a place of mind

UBC

ELEC 341: Systems and Control

Lecture 12

Root locus: Examples

Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - Electrical
  - Electromechanical
  - Mechanical
- Linearization, delay

Analysis

- Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

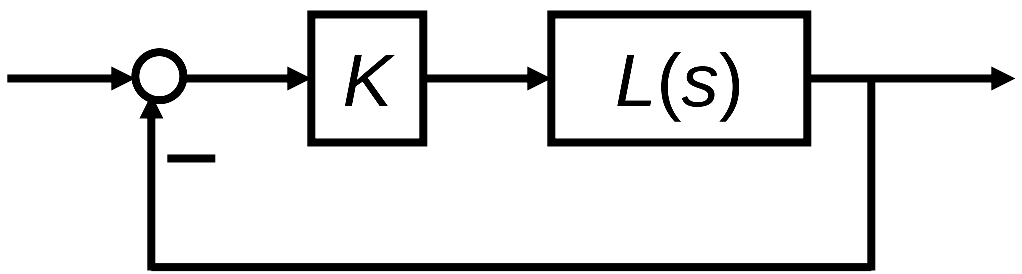
Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

What is Root Locus? (review)

- *Pole locations* of the system characterizes stability and transient properties.
- Consider a feedback system that has one parameter (gain) $K > 0$ to be designed:



$K.L(s)$: open-loop TF

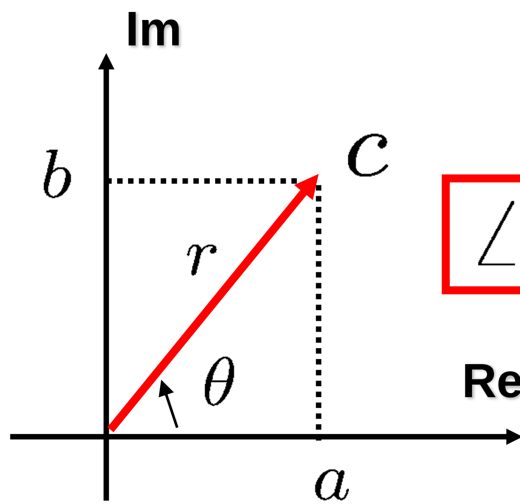
- *Root locus* graphically shows how poles of CL system vary as K varies from 0 to infinity.

RL sketching (review)

- Step 0: Mark open-loop poles and zeros
- Step 1: On the real axis
- Step 2: Asymptotes
- Step 3: Breakaway points
- Step 4: Angles of departures and arrivals (to be explained next)

Complex numbers (review)

- Representation
  - Cartesian form
    $$c = a + bj$$
  - Polar form
    $$c = re^{j\theta}$$



$$\boxed{\angle c = \theta}$$

- Multiplication & division in the polar form

$$\left.\begin{aligned} c_1 &= r_1 e^{j\theta_1} \\ c_2 &= r_2 e^{j\theta_2} \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 c_2 &= r_1 r_2 e^{j(\theta_1+\theta_2)} \\ \frac{c_1}{c_2} &= \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} \end{aligned}$$

Angle condition

- For an open-loop transfer function *L*(*s*),

$$\boxed{s = s_0 \text{ is on the root locus } \Leftrightarrow \angle L(s_0) = 180^\circ}$$

- Why?

$s = s_0$ is on RL

$\Rightarrow$ There exists a $K > 0$ s.t. $1 + KL(s_0) = 0$.

$\Rightarrow L(s_0) = -\frac{1}{K} < 0$

$\Rightarrow \angle L(s_0) = 180^\circ$

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$\Rightarrow$ There exists a $K > 0$ s.t. $L(s_0) = -\frac{1}{K}$.

$\Rightarrow 1 + KL(s_0) = 0$

$\Rightarrow s = s_0$ is on RL

Example with complex poles

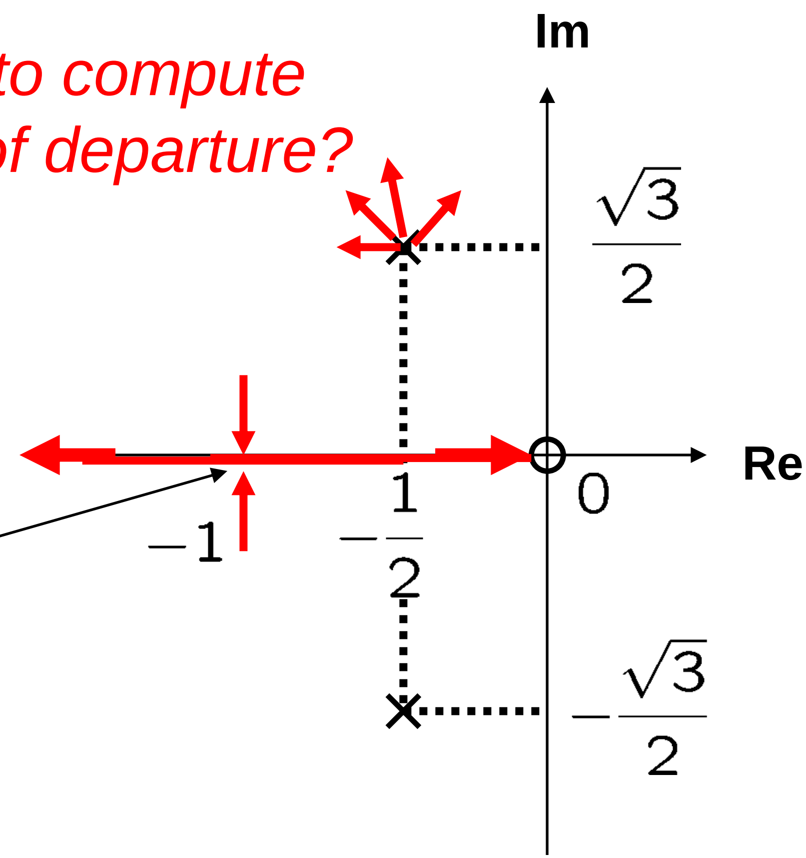
$$L(s) = \frac{s}{s^2 + s + 1}$$

After Steps 0,1,2,3, we obtain

zero $\quad 0$

pole $\quad -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

How to compute angle of departure?



$$\frac{dL(s)}{ds} = 0$$

Breakaway point

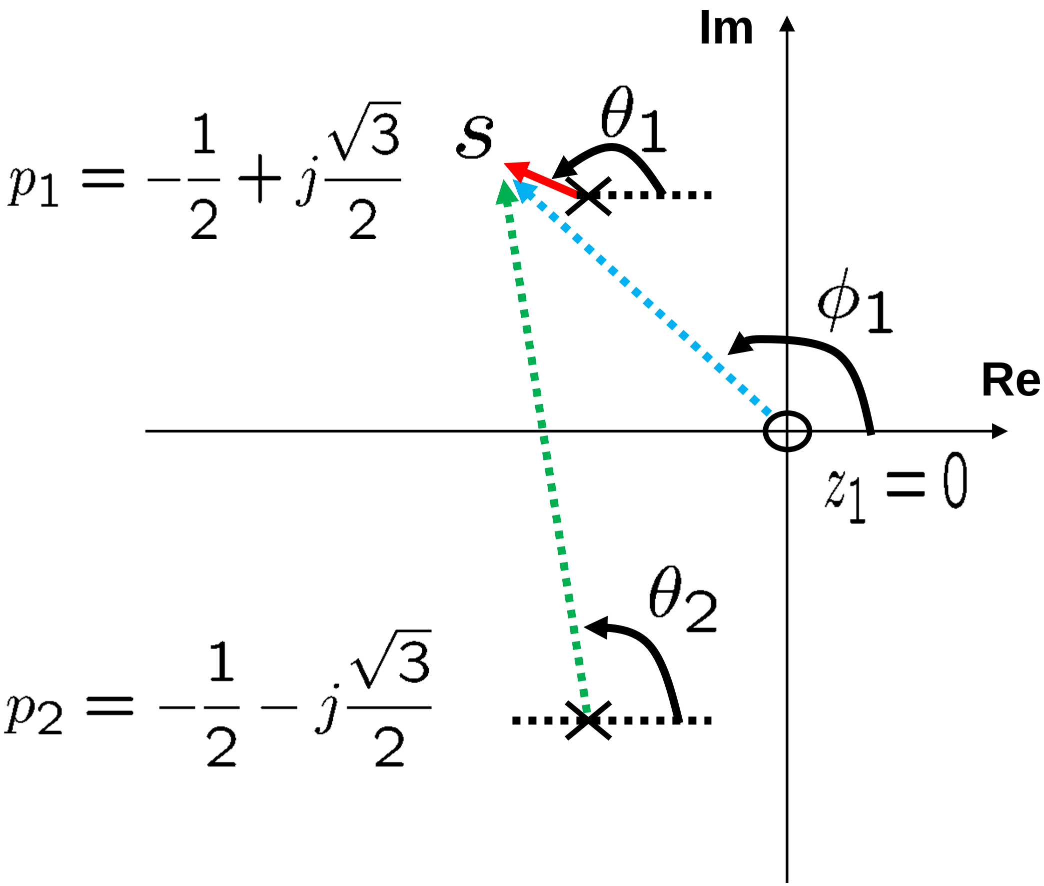
$$s^2 + s + 1 - s(2s+1) = 0$$

$$\Rightarrow s = \pm 1$$

Note that in this case, breakaway point is actually a break-in point. Hence, you can use the term *break-in point*.

Root locus: Step 4. Angle of departure

- Select a point "*s*" near p_1. Use angle condition.



$$\angle L(s) = \angle \frac{s - z_1}{(s - p_1)(s - p_2)}$$

$$= \underline{\angle(s - z_1)} - \underline{\angle(s - p_1)} - \underline{\angle(s - p_2)}$$

$$= \phi_1 - \theta_1 - \theta_2 = 180^\circ$$

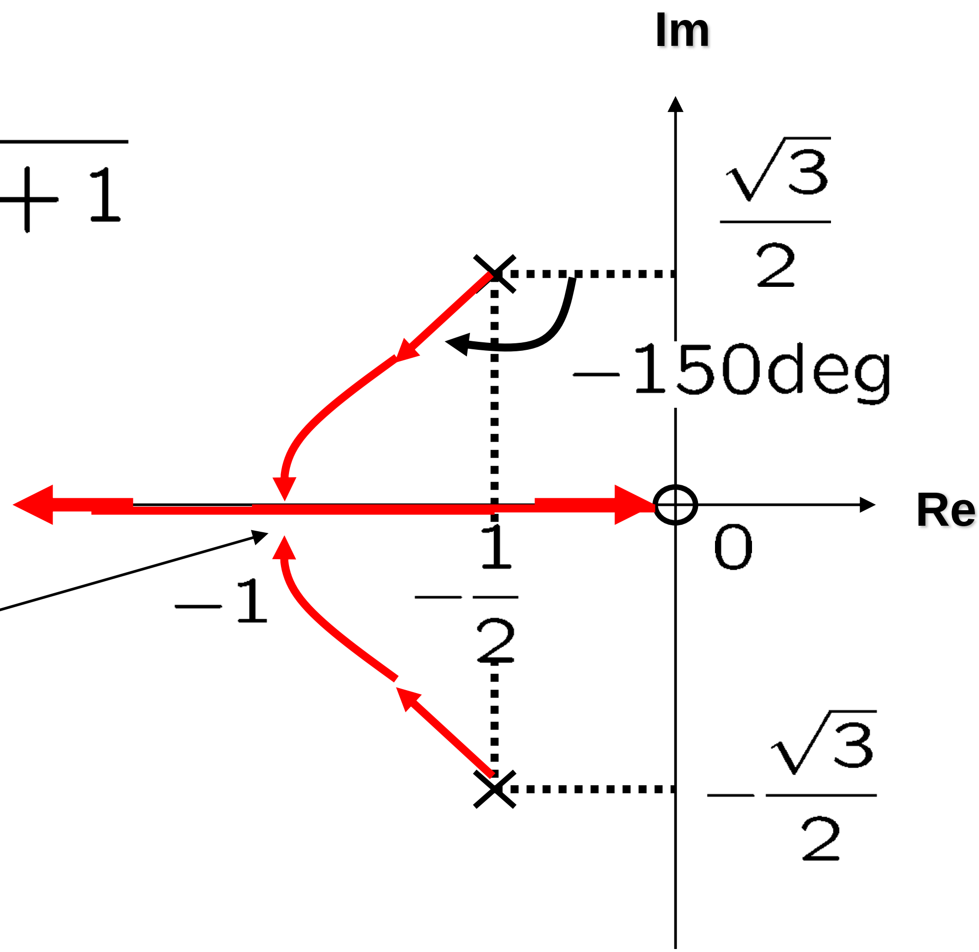
If *s* is close to p_1

$$\phi_1 \approx 120, \;\; \theta_2 \approx 90$$

$$\Rightarrow \;\; \theta_1 = -150$$

Root locus

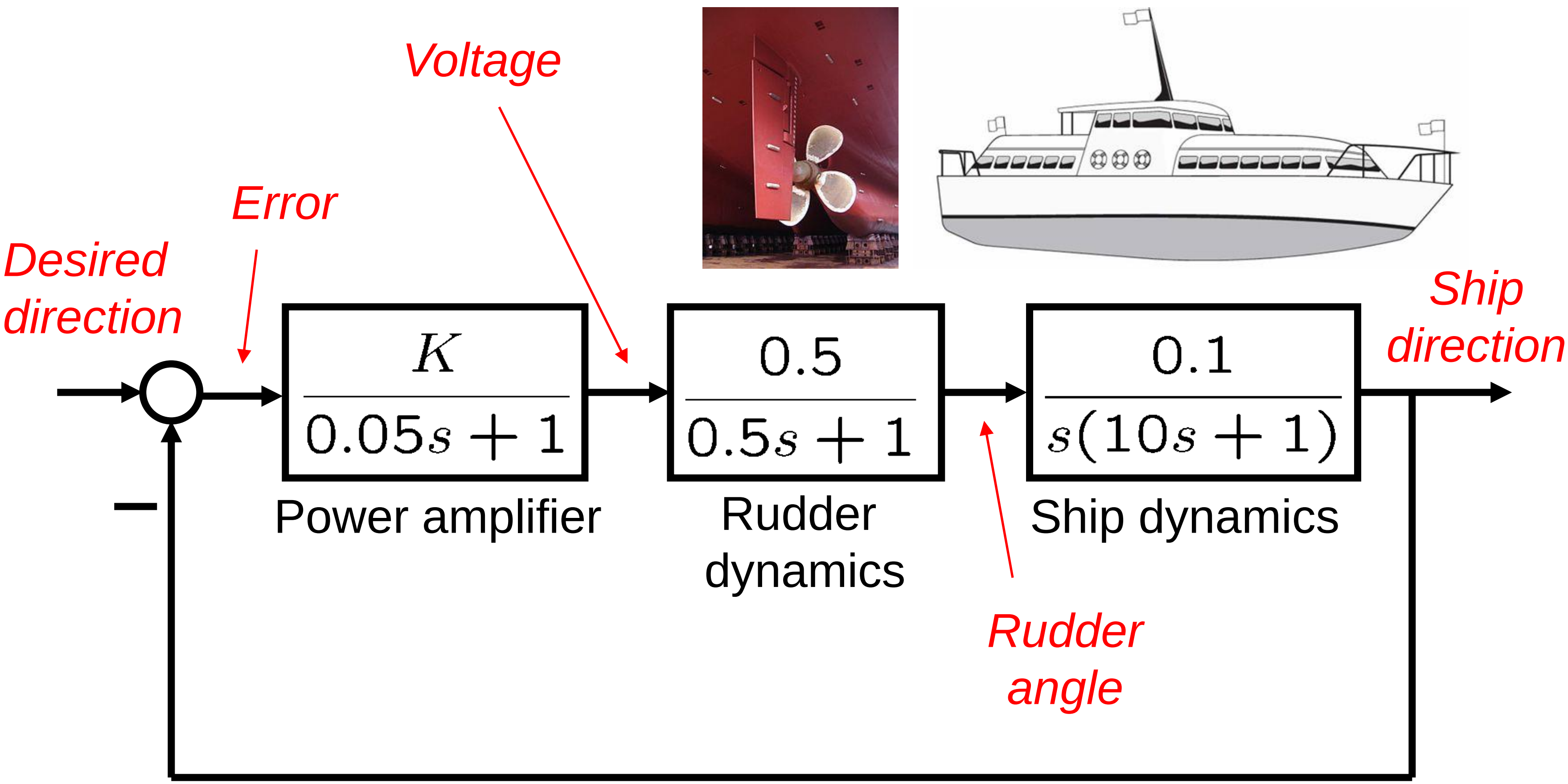
$$L(s) = \frac{s}{s^2 + s + 1}$$



Breakaway point

$$s = -1 \ (K = 1)$$

Example: Ship-steering system



Ship-steering system (cont'd)

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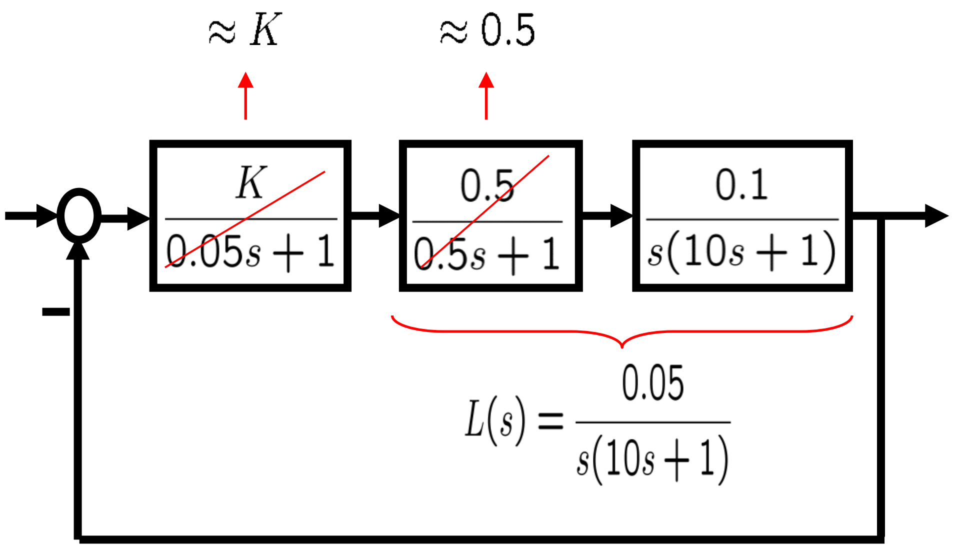
- Time constants
  - Power amplifier: 0.05s (fastest)
  - Rudder dynamics: 0.5s
  - Ship dynamics: 10s (slowest)
- We draw root locus for three cases.
  - We take into account only ship dynamics.
  - We take into account only rudder & ship dynamics.
  - We take into account all three dynamics.

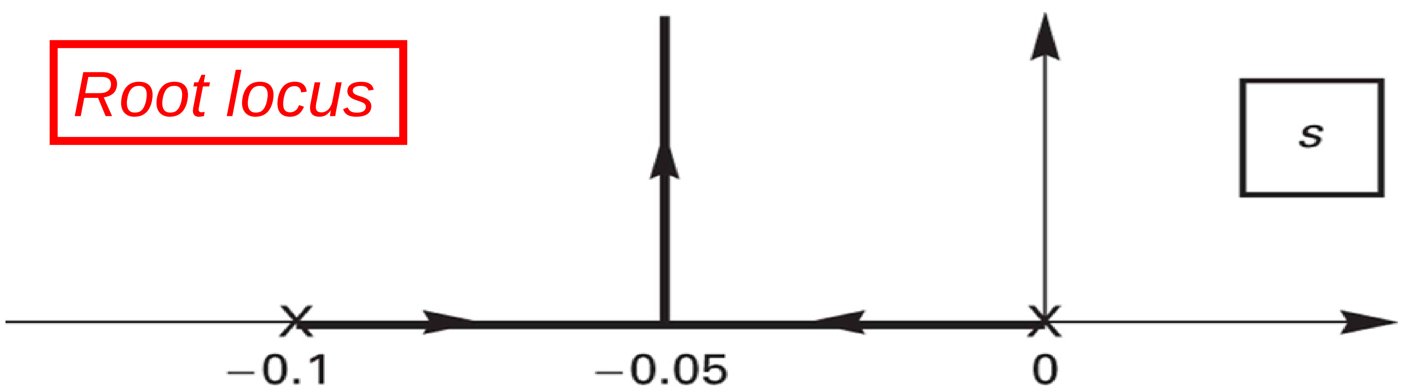
1. Only ship dynamics



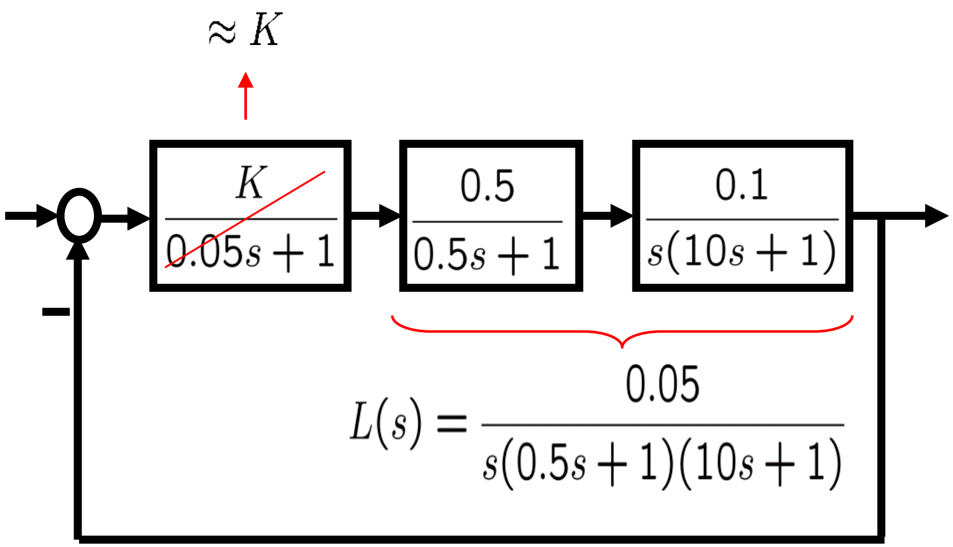
Q: Plot the root locus for the given L(s) and find the 2% settling time.

(Ans: 80 seconds)

Root locus



2. Only rudder & ship dynamics

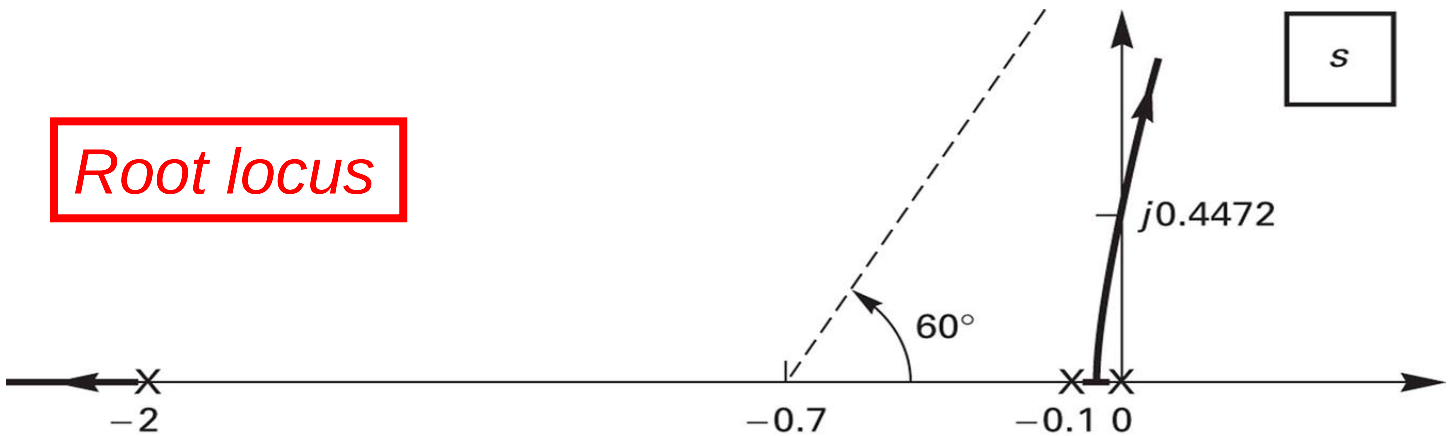


$$L(s) = \frac{0.05}{s(0.5s+1)(10s+1)}$$

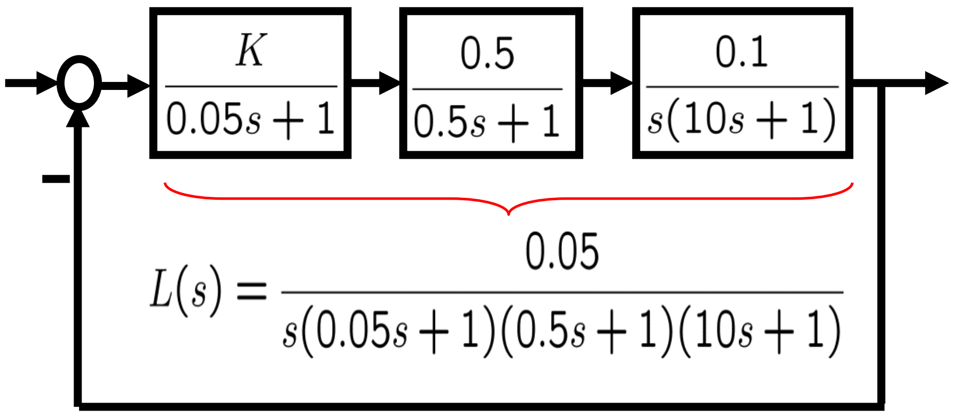
Q: What is the allowable K > 0 for closed-loop stability?

(Ans: 0 < K < 42)

Root locus

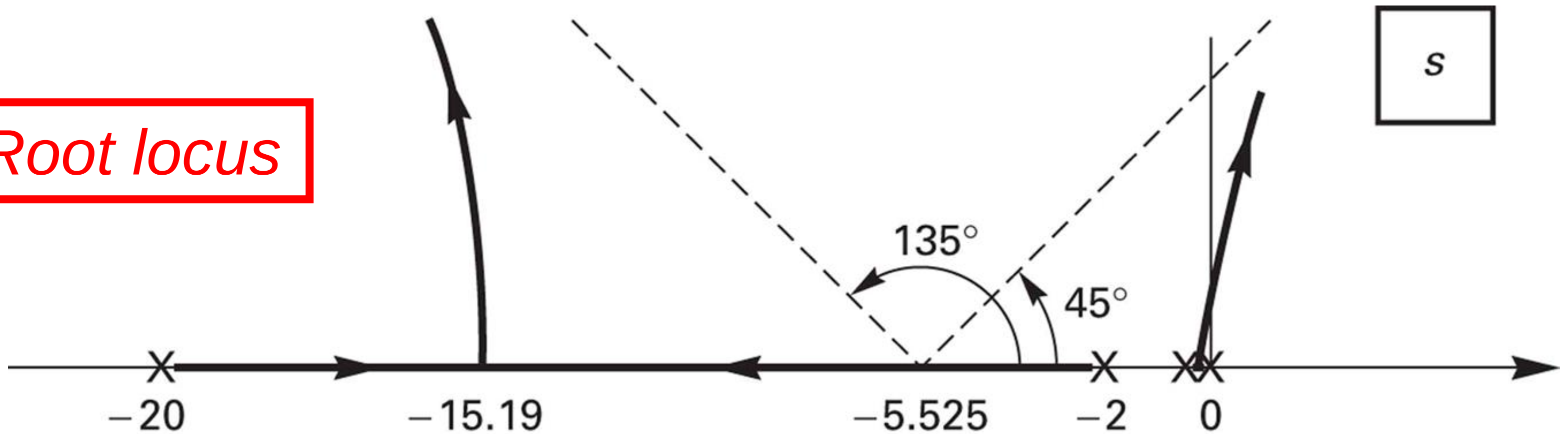


3. All dynamics included



Q: What is the allowable $K > 0$ for closed-loop stability?
(Ans: $0 < K < 38.03$)

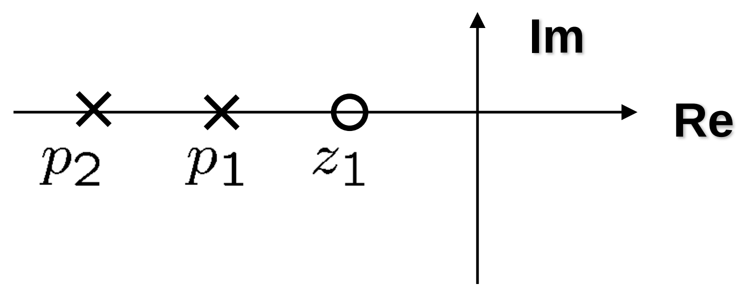
Root locus



Root locus: Step 0

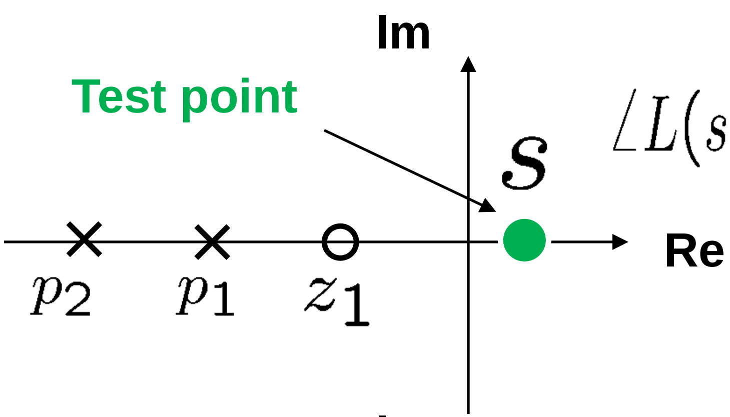
- *Root locus is symmetric w.r.t. the real axis.*
  - Characteristic equation is an equation with real coefficients. Hence, if a complex number is a root, its complex conjugate is also a root.
- *The number of branches = order of L(s)*
  - If L(s) = n(s)/d(s), then Ch. eq. is d(s) + K.n(s) = 0, which has roots as many as the order of d(s).
- *Mark poles of L with "x" and zeros of L with "o".*

$$L(s) = \frac{s - z_1}{(s - p_1)(s - p_2)}$$



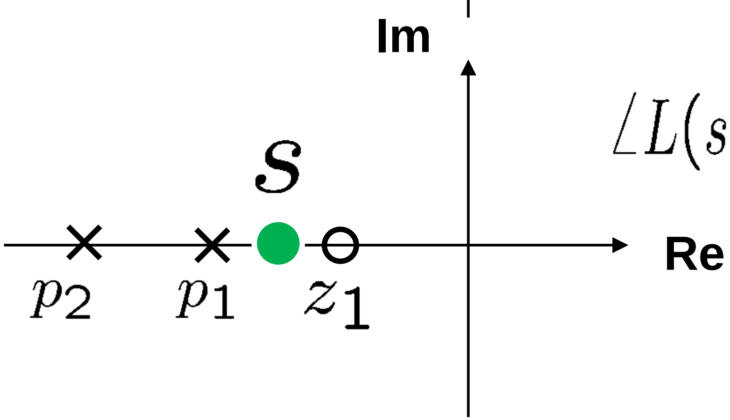
Root locus: Step 1-1

- *RL includes all points on real axis to the left of an odd number of real poles/zeros.*



$$\angle L(s) = \underbrace{\angle(s-z_1)}_{0} - \underbrace{\angle(s-p_1)}_{0} - \underbrace{\angle(s-p_2)}_{0}$$

It does not satisfy angle condition!

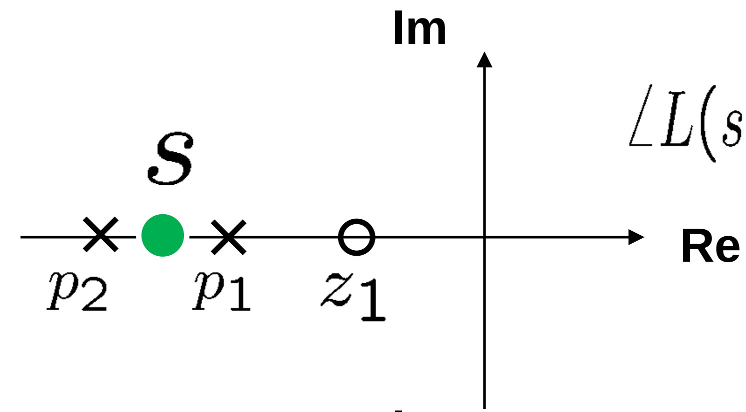


$$\angle L(s) = \underbrace{\angle(s-z_1)}_{180} - \underbrace{\angle(s-p_1)}_{0} - \underbrace{\angle(s-p_2)}_{0}$$

Satisfy angle condition!

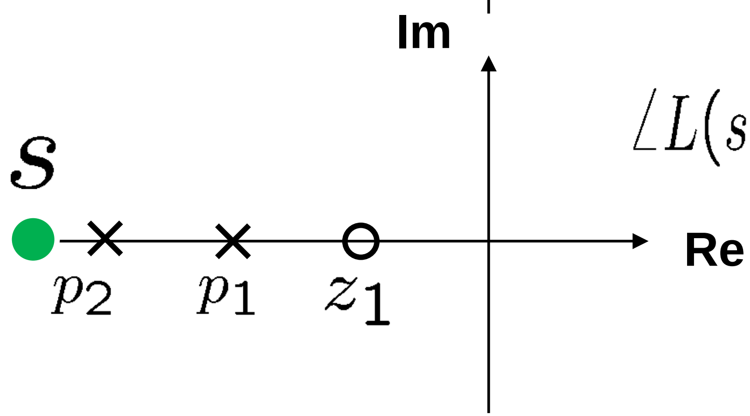
Root locus: Step 1-1 (cont'd)

- *RL includes all points on real axis to the left of an odd number of real poles/zeros.*



$$\angle L(s) = \underbrace{\angle(s - z_1)}_{180} - \underbrace{\angle(s - p_1)}_{180} - \underbrace{\angle(s - p_2)}_{0}$$

It does not satisfy angle condition!



$$\angle L(s) = \underbrace{\angle(s - z_1)}_{180} - \underbrace{\angle(s - p_1)}_{180} - \underbrace{\angle(s - p_2)}_{180}$$

It satisfies angle condition!

Root locus: Step 1-2

- *RL originates from the poles of L, and terminates at the zeros of L, including infinity zeros.*

$$1+K\underbrace{\frac{n(s)}{d(s)}}_{L(s)}=0 \Leftrightarrow d(s)+Kn(s)=0 \Leftrightarrow \frac{1}{K}+\frac{n(s)}{d(s)}=0$$

$$K=0 \Rightarrow d(s)=0$$

$$K=\infty \Rightarrow \frac{n(s)}{d(s)}=0$$

s: Poles of *L*(*s*) **s: Zeros of *L*(*s*)**

Root locus: Step 2-1

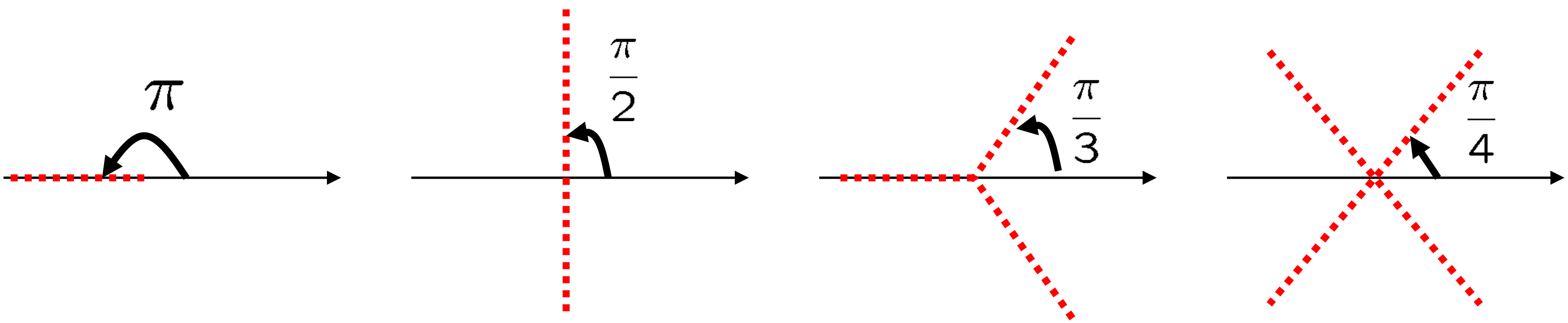
- *Number of asymptotes = relative degree (r) of L:*

$$r = \deg(\text{den}) - \deg(\text{num})$$

- *Angles of asymptotes are*

$$\frac{\pi}{r} \times (2k+1), \; k = 0, 1, \ldots$$

$r = 1$ $\quad$ $r = 2$ $\quad$ $r = 3$ $\quad$ $r = 4$



Root locus: Step 2-1 (cont'd)

- For a very large s, $$L(s) = \frac{n_0 s^{n-r} + \cdots}{s^n + \cdots} \approx \frac{n_0}{s^r}$$
- Ch. eq is approximately

$$1 + KL(s) = 0 \Rightarrow 1 + K\frac{n_0}{s^r} = 0 \Rightarrow s^r + Kn_0 = 0$$

$$\Rightarrow s^r = -Kn_0 < 0 \text{ (we assume } n_0 > 0)$$

$$\Rightarrow \angle s^r = \pi \times (2k+1), \; k = 0, 1, 2, \ldots$$

$$\Rightarrow \boxed{\angle s = \frac{\pi}{r} \times (2k+1), \; k = 0, 1, 2, \ldots}$$

Summary

- Angle condition
- Examples of sketching root locus
- Next
  - Controller design based on root locus