The Modified Nodal Analysis (MNA) Method

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Nodal Formulation

- Matrix ***T*** in the Tableau formulation is very sparse
- The Nodal formulation emerges as a block elimination process on the Tableau formulation
- It yields a very compact formulation, but it is not general

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Nodal Formulation (cont.)

From the Tableau equations,

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n - \boldsymbol{V}_b = \boldsymbol{0} \qquad \boldsymbol{Y}\boldsymbol{V}_b + \boldsymbol{Z}\boldsymbol{I} = \boldsymbol{W} \qquad \boldsymbol{A}\boldsymbol{I} = \boldsymbol{0}$$

Substituting first equation into the second equation,

$$\boldsymbol{Y}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{Z}\boldsymbol{I} = \boldsymbol{W}$$

Combining last two equations,

$$\begin{bmatrix} \boldsymbol{Y}\boldsymbol{A}^{\mathrm{T}} & \boldsymbol{Z} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_n \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{0} \end{bmatrix}$$ an $(b+n) \times (b+n)$ matrix is achieved

Nodal Formulation (cont.)

If every element in the network can be represented by

$$-\boldsymbol{Y}\boldsymbol{V}_b + \boldsymbol{I} = \boldsymbol{J}$$

Using $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n - \boldsymbol{V}_b = \boldsymbol{0}$

$$-\boldsymbol{Y}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{I} = \boldsymbol{J}$$

Since $\boldsymbol{A}\boldsymbol{I} = \boldsymbol{0}$

$$\boldsymbol{A}\boldsymbol{I} = \boldsymbol{A}(\boldsymbol{J} + \boldsymbol{Y}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n) = \boldsymbol{0}$$

$$\boldsymbol{A}\boldsymbol{Y}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n = -\boldsymbol{A}\boldsymbol{J}$$ an $n \times n$ matrix is achieved

$\boldsymbol{Y}_n = \boldsymbol{A}\boldsymbol{Y}\boldsymbol{A}^{\mathrm{T}}$ is the Nodal Admittance Matrix, and $\boldsymbol{J}_n = -\boldsymbol{A}\boldsymbol{J}$

$$\boldsymbol{Y}_n\boldsymbol{V}_n = \boldsymbol{J}_n$$

Nodal Formulation – Final Remarks

- It requires solving an n by n system of equations
- It is restricted to elements that have an admittance representation:

$$-\boldsymbol{Y}\boldsymbol{V}_b + \boldsymbol{I} = \boldsymbol{J}$$

- Independent voltage sources, ideal CCVS, ideal CCCS, transformers, etc, can not be represented in this manner

Modified Nodal Analysis (MNA)

It splits the circuit elements into groups: one group for elements that accept an admittance description, and one group for those which do not

$$-\boldsymbol{Y}_1\boldsymbol{V}_{1b} + \boldsymbol{I}_1 = \boldsymbol{J}_1 \qquad \boldsymbol{Y}_2\boldsymbol{V}_{2b} + \boldsymbol{Z}_2\boldsymbol{I}_2 = \boldsymbol{W}_2$$

If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the corresponding incidence matrices for each group of elements, since $\boldsymbol{AI} = \mathbf{0}$ then

$$\boldsymbol{A}_1\boldsymbol{I}_1 + \boldsymbol{A}_2\boldsymbol{I}_2 = \mathbf{0}$$

Since $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n = \boldsymbol{V}_b$, then

$$\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n = \boldsymbol{V}_{1b} \qquad \boldsymbol{A}_2^{\mathrm{T}}\boldsymbol{V}_n = \boldsymbol{V}_{2b}$$

Modified Nodal Analysis, MNA (cont.)

Substituting last two equations into first two,

$$-\boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{I}_1 = \boldsymbol{J}_1 \qquad \boldsymbol{Y}_2\boldsymbol{A}_2^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{Z}_2\boldsymbol{I}_2 = \boldsymbol{W}_2$$

Pre-multiplying by $\boldsymbol{A}_1$ first previous equation

$$-\boldsymbol{A}_1\boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{A}_1\boldsymbol{I}_1 = \boldsymbol{A}_1\boldsymbol{J}_1$$

Using $\boldsymbol{A}_1\boldsymbol{I}_1 + \boldsymbol{A}_2\boldsymbol{I}_2 = \boldsymbol{0}$

$$-\boldsymbol{A}_1\boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n - \boldsymbol{A}_2\boldsymbol{I}_2 = \boldsymbol{A}_1\boldsymbol{J}_1$$

Then

$$\begin{bmatrix} \boldsymbol{A}_1\boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}} & \boldsymbol{A}_2 \\ \boldsymbol{Y}_2\boldsymbol{A}_2^{\mathrm{T}} & \boldsymbol{Z}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{V}_n \\ \boldsymbol{I}_2 \end{bmatrix} = \begin{bmatrix} -\boldsymbol{A}_1\boldsymbol{J}_1 \\ \boldsymbol{W}_2 \end{bmatrix} \qquad \boldsymbol{HX} = \boldsymbol{W}$$

MNA – Final Remarks

- Once the MNA equation is solved,

$$\begin{bmatrix} \boldsymbol{A}_1\boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}} & \boldsymbol{A}_2 \\ \boldsymbol{Y}_2\boldsymbol{A}_2^{\mathrm{T}} & \boldsymbol{Z}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{V}_n \\ \boldsymbol{I}_2 \end{bmatrix} = \begin{bmatrix} -\boldsymbol{A}_1\boldsymbol{J}_1 \\ \boldsymbol{W}_2 \end{bmatrix}$$

- The rest of the unknowns are calculated using

$$\boldsymbol{I}_1 = \boldsymbol{J}_1 + \boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n \qquad \boldsymbol{V}_{1b} = \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n \qquad \boldsymbol{V}_{2b} = \boldsymbol{A}_2^{\mathrm{T}}\boldsymbol{V}_n$$

- MNA retains the advantages of both the Nodal and the Tableau methods
- The above formulation of MNA can be applied to linear circuits only, but can be extended to nonlinear

A Nodal Formulation to Nonlinear Circuits

Suppose that the branch equations of all elements in a circuit accept a non-linear admittance representation,

$$\boldsymbol{I} = \boldsymbol{g}(\boldsymbol{V}_b)$$

From KVL and KCL: $\boldsymbol{V}_b = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n \qquad \boldsymbol{A}\boldsymbol{I} = \boldsymbol{0}$

Then

$$\boldsymbol{A}(\boldsymbol{g}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n)) = \boldsymbol{0}$$

(a system of n nonlinear equations with n unknowns)

MNA for Linear and Nonlinear Circuits

Split the circuit branches into three groups:

$$-\boldsymbol{Y}_1\boldsymbol{V}_{1b} + \boldsymbol{I}_1 = \boldsymbol{J}_1 \qquad \boldsymbol{Y}_2\boldsymbol{V}_{2b} + \boldsymbol{Z}_2\boldsymbol{I}_2 = \boldsymbol{W}_2 \qquad \boldsymbol{I}_3 = \boldsymbol{g}(\boldsymbol{V}_{3b})$$

If $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{A}_3$ are the corresponding incidence matrices for each group of elements, from KCL

$$\boldsymbol{A}_1\boldsymbol{I}_1 + \boldsymbol{A}_2\boldsymbol{I}_2 + \boldsymbol{A}_3\boldsymbol{I}_3 = \boldsymbol{0}$$

From KVL,

$$\boldsymbol{V}_{1b} = \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n \qquad \boldsymbol{V}_{2b} = \boldsymbol{A}_2^{\mathrm{T}}\boldsymbol{V}_n \qquad \boldsymbol{V}_{3b} = \boldsymbol{A}_3^{\mathrm{T}}\boldsymbol{V}_n$$

Substituting KVL into first three equations,

$$-\boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{I}_1 = \boldsymbol{J}_1 \qquad \boldsymbol{Y}_2\boldsymbol{A}_2^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{Z}_2\boldsymbol{I}_2 = \boldsymbol{W}_2 \qquad \boldsymbol{I}_3 = \boldsymbol{g}(\boldsymbol{A}_3^{\mathrm{T}}\boldsymbol{V}_n)$$

MNA for Linear and Nonlinear Circuits (cont.)

$$-\boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{I}_1 = \boldsymbol{J}_1 \qquad \boldsymbol{Y}_2\boldsymbol{A}_2^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{Z}_2\boldsymbol{I}_2 = \boldsymbol{W}_2 \qquad \boldsymbol{I}_3 = \boldsymbol{g}(\boldsymbol{A}_3^{\mathrm{T}}\boldsymbol{V}_n)$$

Pre-multiplying by $\boldsymbol{A}_1$ first above equation

$$-\boldsymbol{A}_1\boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{A}_1\boldsymbol{I}_1 = \boldsymbol{A}_1\boldsymbol{J}_1$$

Using $\boldsymbol{A}_1\boldsymbol{I}_1 + \boldsymbol{A}_2\boldsymbol{I}_2 + \boldsymbol{A}_3\boldsymbol{I}_3 = \boldsymbol{0}$,

$$\boldsymbol{A}_1\boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{V}_n + \boldsymbol{A}_2\boldsymbol{I}_2 + \boldsymbol{A}_3\boldsymbol{g}(\boldsymbol{A}_3^{\mathrm{T}}\boldsymbol{V}_n) = -\boldsymbol{A}_1\boldsymbol{J}_1$$

Solving $\boldsymbol{Z}_2\boldsymbol{I}_2 = \boldsymbol{W}_2 - \boldsymbol{Y}_2\boldsymbol{A}_2^{\mathrm{T}}\boldsymbol{V}_n$ for $\boldsymbol{I}_2$ at each iteration and substituting into the above equation yields a system of n nonlinear equations that can be solved for $\boldsymbol{V}_n$

MNA Formulation Using Stamps

- The MNA equation

$$\begin{bmatrix} \boldsymbol{A}_1\boldsymbol{Y}_1\boldsymbol{A}_1^{\mathrm{T}} & \boldsymbol{A}_2 \\ \boldsymbol{Y}_2\boldsymbol{A}_2^{\mathrm{T}} & \boldsymbol{Z}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_n \\ \boldsymbol{I}_2 \end{bmatrix} = \begin{bmatrix} -\boldsymbol{A}_1\boldsymbol{J}_1 \\ \boldsymbol{W}_2 \end{bmatrix} \qquad \boldsymbol{H}\boldsymbol{X} = \boldsymbol{W}$$

  can be formulated without using oriented graphs or incidence matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$
- $\boldsymbol{H}$ and $\boldsymbol{W}$ can be directly formulated by inspection, using stamps
- Further, the L and C elements can be separated so that

$$(\boldsymbol{H}_1 + s\boldsymbol{H}_2)\boldsymbol{X} = \boldsymbol{W}$$

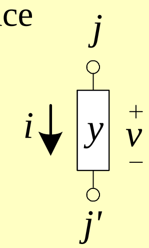
  can also be written directly

MNA Formulation Using Stamps (cont.)

- Start by entering by inspection in a matrix ***H*** all elements which have an admittance description
- Initially ***H*** has an *n* by *n* size, where *n* is the number of ungrounded nodes
- Increase the size of ***H*** whenever we enter an element which does not have an admittance description
- The constitutive voltage equations of an element without admittance description are attached as an extra row of ***H***, while the current flowing into that element is attached as an extra column in ***H***
- ***W*** is generated according to the independent sources

MNA Formulation Using Stamps (cont.)

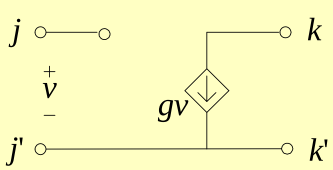
Admittance



$$\begin{array}{c} \\ j \\ j' \end{array}\begin{array}{c} \begin{array}{cc} V_j & V_{j'} \end{array} \\ \begin{bmatrix} y & -y \\ -y & y \end{bmatrix} \end{array}$$

$$I_j = y(V_j - V_{j'})$$
$$I_{j'} = -y(V_j - V_{j'})$$

VCCS

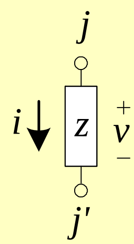


$$\begin{array}{c} \\ k \\ k' \end{array}\begin{array}{c} \begin{array}{cc} V_j & V_{j'} \end{array} \\ \begin{bmatrix} g & -g \\ -g & g \end{bmatrix} \end{array}$$

$$I_j = I_{j'} = 0$$
$$I_k = g(V_j - V_{j'})$$
$$I_{k'} = -g(V_j - V_{j'})$$

MNA Formulation Using Stamps (cont.)

Impedance

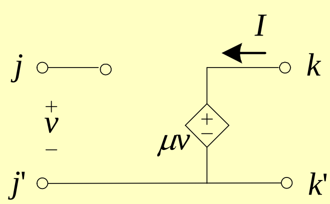


$$\begin{array}{c} \\ j \\ j' \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} V_j & V_{j'} & I \end{array} \\ \left[\begin{array}{ccc} & & 1 \\ & & -1 \\ 1 & -1 & -z \end{array}\right] \end{array}$$

$$I_j = I$$
$$I_{j'} = -I$$
$$V_j - V_{j'} - zI = 0$$

MNA Formulation Using Stamps (cont.)

VCVS

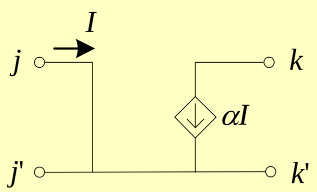


$$\begin{array}{c} \\ j \\ j' \\ k \\ k' \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} V_j & V_{j'} & V_k & V_{k'} & I \end{array} \\ \left[\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & 1 \\ & & & & -1 \\ -\mu & \mu & 1 & -1 & \end{array}\right] \end{array}$$

$$I_j = 0 \qquad I_k = I$$
$$I_{j'} = 0 \qquad I_{k'} = -I \qquad -\mu V_j + \mu V_{j'} + V_k - V_{k'} = 0$$

MNA Formulation Using Stamps (cont.)

CCCS



$$
\begin{array}{c}
 \\ j \\ j' \\ k \\ k' \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccc} V_j & V_{j'} & V_k & V_{k'} & I \end{array} \\
\left[\begin{array}{ccccc}
 & & & & 1 \\
 & & & & -1 \\
 & & & & \alpha \\
 & & & & -\alpha \\
1 & -1 & & &
\end{array}\right]
\end{array}
$$

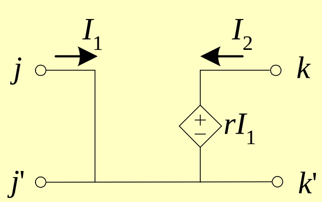
$$I_j = I \qquad I_k = \alpha I$$

$$I_{j'} = -I \qquad I_{k'} = -\alpha I$$

$$V_j - V_{j'} = 0$$

MNA Formulation Using Stamps (cont.)

CCVS



$$
\begin{array}{c}
 \\ j \\ j' \\ k \\ k' \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccccc} V_j & V_{j'} & V_k & V_{k'} & I_1 & I_2 \end{array} \\
\left[\begin{array}{cccccc}
 & & & & 1 & \\
 & & & & -1 & \\
 & & & & & 1 \\
 & & & & & -1 \\
 & & 1 & -1 & -r & \\
1 & -1 & & & &
\end{array}\right]
\end{array}
$$

$$I_j = I_1 \qquad I_k = I_2$$

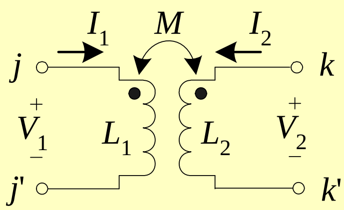
$$I_{j'} = -I_1 \qquad I_{k'} = -I_2$$

$$V_j - V_{j'} = 0$$

$$V_k - V_{k'} - rI_1 = 0$$

MNA Formulation Using Stamps (cont.)

Transformer



$$\begin{array}{cc} & \begin{array}{cccccc} V_j & V_{j'} & V_k & V_{k'} & I_1 & I_2 \end{array} \\ \begin{array}{c} j \\ j' \\ k \\ k' \\ \\ \\ \end{array} & \begin{bmatrix} & & & & 1 & \\ & & & & -1 & \\ & & & & & 1 \\ & & & & & -1 \\ 1 & -1 & & & -sL_1 & -sM \\ & & 1 & -1 & -sM & -sL_2 \end{bmatrix} \end{array}$$

$I_j = I_1 \qquad I_k = I_2$

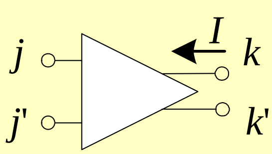
$I_{j'} = -I_1 \qquad I_{k'} = -I_2$

$$V_j - V_{j'} - sL_1 I_1 - sMI_2 = 0$$

$$V_k - V_{k'} - sL_2 I_2 - sMI_1 = 0$$

MNA Formulation Using Stamps (cont.)

Ideal Op-Amp



$$\begin{array}{cc} & \begin{array}{ccccc} V_j & V_{j'} & V_k & V_{k'} & I \end{array} \\ \begin{array}{c} j \\ j' \\ k \\ k' \\ \\ \end{array} & \begin{bmatrix} & & & & \\ & & & & \\ & & & & 1 \\ & & & & -1 \\ 1 & -1 & & & \end{bmatrix} \end{array}$$

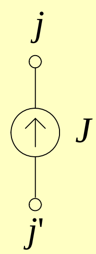
$I_j = 0 \qquad I_k = I$

$I_{j'} = 0 \qquad I_{k'} = -I$

$$V_j - V_{j'} = 0$$

MNA Formulation Using Stamps (cont.)

Current Source



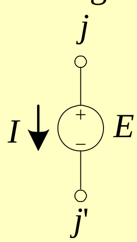
(source vector)

$$\begin{array}{c} j \\ j' \end{array}\begin{bmatrix} J \\ -J \end{bmatrix}$$

$$I_j = -J$$

$$I_{j'} = J$$

Voltage Source



$$\begin{array}{cc} & \begin{array}{ccc} V_j & V_{j'} & I \end{array} \\ \begin{array}{c} j \\ j' \\ \\ \end{array} & \begin{bmatrix} & & 1 \\ & & -1 \\ 1 & -1 & \end{bmatrix} \end{array}$$

(source vector)

$$\begin{bmatrix} \\ \\ E \end{bmatrix}$$

$$I_j = I$$

$$I_{j'} = -I$$

$$V_j - V_{j'} = E$$