Sample Based Determination of Angular Observables

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We study means to determine angular observables for a generic angular distribution of either decays or scattering processes. ...

I. INTRODUCTION

In this letter we study how to obtain a set of angular observables P_i that arise in a generic multi-body processes without the need to carry out an explicit fit of an angular distribution to data. Instead, we present a method that only relies on orthogonality of angular functions, and estimation of integrals by means of Monte Carlo techniques.

The initial motivation for studying what we wish to call the *Method of Moments* is the determination of angular observables in the decay rare, FCNC-mediated decay $\bar{B} \to \bar{K}^*(\to \bar{K}\pi)\ell^+\ell^-$. However, we emphasize that he method we describe in the remainder of this letter is applicable to arbitrary decay or scattering processes.

For the purpose of this work, let P_i be a set of functions of m non-angular kinematic variables $\vec{\nu}$, $P_i \equiv P_i(\vec{\nu})$, with $\vec{\nu} = (\nu_1, \dots \nu_m)$. These include, among others, invariant masses or center of mass energies pertinent to the respective decay or scattering processes. Let us further assume the angular observables to be defined via a suitably factorizing probability distribution (PDF),

$$P(\vec{\nu}, \vec{\vartheta}) = \sum_k P_i(\vec{\nu}) \times f_i(\vec{\vartheta}) \,, \tag{1}$$

Here, the dependence on the n decay angles $\vec{\vartheta} = (\vartheta_1, \dots, \vartheta_n)$ has been explicitly factored out in terms of the angular functions $f_i(\vec{\vartheta})$, which we assume to fulfill the orthonormality relations

$$\int_\Omega f_i(\vec{\vartheta}) f_j(\vec{\vartheta}) \mathrm{d}^n\vartheta = \delta_{ij} \,. \tag{2}$$

(For the purpose of this letter, however, it suffices that the system of angular functions $f_i(\vec{\vartheta})$ can be transformed into an orthonorml basis. The transformations to these bases for a selection of decays has been written out in the appendices A through D.)

For particle decays, we P is expressed in terms of the fully differential decay width,

$$P(\vec{\nu}, \vec{\vartheta}) \equiv \frac{1}{\Gamma} \frac{\mathrm{d}^{n+m}\Gamma}{\mathrm{d}\nu_1 \dots \mathrm{d}\nu_m \, \mathrm{d}\vartheta_1 \dots \vartheta_n} \,, \tag{3}$$

where Γ is the total decay width. For a scattering process, one can similarly use

$$P(\vec{\nu}, \vec{\vartheta}) \equiv \frac{1}{\sigma} \frac{\mathrm{d}^{n+m}\sigma}{\mathrm{d}\nu_1 \dots \mathrm{d}\nu_m \, \mathrm{d}\vartheta_1 \dots \vartheta_n} \,, \tag{4}$$

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where the total cross section σ is used for the normalisation. Since the determination of the total decay width or total cross section can be quite difficult, we emphasize that different normalizations for P can be used. For instance, the total decay width (or cross section) of the process of interest can be replaced by the corresponding quantity of a control-channel process.

In the remainder of this letter we discuss how to obtain the angular observables $P_i(\vec{\nu})$ in an experimental setup where each recorded event is (approximately) as variate of P. We lay down the groundwork in section II. Section III is dedicated to the impact of systematic effects, such as mismodelling of the underlying physics and detector acceptance effects. Numerical studies for one uni-angular and one triple-angular distribution are provided in section IV. In a series of appendices we discuss the relevant orthonormal bases of angular functions at the hand of the several rare b decays.

II. SAMPLE BASED DETERMINATION

From the orthonormality relations eq. (2) follows that a single angular observable P_i can be projected out of the full PDF P by means of

$$P_i(\vec{\nu}) = \int_{\vec{\vartheta}=0}^{2\pi} f_i(\vec{\vartheta}) P(\vec{\nu}, \vec{\vartheta}) \mathrm{d}^n\vartheta \,. \tag{5}$$

In point of fact, P_i is the f_i-moment of the PDF P. Additionally, Integration over the non-angular variables yields

$$\langle P_i \rangle \equiv \int P_i(\vec{\nu}) \mathrm{d}^m\nu = \int \left[\int_{\vec{\vartheta}=0}^{2\pi} f_i(\vec{\vartheta}) P(\vec{\nu}, \vec{\vartheta}) \mathrm{d}^n\vartheta \right] \mathrm{d}^m\nu \,. \tag{6}$$

The remainder of this section is describes the method of moments, in which we replace the analytical integration eq. (5) by Monte Carlo (MC) estimators.

The central tenet of MC integration is the fact that the expectation value $E_P[g]$ of some function $g(x)$ under the probability density $P(x)$,

$$E_P[g] \equiv \int g(x) P(x) \mathrm{d}x \tag{7}$$

can be replaced **[cite needed]**by an MC estimator $\widehat{E_P[g]}$

$$E_P[g] \to \widehat{E_P[g]} \equiv \frac{1}{N} \sum_{k=1} g(x^{(k)}) \,. \tag{8}$$

For the purpose of the estimation, the variates $x^{(k)}$, $k = 1, \ldots, N$ must distributed under P,

$$x^{(k)} \sim P \,. \tag{9}$$

(Throughout this letter we denote all MC estimators with a wide hat.)

We assume, for a moment, that P describes the distribution of the underlying physical process truthfully. In that case, all events detected in an experimental setup are in fact, up to detection efficiencies, distributed as P. Application of eq. (8) then yields

$$\langle P_i \rangle \to \widehat{\langle P_i \rangle} = \frac{1}{N} \sum_{k=1}^{N} f_i(x^{(k)}) \,. \tag{10}$$

However, it is often of interest to obtain the $\vec{\nu}$-integrated observables for certain $\vec{\nu}$ ranges, i.e. binned measurements of $\langle P_i \rangle$[1]. Therefore, we introduce the bin-integrated quantities

$$\langle P_i \rangle_{\vec{a},\vec{b}} = \int_{\vec{a}}^{\vec{b}} P_i(\vec{\nu}) \mathrm{d}^m\nu \tag{11}$$

[1] We refer for example to the determination of the angular observables J_i in $\bar{B} \to \bar{K}^* \ell^+ \ell^-$ decays, see e.g. [1]. For details on the relation between the P_i and the angular observable J_i in the aforementioned decay, we refer to appendix B.

$$= \int_{\vec{a}}^{\vec{b}} \left[\int_{\vec{\vartheta}=0}^{2\pi} f_i(\vec{\vartheta}) P(\vec{\nu}, \vec{\vartheta}) \mathrm{d}^n \vartheta \right] \mathrm{d}^m \nu \tag{12}$$

$$= \int \left[\int_{\vec{\vartheta}=0}^{2\pi} f_i(\vec{\vartheta}) P(\vec{\nu}, \vec{\vartheta}) \vartheta^{(m)}(\vec{\nu} - \vec{a}) \vartheta^{(m)}(\vec{b} - \vec{\nu}) \mathrm{d}^n \vartheta \right] \mathrm{d}^m \nu \,. \tag{13}$$

Application of eq. (8) immediately yields

$$\widehat{\langle P_i \rangle}_{\vec{a},\vec{b}} = \frac{1}{N} \sum_{k=1}^{N} f_i(x^{(k)}) \vartheta^{(m)}(\vec{\nu}^{(k)} - \vec{a}) \vartheta^{(m)}(\vec{b} - \vec{\nu}^{(k)}) \,. \tag{14}$$

This basically reduces the determination of P_i to a (weighted) counting experiment. As a consequence, the statistical error on $\widehat{P}_i$ can be expected to follow a multivariate gaussian distribution **[cite needed]**. In this case, the covariance $\mathrm{Cov}[\widehat{P}_i, \widehat{P}_j]$ is of particular interest. The latter can be replaced by a suitable MC estimator, the sample covariance

$$\mathrm{Cov}[\widehat{P}_i, \widehat{P}_j] \to \widehat{\mathrm{Cov}}[\widehat{P}_i, \widehat{P}_j]_{\vec{a},\vec{b}} = \frac{1}{N-1} \sum_{k=1}^{N} \left[\hat{f}_i^{(k)} - \widehat{\langle P_i \rangle}_{\vec{a},\vec{b}} \right] \left[\hat{f}_j^{(k)} - \widehat{\langle P_j \rangle}_{\vec{a},\vec{b}} \right] \vartheta^{(m)}(\vec{\nu}^{(k)} - \vec{a}) \vartheta^{(m)}(\vec{b} - \vec{\nu}^{(k)}) \,. \tag{15}$$

III. SOURCES OF SYSTEMATIC UNCERTAINTIES

In sec. (II), we assumed that the PDF P described the physical reality accurately, and that the experiment observed each event with perfect accuracy. In order to estimate systematic uncertainties, we lift these assumptions.

A. Mismodelling due to Contributions by Higher Partial Waves

So far, we assumed that our underlying physical process was accurately described by our PDF P. However, in several interesting processes we might only have an approximative result for P. We will now show that this is the case for the interesting class four-body decays $B \to P_1 P_2 \ell_1 \bar{\ell}_2$, which includes the rare $b \to s$-mediated B decay $B \to K\pi\ell^+\ell^-$ and the V_{ub} suppressed decay $B \to \pi\pi\ell^+\bar{\nu}_\ell$. For the sake of clarity and simplicity, we will here discuss the application of our method to the decay $B \to \pi\pi\ell^+\bar{\nu}_\ell$.

We now face the problem that P has a given dependence on the dilepton helicity angle ϑ_1 and the azimuthal angle ϑ_3. However, the dipion system can have an arbitrary large total angular momentum; only the its third component is restricted to $J_z^{\pi\pi} = -1, 0, +1$. Therefore, we will restrict integration to the angles ϑ_1 and ϑ_3. ...

B. Detector Acceptance Effects and Backgrounds

Thanks to accurate detector simulations and control studies, the experimental analyses have become remarkably good at removing detector acceptance effects. In this section we therefore assume that there are only a small, residual acceptance effects left that affect the events. In particular, we assume that the latter can be modelled as a modified probability density,

$$P(\vec{\nu}, \vec{\vartheta}) \to P(\vec{\nu}, \vec{\vartheta}) \times \Omega(\vec{\vartheta}), \tag{16}$$

where Ω is independent of the non-angular kinematics $\vec{\nu}$. The latter assumption is only is invalid in general. However, if residual angular acceptance only weakly depends on $\vec{\nu}$, one can always choose appropriate bin $\vec{\nu}$ so that Ω is approximately constant with respect to $\vec{\nu}$. For such a case we parametrize

$$\Omega_{m,l}(\vec{\vartheta}) = \prod_{i=1}^{m} \left[1 + \sum_{j=1}^{l} \varepsilon_i^{(j)} p_j(\cos\vartheta_i) \right] \tag{17}$$

where $p_j(\cos\vartheta_i)$ denotes the jth Legendre polynomial. Since we only consider residual acceptance effects, we expect $\varepsilon_i^{(j)} \ll 1$. [2]

[2] what are reasonable values/upper limits on the ε?

a. Uniangular Distribution For definiteness and clarity, we consider first an uni-angular distribution $P(\vartheta)$. In this case we can parametrize the PDF P as

$$P(\vartheta) = P_1 + P_2 \cos\vartheta + P_2 \cos^2\vartheta\,, \tag{18}$$

which is a parametrization that has been used in the literature for $\bar{B} \to \bar{K}\ell^+\ell^-$ decays [2]. While this system of angular functions is not orthonormal, it can easily be orthonormalized. For the basis of angular functions, see app. (A). Considering detector acceptance effects up to second order in $\cos\vartheta$, the function $\Omega(\vec{\vartheta})$ reads

$$\Omega_{1,2}(\vartheta_1) = 1 + \varepsilon_1^{(1)} \cos\vartheta + \frac{\varepsilon_1^{(2)}}{2}\left(3\cos^2\vartheta - 1\right) \qquad \varepsilon_1^{(i)} = \mathcal{O}\left(\varepsilon\right)\,. \tag{19}$$

Inclusion of $\Omega_{1,2}$ as part of the PDF as proposed in eq. (16) modifies the MC estimators for the angular observables. We find

$$\begin{aligned}
\widehat{\widehat{P_1}} &= \left(1 + \frac{\varepsilon_1^{(2)}}{2}\right)\widehat{P_1} + \frac{9\varepsilon_1^{(2)}}{70}\widehat{P_3}\\
\widehat{\widehat{P_2}} &= \left(1 - \frac{2\varepsilon_1^{(2)}}{5}\right)\widehat{P_2} - \varepsilon_1^{(1)}\widehat{P_1} - \frac{3\varepsilon_1^{(1)}}{5}\widehat{P_3}\\
\widehat{\widehat{P_3}} &= \left(1 - \frac{11\varepsilon_1^{(2)}}{14}\right)\widehat{P_3} - \frac{3\varepsilon_1^{(2)}}{2}\widehat{P_1} - \varepsilon_1^{(1)}\widehat{P_2}\,.
\end{aligned} \tag{20}$$

Here $\widehat{P}_i$ denotes the uncorrected estimators, while $\widehat{\widehat{P}}_i$ denotes the efficiency-corrected estimators of the angular observables.

b. Triangular Distribution As an example for a triangular distribution we use the decay $\bar{B} \to \bar{K}^*(\to \bar{K}\pi)\ell^+\ell^-$. A parametrization of its angular distribution can be found in app. (B).

For this type of distribution, we expand Ω_3 to linear order in ε,

$$\Omega_{3,2}(\vartheta_1, \vartheta_2, \vartheta_3) = 1 + \sum_{i=1}^{3}\sum_{j=1}^{2} \varepsilon_i^{(j)} p_j(\cos\vartheta_i) + \mathcal{O}\left(\varepsilon^2\right)\,. \tag{21}$$

This leads to a matrix-valued equation that relates the effiency-corrected estimators $\widehat{\widehat{P}}_i$ to the uncorrected estimators $\widehat{P}_j$,

$$\widehat{\widehat{P}}_i = \sum_{j=1\mathrm{s},\dots 9} \left(\delta_{ij} + N_{ij}\right)\widehat{P}_j\,. \tag{22}$$

We provide the matrix N in tab. (I).

c. Recipe to determine the acceptance effects The determination of the coefficients $\varepsilon_i^{(j)}$ that parametrize the detectory effects is generally a difficult task. Here, we will show a systematic method to determine these coefficients. ...

IV. TOY STUDIES

ToDo:

- **Simulate this for one angle ϑ_1 (similar to $B \to K\ell^+\ell^-$) with varying samples sizes.**
- **Simulate this for three angles $\vartheta_{1\dots3}$ (similar to $B \to K^*(\to K\pi)\ell^+\ell^-$) with varying samples sizes.**
- **Also, check explicitly the variance and covariance of the results for all sample sizes considered.**

$$
\begin{pmatrix}
\frac{13}{35}\varepsilon_1^{(2)}-\frac{1}{4}\varepsilon_2^{(2)}-\frac{1}{4}\varepsilon_3^{(2)} & \frac{9}{70}\varepsilon_1^{(2)} & 0 & -\frac{39}{140}\varepsilon_2^{(2)} & 0 & 0 \\
-\frac{12}{35}\varepsilon_1^{(2)} & -\frac{23}{35}\varepsilon_1^{(2)}-\frac{1}{4}\varepsilon_2^{(2)}-\frac{1}{4}\varepsilon_3^{(2)} & -\varepsilon_1^{(1)} & 0 & -\frac{39}{140}\varepsilon_2^{(2)} & 0 \\
-\frac{2}{5}\varepsilon_1^{(1)} & -\frac{3}{5}\varepsilon_1^{(1)} & -\frac{2}{5}\varepsilon_1^{(2)}-\frac{1}{4}\varepsilon_2^{(2)}-\frac{1}{4}\varepsilon_3^{(2)} & 0 & 0 & -\frac{39}{140}\varepsilon_2^{(2)} \\
-\frac{3}{4}\varepsilon_2^{(2)} & 0 & 0 & \frac{13}{35}\varepsilon_1^{(2)}-\frac{1}{28}\varepsilon_2^{(2)}-\frac{1}{4}\varepsilon_3^{(2)} & \frac{9}{70}\varepsilon_1^{(2)} & 0 \\
0 & -\frac{3}{4}\varepsilon_2^{(2)} & 0 & -\frac{12}{35}\varepsilon_1^{(2)} & -\frac{23}{35}\varepsilon_1^{(2)}-\frac{1}{28}\varepsilon_2^{(2)}-\frac{1}{4}\varepsilon_3^{(2)} & -\varepsilon_1^{(1)} \\
0 & 0 & -\frac{3}{4}\varepsilon_2^{(2)} & -\frac{2}{5}\varepsilon_1^{(1)} & -\frac{3}{5}\varepsilon_1^{(1)} & -\frac{2}{5}\varepsilon_1^{(2)}-\frac{1}{28}\varepsilon_2^{(2)}-\frac{1}{4}\varepsilon_3^{(2)} \\
-\frac{15}{16}\varepsilon_3^{(2)} & -\frac{15}{64}\varepsilon_3^{(2)} & 0 & \frac{9}{16}\varepsilon_3^{(2)} & \frac{9}{64}\varepsilon_3^{(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{45}{512}\pi^2\varepsilon_3^{(1)} & 0 & 0 & \frac{45\pi^2\varepsilon_3^{(1)}}{1024} \\
-\frac{27}{256}\pi^2\varepsilon_3^{(1)} & -\frac{9}{256}\pi^2\varepsilon_3^{(1)} & 0 & \frac{27}{512}\pi^2\varepsilon_3^{(1)} & \frac{9}{512}\pi^2\varepsilon_3^{(1)} & 0 \\
-\varepsilon_2^{(1)} & 0 & 0 & -\frac{1}{5}\varepsilon_2^{(1)} & 0 & 0 \\
0 & -\varepsilon_2^{(1)} & 0 & 0 & -\frac{1}{5}\varepsilon_2^{(1)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{3}{16}\varepsilon_3^{(2)} & 0 & 0 & 0 & -\frac{567\pi^2\varepsilon_3^{(1)}}{16384} & -\frac{1}{2}\varepsilon_2^{(1)} \\
0 & 0 & 0 & 0 & -\frac{81\pi^2\varepsilon_3^{(1)}}{8192} & 0 \\
0 & 0 & 0 & -\frac{81\pi^2\varepsilon_3^{(1)}}{2048} & 0 & 0 \\
\frac{3}{16}\varepsilon_3^{(2)} & 0 & 0 & 0 & \frac{315\pi^2\varepsilon_3^{(1)}}{16384} & -\frac{1}{2}\varepsilon_2^{(1)} \\
0 & 0 & 0 & 0 & \frac{45\pi^2\varepsilon_3^{(1)}}{8192} & 0 \\
0 & 0 & 0 & \frac{45\pi^2\varepsilon_3^{(1)}}{2048} & 0 & 0 \\
\frac{2}{7}\varepsilon_1^{(2)}+\frac{2}{7}\varepsilon_2^{(2)}-\frac{1}{4}\varepsilon_3^{(2)} & 0 & 0 & 0 & -\frac{2025\pi^2\varepsilon_3^{(1)}}{32768} & 0 \\
0 & -\frac{1}{7}\varepsilon_1^{(2)}-\frac{1}{7}\varepsilon_2^{(2)}-\frac{5}{8}\varepsilon_3^{(2)} & -\frac{1}{2}\varepsilon_1^{(1)} & -\frac{1}{2}\varepsilon_2^{(1)} & 0 & 0 \\
0 & -\frac{2}{5}\varepsilon_1^{(1)} & \frac{1}{5}\varepsilon_1^{(2)}-\frac{1}{7}\varepsilon_2^{(2)}-\frac{5}{8}\varepsilon_3^{(2)} & 0 & -\frac{1}{2}\varepsilon_2^{(1)} & -\frac{135\pi^2\varepsilon_3^{(1)}}{2048} \\
0 & -\frac{2}{5}\varepsilon_2^{(1)} & 0 & -\frac{1}{7}\varepsilon_1^{(2)}+\frac{1}{5}\varepsilon_2^{(2)}-\frac{5}{8}\varepsilon_3^{(2)} & -\frac{1}{2}\varepsilon_1^{(1)} & 0 \\
-\frac{81\pi^2\varepsilon_3^{(1)}}{2048} & 0 & -\frac{2}{5}\varepsilon_2^{(1)} & -\frac{2}{5}\varepsilon_1^{(1)} & \frac{1}{5}\varepsilon_1^{(2)}+\frac{1}{5}\varepsilon_2^{(2)}-\frac{5}{8}\varepsilon_3^{(2)} & 0 \\
0 & 0 & -\frac{63\pi^2\varepsilon_3^{(1)}}{1024} & 0 & 0 & \frac{13}{35}\varepsilon_1^{(2)}-\frac{2}{5}\varepsilon_2^{(2)}-\frac{1}{4}\varepsilon_3^{(2)} \\
0 & 0 & -\frac{9}{512}\pi^2\varepsilon_3^{(1)} & 0 & 0 & -\frac{12}{35}\varepsilon_1^{(2)} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2}\varepsilon_2^{(1)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2}\varepsilon_2^{(1)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{45\pi^2\varepsilon_3^{(1)}}{2048} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{9}{70}\varepsilon_1^{(2)} & 0 & 0 & 0 & 0 & 0 \\
-\frac{23}{35}\varepsilon_1^{(2)}-\frac{2}{5}\varepsilon_2^{(2)}-\frac{1}{4}\varepsilon_3^{(2)} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{7}\varepsilon_1^{(2)}+\frac{1}{5}\varepsilon_2^{(2)}+\frac{1}{8}\varepsilon_3^{(2)} & -\frac{1}{2}\varepsilon_1^{(1)} & -\frac{2}{5}\varepsilon_2^{(1)} & 0 & 0 \\
0 & -\frac{2}{5}\varepsilon_1^{(1)} & \frac{1}{5}\varepsilon_1^{(2)}+\frac{1}{5}\varepsilon_2^{(2)}+\frac{1}{8}\varepsilon_3^{(2)} & 0 & -\frac{2}{5}\varepsilon_2^{(1)} & -\frac{81\pi^2\varepsilon_3^{(1)}}{2048} \\
0 & -\frac{1}{2}\varepsilon_2^{(1)} & 0 & -\frac{1}{7}\varepsilon_1^{(2)}-\frac{1}{7}\varepsilon_2^{(2)}+\frac{1}{8}\varepsilon_3^{(2)} & -\frac{1}{2}\varepsilon_1^{(1)} & 0 \\
0 & 0 & -\frac{1}{2}\varepsilon_2^{(1)} & -\frac{2}{5}\varepsilon_1^{(1)} & \frac{1}{5}\varepsilon_1^{(2)}-\frac{1}{7}\varepsilon_2^{(2)}+\frac{1}{8}\varepsilon_3^{(2)} & 0 \\
0 & 0 & -\frac{2025\pi^2\varepsilon_3^{(1)}}{32768} & 0 & 0 & \frac{2}{7}\varepsilon_1^{(2)}+\frac{2}{7}\varepsilon_2^{(2)}-\frac{1}{4}\varepsilon_3^{(2)}
\end{pmatrix}
$$

TABLE I: The matrix N_{ij} that appears in eq. (22). Due to spatial constraints, we split its display into columns $j = 1s, 1c, 1i, 2s, 2c, 2i$ (top), $j = 3, 4, 4i, 5, 5i, 6s$ (middle) and $j = 6c, 7, 7i, 8, 8i, 9$ (bottom).

V. CONCLUSION

In this letter ...

1. errors on the means are in very good approximation gaussian.
2. correlation are small and faithfully model a multivariate gaussian.

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Appendix A: Application to $\bar{B} \to \bar{K}\ell^+\ell^-$

The PDF for the decay $\bar{B} \to \bar{K}\ell^+\ell^-$ has been calculated for the most complete basis of dimension-six $b \to s\ell^+\ell^-$ operators. It reads [2, 3]

$$P(q^2, \cos\vartheta_1) = \frac{1}{\Gamma}\frac{\mathrm{d}^4\Gamma}{\mathrm{d}q^2\,\mathrm{d}\cos\vartheta_1} = \frac{a(q^2)}{\Gamma} + \frac{b(q^2)}{\Gamma}\cos\vartheta_1 + \frac{c(q^2)}{\Gamma}\cos^2\vartheta_1 = \sum_i P_i f_i(\cos\vartheta_1)\,, \tag{A1}$$

where the basis of angular functions f_i reads

$$f_1 = 1\,, \quad f_2 = \cos\vartheta_1\,, \quad f_2 = \cos^2\vartheta_1\,. \tag{A2}$$

Here we denote the dilepton mass squared as q^2, and the dilepton helicity angle as $\vartheta_1 \equiv \vartheta_\ell$. We transform the PDF from a $\cos\vartheta_1$-dependence to a ϑ_1-dependence,

$$P(q^2, \vartheta_1) = \sum_i P_i f_i(\cos\vartheta_1)\sin\vartheta_1\,. \tag{A3}$$

We provide dual basis $\hat{f}_i$ of angular functions which is dual to the basis f_i in the sence that

$$\int_0^\pi \mathrm{d}\vartheta_1 \hat{f}_i(\vartheta_1) f_j(\vartheta_1)\sin\vartheta_1 = \delta_{ij}\,, \tag{A4}$$

thereby enabling us to project out exactly one of the angular observables $P_i(q^2)$

$$\int_0^\pi \mathrm{d}\vartheta_1 \hat{f}_i(\vartheta_1) P(q^2, \vartheta_1) = P_i(q^2)\,. \tag{A5}$$

The dual basis reads

$$\hat{f}_1 = \frac{9}{8} - \frac{15}{8}\cos^2\vartheta_1\,, \quad \hat{f}_2 = \frac{3}{2}\cos\vartheta_1\,, \quad \hat{f}_3 = -\frac{15}{8} + \frac{45}{8}\cos^2\vartheta_1\,. \tag{A6}$$

Appendix B: Application to $\bar{B} \to \bar{K}\pi\ell^+\ell^-$ (S-wave and P-wave)

The PDF for the decay $\bar{B} \to \bar{K}\pi\ell^+\ell^-$ – up to and including P-wave contributions – has been calculated for the most general basis of dimension-six $b \to s$ operators. It reads [3, 4]

$$P(q^2, \cos\vartheta_1, \cos\vartheta_2, \phi) = \frac{1}{\Gamma}\frac{\mathrm{d}^4\Gamma}{\mathrm{d}q^2\,\mathrm{d}\cos\vartheta_1\,\mathrm{d}\cos\vartheta_2\,\mathrm{d}\phi} = \frac{3}{8\pi}\sum_i \frac{J_i(q^2)}{\Gamma} f_i(\cos\vartheta_1, \cos\vartheta_2, \phi)\,, \tag{B1}$$

where $\vartheta_1 \equiv \vartheta_\ell$ is the dilepton helicity angle, and $\vartheta_2 \equiv \vartheta_{K^*}$ is the $\bar{K}\pi$ helicity angle. The decay width is

$$\Gamma = \frac{\big(3J_{1c} - J_{2c}\big) + 2\big(3J_{1s} - J_{2s}\big)}{3}\,. \tag{B2}$$

The angular functions read [3, 4]

$$
\begin{aligned}
f_{1s} &= \sin^2\vartheta_2 & f_{1c} &= \cos^2\vartheta_2 & f_{1i} &= \cos\vartheta_2\\
f_{2s} &= \sin^2\vartheta_2\cos 2\vartheta_1 & f_{2c} &= \cos^2\vartheta_2\cos 2\vartheta_1 & f_{2i} &= \cos\vartheta_2\cos 2\vartheta_1\\
f_3 &= \sin^2\vartheta_2\sin^2\vartheta_1\cos 2\phi & f_9 &= \sin^2\vartheta_2\sin^2\vartheta_1\sin 2\phi & &\\
f_4 &= \sin 2\vartheta_2\sin 2\vartheta_1\cos\phi & & & f_{4i} &= \sin\vartheta_2\sin 2\vartheta_1\cos\phi\\
f_5 &= \sin 2\vartheta_2\sin\vartheta_1\cos\phi & & & f_{5i} &= \sin\vartheta_2\sin\vartheta_1\cos\phi\\
f_{6s} &= \sin^2\vartheta_2\cos\vartheta_1 & f_{6c} &= \cos^2\vartheta_2\cos\vartheta_1 & &\\
f_7 &= \sin 2\vartheta_2\sin\vartheta_1\sin\phi & & & f_{7i} &= \sin\vartheta_2\sin\vartheta_1\sin\phi\\
f_8 &= \sin 2\vartheta_2\sin 2\vartheta_1\sin\phi & & & f_{8i} &= \sin\vartheta_2\sin 2\vartheta_1\sin\phi\,.
\end{aligned}
\tag{B3}
$$

(Note here the notation, where indices with an i arise only from S-wave/P-wave interference.) In terms of the angles ϑ_1 and ϑ_2, instead of their cosines, the PDF reads

$$
P(q^2,\vartheta_1,\vartheta_2,\phi) = \frac{3}{8\pi}\sum_i P_i(q^2) f_i(\cos\vartheta_1,\cos\vartheta_2,\phi)\sin\vartheta_1\sin\vartheta_2\,,
\tag{B4}
$$

where we also define $P_i(q^2) = J_i(q^2)/\Gamma$.

We provide dual basis $\hat{f}_i$ of angular functions which is dual to the basis f_j in the sense that

$$
\frac{3}{8\pi}\int_0^\pi \mathrm{d}\vartheta_1 \int_0^\pi \mathrm{d}\vartheta_2 \int_0^{2\pi} \mathrm{d}\phi \hat{f}_i(\vartheta_1,\vartheta_2,\phi) f_j(\vartheta_1,\vartheta_2,\phi)\sin\vartheta_1\sin\vartheta_2 = \delta_{ij}\,,
\tag{B5}
$$

thereby enabling us to project out exactly one of the angular observables $P_i(q^2)$

$$
\int_0^\pi \mathrm{d}\vartheta_1 \int_0^\pi \mathrm{d}\vartheta_2 \int_0^{2\pi} \mathrm{d}\phi \hat{f}_i(\vartheta_1,\vartheta_2,\phi) P(q^2,\vartheta_1,\vartheta_2,\phi) = P_i(q^2)\,.
\tag{B6}
$$

The dual basis reads

$$
\begin{aligned}
\hat{f}_{1s} &= \frac{\big(63\sin^2\vartheta_1 - 42\cos^2\vartheta_1\big) + \big(45\sin^2\vartheta_1 - 30\cos^2\vartheta_1\big)\cos 2\vartheta_2}{64}\,,\\
\hat{f}_{1c} &= \frac{\big(-21\sin^2\vartheta_1 + 84\cos^2\vartheta_1\big) + \big(-15\sin^2\vartheta_1 + 60\cos^2\vartheta_1\big)\cos 2\vartheta_2}{32}\,,\\
\hat{f}_{2s} &= \frac{\big(45\sin^2\vartheta_1 - 30\cos^2\vartheta_1\big) + \big(+135\sin^2\vartheta_1 - 90\cos^2\vartheta_1\big)\cos 2\vartheta_2}{64}\,,\\
\hat{f}_{2c} &= \frac{\big(-15\sin^2\vartheta_1 + 60\cos^2\vartheta_1\big) + \big(-45\sin^2\vartheta_1 + 180\cos^2\vartheta_1\big)\cos 2\vartheta_2}{32}\,,
\end{aligned}
\tag{B7}
$$

as well as

$$
\begin{aligned}
\hat{f}_{1sc} &= \frac{\big(21 + 15\cos 2\vartheta_2\big)\cos\vartheta_1}{16}\,, & \hat{f}_{2sc} &= \frac{\big(15 + 45\cos 2\vartheta_2\big)\cos\vartheta_1}{16}\,,\\
\hat{f}_{6s} &= \frac{\big(9\sin^2\vartheta_1 - 6\cos^2\vartheta_1\big)\cos\vartheta_2}{4}\,, & \hat{f}_{6c} &= \frac{\big(-3\sin^2\vartheta_1 + 12\cos^2\vartheta_1\big)\cos\vartheta_2}{2}\,,\\
\hat{f}_3 &= \frac{75\sin^2\vartheta_1\sin^2\vartheta_2\cos 2\phi}{32}\,, & \hat{f}_9 &= \frac{75\sin^2\vartheta_1\sin^2\vartheta_2\sin 2\phi}{32}\,,\\
\hat{f}_4 &= \frac{75\sin 2\vartheta_1\sin 2\vartheta_2\cos\phi}{32}\,, & \hat{f}_{4i} &= \frac{15\sin\vartheta_1\sin 2\vartheta_2\cos\phi}{8}\,,\\
\hat{f}_5 &= \frac{15\sin 2\vartheta_1\sin\vartheta_2\cos\phi}{8}\,, & \hat{f}_{5i} &= \frac{3\sin\vartheta_1\sin\vartheta_2\cos\phi}{2}\,,\\
\hat{f}_7 &= \frac{15\sin 2\vartheta_1\sin\vartheta_2\sin\phi}{8}\,, & \hat{f}_{7i} &= \frac{3\sin\vartheta_1\sin\vartheta_2\sin\phi}{2}\,,\\
\hat{f}_8 &= \frac{75\sin 2\vartheta_1\sin 2\vartheta_2\sin\phi}{32}\,, & \hat{f}_{8i} &= \frac{15\sin\vartheta_1\sin 2\vartheta_2\sin\phi}{8}\,.
\end{aligned}
\tag{B8}
$$

Appendix C: Application to $\bar{B} \to P_1 P_2 \ell^+ \ell^-$ (all partial waves)

The class of $P \to P_1 P_2 \ell_1 \bar{\ell}_2$ decays, with P, P_1, P_2 pseudoscalar mesons, and ℓ_1, $\bar{\ell}_2$ charge or neutral leptons, includes decays such as e.g. $\bar{B} \to \bar{K}\pi\ell^+\ell^-$ and $\bar{B} \to \pi\pi\ell^-\bar{\nu}_\ell$. Their PDFs can generally be written as [5]

$$P(q^2, k^2, \cos\vartheta_1, \cos\vartheta_2, \phi) = \frac{1}{\Gamma}\frac{\mathrm{d}^5\Gamma}{\mathrm{d}q^2\,\mathrm{d}k^2\,\mathrm{d}\cos\vartheta_1\,\mathrm{d}\cos\vartheta_2\,\mathrm{d}\phi} = \frac{1}{4\pi}\sum_i \frac{I_i(q^2, k^2, \cos\vartheta_2)}{\Gamma} f_i(\cos\vartheta_1, \phi)\,, \tag{C1}$$

where

$$\begin{aligned}
f_1 &= 1\,, & f_2 &= \cos 2\vartheta_1\,, & f_6 &= \cos\vartheta_1\,,\\
f_3 &= \sin^2\vartheta_1 \cos 2\phi\,, & f_4 &= \sin 2\vartheta_1 \cos\phi\,, & f_5 &= \sin\vartheta_1\cos\phi\,,\\
f_9 &= \sin^2\vartheta_1 \sin 2\phi\,. & f_8 &= \sin 2\vartheta_1 \sin\phi\,, & f_7 &= \sin\vartheta_1 \sin\phi\,,
\end{aligned} \tag{C2}$$

Here we denote dilepton mass squared and the dilepton helicity angle as q^2 and ϑ_1, while the dimeson mass squared and the dimeson helicity angles are denoted as k^2 and ϑ_2. The azimuthal angle is denoted as ϕ. The total decay width is

$$\langle\Gamma\rangle = \int_0^\pi \mathrm{d}\vartheta_1\, \frac{\langle 3I_1(\cos\vartheta_1) - I_2(\cos\vartheta_1)\rangle}{3}\,. \tag{C3}$$

As usual, we transform from a $\cos\vartheta_1$-dependence to a ϑ_1-dependence.

$$\begin{aligned}
P(q^2, k^2, \vartheta_1, \vartheta_2, \phi) &= \frac{1}{4\pi}\sum_i \frac{I(q^2, k^2, \cos\vartheta_2)}{\Gamma} f_i(\vartheta_1, \phi)\sin\vartheta_1 \sin\vartheta_2\\
&\equiv \frac{1}{4\pi}\sum_i P_i(q^2, k^2, \vartheta_2) f_i(\vartheta_1, \phi)\sin\vartheta_2\,.
\end{aligned} \tag{C4}$$

where we also define $P_i(q^2, k^2, \vartheta_2) \equiv \vartheta_2 I_i(q^2, k^2, \cos\vartheta_2)/\Gamma$.

We provide a basis $\hat{f}_i$ of angular functions which is dual to the basis f_j in the sense that

$$\frac{1}{4\pi}\int_0^\pi \mathrm{d}\vartheta_1 \int_0^{2\pi} \mathrm{d}\phi \hat{f}_i(\vartheta_1, \phi) f_j(\vartheta_1, \phi)\sin\vartheta_1 = \delta_{ij}\,, \tag{C5}$$

thereby enabling us to project out exactly one of the angular observables $P_i(q^2)$

$$\int_0^\pi \mathrm{d}\vartheta_1 \int_0^{2\pi} \mathrm{d}\phi \hat{f}_i(\vartheta_1, \phi) P(q^2, k^2, \vartheta_1, \vartheta_2, \phi) = P_i(q^2, k^2, \vartheta_2)\,. \tag{C6}$$

The dual basis reads

$$\begin{aligned}
\hat{f}_1 &= \frac{21 + 15\cos 2\vartheta_1}{16}\,, & \hat{f}_2 &= \frac{15 + 45\cos 2\vartheta_1}{16}\,, & \hat{f}_6 &= 3\cos\vartheta_1\,,\\
\hat{f}_3 &= \frac{15}{4}\cos 2\phi\,, & \hat{f}_4 &= \frac{15}{4}\sin 2\vartheta_1\cos\phi\,, & \hat{f}_5 &= 3\sin\vartheta_1\cos\phi\,,\\
\hat{f}_9 &= \frac{15}{4}\sin 2\phi\,, & \hat{f}_8 &= \frac{15}{4}\sin 2\vartheta_1\sin\phi\,, & \hat{f}_7 &= 3\sin\vartheta_1\sin\phi\,.
\end{aligned} \tag{C7}$$

Appendix D: Application to $\Lambda_b \to \Lambda(\to N\pi)\ell^+\ell^-$

The physical PDF for the decay – in the presence of Standard Model operator and their chirality flipped counter parts – reads [6]

$$P_{\Lambda_b}(q^2, \cos\vartheta_1, \cos\vartheta_2, \phi) = \frac{1}{\Gamma}\frac{\mathrm{d}^4\Gamma}{\mathrm{d}q^2\,\mathrm{d}\cos\vartheta_1\,\mathrm{d}\cos\vartheta_2\,\mathrm{d}\phi} = \frac{3}{8\pi}\sum_i \frac{K_i(q^2)}{\Gamma} f_i(\cos\vartheta_1, \cos\vartheta_2, \phi)\,, \tag{D1}$$

where q^2 denotes the dilepton mass squared, and $\vartheta_1 \equiv \vartheta_\ell$ and $\vartheta_2 \equiv \vartheta_\Lambda$ denote the helicity angles in the dilepton and $N\pi$ systems, respectively. The basis of angular functions reads

$$\begin{aligned}
f_{1ss} &= \sin^2\vartheta_1 & f_{1cc} &= \cos^2\vartheta_1 & f_{1c} &= \cos\vartheta_1 \\
f_{2ss} &= \sin^2\vartheta_1\cos\vartheta_2 & f_{2cc} &= \cos^2\vartheta_1\cos\vartheta_2 & f_{2c} &= \cos\vartheta_1\cos\vartheta_2 \\
f_{3sc} &= \sin\vartheta_1\cos\vartheta_1\sin\vartheta_2\cos\phi & f_{3s} &= \sin\vartheta_1\sin\vartheta_2\cos\phi & & \\
f_{4sc} &= \sin\vartheta_1\cos\vartheta_1\sin\vartheta_2\sin\phi & f_{4s} &= \sin\vartheta_1\sin\vartheta_2\sin\phi & &
\end{aligned} \tag{D2}$$

The decay width reads $\langle\Gamma\rangle = \langle 2K_{1ss} + K_{1cc}\rangle$. In terms of the angles ϑ_1 and ϑ_2, instead of the cosines, the PDF reads

$$P_{\Lambda_b}(q^2,\vartheta_1,\vartheta_2,\phi) = \frac{3}{8\pi}\sum_i P_i(q^2) f_i(\cos\vartheta_1,\cos\vartheta_2,\phi)\sin\vartheta_1\sin\vartheta_2\,, \tag{D3}$$

where we also define $P_i(q^2) = K_i(q^2)/\Gamma$.

We provide a dual basis $\hat{f}_i$ of the angular functions which is orthogonal to the basis f_j in the sense that

$$\frac{3}{8\pi}\int_0^\pi \mathrm{d}\vartheta_1 \int_0^\pi \mathrm{d}\vartheta_2 \int_0^{2\pi} \mathrm{d}\phi \hat{f}_i(\vartheta_1,\vartheta_2,\phi) f_j(\vartheta_1,\vartheta_2,\phi)\sin\vartheta_1\sin\vartheta_2 = \delta_{ij}\,, \tag{D4}$$

thereby enabling us to project out exactly one of the angular observables $P_i(q^2)$ using

$$\int_0^\pi \mathrm{d}\vartheta_1 \int_0^\pi \mathrm{d}\vartheta_2 \int_0^{2\pi} \mathrm{d}\phi P(q^2,\vartheta_1,\vartheta_2,\phi)\hat{f}_i(\vartheta_1,\vartheta_2,\phi) = P_i(q^2)\,. \tag{D5}$$

The dual basis reads

$$\begin{aligned}
\hat{f}_{1ss} &= \frac{\pi^3}{4}\big(3\sin^2\vartheta_1 - 2\cos^2\vartheta_1\big) & \hat{f}_{1cc} &= \frac{\pi^3}{2}\big(-\sin^2\vartheta_1 + 4\cos^2\vartheta_1\big) & \hat{f}_{1c} &= \pi^3\cos\vartheta_1 \\
\hat{f}_{2ss} &= \frac{3\pi^3}{4}\big(3\sin^2\vartheta_1 - 2\cos^2\vartheta_1\big)\cos\vartheta_2 & \hat{f}_{2cc} &= \frac{3\pi^3}{2}\big(-\sin^2\vartheta_1 + 4\cos^2\vartheta_1\big)\cos\vartheta_2 & \hat{f}_{2c} &= 3\pi^3\cos\vartheta_1\cos\vartheta_2 \\
\hat{f}_{3sc} &= \frac{15\pi^3}{2}\sin\vartheta_1\cos\vartheta_1\sin\vartheta_2\cos\phi & \hat{f}_{3s} &= \frac{3\pi^3}{2}\sin\vartheta_1\sin\vartheta_2\cos\phi & & \\
\hat{f}_{4sc} &= \frac{15\pi^3}{2}\sin\vartheta_1\cos\vartheta_1\sin\vartheta_2\sin\phi & \hat{f}_{4s} &= \frac{3\pi^3}{2}\sin\vartheta_1\sin\vartheta_2\sin\phi & &
\end{aligned} \tag{D6}$$

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