Calculation of the A_1,A_2,A_3 filter coefficients in the time domain

$$ENC^2 = \frac{\langle v_{no}^2 \rangle}{Max^2\left(v_{sou\delta}\right)} = \frac{\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}\left(a + \frac{2\pi a_f}{|\omega|} + \frac{b}{\omega^2\left(C_D + C_G\right)^2}\right) g_m^2 \left|T(j\omega)\right|^2 d\omega}{\dfrac{g_m^2}{\left(C_D + C_G\right)^2} Max^2\left(L^{-1}\left[\dfrac{1}{s}T(s)\right]\right)}$$

$$ENC^2 = \left(C_D + C_G\right)^2 a \frac{1}{\tau} A_1 + \left(C_D + C_G\right)^2 2\pi a_f A_2 + b\,\tau A_3$$

series 1/f parallel

$$A_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|T(x)\right|^2 dx \qquad A_2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{|x|}\left|T(x)\right|^2 dx \qquad A_3 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{x^2}\left|T(x)\right|^2 dx$$

$$ENC^2{}_{series} = aC_T{}^2 1/2\pi \int_{-\infty}^{+\infty} |T(j\omega)|^2 d\omega =$$

$$= aC_T{}^2 1/2\pi \int_{-\infty}^{+\infty} |H(j\omega)|^2 \omega^2 d\omega =$$

$H(j\omega)$

is the overall output response (preamp + filter T(jω))

$$= [Parseval\ theorem] = aC_T{}^2 \int_{-\infty}^{+\infty} [h'(t)]^2 dt$$

$$h(t) = L^{-1}[H(s)] = L^{-1}[T(s)/s]$$

$h(t)$ is the output response in the time domain, $h'(t)$ its derivative

using $t' = t/\tau$ as dimensionless variable, i.e. the time normalized to the shaping time τ :

$$dt = dt' \cdot \tau \qquad\qquad h'(t) = h'(t') \cdot 1/\tau$$

$$ENC^2{}_{series} = aC_T{}^2 \frac{1}{\tau} \int_{-\infty}^{+\infty} [h'(t')]^2 dt'$$

introducing the shaping factor A_1 as done previously:

$$A_1 = 1/2\pi \int_{-\infty}^{+\infty} |T(j\omega)|^2 d\omega = \int_{-\infty}^{+\infty} [h'(t')]^2 dt'$$

$\Rightarrow$ A_1 is related to the area of the derivative of h(t')

$$ENC^2{}_{parallel} = b1/2\pi \int_{-\infty}^{+\infty} \frac{|T(j\omega)|^2}{\omega^2} d\omega =$$

$$= b1/2\pi \int_{-\infty}^{+\infty} |H(j\omega)|^2 d\omega = [Parseval\ theorem] = b \int_{-\infty}^{+\infty} [h(t)]^2 dt$$

$$t' = t/\tau \qquad dt = dt' \cdot \tau \qquad ENC^2{}_{parallel} = b\,\tau \int_{-\infty}^{+\infty} [h(t')]^2 dt'$$

$$\mathrm{A}_3 = 1/2\pi \int_{-\infty}^{+\infty} \frac{|T(j\omega)|^2}{\omega^2} d\omega = \int_{-\infty}^{+\infty} [h(t')]^2 dt'$$

$\Rightarrow$ A_3 is related to the area of h(t')

$$ENC^2{}_{1/f} = a_f {C_T}^2 \int_{-\infty}^{+\infty} \frac{|T(j\omega)|^2}{|\omega|} d\omega = a_f {C_T}^2 \int_{-\infty}^{+\infty} |\omega||H(j\omega)|^2 d\omega$$

$$|\omega| = \sqrt{j\omega}\sqrt{-j\omega}$$

$$|\omega||H(j\omega)|^2 = \sqrt{j\omega}H(j\omega)\overline{\sqrt{j\omega}H(j\omega)} = Z(j\omega)\overline{Z(j\omega)} = |Z(j\omega)|^2$$

$$with\;\; Z(j\omega) = \sqrt{j\omega}H(j\omega)$$

$$we\ will\ apply\ the\ Parseval\ theorem\ to\ Z(j\omega)$$

$note$: $the\ counterpart\ of\ operator\ j\omega\ in\ the\ time\ domain\ is\ the\ derivative\ of\ order$ 1:

$$F^{-1}\left(\mathrm{j\omega}H(j\omega)\right) = h'(t)$$

$$Z(j\omega) = \sqrt{j\omega}H(j\omega)$$

$the\ counterpart\ of\ operator\ \sqrt{j\omega}\ in\ the\ time\ domain\ is\ the\ derivative\ of\ order$ ½

$$\Rightarrow F^{-1}\left(\sqrt{j\omega}H(j\omega)\right) = h^{(1/2)}(t)$$

$$ENC_{1/f} = a_f C_T{}^2 \int_{-\infty}^{+\infty} |\omega||H(j\omega)|^2 d\omega = [Parseval\ theorem] =$$

$$= a_f C_T{}^2 2\pi \int_{-\infty}^{+\infty} [h^{(1/2)}]^2 dt$$

considering:

$$ENC^2 = (C_D + C_G)^2 a \frac{1}{\tau} A_1 + (C_D + C_G)^2 2\pi a_f A_2 + b\tau A_3$$

$$\Rightarrow A_2 = \int_{-\infty}^{+\infty} [h^{(1/2)}]^2 dt$$

(it can be verified that the integral does not depend from the time scale, i.e. $t' = t/\tau$)

$Simple\ derivation\ of\ h^{(1/2)}(t)$:

$$Z(j\omega) = \sqrt{j\omega}H(j\omega) = \frac{1}{\sqrt{j\omega}}j\omega H(j\omega)$$

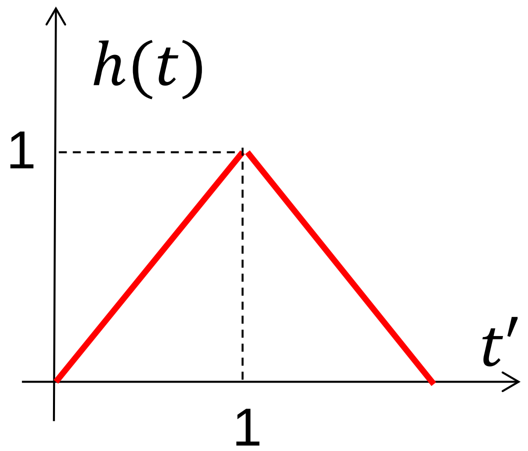
$$h^{(1/2)}(t) = F^{-1}\left(\frac{1}{\sqrt{j\omega}}\right) * F^{-1}\big(j\omega H(j\omega)\big) = g(t) * h'(t)$$

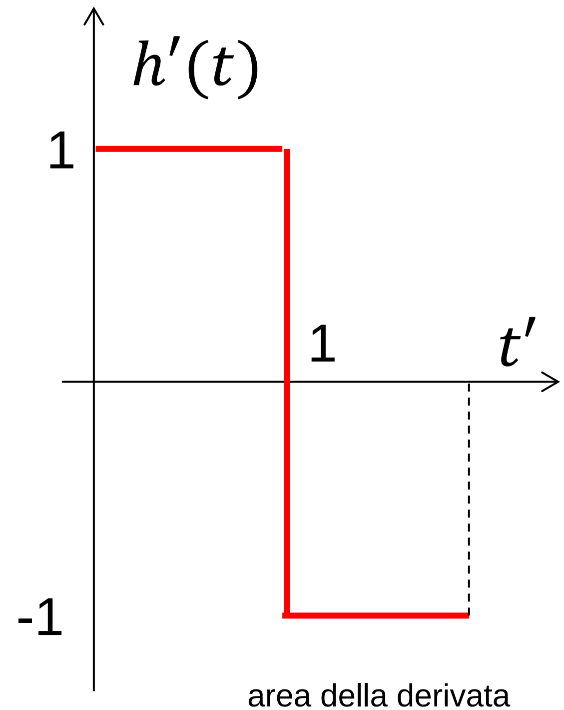
$with$ $$g(t) = \begin{cases} 1/\sqrt{\pi t}, & t > 0, \\ 0, & t < 0, \end{cases}$$

(see A.Pullia, Nucl. Instr. and Meth. in Phys. Res. A 405, (1998), pp.121-125)

Example: triangular filter

esempio di tutto questo col triangolo





area

$$A_3 = \int_{-\infty}^{+\infty} [h(t')]^2 dt' =$$

$$= 2\int_0^1 t'^2 dt' = \frac{2}{3}$$

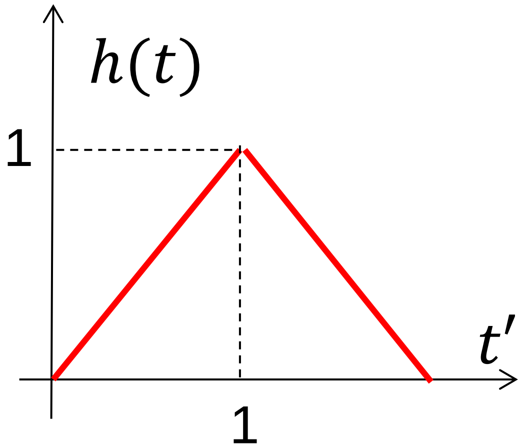
area della derivata

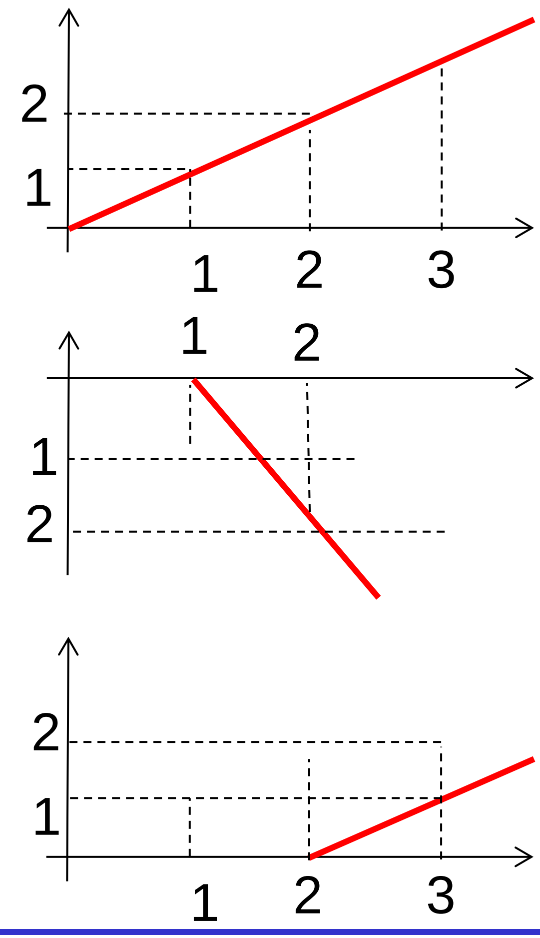
$$A_1 = \int_{-\infty}^{+\infty} [h'(t')]^2 dt' = 2$$

$calculation\ of\ A_2$:

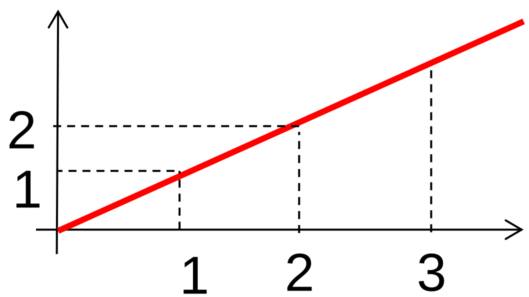
$the\ triangle\ can\ be\ represented\ as\ the\ sum\ of\ \ three\ ramps\ suitably\ delayed\ and\ weighted$:

triangolo diviso come somma
di tre linee, poi applichiamo la convoluzione
e sommiamo le 3 contrib



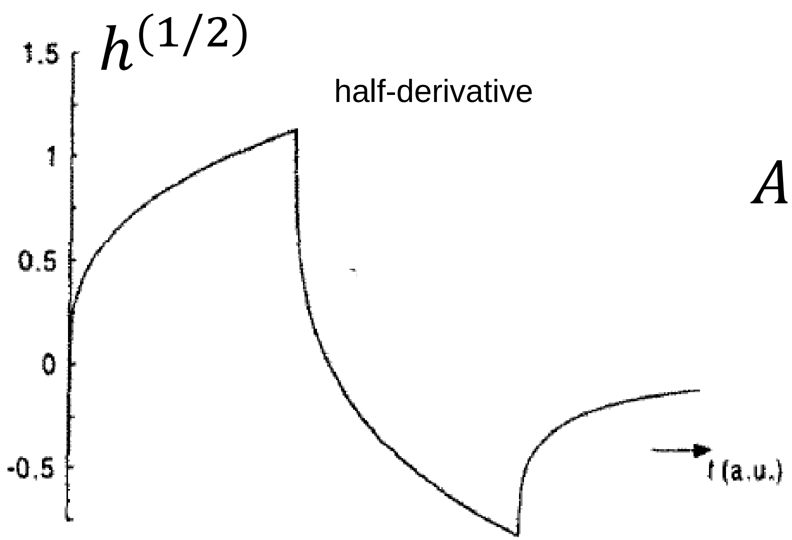


$calculation\ of\ h^{(1/2)}\ for\ a\ single\ ramp$:



$$\begin{cases} \sqrt{4\frac{t}{\pi}}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

$calculation\ of\ h^{(1/2)}\ for\ the\ triangular\ filter$:



$$A_2 = \int_{-\infty}^{+\infty} [h^{(1/2)}]^2 dt = \frac{4}{\pi} ln2 = 0.88$$