HOMEWORK # 4

QUESTION 1

(a) $M_{t+1} = \beta \exp(-\gamma \Delta c_{t+1})$

The Euler equation states: $E_t[M_{t+1} R_{t+1}] = 1$,
with R_{t+1}: gross stock return

Thus, $E_t[\beta \exp(-\gamma \Delta c_{t+1} + r_{t+1})] = 1$

By log-normality:

$$E_t[r_{t+1}] = -\log(\beta) + \gamma E_t[\Delta c_{t+1}] + V$$

where $V = -\frac{1}{2} var_t[r_{t+1}] - \frac{1}{2}\gamma^2 VAR_t[\Delta c_{t+1}]$
$+ \gamma\, cov_t[\Delta c_{t+1}, r_{t+1}]$

Using obvious notation and imposing constant variances,

$$\mu_r = -\log(\beta) + V = -\log(\beta) - \frac{1}{2}\sigma_r^2 - \frac{1}{2}\gamma^2\sigma_c^2 + \gamma\sigma_{cr}$$

Thus, $E_t[r_{t+1}] = \underbrace{\mu_r + \gamma\mu_c}_{\bar{\mu}_r} + \gamma\phi\Delta c_t$

The time variation in the expected return process is fully driven by consumption growth.

(b) By definition, $r_{t+1} = E_t[r_{t+1}] + \eta_{t+1}$
with $E_t[\eta_{t+1}] = 0$

The process for r_t becomes:

$$r_{t+1} = \bar{\mu}_r + \gamma\phi\Delta c_t + \eta_{t+1}$$

or: $r_{t+1} = \bar{\mu}_r + \gamma\phi(1-\phi L)^{-1}[\mu_c + \varepsilon_t] + \eta_{t+1}$

or: $r_{t+1} = \bar{\mu}_r(1-\phi) + \gamma\phi\mu_c + \underline{\phi r_t} + \gamma\phi\varepsilon_t + \eta_{t+1} - \phi\eta_t$ ②

AR Component

MA component

The MA component's autocorrelation vanishes after one lag, so it is effectively an MA(1) the process is ARMA(1,1).

Note that most parameters in the ARMA-representation for stock returns are implied by the consumption process. [Exceptions are the variance of η_t and its covariance with ε_t]

(c) Given normally distributed shocks, MLE is the natural approach.

The process for Δc_t is given, so we are left to determine the process for r_t.

Combining our findings in (a) and (b), we have

$$r_{t+1} = \bar{\mu}_r + \gamma\phi\Delta c_t + \eta_{t+1}$$

the second equation is

$$\Delta c_{t+1} = \mu_c + \phi\Delta c_t + \varepsilon_{t+1}$$

The covariance matrix is by definition:

$$\Sigma = \begin{bmatrix} \sigma_r^2 & \sigma_{cr} \\ \sigma_{cr} & \sigma_c^2 \end{bmatrix}$$

Let's denote $u_{t+1} = \begin{bmatrix} \eta_{t+1} \\ \varepsilon_{t+1} \end{bmatrix}$

and $y_{t+1} = [r_{t+1}, \Delta c_{t+1}]'$

It is straightforward to write down the joint likelihood

$$L(y_T ; \theta) = -T/2 \log|\Sigma| - 1/2 \sum_{t=1}^{T} (u_{t+1}' \Sigma^{-1} u_{t+1}) \quad (*)$$

Importantly, what is θ?

$$\theta = [\beta, \delta; \mu_c ; \phi, \sigma_c, \sigma_r, \sigma_{cr}]'$$

Note that there are no conditional mean parameters for the stock return equation as they are constrained by the Lucas model.

A natural alternative model is an unconstrained bivariate dynamic system (that is, a "vector autoregression"):

$$Y_{t+1} = \mu_Y + \underset{\substack{\uparrow \\ 2\times 2 \text{ matrix}}}{A_Y} Y_t + u_{t+1}$$

$$\theta = [\mu_Y', \text{vec}(A_Y)', \text{vech}(\Sigma)']'$$

The likelihood function is as in (*).

$\Rightarrow$ Likelihood ratio test:

$$2(\underbrace{L_{UC}}_{\text{Unconstrained likelihood}} - \underbrace{L_{CONS}}_{\text{Constrained likelihood}}) \sim \chi^2(\text{\# of restrictions})$$

(d) We do have:

$$E_t[r_{t+1}] = \mu_r + \gamma E_t[\Delta c_{t+1}]$$

Thus, $E_t[r_{t+1} - \mu_r - \gamma \Delta c_{t+1}] = 0$

We can thus write

$$r_{t+1} = \mu_r + \gamma \Delta c_{t+1} + u_{t+1}$$

with $E_t[u_{t+1}] = 0$

However, it is not true that

$E[u_{t+1} \Delta c_{t+1}] = 0$

Thus, the OLS conditions are not satisfied and OLS will not provide a consistent estimate of γ.

An estimator that would work is an IV (instrumental variables) with instruments that are:

- uncorrelated with u_{t+1}
- correlated with Δc_{t+1}

Lagged ($\in I_t$) values of Δc_{t+1}, r_{t+1} would make sense (see e.g., Hall, 1985)