CONFORMAL MAPPINGS OF AIRFOILS

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Acknowledgments
I would like to give special thanks to my thesis advisor, Joshua Sabloff, for the outstanding guidance in conducting the research and writing this paper. Moreover, I would like to thank the Haverford Mathematics department for their time and support.

1. Introduction

The underlying motivation behind this paper is to model the forces exerted on the particles surrounding an airfoil. In particular, we will focus on a geometric transformation of an airfoil to a simpler space, where the forces on the particles surrounding an airfoil can be more easily modeled.

On Sforza page 126, airfoil sections are defined as the level set of a wing platform (see picture below) [27].

Figure 1. Airfoil

We shall focus on arguably the most important force: lift. It is lift that enables an airplane to fly. If one imagines a stream of air particles, lift acts in the direction normal to the flow. The other major force on an airfoil, drag, acts in the direction opposite the direction of the flow. Drag will not be discussed here (see page 362, [4]). In this paper, we are mostly concerned with the computation of the geometric transformation. Nevertheless, we shall introduce the physical model for lift as the larger problem. The mapping will be a crucial component of this model.

1.1. Assumptions for Model.

The overall approach to approximating lift is to define the concept of a potential function: a function whose gradient is the complex velocity field (the conjugate of the velocity field) [10]. By estimating the velocity, the force can be calculated by means of the Kutta-Joukowski

Theorem. We must make a few assumptions to ensure the potential exists and that the Kutta-Joukowski Theorem applies.

First, we shall assume that the flow of the particles surrounding an airfoil takes place in the xz plane, as opposed to modeling the flow in three dimensions. This assumption has been made elsewhere, provided the airfoil is reasonably thin (see Sforza, page 499) [27]. This will eventually enable us to use complex analysis.

If we assume that the velocity field is incompressible and irrotational, then its conjugate will satisfy the Cauchy Riemann equations and will thus be analytic. The significance of complex potential will become explained at the end of the introduction.

Niksch et al. confirm that the assumption of incompressibility is valid at realistic velocities [24]. For distances sufficiently large from the airfoil, it is also reasonable to assume irrotational flow [3]. The lack of free vorticity (irrotationality, [6]) on the exterior of the airfoil is an essential hypothesis for the Kutta-Joukowski Theorem [20]. The boundary layer is defined by Buresti, page 470, to be the boundary of the region on which the velocity field is not irrotational, and therefore outside of which the potential function exists [3].

It is also important to assume that motion is "steady:" the velocity field is constant with respect to time [3, 19]. Furthermore, we must assume an absence of net external forces and unseparated flow; separation of flow is described in detail by Chang [5]. Although zero viscosity is stated as a hypothesis for the most well-known form of the Kutta-Joukowsii Theorem, it is essential that we avoid this restriction; otherwise, the above assumptions together would imply that $\mathbf{V} = 0$ (see Buresti page 474) [3]. Fortunately, there is a form of that theorem that holds for viscid but incompressible and irrotational fluids [19].

1.2. **Description of Model.** The following section is based on an article by Li Juan [20].

As a first step towards approximating lift, one typically examines the special case of a cylindrical airfoil, thus having circular cross-section. The reason for this initial estimate will become clear in the next subsection.

Suppose the velocity field is known. To understand the computation of lift, we will need the notions of a free stream and a stream involving singularities.

A free-stream can be defined as "an incompressible two-dimensional flow with a constant density" (see Li Juan, page 1038, [20]). This type of flow is deviated from due to the presence of isolated singularities in the flow.

If the assumptions stated in the previous subsection hold (viscosity may be nonzero), then the lift can be determined from knowledge of the velocity field, the discrete set of singularities for the flow of the particles surrounding an airfoil, and the constant density of these particles. This relationship is the goal of the Kutta-Joukowski Theorem. We don't provide an explicit formula here because the equations are complicated and depend on the singularities. A thorough discussion of this theorem can be found in Li Juan [20].

1.3. **Using the Conformal Map to Approximate Lift.** If the geometry of an airfoil were simply a unit disk, the lift could be computed using the Kutta-Joukowski Theorem. However, airfoils are very different from disks, so it is necessary to first transform the airfoil to the disk. It will be our primary goal to approximate this function.

First, we will see exactly how this transformation is used. Subsequently, we will summarize an approximation algorithm for this transformation.

We describe the exact sequence of steps which can be used to approximate the lift on the particles surrounding the airfoil.

First, one should approximate the velocity on a cylindrical airfoil, as in [2]. Suppose p_z is the potential function on the disk, p_w is the potential function on the airfoil, and w is a point in the airfoil. If f is the transformation from the airfoil to the disk, then the complex potential on the airfoil is related to potential on the disk by $p_w(w) = p_z(f(w))$ [4].

Thus, we may determine the velocity field on an airfoil. This enables us to use the Kutta-Joukowski Theorem to find the lift, provided we can approximate the transformation.

There are many functions which map the airfoil onto the disk, but it is difficult to write down formulas when we don't even have an explicit representation for the airfoil as a set. The advantages of using analytic maps are that they can be approximated by their boundary values, and assuming the original complex velocity function is analytic, the only way we can hope for the composition of the transformation with complex velocity to be analytic is for the transformation itself to be analytic. (If the composition were not analytic, we would not necessarily be able to apply the Kutta-Joukowski Theorem.)

In order to carry out the transformation, we will first need to know that such a mapping exists, which we establish with the Riemann Mapping Theorem. Our algorithm will approximate the boundary of a conformal map rather than the values in the interior. (By conformal, we mean analytic with nonzero derivative.) These considerations motivate a theorem guaranteeing "nice" behavior for the boundary. In particular, we need a continuous extension to even consider approximating the conformal map on the boundary. Hence we will subsequently prove the Osgood Caratheodory Theorem. Since the mapping we want to approximate is actually from the airfoil to the unit disk, but the function we will eventually approximate maps the unit disk to the airfoil, we need our conformal map to be injective along the boundary to be able to compute the desired transformation by inverting this map.

The reason for inverting the map from the airfoil to the disk is so that we can use Poisson's formula to derive an expression for the boundary values. In particular, we shall show that the restriction of f to the unit circle can be related to the solution to Theodorsen's Integral Equation. We will subsequently define a sequence of functions to approximate the solution and prove that the sequence converges uniformly to the solution. These approximations can be rewritten as a fast Fourier transform, and can therefore be evaluated on a computer program.

2. Foundational Theorems

2.1. **Riemann Mapping Theorem (Existence).**

Theorem 2.1 (Riemann Mapping Theorem)**.** *Given a simply connected region $R \subsetneq \mathbb{C}$ and $z_0 \in \mathbb{C}$, there exists a unique holomorphic map $f : R \to \mathring{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

The following proof is partially based on Ahlfors's proof [1]. I did not use Ahlfors for lemmas 2.3-2.5.

Proof. It will help us later in the proof to assume $f(0) = 0$. By introducing translations, we can assume $z_0 = 0$. First, translate R until zero is not in the translation of R (which will allow us to use square root, see below). Also, it suffices to find a conformal map onto the disk satisfying $f'(0) \neq 0$, because there exists a rotation r such that $(r \circ f)'(0) > 0$. Define

$$\mathcal{F} = \{h : R \to D : h \textit{ is analytic and injective}, h(0) = 0, h'(0) \neq 0\}.$$

Outline : First, we show that $\mathcal{F} \neq \emptyset$. Then, we use Arzela Ascoli to show there exists $f \in \mathcal{F}$ such that $|f'(0)| = max\{|f'(0)| : f \in \mathcal{F}\}$. This is the hardest part of the proof. Finally, we show that this particular function is surjective onto the disk.

Lemma 2.2. $\mathcal{F} \neq \emptyset$.

Proof. We explicitly produce an element. To do this, we use simple connectivity and the assumption that R is not the entire plane to define the square root function on the region.

By Proposition 2.2.6 in Marsden on page 117, and definition of the complex square root, there exists an analytic function $F : R \to \mathbb{C}$ such that $(F(z))^2 = z$ [21].

The result of taking the square root is to halve the arguments. In particular, A, the image under the square root of R, will not contain a neighborhood around some point in the complex plane. It is then easy to show that there is a Mobius transformation from A to a subset of the open unit disk which has nonzero derivative. The overall transformation is a composition of taking the square root followed by the Mobius transformation. Such a function is certainly analytic and injective.

Thus, we have $\mathcal{F} \neq \emptyset$. $\square$

We now have a map with all the desired properties except possibly surjectivity. To find a surjective function in $\mathcal{F}$, we first show there is a function in $\mathcal{F}$ for which $|h'(0)|$ is maximized.

Let $S = \{|h'(0)| : h \in \mathcal{F}\}$. Let $f \in \mathcal{F}$ be arbitrary. Since $|f(z)| \leq 1$, and since 0 is a positive distance from the boundary of R, by Cauchy's Inequalities (page 150 of Marsden), S is bounded above [21].

So $sup(S)$ exists by the axiom of completeness.

We want to show $sup(S) \in S$. For that, we want equicontinuity on compact subsets of R so we may get a uniformly convergent subsequence. We stress that the subsequence needs to be the same for all compact subsets so that we may apply lemma 2.5 to get injectivity (the same subsequence converges uniformly on all compact subsets of R). We will also need to prove that the limit function is injective (lemma 2.5 and lemma 2.6).

If the functions were instead of real-variables, and we had a uniform bound on f_n', we could immediately obtain equicontinuity by applying the mean value theorem to each component. Working with functions of two variables prevents us from using this strategy directly. But there is a related result for analytic functions which relates the derivative to the complex difference quotients.

The next four lemmas are preliminary results:

Lemma 2.3 (Criterion for Equicontinuity)**.** *If a sequence of analytic functions (f_n) is uniformly bounded on a region A by some $M > 0$, then (f_n) is equicontinuous on all compact subsets.*

Proof. Let C be an arbitrary compact subset of A. The goal is to use Cauchy's Inequalities to show (f_n) is uniformly Lipschitz on S, which immediately implies equicontinuity. To do this, we shall use the relationship between the derivative and difference quotient for analytic functions.

Define the collection of open disks in C with the property that the disks centered at the same points with twice the radius are contained in A. This collection clearly covers C, so it has a finite subcover $\{D_k\}_{k=1}^m$. Let r be the minimum radius of the disks, and R the maximum. Let r_k be the radius of each disk.

For each k, let γ_k be the circle of radius $2r_k$ centered at the center of the disk D_k, denoted by $z_{0,k}$. Suppose $z \in \gamma_k$ and z_1, z_2 are such that $|z_1 - z_{0,k}|,\ |z_2 - z_{0,k}| < r_k$. By Cauchy's Integral Formula, we may write $\frac{f(z_1)-f(z_2)}{z_1-z_2} - f'(z_{0,k}) = \frac{1}{2\pi i}\int_{\gamma_k}[\frac{1}{(z-z_1)(z-z_2)} - \frac{1}{(z-z_{0,k})^2}]f(z)dz$. (See Marsden page 163) [21].

Since z is on the boundary of a disk of radius $2r_k$, and z_1, z_2 are in the interiors of the disk of radius r_k, centered at the same point $z_{0,k}$, it follows that $|f(z)[\frac{1}{(z-z_1)(z-z_2)} - \frac{1}{(z-z_{0,k})^2}]| \le |f(z) * \frac{2}{r^2}| < \frac{2M}{r^2}$.

Thus, $|\frac{f_n(z_1)-f_n(z_2)}{z_1-z_2}| < |f_n'(z_{0,k})| + |\frac{1}{2\pi i} * \frac{2\pi R*2M}{r^2}| = |f_n'(z_{0,k})| + |\frac{2MR}{r^2}|$.

By Cauchy's Inequalities, $\{f_n'(z_{0,k}) : k, n \in \mathbb{N}, k \le m\}$ is bounded by some $M_2 > 0$.

Equicontinuity easily follows. □

To get injectivity for the limit of the sequence of analytic functions we will shortly be constructing with Arzela Ascoli, we will need the following three lemmas (where the first lemma is needed only for proving the second, and the second for proving the third). See Marsden pages 203, 389, and 390 for the statements. [21]

Lemma 2.4. *Suppose (g_n) is a sequence of analytic (or harmonic) functions on a bounded region A which extends continuously onto its boundary. Suppose further that g is analytic (or harmonic) on A and continuous on its boundary, and that $g_n|_{Bd(A)} \to g|_{Bd(A)}$ uniformly. Then $g_n \to g$ uniformly* [21].

Proof. By the Maximum Modulus Principle, applied to $g_n - g$, $sup_{z\in\overline{A}}|g_n - g| = sup_{z\in Bd(A)}|g_n - g| \to 0$ as $n \to \infty$. □

Lemma 2.5 (Hurwitz's Theorem)**.** *Suppose (f_n) is a sequence of analytic functions on A which converges pointwise to f and uniformly on compact subsets, and $f_n(z) \neq 0$ for all $n \in \mathbb{N}, z \in A$. Then either 1) $f = 0$ or 2) $f(z) \neq 0$ for all $z \in A$.*

Proof. Suppose not. Then f is not constant but has a zero z_0. By the analytic convergence theorem, f is analytic on A.

Our goal is to define an auxiliary sequence of functions that, together with its limit, has the "opposite" behavior to f_n and f: the sequence will be zero at z_0 for all n, but the limit function will be nonzero at z_0. To construct such a function, we isolate the zero at z_0, use the Power Series representation for (f_n) and f, and then show convergence at z_0 by relating the values of f_n and f at z_0 to the boundary of a disk.

By the identity theorem, the zeros of f must be nowhere dense in A. There exists $R_1 > 0$ such that $f(z) \neq 0$ on $D(z_0, R_1)\backslash\{z_0\}$. By possibly finding a smaller open ball, we can choose R_1 such that $\overline{D(z_0, R_1)} \subset A$, so (f_n) converges uniformly on $D(z_0, R_1)$.

Let f have a zero of order $k > 0$ at z_0 (it cannot be infinite order, or else f would be locally constant).

Define the auxiliary sequence $(g_n) : A \to \mathbb{C}$ and $g : A \to \mathbb{C}$ by $g_n(z) = \frac{(z-z_0)^k}{f_n(z)}$ and $g(z) = \frac{(z-z_0)^k}{f(z)}$.

Then g_n and g are analytic since $\frac{1}{f(z)}$ has a pole of order k at z_0. $g_n(z_0) = 0$ for all n, but $g(z_0) \neq 0$. Thus we will have a contradiction if we show that (g_n) converges to g at z_0.

By lemma 2.4, it suffices to show uniform convergence on $Bd(D(z_0, R_1))$.

Since $f(z) \neq 0$ on the boundary, f_n is uniformly bounded below by some $m > 0$.

Because of this fact and the uniform convergence of (f_n) to f, it is easy to show that $(g_n) \to g$ uniformly on the boundary of the disk. Thus $0 = g_n(z_0) \to g(z_0) \neq 0$, a contradiction.

$\square$

Lemma 2.6. *Suppose (f_n) is a sequence of analytic functions on A which converges pointwise to f and uniformly on compact subsets, and f_n is injective for all n. Then f is either constant or injective.*

Proof. Let $z_1, z_2 \in A$ be two distinct points such that $f(z_1) = f(z_2)$.

The idea is to restrict the problem to a small neighborhood in which an auxiliary sequence of functions is nonzero.

Define $g_n(z) = f_n(z) - f_n(z_2)$ and $g(z) = f(z) - f(z_1)$. Clearly $g_n(z)$ is also a sequence of analytic injective functions; it converges uniformly to g. Moreover $g(z_1) = 0$. Now let B be a non-empty region contained in $A \setminus \{z_2\}$ such that $z_1 \in B$. Then $g_n|_B \neq 0$.

It is easy to show that uniform convergence on closed disks implies uniform convergence on compact subsets (in fact by a similar argument to lemma 2.3).

By Hurwitz's Theorem, $g|_B = 0$. By the identity theorem, $g = 0$. Thus f is constant. $\square$

We now return to the main proof that $sup(S) \in S$.

We know there exists a sequence x_n which converges to $sup(S)$. To this sequence, there is an associated sequence of functions (f_n) for which $|f_n'(0)| = x_n$.

Since $|f_n(z)| \leq 1$, (f_n) is uniformly bounded on compact subsets of R. By lemma 2.3, (f_n) is equicontinuous on all compact subsets in R.

By Arzela Ascoli, there exists a subsequence (f_{n_k}) which converges uniformly to a function f on all closed disks in R.

We claim $f \in \mathcal{F}$, which if shown will imply $|f'(0)| \in S$. Since each f_n is analytic on R, f is analytic by the analytic convergence theorem. It is injective by lemma 2.6 (since $|f'(0)| \geq |f_n'(0)| > 0$ ensures f is not constant). Since $f_n(0) = 0$ for all n, $f(0) = 0$. So $f \in \mathcal{F}$, which implies $|f'(0)| \in S$.

Let f have the maximal derivative. We must show f is surjective.

Suppose not. Let $w \in \mathring{D} \setminus f(R)$. Let f_1 be a fractional linear transformation $\mathring{D} \to \mathring{D}$ which takes w to zero. Then $(f_1 \circ f)$ is analytic and nonzero (for $w \notin f(R)$) and R is simply connected. (This is a second key use of simple connectivity; there might be a conformal map from R into the disk, but there might not be one onto the disk. Using f_1 prior to taking the square root is important because branches of analyticity for the square root can only be found for regions which exclude some path from zero to infinity.) Its being a linear fractional transformation automatically gives injectivity.)

So we can define a branch of the square root function which makes $\sqrt{f_1 \cdot f}$ analytic.

Now let f_2 be a fractional linear transformation from $\mathring{D}$ to $\mathring{D}$ such that $(f_2 \circ \sqrt{f_1 \circ f})(0) = 0$. Clearly this composition is in $\mathcal{F}$, for it is analytic and injective, and this implies nonzero derivative.

Now we note that the function $g : \mathring{D} \to \mathring{D}$ defined by $g(z) = (f_1^{-1} \circ (f_2^{-1})^2)(z)$ is analytic on all of $\mathring{D}$, and its restriction to the image $(f_2 \circ \sqrt{f_1 \circ f})(R)$ is the inverse of $(f_2 \circ \sqrt{f_1})$. Moreover, it satisfies $g(0) = 0$ by choice of functions f_1, f_2 and the fact that $f(0) = 0$. g is invertible on the aforementioned set, and this set contains zero, so $|g'(0)| > 0$. Clearly g is not a rotation; the composition includes exactly one non-Mobius transformation. By the Schwarz Lemma, we obtain $|g'(0)| < 1$. By the inverse function theorem, applied to a small neighborhood of zero, $|{g^{-1}}'(0)| = \frac{1}{|g'(0)|} > 1$. We conclude $|(f_2 \circ$

$\sqrt{f_1 \circ f})'(0)| = |(f_2 \circ \sqrt{f_1})'(f(0)) * f'(0)| > |f'(0)|$, a contradiction. $\square$

2.2. **Osgood Caratheodory Theorem.**

Theorem 2.7 (Osgood Caratheodory Theorem)**.** *If A is simply connected, f is conformal and injective on A, and both A and $f(A)$ are bounded by simple curves, then f extends to a homeomorphism from $\overline{A}$ onto $\overline{f(A)}$.*

Remark 2.8. *If f is not assumed to be analytic, then f need not extend continuously to the boundary.*

Proof of remark: We construct an "almost linear homotopy" between the graph of $sin(\frac{1}{x})$ and two lines parallel to the x-axis (almost means that the resulting function is continuous except for points such that $x = 0$). The key is that although this function will oscillate as $x \to 0$, the boundary will be continuous because of the use of the horizontal curves.

Define $f : (0, 1) \times (0, 1) \to (0, 1) \times (-2, 2)$ by

$$f(x, y) = \begin{cases} \\ (x, sin(\frac{1}{x}) + (-2 - sin(\frac{1}{x}))(1 - 2y)) & y \leq \frac{1}{2} \\ (x, sin(\frac{1}{x}) + (2 - sin(\frac{1}{x}))(2y - 1) & y \geq \frac{1}{2}) \end{cases}.$$

The function maps vertical line segments between the lines $y = 0, 1$ to vertical line segments between $y = -2, 2$ (the endpoints of the line segments come from the fact that the y-component is an almost linear homotopy between ± 2 and $sin(\frac{1}{x})$). So the image under f of $(0, 1) \times (0, 1)$ is exactly $(0, 1) \times (-2, 2)$. This means that the boundary of the image is a rectangle, and thus can be parametrized by a simple closed curve.

We check that f satisfies all the hypotheses of the Osgood Caratheodory Theorem except analyticity. The functions for $f|_{y \leq \frac{1}{2}}$ and $f|_{y \geq \frac{1}{2}}$ agree for $y = \frac{1}{2}$, so f is continuous. If $(x_1, y_1) \neq (x_2, y_2)$ but $x_1 = x_2$, we have the following two cases:

Case 1: If $y_1, y_2 \leq \frac{1}{2}$ or $y_1, y_2 \geq \frac{1}{2}$, then the factor $(1-2y)$ or $(2y-1)$ is injective, and the factor involving x is nonzero.

Case 2: Otherwise, the factors involving x in the upper and lower halves of the piecewise definition must have opposite sign, and the factors involving y are strictly positive. Thus f is injective.

Finally, due to the $sin(\frac{1}{x})$, f clearly does not extend continuously onto the boundary of the unit square.

The idea for this counterexample was to find a function which approached a nice boundary uniformly but which oscillated. As we shall see in the proof of Osgood, it is essential that the partial derivatives of f with respect to r and theta be related by the Cauchy Riemann equations. $\square$

Proof of Theorem 2.6. By the Riemann Mapping Theorem, we may assume without loss of generality that A is the open unit disk.

We first define the extension of f to S^1 and then show continuity, surjectivity, and injectivity.

For each point z_0 in the boundary of $\mathring{D}$, let (x_n) be an arbitrary sequence converging to z_0. Since f is a homeomorphism from $\mathring{D}$ onto a bounded region, a subsequence of $f(x_n)$ converges, and its limit must be contained in the boundary. (This latter fact, proved by Ahlfors [1], is crucial; it ensures that all convergent sequences actually approach the boundary in the image.) Let $f(z_0)$ be this limit. Also, let $\Gamma : [0,1] \to \mathbb{C}$ be the parametrization for the boundary of the image, so that Γ is a simple, closed curve.

Part 1 : **Continuity**:

The proof is based on McShane [22]. Fix $z_0 \in S^1$.

We shall use the existence of Lebesgue area to construct a sequence of circular arcs around the point z_0 such that the lengths of the images converge to zero. (The reason for using Lebesgue integration, rather than Riemann Integration, is that the characteristic function of the interior is not necessarily Riemann integrable. Moskowitz (corollary 4.1.35) shows that a necessary and sufficient condition for the double integral to exist is for the boundary to have "zero volume" [23]. However, this condition need not hold in general for arbitrary Jordan curves [31]). We will then use this fact to show that values near z_0 are mapped to an arbitrarily small neighborhood of $f(z_0)$.

Lemma 2.9. *$f(\mathring{D})$ has finite Lebesgue area.*

Proof. : $f(\mathring{D})$ is open and bounded, so 1 is Lebesgue measurable on $f(A)$. $\square$

Assuming area exists, we next represent the area formulaically. The main computational tools for computing Riemann double integrals are

Fubini's Theorem and the Change of Variables formula [23]. We shall use both techniques to compute the Lebesgue integral.

We shall be applying these two theorems in the region enclosed by circular arcs of the point z_0 and S^1. By using characteristic functions, we can integrate over regions which are rectangles in the polar coordinate system centered at z_0. While it is clear that both of these theorems can be applied to appropriate compact subsets of this region within the disk, it is difficult to examine the limit of arc-length integrals over a sequence of curves if the minimum distance between these curves and the boundary is vanishing. The integrands will involve the derivative of our conformal map and could easily approach infinity. Thus we shall need to apply versions of Fubini's theorem and the Change of Variables theorem which apply to open sets. See the image on the left side of figure 2.2 for intuition (ignoring the C_{R_n} and w for now).

Before we can even state the theorems, we shall need a definition. The following definition is based on Levin and Stiles page 202 [17].

Definition 2.10 (Borel regular set)**.** A set B is Borel regular if its measure is both the supremum of the measures of compact subsets of B and the infimum of the measures of open supersets of B.

We can now state without proof the following generalization of Fubini's Theorem (slightly weakened for our particular application). See Johnson page 126, theorem 5.8.

Theorem 2.11 (Fubini's Theorem)**.** *Suppose $O = X \times Y$, where X and Y are measurable, bounded, and open. Assume that a function $f : O \to \mathbb{R}$ is Lebesgue double integrable. Then $\int_X f dx$ and $\int_Y f dy$ exist almost everywhere and satisfy $\int_X (\int_Y f(x, y)dy)dx = \int\int_O f(x, y)dz = \int_Y (\int_X f(x, y)dx)dy$.*

(To see that this form of the theorem follows from theorem 5.8 in Johnson, we note that all open, bounded sets are locally compact Borel sets, and that therefore f is a Borel measurable function by lemma 3 in Johnson ([14].)

We now state a version of Change in Variables:

Theorem 2.12. *If $g : M \subset \mathbb{R}^2 \to \mathbb{R}^2$ has finite partial derivatives (having Jacobian J), is injective, and if M can be written as a countable union of sets on which g is Lipschitz, then $\int_{g(M)} 1dz = \int_M J(w)dw$.*

(See Federer, theorem 4.5 on page 448, [11].)

With these two technical results, we now return to the main proof.

Lemma 2.13. *If f is univalent, then the area of the interior, or of any region enclosed by a circular arc around the point z_0, is given by the double integral of $|f'(z)|^2$.*

Proof. It suffices to consider the case of the disk, for the proof will be the same in the case of a region bounded by a circular arc. Since f is analytic on $\mathring{D}$, $f|_{D(0,1-\frac{1}{n})}$ is Lipschitz for all n, so M is the countable union of sets on which f is Lipschitz. It is clear that the hypotheses of the change of variables theorem (theorem 2.12) hold for the function f and the set $\mathring{D}$.

We have *area of f* $= \int_{f(\mathring{D})} 1 = \int_{\mathring{D}} |u_x v_y - u_y v_x|$. By the Cauchy-Riemann Equations, this equals $\int_{\mathring{D}} |f'(z)|^2$. □

Let us represent f in polar coordinates centered at z_0. That is, let $r = |z - z_0|$ and $\theta = arg(z - z_0)$, where $0 \leq arg(z) < 2\pi$.

Then it is easy to show from the Cauchy-Riemann equations and the chain rule that $|f'(re^{i\theta})|^2 = \frac{1}{r^2}(u_\theta^2 + v_\theta^2)$.

We have *area of* $f(\mathring{D}) = \int\int_{\mathring{D}} [(\frac{\partial u}{\partial \theta})^2 + (\frac{\partial v}{\partial \theta})^2] * \frac{1}{r^2} r dr d\theta$. (This follows from the proof of Example 11 in Colley, page 360, by use of the ordinary change of variables formula [9].)

Next, we use Fubini's Theorem to integrate first with respect to θ, obtaining an expression related to length. Integrating the length with respect to r will yield the area, providing a bound on the length on circular arcs.

We set notation. Let (R_n) represent any sequence that is strictly decreasing and converging to zero (the reason for using strictly decreasing sequences will become evident near the end of the proof of continuity). By the notation $\int_{C_{R_n}}$, we mean the Lebesgue interval over the interval of theta values $(\theta_{1,n}, \theta_{2,n})$ which correspond to the arc $C_{R_n} = \{z \in \mathring{D} : |z - z_0| = R_n\}$.

We shall apply Fubini's Theorem (theorem 2.11) on the set $D(2, z_0) \supset \mathring{D}$. On this set, the boundary is a rectangle in polar coordinates centered at z_0.

By Fubini's Theorem, the Lebesgue integral $\int_{C_{R_n}} (u_\theta^2 + v_\theta^2) d\theta$, as well as the iterated integral with respect to r, exists except for null sets N_r and $N_{R_0,\theta}$, where $N_{R_0,\theta}$ is a set of points with fixed $R_0 > 0$ and values of θ ranging over $(\theta_{1,R}, \theta_{2,R})$.

Lemma 2.14. *For some strictly decreasing sequence* (R_n), $\lim_{n\to\infty} \int_{C_{R_n}} (u_\theta^2 + v_\theta^2) d\theta = 0$.

Proof. Suppose not. Then there exists $r_0 > 0$ and $\epsilon > 0$ such that for all curves C_{R_n} such that $R_n \leq r_0$, $|\int_{C_{R_n}} (u_\theta^2 + v_\theta^2) d\theta| \geq \epsilon$ (or is infinity).

The goal is to show that the area of the image is infinity. Because of the factor $\frac{1}{r}$ in the integration with respect to r, we shall derive a contradiction from assuming there are not curves whose lengths approach zero as $r \to 0$.

To obtain a function integrable on all of $D(z_0, 2)$, define $h : D(z_0, 2) \to \mathbb{C}$ by

$h(z) = \begin{cases} u_\theta^2 + v_\theta^2 & z \in \{w \in \mathring{D} : |w| \notin N_r, arg(w) \notin N_{|w|,\theta}\} \\ 0 & otherwise \end{cases}.$

For any $r \in (0, r_0]$, let $\theta_1(r)$, $\theta_2(r)$ be the endpoints of the circular arcs of z_0 with radius r. By Fubini's Theorem, if B_{r_0} is the region bounded by C_{r_0} and S^1, we have $area\ of\ f(B_{r_0}) \geq \int\int_{B_{r_0}} h(\theta)\frac{1}{r} = \int_0^{r_0}[\int_{\theta_1(r)}^{\theta_2(r)} h(\theta)d\theta] * \frac{1}{r}dr$ (we are underestimating the area of B_{r_0}, justifying the first inequality). Applying the contradiction hypothesis to the inner integral of the preceding equation, $area\ of\ f(B_{r_0}) \geq \int_0^{r_0} \frac{\epsilon}{r}\mathcal{X}_{[0,\epsilon]\setminus N_r}dr$. (The last integral exits since $\int_{\theta_{1,n}}^{\theta_{2,n}} h(\theta)d\theta \geq \epsilon$.) The constant sequence $(\frac{\epsilon}{r}\mathcal{X}_{[0,\epsilon]})$ of non-negative, integrable functions converges to $\frac{\epsilon}{r}$ almost everywhere, so by the non-negative Lebesgue convergence theorem, $\int_0^\epsilon \frac{\epsilon}{r}dr$ exists [12]. This is a contradiction. $\square$

Let (R_n) be defined by the previous lemma. By Cauchy Schwarz (for L^2), we may bound $\int_{C_{R_n}} 1 * \sqrt{u_\theta^2 + v_\theta^2}d\theta$ arbitrarily close to zero by using the same sequence that bounded $\int_{C_{R_n}}(u_\theta^2 + v_\theta^2)d\theta$ arbitrarily close to zero. Hence we have found a sequence of circular arcs centered at the point z_0 such that the lengths of the images of the arcs approach zero as n approaches infinity.

It may seem odd to the reader that we have not actually used continuity of the parametrization of the boundary in getting curves of finite length. But the existence of such curves is reasonable because not all curves approaching a point on the boundary need to have as bad behavior as the actual parametrization of the boundary might have.

The following lemma is easily verified from the previous lemma and the definition of a limit.

Lemma 2.15. *Assume $C_{R_n}(t)$ is the parametrization of C_{R_n}. Then $\lim_{t\to 0,1} f(C_{R_n}(t))$ exists.*

Thus f exhibits nice behavior along these curves, even near the boundary.

Let these limits be w_n, W_n.

Now the image of the interior of the region bounded by C_{R_n} and S^1, denoted by B_n, is the image of a connected set under the continuous function f, and is thus connected. Its closure will also be connected. And by the fact from Ahlfors, the boundary must consist of $f(C_{R_n})$ and Γ. Define $S_n = f(B_n)$.

We now obtain a bound on the distance between any two points in the regions:

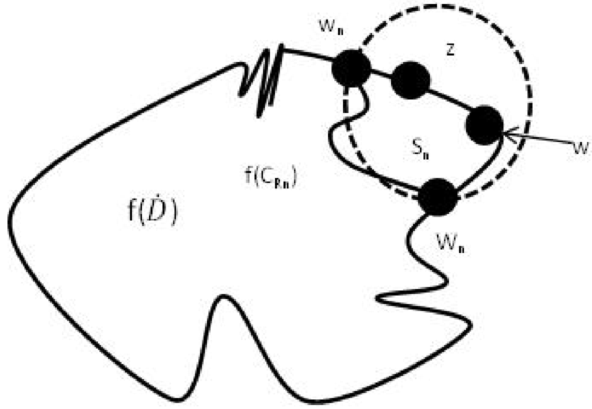

FIGURE 2. Regions Shrinking to a Point

Lemma 2.16. $\lim_{n\to\infty} \sup_{z,w\in Bd(S_n)} |z-w| = 0$.

Proof. The idea is that the images of C_{R_n} have arbitrarily small length for large n, meaning that the w_n and W_n must be close to each other, and that the fact that Γ is continuous allows us to bound the distance between any two points, z and w, on the portion of Γ between w_n and W_n.

Let us parametrize $\Gamma : [0,1] \to Bd(f(\mathring{D}))$ in such a way that $\Gamma(0) \notin \bigcup_{n\in\mathbb{N}} \overline{S_n}$ (i.e. it is out of the way and we can invert Γ elsewhere). Since Γ is an open map, if we let $t_{1,n} = \Gamma^{-1}(w_n)$ and $t_{2,n} = \Gamma^{-1}(W_n)$, we know that $|t_{1,n} - t_{2,n}| \to 0$. Since Γ is simple, $\Gamma^{-1}\{z\}, \Gamma^{-1}\{w\} \in [min\{t_{1,n}, t_{2,n}\}, max\{t_{1,n}, t_{2,n}\}]$ (the order of $w_n, z,$ *and* W_n is preserved under Γ^{-1}). Since Γ is continuous, z and w can be contained in an arbitrarily small ball by making w_n and W_n arbitrarily close. Thus $\sup_{z,w\in Bd(S_n)}$ is bounded by the sum of the length of C_{R_n} and the diameter of this ball. ☐

Note that it immediately follows that $\lim_{n\to\infty} \sup_{z,w\in\overline{S_n}} |z-w| = 0$ (since a bound on the points furthest from each other in the boundary is also a bound on the points furthest from each other in the interior).

We are ready to prove continuity at z_0 using an $\epsilon - \delta$ argument. See the picture below for intuition:

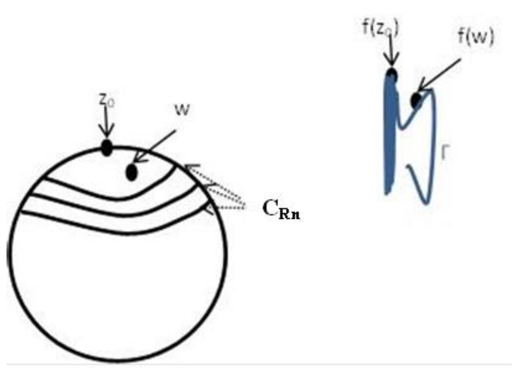

FIGURE 3. Proof of Continuity

Given $\epsilon > 0$, by lemma 2.16, there exists $N_1 \in \mathbb{N}$ such that for $n \geq N$, $\sup_{z,w \in \overline{S_n}} |z - w| < \frac{\epsilon}{2}$.

Since $(f(z_n))$ is the convergent subsequence used to define f at z_0, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, $|f(z_n) - f(z_0)| < \frac{\epsilon}{2}$.

Choose $\delta = min\{R_{N_1}, |z_{N_2} - z_0|\}$. Given $w \in B(z_0, \delta)$, since (R_n) is strictly decreasing and is the set of points in the unit disk exactly R_n from z_0, it follows that w is in the region bounded by $C_{R_{N_1}}$ and S^1, so $f(w) \in \overline{S_{N_1}}$. Thus we have $|f(z_0) - f(w)| < \epsilon$ by the triangle inequality.

We have only considered the case where $(z_i) \subset \mathring{D}$. Continuity follows from the fact that the set of limit points for a set is closed. We leave it to the reader to supply the details of the routine argument.

Part 2 of Osgood Caratheodory Proof : **Bijectivity**:

Lemma 2.17. *f is surjective.*

Proof. By definition of closure, $f(\overline{D})$ is dense in the compact set $\overline{f(D)}$. Since f is continuous, $f(\overline{D})$ is compact.

The only compact, dense subset of a compact set is the entire set. So f is surjective.

$\square$

Therefore f is continuous and surjective.

To prove injectivity, we first want to show that if the preimage of a point is non-singular, it is an arc. This fact will allow us to analytically extend f along this arc and derive a contradiction.

The framework for this part of the proof comes from McShane [12], but the details of the argument given here are significantly different.

Lemma 2.18. *If O is open, γ is a continuous curve with $t_1 < t_2$, $\gamma(t_1) \in O$ and $\gamma(t_2) \in \mathbb{C} \setminus \overline{O}$, then there exists $t_3 \in (t_1, t_2)$ such that $\gamma(t_3) \in Bd(O)$.*

Proof. Let $t_3 = \sup\{t \in (t_1, t_2) : \gamma(t) \in O\}$. Then there exists a sequence $(t_n) \subset \gamma^{-1}(O)$ with $t_n \to t_3$, and so continuity implies $\gamma(t_n) \to \gamma(t_3)$. Hence $\gamma(t_3) \in \overline{O}$. But since O is open, $\gamma(t_3) \notin O$ (since there is a decreasing sequence of points T_n converging to t_3, and $f(T_n) \notin O$), so $\gamma(t_3) \in Bd(O)$. $\square$

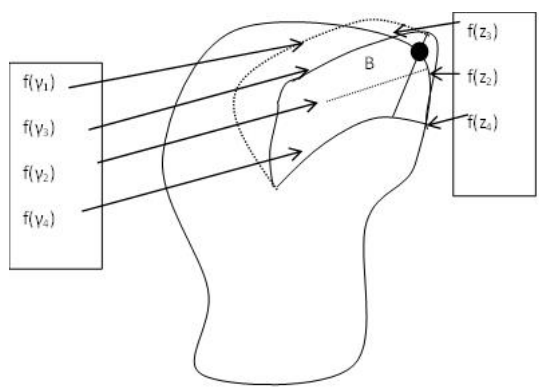

Figure 4. Proof of Preimage Connected

Lemma 2.19. *Assume the preceding notation and hypotheses. For all $z \in f(S^1)$, $f^{-1}(\{z\})$ is connected.*

Proof. Suppose not.

The idea is to use the contradiction hypothesis to find a pair of curves approaching distinct points in S^1 such that the images approach the same point on Γ. Moreover, in the image, one of the curves will be bounded away from the other by an open set; the limit point of the two curves will be on the boundary of this open set. We shall find that there will be no way for this other curve to still approach this point and simultaneously for Γ to be simple.

In view of non-path-connectivity, there exists $z_1, z_2 \in f^{-1}(\{z\})$ and $z_3, z_4 \in S^1 \setminus f^{-1}(\{z\})$ such that if γ is parametrized to start at z_1, and it intersects z_2, z_3, z_4 at t_2, t_3, t_4, then $0 < t_3 < t_2 < t_4 < 1$.

(If no such points existed, then one of the two arcs between z_1 and z_2 would be contained in $f^{-1}(\{z\})$).

For $i = 1, ..., 4$, let γ_i be the straight line from 0 to z_i.

Let B be the region bounded by $f(\gamma_3)$, $f(\gamma_4)$, and Γ (which exists by the Jordan Curve Theorem).

Since $f|_{\mathring{D}}$ is injective, $f(\gamma_1)$ does not enter B except possibly on the boundary, but $f(\gamma_2) \subset \overline{B}$.

We now hope to contradict the boundary being simple, which suggests producing points along the boundary approaching $f(z_2)$ from a different direction.

Now superimpose a sequence of straight line segments L_n from $f(\gamma_1)(1-\frac{1}{n})$ to $f(\gamma_2)(1-\frac{1}{n})$ (i.e. between pairs of points on the two paths which are approaching the same limit). We shall find that because of the distance between $f(z_2)$ and the points $f(z_3), f(z_4)$, for large n, these lines can neither approach the limit point $f(z_2)$ from the inside of the image nor the outside (because Γ is simple).

By lemma 2.18, for all n, there exists $p_n \in Bd(B)$ such that p_n lies on L_n.

Now either $p_n \in f(\mathring{D})$ or $p_n \in f(S^1)$. Eventually, however, we must have $p_n \in f(S^1)$. To see this, note that $\{f(z_2)\}$ and $S = (Bd(B) \setminus \Gamma([0,1])) \bigcup(\{f(z_3), f(z_4)\})$ are disjoint and compact, so by the distance lemma, we may define ρ to be a minimum distance. Once the length of L_n is less than ρ, L_n cannot intersect S. Thus $p_n \in \Gamma([0,1])$. This fact is critical; since this lemma does not use analyticity, we can only be proving the preimage is connected, not a single point. But the points z_3, z_4 provide a "buffer" against $f(\gamma_1)$.

Thus we have found a sequence of points p_n in the boundary of the image, but not in B, which converges to $f(z_2)$. Since we already know there is a portion of Γ in B which approaches $f(z_2)$, this contradicts the boundary being simple. $\square$

We now show f is injective.

Suppose not. Let $|f^{-1}(\{k\})| > 1$. By lemma 2.19, the preimage of k is a connected subset of S^1, and is therefore an arc σ along the unit circle.

The goal is to apply the Analytic Continuation by Continuity theorem, which says that if A,B are simply connected regions which intersect on a simple, smooth curve σ, if $A \bigcup B \bigcup \sigma((0,1))$ is open, if f and g are analytic on A and B respectively, and if both functions extend continuously to the same function on σ, then there exists an analytic function h such that $h|_A = f$ and $h|_B = g$ (see Marsden page 371) [21].

Since the domain is the disk, there exists a simply connected region Y in the outside of S^1 such that $Bd(Y) \bigcap S^1 = \sigma$.

Define $g : \overline{Y} \to \mathbb{C}$ by $g(z) = k$.

Parametrize σ so that $\sigma \in C^1$.

All the requirements of the analytic continuation by continuity theorem are clearly met. Thus f extends to $\mathring{D} \bigcup \sigma \bigcup Y$ in such a way that it agrees with g on Y.

By the identity theorem, f is constant, a contradiction. $\square$

We now have proven that there exists a conformal transformation between the airfoil and the disk, and that it is possible to extend this map continuously one-to-one onto the boundary. (Clearly, the boundary of an airfoil is a simple closed curve.) We now move on to computing it.

3. Boundary Representation for Conformal Maps

The following section is based on Kythe pages 207-208 [16] and Warschawski pages 12-13 [29].

We now derive a way of expressing the boundary of a conformal map from the unit disk to the airfoil in a form amenable to the techniques of harmonic analysis. Using Fourier series will take us closer towards an iterative procedure for approximating the restriction to the boundary of the unique conformal transformation from an airfoil to the unit disk (assuming a zero of order one at the origin and a positive derivative). In turn, analyticity will allow us to determine the conformal map from its boundary values.

In fact, we shall be writing out the transformation in the opposite direction. The reason for this is because we can represent the extension of the conformal map to the boundary using Poisson's formula, but only if the domain is the interior of the disk. Fortunately, there is a method for approximating the inverse of a conformal map (see the end of section 4).

We shall first derive Theodorsen's integral equation for a relatively general conformal map on the disk (subject to a few assumptions). In section 3, we will construct a sequence of approximations of the conformal map.

Let R be the interior of the airfoil.

Use the Riemann Mapping Theorem to find a conformal map $f : \mathring{D} \to R$ with $f(0) = 0$ and $f'(0) > 0$. (The reason for assuming $f'(0) > 0$ rather than merely $f'(0) \neq 0$ is technical; see the proof of lemma 4.10.)

We assume that the airfoil satisfies the following two definitions. Justifications for why these assumptions are reasonable are given immediately after the definitions.

Convention: Unless otherwise stated, we shall assume that an angle θ may take on any real number. The reason for this is to allow for substitutions involving integrals which may require angles outside of the range $[0, 2\pi)$.

3.1. **Definitions and Assumptions.**

Definition 3.1. A parametrized curve σ is starlike with respect to the origin if it is injective with respect to $arg(z)$. Also, a function $f : \mathring{D} \to \mathbb{C}$ with $f(0) = 0$ is starlike with respect the origin if the image contains the line segment from the origin to every other point in the image (the image is a starlike set) [28].

The assumption of the boundary of an airfoil being a starlike curve is very reasonable, as evident from the image of an airfoil in the introduction. The assumption is important because we want to represent the modulus of the conformal map on the boundary as a function of the input $arg(z)$. Once the boundary is represented in this form, we can use real analysis. Notice that whenever a region is bounded by a starlike curve, it is easy to show using the Jordan Curve Theorem that the interior is a starlike set. So the starlike curve assumption is stronger.

It trivially follows from Osgood Caratheodory, given the assumption that the parametrization of the boundary is starlike, that the curve $f(e^{i\theta})$ is also starlike. This means there exists a well-defined function $\rho : [0, 2\pi] \to [0, \infty)$ such that $\rho(arg(f(z))) = |f(z)|$ for $z \in S^1$. We now denote the extension by $\rho(\phi(\theta))e^{i\phi(\theta)}$, where $\phi : \mathbb{R} \to [0, 2\pi)$ is 2π periodic.

Note that the following definition is adapted from the definition given by Kythe pages 207-208 [16]. We have chosen to slightly change the definition to simplify the argument. In particular, we are explicit about exactly which curve on the boundary the definition will be applied to.

Definition 3.2. Given a bounded simply connected region R whose boundary is parametrized by a curve that is starlike with respect to the origin, let f be the unique conformal map $f : \mathring{D} \to R$ with a first-order zero at the origin. R is a nearly circular region if $\frac{d}{d\phi}\rho(\phi(\theta))$ exists for all ϕ and there exists $a \in (0, \infty)$ and $\epsilon \in (0, 1)$ such that for all $\theta \in \mathbb{R}$, $\frac{a}{\epsilon+1} \leq \rho(\phi(\theta)) \leq a(\epsilon + 1)$ and $|\frac{\frac{d}{d\phi}[\rho(\phi(\theta))]}{\rho(\phi(\theta))}| \leq \epsilon$ The boundary of a conformal map is a **nearly circular contour** if it satisfies these conditions.

Intuitively, this means that the boundary curve cannot oscillate too rapidly and must remain in an annulus of inner radius $\frac{a}{\epsilon+1}$ and outer radius $a(\epsilon + 1)$ (the constant a is a scaling factor). Thus, the contour cannot be extremely different from a circle.

The differentiability hypothesis is reasonable for an airfoil since even if we have a corner or cusp (such as for the unit square), the sudden change in the derivative of ρ will be a result of a rapid change in direction. Since in most cases, both ρ and $arg(f(z))$ experience the same corners and cusps, ρ will be smooth relative to $arg(f(z))$. For example, this assumption holds if the airfoil were the unit square.

The bounds do not actually hold for an airfoil. Thus it is necessary to first transform the airfoil to a nearly circular region, and only then onto the disk [16]. This assumption will be important for us mainly in the next section; it will significantly help us prove the convergence of our approximations.

We now describe the tools needed to derive a representation for our conformal map on the boundary:

Definition 3.3. Given $f \in L^2$, define the harmonic conjugate of f to be $\mathcal{C}[f] = \frac{-1}{2\pi} \int_0^{\pi} [f(x + t) - f(x - t)] cot(\frac{t}{2}) dt$, provided this integral exists.

This definition comes from Zygmund page 22 [32].

Existence of this integral will not be an obstacle for us as long as we are working with 2π periodic functions because of the following theorem from Zygmund page 76, which we state without proof:

Theorem 3.4. *If $f \in L^2(0, 2\pi)$ is 2π periodic, then $\mathcal{C}[f]$ exists almost everywhere* [32].

Since the countable union of null sets is null by the countable additivity axiom, this theorem holds if the domain $(0, 2\pi)$ is replaced by $\mathbb{R}$.

3.2. **Theodorsen's Integral Equation.**

We shall sketch the proof that conformal maps satisfying the hypotheses stated above satisfy Theodorsen's integral equation. In particular, we shall not prove Schwarz's formula and omit the details of a couple of other computations. The proof is based on Kythe pages 207-208 [16].

Theorem 3.5 (Theodorsen's Integral Equation)**.** *If f is a bijective conformal map of the unit disk whose extension to the boundary is a starlike curve with respect to the origin, $f(0) = 0$, and $f'(0) > 0$, then $f|_{S^1}$ satisfies the equation $\phi(\theta) - \theta = \mathcal{C}[log(\rho(\phi(\theta)))]$.*

Proof. The key idea behind this theorem is that we can represent the boundary values of a conformal map by its values in the interior.

We divide f by z so we can take the log. It is essential that we in fact be able to use the principal branch of the log; the reason for this will be given immediately following Schwarz's Formula (below).Thus we will need to prove the following:

Lemma 3.6. *Let $f : \overline{D} \to \mathbb{C}$ be analytic and injective on $\mathring{D}$ and continuous on S^1. Furthermore, assume that f is starlike with respect to the origin. If $f(0) = 0$ and $f'(0) > 0$, then there does not exist $z_0 \in \mathring{D}$ such that $f(z_0) = -kz_0$ for some $k > 0$.*

Proof. Suppose not. Let $z_0 \neq 0$ is such that $f(z_0) = -kz_0$.

We shall need the following definition and result, which we state without proof:

Definition 3.7. $f : \overline{D} \to \mathbb{C}$ is fully starlike if for all $r \in (0, 1)$, the curve $f(re^{i\theta})$ is a starlike curve.

See [18] for the statement.

Lemma 3.8. *If f is univalent on $\overline{D}$, and if $f|_{S^1}$ is a starlike curve with respect to the origin, then f is fully starlike* [25].

Our goal is to consider the derivative of the argument of f(z) as z traverses a circle of radius $|z_0|$. We shall see that it must be negative. This fact will later be used, in conjunction with the other hypotheses, to obtain a contradiction.

Let all angles be assigned a value in the interval $[-\pi, \pi)$ (as long as we do not use logarithms for this proof, assigning this interval is valid).

Since $\frac{f(z)}{z}$ is an open map, there exists a sequence $(z_n) \to z_0$ such that $\frac{f(z_n)}{z_n} < 0$. (The image must contain an open ball about the point $\frac{f(z_0)}{z_0}$, which includes points along the negative real axis.)

We have $\frac{\partial arg(f(z))}{\partial arg(z)}|_{z=z_0} = \lim_{n\to\infty} \frac{arg(f(z_n))-arg(f(z_0))}{arg(z_n)-arg(z_0)} = -1$. Notice that having equality, as opposed to approximate equality, was critical to being able to analyze the difference quotient, for we would otherwise we would have to consider a constant ϵ in the numerator and a denominator converging to zero.

However, since $f(z_0) \neq 0$, the same limit behavior must be observed for points along the circle $|z| = |z_0|$ approaching z_0.

This fact will, however, be contradicted. Since $\lim_{z\to 0} \frac{f(z)}{z} = f'(0) > 0$, and the function $\frac{f(z)}{z}$ is an open map, there exists z_1 with $|z_1| < |z_0|$ and $\frac{f(z_1)}{z_1} > 0$.

By symmetry and the fact that $z_1 \neq 0$, we have $\frac{\partial arg f(z)}{\partial arg(z)}|_{z=z_1} = 1$.

Since f is fully starlike, on any given circle, the derivative $\frac{\partial arg(f(z))}{\partial arg(z)}$ must be non-negative. (Otherwise, since $\frac{\partial arg(f(z))}{\partial arg(z)}$ is continuous except at zero, there would be some circle on which this partial derivative were either identically zero or negative at one point on the circle and positive at another.) This contradicts $\frac{\partial arg(f(z))}{\partial arg(z)}|_{z_0} = -1$. $\square$

We also note how the formula for the argument of log resembles the left side of the equation we are trying to derive. Thus it makes sense to consider the function $F(z) = log(\frac{f(z)}{z})$. (Note that it is easy to show with the complex analogue of L'Hopital's rule that $\frac{f(z)}{z}$ has a removable discontinuity; this applies since f is assumed to have a zero of order one.) By a corollary to Morera's Theorem, F is analytic in R.)

Then, we write F in terms of its real and imaginary components. We have $F(z) = log|\frac{f(z)}{z}| + iarg(\frac{f(z)}{z})$. Since $z \in S^1$, $|\frac{f(z)}{z}| = \rho(\phi(\theta))$. Also, $arg(F(z)) = arg(f(z)) + arg(\frac{1}{z}) = \phi(\theta) - \theta$. So $F(e^{i\theta}) = log(\rho(\phi(\theta))) + i(\phi(\theta) - \theta)$.

We then use the following result about the boundary values of a conformal map, which is proved via computation and Poisson's Formula. See Derrick page 251 for the statement [10].

Theorem 3.9 (Schwarz's Formula). *Let $f = u + iv$ be conformal on $D(0, r)$, continuous on $\overline{D(0, r)}$, and $z \in D(0, r)$.*
Then $f(re^{iarg(z)}) = iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\phi})\frac{re^{i\phi}+|z|e^{iarg(z)}}{re^{i\phi}-|z|e^{iarg(z)}}d\phi$.

All definite integrals in the rest of the thesis are assumed to be Cauchy principal values. The proof of Schwarz's Formula depends only on the ability to use Poisson's formula, By Schwarz's Theorem, if a harmonic function extends continuously to the unit circle, the Poisson integrals for $u(re^{i\theta})$ will also converge to the values $u(e^{i\theta})$ [1]. Thus we may relax the assumption that $z \in D(0,r)$, as long as we are referring to Cauchy principal values.

This formula secretly allows us to rewrite F as an integral. Fix $\theta_0 \in [-\pi, \pi)$ arbitrarily. Computation shows that $\frac{e^{i\theta}+e^{i\theta_0}}{e^{i\theta}-e^{i\theta_0}} = -icot(\frac{\theta-\theta_0}{2})$. Let $V(z) = Im(F(z))$. We now apply the Schwarz formula with $|z| = r = 1$ and $V(0) = 0$, where $V(0) = 0$ follows from the hypothesis that $f'(0) > 0 \implies V(0) = Im(log(f'(0))) = 0$. This is only true because we were able to use the principal branch.

We have $F(e^{i\theta_0}) = \frac{-i}{2\pi}\int_0^{2\pi} log(\rho(\phi(\theta)))cot(\frac{\theta-\theta_0}{2})d\theta$.

Since $\phi(\theta_0) - \theta_0 = Im(F(e^{i\theta_0}))$, we can write

$$\phi(\theta_0) - \theta_0 = \frac{-1}{2\pi}\int_0^{2\pi} log(\rho(\phi(\theta)))cot(\frac{\theta-\theta_0}{2})d\theta.$$

To complete the proof of theorem 3.5, we will use substitution.

Let $t = \theta - \theta_0$, so $dt = d\theta$.

Then computation shows that

$$\phi(\theta_0) - \theta_0 = -\frac{1}{2\pi}\int_0^{\pi}[(log(\rho(\phi(\theta_0+t))) - log(\rho(\phi(\theta_0-t))))cot(\frac{t}{2})]dt.$$

Since the right side of this equation is 2π periodic, this equation holds for any $\theta_0 \in \mathbb{R}$, and in particular, for $\theta_0 \in [0, 2\pi)$. □

4. Proof of an Algorithm to Approximate Conformal Transformations

We have just derived a representation for a conformal map on the boundary. We now show that this equation can be approximated by a sequence of iterations.

In general, the algorithm can be described as follows: start with several measurements of the values of $\phi(\theta)$ and $\rho(\phi(\theta))$, where θ ranges over $[0, 2\pi)$. Use these inputs to create a rough Fourier series approximation. Then iteratively take harmonic conjugates (see Kythe, pages 228-229, [16]). The basic relationship between a Fourier series and its conjugate is that if $f = \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k cos(kx) + b_k sin(kx)]$, then $\mathcal{C}[f] = \sum_{k=1}^{\infty}[b_k cos(kx) - a_k sin(kx)]$. If f is the boundary of a harmonic function on the disk, then it follows that $f + \mathcal{C}[f]i$ defines the extension of a power series on the disk (See Zygmund, 1, for a quick proof) [32]. The algorithm does not simply iterate between a harmonic function and its conjugate, however, because we will be adding in a $log(\rho)$ prior to taking the conjugate (exactly as is done in Theodorsen's Integral Equation). The reason for taking conjugates will be explained in the proof of the validity of this algorithm, theorem 4.4.

We now define a sequence of approximations, writing the formulas to exactly parallel Theodorsen's Integral Equation. While the approximations are defined for the boundary, we must also demonstrate that we may approximate the interior by means of the boundary values.

Definition 4.1. Assume f is nearly circular with constants a, ϵ. Define the following recursive sequence of functions $\mathbb{R} \to [0, 2\pi)$ and $S^1 \to \mathbb{C}$:

$$\phi_0(\theta) = \theta \ and\ \phi_n(\theta) = \theta + \mathcal{C}[log(\rho(\theta_{n-1}(\theta)))]$$
$$F_0(e^{i\theta}) = (log(a), \theta)$$
$$With\ \phi_{n-1},\ let\ F_n(e^{i\theta}) = log(\rho(\phi_{n-1}(\theta))) + i(\phi_{n-1}(\theta) - \theta)$$

. [29]

Note that this ρ is the same ρ which defined F; thus the terms in the sequence are related to F and we can hope for convergence. (For the rest of the paper, ρ always has this definition.)

We ultimately want to approximate f^{-1}. First, we reduce this task to the approximation of ϕ by ϕ_n as follows:

Remark 4.2. *If $\phi_n \to \phi$ uniformly, and if we have an explicit formula for ϕ_n, then we may approximate f^{-1}* [29].

It is easy to show that if (f_n) converges uniformly to f and g is uniformly continuous, $(g \circ f_n)$ converges uniformly to $(g \circ f)$. Since $0 \notin \rho([0, 2\pi])$ is compact, $log|_{\rho([0,2\pi])}$ is uniformly continuous. Thus $log(\rho(\phi_n(\theta))) \to log(\rho(\phi(\theta)))$ uniformly.

By the existence of solutions to the Dirichlet Problem on the disk (see Ahlfors, page 251) [1], the two component functions of F_n are continuous extensions of harmonic functions (u_n, v_n) on the unit disk. By lemma 2.4, $(u_n, v_n) \to F$ uniformly.

Finally, we note that f may be approximated using Poisson's formula. See the end of the section for how to approximate f^{-1}.

We also need the following technical result, which we state without proof.

Lemma 4.3. *Assume the parametrization of $f(\mathring{D})$ is rectifiable. For all n, ϕ_n' exists almost everywhere and is square-integrable. Moreover, ϕ' exists almost everywhere and is integrable* [29].

We now prove the following key theorem:

Theorem 4.4. *$\phi_n \to \phi$ uniformly.*

The proof is based on Warschawski [29].

Proof. Let $M_n = \frac{1}{2\pi}||\phi_n(\theta) - \phi(\theta)||_{L_2}$ and $M'_n = \frac{1}{2\pi}||\phi'_n(\theta) - \phi'(\theta)||_{L_2}$ (that M'_n exists will be established in lemma 4.10).

Outline: The proof consists of three steps:
(i) show that $\lim_{n\to\infty} M_n \to 0$
(ii) show that $\lim_{n\to\infty} M'_n \le M$ for a positive constant M
(iii) show that for all $n \in \mathbb{N}$, $|\phi_n(\theta) - \phi(\theta)| \le \sqrt{2\pi M_n M'_n}$, which clearly implies the conclusion.

We need the following crucial result. Only the main ideas for proving the following theorem will be explained here; the parts omitted are computations.

Lemma 4.5 (Conjugacy Theorem)**.** *If $f \in L^2(0, 2\pi)$ and is 2π periodic, then $\frac{1}{2\pi}\int_0^{2\pi} f^2 = (\frac{1}{2\pi}\int_0^{2\pi} f)^2 + \frac{1}{2\pi}\int_0^{2\pi}(\mathcal{C}[f])^2$.*

Zygmund (pages 20-21) proves this fact by showing that the Fourier series of the conjugate is the conjugate Fourier series, and then noting that the conclusion immediately follows from Parseval's relation. The partial sums for the Fourier series for f and $\overline{f}$ are simplified using well-known formulas for $\sum^n cos(kx)$ and $\sum^n sin(kx)$. Suppose $\mathcal{S}[f]$ represents the Fourier series for f. Then it is shown that $\mathcal{S}[f] - f$ and $\mathcal{S}[\overline{f}] - \overline{\mathcal{S}[f]}$ are Fourier coefficients for some integrable function. This means that the coefficients converge to zero in the limit; hence $\mathcal{S}[\overline{f}] \to \overline{\mathcal{S}[f]}$. [32]

Note that this lemma is the reason why we wanted to write Theodorsen's Integral Equation as a Cauchy principal Value involving cotangent. We use this lemma along with the nearly circular hypothesis to derive a mechanism by which we may define bounds on functions related to each other by conjugation. This mechanism is a two-step process: relate a bound on $||\phi_{n-1}(\theta) - \phi(\theta)||_{L_2}$ to a bound on $||log(\rho(\phi_{n-1}(\theta))) - log(\rho(\phi(\theta)))||_{L_2}$ using the second nearly circular hypothesis, and then relate that to a bound on $||\phi_n(\theta) - \phi(\theta)||_{L_2}$ by means of the conjugacy theorem.

Lemma 4.6. *Suppose $G, H : \mathbb{R} \to [0, 2\pi)$ are square integrable with $(G - H)$ 2π periodic. Let $h \in L^2[\mathbb{R}]$, assume $\phi - h$ is 2π periodic, and assume $G - H$ is the harmonic conjugate for $log(\rho(\phi)) - log(\rho(h))$. Then*

(1) $\frac{1}{2\pi}\int_0^{2\pi}(G(\theta)-H(\theta))^2 d\theta \le \epsilon^2$ *if* $h(\theta) = log(a)$ *and (2)* $\frac{1}{2\pi}\int_0^{2\pi}(G(\theta)-H(\theta))^2 d\theta \le \frac{\epsilon^2}{2\pi}\int_0^{2\pi}(\phi(\theta)-h(\theta))^2 d\theta.$

Proof. This proof combines lemma 4.5 and the nearly circular hypothesis into a compact formula. By the conjugacy theorem (lemma 4.5), as applied to the function $\overline{f} = G - H$,

$$\frac{1}{2\pi}\int_0^{2\pi}(G(\theta)-H(\theta))^2 = \frac{-1}{2\pi}[\int_0^{2\pi} G(\theta)-H(\theta)d\theta]^2 + \frac{1}{2\pi}\int_0^{2\pi}(log(\rho(\phi(\theta))) - log(\rho(h(\theta))))^2$$

$\le \frac{1}{2\pi}\int_0^{2\pi}(log(\rho(\phi(\theta))) - log(\rho(h(\theta))))^2$.

We are using 2π periodicity of $G-H$ and $\phi - h$ to invoke this theorem.

Note that the first nearly circular hypothesis and $Re(F_0(e^{i\theta})) = log(a)$ implies $\frac{1}{2\pi}\int_0^{2\pi}(log(\frac{\rho(\phi(\theta))}{\rho(h(\theta))}))^2 \le \frac{1}{2\pi}\int_0^{2\pi} log^2(\epsilon+1)$. It is easy to show by calculus that for $x > 0$, $log(x+1) \le x$. The first assertion follows.

We observe that $\frac{\rho'(\phi(\theta))}{\rho(\phi(\theta))} = \frac{d}{d\phi(\theta)}[log(\rho(\phi(\theta)))]$.

By the Mean Value Theorem, the bound on $\frac{d}{d\phi(\theta)}[log(\rho(\phi(\theta)))]$ implies the same bound, ϵ, on the difference quotient $|\frac{log(\rho(\phi(\theta)))-log(\rho(h(\theta)))}{\phi(\theta)-h(\theta)}|$.

We see how important it was to assume differentiability at all values ϕ.

Thus $\frac{1}{2\pi}\int_0^{2\pi}(G(\theta)-H(\theta))^2 \le \frac{\epsilon^2}{2\pi}\int_0^{2\pi}(\phi(\theta)-h(\theta))^2$. □

In fact, the proof of part one is to apply lemma 4.6 in an induction. The first half will imply the base case, while the second will imply the inductive step. The second part requires first obtaining the bound $\frac{1}{2\pi}||\phi'(\theta)||_{L_2} \le \frac{1}{1-\epsilon^2}$ and then applying lemma 4.5 to the derivatives. This is the most difficult stage. Part three is shown by the Cauchy Schwarz inequality and integration by parts (which is justified by absolute continuity).

It follows immediately from Theodorsen's integral equation that $\phi_n - \phi$ is 2π periodic for all n. By lemma 4.6 (which applies because of the periodicity of $\phi_n - \phi$, it is trivial to prove by induction that $M_n \le \epsilon^{n+1}$.

Estimating $\mathbf{M'_n}$:

Theorem 4.7. $M_n'^2 \le \frac{4\epsilon^2}{1-\epsilon^2}$.

Proof. To prove this theorem, we will first use the conjugacy theorem. Next, we will break up the term $(\phi'_{n-1}(\theta)-\phi'(\theta))^2$ into its two terms via the triangle inequality. The term $||\phi'(\theta)||_{L_2}$ is bounded by lemma 4.10 below. In particular, this bound is stronger than merely an arbitrary

M we obtain from square-integrability; without this tighter bound, the recurrence relation on M_n' might not guarantee a uniform bound. The second term will be bounded by lemma 4.6 techniques.

The key to being able to bound $\phi'(\theta)$ is to represent it by its values on the interior, using Poisson's formula. This is justified in the following lemma, which we show to be a consequence of results proven by Katznelson. Unlike on the boundary, we are a-priori guaranteed square-integrability on the interior.

Definition 4.8 (Hardy Space)**.** An analytic function $g : \mathring{D} \to \mathbb{C}$ is in the hardy space H^n if $sup_{r\in(0,1)}||g||_{L_n} < \infty$ [15].

Lemma 4.9. *Provided that the parametrization of the boundary is rectifiable, $f'(re^{i\theta})$ admits a Poisson integral representation in terms of its boundary values $f'(e^{i\theta})$. Furthermore, $f'(re^{i\theta}) \to f'(e^{i\theta})$ almost everywhere.*

Proof. : By lemma 4.3, f' is integrable on the boundary. By a corollary given on page 87 of Katznelson, $f' \in H^1$ [15]. By theorems 3.8 and 3.11 of Katznelson, $f'(re^{i\theta})$ is the Poisson integral for its boundary values, and it converges almost everywhere [15]. □

Lemma 4.10. $\frac{1}{2\pi}\int_0^{2\pi}\phi'^2(\theta)d\theta \leq \frac{1}{1-\epsilon^2}$.

Proof. Consider the function $u + iv = \frac{dF[(e^{i\theta})]}{d\theta} - i = \phi'(\theta)(\frac{\rho'}{\rho}[\phi(\theta)] + i)$.

We shall apply the previous lemma to the function $\frac{dF}{d\theta}$.

Since the boundary is nearly circular, and since $\phi'(\theta) > 0$, we have $v \pm u \geq \phi'(\theta)\, min\{1+\epsilon, 1-\epsilon\} = \phi'(\theta)(1-\epsilon) \geq 0$. Therefore, $v^2 - u^2 \geq 0$. This is significant because we will eventually be using Fatou's lemma, which only applies if the integrand is non-negative.

We claim the same bound holds on $\mathring{D}$. This follows from taking the real and imaginary components of Poisson's formula:

$u(re^{i\theta}) + iv(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}[u(e^{it}) + iv(e^{it})]\frac{1-r^2}{1+r^2-2rcos(t-\theta)}dt$

The factor $\frac{1-r^2}{1+r^2-2rcos(t-\theta)}$ is strictly positive for $r < 1$ since the minimum of $r^2 - 2r$ is -1 and occurs at $r = 1$. Poisson's formula is valid since we may choose $arg(\frac{f(z)}{z})$ to lie in $[-\pi, \pi)$ and have observed in the previous section that the argument of this function is not $-\pi$.

Claim 4.11. $\frac{1}{2\pi}\int_0^{2\pi}[v^2(re^{i\theta}) - u^2(re^{i\theta})]d\theta = 1.$

Proof. Let $V(z) = Im(F(z))$.

Then $F|_{\mathring{D}} \in L^2$. Moreover, along any circle centered at the origin with radius $0 < r < 1$, by the fundamental theorem of calculus and the fact that $V(re^{i\theta})$ is continuous in θ, we have $0 = V(\pi) - V(-\pi) = \int_{-\pi}^{\pi} V_\theta d\theta = \int_0^{2\pi} V_\theta d\theta$

Hence we may apply the conjugacy theorem to any integral along such a circle to the function V_θ.

We have $\int_0^{2\pi} V_\theta^2 = \int_0^{2\pi} u_\theta^2 + \frac{1}{2\pi}[\int_0^{2\pi} V_\theta]^2$.

Recall that $v_\theta = Im(\frac{dF}{d\theta}) = V_\theta - 1$.

Thus $-2\pi + \int_0^{2\pi}(v^2 - u^2) = -2\pi + \int_0^{2\pi}(V_\theta - 1)^2 - \int_0^{2\pi} u_\theta^2 = \int_0^{2\pi} V_\theta^2 - \int_0^{2\pi} u_\theta^2 - 2\int_0^{2\pi} V_\theta = \frac{1}{2\pi}(\int_0^{2\pi} V_\theta)^2 - 2\int_0^{2\pi} V_\theta = 0$ (both integrals are zero in the final expression). $\square$

We want to apply Fatou's lemma, and for that we want a sequence of functions of real variables.

Define $(I_n) : [-\pi, \pi) \to \mathbb{R}$ by $I_n(\theta) = v^2((1-\frac{1}{n})e^{i\theta}) - u^2((1-\frac{1}{n})e^{i\theta})$.

Then $I_n \geq 0$ and $I_n \to f(e^{i\theta})$ almost everywhere.

By Fatou's lemma [12], $\phi' \in L^2$ and we have $\int_0^{2\pi}(v^2(e^{i\theta}) - u^2(e^{i\theta}))d\theta = \lim_{n\to\infty} \int_0^{2\pi} I_n(\theta)d\theta \leq \liminf(\int_0^{2\pi}(v^2(re^{i\theta}) - u^2(re^{i\theta}))d\theta) = 1$.

Thus $\frac{1}{2\pi}\int_0^{2\pi} \phi'^2(\theta) * [1 - \frac{\rho'}{\rho}[\phi(\theta)]]^2 d\theta \leq 1$.

Since the boundary is nearly circular, $\frac{\rho'}{\rho}[\phi\theta] < \epsilon$, which implies the conclusion.

$\square$

For the moment, assume n is fixed.

We will transform M_n' into an expression involving ϕ' and ϕ_{n-1}', obtaining separate bounds for each. The bound on the former term will come from lemma 4.10, while we will prove a bound on the latter via induction.

Since the boundary is a closed curve, we have $F(0) = F(2\pi)$, so that $\int_0^{2\pi} \frac{dF}{d\theta} d\theta = \int_0^{2\pi} \frac{dF_n}{d\theta} d\theta = 0$.

We cannot use lemma 4.6 directly since we are working with ϕ' rather than ϕ, but we repeat the same scheme.

By the conjugacy theorem, with $f(\theta) = \phi_n'(\theta) - \phi'(\theta)$,

$\frac{1}{2\pi}\int_0^{2\pi}[\phi_n'(\theta) - \phi'(\theta)]^2 \leq \frac{1}{2\pi}(\int_0^{2\pi}[\frac{\rho'(\phi_n(\theta))}{\rho(\phi_{n-1}(\theta))} * \phi_{n-1}'(\theta) - [\frac{\rho'(\phi(\theta))}{\rho(\phi(\theta))} * \phi'(\theta)]]^2 d\theta$

$\leq \frac{1}{2\pi}(\int_0^{2\pi}\{[\frac{\rho'(\phi_n(\theta))}{\rho(\phi_{n-1}(\theta))} * \phi_{n-1}'(\theta)]^2 + [\frac{\rho'(\phi(\theta))}{\rho(\phi(\theta))} * \phi'(\theta)]^2\}d\theta$.

Since f is nearly circular, this is at most $\frac{\epsilon^2}{\pi}\int_0^{2\pi}([\phi_{n-1}'(\theta)]^2 + [\phi'(\theta)]^2)d\theta$

$\leq \frac{\epsilon^2}{\pi}\int_0^{2\pi}[\phi_{n-1}'(\theta)]^2 d\theta + \frac{2\epsilon^2}{(1-\epsilon^2)}$ (by lemma 4.10).

We next claim that for all n, $\frac{1}{2\pi}\int_0^{2\pi}(\phi_n'(\theta))^2 d\theta \leq \frac{1}{1-\epsilon^2}$.

Again, we use induction on $n+1$.

Base case: If $n = 0$, $\frac{1}{2\pi}\int_0^{2\pi}(\phi_n'(\theta))^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} 1 d\theta = 1$.

Inductive step: Suppose $\frac{1}{2\pi}\int_0^{2\pi}(\phi'_{n-1}(\theta))^2 d\theta \leq \frac{1}{1-\epsilon^2}$.

The two components of $\frac{dF_n}{d\theta}$ are $\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho'(\phi_{n-1}(\theta))\phi'_{n-1}(\theta)}{\rho(\phi_{n-1}(\theta))}$ and $\phi'_n(\theta)-1$. Thus the harmonic conjugate of $log(\rho(\phi'_{n-1}(\theta)))$ is $\phi'_n(\theta)-1$.

We bound ϕ'_n in terms of ϕ'_{n-1} and then use induction. The strategy is the usual combination of the conjugacy theorem and the nearly circular hypothesis to obtain a recursive bound on $||\phi'_n||_{L_2}$.

It is easy to show based on the conjugacy theorem and the nearly circular hypothesis that

$\frac{1}{2\pi}\int_0^{2\pi}((\phi'_n(\theta)-1))^2 d\theta \leq \frac{\epsilon^2}{2\pi}\int_0^{2\pi}(\phi'_{n-1}(\theta))^2 d\theta$.

To simplify the left side of inequality (2), we note $\int_0^{2\pi}\phi'_n(\theta)d\theta = 2\pi$ since the absolute continuity of ϕ, the 2π periodicity of $\phi_n - \phi$, and the fact that $f(e^{i\theta})$ is a starlike curve implies $\int_0^{2\pi}\phi'_n(\theta)d\theta = \phi_n(2\pi) - \phi_n(0) = [(\phi_n(2\pi)-\phi(2\pi)) - (\phi_n(0)-\phi(0))] + (\phi(2\pi)-\phi(0)) = 0 + 2\pi$.

So $\frac{1}{2\pi}\int_0^{2\pi}(\phi'_n(\theta)-1)^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}\phi'^2_n(\theta)d\theta - 2 + 1 \leq \frac{\epsilon^2}{2\pi}\int_0^{2\pi}(\phi'_{n-1}(\theta))^2 d\theta$.

Thus $\frac{1}{2\pi}\int_0^{2\pi}(\phi'_n(\theta))^2 d\theta \leq 1 + \frac{\epsilon^2}{1-\epsilon^2} = \frac{1}{1-\epsilon^2}$.

This completes the induction.

Therefore, $\frac{\epsilon^2}{\pi}\int_0^{2\pi}\phi'^2_{n-1} \leq \frac{2\epsilon^2}{1-\epsilon^2}$, which implies $M'^2_n \leq \frac{4\epsilon^2}{1-\epsilon^2}$.

□

Relate $|\phi_{\mathbf{n}}(\theta) - \phi(\theta)|$ to $\mathbf{M_n}$ and $\mathbf{M'_n}$:

Lemma 4.12. $|\phi_n(\theta) - \phi(\theta)| \leq \sqrt{2\pi M_n M'_n}$ *(1).*

Proof. The general scheme is to use Cauchy-Schwarz to simplify the right-hand side of the equation.

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(\theta) = \phi_n(\theta) - \phi(\theta)$.

Since $\int_0^{2\pi} g(\theta)d\theta = 0$, and the integral of a continuous, strictly positive or negative function is nonzero, $g(\theta_0) = 0$ for some $\theta_0 \in [0, 2\pi]$.

Fix this value of $\theta_0 \in [0, 2\pi]$ and $n \in \mathbb{N}$.

Then $2\pi M_n M'_n = \sqrt{\int_0^{2\pi} g^2(t)dt * \int_0^{2\pi} g'^2(t)dt}$

which by Cauchy-Schwarz is greater than or equal to $\int_0^{2\pi}|g(t) * g'(t)|dt$. (2)

Suppose $\theta \in [0, 2\pi]$ is arbitrary. We now manipulate the left side of inequality (1) which we are trying to establish; given the form of the right side that we have just derived, as an integral, it makes sense to use integration by parts.

Note that integration by parts is valid by absolute continuity of g. Thus

$$g^2(\theta) = g^2(\theta) - g^2(\theta_0) = 2\int_{\theta_0}^{\theta} g(t)g'(t)dt \text{ (3).}$$

Also, considering the left side of this equation, it follows that $\int_{\theta-2\pi}^{\theta} g(t)g'(t)dt = 0$ since g is 2π periodic.

We get $0 = -\int_{\theta_0}^{\theta} g(t)g'(t)dt + \int_{\theta_0}^{\theta-2\pi} g(t)g'(t)dt$, so that the two integrals are equal.

Thus $g^2(\theta) - g^2(\theta_0) = 2\int_{\theta_0}^{\theta-2\pi} g(t)g'(t)dt$
$\leq \int_{\theta_0}^{\theta} |g(t)g'(t)|dt + \int_{\theta_0}^{\theta-2\pi} |g(t)g'(t)|dt$
$= \int_{\theta}^{\theta-2\pi} |g(t)g'(t)|dt = \int_{0}^{2\pi} |g(t)g'(t)|dt$ (the absolute value of a 2π periodic function is 2π periodic).

Combining this with (2) and (3), we have $g(\theta) \leq \int_0^{2\pi} |g(t)g'(t)|dt \leq \sqrt{2M_n M_n'}$.

□

Thus $|\phi_n(\theta) - \phi(\theta)| \leq \sqrt{2\pi \frac{2\epsilon}{1-\epsilon^2}\epsilon^{n+1}}$.

The right side converges to zero. So $\phi_n \to \phi$ uniformly.

□

Theorem 4.13. *Theodorsen's Integral Equation has a unique solution that is continuous and 2π periodic and whose image is the boundary $f(D)$* [29].

Proof. Let ϕ_1 be the solution derived from the Riemann Mapping Theorem, and ϕ_2 be any continuous, 2π periodic solution.

We want to apply lemma 4.6. Set $H = h = \phi_2(\theta)$ and $G = \phi_1(\theta)$. The harmonic conjugate of $G - H = \phi_1(\theta) - \phi_2(\theta)$ is $log(\rho(\phi_1(\theta))) - log(\rho(\phi_2(\theta)))$ by Theodorsen's Integral Equation. Therefore, lemma 4.6 applies and we have $||\phi_1(\theta) - \phi_2(\theta)||_{L_2} \leq \epsilon^2 ||\phi_1(\theta) - \phi_2(\theta)||_{L_2} = \epsilon^2 ||G(z)||_{L_2}$. The only solution to the equation $x \leq kx$ with $k \in (0, 1)$ is $x = 0$. Since $|| \; ||_{L_2}$ is positive definite, $\phi_1 = \phi_2$. □

4.1. Inverting and Generalizing Approximations for Exterior Rather than Interior.

We shall be content with briefly summarizing the techniques for dealing with these two issues rather than rigorously justifying them.

A) Interior vs. Exterior

Since the ultimate goal is to study the airfoil particles outside the boundary layer, the desired mapping should be a mapping between the exterior of the disk and the exterior of the airfoil.

Suppose f maps the exterior of the disk conformally onto the exterior of the airfoil, provided we assume that the conformal map f from the exterior of the disk onto the exterior of the airfoil satisfies $\lim_{z\to\infty} \frac{f(z)}{z} = 1$. By making coordinate transforms, one can derive an analogous equation to Theodorsen's integral equation. See Kythe pages 226-227 [16]

B) Inversion

This can be approximated via Schwarz-Christoffel transformations [8].

5. Implementing the Algorithm

The algorithm is based on Kythe, pages 228-229 [16], but the implementation is my own.

I have implemented a program which approximates a conformal map using fast Fourier transforms in such a way that the iterations have the form of the sequence of functions constructed in the previous section. Because of time limitations, instead of allowing the program to be given a simply connected region and actually using the computer to approximate the map, my code requires that it be given a conformal map on the disk satisfying the assumptions laid out earlier in the thesis. In order to actually compute the mapping without knowing it ahead of time, one would have to approximate $\rho(\phi)$, which has usually been done using the Fourier coefficients found in each iteration (see Kythe page 281, [16]). As a result, the convergence depends on whether the functions approximated in the ith iteration are starlike curves with respect to the origin. If even one function F_n was not a starlike curve, $\rho(\phi_n)$ would be multi-valued. Warschawski (page 16) establishes stronger conditions which guarantee that the iterations are starlike curves [29], but the hypotheses are more restrictive, and it would be difficult to confirm that they are actually met for any particular example without knowing the conformal map.

The conformal map is the only input required for the approximation. My program uses the function to estimate the values at single points. Once the Fourier coefficients are found, they are used to generate the approximation of the given function.

The key computations are performed in the function "compute." The program initially sets $\phi_k = \theta_k$. For each iteration, the computer first updates the auxiliary variables ω by evaluating $log(\rho(\phi(\theta)))$. To compute this, the given function is used to approximate the inverse of the angle ϕ under the function $\phi(\theta)$. Subsequently, the Fourier coefficients are updated by an approximation for the inverse Fourier transform (approximation of an integral by a lower sum). It can be shown that the coefficients of any complex-valued Fourier polynomial $P(t)$ are of the form $a_n = \frac{1}{2\pi}\int_0^{2\pi} P(t)e^{-int}dt$; see Katznelson page 2 [15]. This explains the equations used below to update the coefficients.

The computer then uses the Fourier coefficients to take the conjugate series to ω. This process repeats until sufficiently accurate results are obtained. To test the accuracy of the algorithm, my code evaluates the function obtained by substituting the evaluations of the Fourier coefficients. This approximation function is evaluated at equally spaced test points (see the analysis of the results for why the points are equally spaced). The results are compared against the evaluation of the given conformal map at these points. My code therefore attempts to verify the algorithm by simulation. These iterations will be shown to have the correct limit.

We now write out the formulas for the iterations. Suppose we are given a function, a number M of iterations to perform, and a number 2N of equally spaced points on the boundary of the disk to use. We use three indices for our variables: v is the number of iterations, k is the index of the value θ we are estimating the function at currently, and n indexes the Fourier coefficients we are computing. The values that these indices can have are $v \in \mathbb{Z} \bigcap [0, M]$, $k \in \mathbb{Z} \bigcap [0, 2N]$, and $n \in \mathbb{Z} \bigcap [0, N]$.

Let $\theta_k = \frac{\theta \pi}{N}$.

Base case:

$\phi(\theta_k) = \theta_k$ (exactly as ϕ_n was defined in theorem 4.4).

$a = \frac{\rho(\theta_k)}{2N}$ (use the average magnitude of the sample points as an estimate for the value a)

Recursion:

$\omega_k^{(v)} = log(\rho(\phi^{(v)}(\theta_k)))$

$\alpha_n^{(v)} = \frac{1}{N} \sum_{n=1}^{2N-1} \omega_n^{(v)} sin(n\theta_k)$,

$\beta_n^{(v)} = \begin{cases} \frac{1}{N} \sum_{n=1}^{2N-1} \omega_n^{(v)} cos(n\theta_k)) & n < N \\ 0 & n = N \end{cases}$

$\phi_k^{(v+1)} - \theta_k = \sum_{n=1}^{N} (\alpha_n^{(v)} sin(n\theta_k) - \beta_n^{(v)} cos(n\theta_k)).$

At the end of the iterations, define the approximation to F

$G_N^{(v)}(z) = \frac{\alpha_0^v}{2} + \sum_{n=1}^{N-1} (\alpha_n^v - i\beta_n^v) z^n + \alpha_N^v z^N$

Let us now verify that this procedure approximates the iterations defined in definition 4.1 (limited only by the number of equally spaced points used and the validity of our approximation for a). First, we define a sequence of interpolations to ϕ_v by using the above algorithm with increasing N and estimating the values at all other points in $[0, 2\pi)$. We will then show the resulting sequence of functions converges pointwise to the iterations in definition 4.1. We will also justify the equation for $G_N^{(v)}$.

Define $S_N = \{\frac{\pi k}{2^N} : k = 0, ..., 4^N - 1\}$.

For any $v, N \in \mathbb{N}$, define the step function $\psi_N^{(v)} : [0, 2\pi) \to \mathbb{R}$ by

$$\psi_N^{(v)}(\theta) = \phi^{(v)}(max\{\theta_0 \le \theta : \theta_0 \in S_N\})$$

.

Theorem 5.1. *For any* $v \in \mathbb{N}\bigcup\{0\}$*,* $\lim_{N\to\infty} \psi_N^{(v)} = \phi_v(\theta)$ *pointwise. Also,* $\lim_{v\to\infty}[\lim_{N\to\infty} G_N^{(v)}(e^{i\theta})] = F(e^{i\theta})$*.*

Proof. Suppose we are given $\theta \in [0, 2\pi)$. Since ϕ_v is continuous, and $\bigcup_{N\in\mathbb{N}} S_N$ is dense in $[0, 2\pi)$, we may assume without loss of generality that $\theta \in \bigcup_{N\in\mathbb{N}} S_N$. Suppose $\theta \in S_N$ for $N \ge N_0$ (this is why we are only hoping for pointwise convergence). To simplify the notation, reindex so that $N_0 = 1$.

To establish the first claim for all v, use induction on $v - 1$.

Base case: if $v = 0$, then by definition, $\psi_N^{(0)}(\theta) = \theta = \phi_0(\theta)$ for $N \ge N_0$.

Inductive step:

Suppose that $\lim_{N\to\infty} \psi_N^{(v)}(\theta) = \phi_v(\theta)$.

The goal is to show that $\lim_{N\to\infty} \psi_N^{(v+1)}$ satisfies the recurrence relation for ϕ_n in definition 4.1. To achieve this goal, we will exploit the close connection between conjugation of Fourier series and conjugate functions to compare the algorithm in this section with the sequence from the previous section.

Let $k(N)$ be defined so that for all N, $\theta_{k(N)} = \theta$. As we have remarked before, $\lim_{N\to\infty} \alpha_n^{(v)}$ and $\lim_{N\to\infty} \beta_n^{(v)}$ are Fourier coefficients for $\lim_{N\to\infty} \omega_{k(N)}$. (We are using lower sums to approximate an integral.)

As a result, we see from the proof of the Conjugacy Theorem in Zygmund that in the limit as $N \to \infty$, taking the conjugate series is the same as taking the harmonic conjugate of the original Fourier series. That is, $\lim_{N\to\infty} \psi_N^{(v+1)}(\theta) - \theta = \lim_{N\to\infty} \overline{\omega_{k(N)}^{(v)}} = \lim_{N\to\infty} \overline{log(\rho(\psi_N^{(v)}))}(\theta)$. By induction and the continuity of $\overline{log(\rho)}$, this equals $\overline{log(\rho(\phi_v(\theta)))}$. By definition 4.1, this equals $\phi_{v+1}(\theta) - \theta$. This proves the first assertion.

By the result just shown and the convergence of F_n to F, $Re[F(e^{i\theta})] = log(\rho(\phi(\theta))) = \lim_{v\to\infty} log(\rho(\phi_v(\theta))) = \lim_{v\to\infty}[\lim_{N\to\infty} \omega_{k(N)}^{(v)}]$. The formulas for $\alpha_n^{(v)}$ and $\beta_n^{(v)}$ defined above apply to the Fourier coefficients for F if we replace the approximation of the integral with the actual integral (i.e. if we let N approach infinity). Moreover, by Katznelson page 13, Fourier series are unique if they converge [15]. Combining these three observations, it follows that the Fourier coefficients $\alpha_n^{(v)}, \beta_n^{(v)}$ will converge to the coefficients for F. If we consider the Fourier series for F, since F is the extension of an analytic function to the boundary, it has the form $\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k - ib_k)z^k$ [32]. Even though we don't know this is true for F_v, we know from the convergence of the Fourier coefficients that the partial sums involving negative powers of e must converge to zero, so we might as well let them be zero for $G_N^{(v)}$. We still have $\lim_{v\to\infty}[lim_{N\to\infty}G_N^{(v)}] = F$. □

6. Results of Computer Simulation

I have implemented the above algorithm using the functions $-i * (log(z-3i) - log(-3i))$, $sin(z)$, $tan^{-1}(0.5 * z)$, and $\frac{1}{z+3i} - \frac{1}{3i}$, using at least 20 points and iterations and accuracy of 10^{-5} for "fast_approximate_theta." The contour plot of sin(z) is shown below, along with the values of six equally spaced points under the approximation function.

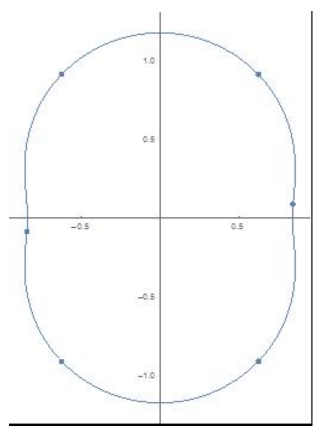

Figure 5. Parametric Plot of Sin(z) and Sample Points

[30]

The reason for these choices of functions is that they are analytic on the disk and appear to be reasonably close to satisfying the starlike and nearly circular hypotheses. In the case of the function involving log, I have tested the value of $|\frac{f'(z)}{f(z)}| \geq \frac{\rho'(\theta)}{\rho(\theta)}$ at several points and found it to be significantly less than one. (The intuition behind this function is that if we replace 3i with ni, for large n, and compute $|\frac{f'(z)}{f(z)}|$, we see algebraically that it must be nearly circular for sufficiently large n.) If the functions were not starlike curves, the function "fast_approximate_theta," which depends on the function being starlike to approximate the inverse of the function $\theta(\phi)$ using binary search, should fail to find an inverse. Note that in all four cases, we have made sure that $f(0) = 0$ and $f'(0) > 0$; the algorithm is only valid for functions satisfying these conditions.

In all four cases, the percent error between the approximation of the value of the function and the actual value is less than .01%. The errors can be decreased to some extent by improving the number of equally-spaced points. Moreover, the convergence is exponentially fast, as proven in theorem 4.4.

Despite being a remarkably accurate approximation for the boundary of our analytic map, Theodorsen's method has limitations. First, we have had to impose strong assumptions on the mapping function. We assumed it was nearly circular, and that its boundary is starlike with respect to the origin. It is clear that the boundary layer is starlike, but it is debatable whether the airfoil is nearly circular. Kythe describes a method for first transforming the airfoil onto a nearly circular region, and only then using Theodorsen's method (see page 284, [16]).

Moreover, if the number of points (2N) is too large, the accuracy of the approximations will decrease. This is because if the values θ_k are too close together, the order of the approximations of the values ϕ_k might be out of order. As a result, the approximations will actually get worse as v increases and converge to the incorrect values [13]. Notice in the above image that the six equally spaced points on the disk are mapped to points which are spread apart from each other, so this issue is not likely to occur for sin(z) when the number of points used is small. Halsey discusses a different method, James Method. James's method is still based on the same theorem, but it uses values for the derivative of the conformal map on the boundary. This method will converge in many cases where the nearly circular and starlike hypotheses are relaxed [13].

7. Conclusion

We have taken an important step towards approximating the lift on the particles surrounding the airfoil. In particular, we have approximated an analytic transformation from the airfoil onto the disk. We have proved an algorithm for approximating this map, and then discussed its implementation. We have found through simulation that the approximation method is accurate and efficient, provided the map satisfies the assumptions described in section 3. Following this transformation, the lift can be computed using the Kutta-Joukowski Theorem.

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