SPRING 2023 FINC B 9325 Financial Econometrics – Time Series Homework 4 Q2 Sample Solution

Question 2

This uses variables and notation from the class. Assume all the assumptions from Hamilton Proposition 14.1 on p. 414.

We want to show:

$$T[g_T(\hat{b}_T)]'\hat{S}_T^{-1}[g_T(\hat{b}_T)] \xrightarrow{L} \chi^2(r-q) \tag{1}$$

By the Mean Value Theorem, there exists $\bar{b}_T$ between b_0 and $\hat{b}_T$ such that

$$g_T(\hat{b}_T) = g_T(b_0) + \bar{D}_T(\hat{b}_T - b_0) \tag{2}$$

where

$$\bar{D}_T = \frac{\partial g_T(b)}{\partial b'}\Big|_{b=\bar{b}_T} \tag{3}$$

Let $A_T \equiv D_T'W_T$. The first order condition implies $A_T g_T(\hat{b}_T) = 0$. Pre-multiplying (2) by A_T, we have

$$A_T g_T(\hat{b}_T) = A_T g_T(b_0) + A_T\bar{D}_T(\hat{b}_T - b_0)$$

Re-arranging and then combining with (2),

$$\hat{b}_T - b_0 = -(A_T\bar{D}_T)^{-1}A_T g_T(b_0) \tag{4}$$
$$\sqrt{T}g_T(\hat{b}_T) = [I - \bar{D}_T(A_T\bar{D}_T)^{-1}A_T]\sqrt{T}g_T(b_0) \tag{5}$$

Factor $S = CC'$ and factor $\hat{S}_T = C_T C_T'$ such that $C_T \xrightarrow{p} C$, with C full rank. This implies

$$C_T^{-1}\sqrt{T}g_T(\hat{b}_T) = C_T^{-1}[I - \bar{D}_T(A_T\bar{D}_T)^{-1}A_T]C_T C_T^{-1}\sqrt{T}g_T(b_0) \tag{6}$$

Define

$$M_T \equiv C_T^{-1}[I - \bar{D}_T(A_T\bar{D}_T)^{-1}A_T]C_T$$
$$M \equiv C^{-1}[I - D_0(A_0 D_0)^{-1}A_0]C$$

When $W = S^{-1}$, and thus $W = (C')^{-1}C^{-1}$, M can be written as

$$M = I - X(X'X)^{-1}X', \quad \text{with } X = C^{-1}D_0 \tag{7}$$

and M is idempotent.

Thus, $C_T^{-1}\sqrt{T}g_T(b_T) = M_T C_T^{-1}\sqrt{T}g_T(b_0)$ and
$T[g_T(\hat{b}_T)]'\hat{S}_T^{-1}[g_T(\hat{b}_T)] = T[g_T(b_0)]'[C_T^{-1}]'M_T^2 C_T^{-1}g_T(b_0)$. Using the following,

$$\sqrt{T}g_T(b_0) \xrightarrow{L} N(0, S) \tag{8}$$

$$\bar{D}_T \xrightarrow{p} D_0 \tag{9}$$

$$A_T \xrightarrow{p} A_0 \;\; (= D_0'W) \tag{10}$$

we have $M_T \xrightarrow{p} M$ and by Slutsky's Theorem[1], we have

$$T[g_T(\hat{b}_T)]'\hat{S}_T^{-1}[g_T(\hat{b}_T)] \xrightarrow{L} z'M^2z = z'Mz \tag{11}$$

for $z \sim N(0, I)$ (from (8)). Also,

$$rank(M) = trace(M) \tag{12}$$

$$= trace(I_r) - trace(X(X'X)^{-1}X') \tag{13}$$

$$= r - trace(I_q) \tag{14}$$

$$= r - q \tag{15}$$

where in (14) we have used $trace(AB) = trace(BA)$ with $A = X$ and $B = (X'X)^{-1}X$. A useful result we can apply here: for $z \sim N(0, I)$ and an idempotent matrix A of rank m, we have $z'Az \sim \chi^2(m)$. Applying this result, we are done.

[1]These properties are reviewed in Hamilton, Chapter 7 (see pp. 184-185).