**Assignment 5 (ELEC 341 L5_Model_DCMotor)**

**Problem 1:**

Consider the differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = f(x)$$

where $f(x)$ is the input and is a function of the output, $x$. If $f(x) = \sin x$, linearize the differential equation for small excursions.

**a.** $x = 0$

**b.** $x = \pi$

**Solution:**

$$\ddot{x}+3\dot{x}+2x=f(x)\ ;\ \text{let } f(x)=F(x)\ \longrightarrow$$

$$\underline{\ddot{x}+3\dot{x}+2x=F(x)}\ ;\ \begin{cases} f(x)=F(x)=\text{input} \\ x=\text{output} \end{cases}$$

a) $x_0=0$

b) $x_0=\pi$

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a) $x_0=0$; $f(\ddot{x},\dot{x},x,F(x))=0 \rightarrow$

$$f(\ddot{x},\dot{x},x,F)=\ddot{x}+3\dot{x}+2x-F(x)=0$$

Linearize at $x=x_0=0$:

$$\cancelto{0}{\ddot{x}\big|_{x_0=0}}+3\cancelto{0}{\dot{x}\big|_{x_0=0}}+2\cancelto{0}{x_0}-F(x_0)=0 \longrightarrow$$

$$F(x_0)=F(0)=0$$

New coordinates:

$$\begin{cases} x=x_0+\delta x \\ F=F_0+\delta F \\ \dot{x}=\dot{x}_0+\delta\dot{x} \\ \ddot{x}=\ddot{x}_0+\delta\ddot{x} \end{cases}$$

Taylor Expansion at $x_0, F_0$:

$$f(\ddot{x},\dot{x},x,F(x))=f(\ddot{x}_0,\dot{x}_0,x_0,F(x_0))+\frac{\partial f}{\partial x}\bigg|_{eq.}\delta x+\frac{\partial f}{\partial F}\bigg|_{eq.}\delta F+\frac{\partial f}{\partial \dot{x}}\bigg|_{eq.}\delta\dot{x}+\frac{\partial f}{\partial \ddot{x}}\bigg|_{eq.}\delta\ddot{x}$$

$$=0+(2)\,\delta x+(-1)\,\delta F+(3)\,\delta\dot{x}+(1)\,\delta\ddot{x}\ \longrightarrow$$

$$\rightarrow f(\ddot{x}, \dot{x}, x, F(x)) = 2\delta x - \delta F + 3\delta\dot{x} + \delta\ddot{x} = 0 \rightarrow$$

$$\delta\ddot{x} + 3\delta\dot{x} + 2\delta x - \delta F = 0 \quad (1)$$

Note: $\delta F = F(x_0 + \delta x)$

$$\hookrightarrow \delta F = \sin(x_0 + \delta x); \quad \sin(x_0 + \delta x) = \cancelto{0}{\sin x_0} + \left.\frac{d \sin x}{dx}\right|_{x = x_0} \cdot \delta x$$

$$= \cos x \Big|_{x = x_0 = 0} \cdot \delta x = 1 \cdot \delta x \rightarrow \delta F = \delta x$$

$$\xrightarrow[\text{in } (1)]{\text{sub}} \delta\ddot{x} + 3\delta\dot{x} + 2\delta x - \delta x = 0 \longrightarrow$$

$$\boxed{\delta\ddot{x} + 3\delta\dot{x} + \delta x = 0}$$

$$b)\ x_0=\pi\ ;\ f(\ddot{x},\dot{x},x,F)=\ddot{x}+3\dot{x}+2x-F(x)=0$$

Linearize at $x=x_0=\pi$:

$$\cancel{\left.\ddot{x}\right|_{x_0=\pi}}^{\,0}+3\cancel{\left.\dot{x}\right|_{x_0=\pi}}^{\,0}+2\cancel{\left.x\right|_{x_0=\pi}}^{\,\pi}-F(x_0)=0\ \rightarrow$$

$$2\pi-F(x_0=\pi)=0\rightarrow F(x_0=\pi)=2\pi$$

$$f(\ddot{x},\dot{x}_0,x_0,F)=0+3\times 0+2\pi-F(x_0=2\pi)=$$

$$=2\pi-2\pi=0\ \rightarrow$$

$$f(\ddot{x},\dot{x}_0,x_0,F_0)=0$$

New coordinates:

$$\begin{cases} x=x_0+\delta x \\ F=F_0+\delta F \\ \dot{x}=\dot{x}_0+\delta\dot{x} \\ \ddot{x}=\ddot{x}_0+\delta\ddot{x} \end{cases}$$

Taylor expansion @ $x_0, F_0$:

$$f(\ddot{x},\dot{x},x,F(x))=f(\ddot{x}_0,\dot{x}_0,x_0,x_0,F(x_0))^{\nearrow 0}+\dots$$

$$=0+(2)\cdot\delta x+(-1)\cdot\delta F+(3)\cdot\delta\dot{x}+(1)\cdot\delta\ddot{x}=0$$

$$\rightarrow\ \delta\ddot{x}+3\delta\dot{x}+2\delta x-\delta F=0\qquad(1)$$

$$\delta F=F(x_0+\delta x)=\sin(x_0+\delta x)$$

$$\sin(x_0+\delta x)=\cancel{\sin\pi}^{\,0}+\left.\frac{d\sin x}{dx}\right|_{x=\pi}\cdot\delta x$$

$$=0+\cos\pi\cdot\delta x=\delta x\ \rightarrow\ \delta F=-\delta x\ \xrightarrow[(1)]{\text{sub in}}$$

$$\delta\ddot{x}+3\delta\dot{x}+2\delta x-(-\delta x)=0\ \rightarrow$$

$$\boxed{\delta\ddot{x}+3\delta\dot{x}+3\delta x=0}$$