Multi-Coil Arrays and Sub-Nyquist Imaging

Linear Systems and Parallel Imaging

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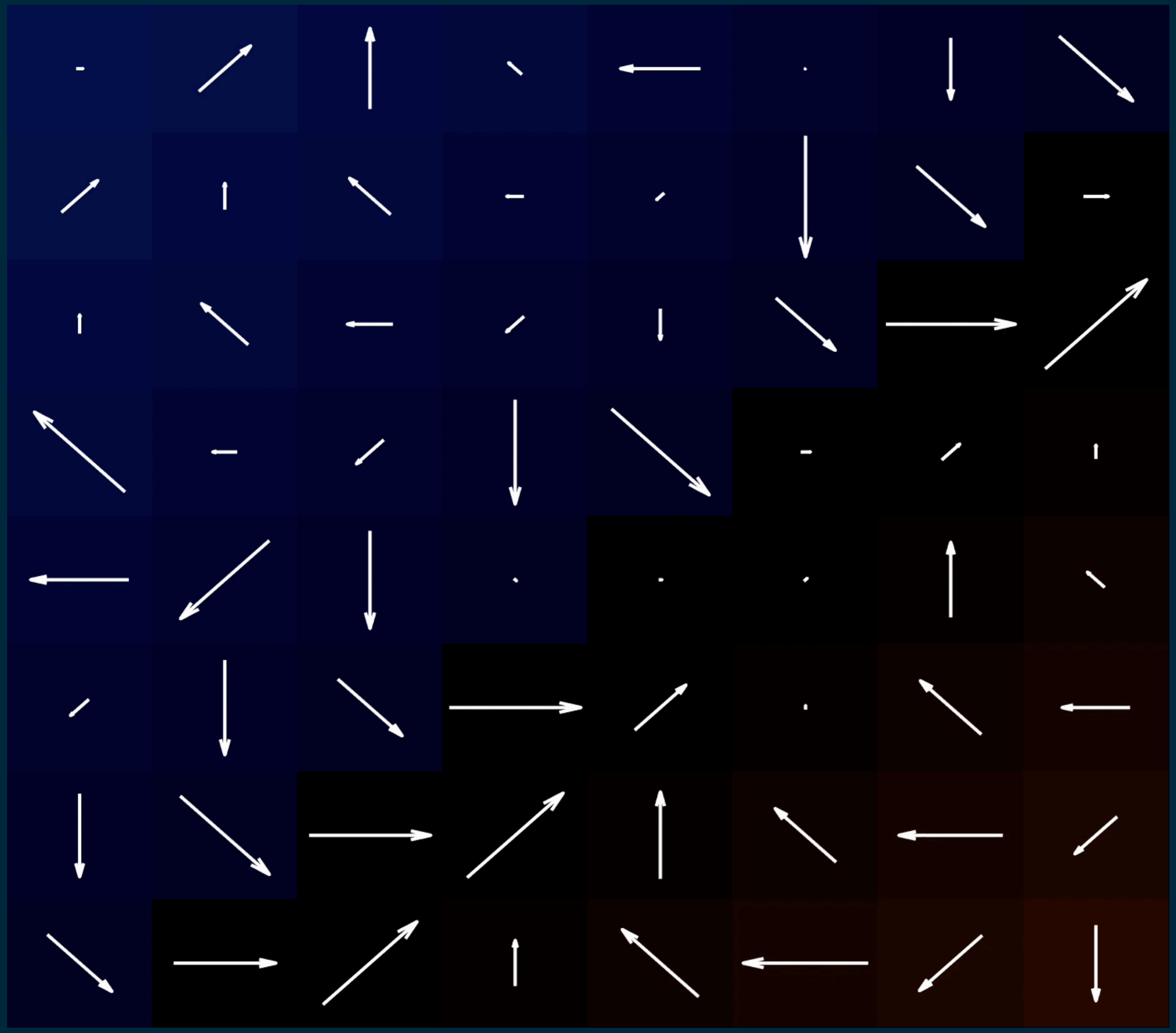
Outline

- Imaging as a Linear System
- Receive Coil Arrays
- Parallel Imaging

Imaging as a Linear System

Magnetisation has magnitude and phase

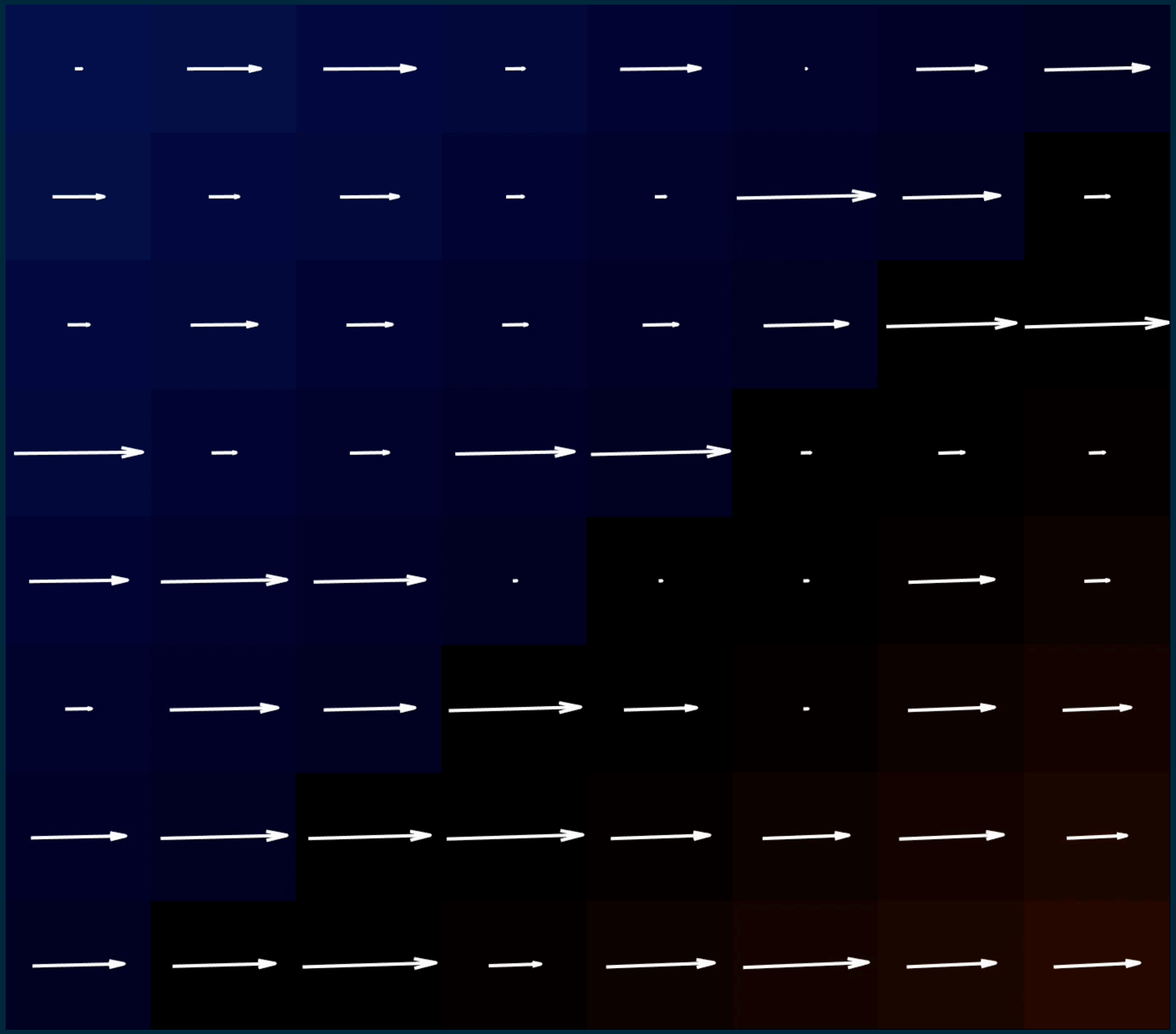
$$M(x,t) = \rho(x)e^{-i\theta(x,t)} \qquad [0]$$



Magnetisation Precesses

Magnetisation has magnitude and phase

$$M(x,t) = \rho(x)e^{-i\omega(x)t} \qquad [1]$$



Magnetisation Precesses

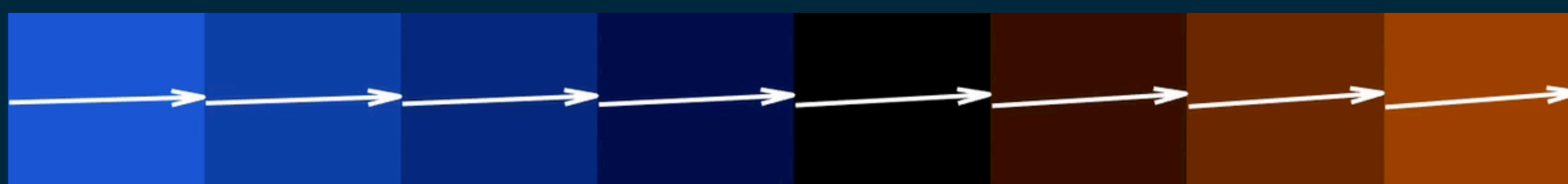
Larmor Equation

$$\omega = \gamma B$$

[2]

Lower Field

Higher Field



Lower Frequency

Higher Frequency

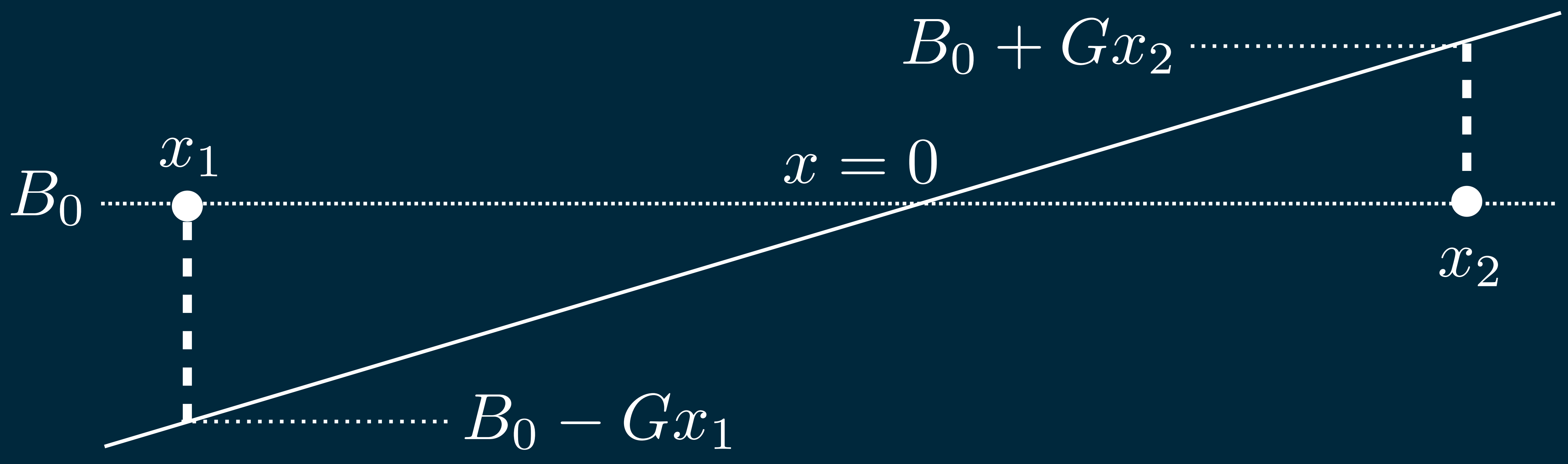
Slower precession/rotation

Faster precession/rotation

Frequency scales with field strength

Linear Magnetic Fields

$$B = B_0 + Gx \qquad [3]$$



"Gradients" vary field strength linearly with distance from isocentre

Magnetisation

$$M(x,t) = \rho(x) e^{-i\omega(x)t} \qquad [1]$$

Larmor Equation

$$\omega = \gamma B \qquad [2]$$

Linear Magnetic Fields

$$B = B_0 + Gx \qquad [3]$$

Magnetisation

$$M(x,t) = \rho(x) e^{-i\omega(x)t} \tag{1}$$

Larmor Equation

$$\omega = \gamma B \tag{2}$$

Linear Magnetic Fields

$$B = B_0 + Gx \tag{3}$$

Magnetisation

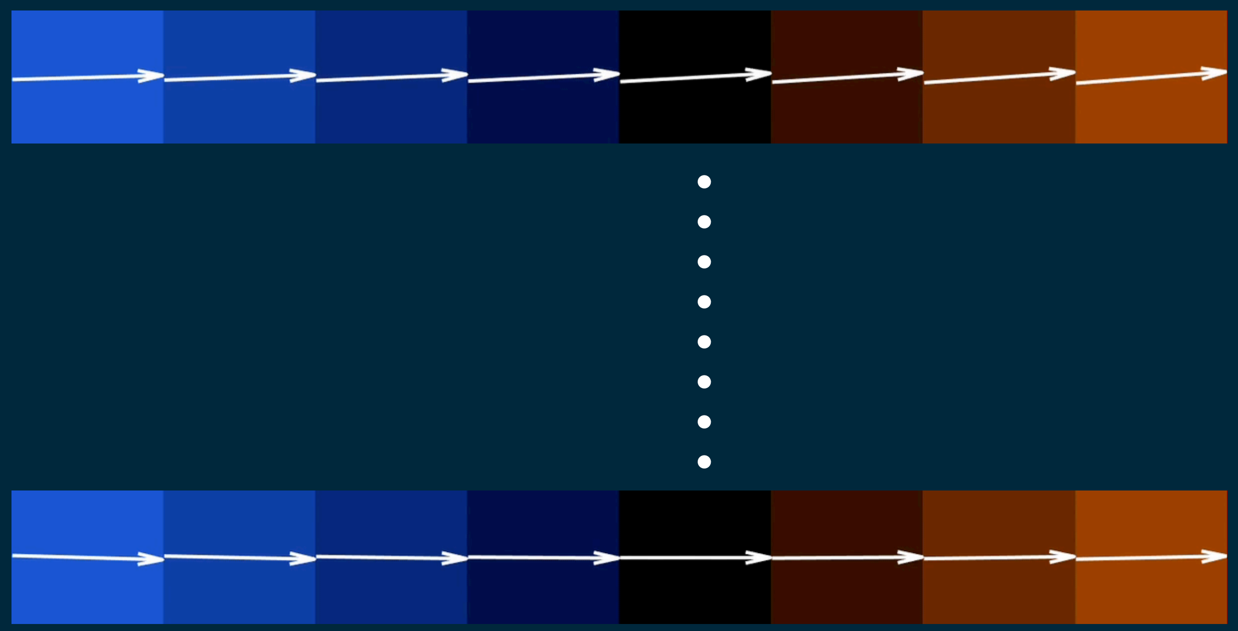
$$M(x,t) = \rho(x)e^{-i\omega(x)t} \qquad [1]$$

Frequency varies linearly with position

$$\omega(x) = \gamma B_0 + \gamma(Gx) \qquad [4]$$

Magnetisation

$$M(x,t) = \rho(x) e^{-i\gamma B_0 t} e^{-i\gamma G x t} \qquad [5]$$



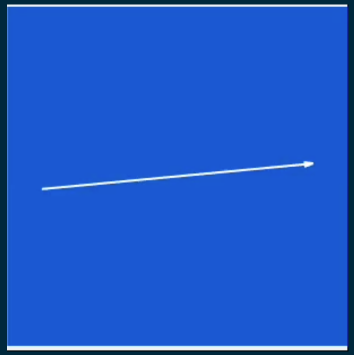
Rotating Frame Magnetisation

$$M(x,t) = M_{rot}(x,t) e^{-i\gamma B_0 t}$$

$$M_{rot}(x,t) = \rho(x) e^{-i\gamma G x t} \qquad [6]$$

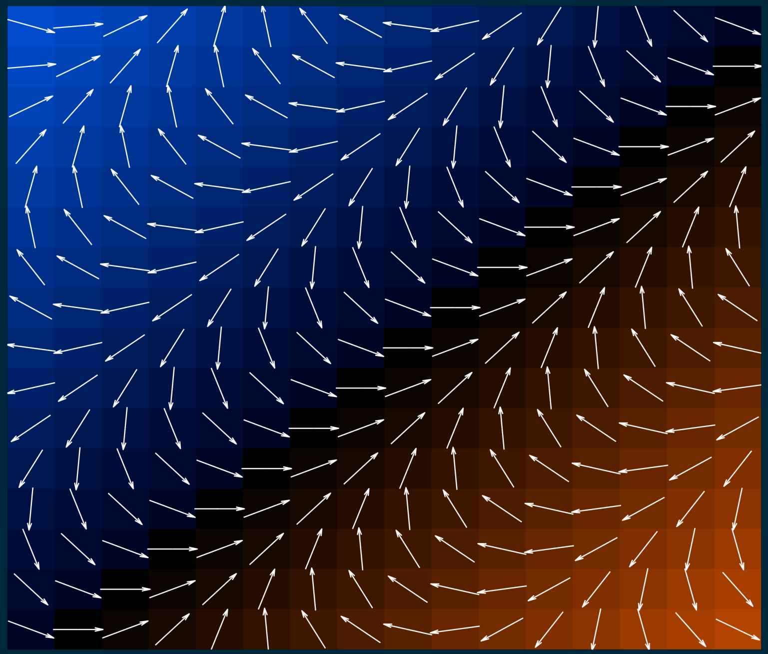
Rotating Frame Magnetisation: **Temporal** Frequency

$$M_{rot}(x_0, t) = \rho(x_0) e^{-i(\gamma G x_0)t} \quad [6a]$$



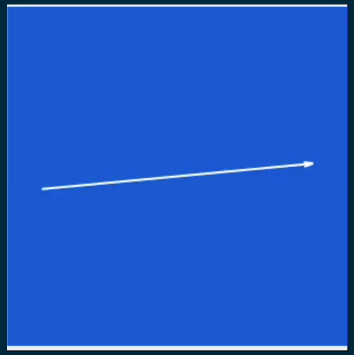
Rotating Frame Magnetisation: **Spatial** Frequency

$$M_{rot}(x, t_0) = \rho(x) e^{-i(\gamma G t_0)x} \quad [6b]$$



Rotating Frame Magnetisation: **Temporal** Frequency

$$M_{rot}(x_0,t) = \rho(x_0)e^{-i\underline{(\gamma G x_0)}t} \qquad [6a]$$

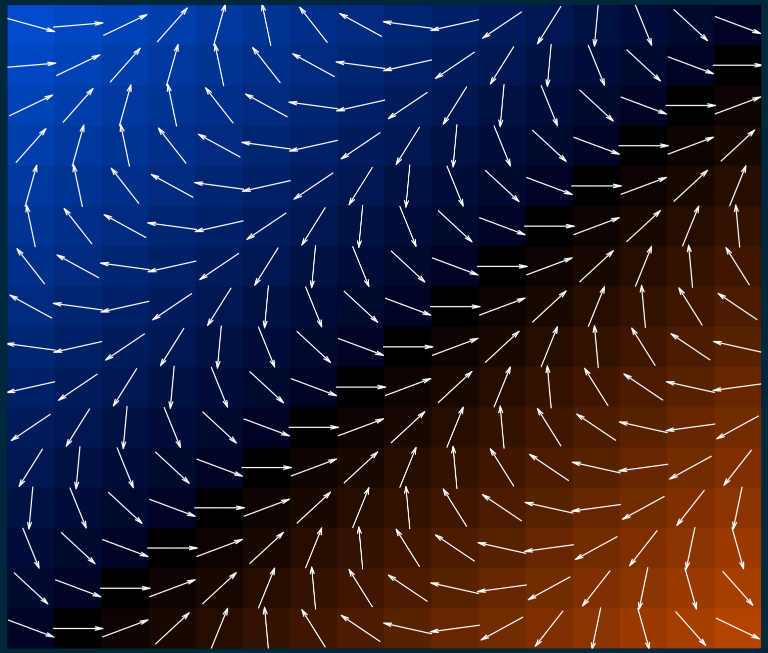


Phase rate of change
w.r.t. time

$$\frac{d\theta}{dt}$$

Rotating Frame Magnetisation: **Spatial** Frequency

$$M_{rot}(x,t_0) = \rho(x)e^{-i\underline{(\gamma G t_0)}x} \qquad [6b]$$

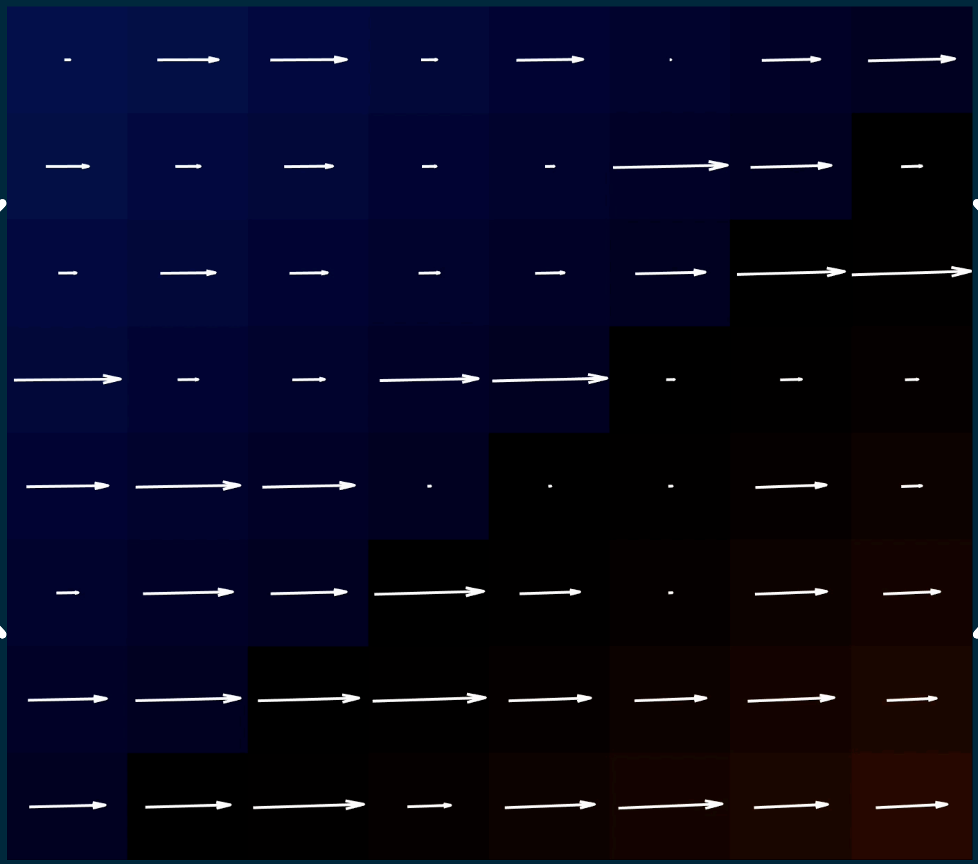


Phase rate of change
w.r.t. space

$$\frac{d\theta}{dx}$$

Receive Coils

$$s(t) = \int M_{rot}(x,t)dx \qquad [7]$$

$$= \int \Big($$  $$\Big)$$

Coils integrate signal across all space

Magnetisation

$$M_{rot}(x,t) = \rho(x)e^{-i(\gamma G t)x} \quad [6]$$

Coil Integration

$$s(t) = \int M_{rot}(x,t)dx \quad [7]$$

Signal Equation

$$s(t) = \int \rho(x)e^{-i(\gamma G t)x}dx \quad [8]$$

Signal Equation

$$s(t) = \int \rho(x) e^{-i(\gamma G t)x} dx \qquad [8]$$

Definition [edit]

https://en.wikipedia.org/wiki/Fourier_transform

The Fourier transform of the function f is traditionally deno

$f: \mathbb{R} \to \mathbb{C}$.[1][2] Here we will use the following definition:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx,$$

Signal Equation

$$s(t) = \int \rho(x) e^{-i(\gamma G t)x} dx \qquad [8]$$

Fourier Transform

Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

Signal Equation

$$s(t) = \int \rho(x) e^{-i(\gamma G t)x} dx \qquad [8]$$

Change of variables

$$k = \frac{\gamma}{2\pi} G t = \frac{\gamma}{2\pi} \int_0^t G(\tau) d\tau \qquad [9]$$

Fourier Transform Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

Physical Perspective

MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

What the scanner "measures"

MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

Magnetisation magnitude at every point in space

MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

Magnetisation phase at every point in space

Dictated by Bloch equation and linear magnetic fields

Unfortunately we can't make these like "delta" functions

MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

Coil integrates vector magnetisation over all space

We can't prevent the coil from seeing "everything"

Unfortunately we can't "pick out" specific x-locations

MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

Fourier Transform Perspective

MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

Our signal in the "standard" pixel basis:

MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

New basis of complex exponentials

MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

Inner product to compute Fourier coefficients

i.e. change of basis from standard to Fourier

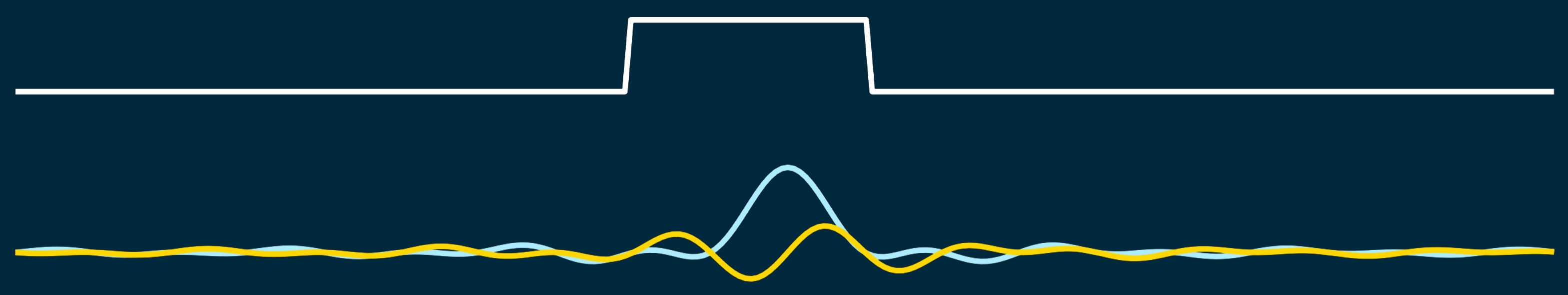
MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$

Same signal in new (Fourier) basis

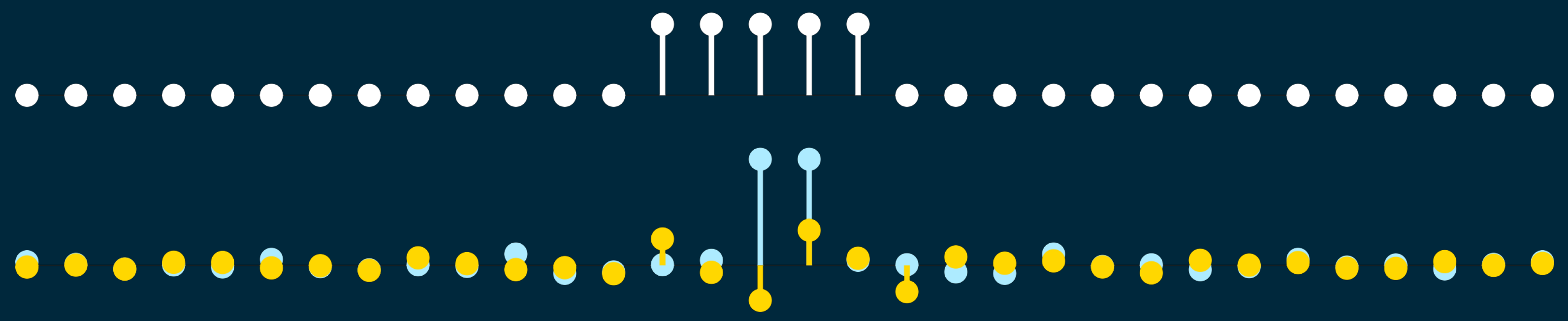
MR Signal Equation

$$\hat{s}(k) = \int \rho(x) e^{-i(2\pi k)x} dx \qquad [10]$$



Discretised Signal Equation

$$\hat{s}(k) = \sum_{n=0}^{N-1} \rho(x_n) e^{-i2\pi kn/N} \qquad [11]$$



Discretised Signal Equation

$$\hat{s}(k) = \sum_{n=0}^{N-1} \rho(x_n) e^{-i2\pi kn/N} \qquad [11]$$

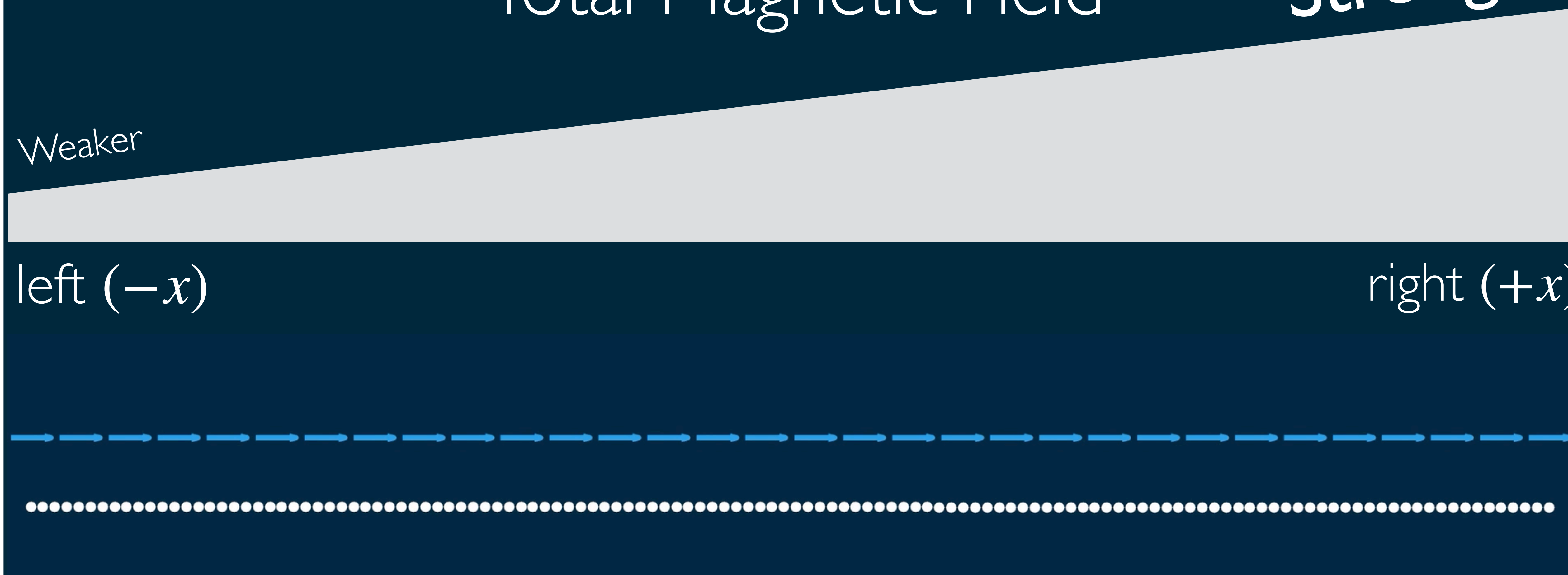
$$\begin{bmatrix} \hat{s}(k_0) \\ \hat{s}(k_1) \\ \vdots \\ \hat{s}(k_{N-1}) \end{bmatrix} = \begin{bmatrix} e^{-i2\pi k_0(0)/N} & e^{-i2\pi k_0(1)/N} & \dots & e^{-i2\pi k_0(N-1)/N} \\ e^{-i2\pi k_1(0)/N} & e^{-i2\pi k_1(1)/N} & \dots & e^{-i2\pi k_1(N-1)/N} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-i2\pi k_{N-1}(0)/N} & e^{-i2\pi k_{N-1}(1)/N} & \dots & e^{-i2\pi k_{N-1}(N-1)/N} \end{bmatrix} \begin{bmatrix} \rho(x_0) \\ \rho(x_1) \\ \vdots \\ \rho(x_{N-1}) \end{bmatrix}$$

Matrix Signal Equation

$$\hat{s} = F\rho \qquad [12]$$

Total Magnetic Field

Stronger



Slower

Faster

Gradients cause precession rate to change across space

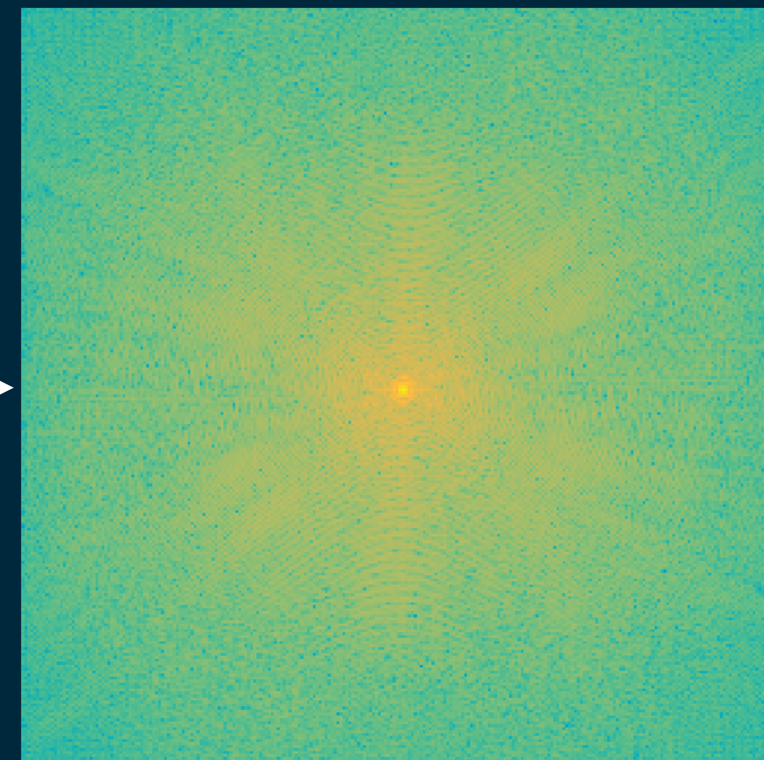
"Wave-like" spatial patterns of phase emerge at any given time

Youtube Video

https://www.youtube.com/watch?v=yVkdfJ9PkRQ&t=10

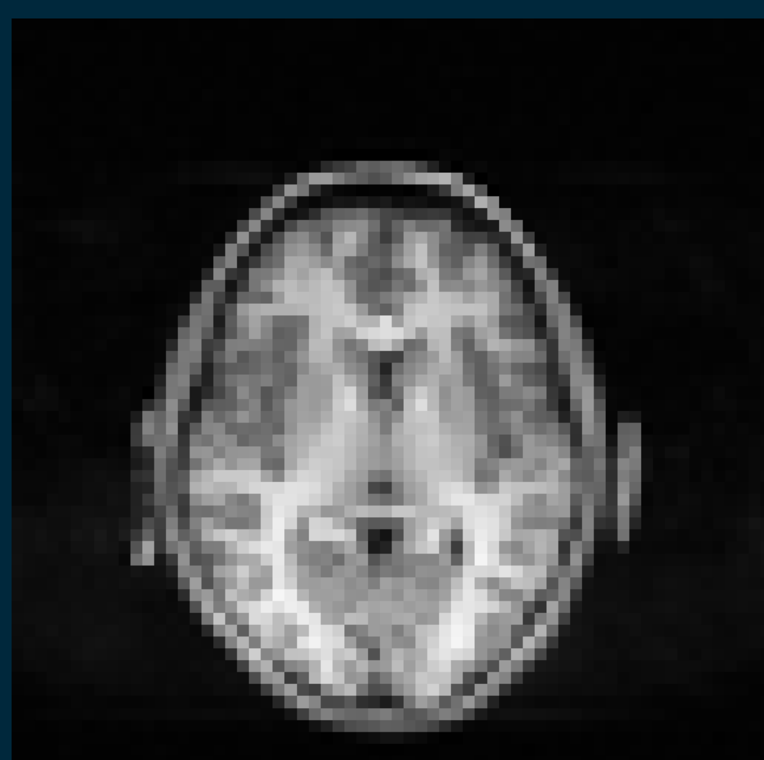
Continuous vs. Discrete

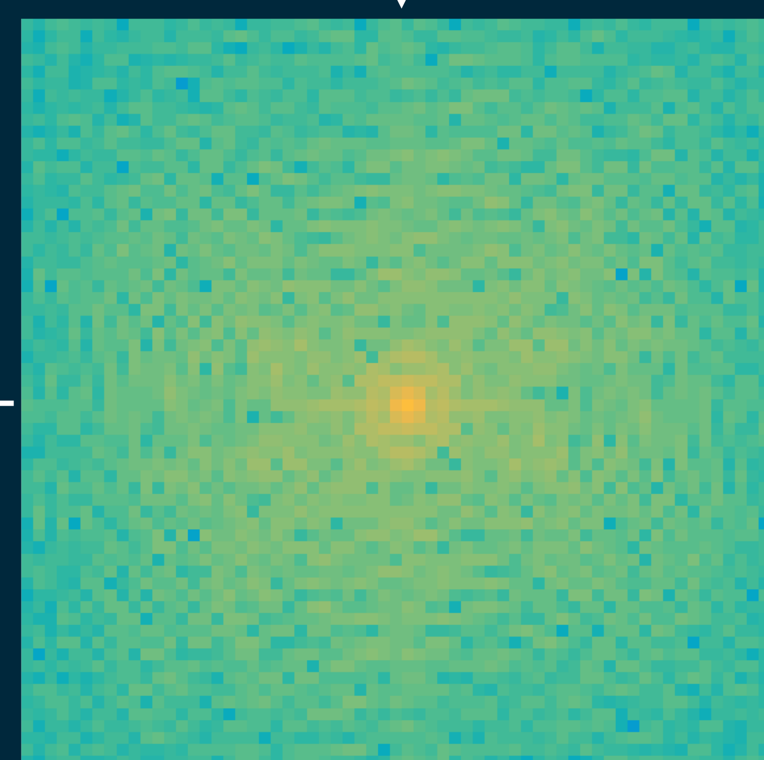
Continuous
source space



Continuous
Fourier (k)-space

Discrete image
space

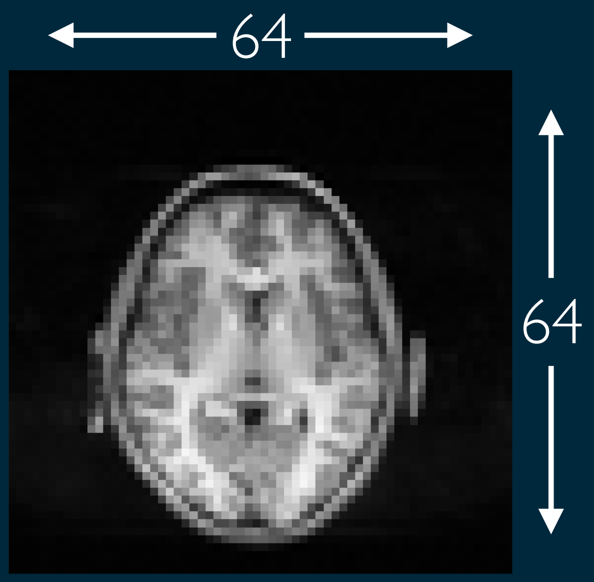




Discrete
Fourier (k)-space

The Image as an Array of Voxels

- An ***m***×***n*** voxel image contains ***mn*** voxels

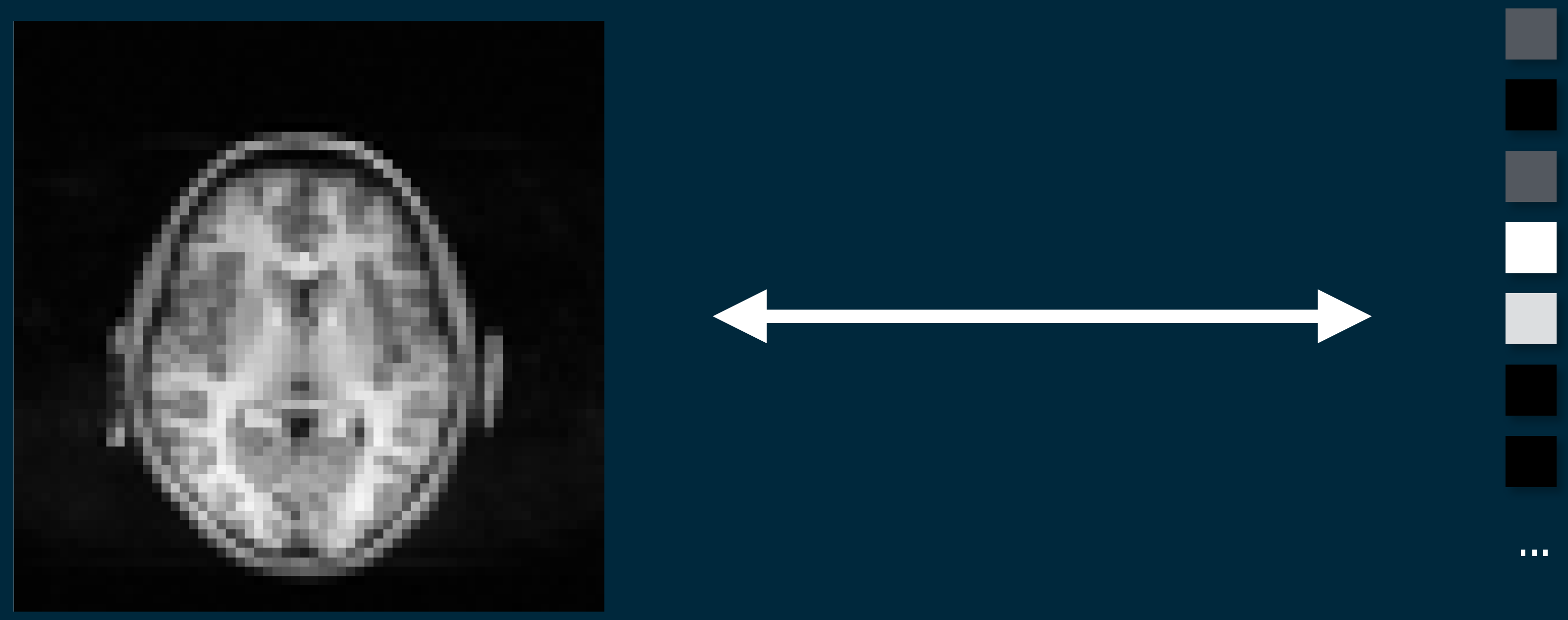


- With no knowledge or assumptions, each voxel's value is independent of the others

- Therefore the image *data* itself lies in $\mathbb{C}^{mn}$ (e.g. 4096 dimensional complex vector space)

The Image as a Vector

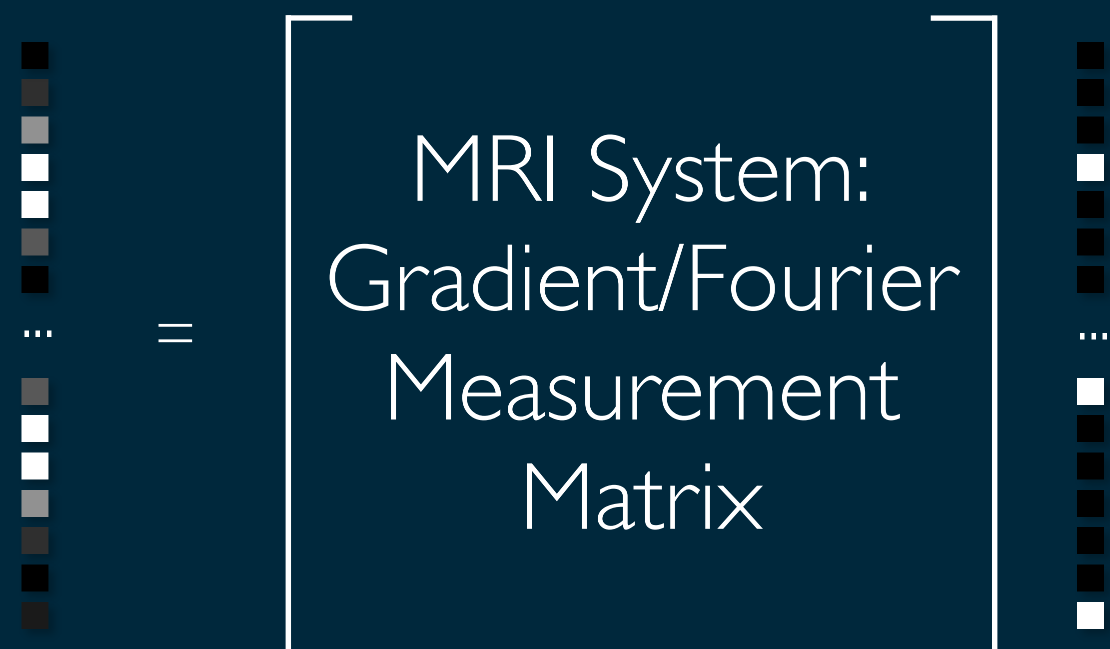
- Image data are vectors in an ***mn***-dimensional vector space
- Consider the data in conventional [***mn*** × *1*] vector form
  - We know exactly how to map the data vector to an “image”



- To find the image, we need to **solve for this vector**

Image Encoding (Measurement)

1. MRI measurements are encoded via magnetic field gradients
2. Gradients impose sinusoidal phase variation prior to integration
3. This process is linear, and physically encodes a Fourier transform



$$k = E \cdot x$$

Our measurement can be described as a discrete linear system

Imaging as a Linear System

- Imaging becomes a simple matter of solving:

$$k = Ex$$

  - where ***k*** is some vector of measured k-space values, ***E*** is a Fourier *encoding matrix* (or *measurement transform*) modelling the action of the MRI hardware, and ***x*** is the unknown image

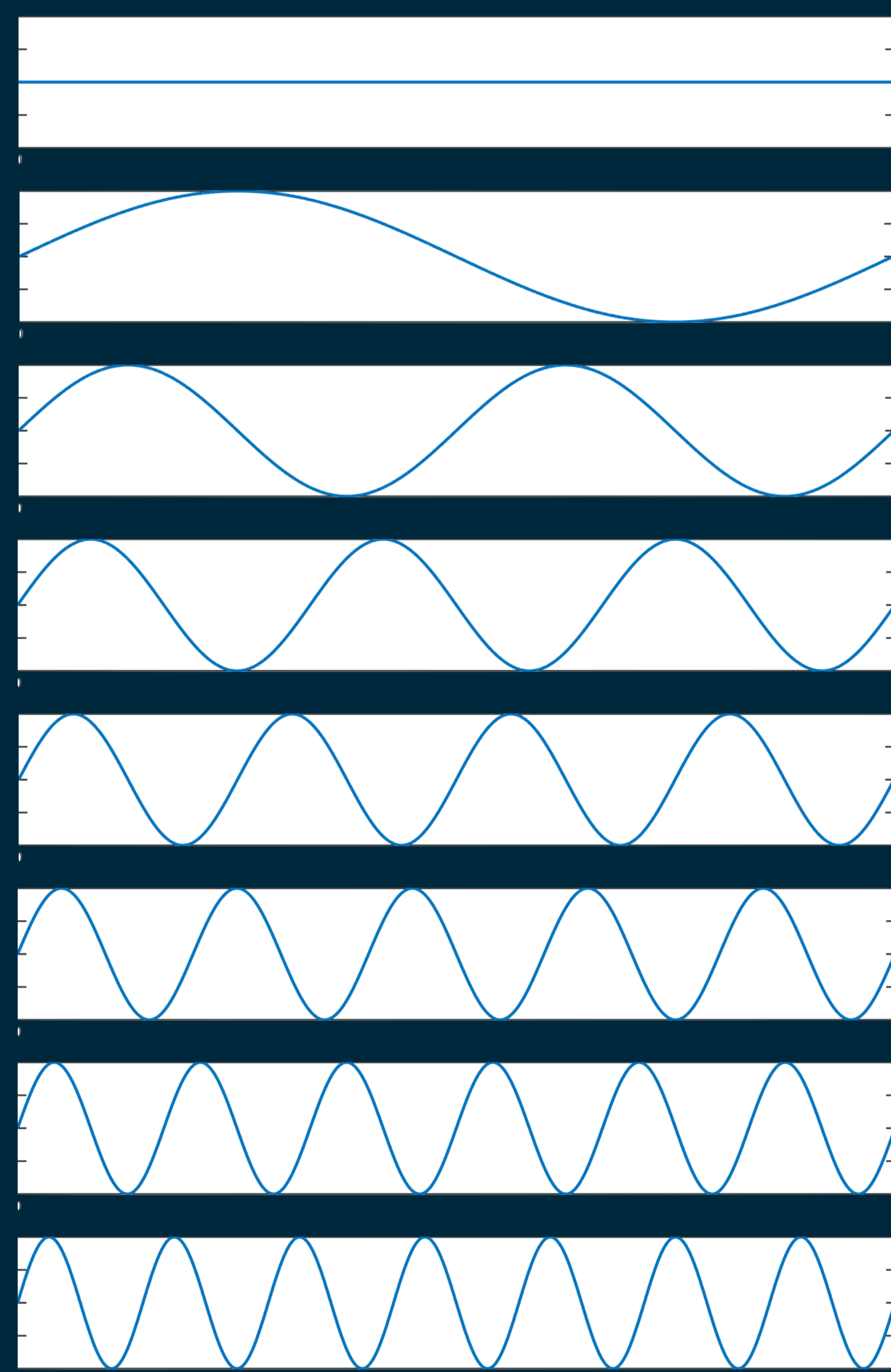
- Abstraction of the imaging problem offers flexibility and utility afforded by existing mathematics

- Much of the discussion in the rest of this lecture regards the design of the linear measurement transform (matrix) ***E***, and/or how to solve for ***x*** when ***E*** is not easily invertible

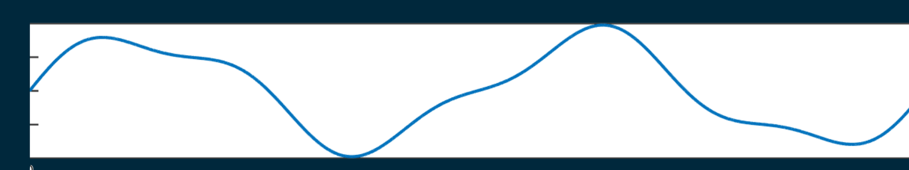
- No way to directly probe the values of each voxel (***E*** is constrained by what is physically realisable by the MRI system)

The Fourier Basis

Fourier Basis Set



Signal



The Fourier Basis

Basis Coefficients

$$\sum b_0 \cdot s$$

$$\sum b_1 \cdot s$$

$$\sum b_2 \cdot s$$

$$\sum b_3 \cdot s$$

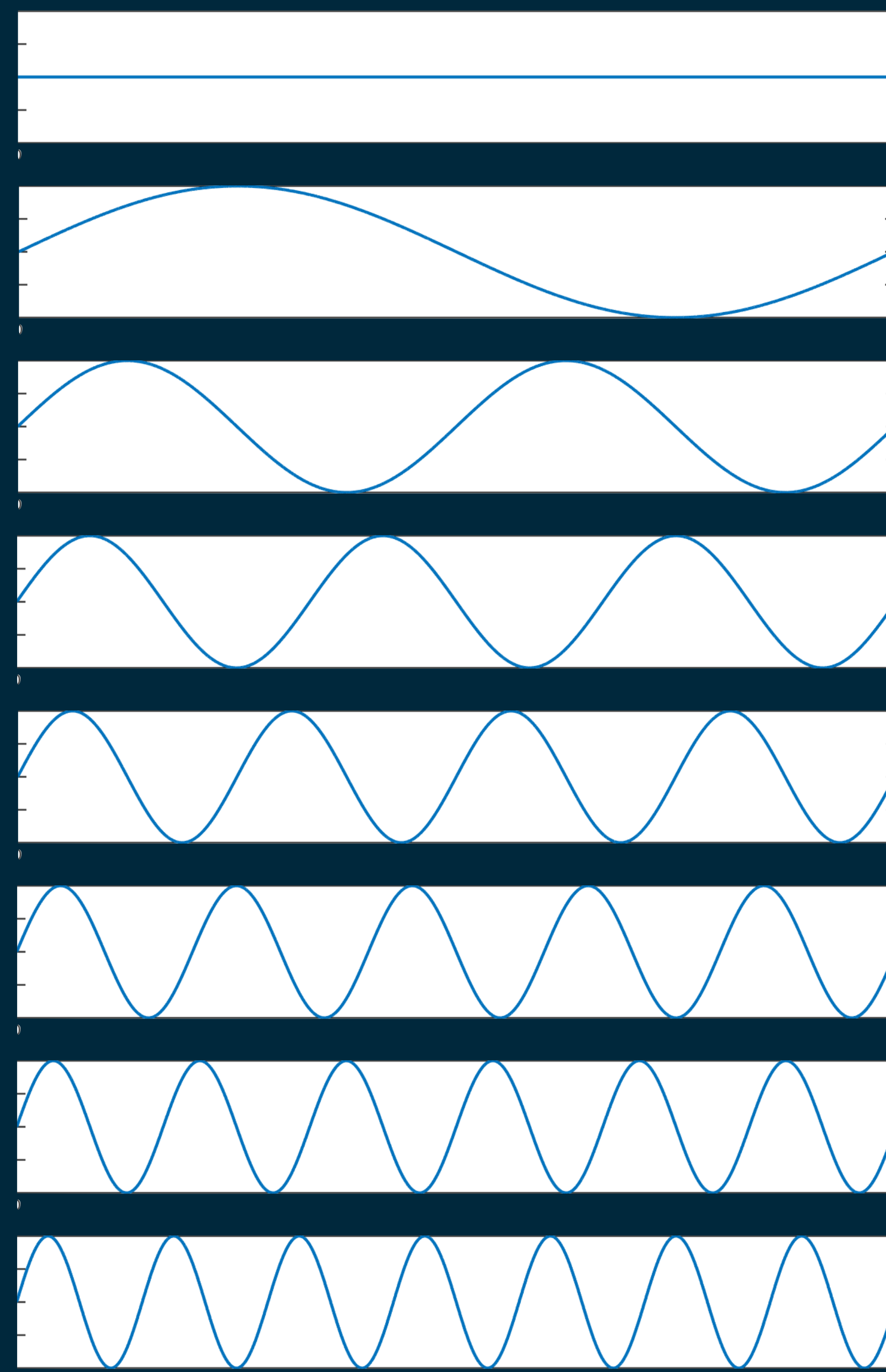
$$\sum b_4 \cdot s$$

$$\sum b_5 \cdot s$$

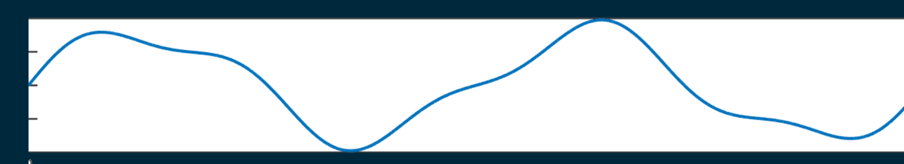
$$\sum b_6 \cdot s$$

$$\sum b_7 \cdot s$$

Fourier Basis Set



Signal



The Fourier Basis

Basis Coefficients

0

0

0.8

0

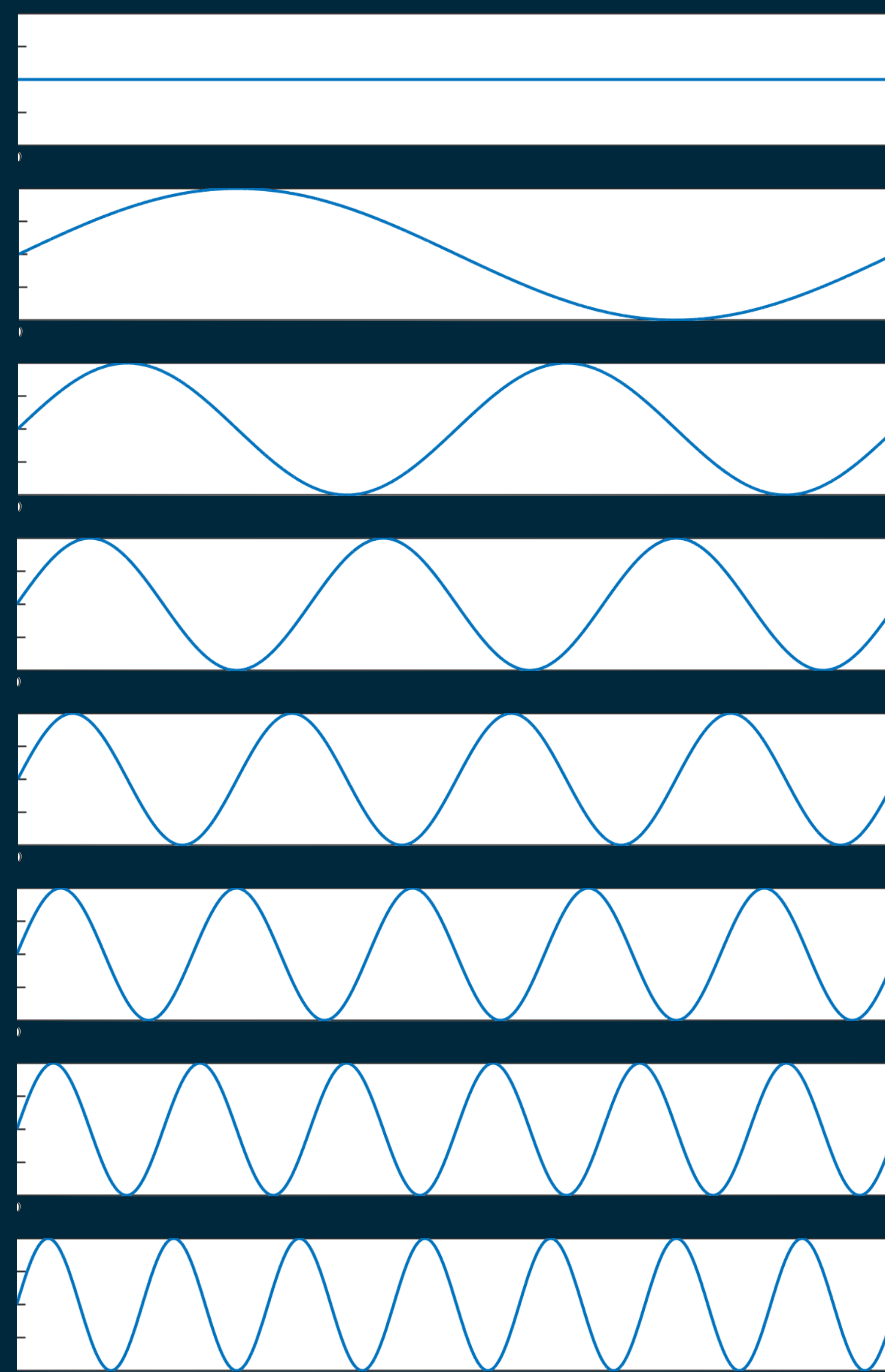
0

0.2

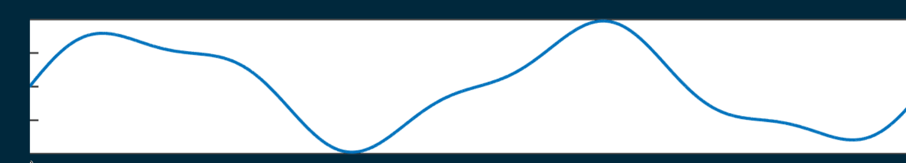
0

0

Fourier Basis Set



Signal

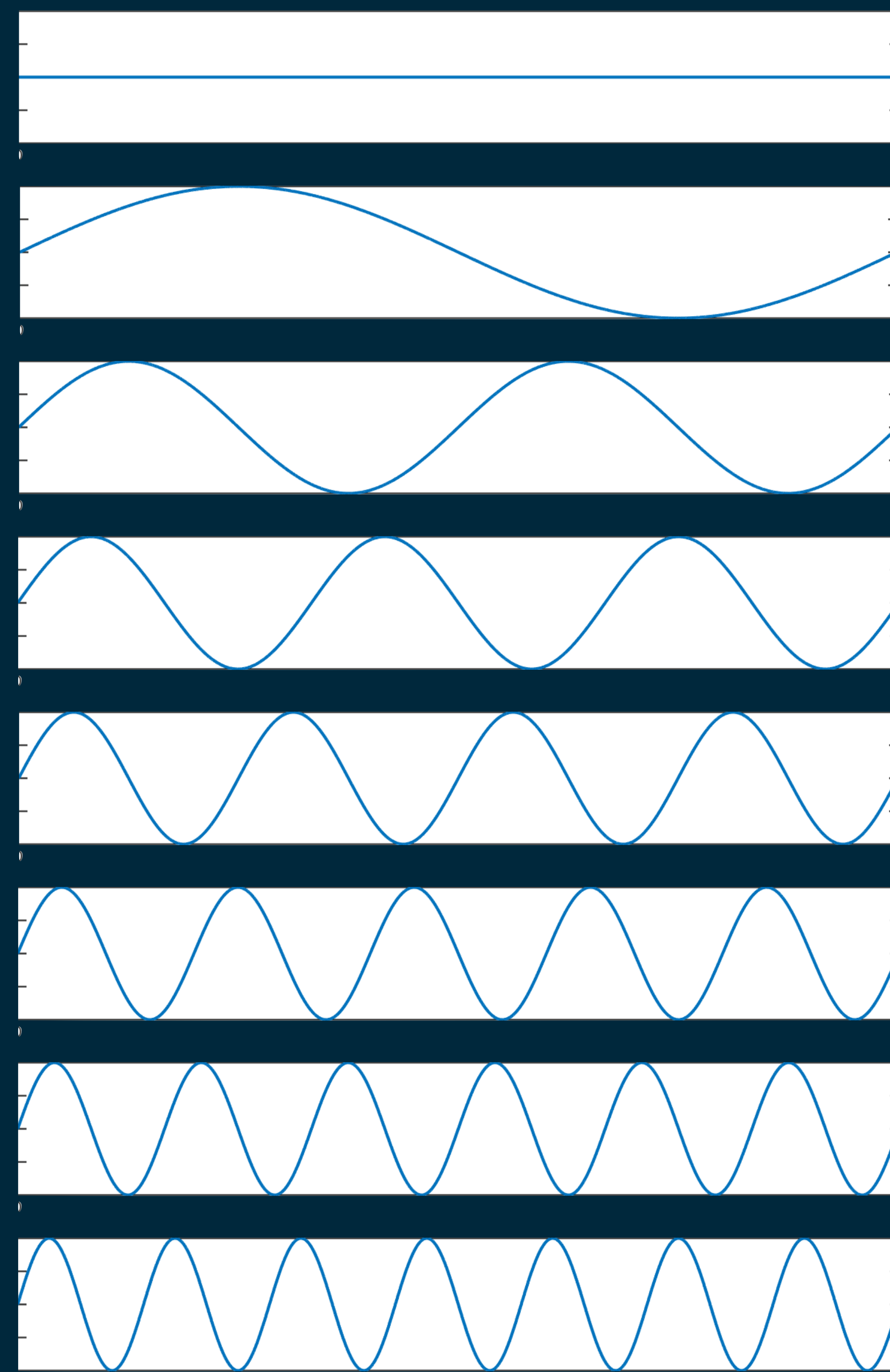


The Fourier Basis

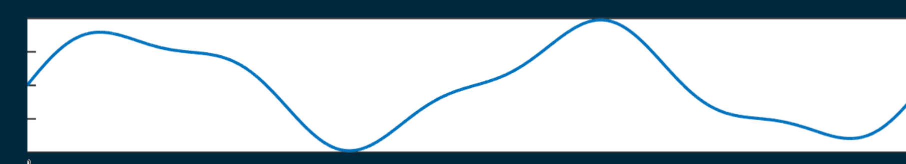
Signal in New Basis

$$\begin{bmatrix} 0 \\ 0 \\ 0.8 \\ 0 \\ 0 \\ 0.2 \\ 0 \\ 0 \end{bmatrix}$$

Fourier Basis Set



Signal

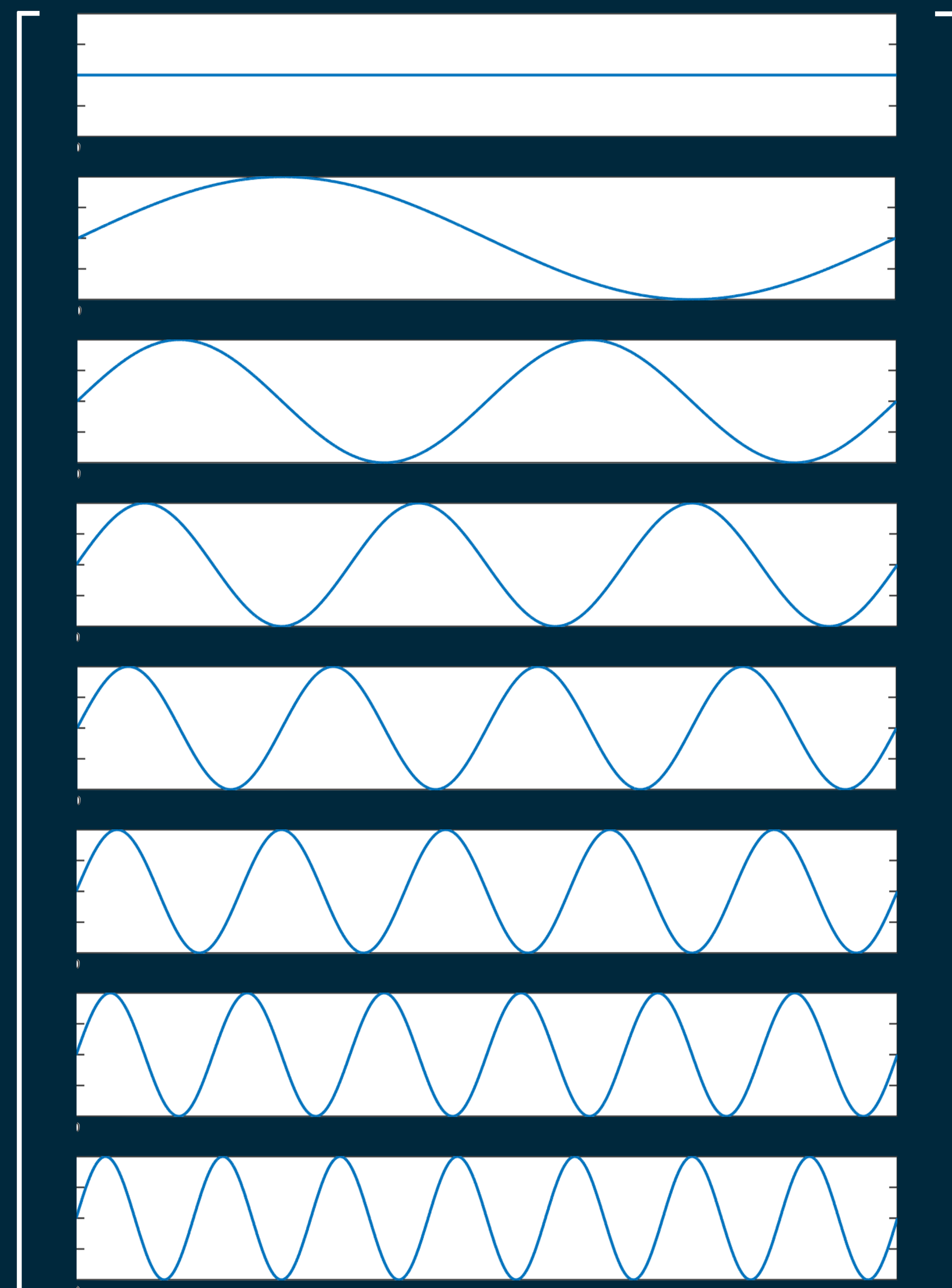


The Fourier Basis

Signal in Fourier Basis

$$\begin{bmatrix} 0 \\ 0 \\ 0.8 \\ 0 \\ 0 \\ 0.2 \\ 0 \\ 0 \end{bmatrix} =$$

Fourier Encoding Matrix



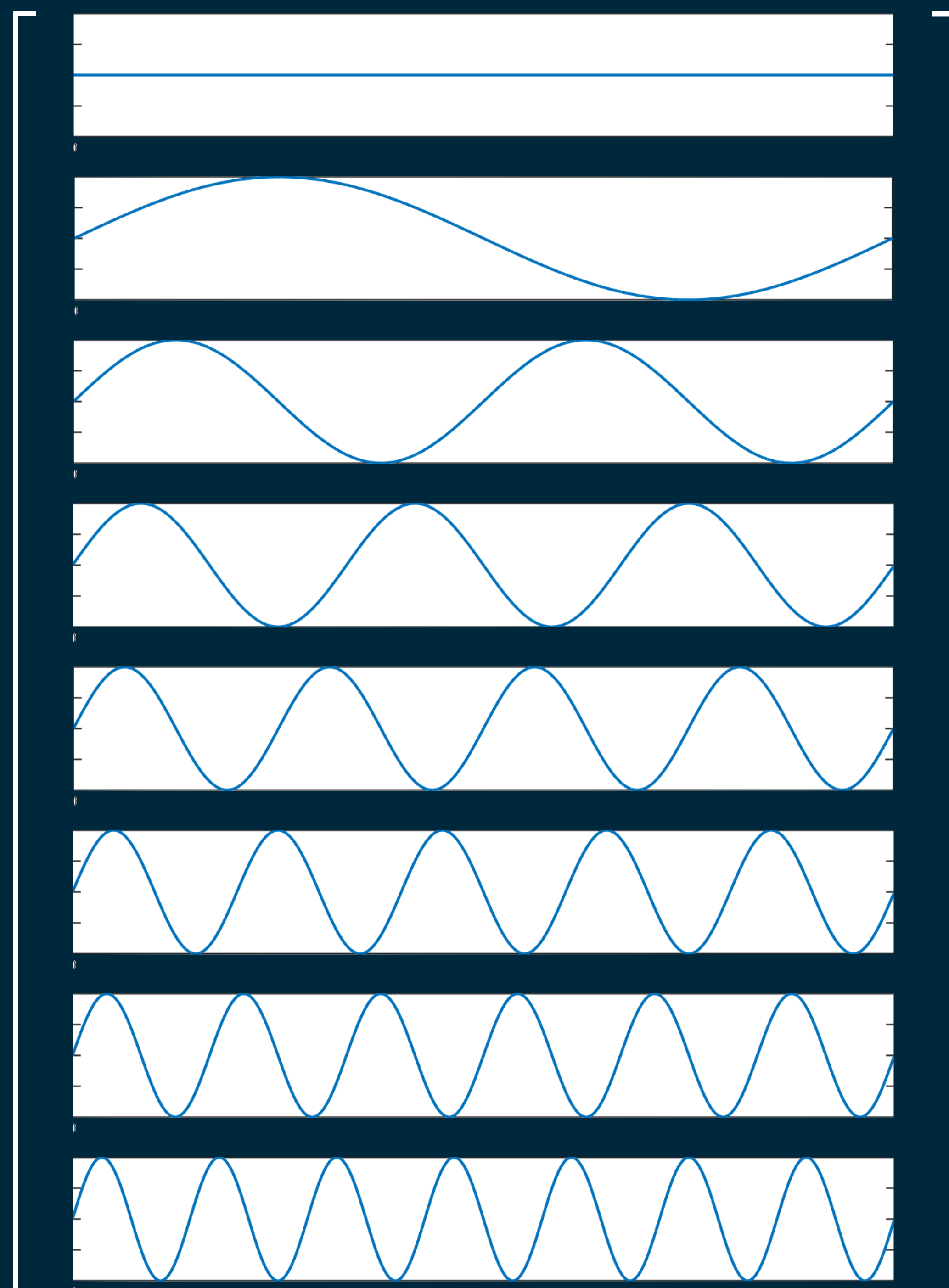
$*$

Signal

The Fourier Encoding Matrix

Signal in k-space | Fourier Encoding Matrix | Signal

$$\begin{bmatrix} 0 \\ 0 \\ 0.8 \\ 0 \\ 0 \\ 0.2 \\ 0 \\ 0 \end{bmatrix} =$$



$*$

The Fourier Encoding Matrix

Signal in k-space — Fourier Encoding Matrix — Signal

$$\begin{bmatrix} k \end{bmatrix} = \begin{bmatrix} E \end{bmatrix} * \begin{bmatrix} x \end{bmatrix}$$

The Measurement Process

In MRI, we use the Fourier (k-space) basis to represent and collect our data

This is necessary* because we *cannot* encode our data with a standard basis

Measurements must be made as an integral/sum over the entire source, due to the physics of electromagnetic induction

The Fourier basis is a natural choice (given how magnetic field gradients work in light of the Larmor relationship of field strength and frequency)

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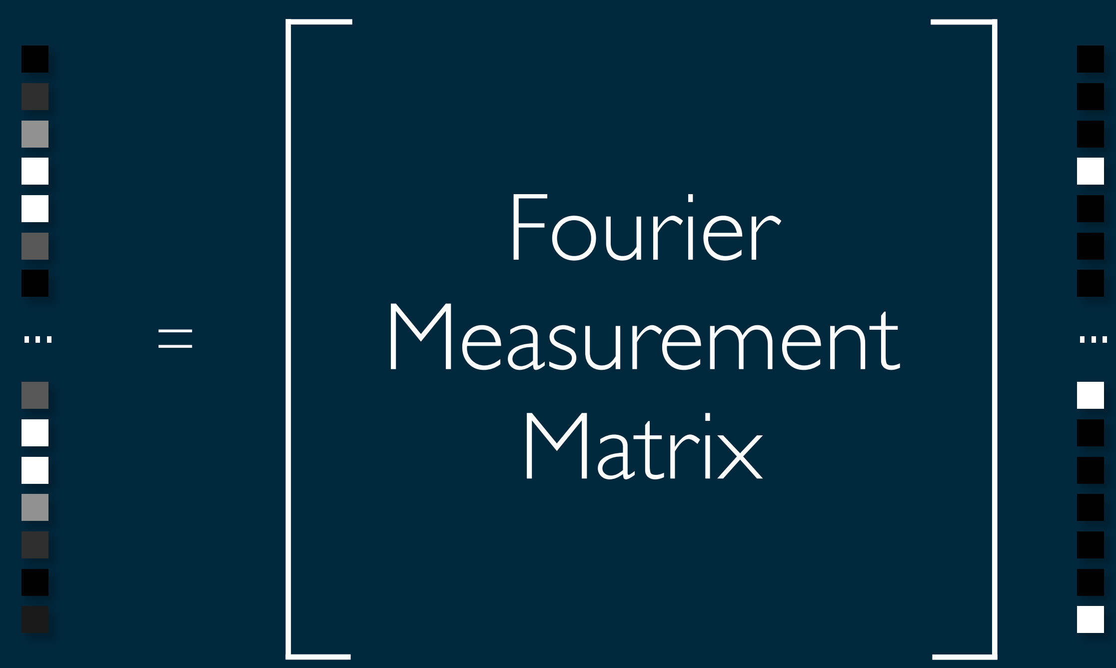
*Actually only strictly linearly varying magnetic fields generate Fourier encodings
Encoding with more general, non-linear fields is called "non-linear encoding"

The Inverse Problem

- Forward problem: "Given ***E*** and ***x***, generate ***k***"
- Inverse problem: "Given ***E*** and ***k***, what should ***x*** be"



$$k = E \cdot x$$

Solving for the Image

- "Apply the inverse Fourier transform to the k-space data"

$$k = Fx \longrightarrow \hat{x} = F^{-1}k$$

- We know that the *inverse* Fourier transform means the *inverse* or *conjugate transpose* of the (unitary) Fourier transform matrix
- A square matrix is necessary (but not sufficient) for invertibility
  - Square means equal numbers of measurements and unknowns
  - Need as many k-space samples as there are image points

Practically Finding the Image

- ~~"Apply the inverse Fourier transform to the k-space data"~~

$$\cancel{k = Fx \longrightarrow \hat{x} = F^{-1}k}$$

In practice, we don't form DFT encoding matrices and invert them

A 3D (64×64×64) image would require a 262144×262144 encoding matrix which would require 512 GB of memory to represent in single precision (64^6×2×4 bytes) and O(N^2) operations to multiply, O(N^3) to invert

More commonly, fast Fourier transform (FFT) algorithms (O(N log N)) are used

$$k = Fx \longrightarrow \hat{x} = iFFT(k)$$

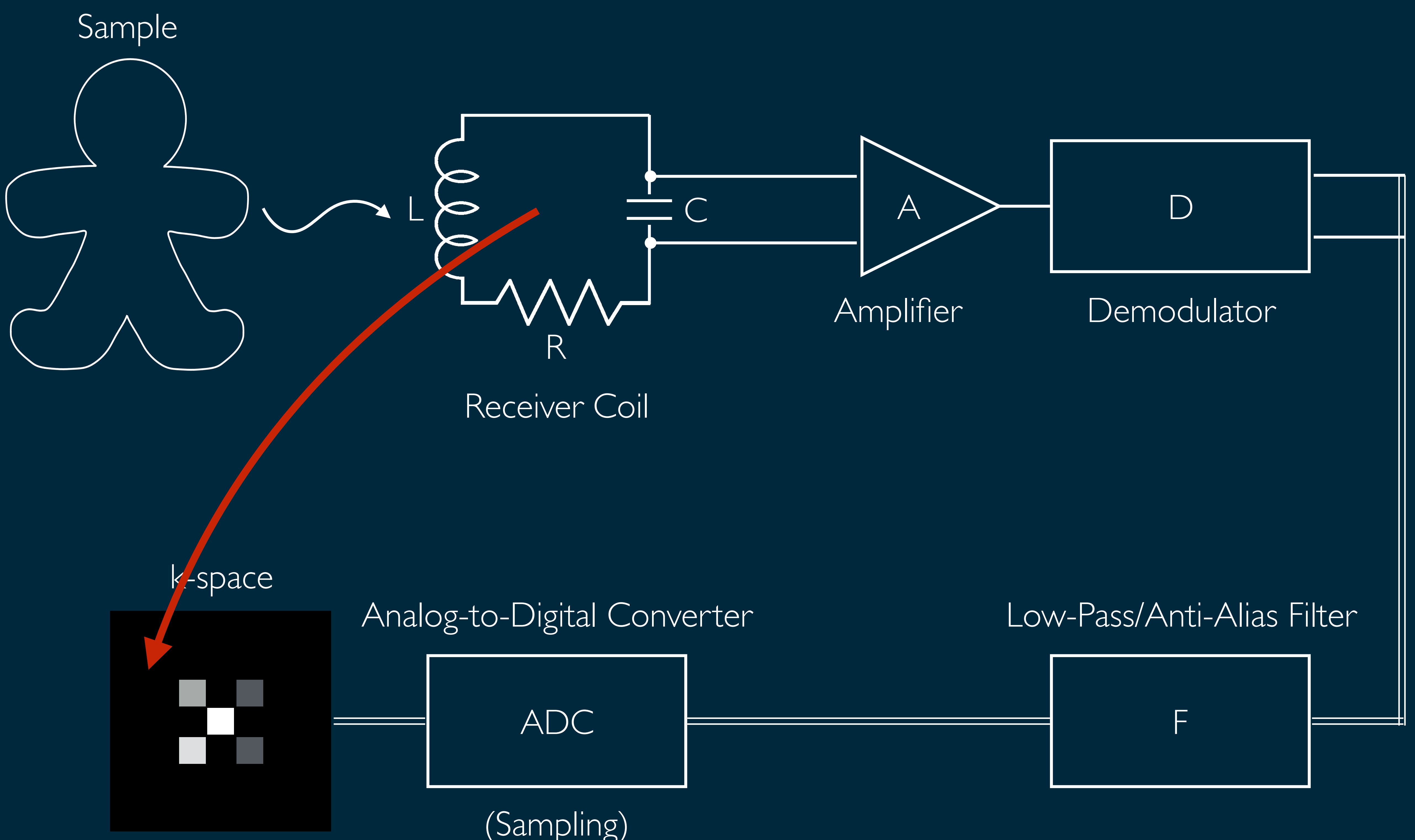
Parallel Imaging

Acceleration

- Measurements we make in k-space are sequential, and take time
- To speed up data acquisition, we can reduce the # of measurements
  - With fewer measurements, total measurement time is reduced,, but our linear system is now underdetermined (#unknowns > #measurements)

- How do we solve for our unknown image given a reduced number of measurements?

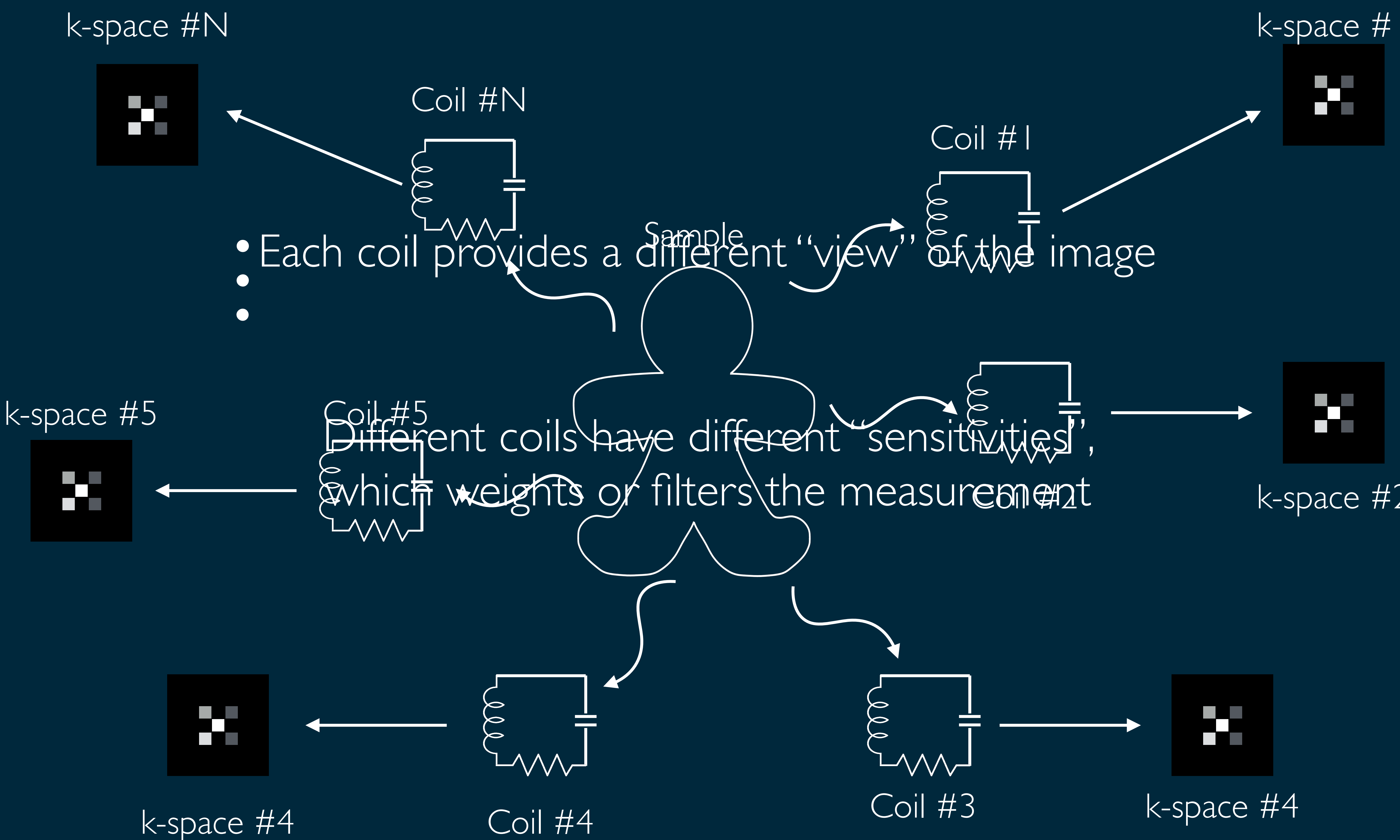
The MRI Receiver Chain



Parallel Receivers

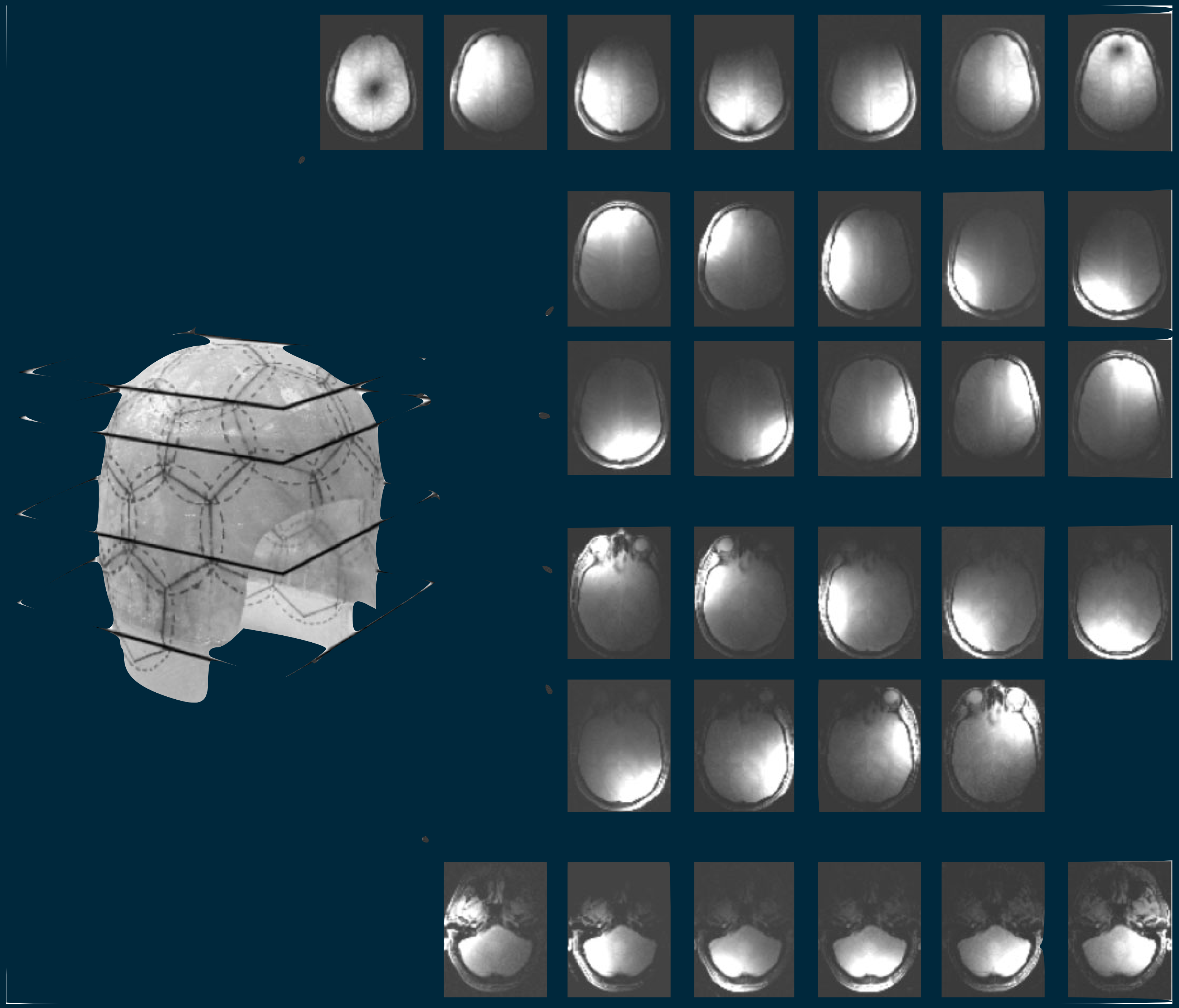
- Each coil provides a different "view" of the image

Different coils have different "sensitivities", which weights or filters the measurement

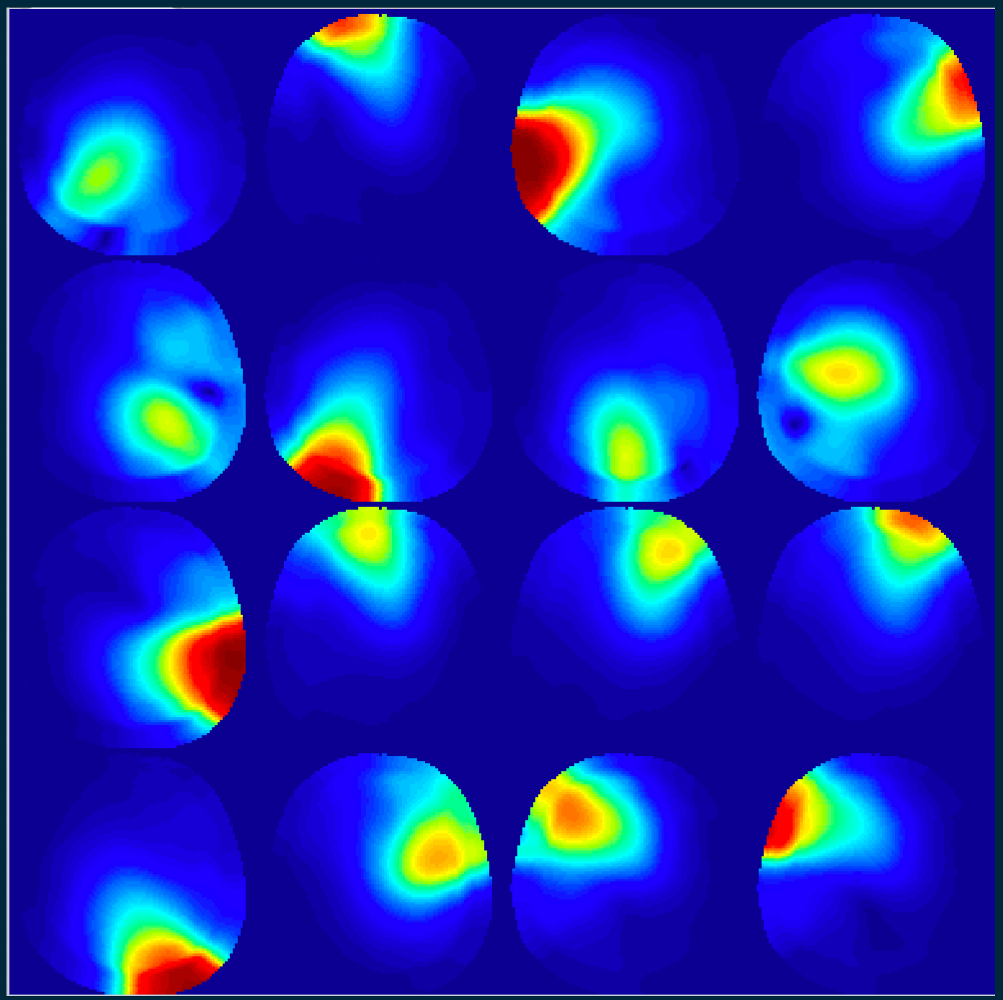


Example Coil Sensitivities

Example 32-channel head coil phased array



Wiggins et al., MRM 2006



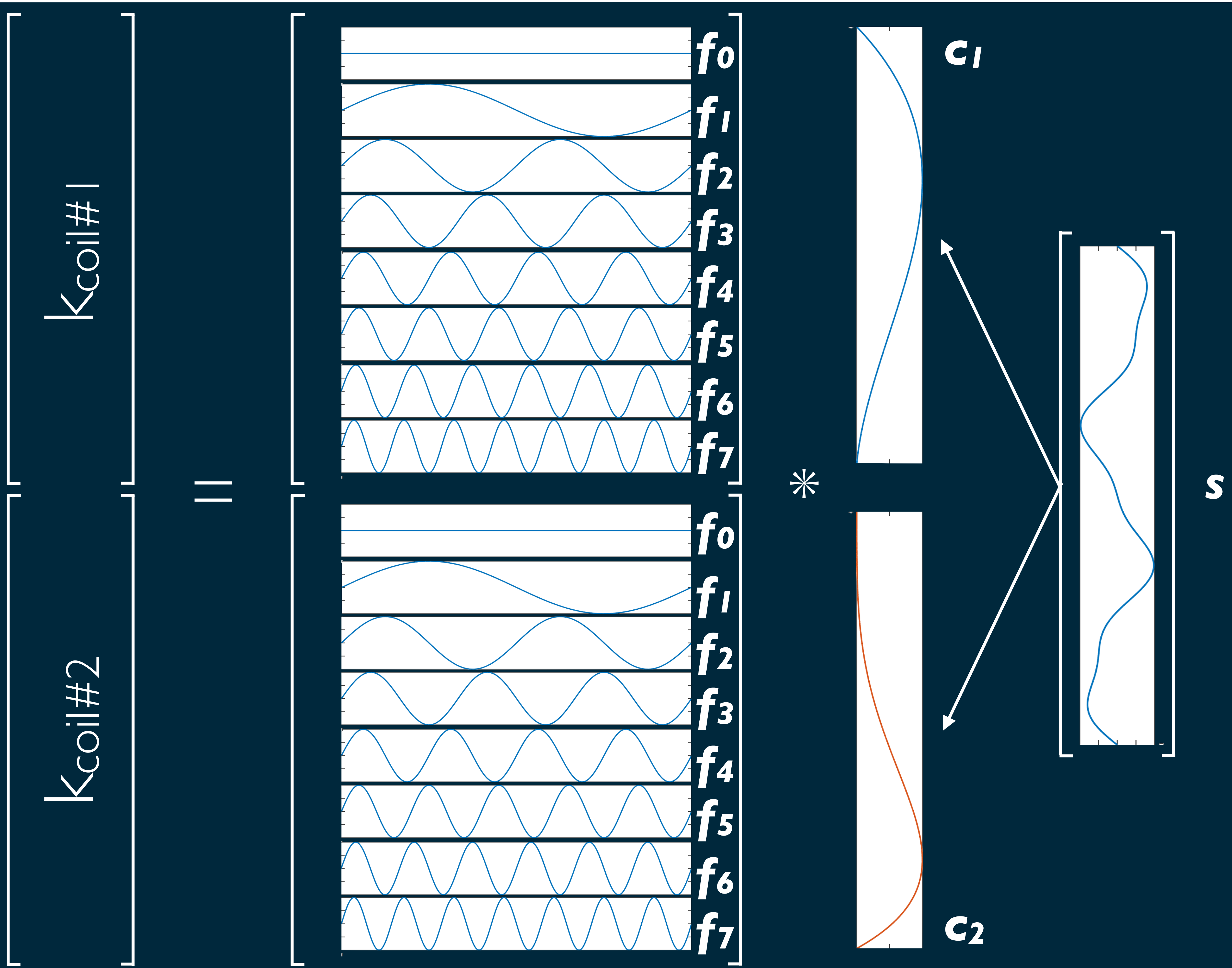
What is a coil's "sensitivity"?

- We interact with receive coils via Faraday induction
- EMF in the coil due to changing flux
- Coil is fixed, so flux change is due to magnetic field change
- Magnetic field is due to magnetisation at each voxel
- Now consider each voxel's magnetization to be a unit magnetic dipole
- Compute the flux contribution from that voxel source only
- That unit flux is the sensitivity of the coil at that voxel's location in space

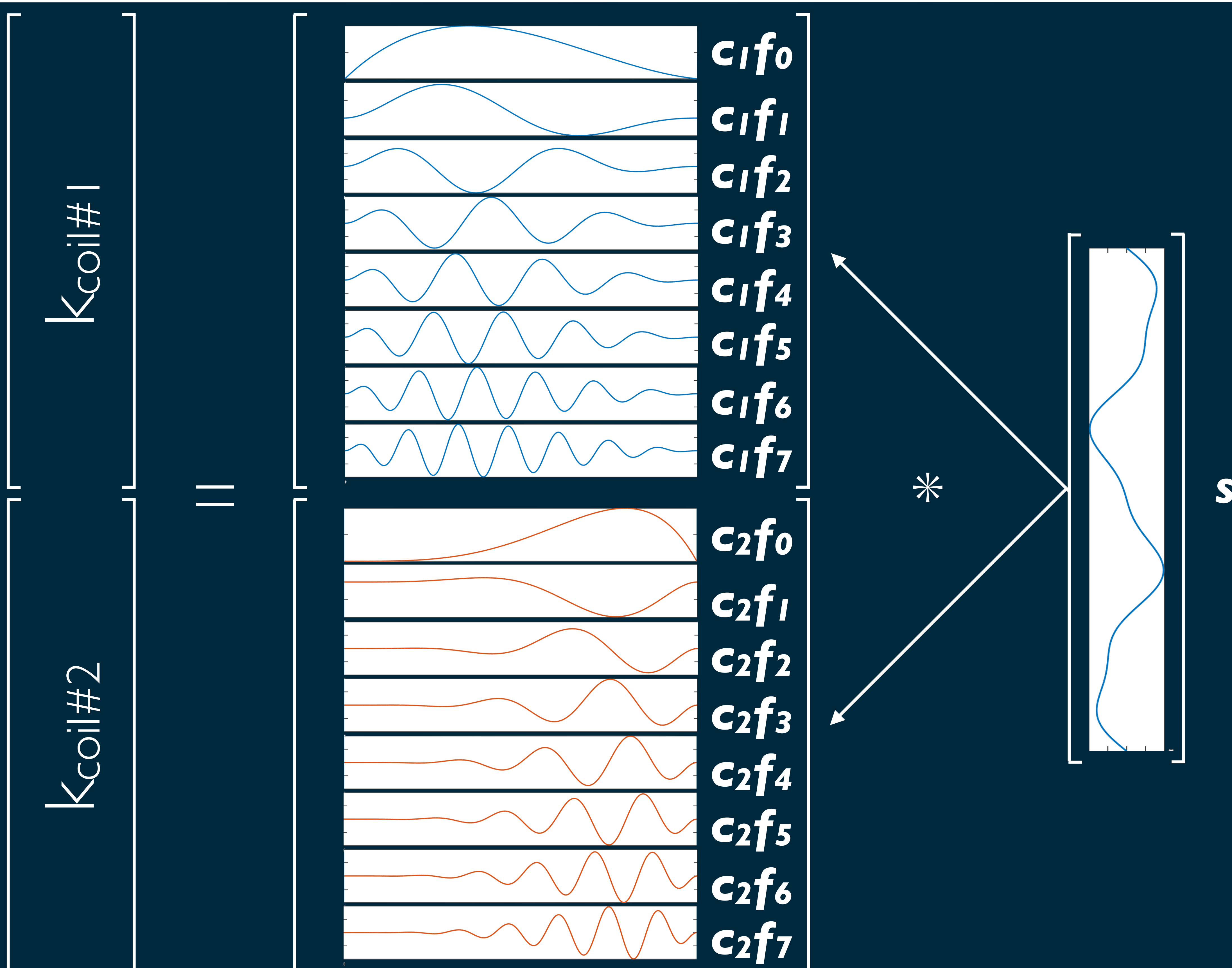
Using Coils for Parallel Imaging

- Each coil or receiver element represents a separate, parallel measurement
- Because every coil has a spatial sensitivity profile, this effectively modifies what it “sees”
- We use these properties to, in essence, increase the number of measurements we observe with no time penalty

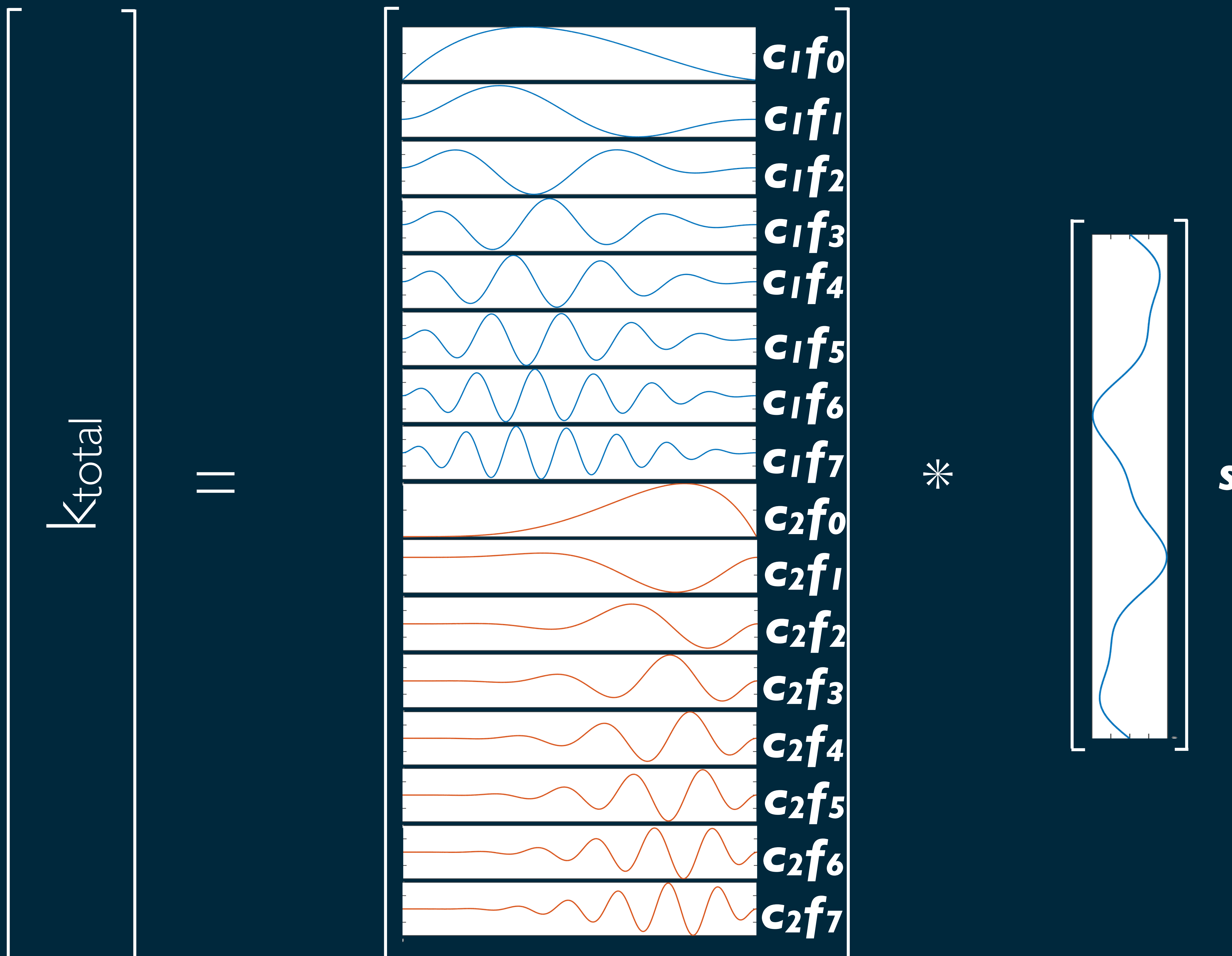
Coil Sensitivities



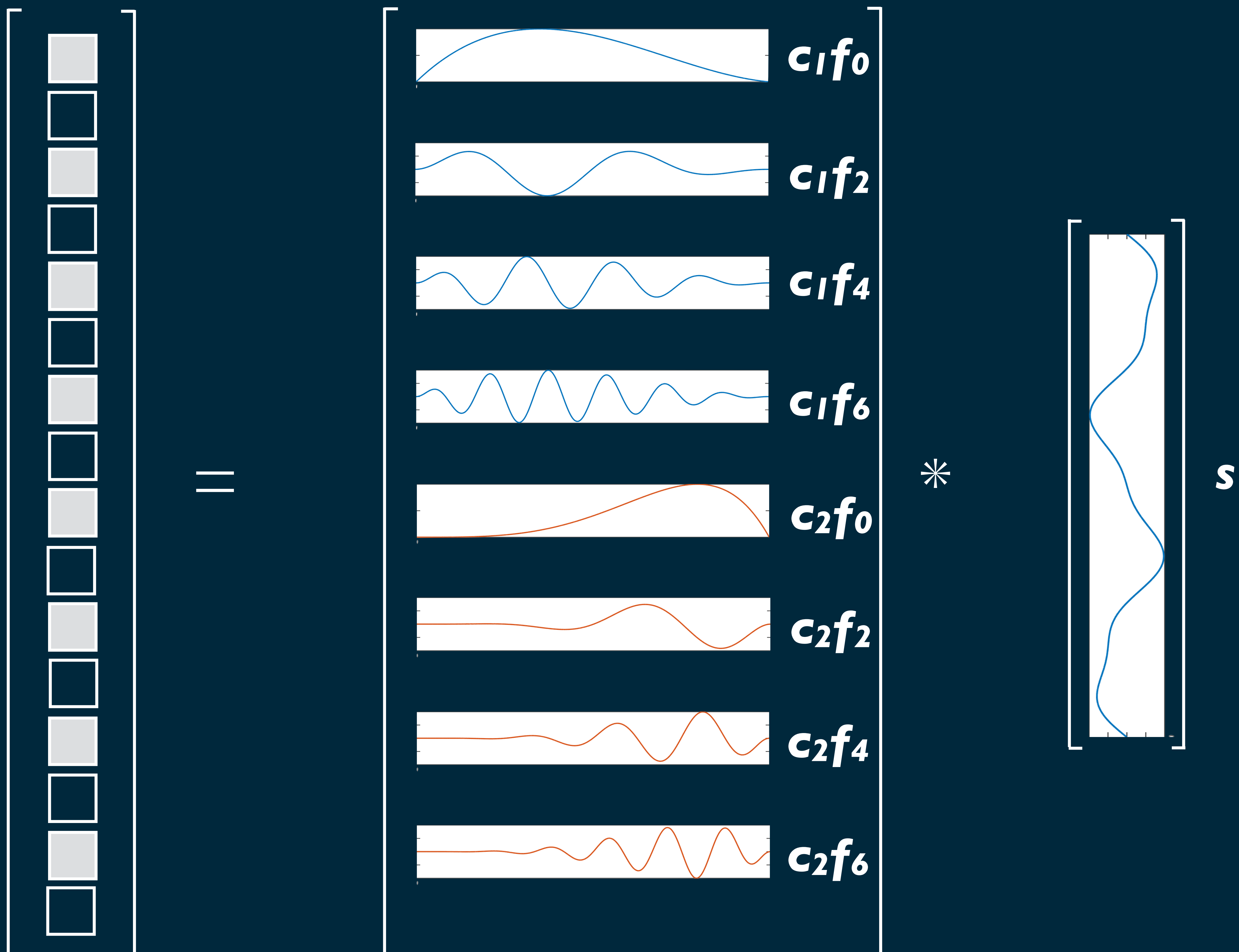
Sensitivity Encoding Matrix



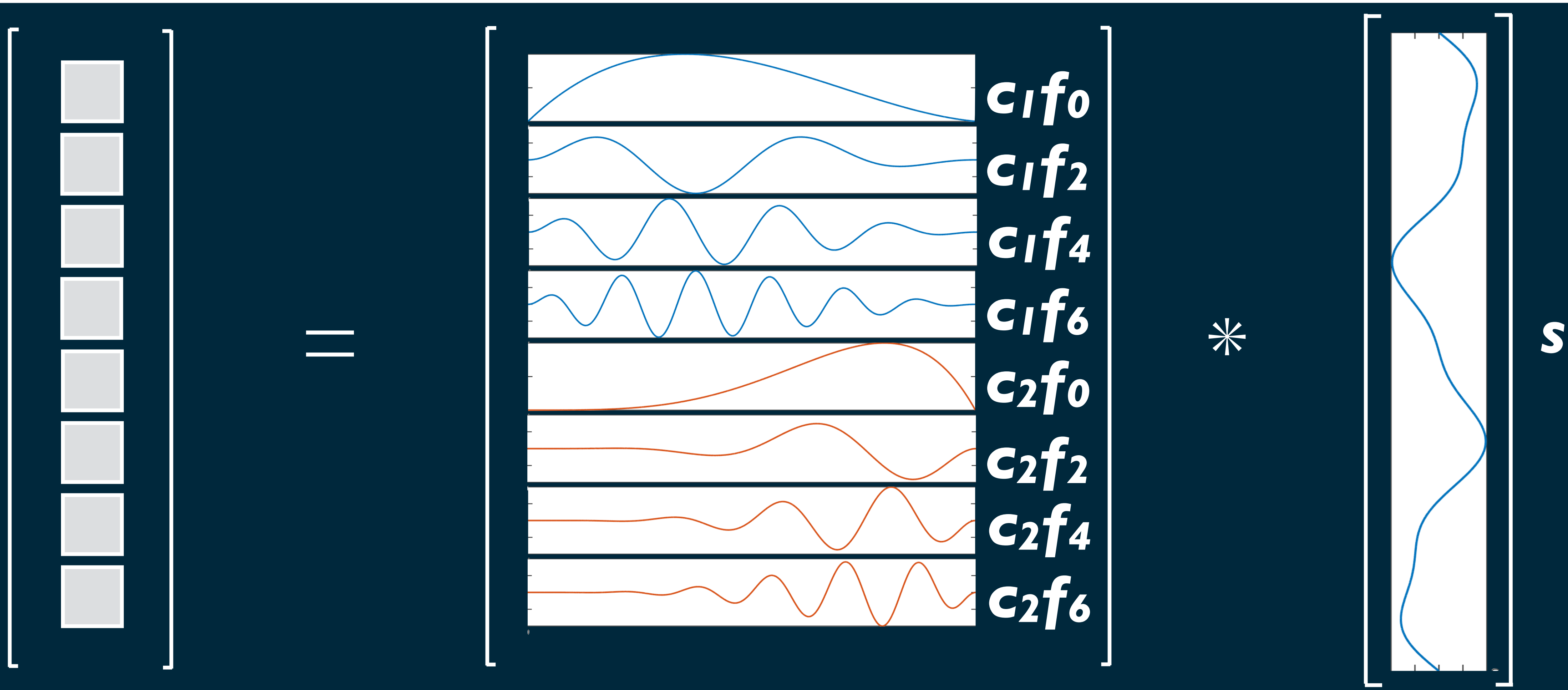
Extended Encoding Matrix



Parallel Acceleration



Parallel Acceleration Concept



This set of measurements takes half the amount of time (*i.e.* only half of the gradient-induced Fourier encoding steps are performed)

However, conditions on invertibility still hold (we need coil sensitivities to be different enough so that rows of the encoding matrix are linearly independent)

Parallel Acceleration Concept

- If there are insufficient numbers of coils, or not enough coil sensitivity differentiation (linear independence) for the amount of undersampling
  - The encoding matrix is still underdetermined (or not full rank)
  - In other words, our encoding basis does not span the full signal space
  - Without extra information, there are too many degrees of freedom to identify a unique solution to the inverse problem

- If there are more coils than necessary, the problem is overdetermined
  - The encoding matrix however, is full rank and our encoding basis is an over-complete representation of the signal vector space
  - We commonly use least squares criterion to select an optimal solution

Parallel Acceleration Costs

- Reduced Readout Time
  - Signal-to-Noise Ratio (SNR) is proportional to the square root of the amount of time spent sampling signal
  - With a under-sampling factor of R, SNR decreases by a factor of $\sqrt{R}$
- Noise Amplification
  - The condition number of a matrix dictates how robust the transform is to noise, and is a measure of how "nearly rank deficient" it is
  - Unitary transforms, like the DFT, have condition number = 1 (i.e. no noise amplification, perfectly conditioned)
  - Non-unitary transforms, like parallel imaging encoding matrices, have condition numbers > 1 (i.e. noise amplification)

Conditioning

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0.707 \\ 0 & 0.707 \end{pmatrix} \quad \begin{pmatrix} 1 & 0.866 \\ 0 & 0.5 \end{pmatrix} \quad \begin{pmatrix} 1 & 0.985 \\ 0 & 0.174 \end{pmatrix}$$

1.0 2.414 3.732 11.430

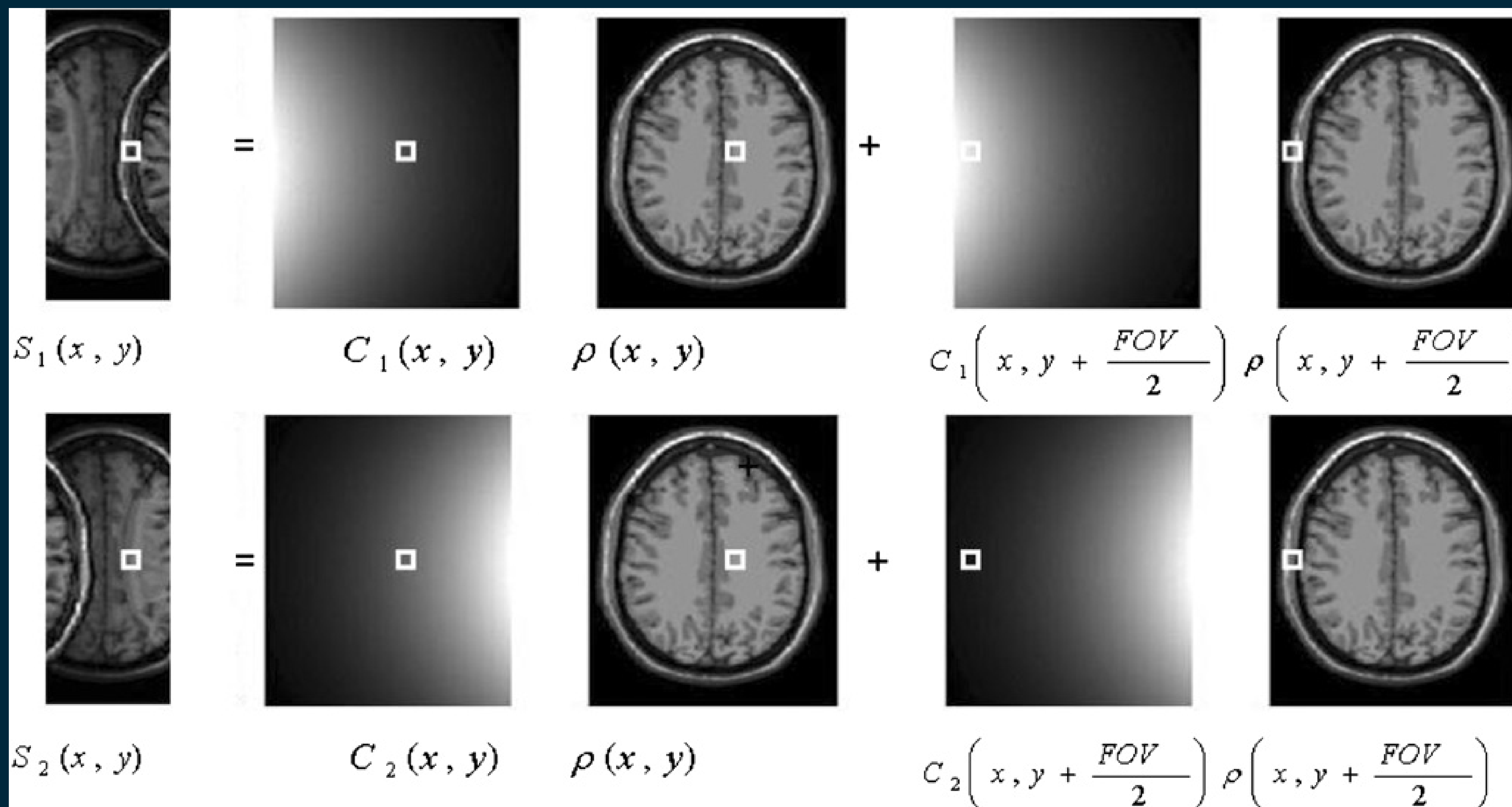
Matrices with columns (or rows) that are highly co-linear have high condition numbers, and can produce high degrees of inaccuracy in reconstruction

Similarly, while coil sensitivities that are similar but still linearly independent technically produce full rank encoding matrices, it is optimal from an SNR perspective to have them as orthogonal as possible

Note that Fourier basis functions are completely orthogonal by construction

SENSE (Pruessmann et al., MRM 1999)

Inverse Fourier Transform before using sensitivity encoding information



(Larkman et al., Phys Med Biol 2007)

coil #1 $y_1 = c_{11}x_1 + c_{12}x_2$

coil #2 $y_2 = c_{21}x_1 + c_{12}x_2$

$\longrightarrow$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\text{Aliased coil measurements}} = \underbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}_{\text{Coil sensitivities}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Voxels}}$$

SENSE (Pruessmann et al., MRM 1999)

Inverse Fourier Transform before using sensitivity encoding information

This is advantageous because it greatly simplifies the inverse problem

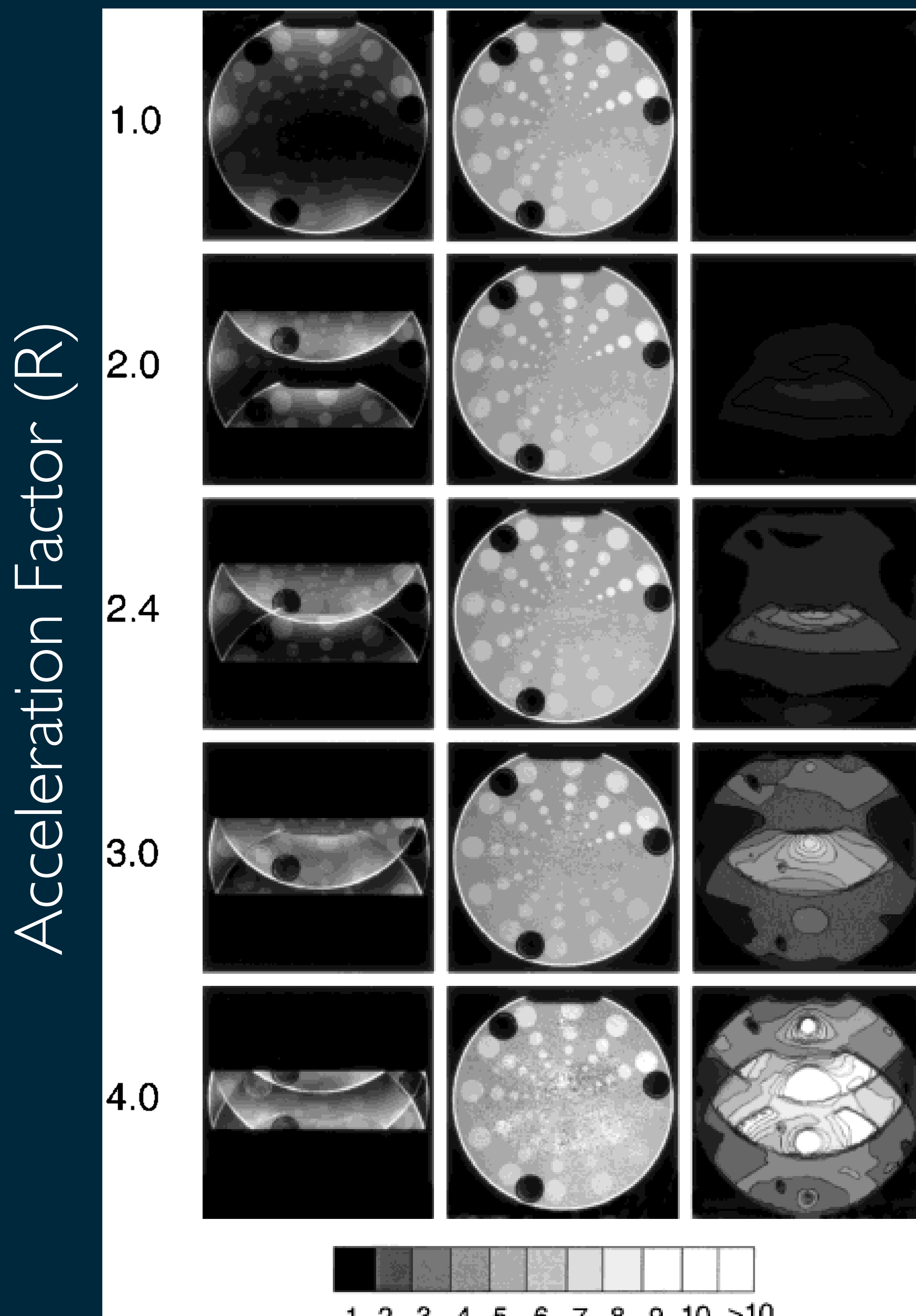
- It becomes spatially separable, meaning that the reconstruction problem is reduced to sets of *R* voxels, where *R* is the acceleration factor, and also the aliasing factor
- Solve many trivially small problems, instead of one giant one

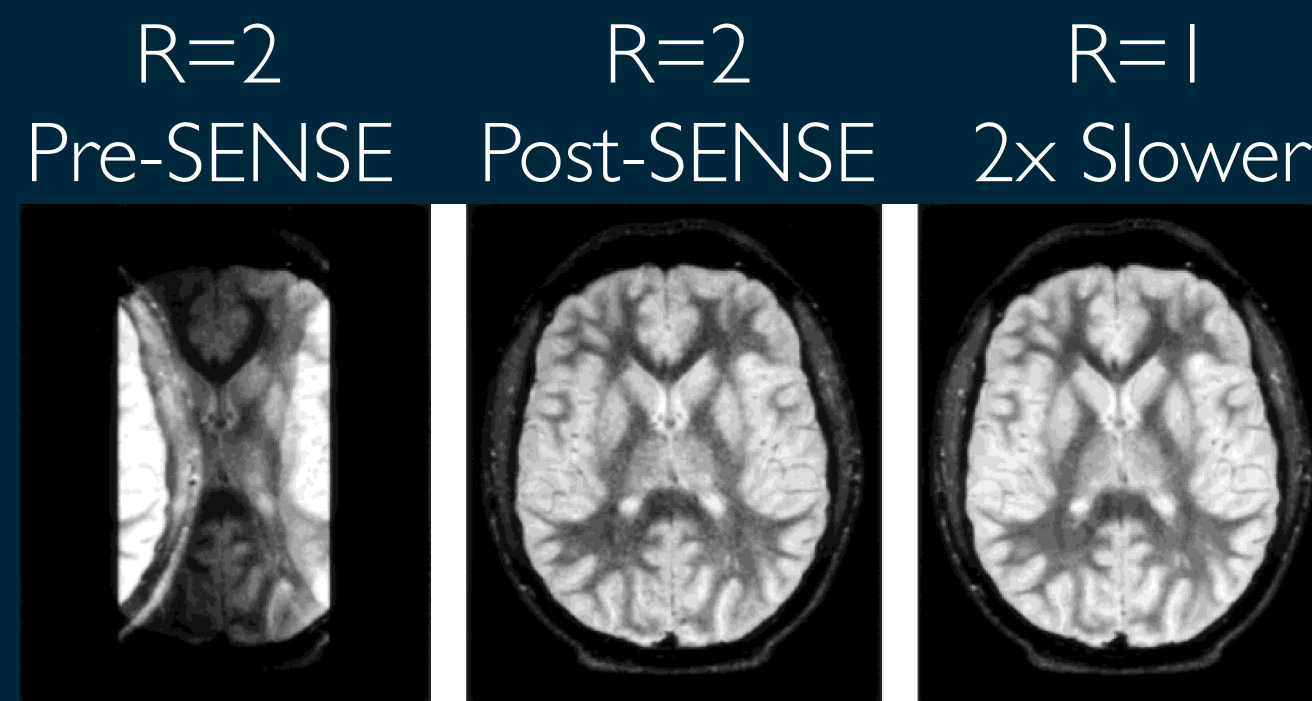
This produces noise amplification that is spatially varying, because the encoding matrices (and hence conditioning) is different for different sets of voxels

This noise amplification is often called the geometry factor, or "g-factor"

SENSE also requires explicit knowledge and/or mapping of the coil sensitivities

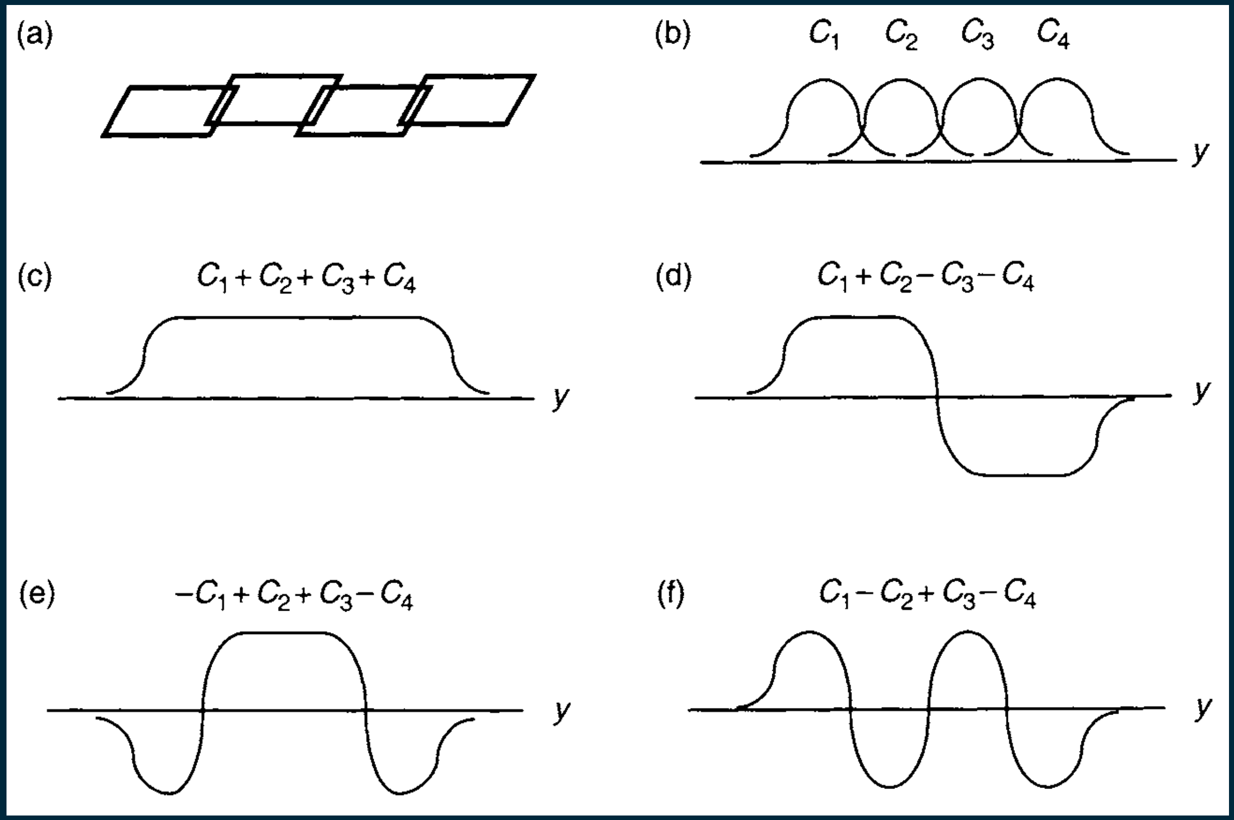
SENSE (Pruessmann et al., MRM 1999)





SMASH (Sodickson et al., MRM 1997)

"SiMultaneous Acquisition of Spatial Harmonics"

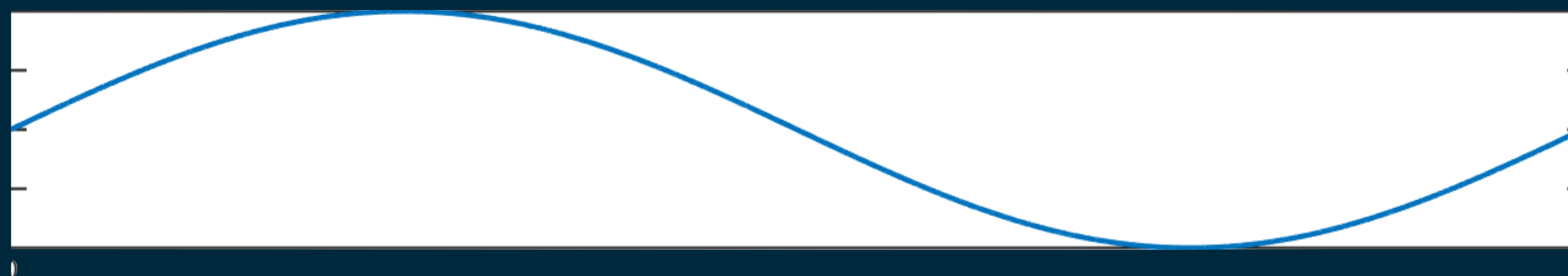


(Bernstein et al., 2004)

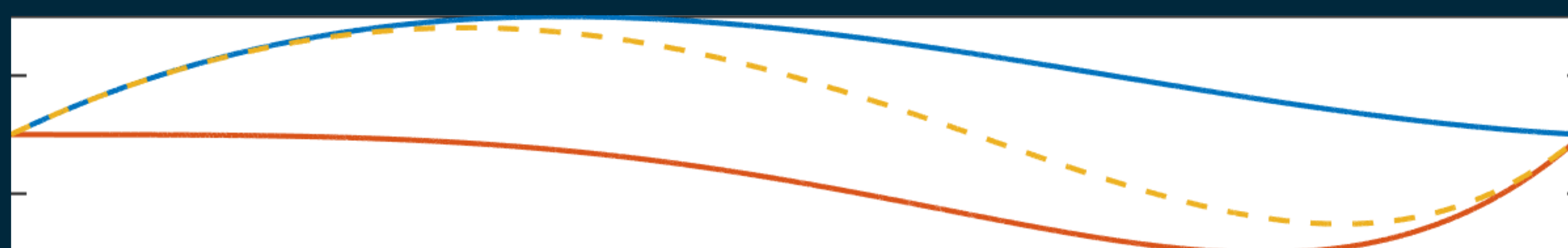
The basic idea is to explicitly replace Fourier encoding steps with linear combinations of coil sensitivities

SMASH (Sodickson et al., MRM 1997)

f_1



$c_1 - c_2$



With appropriate linear combinations weights (e.g. [1,-1] above), missing k-space samples can be estimated from acquired samples by post-hoc synthesis of Fourier encodings

SMASH (Sodickson et al., MRM 1997)

$$s_j(k) = \int c_j \rho(x) e^{-ikx} dx \qquad \sum w_j c_j = e^{-i\Delta kx}$$

$$\sum w_j s_j(k) = \sum \int w_j c_j \rho(x) e^{-ikx} dx$$

$$\sum w_j s_j(k) = \int (\sum w_j c_j) \rho(x) e^{-ikx} dx$$

$$\sum w_j s_j(k) = \int \rho(x) e^{-i(k+\Delta k)x} dx$$

$$\sum w_j s_j(k) = s(k + \Delta k)$$

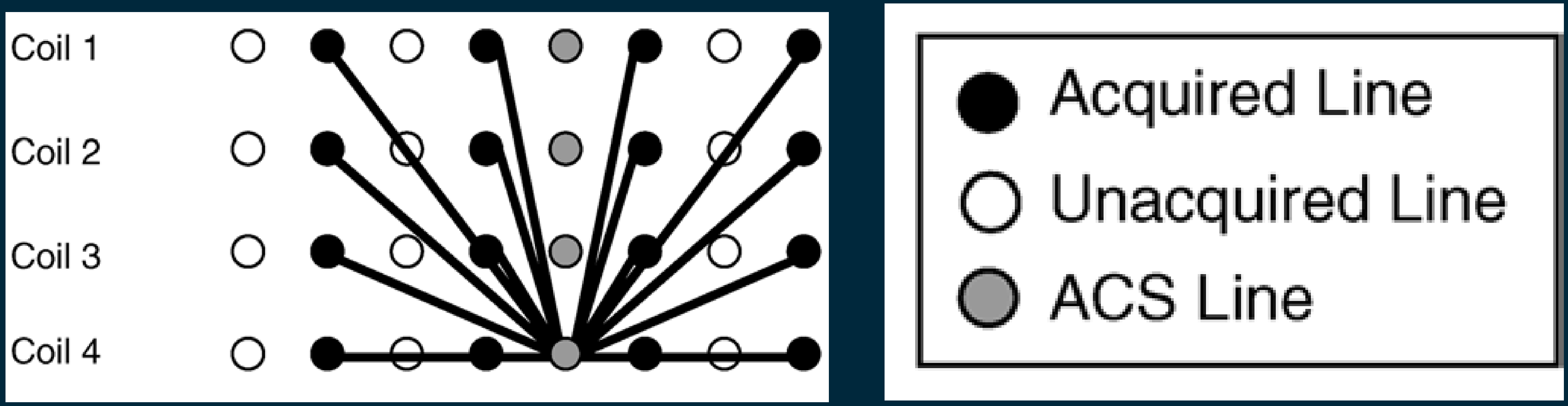
Originally, required coil sensitivities (for fitting to Fourier basis), and poor synthesis led to heavy artefacts in reconstruction

GRAPPA (Griswold et al., MRM 2002)

"GeneRalised Auto-calibrating Partially Parallel Acquisition"

Treats missing k-space data as a k-spatial autocorrelation problem

An interpolation kernel must be found that can synthesize missing k-space points from its sampled neighbourhood



GRAPPA (Griswold et al., MRM 2002)

- GRAPPA suffers from the same noise amplification and reduction factor SNR penalties, although it is not as obvious to derive
- Although SENSE and GRAPPA seem totally different, it can be shown that they rely on the same underlying principles, and differ largely in part from the Fourier transform in SENSE

The multiplication of sensitivity maps becomes a convolution operation in k-space, which is a basic Fourier transform identity

SENSE Practical

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