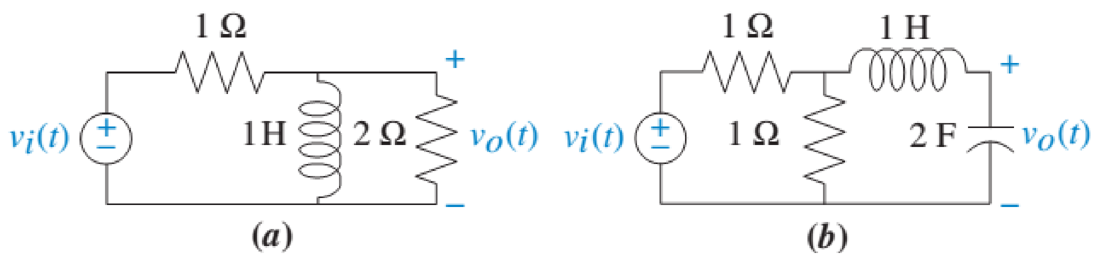
Assignment 4 (ELEC 341 L4_Model_ElecMech)

Problem 1:

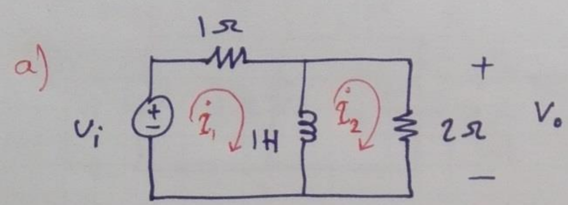
Find the transfer function, $G(s) = V_o(s)/V_i(s)$ for each network shown in the following figure:



Solution:

problem 1 — Assignment 4

a)



We have two methods for solving this problem: { mesh analysis / node analysis

this solution is with mesh analysis.

For writing the mesh equations, (1) we can write them in time domain and then transfer them to Laplace or (2) write them directly in Laplace domain. This solution is based on First method.

$$\text{KVL}\ (1):\ 1 \times i_1(t) + 1 \times \frac{d(i_1 - i_2)}{dt} = V_i(t)$$

$$\text{KVL}\ (2):\ 2 \times i_2(t) + 1 \times \frac{d}{dt}(i_2 - i_1) = 0$$

$$\xrightarrow{\text{Laplace}} \begin{cases} I_1(s) + S(I_1(s) - I_2(s)) = V_i(s) \\ 2I_2(s) + S(I_2(s) - I_1(s)) = 0 \end{cases}$$

← Note that initial condition ($i_{1,2}(0)$) is zero because we want to calculate transfer function.

$$\rightarrow \begin{cases} I_1(S+1) + I_2(-S) = V_i \\ I_1(-S) + I_2(2+S) = 0 \end{cases} \rightarrow \begin{bmatrix} S+1 & -S \\ -S & 2+S \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_i \\ 0 \end{bmatrix}$$

$$I_2 = \frac{\begin{vmatrix} S+1 & V_i \\ -S & 0 \end{vmatrix}}{(S+1)(S+2) - S^2} = \frac{S V_i}{3S + 2}$$

$$V_o = 2I_2 = \frac{S}{1.5S+1} V_i \rightarrow \frac{V_o(S)}{V_i(S)} = \frac{S}{1.5S+1}$$

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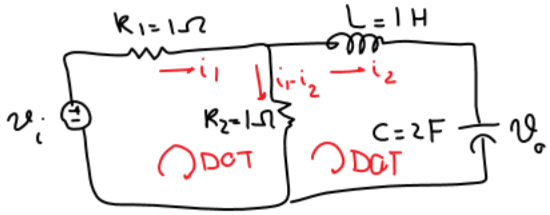
$\frac{V_o(s)}{V_i(s)} = ?$



$i_1 = i_2 + i_3 \rightarrow i_3 = i_1 - i_2$

$v_L = L\frac{di}{dt}$, $v_o = v_c = \frac{1}{C}\int_0^t i\,dt$

$R_1 i_1 + R_2(i_1 - i_2) - v_i = 0 \rightarrow i_1 + i_1 - i_2 - v_i = 0 \rightarrow \underline{2i_1 - i_2 = v_i} \xrightarrow{\mathcal{L}}$

$L\frac{di_2}{dt} + \frac{1}{C}\int_0^t i_2\,dt - R_2(i_1 - i_2) = 0 \rightarrow$ $\underline{2I_1 - I_2 = V_i \quad (1)}$

$\frac{di_2}{dt} + \frac{1}{2}\int_0^t i_2\,dt - i_1 + i_2 = 0 \xrightarrow{\mathcal{L}} sI_2 + \frac{1}{2s}I_2 - I_1 + I_2 = 0 \rightarrow$

$(s + \frac{0.5}{s})I_2 + I_2 = I_1 \xrightarrow[(1)]{\text{sub in}} 2\{(s + \frac{0.5}{s})I_2 + I_2\} - I_2 = V_i \rightarrow$

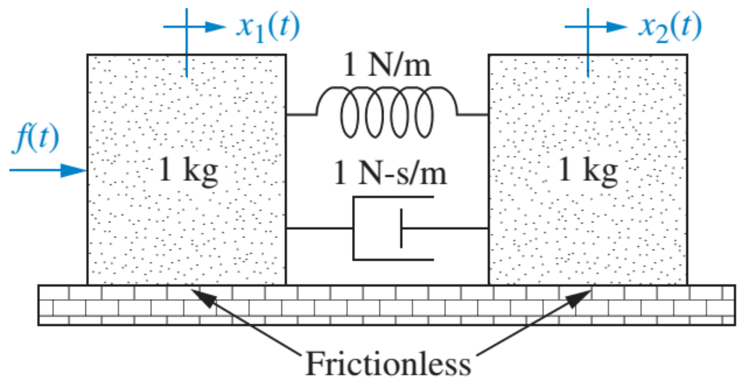
$2I_2(s + \frac{0.5}{s}) + 2I_2 - I_2 = V_i \rightarrow \underline{I_2 = \frac{s.V_i}{2s^2 + s + 1}} \xrightarrow[\text{in (1)}]{\text{sub}} 2I_1 - \frac{sV_i}{2s^2 + s + 1} = V_i \rightarrow$

$\underline{I_1 = \frac{V_i}{2}\left(\frac{2s^2 + 2s + 1}{2s^2 + s + 1}\right)}$; $v_o = \frac{1}{2}\int_0^t i_2\,dt \xrightarrow{\mathcal{L}} V_o = \frac{I_2}{2s} \rightarrow V_o = \frac{1}{2\cancel{s}} \cdot \frac{\cancel{s}V_i}{2s^2 + s + 1}$

$\rightarrow \boxed{\frac{V_o}{V_i} = \frac{1}{4s^2 + 2s + 2}}$

Problem 2:

Find the transfer function, $G(s) = X_2(s)/F(s)$, for the translational mechanical network shown in figure:



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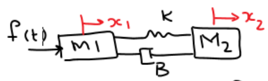
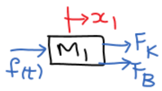
Solution:



M_1:



$$M_1\ddot{x}_1 = F_K + F_B + f(t)$$

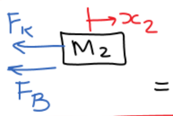
F_K is always: $F_K = -K(x_{left} - x_{right})$

F_B " " : $F_B = -B(\dot{x}_{left} - \dot{x}_{right})$ $\Rightarrow$

$$M_1\ddot{x}_1 = -K(x_1 - x_2) - B(\dot{x}_1 - \dot{x}_2) + f(t) \longrightarrow$$

$$\boxed{M_1\ddot{x}_1 = f(t) - B(\dot{x}_1 - \dot{x}_2) - K(x_1 - x_2)}$$

M_2:



$$M_2\ddot{x}_2 = -F_K - F_B =$$

$$= -\{(-K)(x_1 - x_2) + (-B)(\dot{x}_1 - \dot{x}_2)\}$$

$$\longrightarrow \boxed{M_2\ddot{x}_2 = -B(\dot{x}_2 - \dot{x}_1) - K(x_2 - x_1)}$$

$$\longrightarrow \begin{cases} \ddot{x}_1 = f(t) - \dot{x}_1 + \dot{x}_2 - x_1 - x_2 \\ \ddot{x}_2 = -\dot{x}_2 + \dot{x}_1 - (x_2 - x_1) \end{cases} \longrightarrow \begin{cases} \ddot{x}_1 + \dot{x}_1 - \dot{x}_2 + x_1 - x_2 = f(t) \\ \ddot{x}_2 + \dot{x}_2 - \dot{x}_1 + x_2 - x_1 = 0 \end{cases}$$

$$\xrightarrow{\mathcal{L}} \begin{cases} s^2X_1 + sX_1 - sX_2 + X_1 - X_2 = F(s) \\ s^2X_2 + sX_2 - sX_1 + X_2 - X_1 = 0 \end{cases} \longrightarrow$$

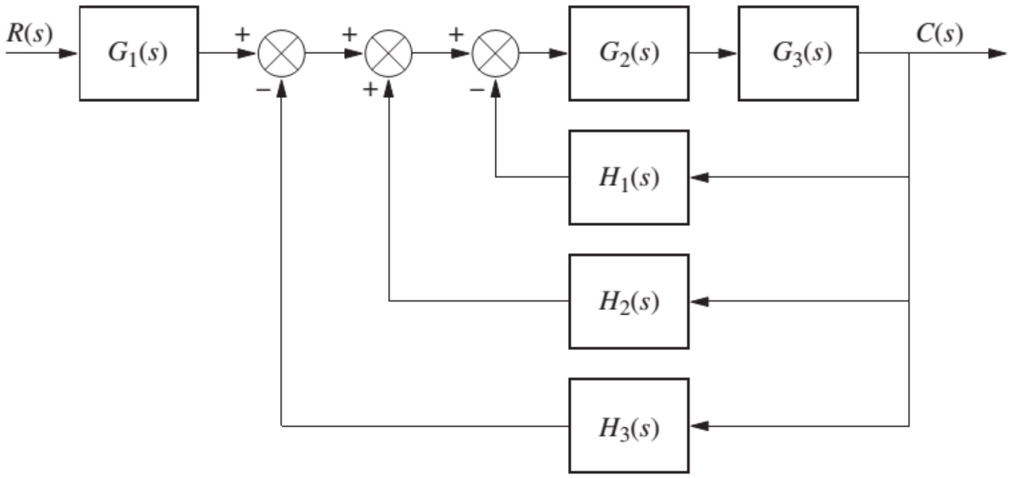
$$\begin{cases} (s^2+s+1)X_1 - (s+1)X_2 = F(s) \\ -(s+1)X_1 + (s^2+s+1)X_2 = 0 \end{cases} \longrightarrow$$

$$X_2 = \frac{\begin{vmatrix} s^2+s+1 & F(s) \\ -(s+1) & 0 \end{vmatrix}}{\begin{vmatrix} s^2+s+1 & -(s+1) \\ -(s+1) & s^2+s+1 \end{vmatrix}} = \frac{+F(s)(s+1)}{(s^2+s+1)^2 - (s+1)^2} \longrightarrow$$

$$X_2 = \frac{(s+1)F(s)}{s^2(s^2+2s+2)} \longrightarrow \boxed{\frac{X_2}{F(s)} = \frac{(s+1)}{s^2(s^2+2s+2)}}$$

Problem 3:

Reduce the block diagram shown in figure to a single transfer function:



Solution:

We solve the problem by following the steps in Figure . First, the three summing junctions can be collapsed into a single summing junction, as shown in Figure (*a*).

Second, recognize that the three feedback functions, $H_1(s)$, $H_2(s)$, and $H_3(s)$, are connected in parallel. They are fed from a common signal source, and their outputs are summed. The equivalent function is $H_1(s) - H_2(s) + H_3(s)$. Also recognize that $G_2(s)$ and $G_3(s)$ are connected in cascade. Thus, the equivalent transfer function is the product, $G_3(s)G_2(s)$. The results of these steps are shown in Figure (*b*).

Finally, the feedback system is reduced and multiplied by $G_1(s)$ to yield the equivalent transfer function shown in Figure (*c*).

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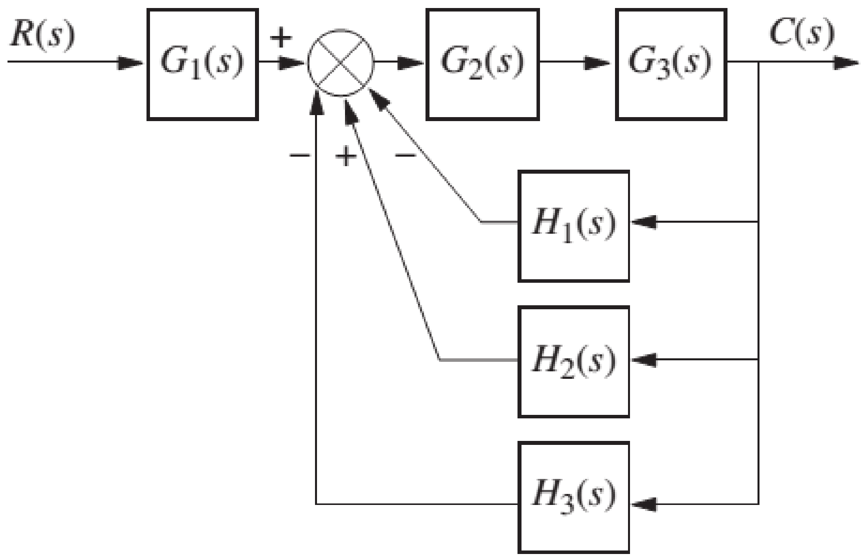
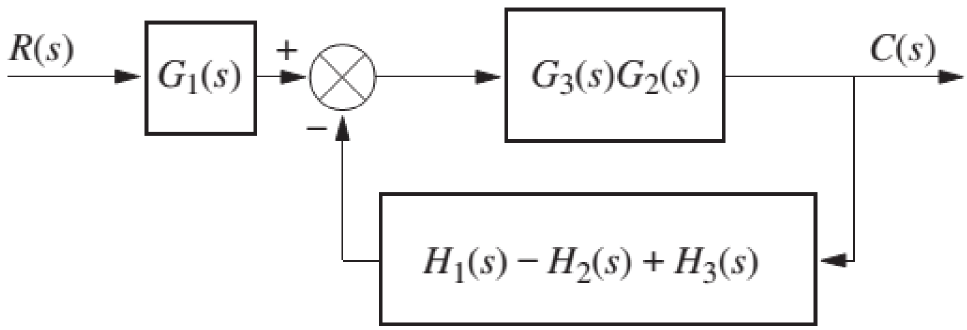
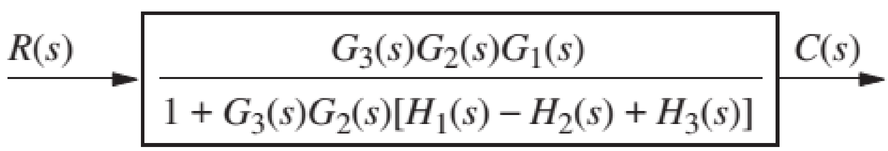


(a)



(b)



(c)

Problem 4:

Calculate the transfer function of the system represented in the state space model as

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad and \quad D = [0]$$

The formula for the transfer function when $D = [0]$ is -

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Substitute, A, B & C matrices in the above equation.

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}}{(s+1)s - 1(-1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix}}{s^2 + s + 1} = \frac{1}{s^2 + s + 1}$$

Therefore, the transfer function of the system for the given state space model is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

Find $(sI-A)^{-1} = ?$

$$s\begin{bmatrix}1&0\\0&1\end{bmatrix} - \begin{bmatrix}-1&-1\\1&0\end{bmatrix} = \begin{bmatrix}s&0\\0&s\end{bmatrix} - \begin{bmatrix}-1&-1\\1&0\end{bmatrix}$$

$$= \begin{bmatrix}s&0\\0&s\end{bmatrix} + \begin{bmatrix}1&1\\-1&0\end{bmatrix} \rightarrow sI-A = \begin{bmatrix}s+1&1\\-1&s\end{bmatrix}$$

$(sI-A)^{-1} = ?$; $Q = \begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \dfrac{1}{\det Q}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$$\det(sI-A) = s(s+1)+1 = s^2+s+1 \longrightarrow$$

$$(sI-A)^{-1} = \frac{1}{s^2+s+1}\begin{bmatrix}s&-1\\1&s+1\end{bmatrix}$$

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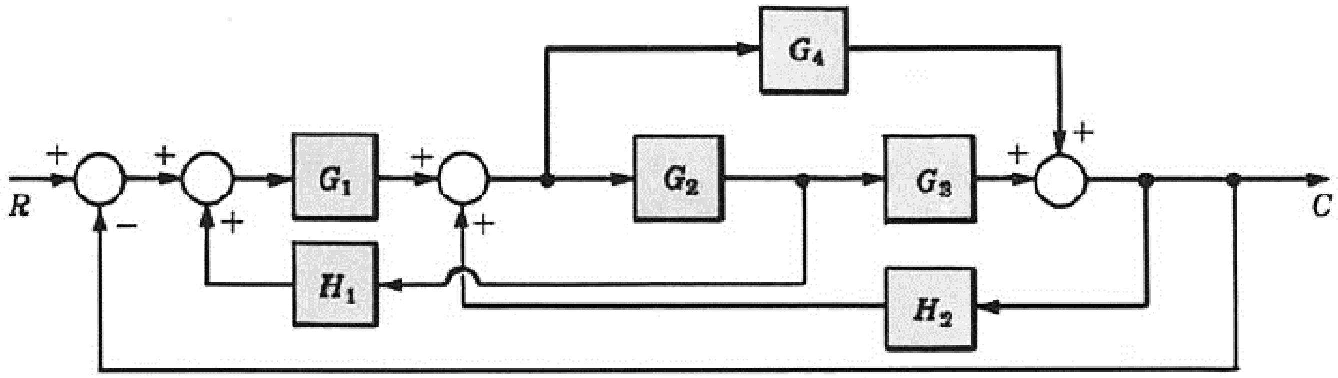
Problem 5:

Find the signal flow graph for the following block diagram:



Solution: