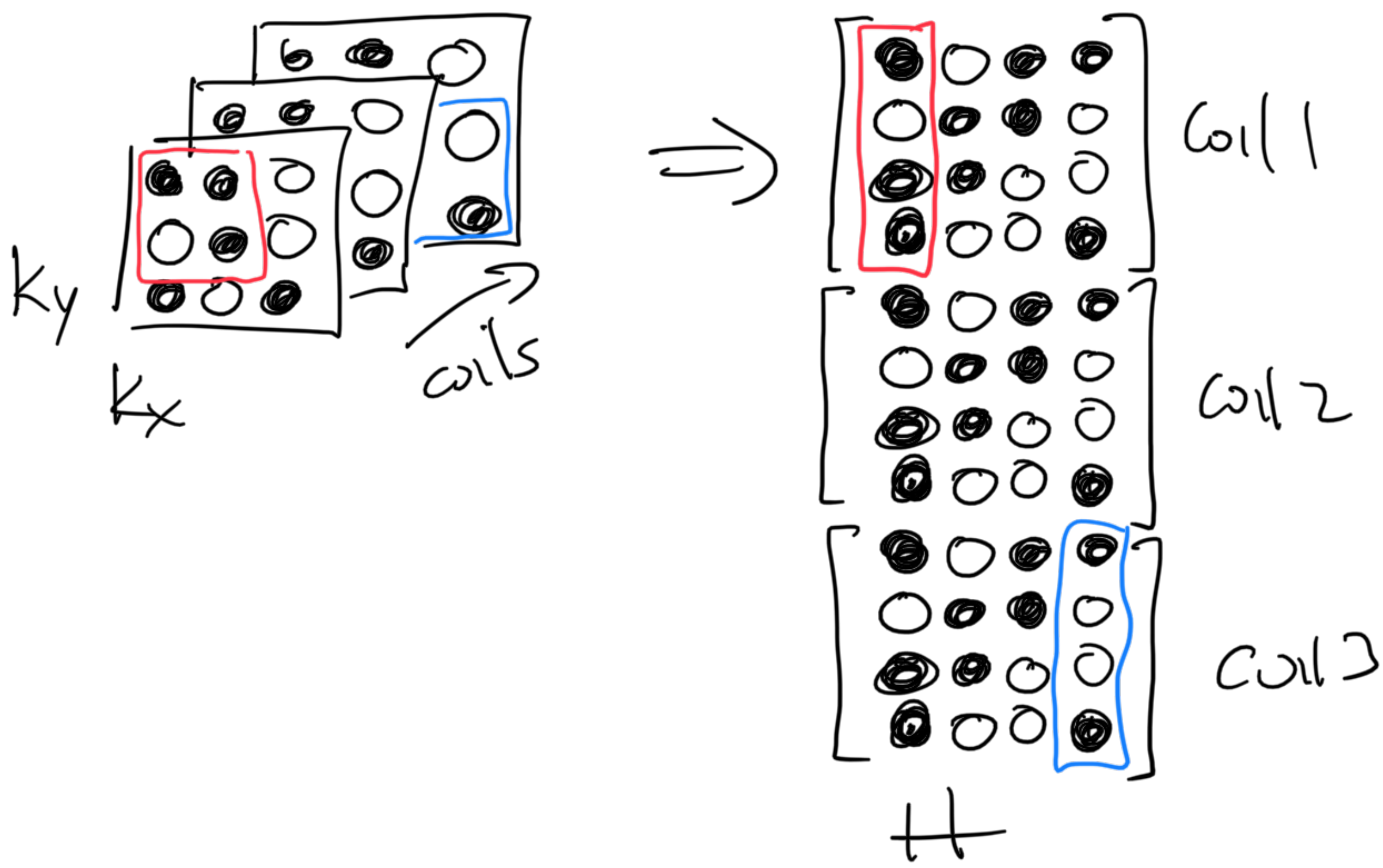
ESPIRiT

Step 1: Form Hankel matrix

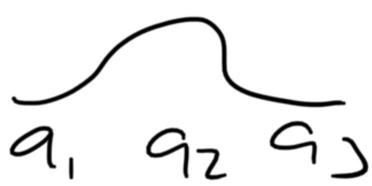


Step 2: Take SVD of H

Question: Why is Hankel matrix low rank?

Because points in k-coil space are correlated / linearly-related

$x_1 \quad x_2 \quad x_3 \quad \circledast$



$$\begin{pmatrix} a_2 & a_3 & a_1 \\ b_2 & b_3 & b_1 \end{pmatrix}$$

$\begin{pmatrix} x_1 \\ \ldots \end{pmatrix}$

These 6 pts in k-coil space are ...

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$ are linear combinations of the underlying magnetization – so they are also linearly dependent on each other

$$\begin{pmatrix} a_3 & a_1 & a_2 \\ b_3 & b_1 & b_2 \end{pmatrix}$$

there are at most 3 lin. indep. points in the k-coil neighborhood

---

Alterate explanation: $\boxed{\text{Physics intuition}}$

if $e^{-i2\pi \Delta k}$ can be synthesized by a linear combination of coil sensitivities that directly implies that k-coil neighborhood points are linearly related

Let $S_1(k) = \int c_1 \cdot \rho \cdot e^{-i2\pi kx} dx$

$S_2(k) = \int c_2 \cdot \rho \cdot e^{-i2\pi kx} dx$

consider $w_1 S_1(k) + w_2 S_2(k)$

$$= w_1 \int c_1 \rho e^{-i2\pi kx} dx + w_2 \int c_2 \rho e^{-i2\pi kx} dx$$

$$= \int (w_1 c_1 + w_2 c_2) \rho e^{-i2\pi kx} dx$$

if $w_1 c_1 + w_2 c_2 = c_1 e^{-i2\pi \Delta k x}$

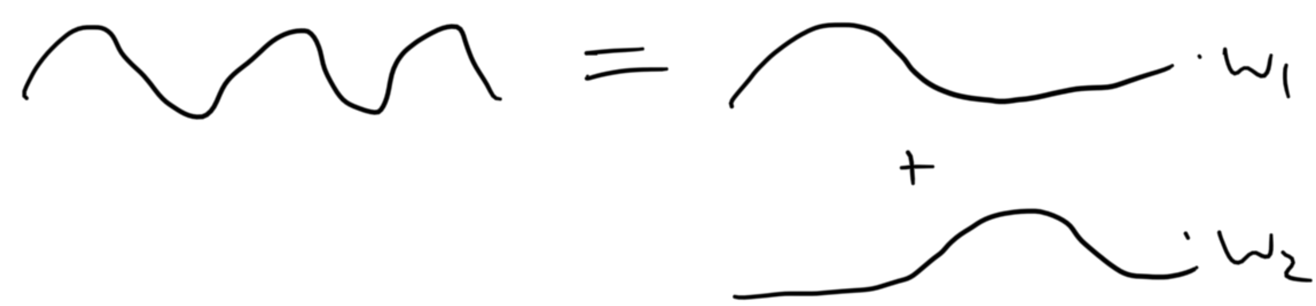
$\Rightarrow$

$$= \int c_1 e^{-i2\pi \Delta k x} \rho e^{-i2\pi kx} dx$$

$$= \int c_1 \rho \, e^{-i2\pi (k+\Delta k)x} \, dx$$

$$= S_1(k+\Delta k) \quad \blacksquare$$

ie



Question: Why is U (from $H = U\Sigma V'$) contrast and shift invariant?

Look at the construction of H
contrast information is position dependent
so the only way a basis can be formed
in U based on shifted local k-patches
is if it encodes contrast-independent
information, ie. sensitivity information

More concretely, look at construction of H:

$$H = \begin{bmatrix} a_1 a_2 a_3 & a_2 a_3 a_1 & a_3 a_1 a_2 \\ b_1 b_2 b_3 & b_2 b_3 b_1 & b_3 b_1 b_2 \\ a_2 a_3 a_1 & a_3 a_1 a_2 & a_1 a_2 a_3 \\ b_2 b_3 b_1 & b_3 b_1 b_2 & b_1 b_2 b_3 \\ \vdots & & \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 0 \\ \vdots & & \end{bmatrix}$$

$$\begin{bmatrix} a_3 a_1 a_2 & a_1 a_2 a_3 & a_2 a_3 a_1 \\ b_3 b_1 b_2 & b_1 b_2 b_3 & b_2 b_3 b_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ 0 & 0 & x_3 \end{bmatrix}$$

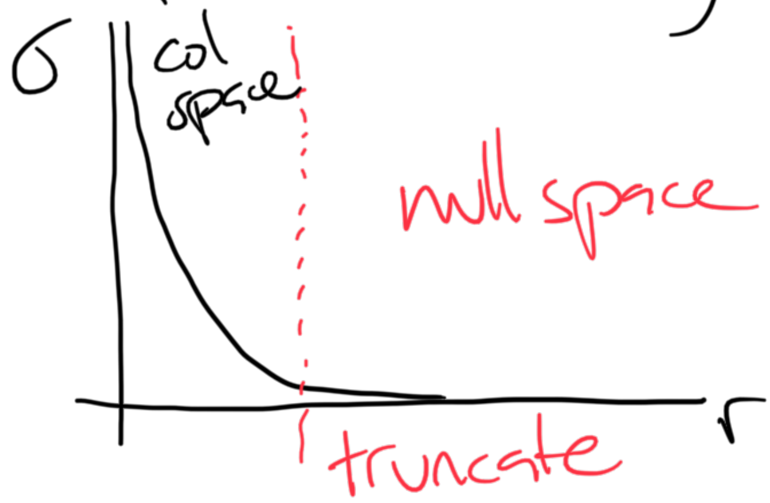
$$= \tilde{U} \cdot \tilde{V}^H$$

So the column space of H : $C(H)$ is related to the column space of $\tilde{U}$: $C(\tilde{U})$

and we can see that $\tilde{U}$ is determined only by coil sensitivity information.

$\therefore$ $U \sim C(H)$ is contrast independent

Step 3: Identify $U \sim C(H)$



$W = UU^H$ is the projection onto U, the subspace of signals that are consistent with having been multiplied by the same sensitivities as the data used to form H (ie the calibration data)

Even for an input signal with a

very different contrast, it is easy to see that the local k-coil patch lies in U:

Recall that $c(H) = c(U)$ is spanned by something like →

$$\begin{bmatrix} a_1 a_2 a_3 & a_2 a_3 a_1 & a_3 a_1 a_2 \\ b_1 b_2 b_3 & b_2 b_3 b_1 & b_3 b_1 b_2 \\ a_2 a_3 a_1 & a_3 a_1 a_2 & a_1 a_2 a_3 \\ b_2 b_3 b_1 & b_3 b_1 b_2 & b_1 b_2 b_3 \\ a_3 a_1 a_2 & a_1 a_2 a_3 & a_2 a_3 a_1 \\ b_3 b_1 b_2 & b_1 b_2 b_3 & b_2 b_3 b_1 \end{bmatrix}$$

Now consider a new k-space formed by a different, arbitrary input contrast:

$$\begin{matrix} a_1 z_1 + a_2 z_2 + a_3 z_3 \\ b_1 z_1 + b_2 z_2 + b_3 z_3 \\ a_2 z_1 + a_3 z_2 + a_1 z_3 \\ b_2 z_1 + b_3 z_2 + b_1 z_3 \\ a_3 z_1 + a_1 z_2 + a_2 z_3 \\ b_3 z_1 + b_1 z_2 + b_2 z_3 \end{matrix} = \begin{bmatrix} a_1 a_2 a_3 \\ b_1 b_2 b_3 \\ a_2 a_3 a_1 \\ b_2 b_3 b_1 \\ a_3 a_1 a_2 \\ b_3 b_1 b_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

So we can see that no matter what the input contrast is, the k-coil patch vector will be in $c(H)$

Step 4: Zero pad and $\mathcal{F}^{-1}(U)$

IF W is the projection operator for a patch, let $\tilde{W}$ be the corresponding convolution operator that convolves W

across k-space.

Then: $\tilde{W} \cdot \mathcal{F} \cdot S \cdot x = \mathcal{F} \cdot S \cdot x$

and $\mathcal{F}^{-1} \tilde{W} \mathcal{F} S x = S x$

so Sx are eigenvectors of $\mathcal{F}^{-1} \tilde{W} \mathcal{F}$ with eigenvalue $= 1$

and where x is non-zero, this means $\mathcal{F}^{-1} \tilde{W} \mathcal{F} S = S$

So S are also eigenvectors of $\mathcal{F}^{-1} \tilde{W} \mathcal{F}$ with $\lambda = 1$.

so, if $\tilde{W} \sim UU^H$, then $\mathcal{F}^{-1} U U^H \mathcal{F} = G G^H$, with $G = \mathcal{F}^{-1} U$

Step 5: Find eigenvectors of GG^H

Question: why can this be done voxel-wise independently?

Because $GG^H = \mathcal{F}^{-1} \tilde{W} \mathcal{F}$, and $\tilde{W}$ corresponds to a convolution in K-space, then GG^H must be a point-wise multiplication in image space.

Then for each voxel in (N_x, N_y)

we have $G_{x,y} \sim$ (coils, components)
and therefore $GG^H_{x,y} \sim$ (coils, coils)

Finding the eigenvectors of $GG^H_{x,y}$
and keep the $\lambda=1$ vectors as the
coil sensitivities (relative).

Notes:

$\boxed{1}$ For the Hankel calibration matrix,
$H = U \Sigma V^H$, we noted that U
is contrast-independent, so V actually
contains all the contrast information.

$\boxed{2}$ There is a little sleight of hand
occuring at step 4, because
the k-space convolution operator
$\tilde{W} \neq UU^H$, because UU^H directly
only performs the projection of a single
patch, but it isn't too hard to
make this more rigorous
: consider that $\tilde{W}$ is a circular
convolution, which is described by a
circulant matrix.

we know circulant matrices are diagonalized by the Fourier Transform:

$$\tilde{W} = \mathcal{F} \cdot X \cdot \mathcal{F}^{-1} \quad \text{where } X \text{ is diagonal}$$

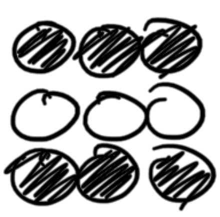
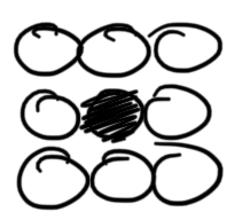
then

$$GG^H = \mathcal{F}^{-1} \tilde{W} \mathcal{F} = \mathcal{F}^{-1} \mathcal{F} X \mathcal{F}^{-1} \mathcal{F}$$
$$= X$$

and we know the diagonal entries of X are determined by a single row or column of $\tilde{W}$.

$\boxed{3}$ Connection to GRAPPA

GRAPPA can be described by the same framework as ESPIRiT, with 1 major difference:
The under-sampling pattern is explicitly encoded into the problem.
Consider the following operators that enforce specific kernel geometries:

R_{source}  and R_{Target} 

Then using the same Hankel calibration matrix H, we get the GRAPPA weight equation as:

$$R_T \cdot H = w \cdot R_S \cdot H$$

and if $H = U \Sigma V^H$ as before,

$$\Rightarrow R_T \cdot U \Sigma V^H = w R_S \cdot U \Sigma V^H$$

so $W = R_T U \Sigma V^H \left[R_S U \Sigma V^H \right]^+$

$$= R_T U \Sigma V^H V \Sigma U^H R_S \left(R_S U \Sigma V^H V \Sigma U^H R_S \right)^{-1}$$

$$= \underbrace{R_T U \Sigma^2 U^H R_S \left(R_S U \Sigma^2 U^H R_S \right)^{-1}}_{\text{expression is independent of contact information (ie, no V dependence).}}$$