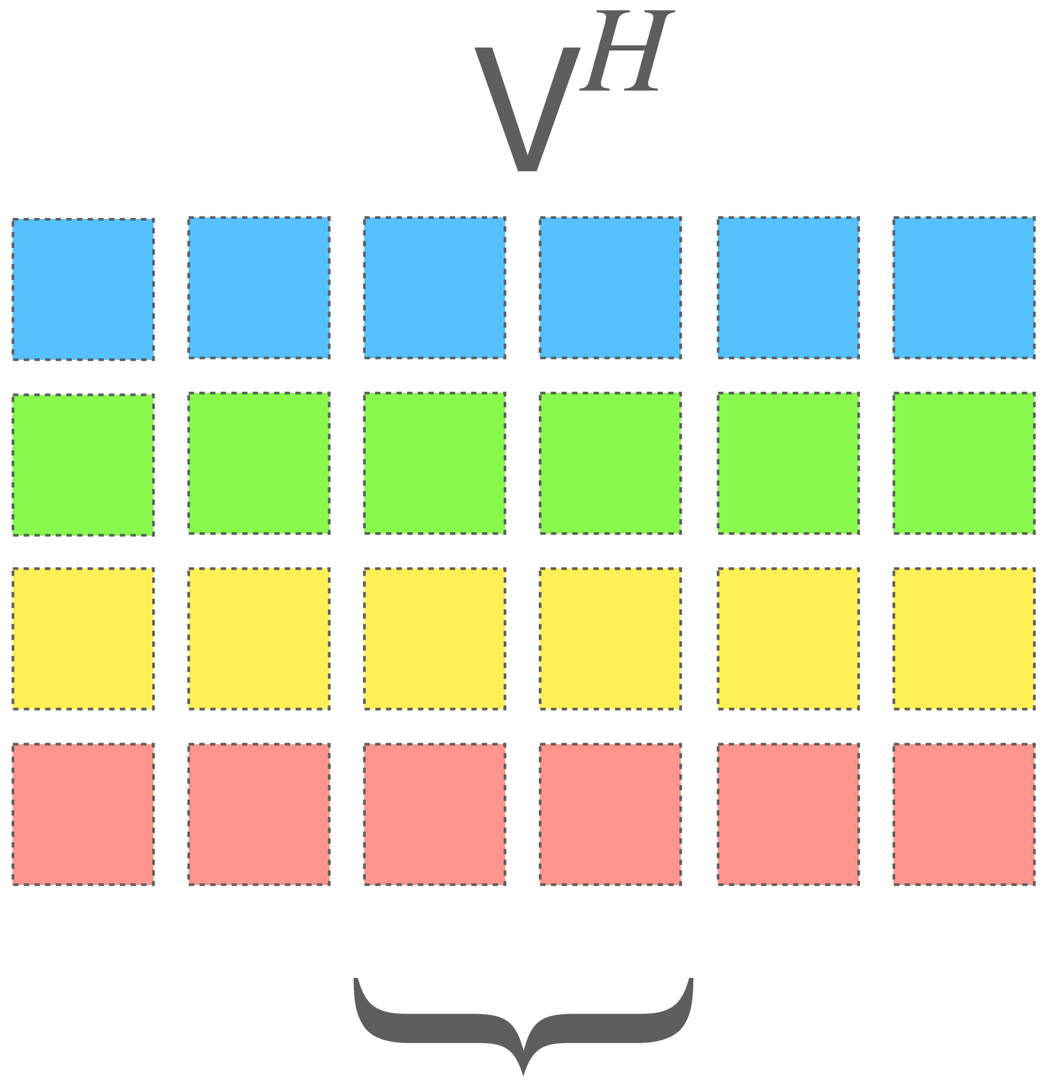
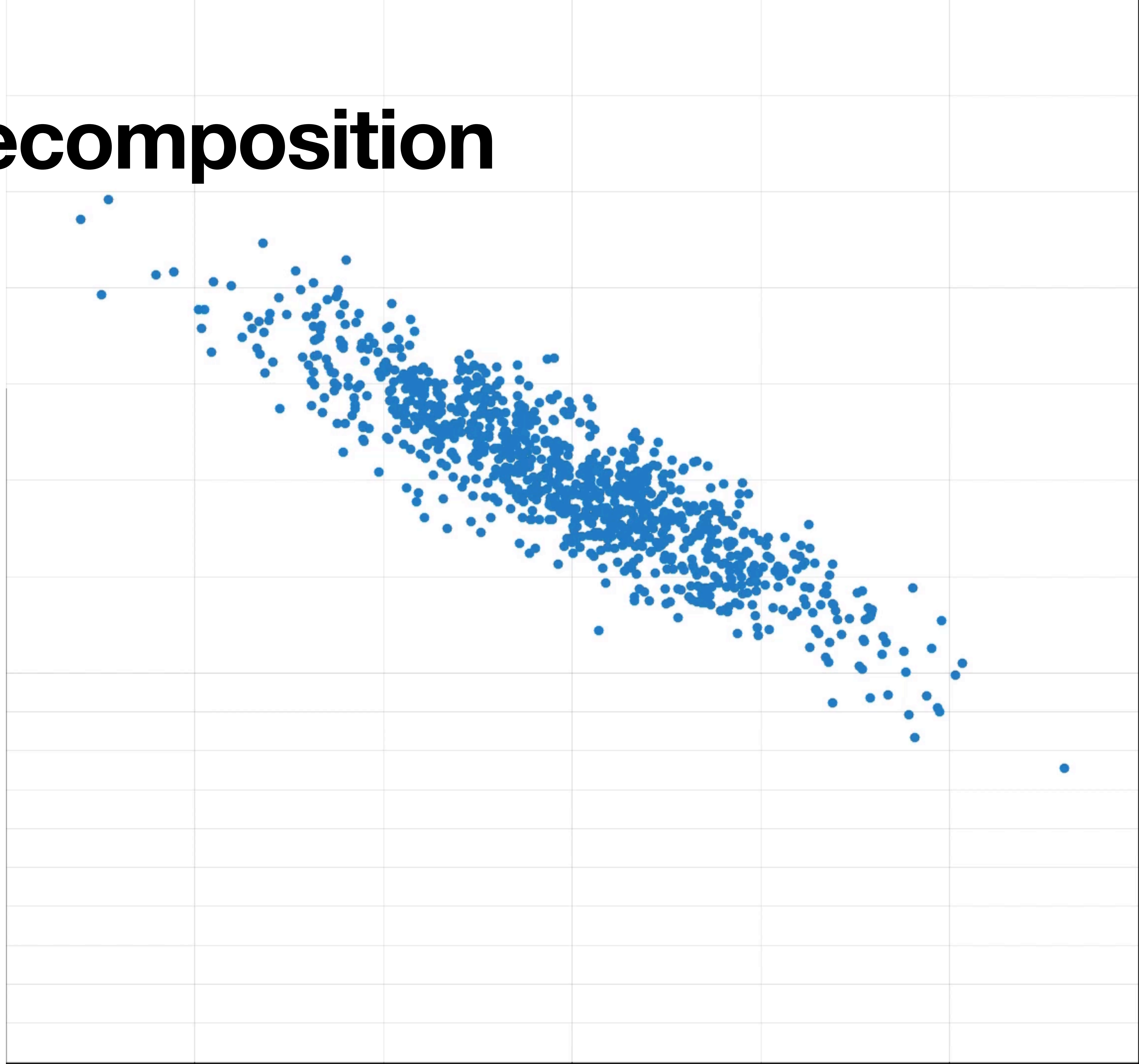


Singular Value Decomposition

Subspaces

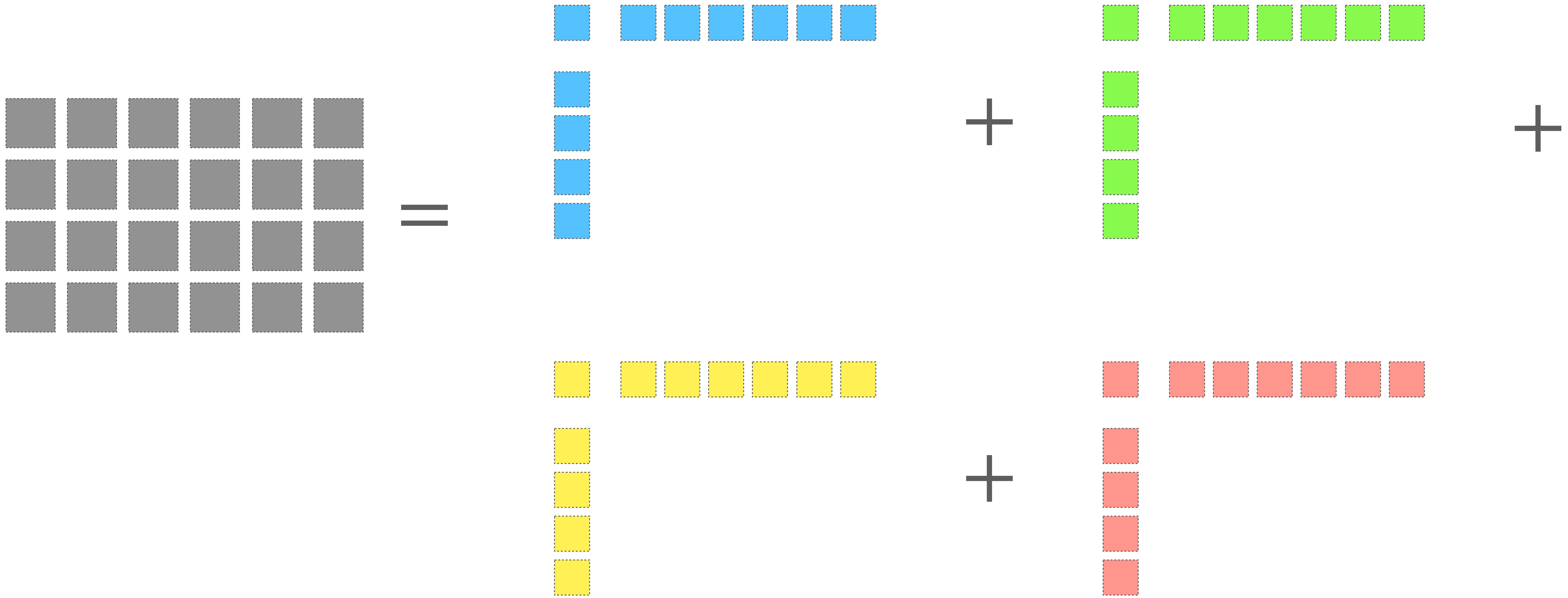


Row space basis,
defines the
row subspace



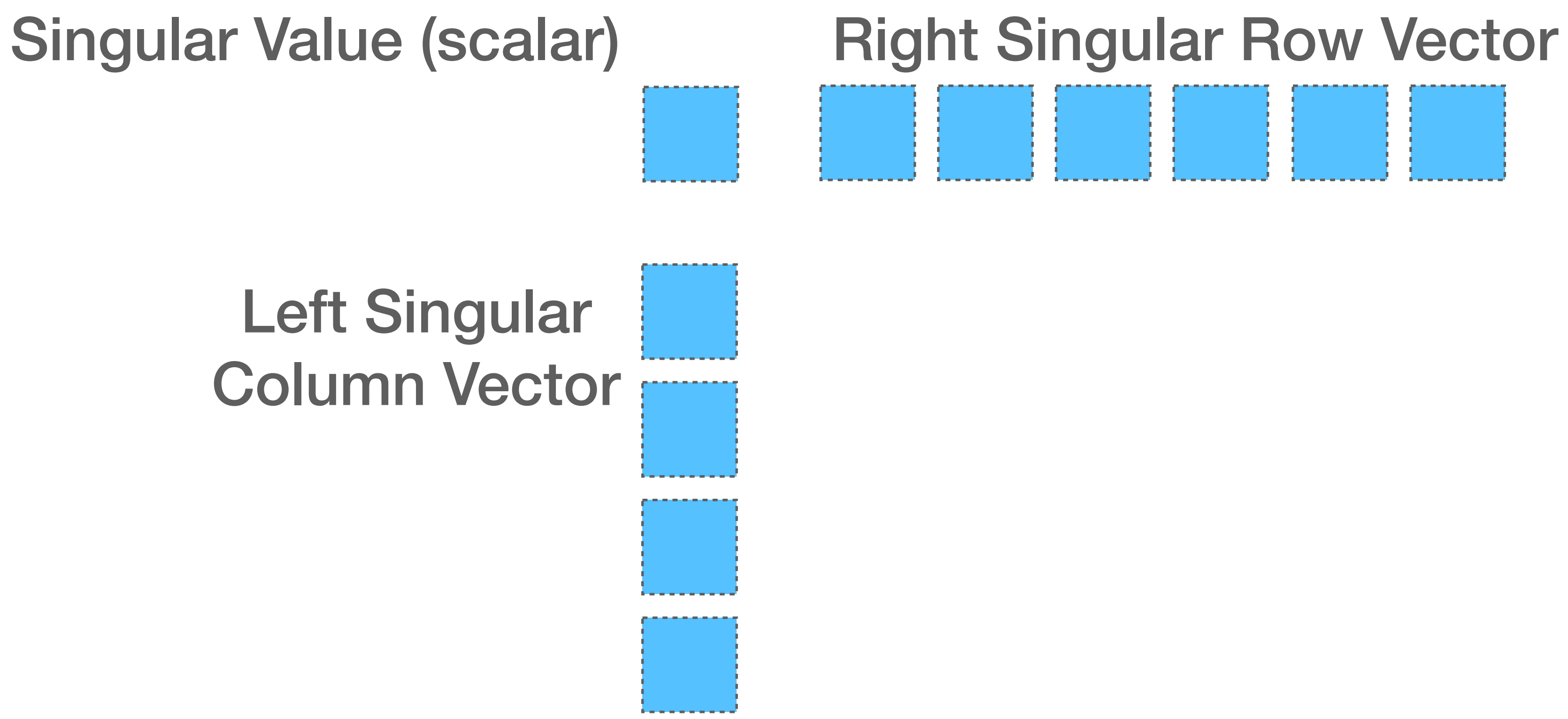
Singular Value Decomposition

Decomposition into sum of rank-1 matrices

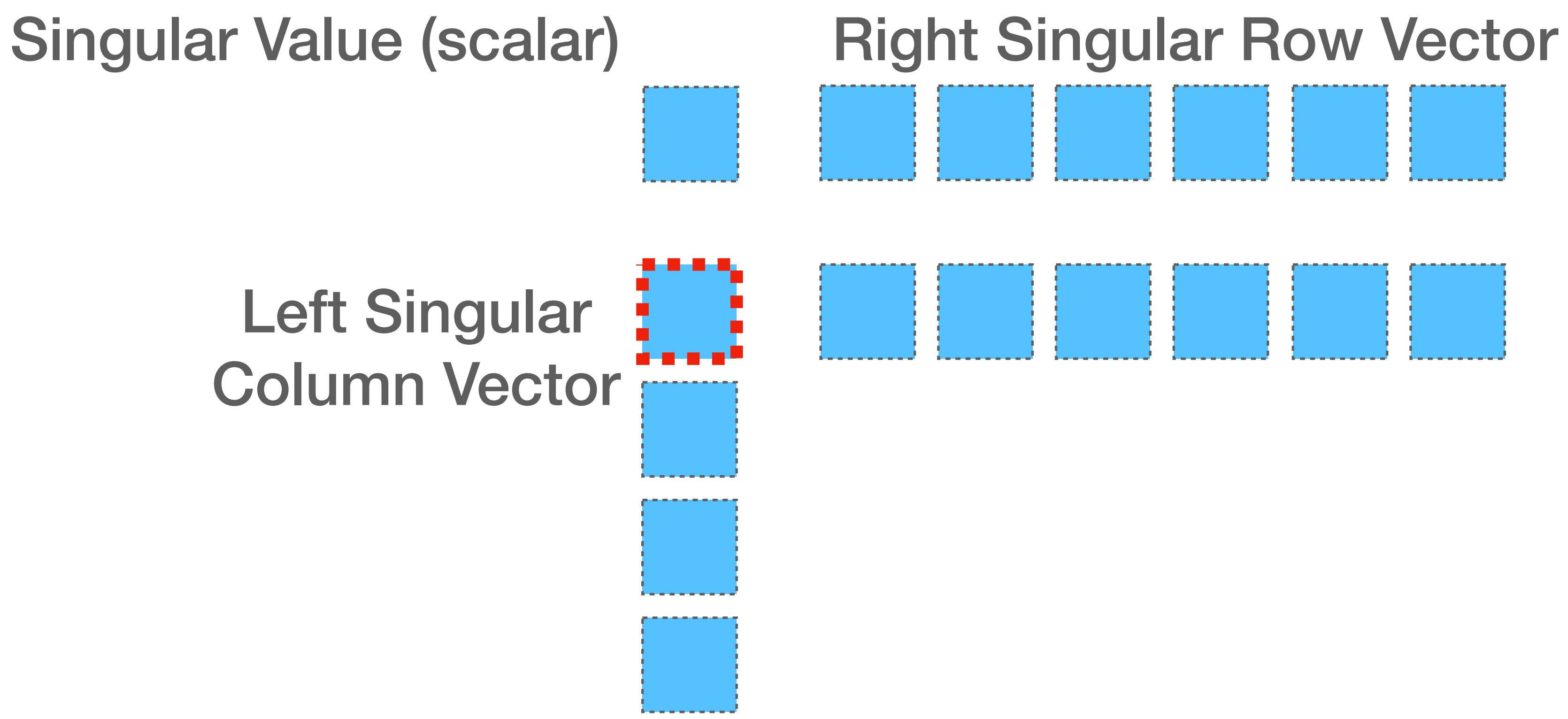


Singular Value Decomposition

Rank-1 matrix

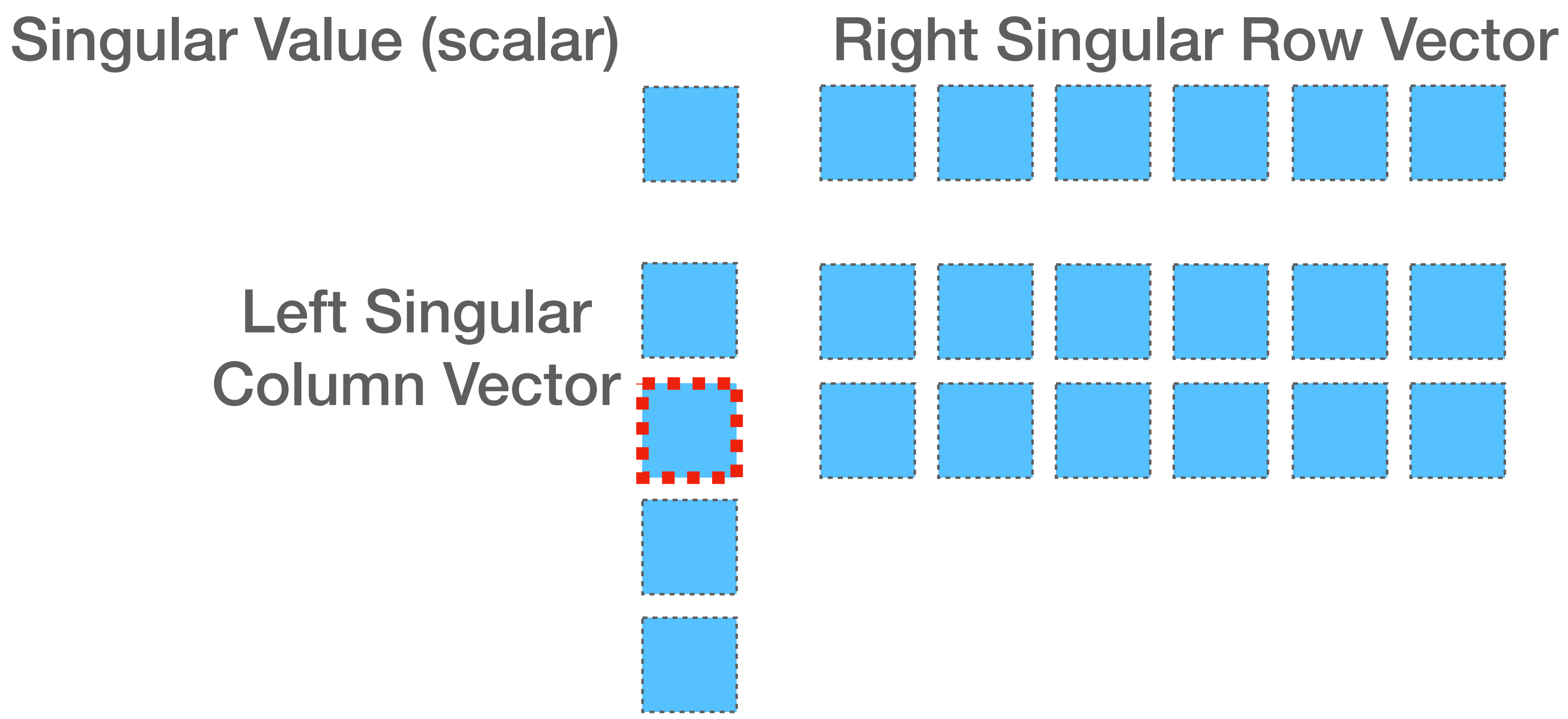


Singular Value Decomposition
Rank-1 matrix



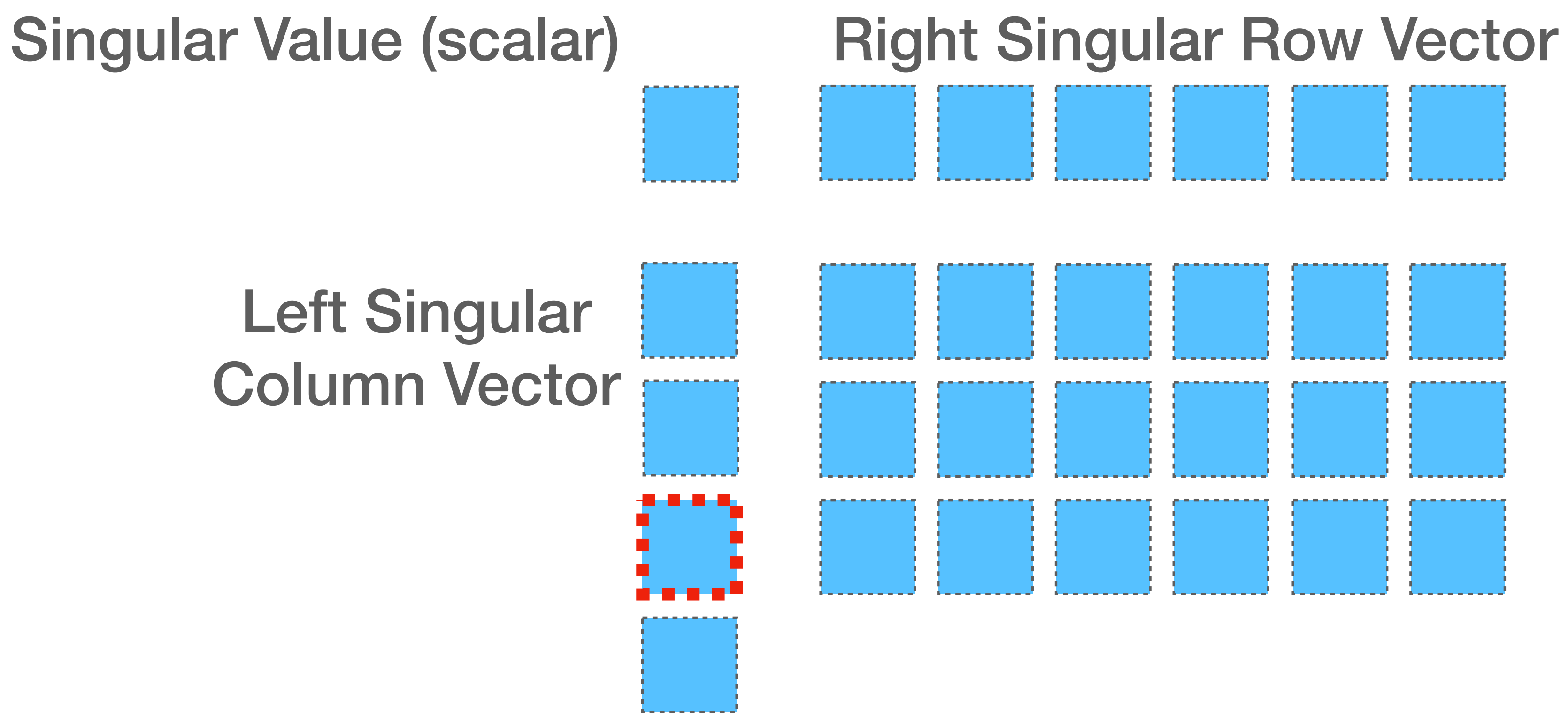
Singular Value Decomposition

Rank-1 matrix



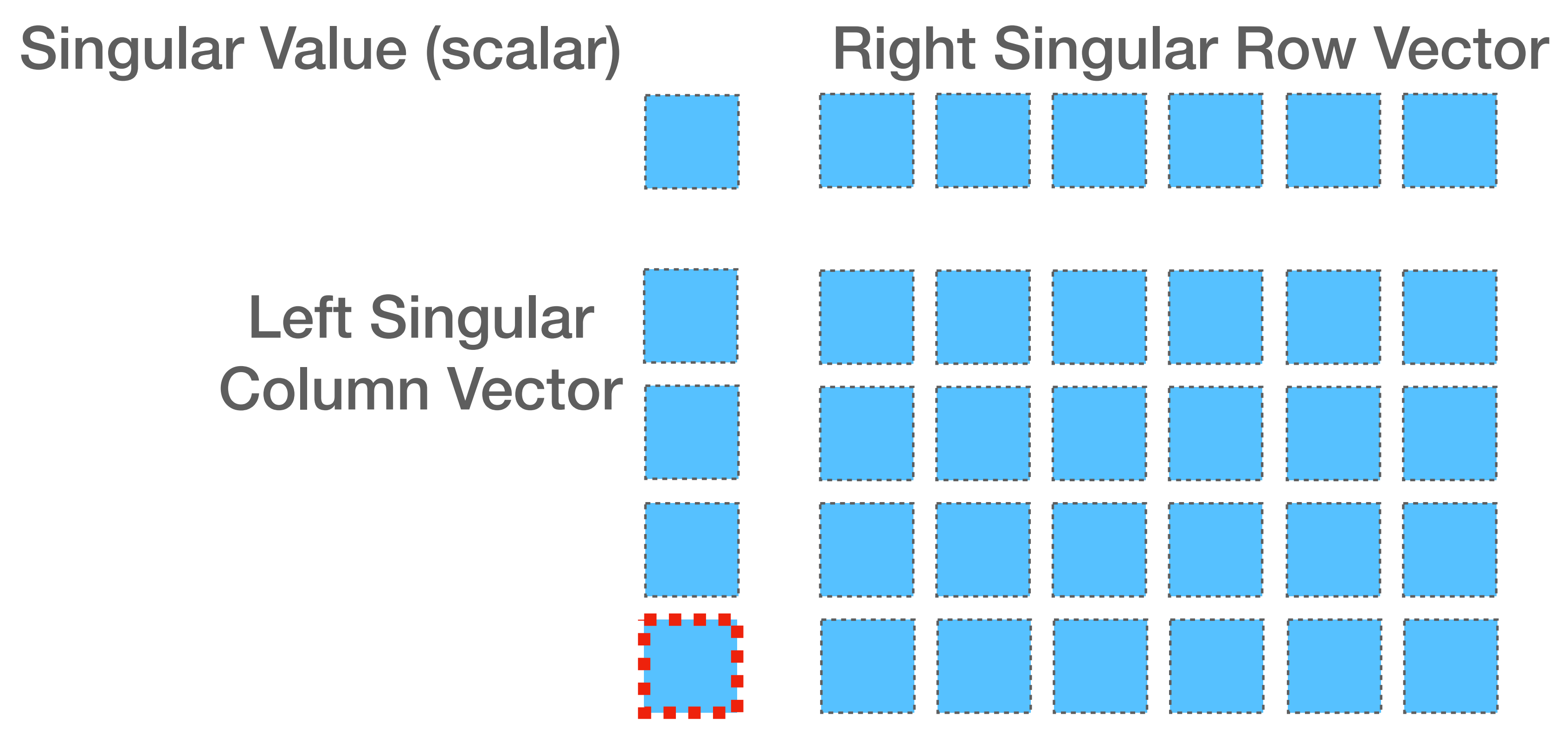
Singular Value Decomposition

Rank-1 matrix



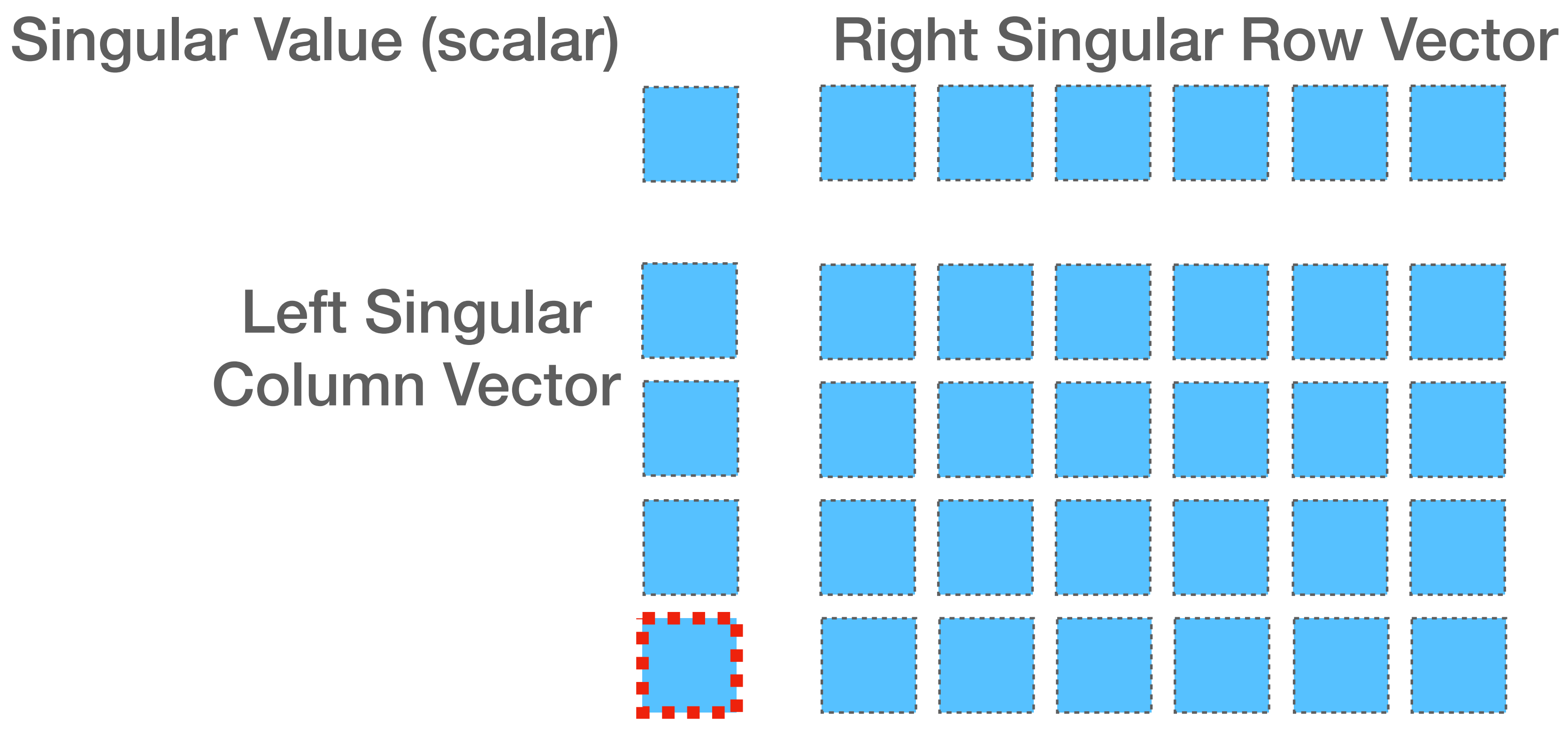
Singular Value Decomposition

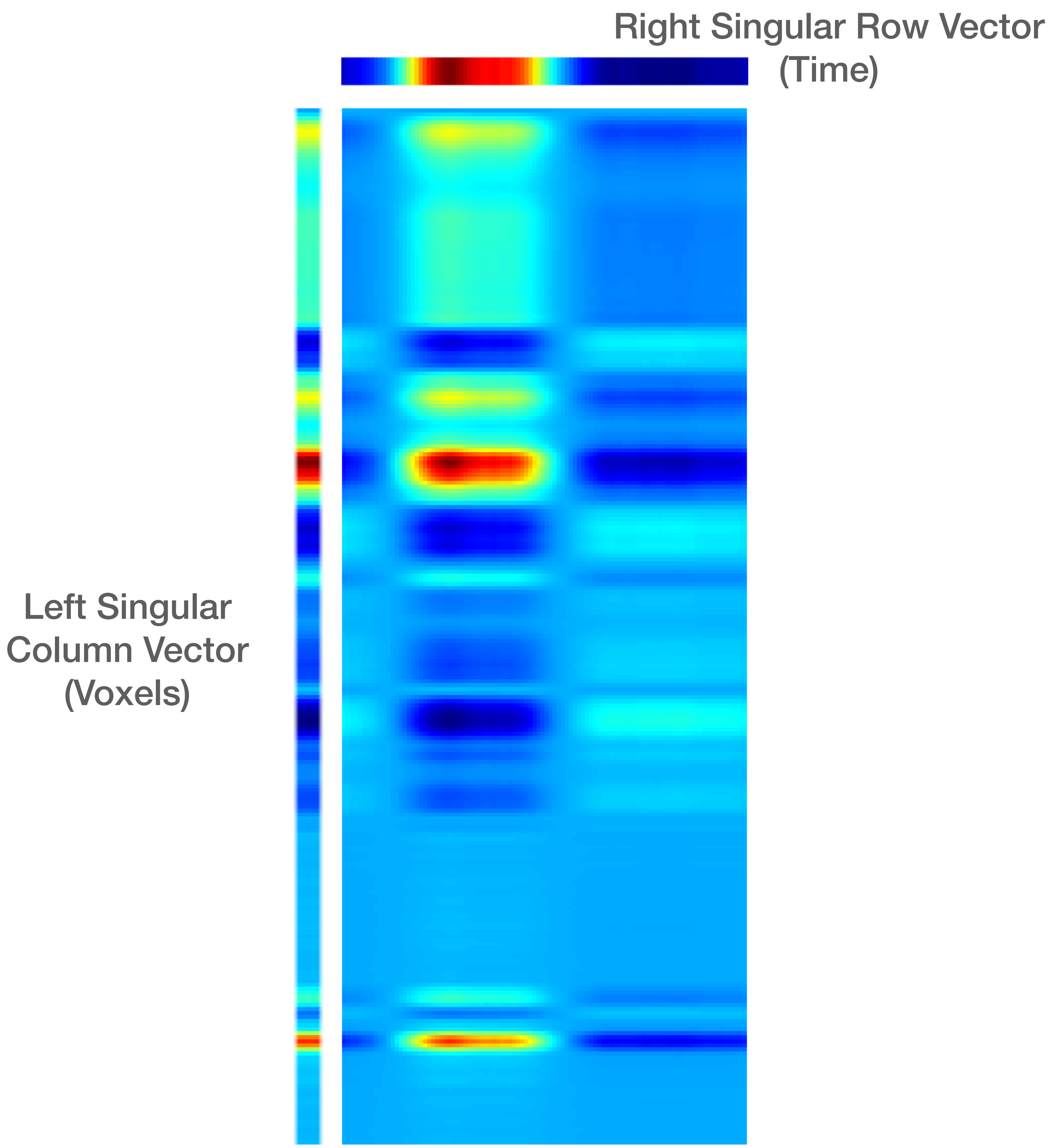
Rank-1 matrix



Singular Value Decomposition

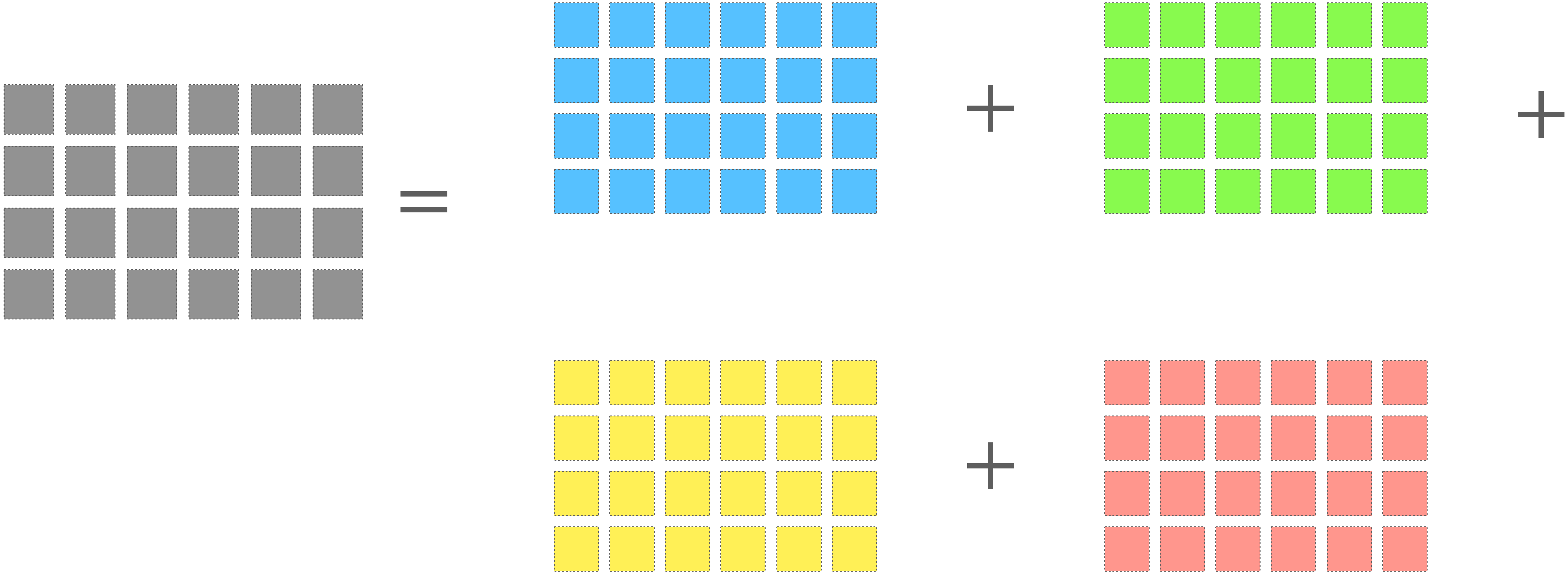
Rank-1 matrix



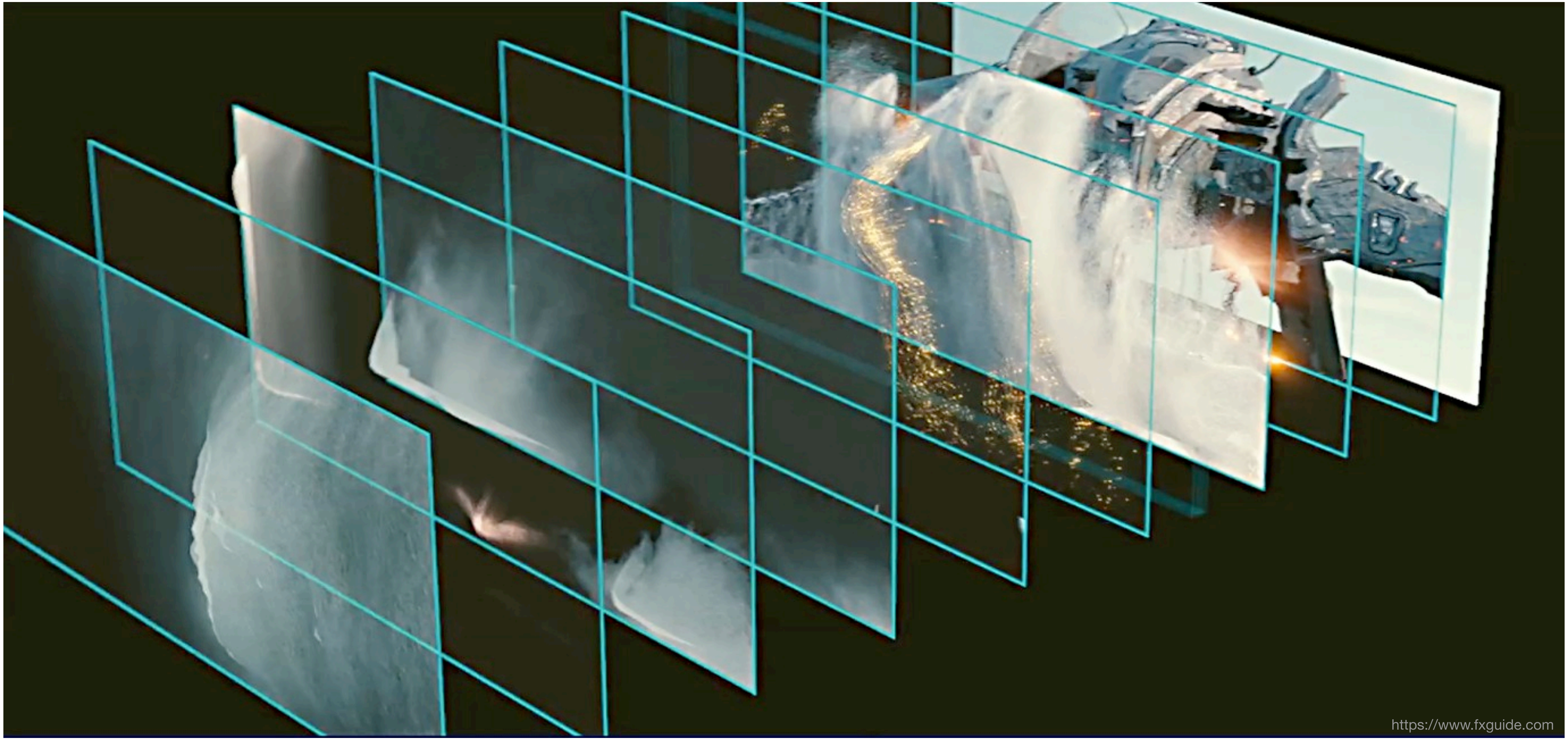


Singular Value Decomposition

Decomposition into sum of rank-1 matrices

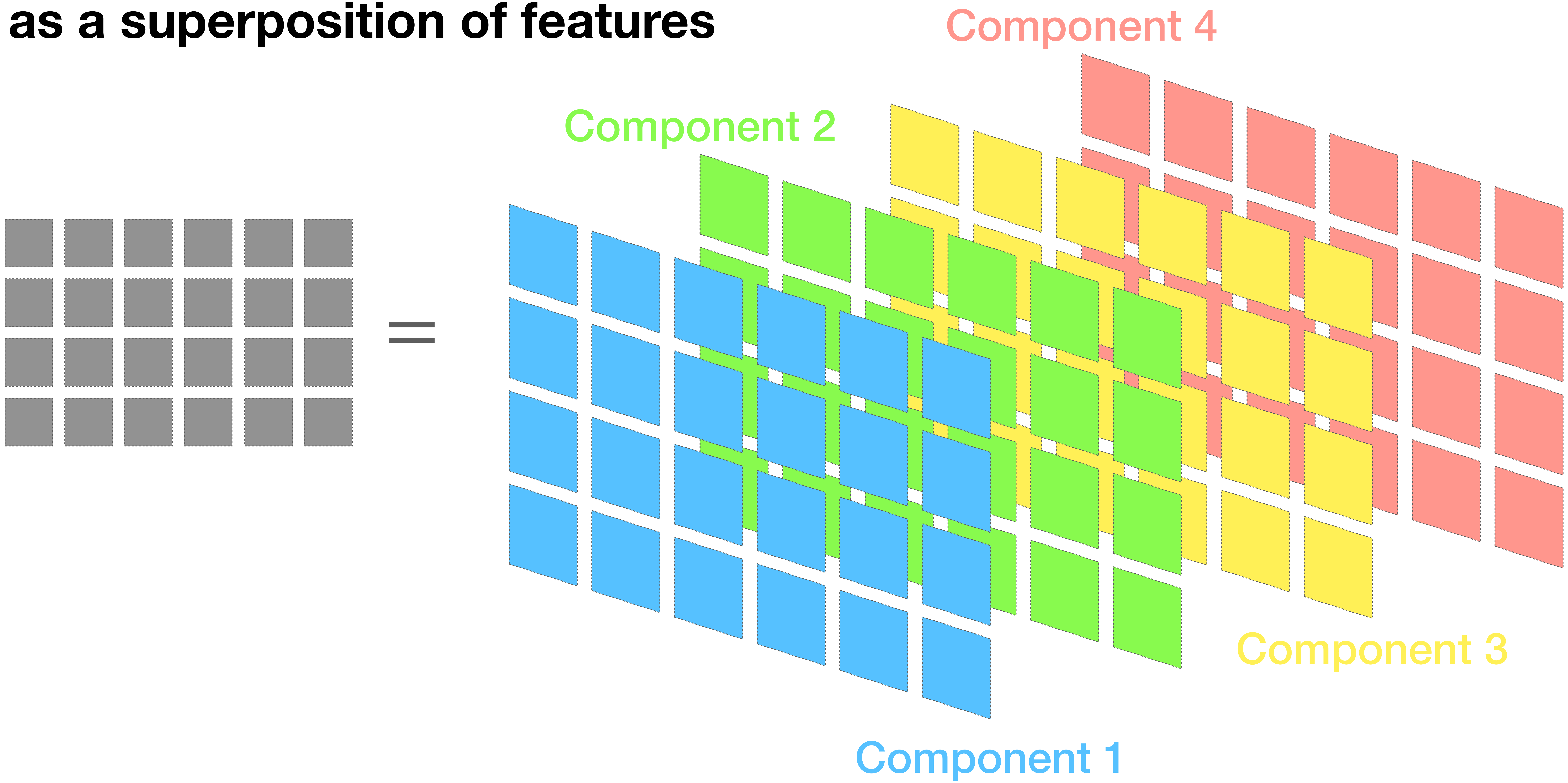


Singular Value Decomposition



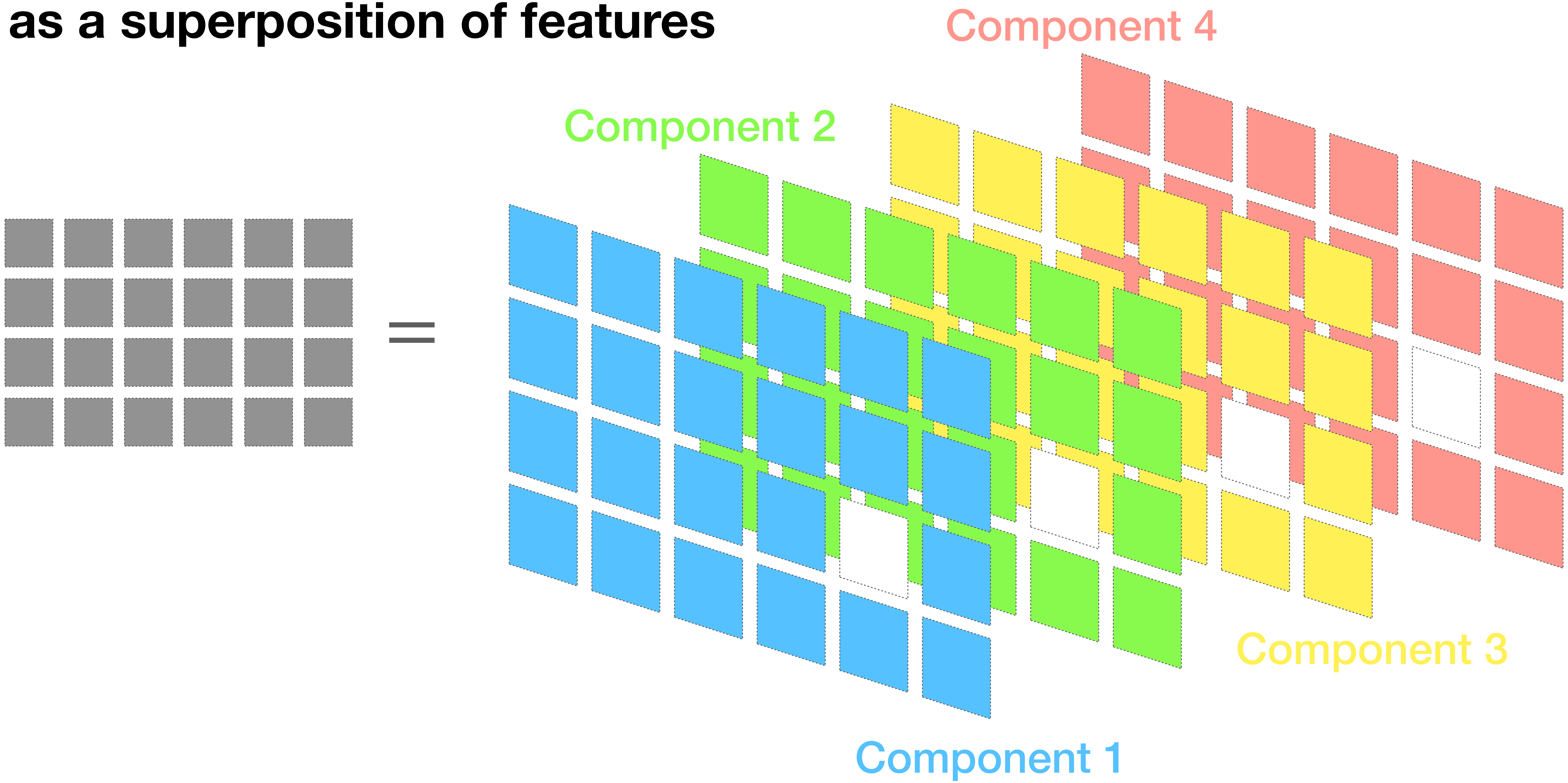
Singular Value Decomposition

as a superposition of features



Singular Value Decomposition

as a superposition of features



Rank, Redundancy and Linear Dependence

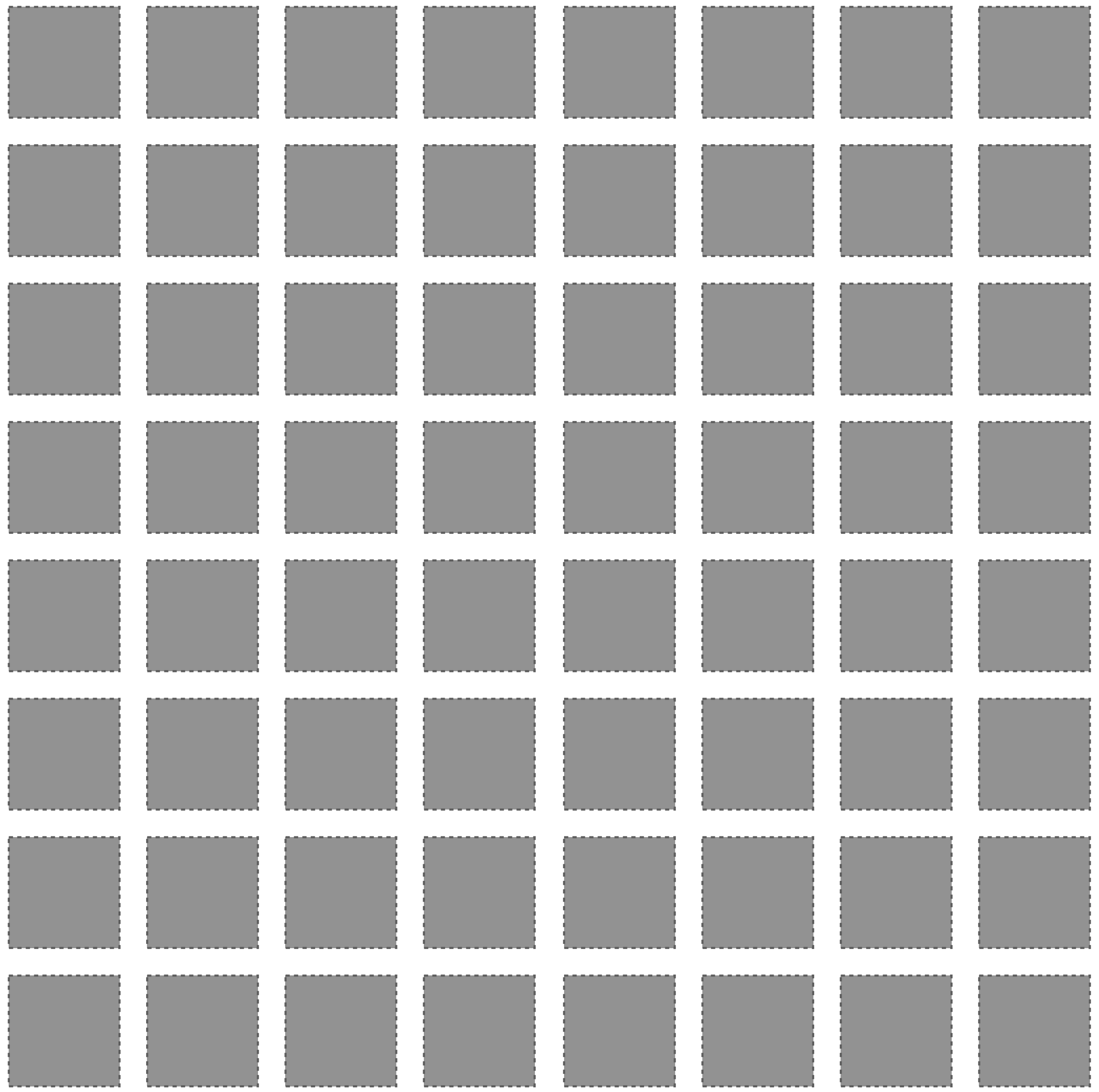
- The **rank** of a matrix is:
  - the number of non-zero singular values
  - the number of linearly independent rows and columns in the data
  - the dimension of the row and column spaces

Rank, Redundancy and Linear Dependence

- When a matrix is **low-rank**, we can say that:
  - there is information shared across entries of the matrix
  - the rows and columns of the matrix are not linearly independent
  - the rows and columns of the matrix are correlated
  - the rows and column vectors lie in low-dimensional subspaces
  - the matrix is redundant

Matrix Degrees of Freedom

How much information is in a matrix?

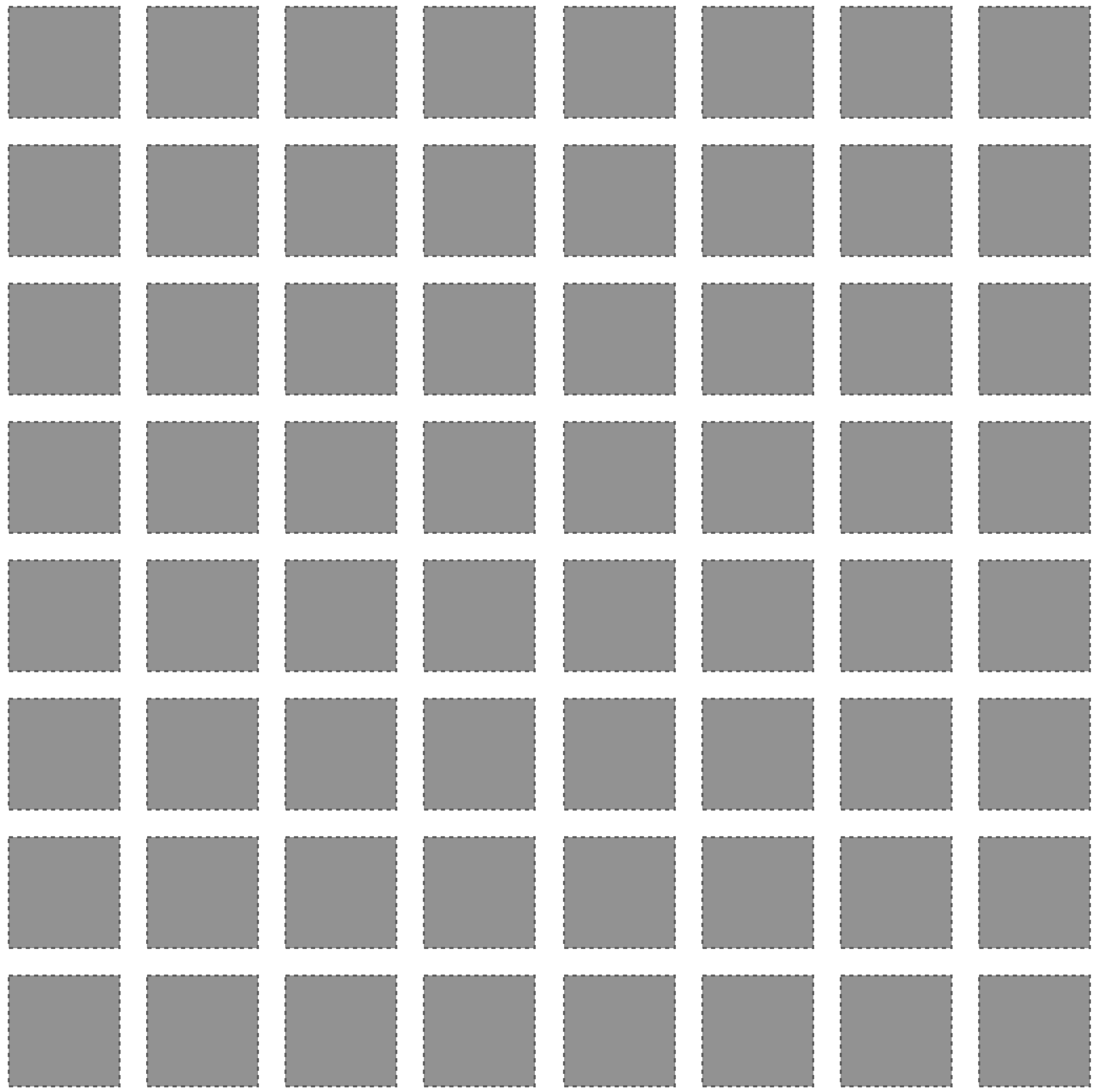


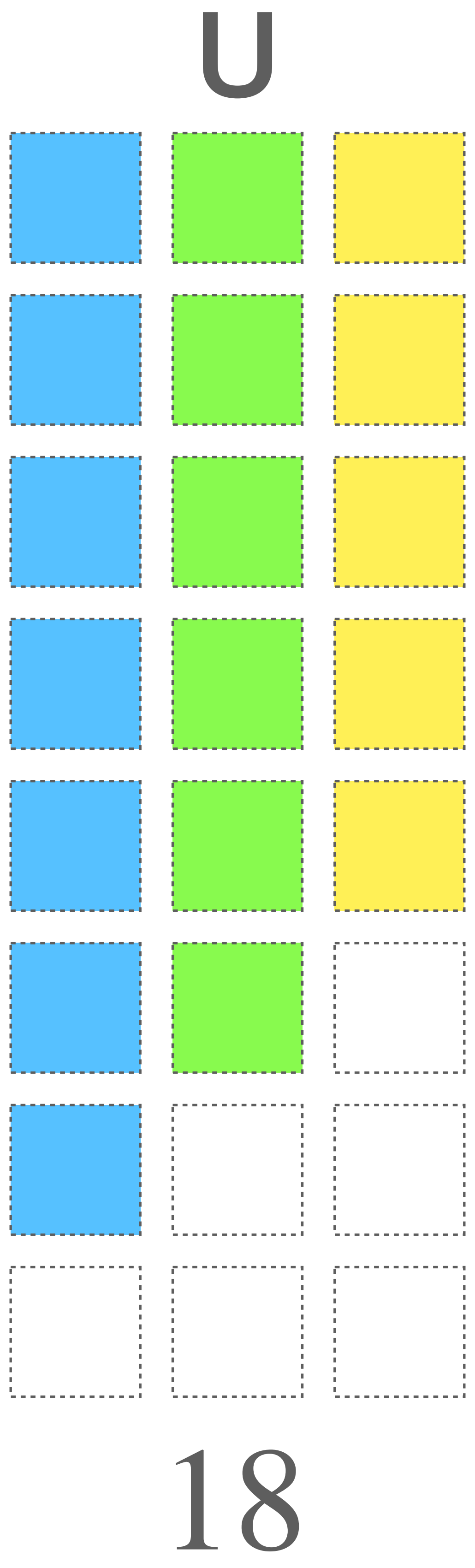
8×8 matrix

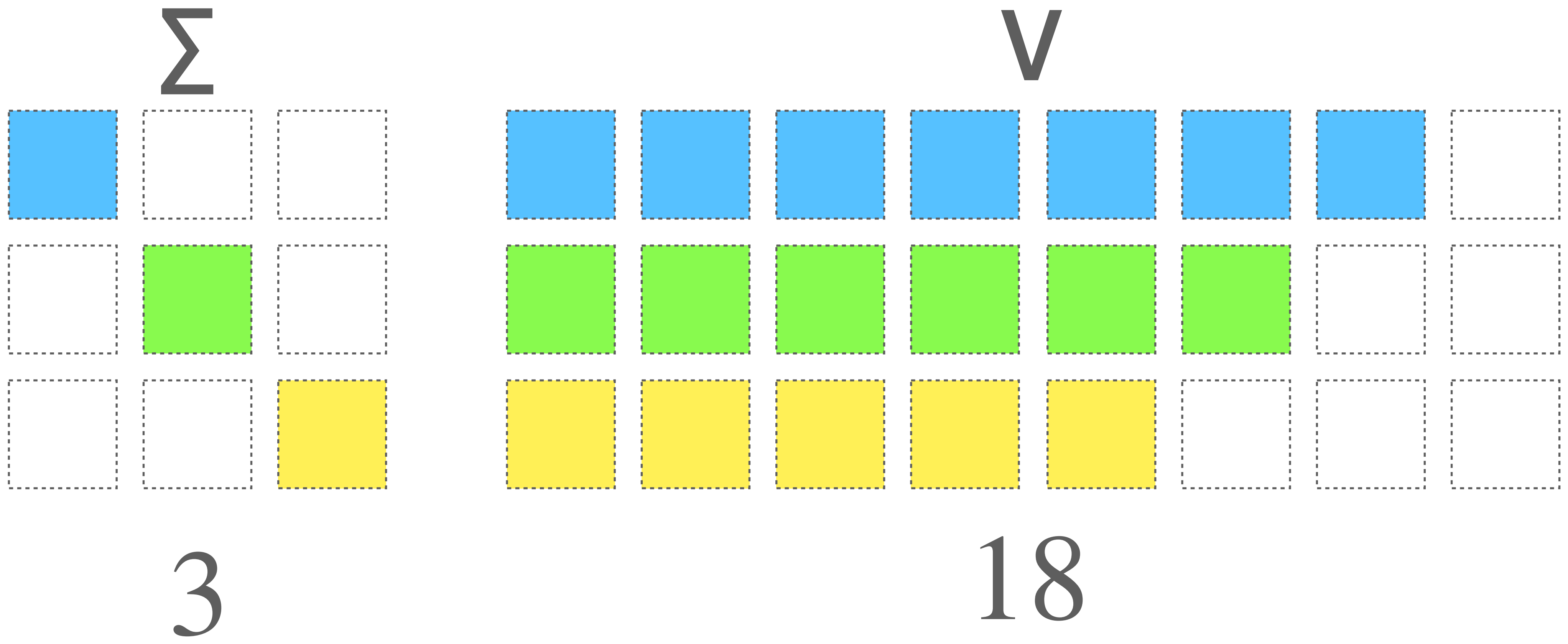
64 DOF

Matrix Degrees of Freedom

How much information is in a matrix?





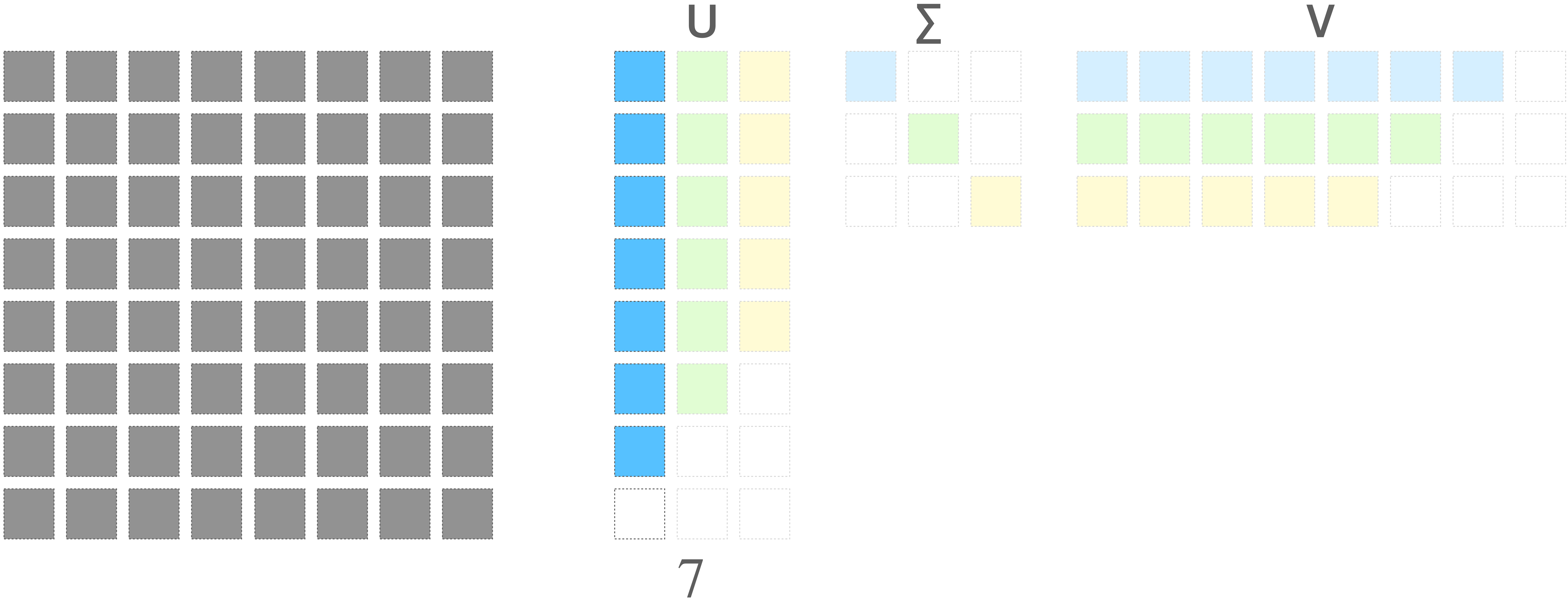


$$DOF = r(m + n - r)$$

39 DOF

Matrix Degrees of Freedom

How much information is in a matrix?



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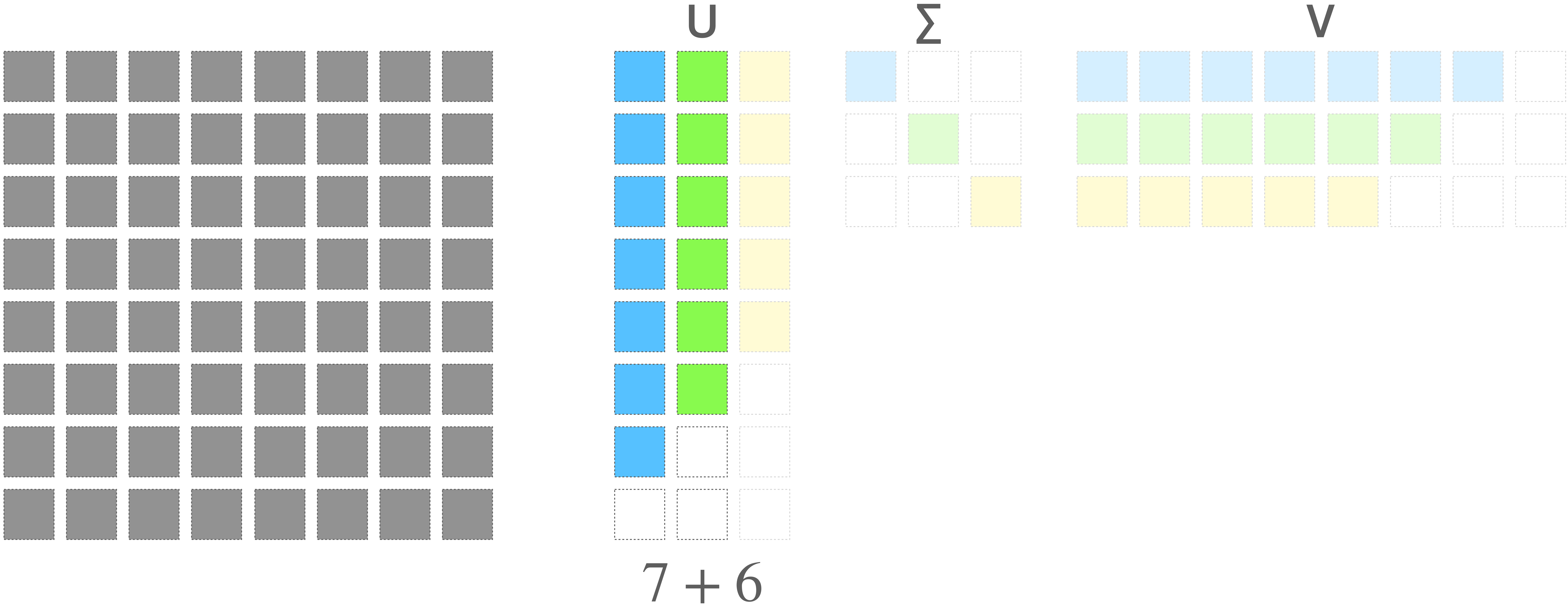

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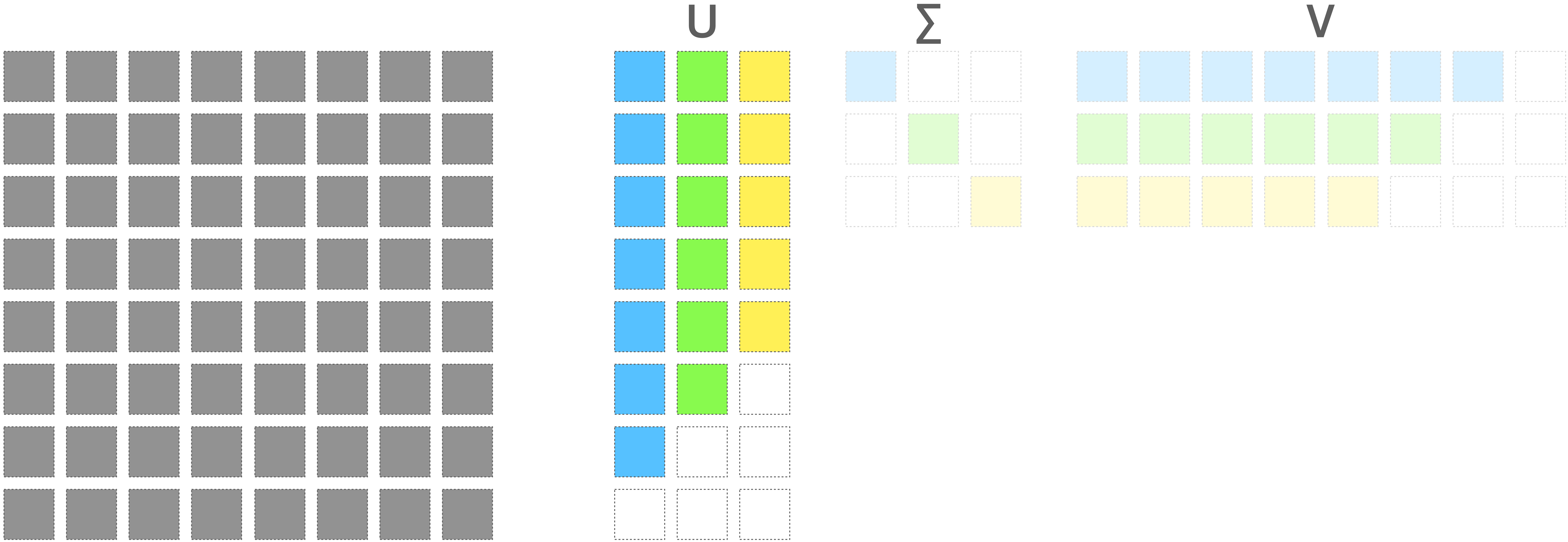
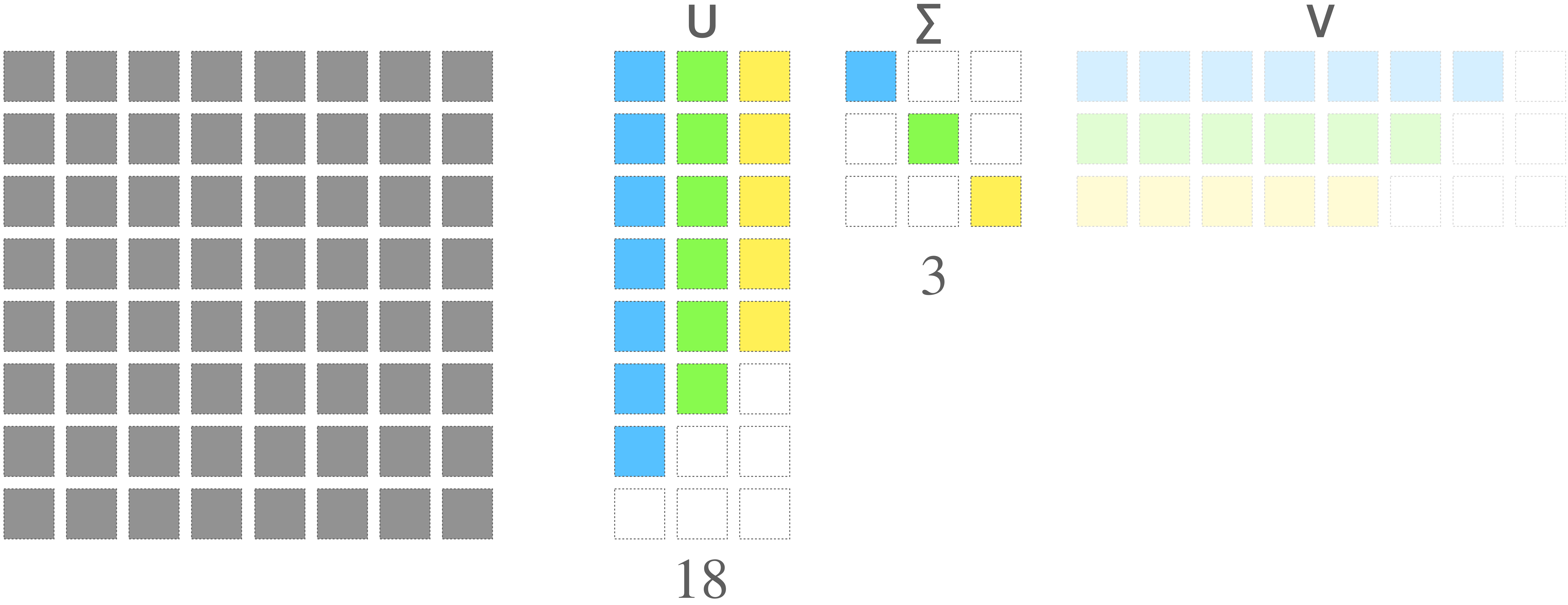


$7 + 6 + 5$

Matrix Degrees of Freedom

How much information is in a matrix?



Matrix Degrees of Freedom

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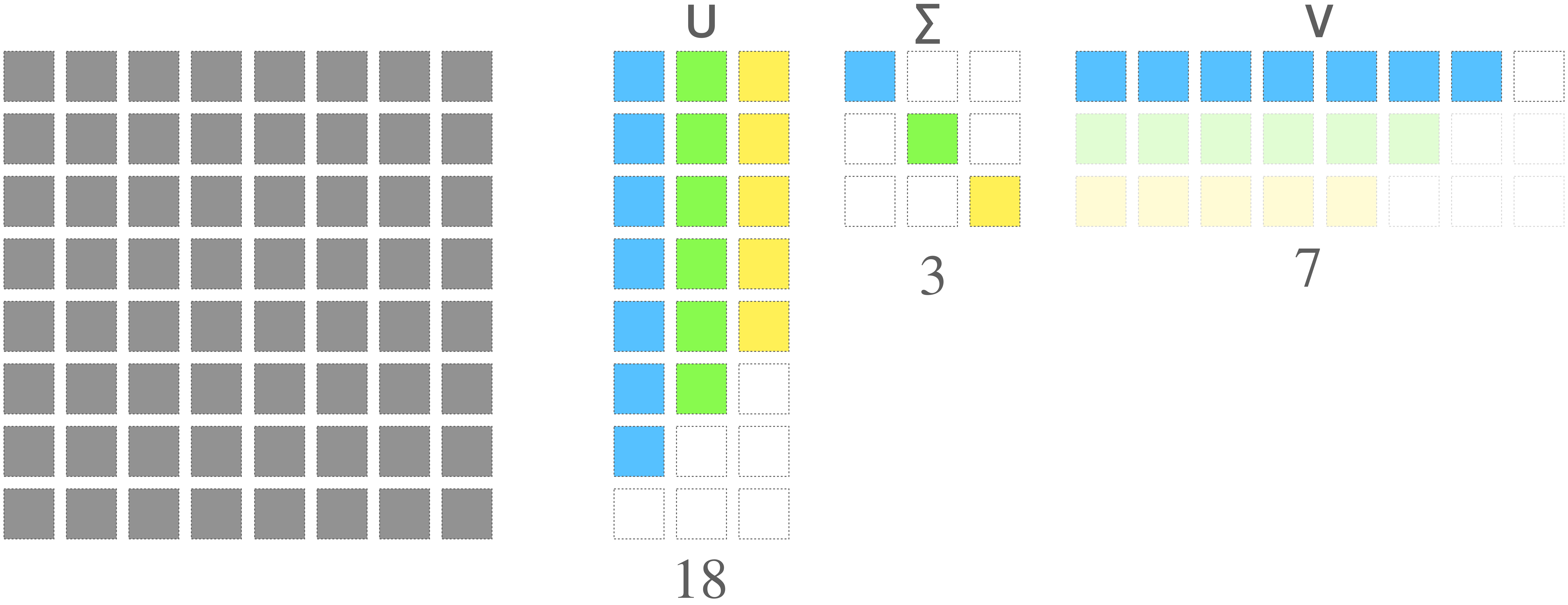

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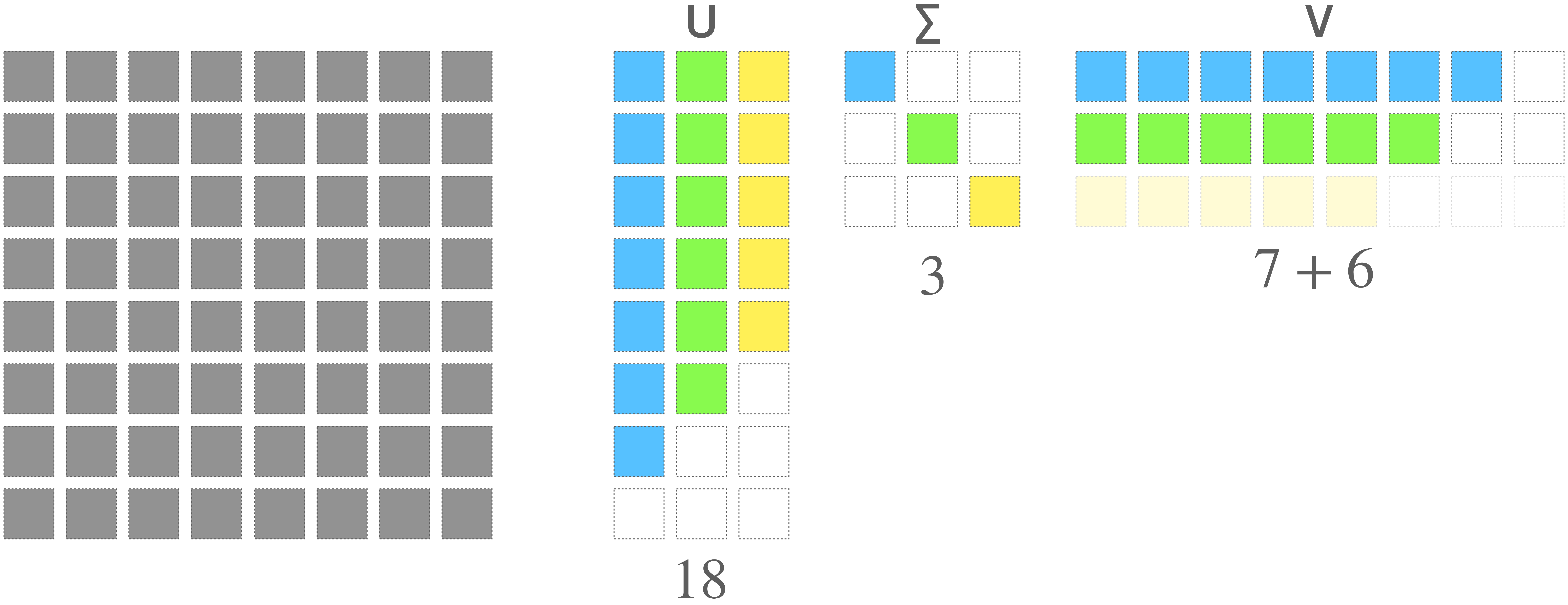

Matrix Degrees of Freedom

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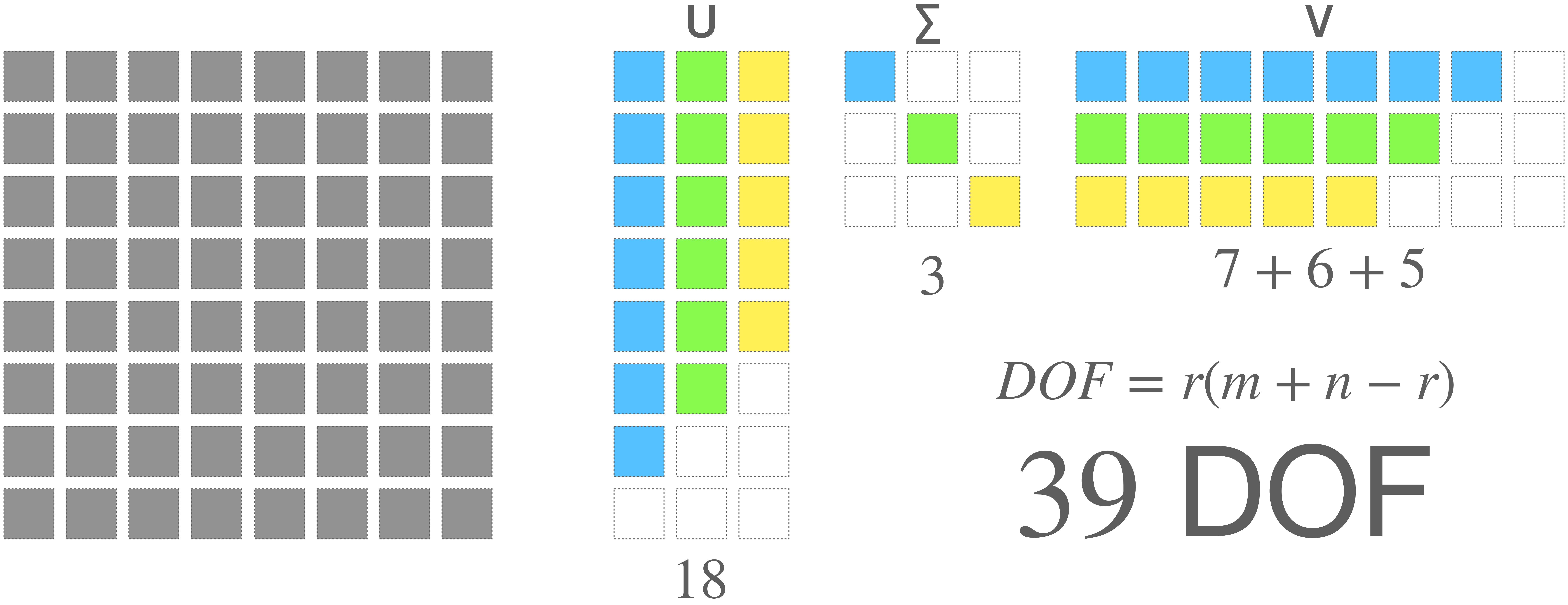
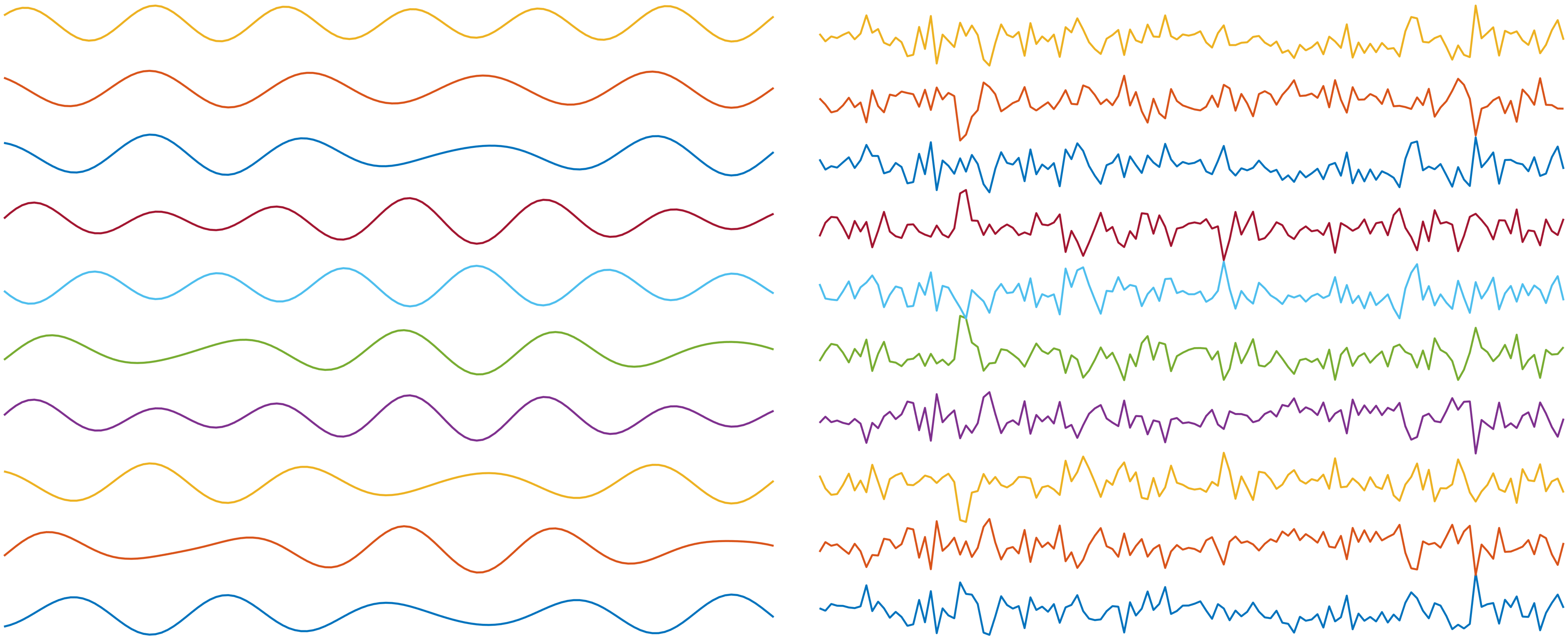
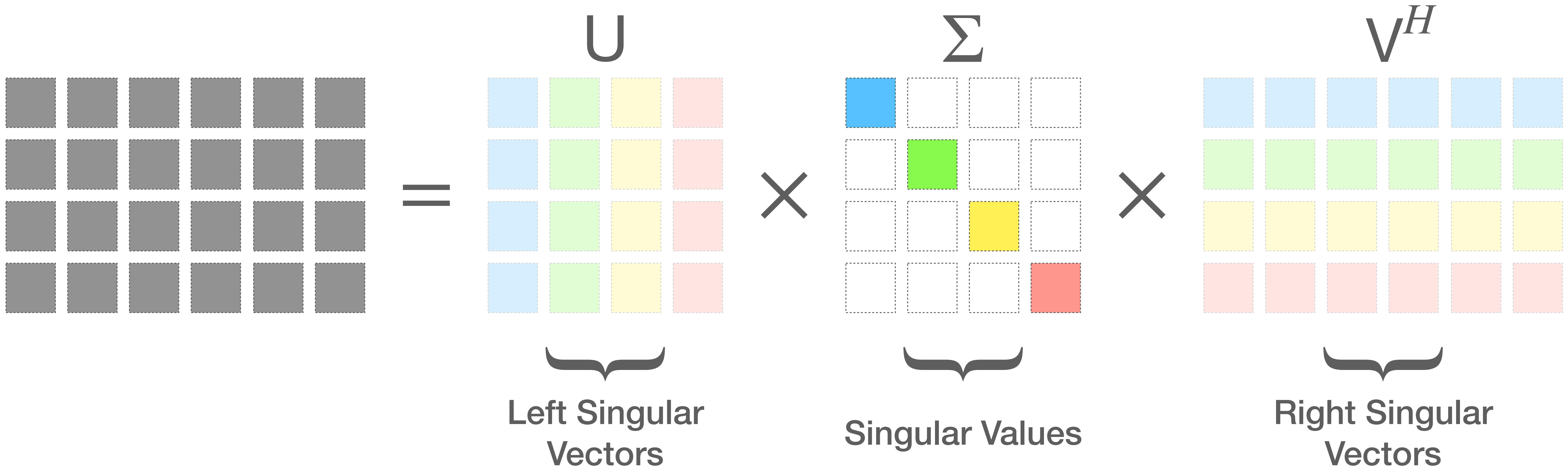


Representational Power

Examples of signals formed by linear combinations of 2 vectors



Connection to Sparsity



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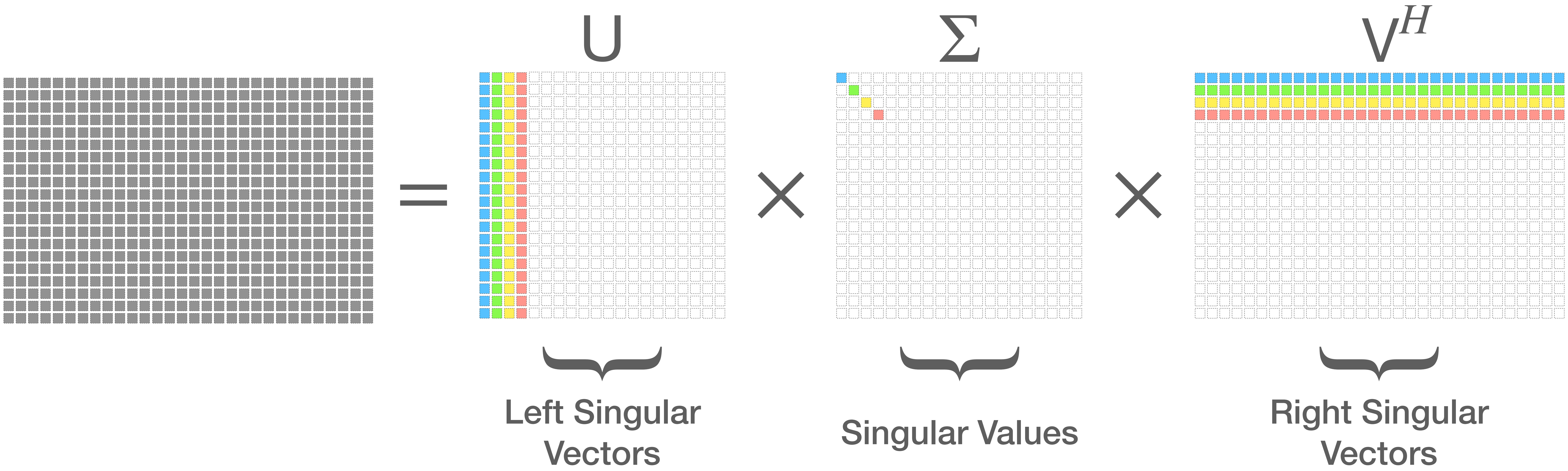

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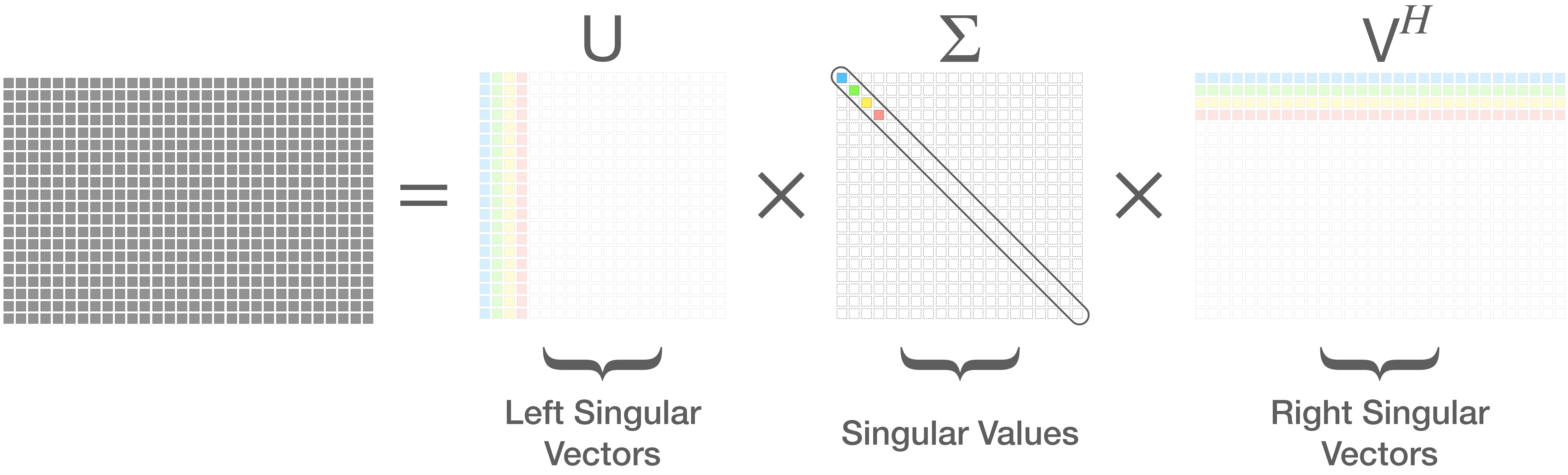
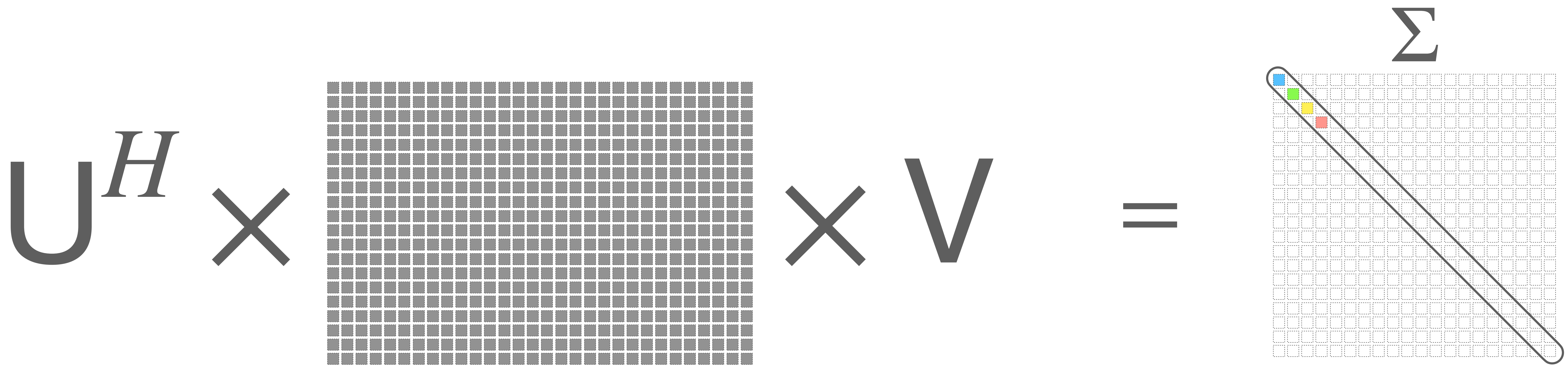


Diagonal Vector of Singular Values is Sparse

Connection to Sparsity



U^H and V are like sparsifying transforms

But we don't need to know them *a priori*, only that they *exist*

Low-Rank Methods

Basic Premise

- You want to reconstruct your data in the presence of under-sampling, noise, or other corruption/perturbation
- Conventional reconstruction models find the image(s) that best fit the measured k-space, in a least squares sense
- However, this may not produce the "best" reconstruction in terms of metrics such as MSE, or the problem might be under-determined
- If your data can be represented as a low-rank matrix, you can use that prior knowledge to additionally constrain/regularize the image reconstruction

Recipe
Low-Rank Methods

1. Identify the low-rank matrix data
    - e.g., space by time
2. Construct your forward measurement model
    - i.e., information about sampling, coil sensitivities, etc.
3. Choose a suitable cost function with low-rank constraint
    - e.g., convex of non-convex, local or global low-rank constraints
4. Solve the cost function using an appropriate algorithm
    - typically a first order, iterative algorithm

1. Identify appropriate matrix data

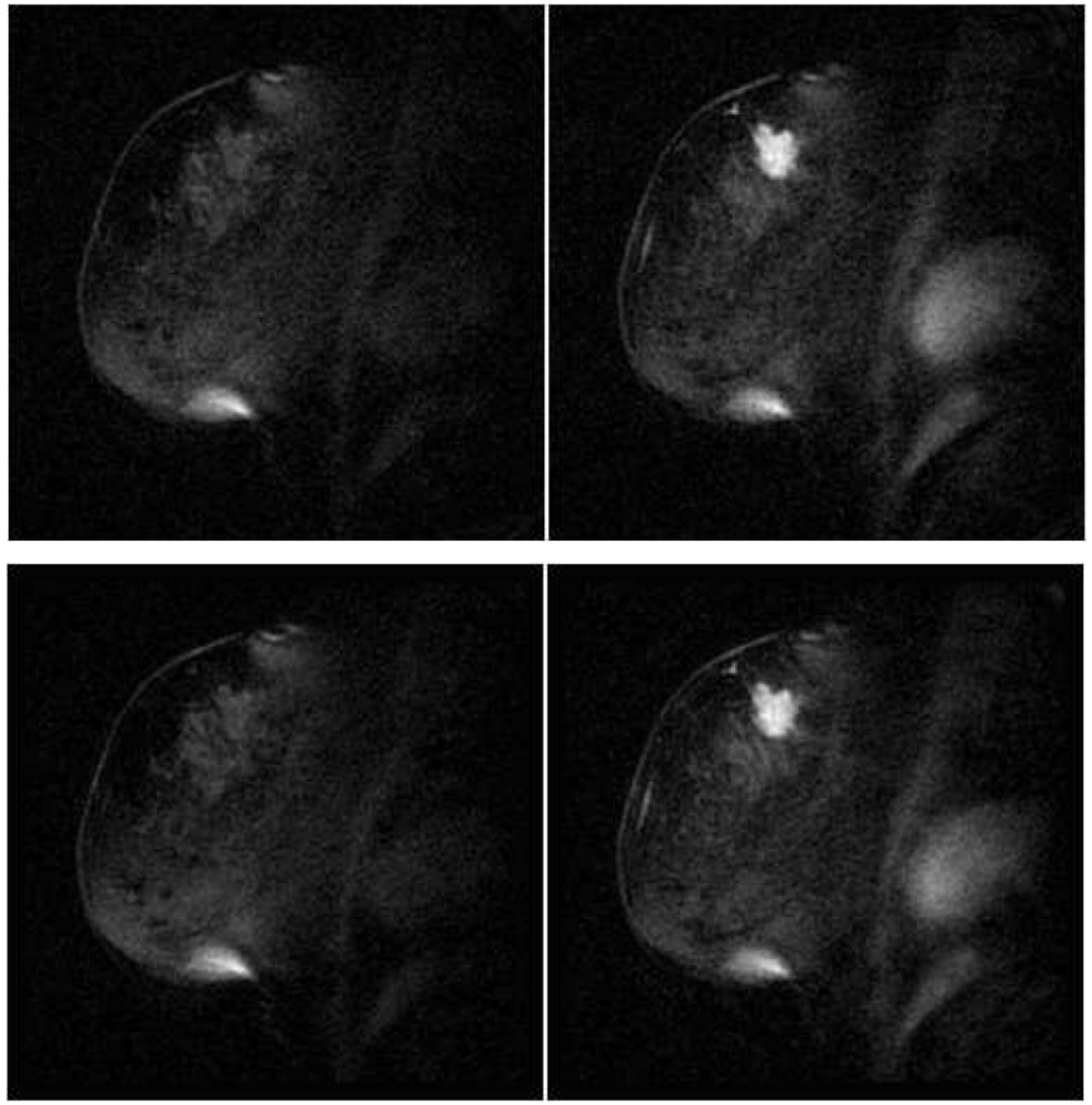
e.g. (Space × Time)

- Space-time matrices are probably the most intuitive
- Signal time-courses are correlated across space
- Many voxels exhibit the same contrast dynamics

Dynamic Contrast Enhanced Breast

Ground Truth

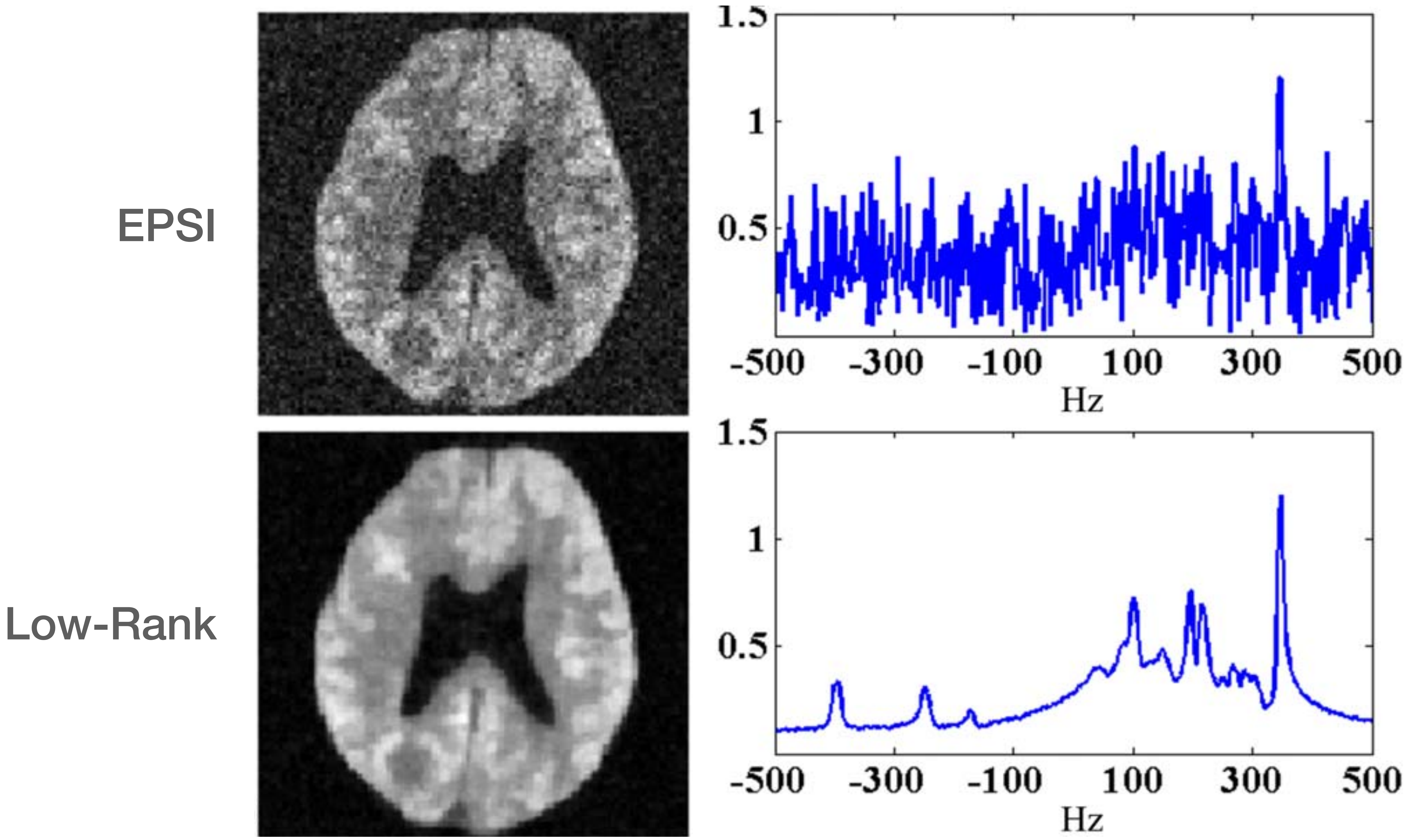
Low-Rank



Liang, “Spatiotemporal imaging with partially separable functions”ISBI 2007

1. Identify appropriate matrix data

e.g. (Space × Spectra)

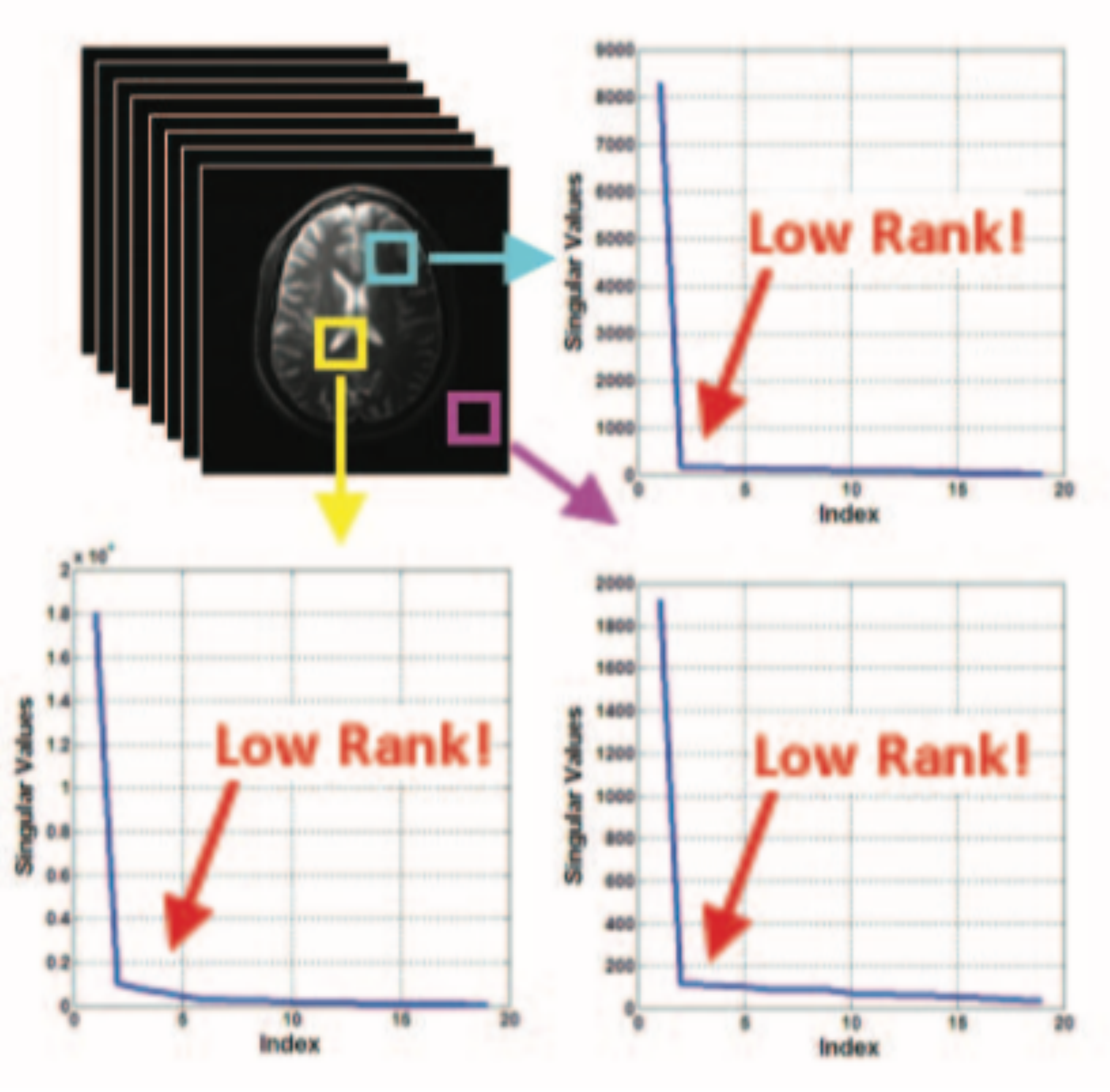


- Spectral content is similar across voxels

Lam & Liang, "A subspace approach to high-resolution spectroscopic imaging" MRM 2014

1. Identify appropriate matrix data

e.g. (Space × Coils)

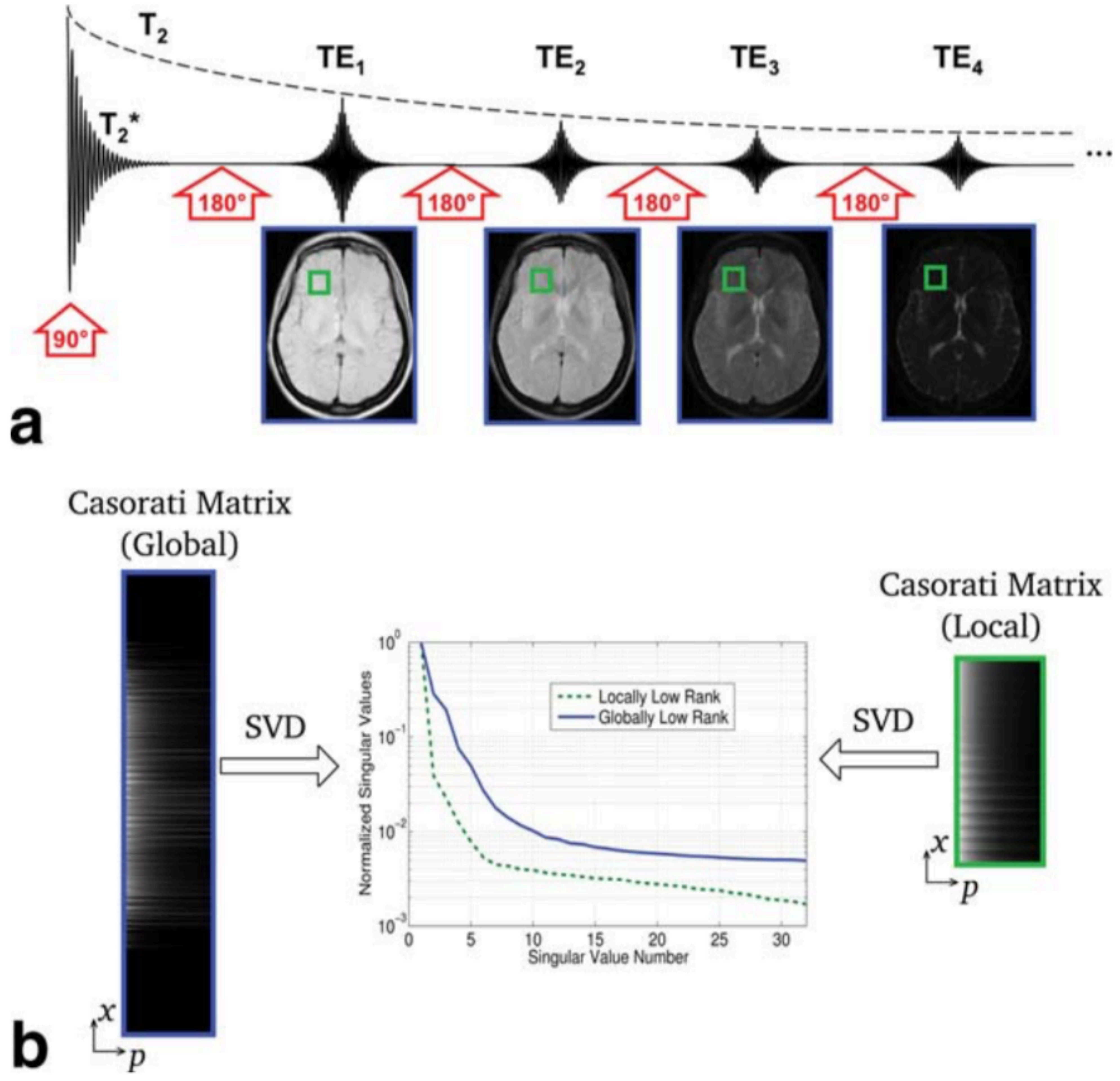


- Local spatial information is shared across coils

Trzasko & Manduka, "Calibrationless parallel MRI using CLEAR" ASILOMAR 2011

1. Identify appropriate matrix data

e.g. (Space × TE)



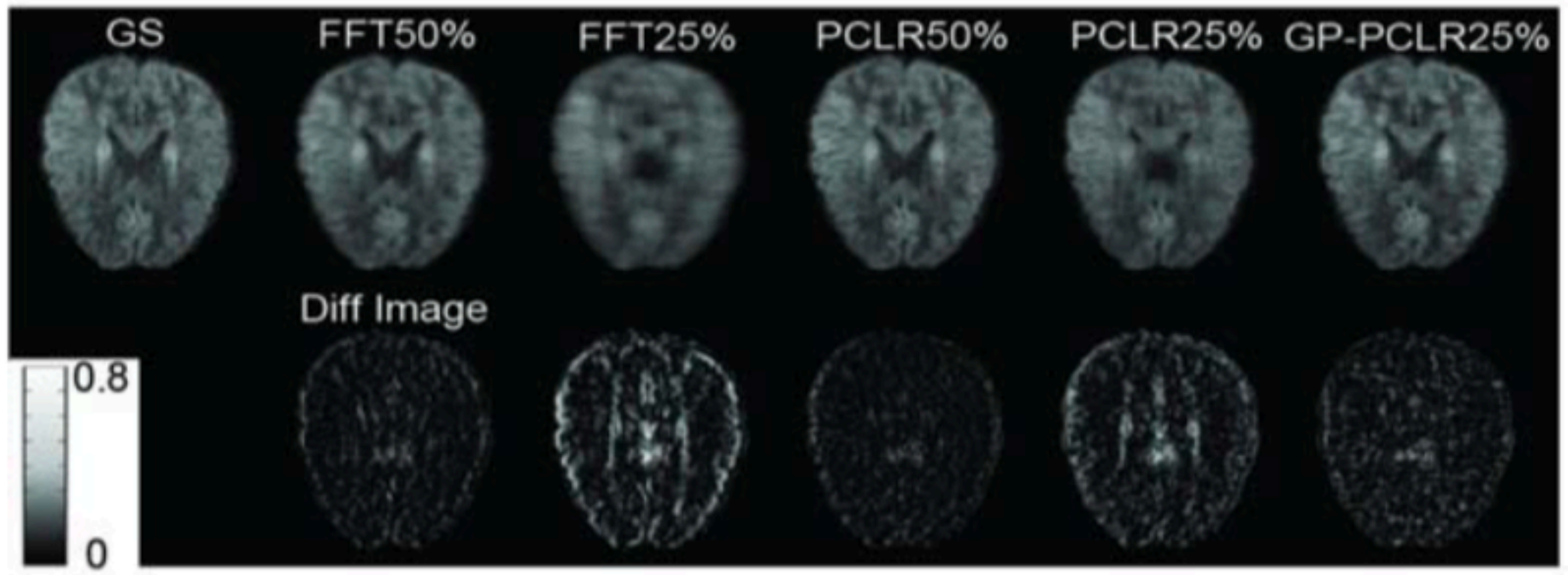
- Voxels or spatial features are correlated across echo time

Zhang et al., "Accelerating parameter mapping with a locally low rank constraint" MRM 2014

1. Identify appropriate matrix data

e.g. (Space × Diffusion Encoding)

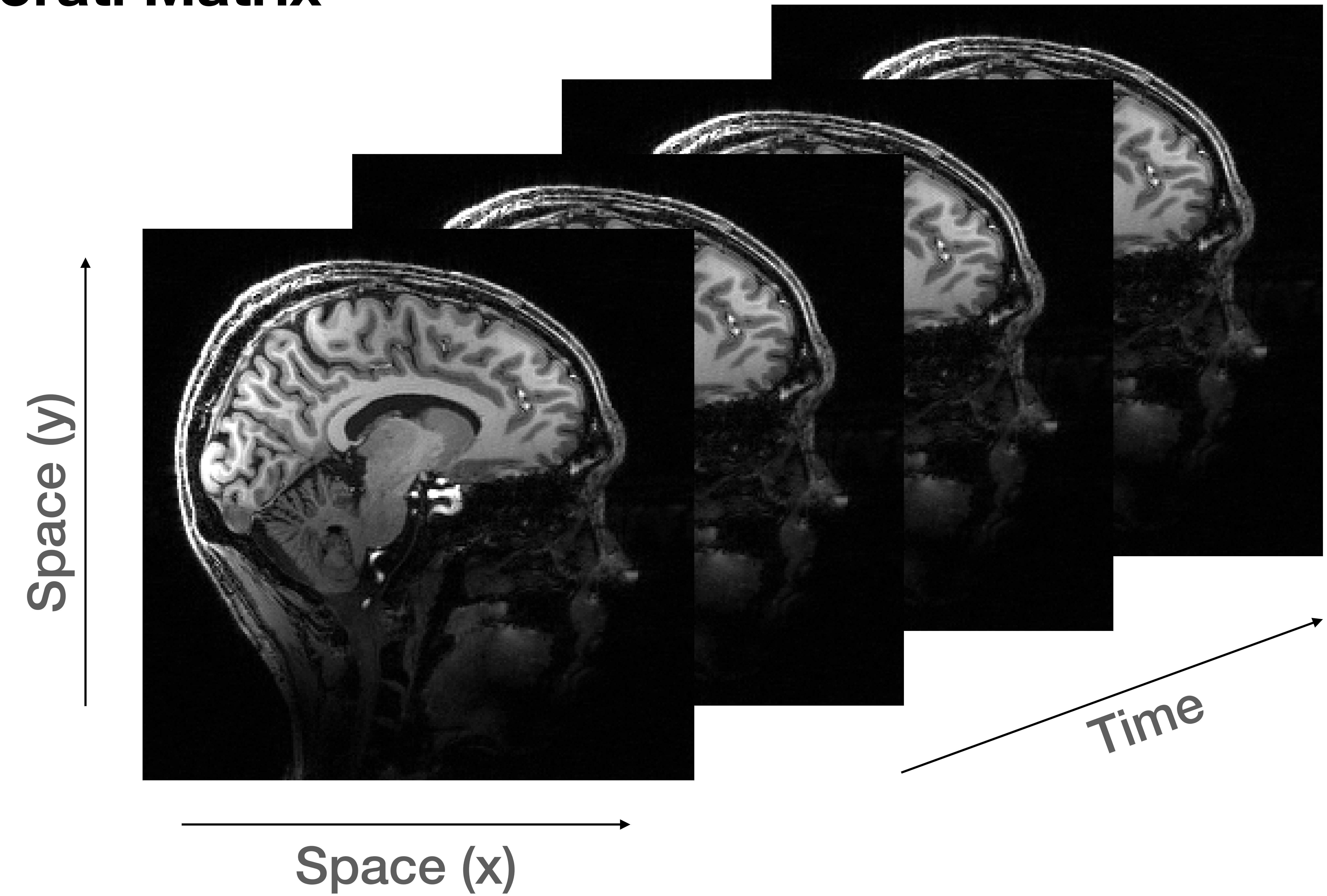
- Voxel contrasts are correlated across diffusion encodings



Gao et al., “PCLR: Phase-constrained low-rank model for compressive diffusion-weighted MRI” MRM 2013

1. Identify appropriate matrix data

n.b. “Casorati Matrix”



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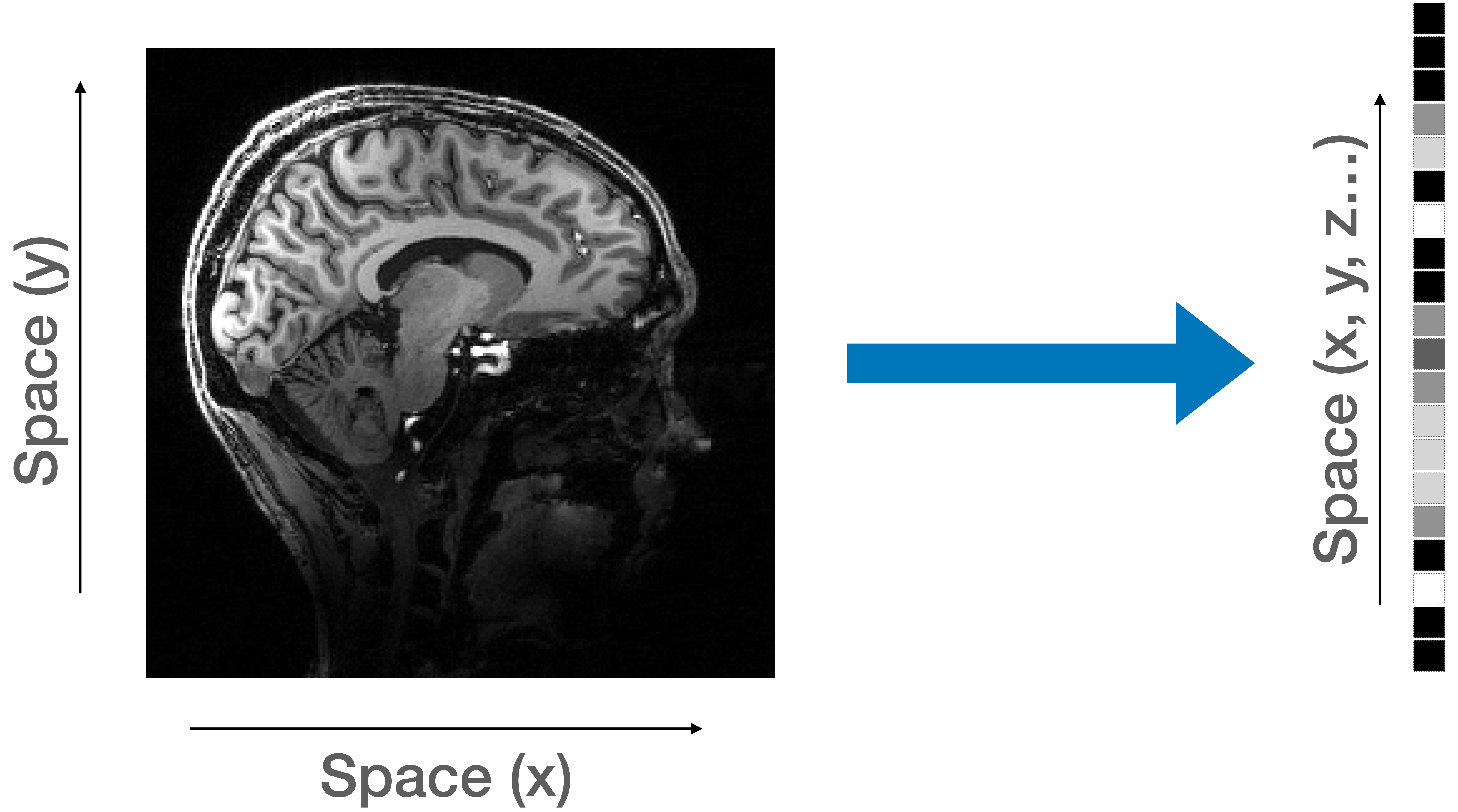

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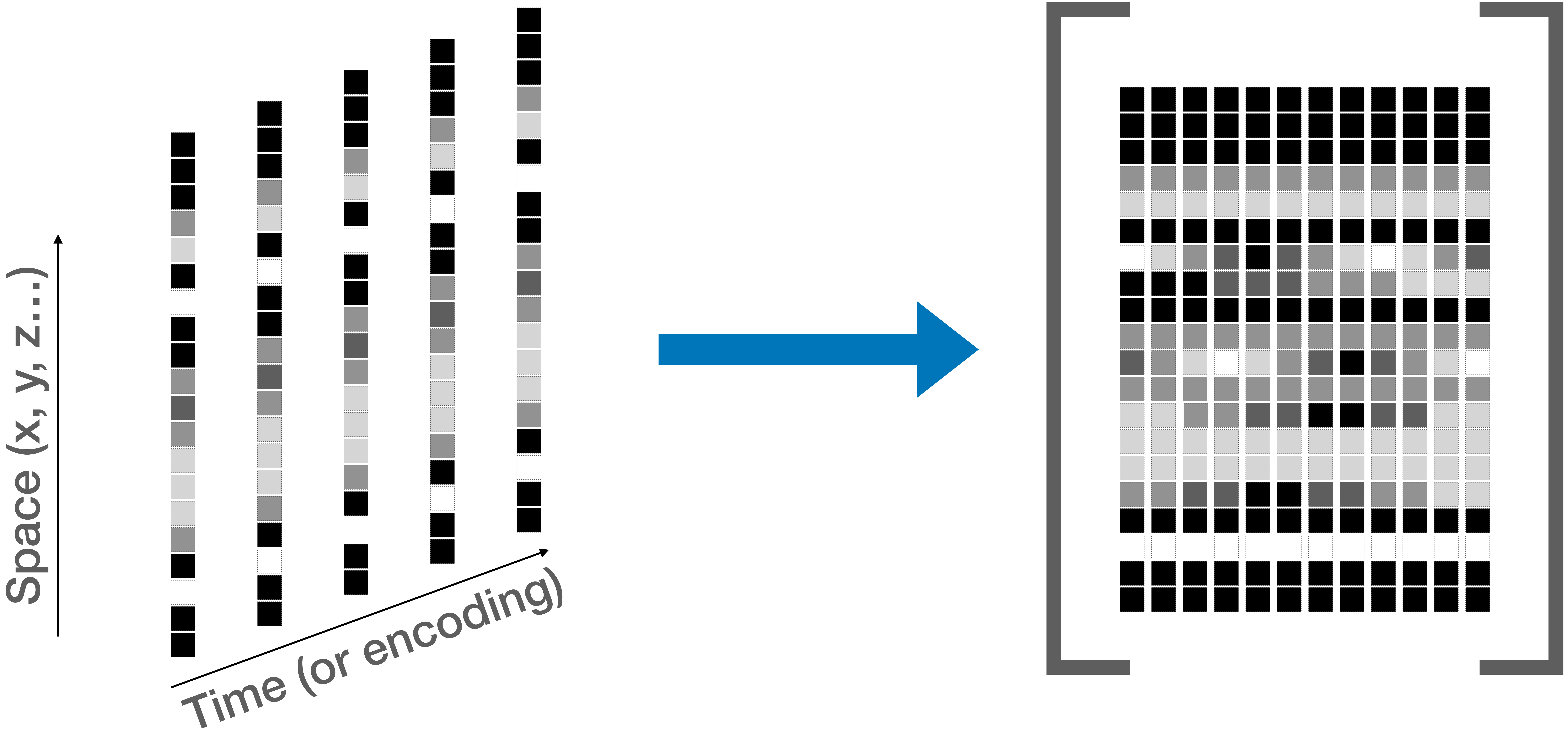
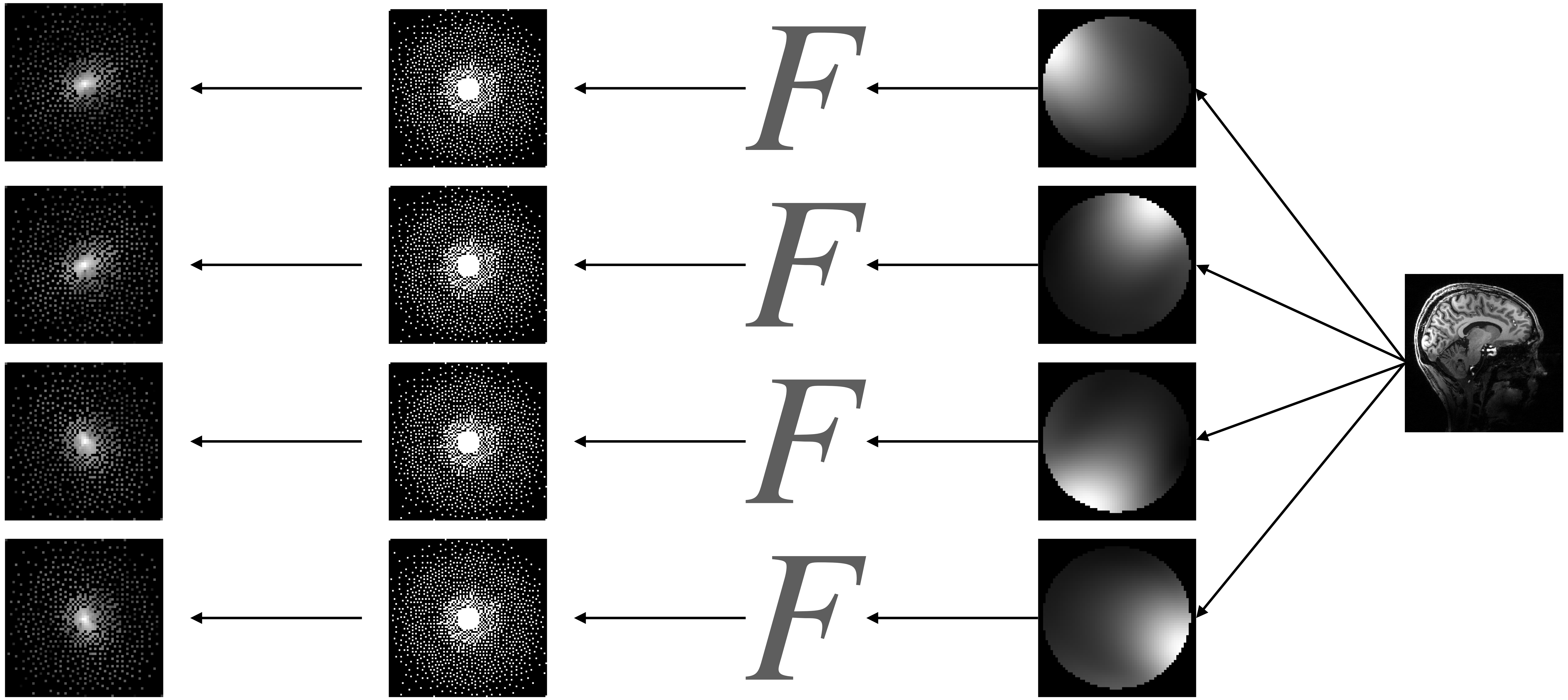
n.b. “Casorati Matrix”



2. Construct the Forward Operator

Maps your "image" to the sampled k-space measurements



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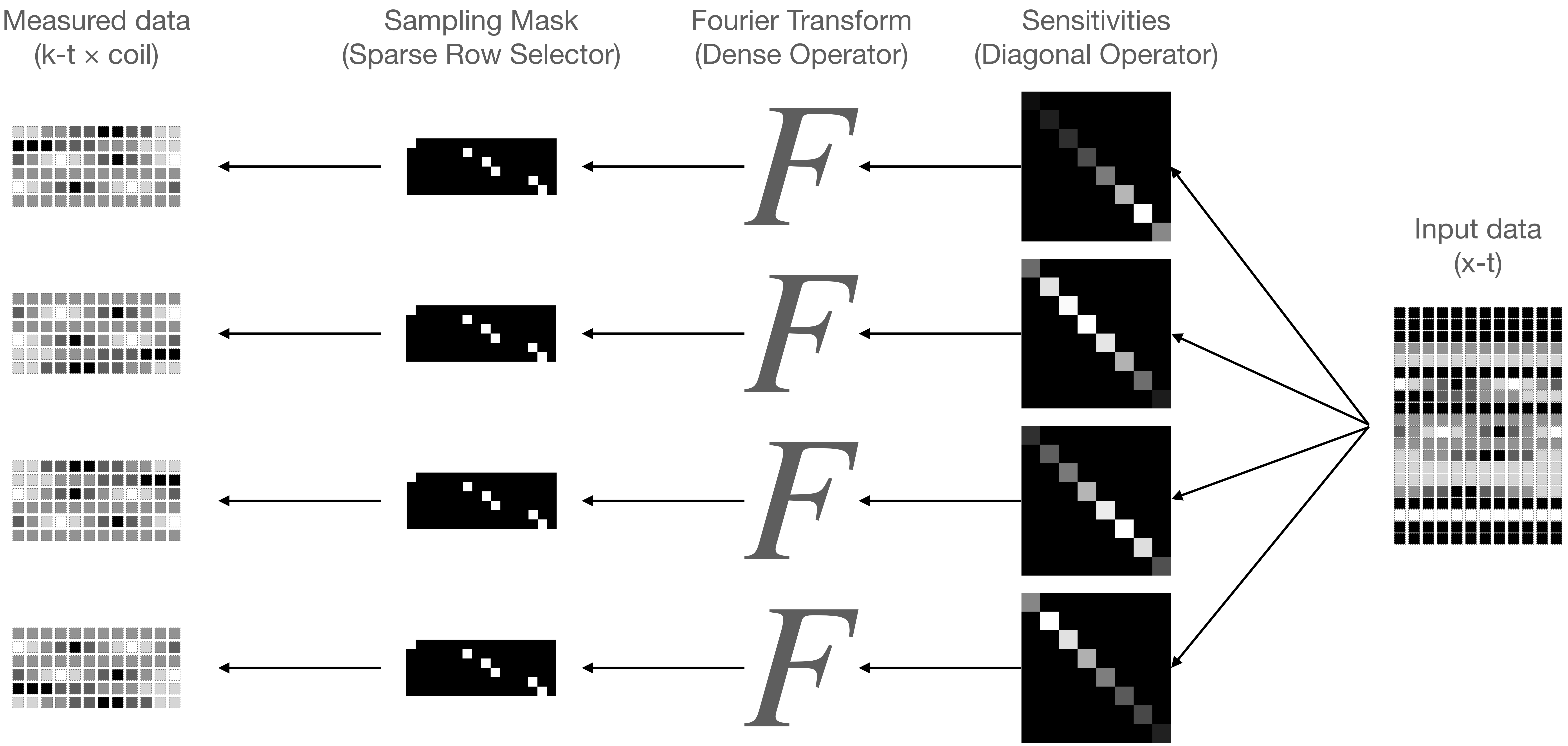
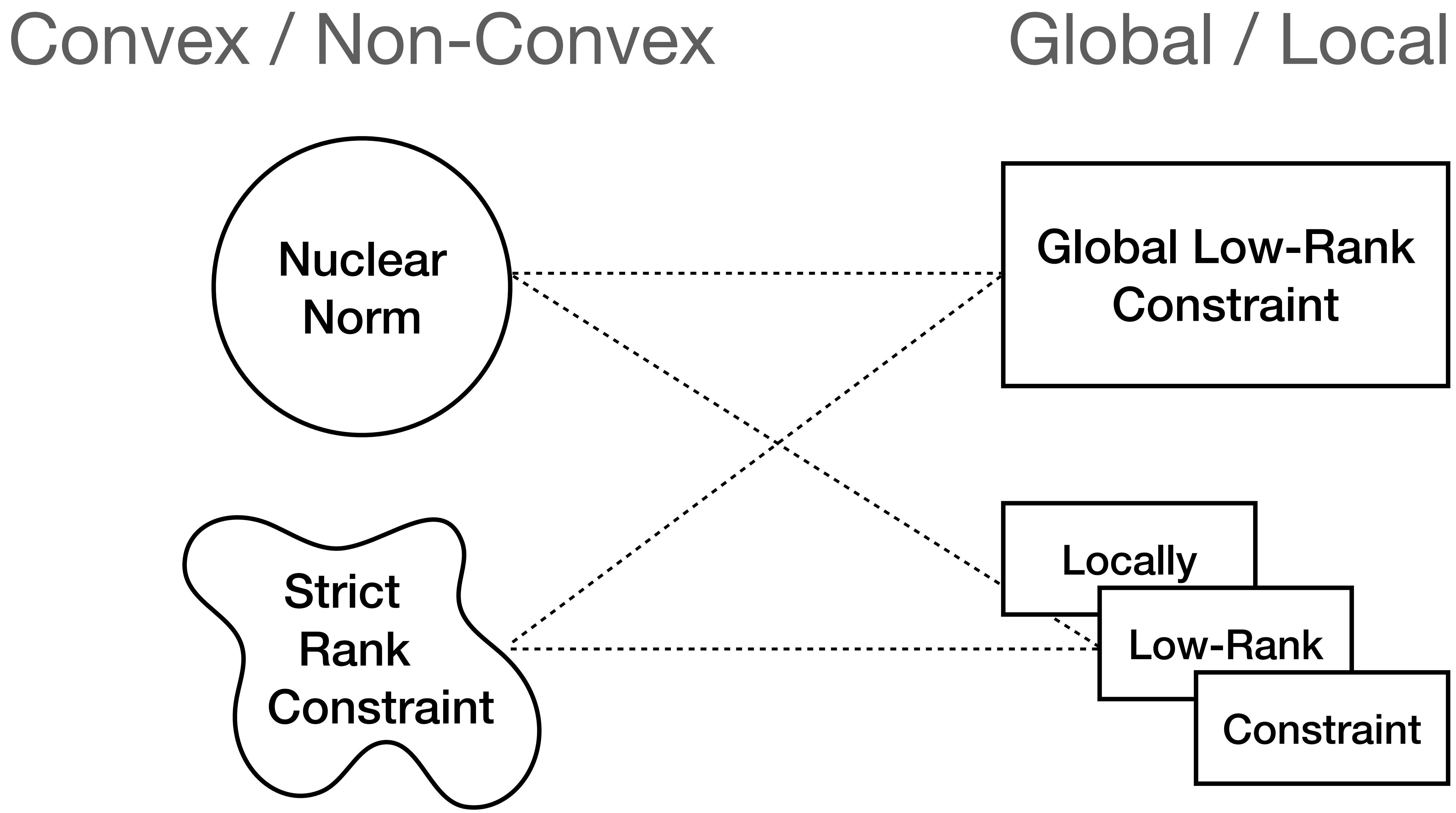


3. Choose a Low-Rank Constraint

How will the low-rank constraint be enforced?

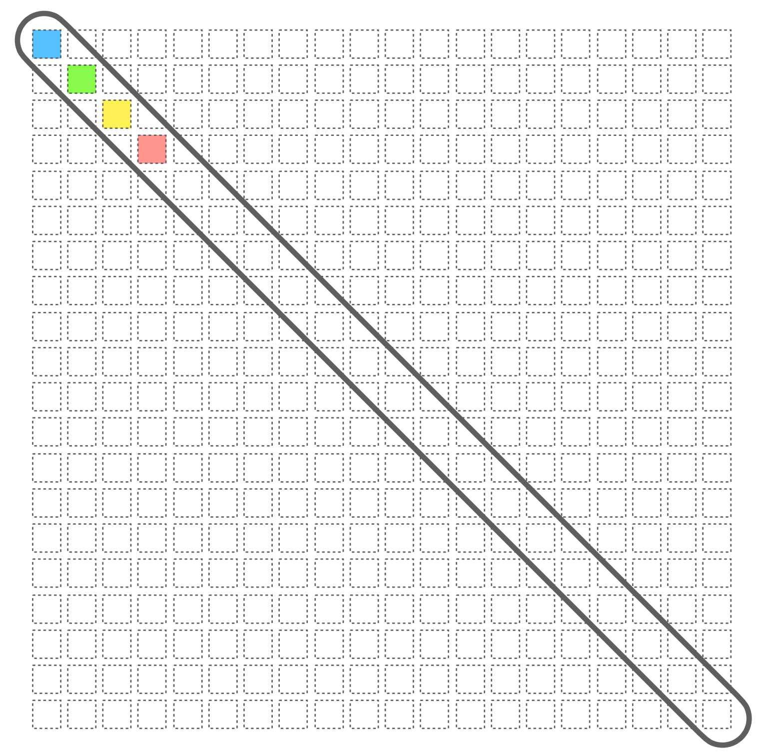
- Convex Nuclear Norm
  - Sum of singular values (analogous to L1)
- Non-Convex rank constraint
  - Matrix has exactly (or at most) rank “**r**”
- Global Low-Rank constraint
  - Entire matrix is low-rank
- Locally low-rank constraint
  - Matrix is “patch-wise” low-rank

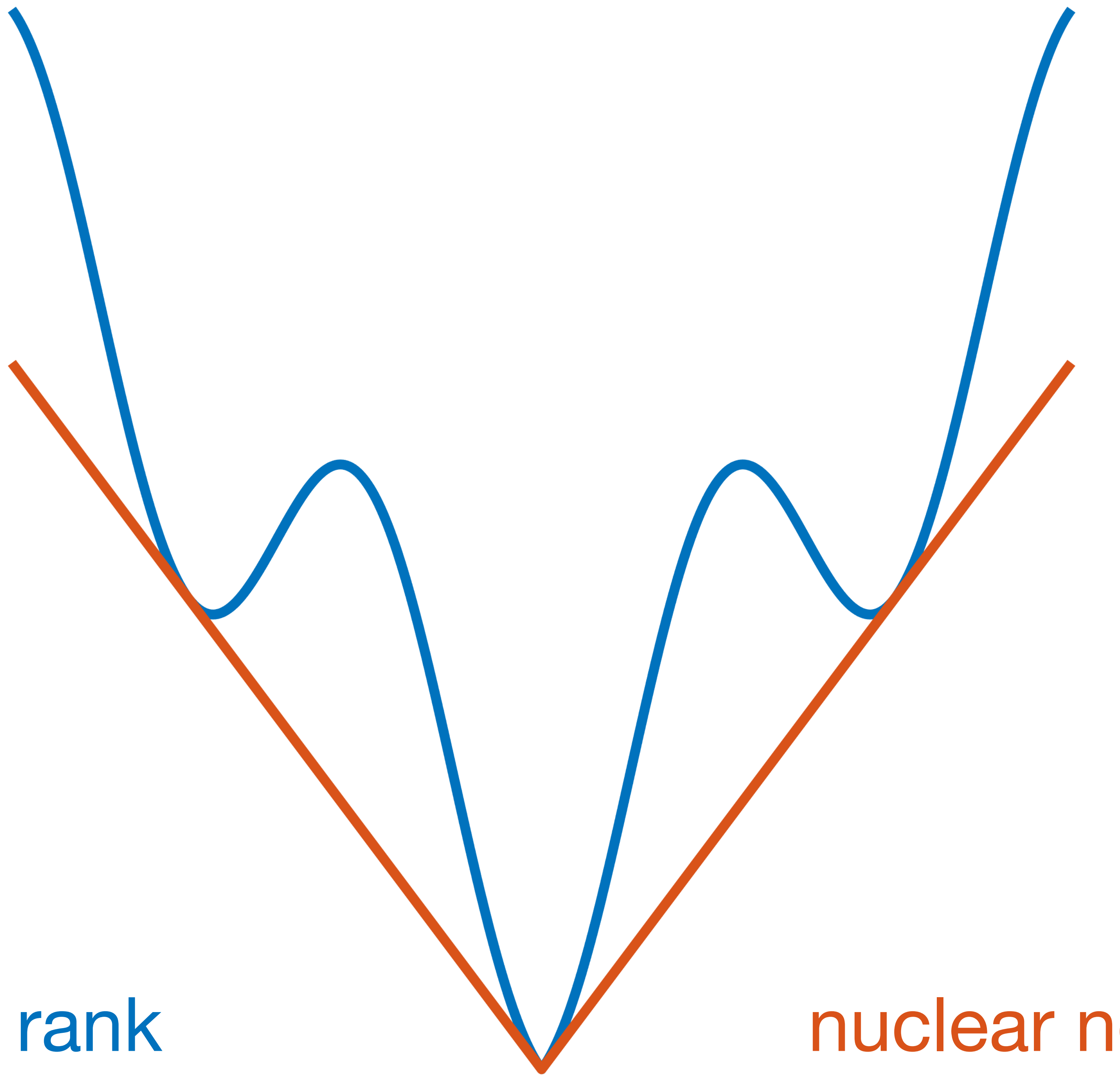


3. Choose a Low-Rank Constraint

Nuclear Norm (Convex)

$$\|x\|_* = \left\| \quad \right\|_1 = \sum_i \sigma_i$$

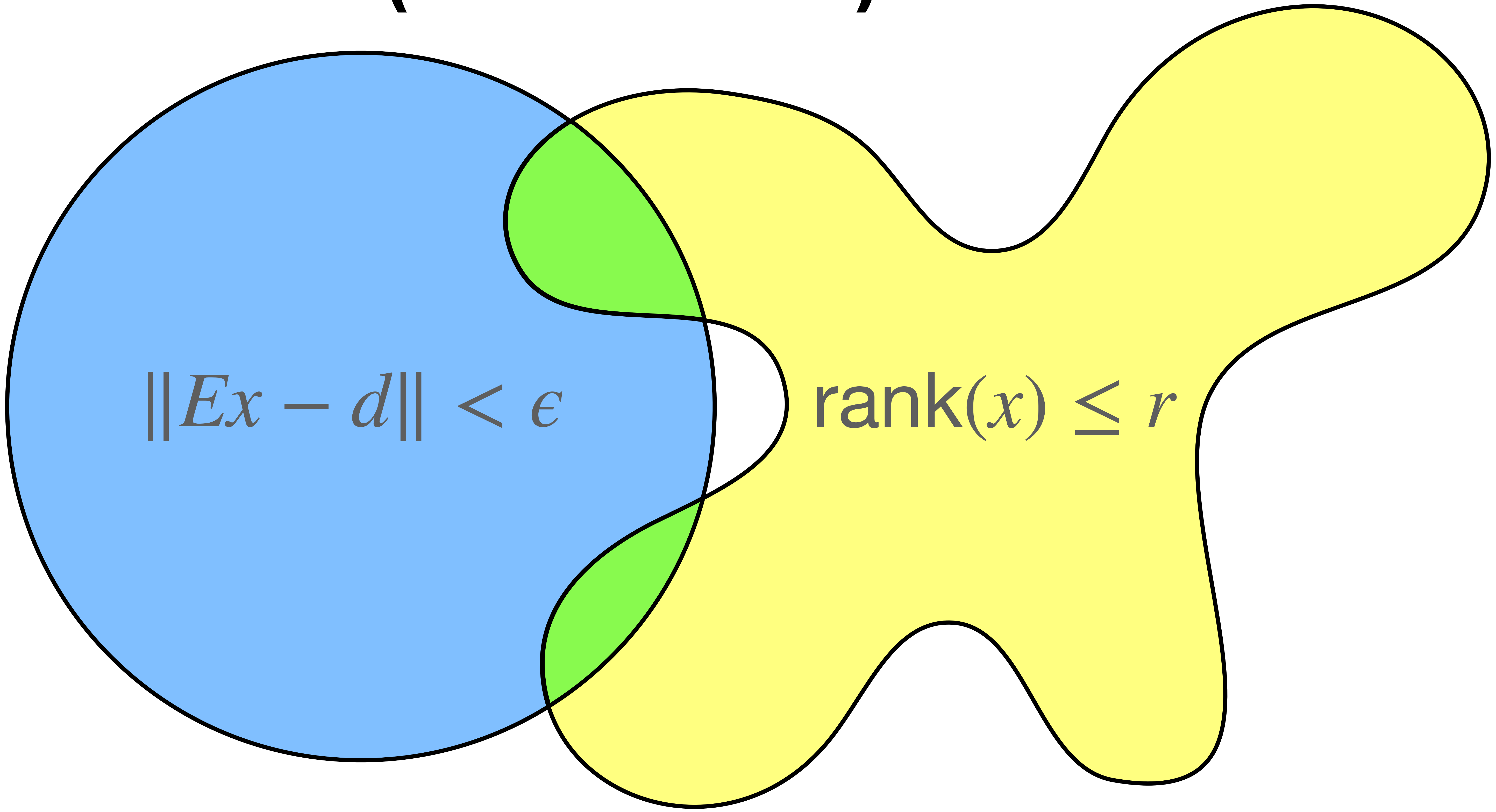




$$\min_x \|Ex - d\| + \lambda \|x\|_*$$

3. Choose a Low-Rank Constraint

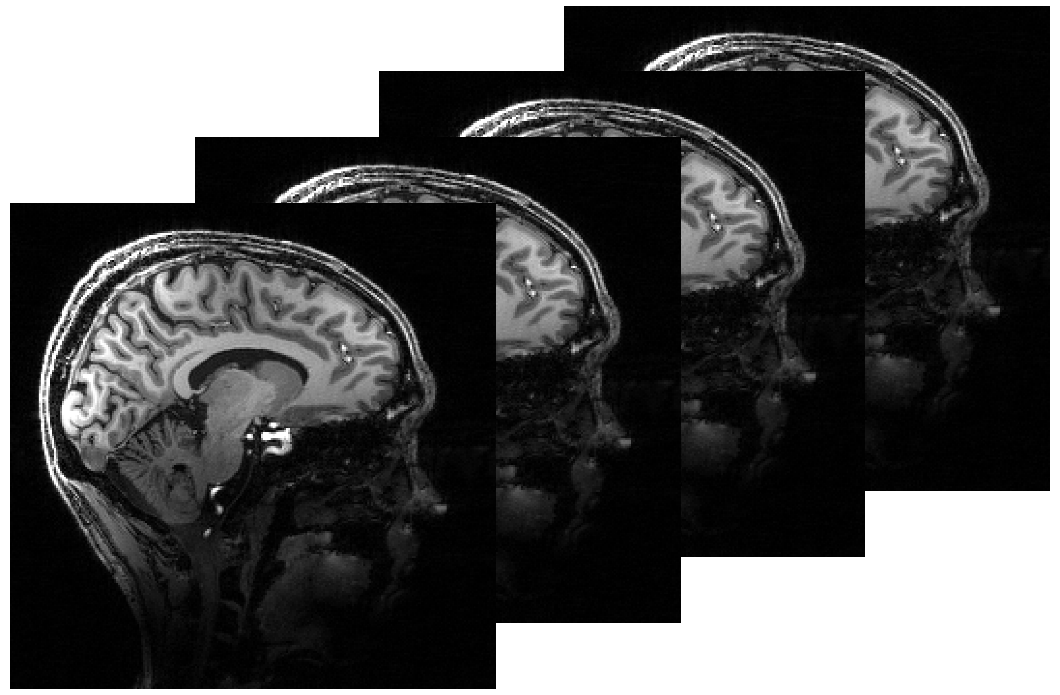
Strict Rank Constraint (Non-Convex)

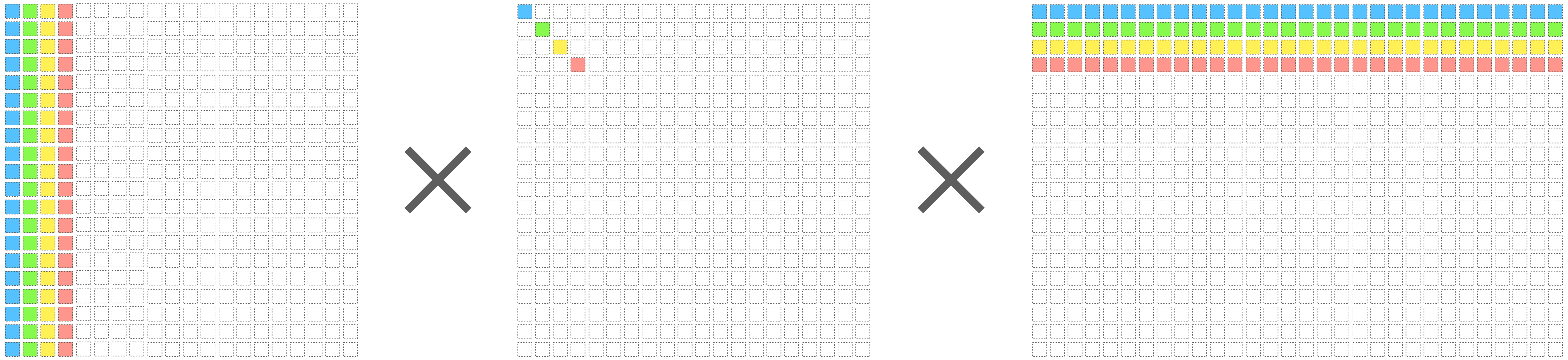


$$\min_{rank(x) \leq r} \|Ex - d\|$$

3. Choose a Low-Rank Constraint

Global Low-Rank Constraint

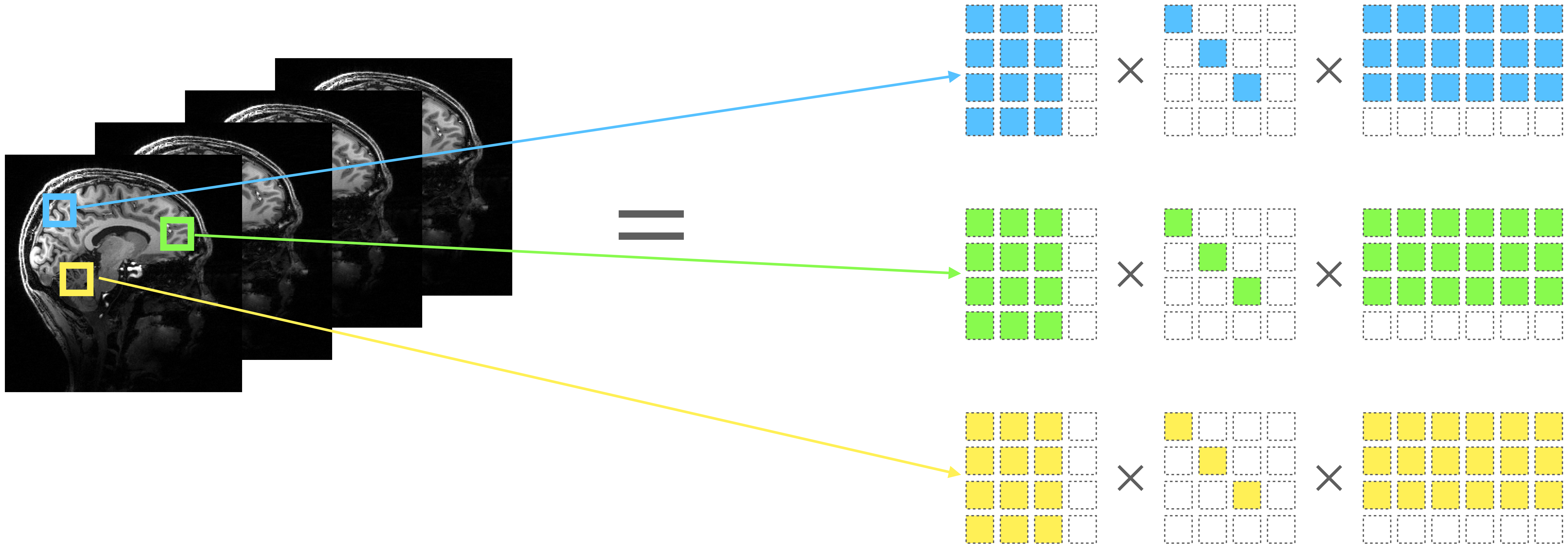




$$\min_{x} ||Ex - d|| + \lambda ||x||_*$$

3. Choose a Low-Rank Constraint

Locally Low-Rank Constraint



$$\min_x \|Ex - d\| + \lambda \sum_p \|R_p x\|_*$$

3. Choose a Low-Rank Constraint

Extensions with other constraints or formulations

Low Rank + Sparse Decomposition[1]

$$\min_{L,S} \|E(L+S)-d\| + \lambda_L \|L\|_* + \lambda_S \|\Psi S\|_1$$

Low Rank & Sparse Penalties[2]

$$\min_{x} \|Ex-d\| + \lambda_L \|x\|_* + \lambda_S \|\Psi x\|_1$$

Low Rank Tensor Reconstruction[3]

$$\min_{x} \|Ex-d\| + \lambda \sum_i \|x_{(i)}\|_*$$

1. Otazo et al., “Low-rank plus sparse matrix decomposition for accelerated dynamic MRI with separation of background and dynamic components.” MRM 2015
2. Lingala et al., “Accelerated Dynamic MRI Exploiting Sparsity and Low-Rank Structure: k-t SLR” IEEE-TMI 2011
3. Trzasko & Manduca, “A Unified Tensor Regression Framework for Calibrationless Dynamic, Multi-Channel MRI Reconstruction” ISMRM 2013

4. Algorithm

Optimize the cost function with some algorithm

- SVD-free: e.g. Matrix Factorization

- SVD-based: e.g. Proximal gradient method (Singular Value Thresholding)

4. Algorithm

Matrix Factorization - Low Rank by Construction

$$\min_{\substack{U \in \mathbf{C}^{m\times r} \\ V \in \mathbf{C}^{n\times r}}} \|E(UV^H) - d\|$$

$$U_{k+1} = \arg\min_{U} \|E(UV_k^H) - d\|_F$$
$$V_{k+1} = \arg\min_{V} \|E(U_{k+1}V^H) - d\|_F$$

Haldar & Hernando, "Rank-Constrained Solutions to Linear Matrix Equations Using PowerFactorization" IEEE Sig Proc Lett 2009

4. Algorithm

Matrix Factorization - Training data

$$\min_{U \in \mathbf{C}^{m \times r}} \|E(UV^H) - d\|$$

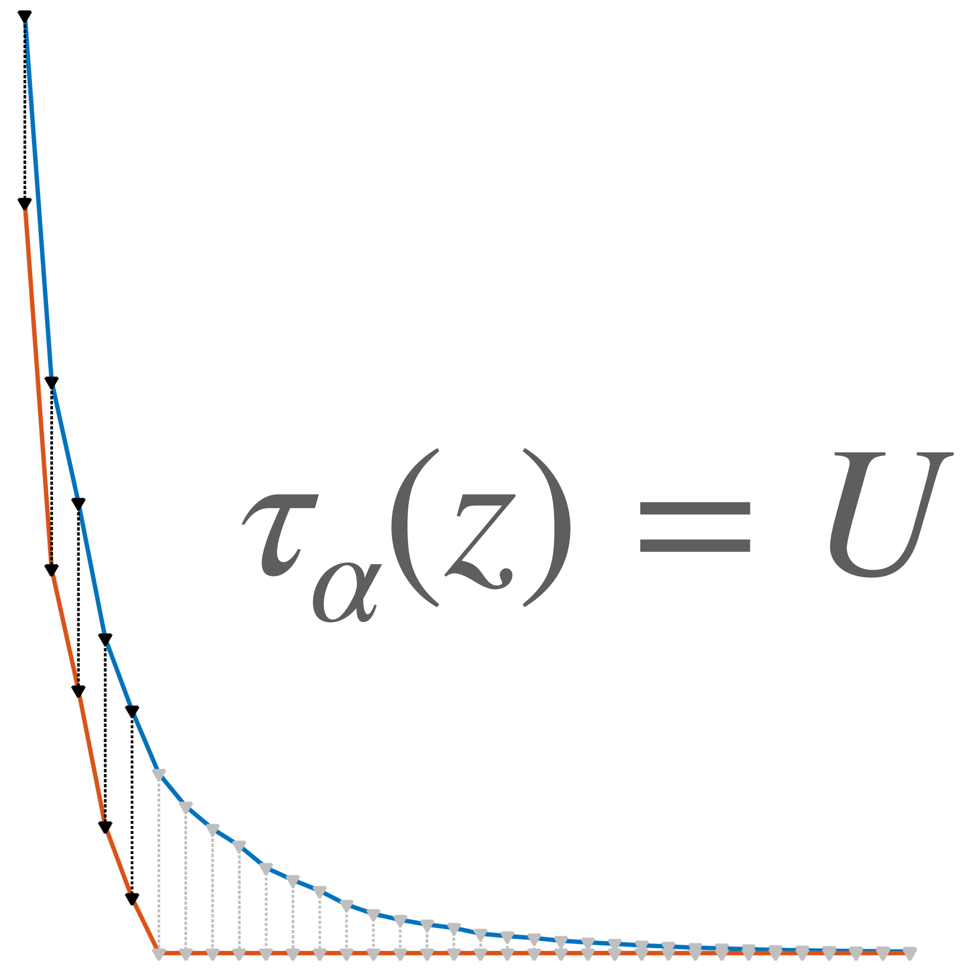
- ***V*** can be determined from training data
- e.g. low-resolution, fully-sampled k-space to estimate temporal basis vectors

Liang, "Spatiotemporal imaging with partially separable functions" ISBI 2007

4. Algorithm
SVD-based thresholding

- Analogous to soft-thresholding algorithms for sparse reconstruction problems
- e.g. Alternating between gradient step w.r.t. data consistency and a singular value soft thresholding proximal projection step to promote low-rankness

$$y_{k+1} = x_k + \mu E^H(Ex_k - d)$$
$$x_{k+1} = \tau_{\lambda\mu}(y_{k+1})$$



$$\tau_\alpha(z) = U \cdot \max(\Sigma - \alpha, 0) \cdot V^H$$

Cai et al., "A Singular Value Thresholding Algorithm for Matrix Completion" Siam J Opt 2010
Ma et al., "Fixed point and Bregman iterative methods for matrix rank minimization" Math Prog 2011

Other Considerations
Sampling and Parameter Selection

- Sampling
  - With training data (e.g. low-resolution fully-sampled reference), under-sampling can be regular
  - Otherwise, typically more “random” or incoherent sampling, similar to compressed sensing
- Parameter Selection
  - Can be estimated from training data or other reference data, if available
  - Otherwise often empirically chosen, although model-order penalties can also be used

Haldar & Liang, “Spatiotemporal imaging with partially separable functions: A matrix recovery approach” ISBI 2010

Structured Low-Rank Methods

Structured Low Rank Methods

- Similar to general low-rank methods, but now it is not the data matrix *itself* that is low-rank, but rather a transformation of the input data

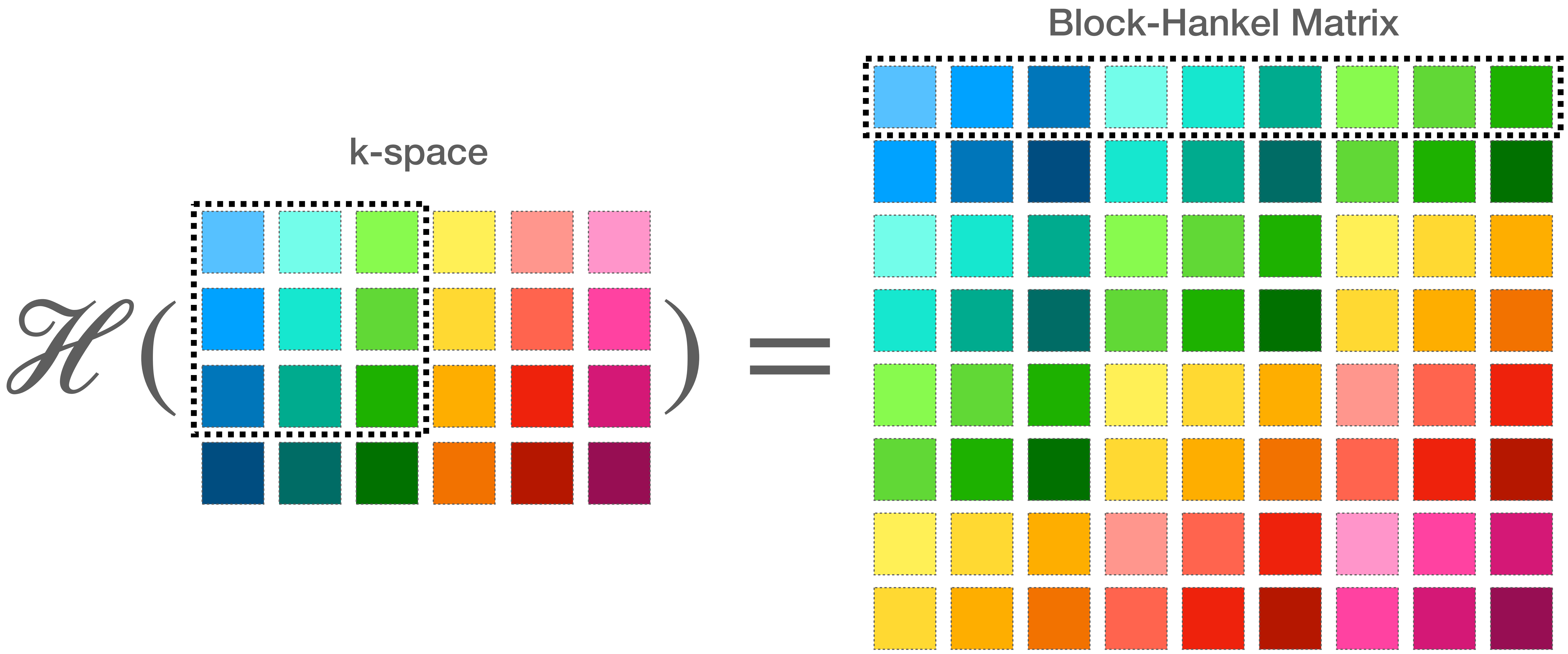
- Don't require multi-dimensional data, you can generate a structured low-rank matrix from a single image

- Typically involves collecting points of k-space using a kernel and concatenating them to form a Hankel structured matrix

Structured Low Rank Methods

The Hankel Matrix

- Take small windows or kernels and unravel them into vectors



Structured Low Rank Methods

The Hankel Matrix

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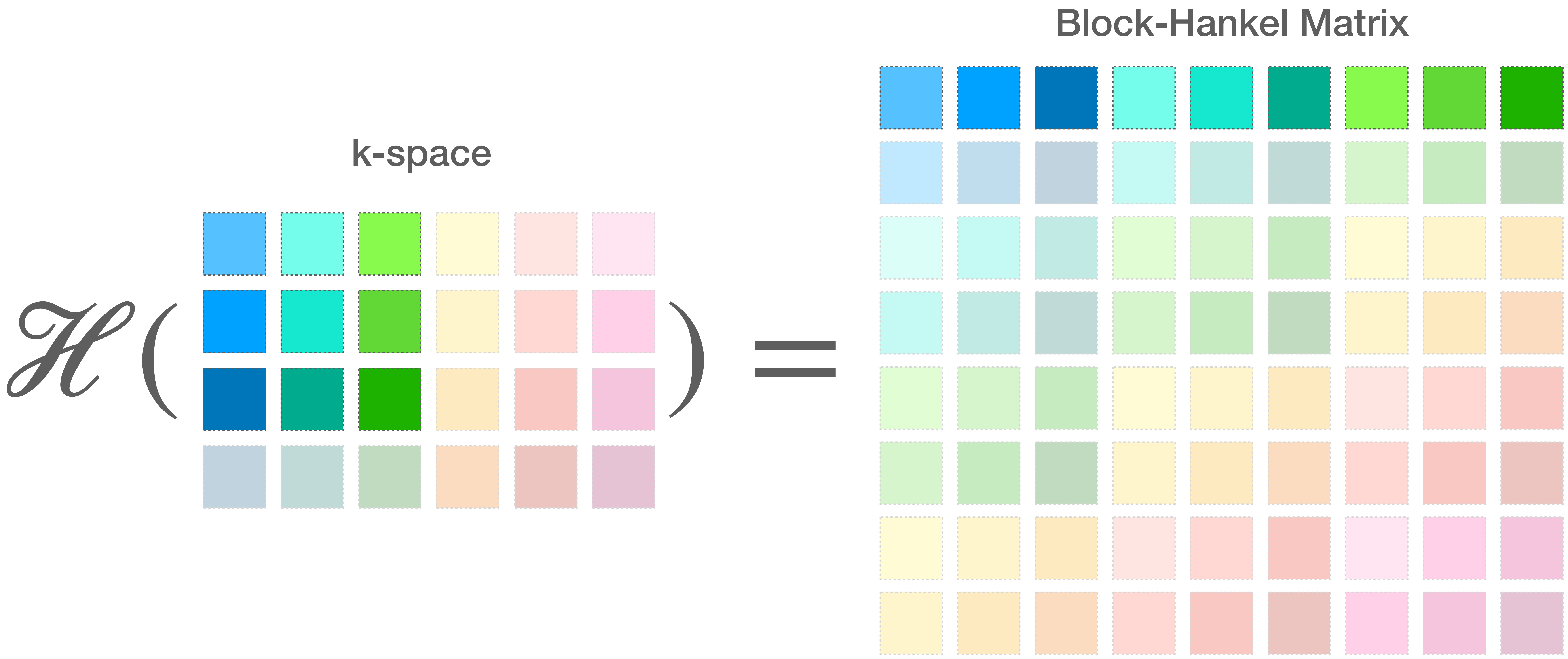
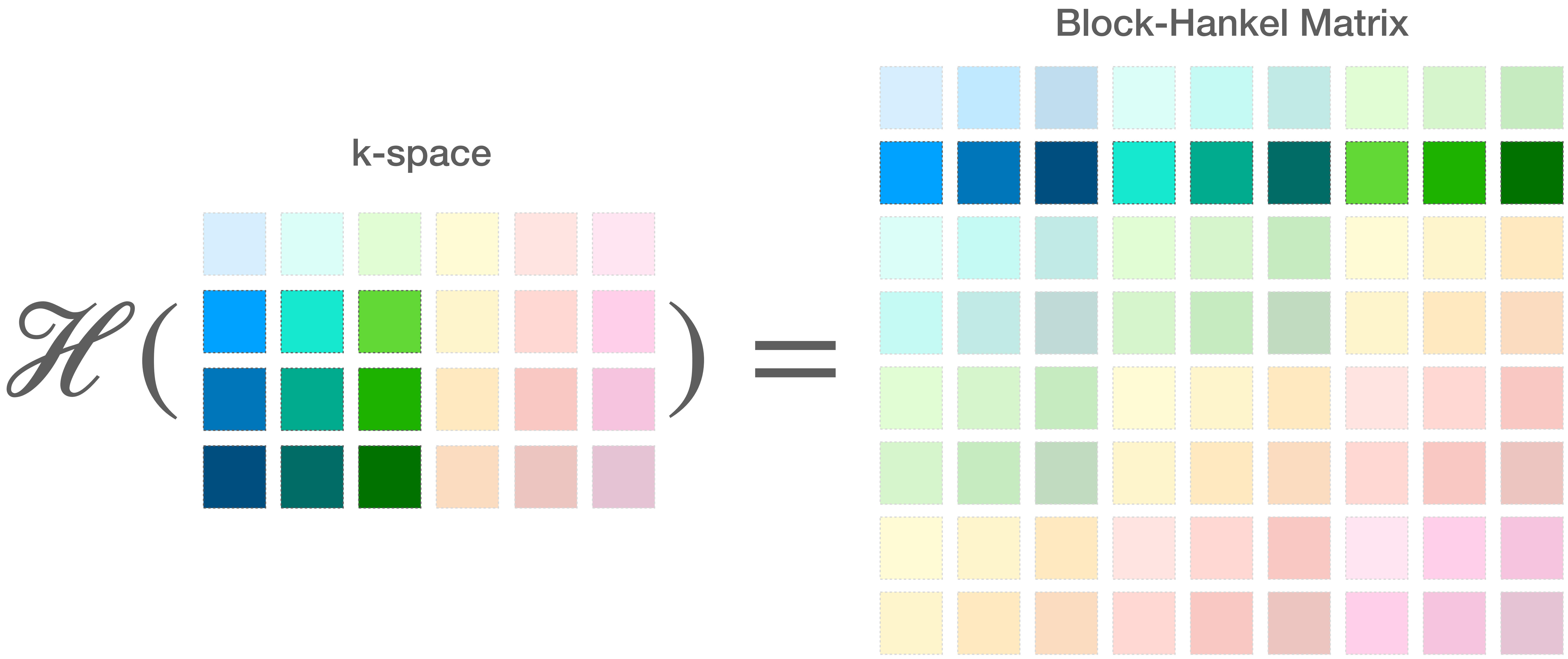


Structured Low Rank Methods
The Hankel Matrix

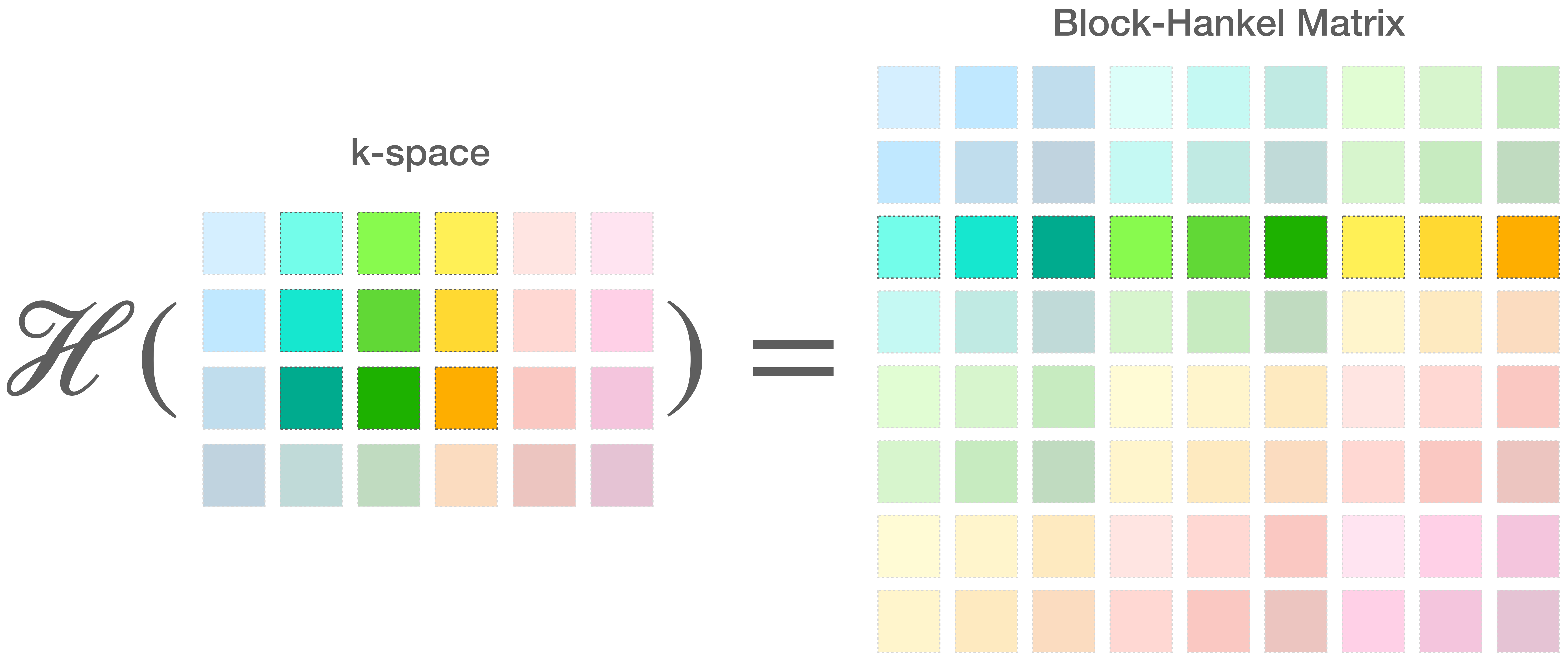
- Stack, or concatenate the vectorized kernels



Structured Low Rank Methods

The Hankel Matrix

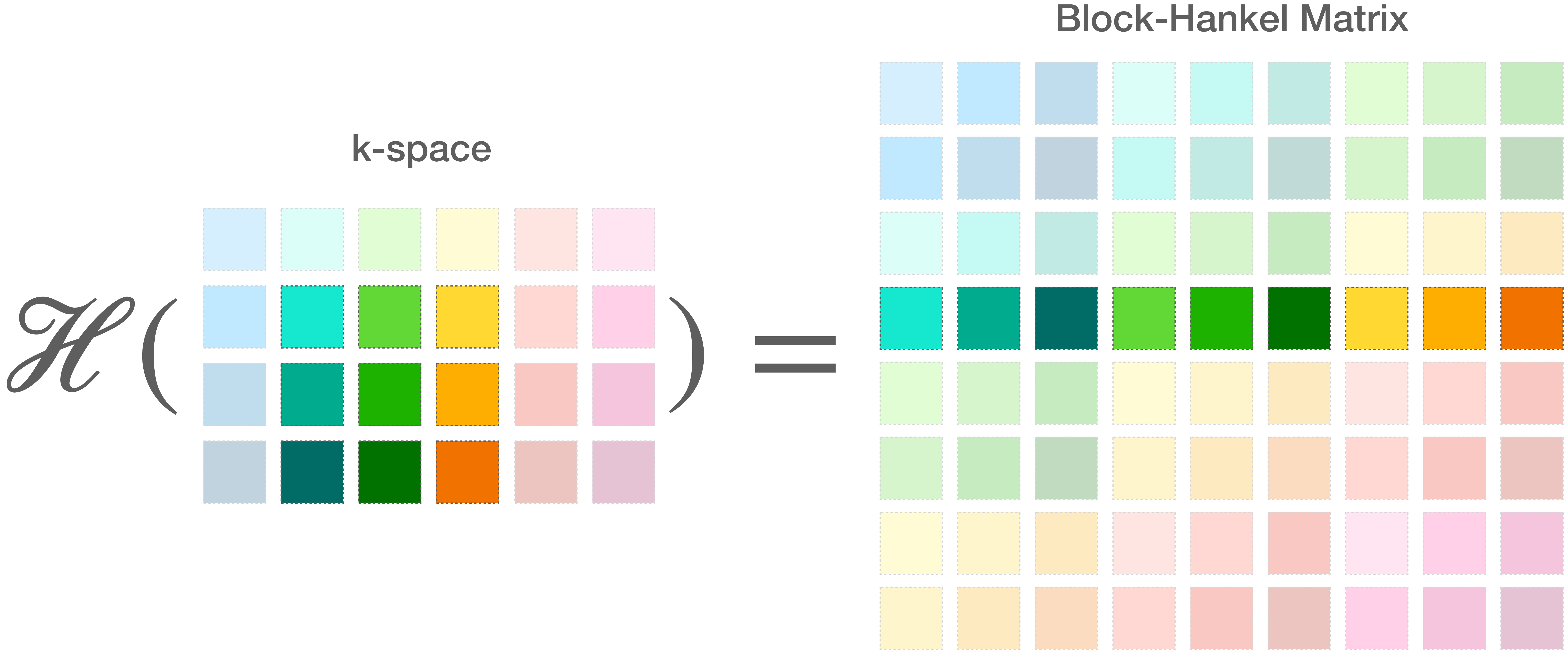
- Stack, or concatenate the vectorized kernels



Structured Low Rank Methods

The Hankel Matrix

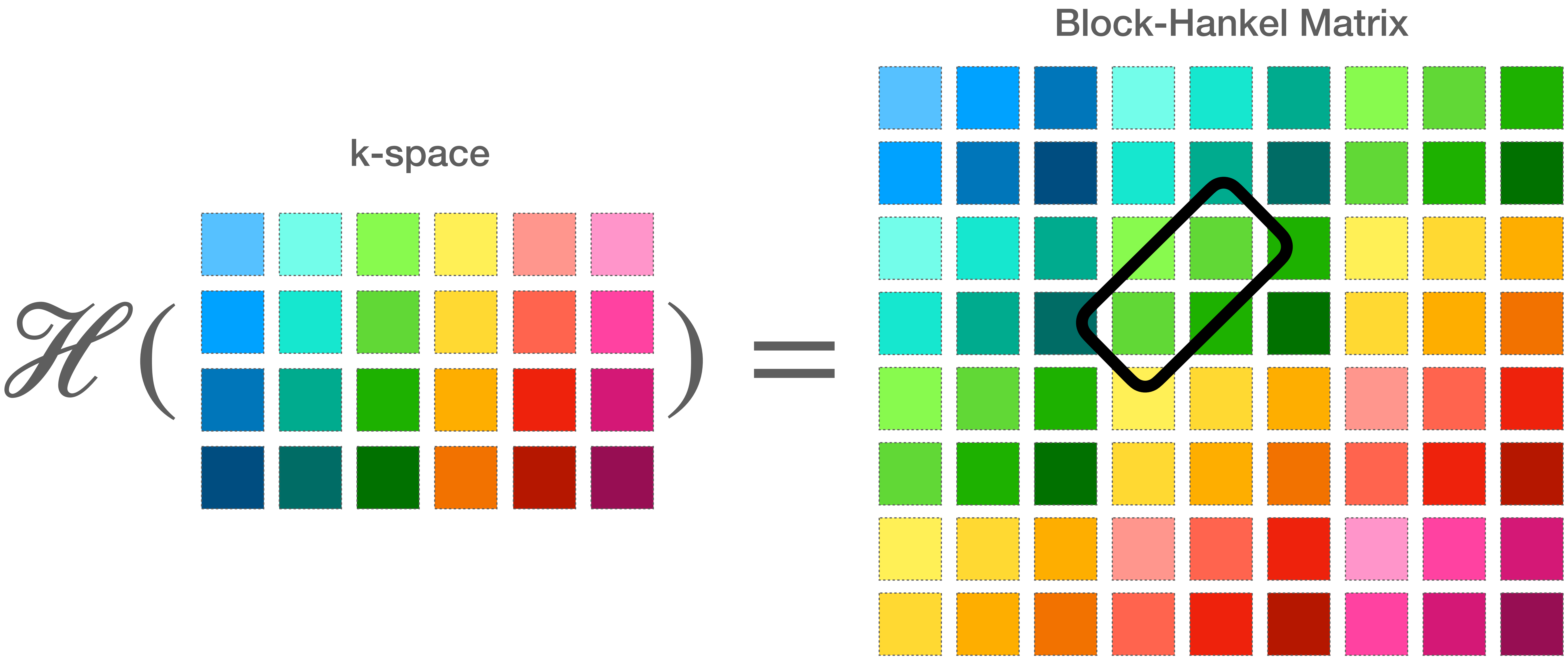
- Stack, or concatenate the vectorized kernels



Structured Low Rank Methods

The Hankel Matrix

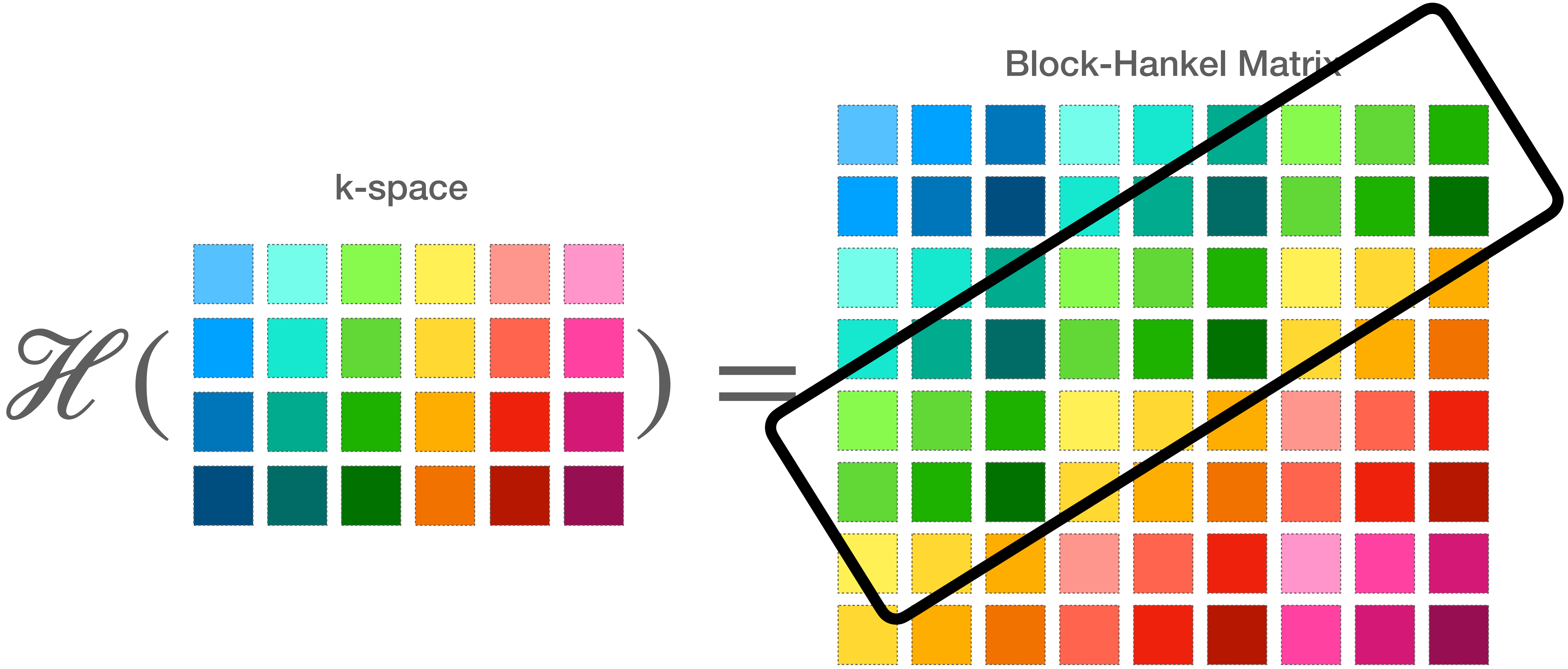
- This procedure generates a block-Hankel structured matrix



Structured Low Rank Methods

The Hankel Matrix

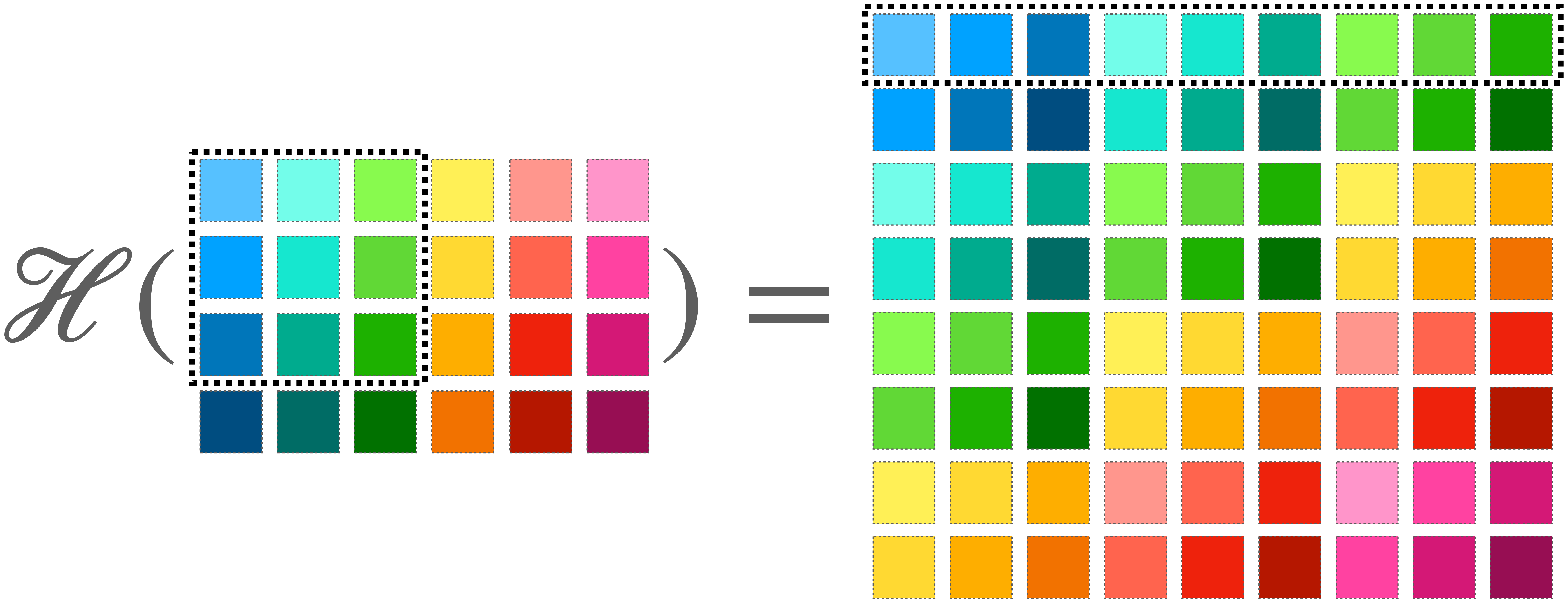
- This procedure generates a block-Hankel structured matrix



Structured Low Rank Methods

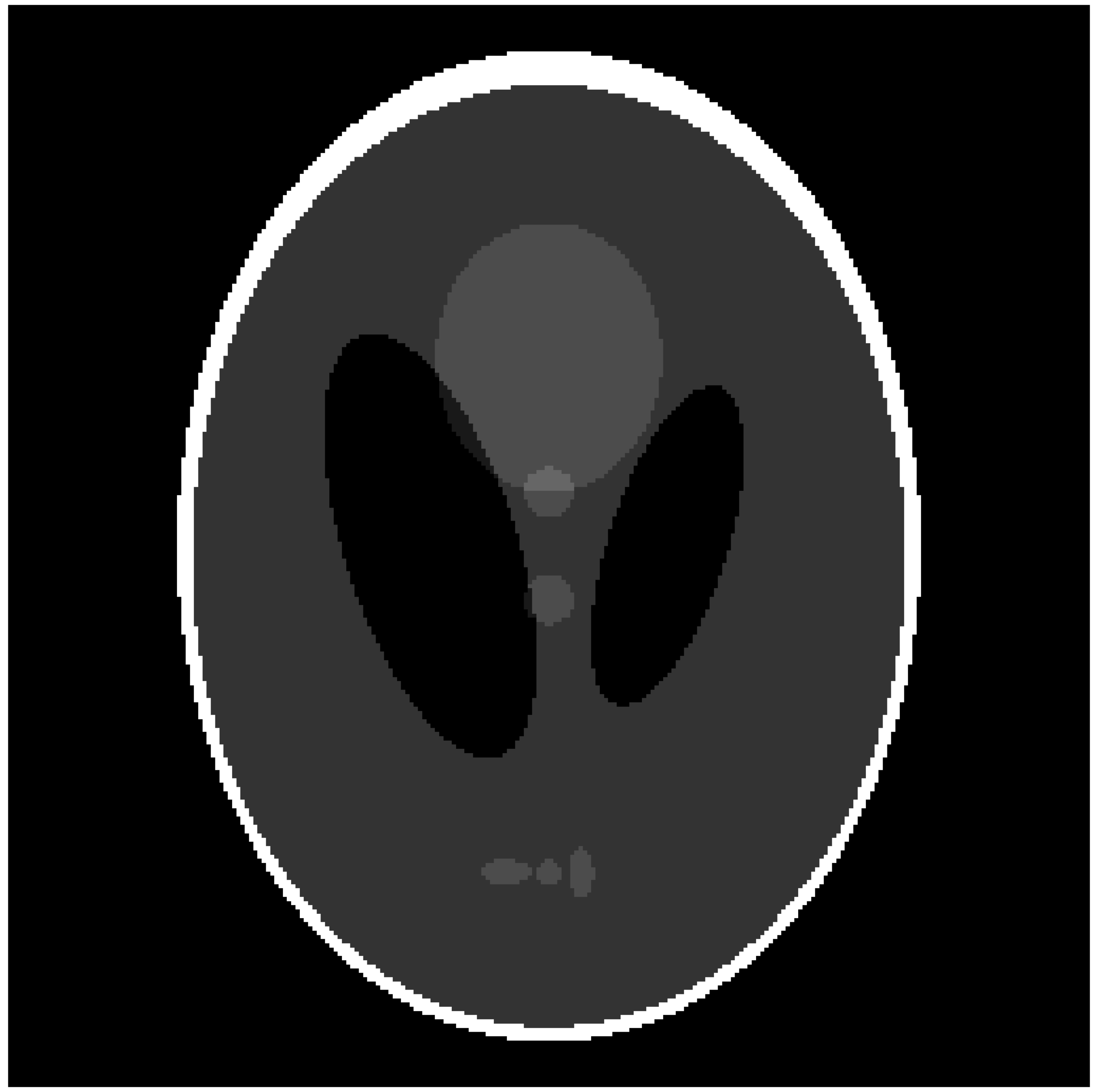
Convolution Perspective

$$x \circledast y = \mathscr{H}(x)\, y$$



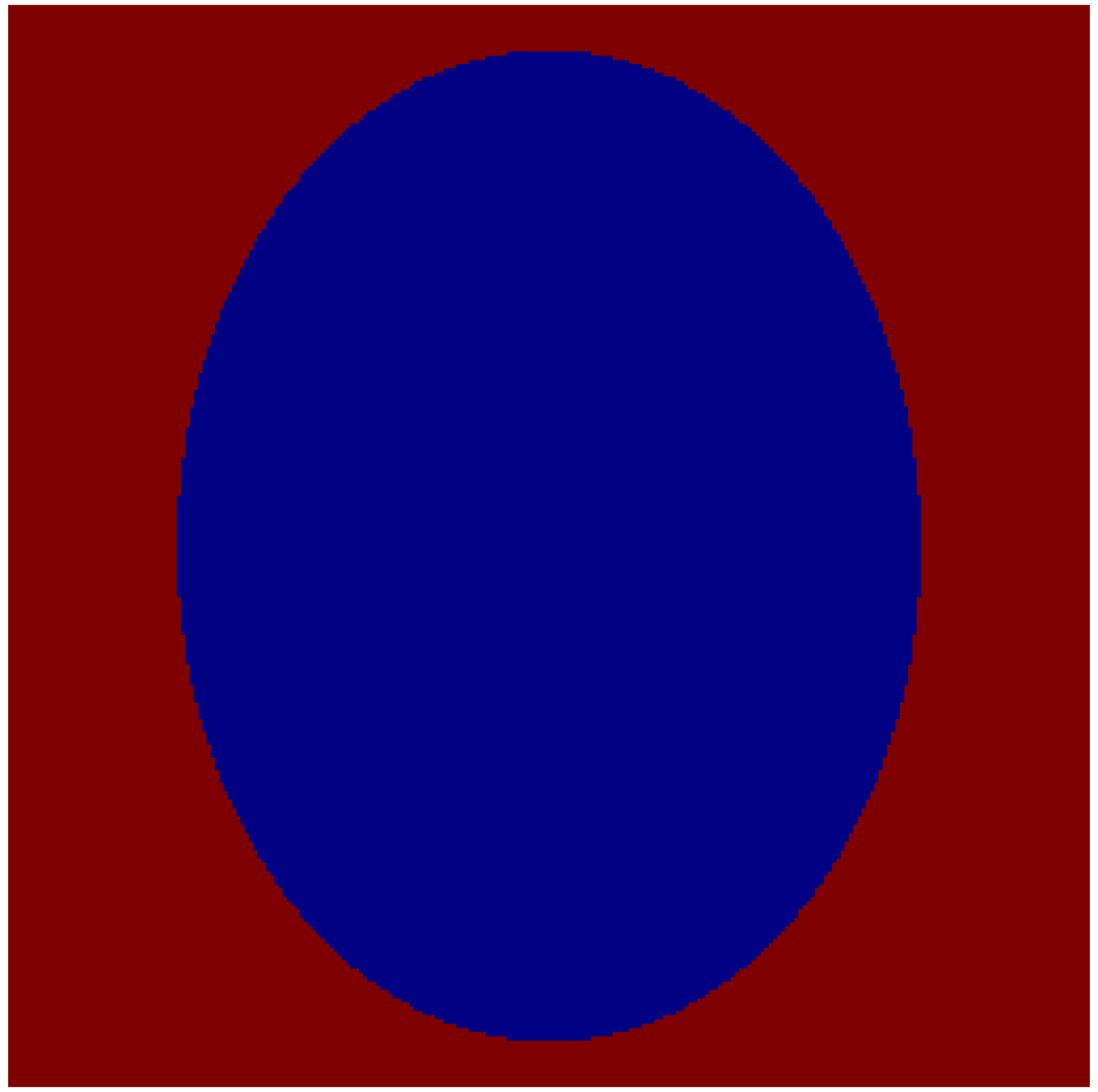
Structured Low Rank Methods

Limited Support Intuition



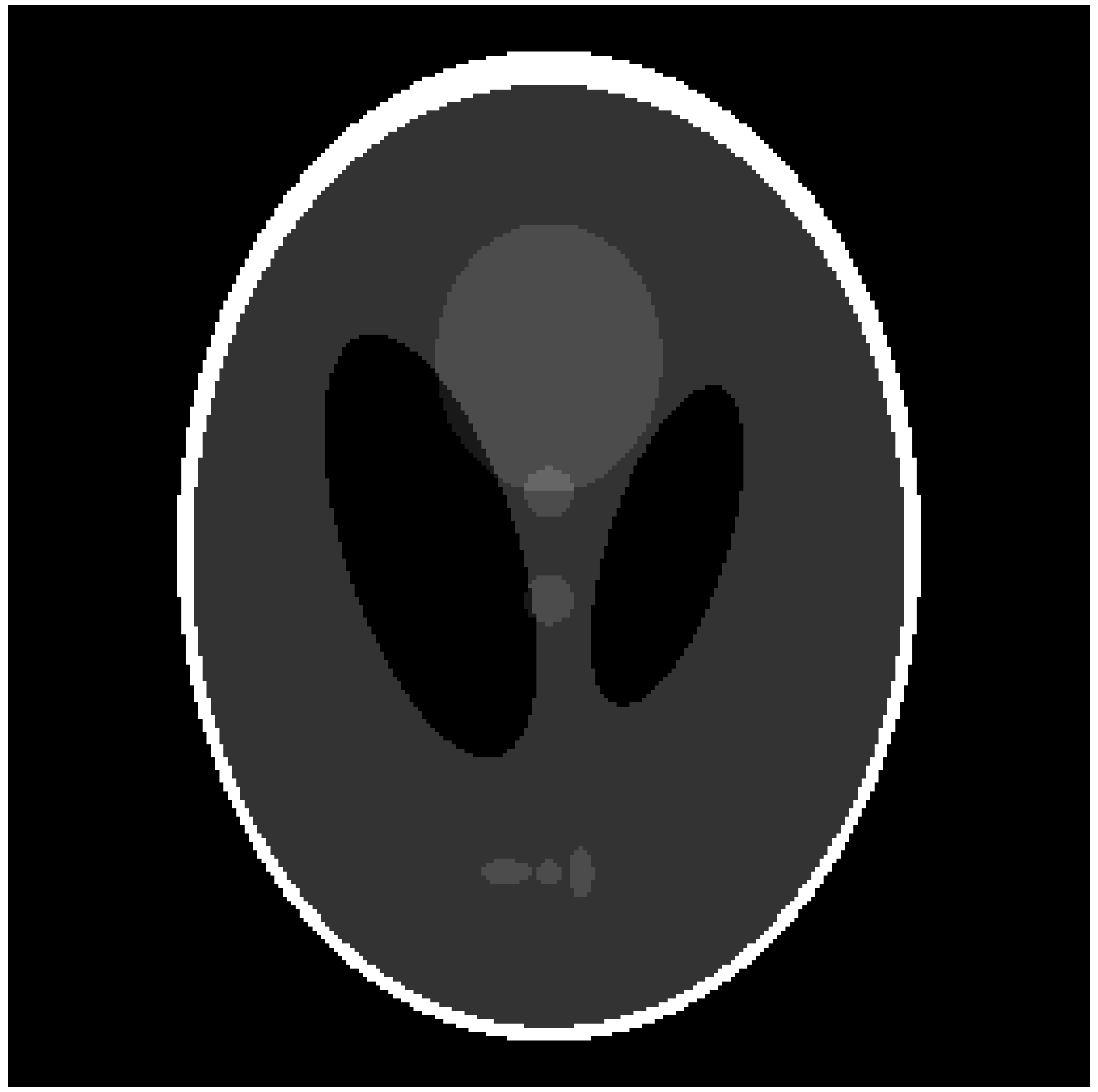
A

$\times$



w

$=$



A

Haldar, "Low-Rank Modeling of Local k-Space Neighborhoods (LORAKS) for Constrained MRI" IEEE-TMI 2014

Structured Low Rank Methods

Limited Support Intuition

$$A \times w - A = 0$$

$$A \times (w - 1) = 0$$

$$A \times \tilde{w} = 0$$

$$\mathscr{F}[A] \circledast \mathscr{F}[\tilde{w}] = 0$$

$$\boxed{\mathscr{H}(\mathscr{F}[A]) \cdot \mathrm{vec}(\mathscr{F}[\tilde{w}]) = 0}$$

Implies the Hankel-structured matrix $\mathscr{H}(\mathscr{F}[A])$ has a non-trivial null-space

Which in turn means that $\mathscr{H}(\mathscr{F}[A])$ has low-rank

Structured Low-Rank Methods

Other Constraints

- Phase Constrained
    - Haldar, “Low-Rank Modeling of Local k-Space Neighborhoods (LORAKS) for Constrained MRI” IEEE-TMI 2014
    - Lee et al., “Reference-Free Single-Pass EPI Nyquist Ghost Correction Using Annihilating Filter-Based Low Rank Hankel Matrix (ALOHA)” MRM 2016
    - Mani et al., “Multi-Shot Sensitivity-Encoded Diffusion Data Recovery Using Structured Low-Rank Matrix Completion (MUSSELS)” MRM 2017
- Coil Sensitivity Constrained
    - Zhang et al., “Parallel Reconstruction Using Null Operations” MRM 2011
    - Uecker et al., “ESPIRiT - An Eigenvalue Approach to Autocalibrating Parallel MRI: Where SENSE Meets GRAPPA” MRM 2014
    - Shin et al., “Calibrationless Parallel Imaging Reconstruction Based on Structured Low-Rank Matrix Completion” MRM 2014
    - Haldar & Zhuo, “P-LORAKS: Low-Rank Modeling of Local k-Space Neighborhoods with Parallel Imaging Data” MRM 2016
- Transform Sparsity Constrained
    - Ongie & Jacob, “Off-the-Grid Recovery of Piecewise Constant Images from Few Fourier Samples” SIAM J Im Sci, 2016
    - Jin et al., “A General Framework for Compressed Sensing and Parallel MRI Using Annihilating Filter Based Low-Rank Hankel Matrix’. IEEE Trans Comp Imag 2016

Structured Low Rank Methods

Reconstruction

$$\min_{x} \|Ex - d\| + \lambda \|x\|_*$$

$$\min_{rank(x) \leq r} \|Ex - d\|$$

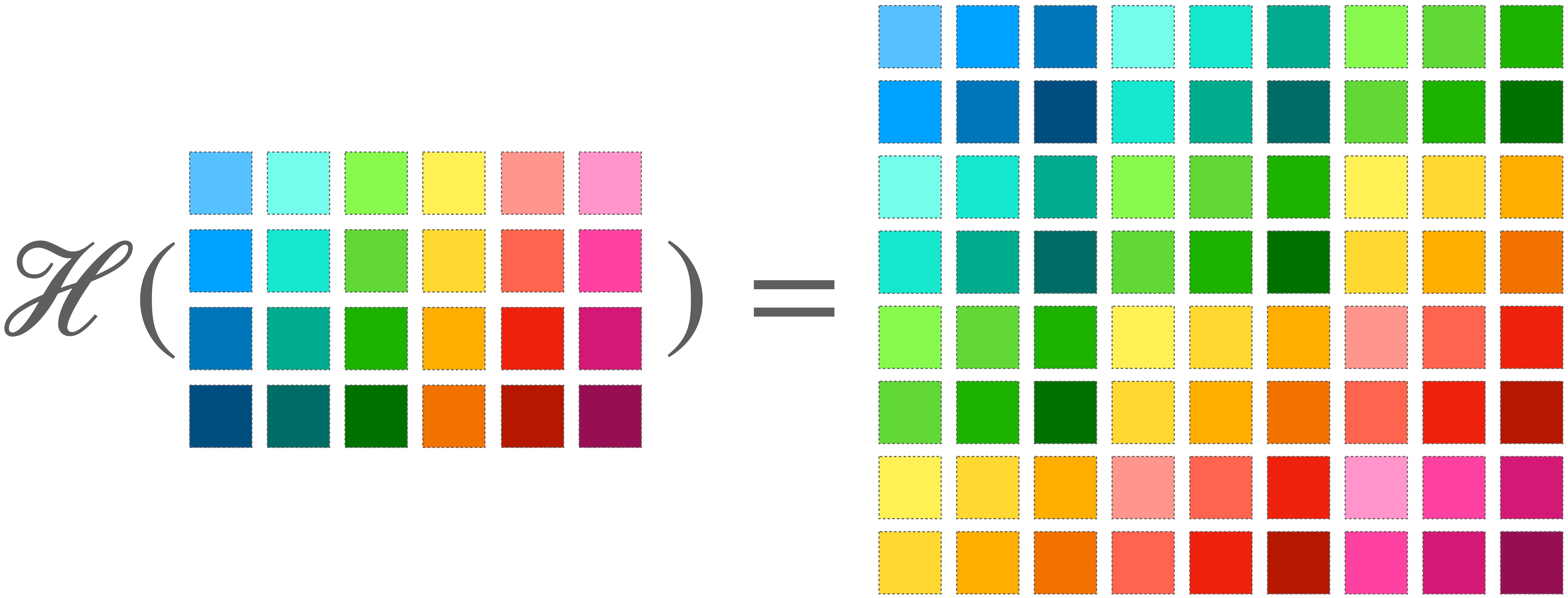
$$\min_{x} \|Ex - d\| + \lambda \|\mathscr{H}(\mathscr{F}[x])\|_*$$

$$\min_{rank(\mathscr{H}(\mathscr{F}[x])) \leq r} \|Ex - d\|$$

Ongie & Jacob, "A Fast Algorithm for Convolutional Structured Low-Rank Matrix Recovery" IEEE Trans Comp Imag 2017

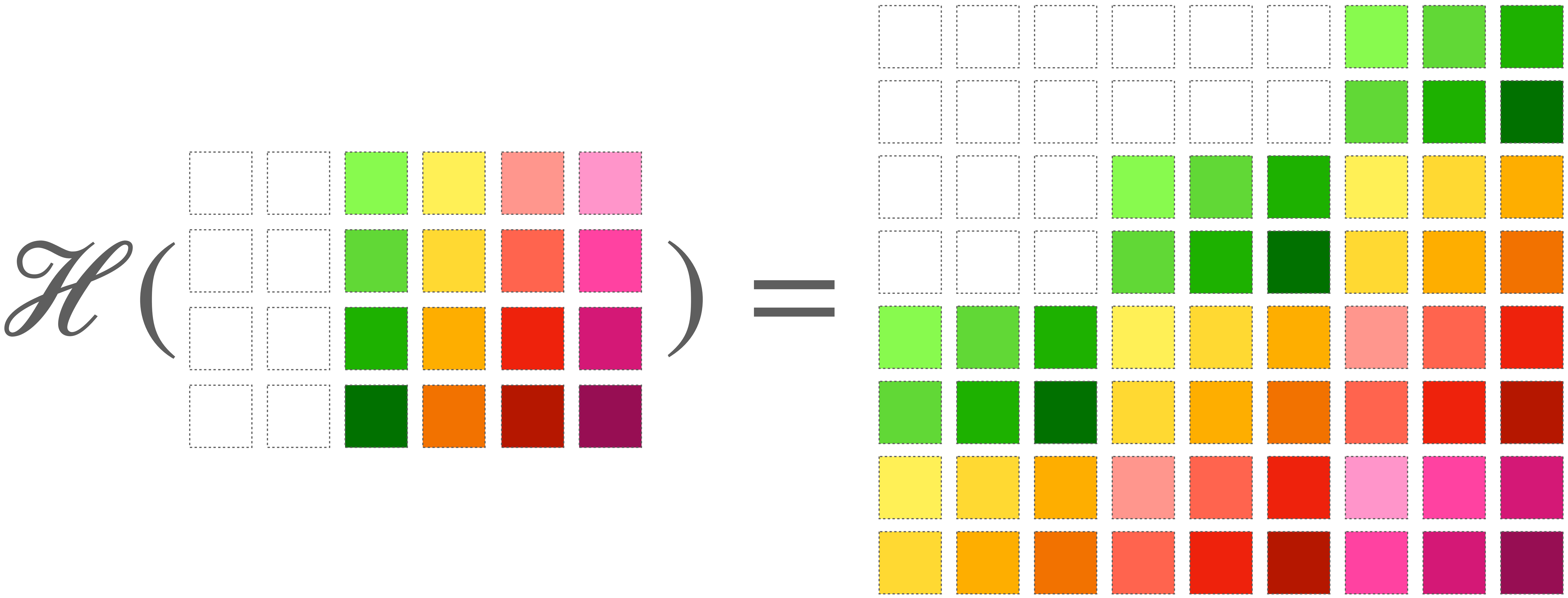
Structured Low-Rank Methods

Sampling Considerations



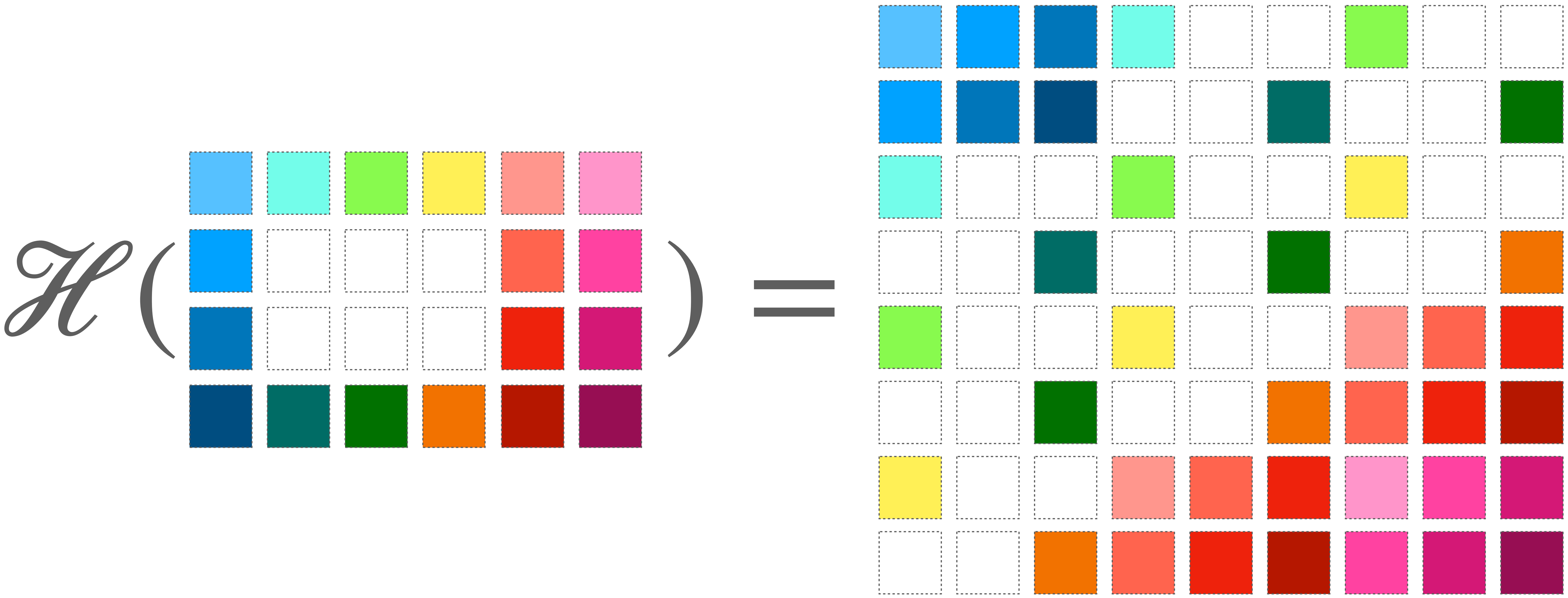
Structured Low-Rank Methods

Partial-Fourier Sampling



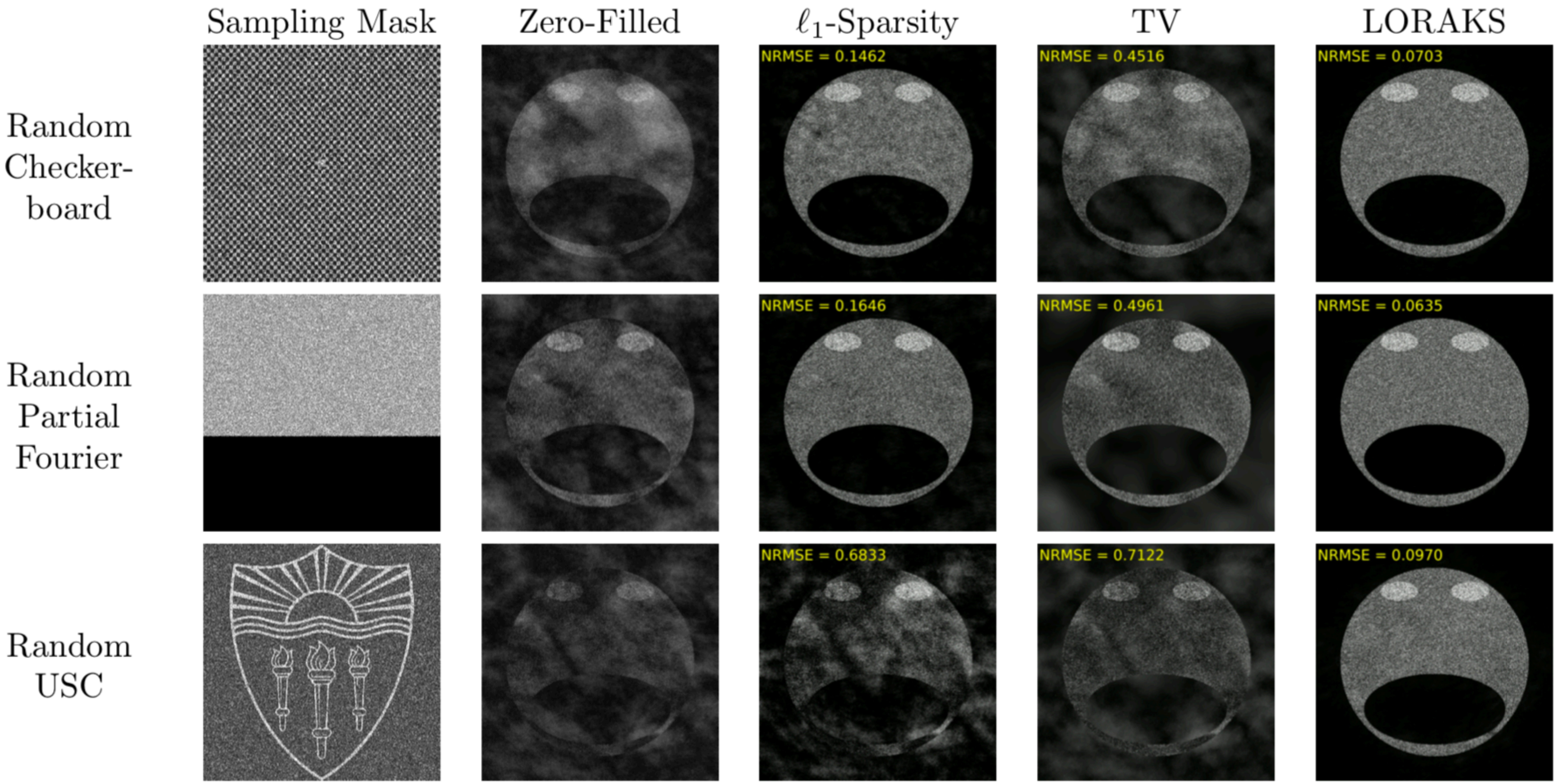
Structured Low-Rank Methods

Random Contiguous Missing Chunk



Structured Low-Rank Methods

Arbitrary Sampling



Haldar, "Low-rank modeling of local k-space neighborhoods: from phase and support constraints to structured sparsity," Proc. SPIE 9597, Wavelets and Sparsity XVI, 2015

Other Resources

- IEEE Signal Processing Magazine, vol 37 no 1, 2020: "Computational MRI"
  - *Christodoulou & Lingala, "Accelerated Dynamic Magnetic Resonance Imaging Using Learned Representations: A New Frontier in Biomedical Imaging"*
  - *Haldar & Setsompop, "Linear Predictability in Magnetic Resonance Imaging Reconstruction: Leveraging Shift-Invariant Fourier Structure for Faster and Better Imaging"*
  - *Jacob, Mani, & Ye, "Structured Low-Rank Algorithms: Theory, Magnetic Resonance Applications, and Links to Machine Learning"*