a place of mind

ELEC 341: Systems and Control

Lecture 10

Time response: Examples

Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - Electrical
  - Electromechanical
  - Mechanical
- Linearization, delay

Analysis

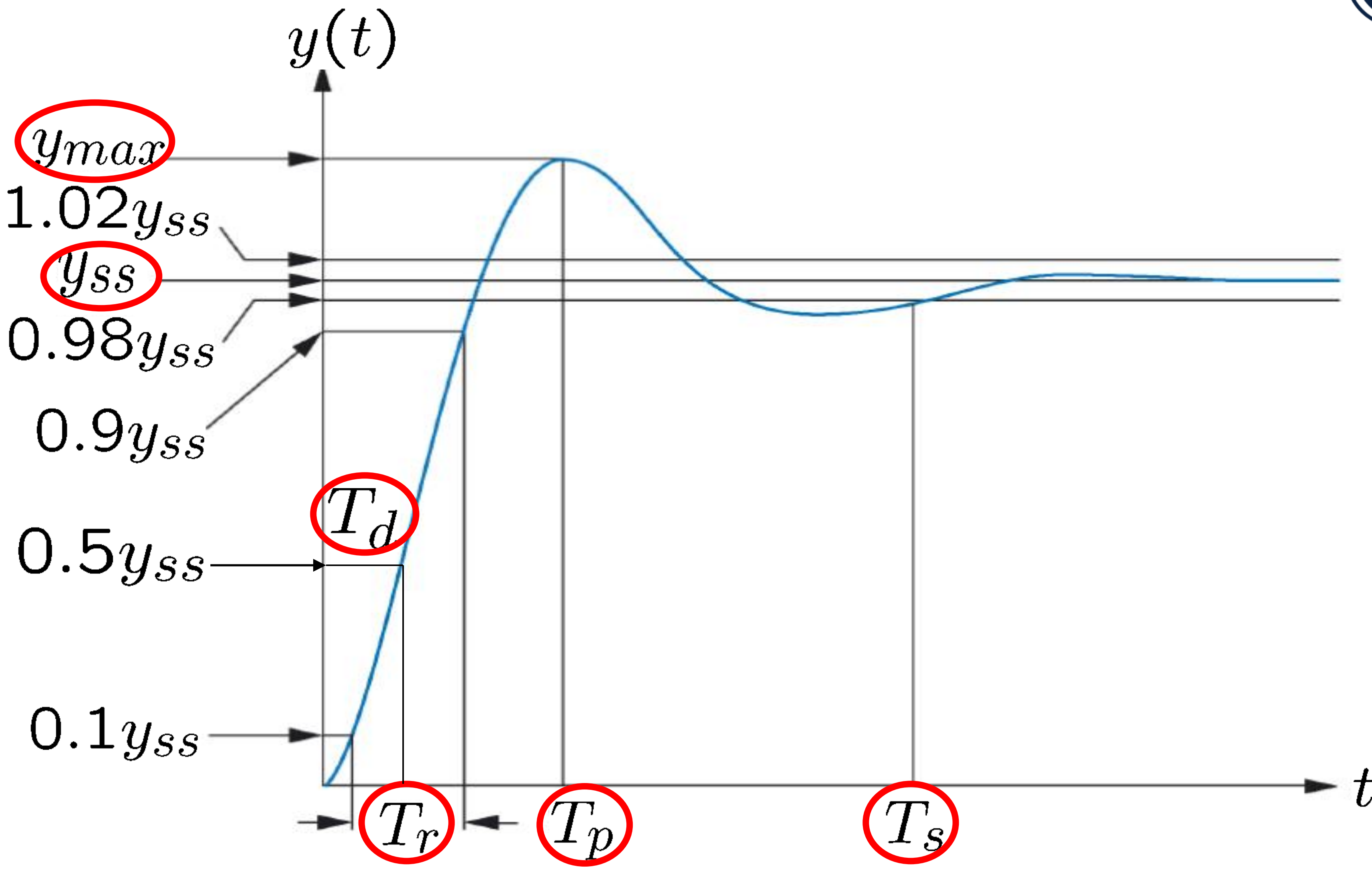
- Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

Typical step response (review)



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Performance measures

- Transient response
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time
- Steady state response
  - Steady state error

(Done for 1st & 2nd order systems)

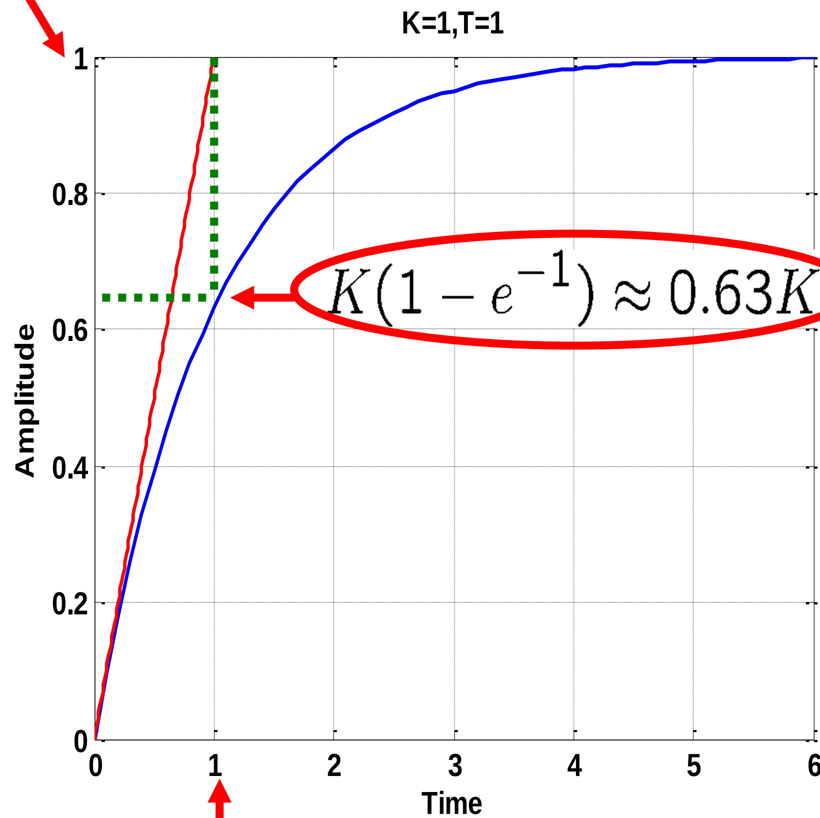
Next, we will connect these performance measures with s-domain.

(Done)

Step response for 1st-order system (review)

K *DC gain*

$$G(s) = \frac{K}{Ts+1}$$



$$K(1-e^{-1}) \approx 0.63K$$

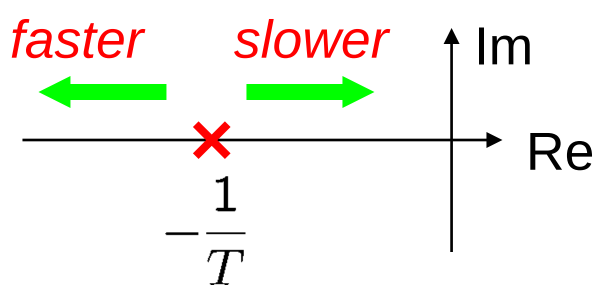
T *Time constant*

Settling time

5%: 3*T* = 3 / |Re|

2%: 4*T* = 4 / |Re|

× Pole



Step response of 2nd-order system for various damping ratios (review)

- Undamped

  $\zeta = 0$

- Underdamped

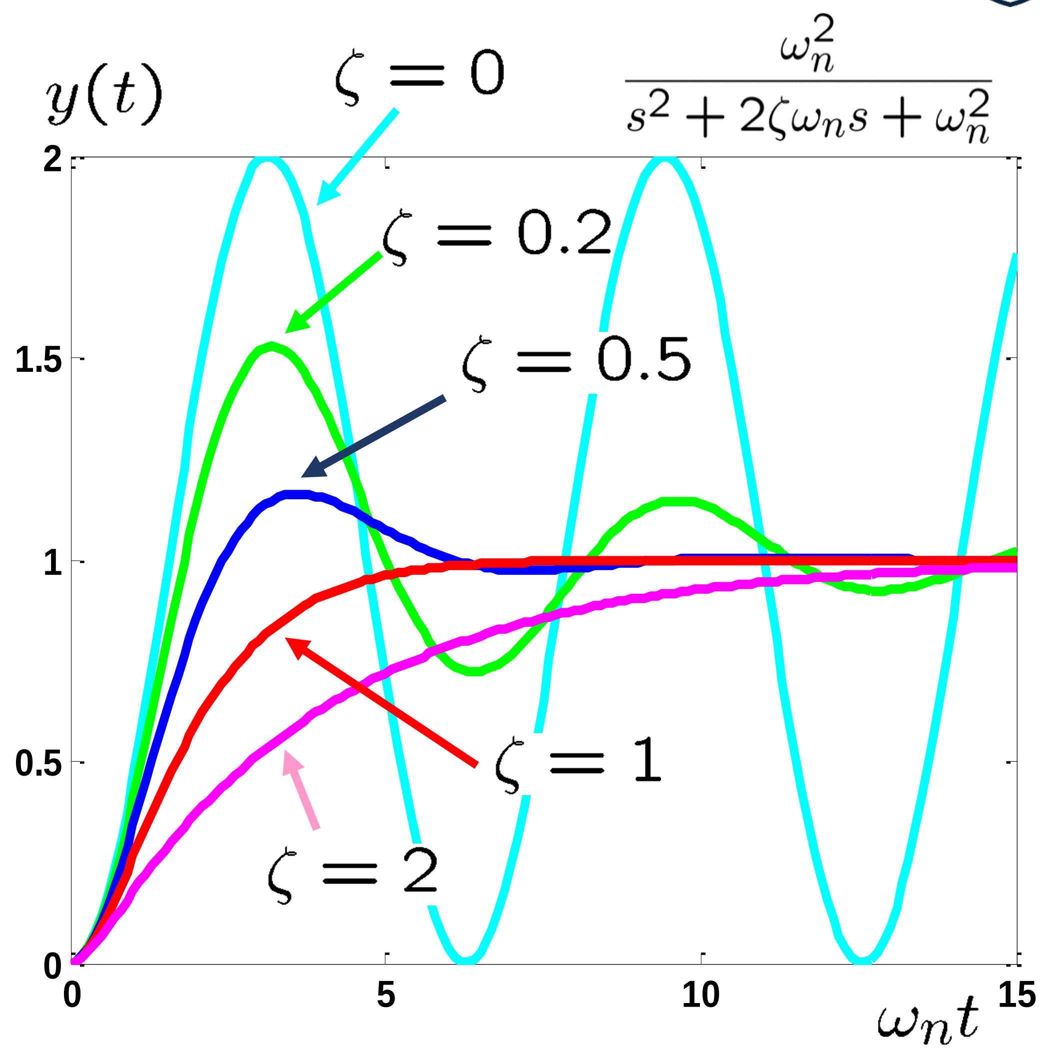
  $0 < \zeta < 1$

- Critically damped

  $\zeta = 1$

- Overdamped

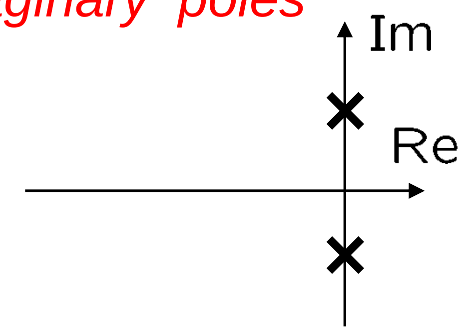
  $\zeta > 1$



Pole locations & damping

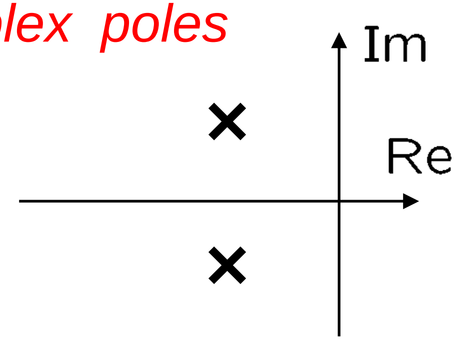
- Undamped

Imaginary poles



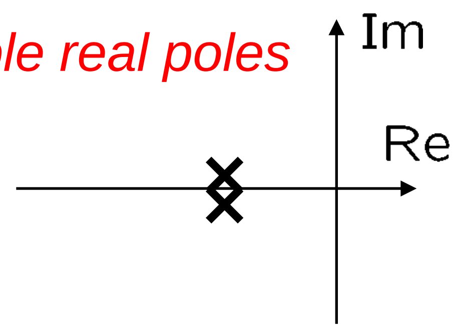
- Underdamped

Complex poles



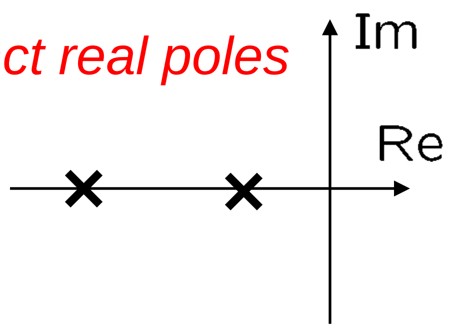
- Critically damped

Double real poles



- Overdamped

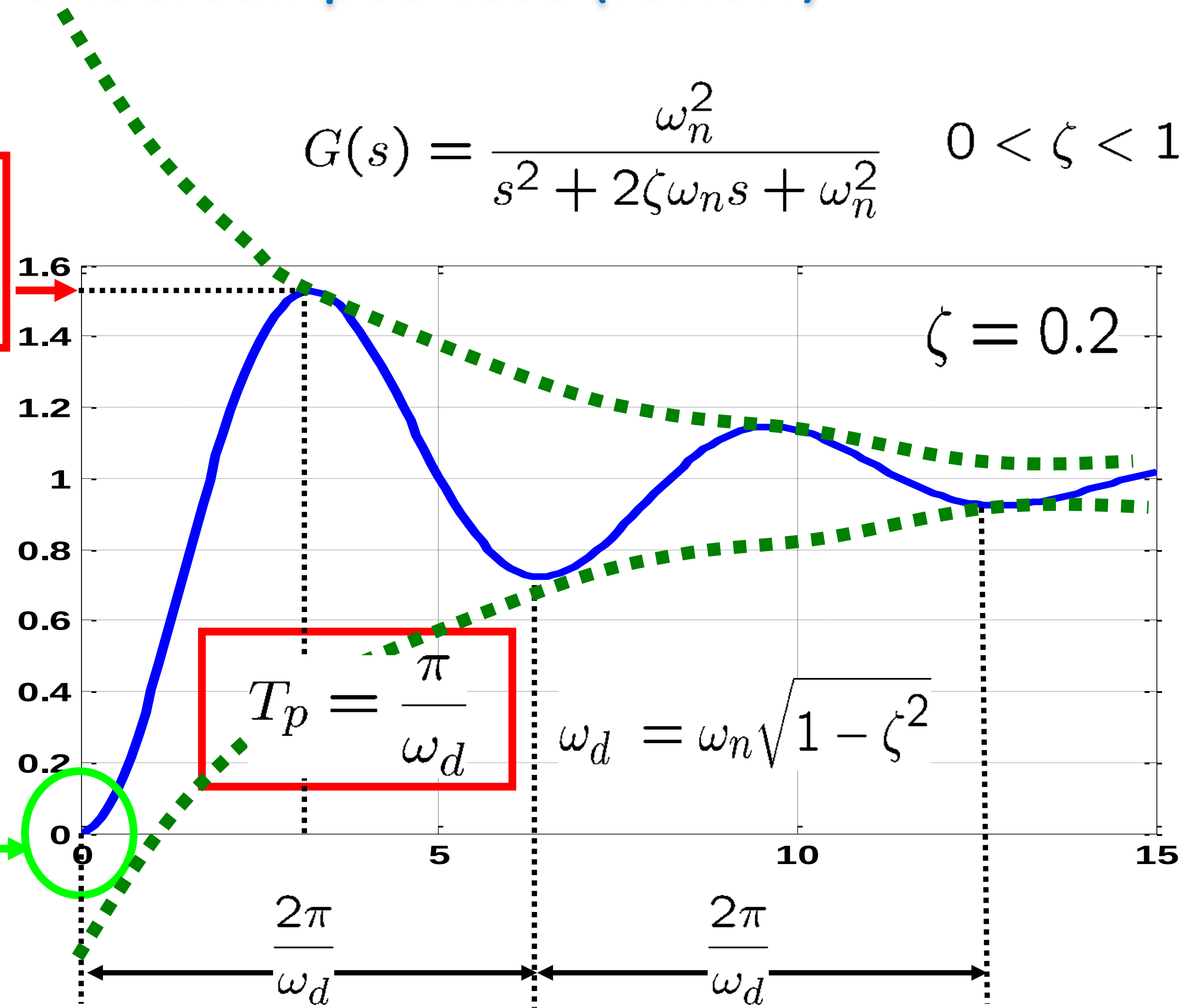
Distinct real poles



Step response for 2nd order system: Underdamped case (review)

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 0 < \zeta < 1$$

$$y_{max} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$



$$T_p = \frac{\pi}{\omega_d}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$y(0) = 0$$
$$y'(0) = 0$$

Properties of underdamped 2nd order system in terms of ζ and ω_n (review)

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 0 < \zeta < 1$$

	(5%) (2%)
Settling time	$\approx \frac{3}{\zeta\omega_n}$ or $\frac{4}{\zeta\omega_n}$
Peak time	$\frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
Peak value	$1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$
Percent overshoot	$100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

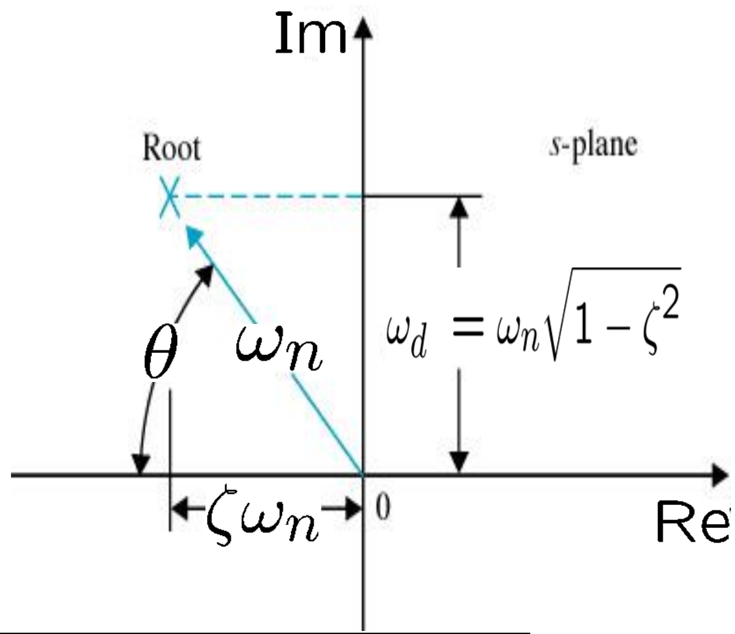
Step response properties of underdamped 2nd order system in terms of pole locations (review)

- Poles ($0 < \zeta < 1$)

$$s = -\zeta\omega_n \pm j\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

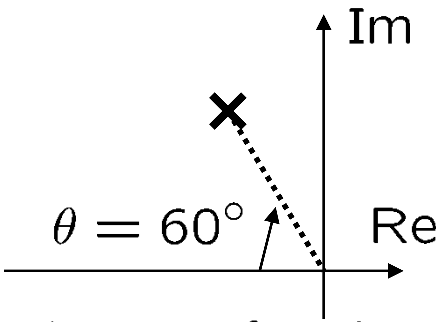
$$\zeta = \cos\theta$$

T = Time Constant = $1/\zeta\omega_n$

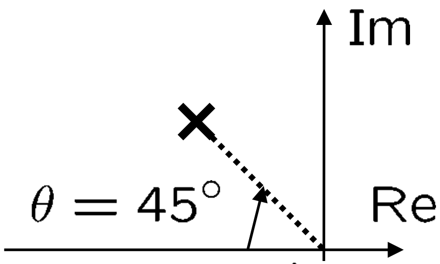


Pole			Performance	
Real part	$\zeta\omega_n$	determines	$T_s = \frac{3}{\zeta\omega_n}, \frac{4}{\zeta\omega_n}$	$= \frac{3}{\lvert\mathrm{Re}\rvert}, \frac{4}{\lvert\mathrm{Re}\rvert}$
Imag. part	ω_d	determines	$T_p = \frac{\pi}{\omega_d}$	$= \frac{\pi}{\lvert\mathrm{Im}\rvert}$
Angle	θ	determines	overshoot	

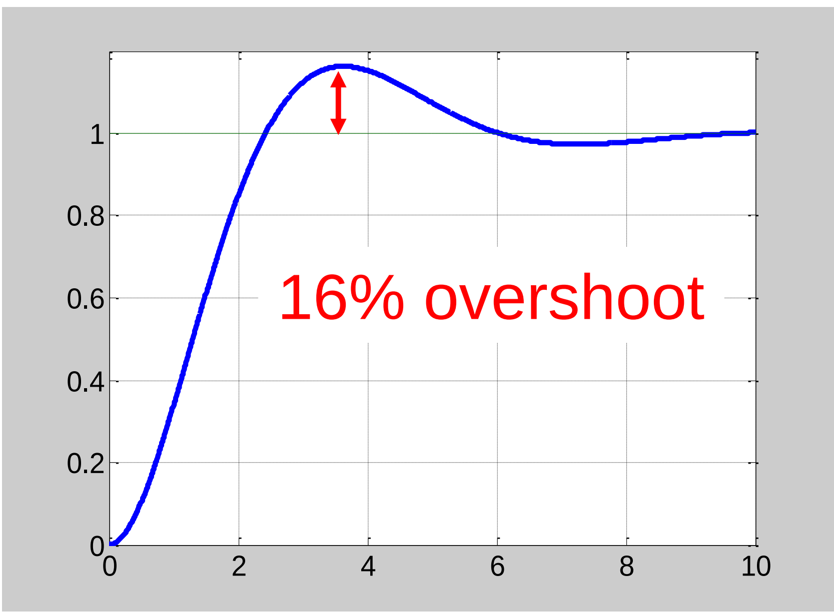
Angle θ and the overshoot (review)

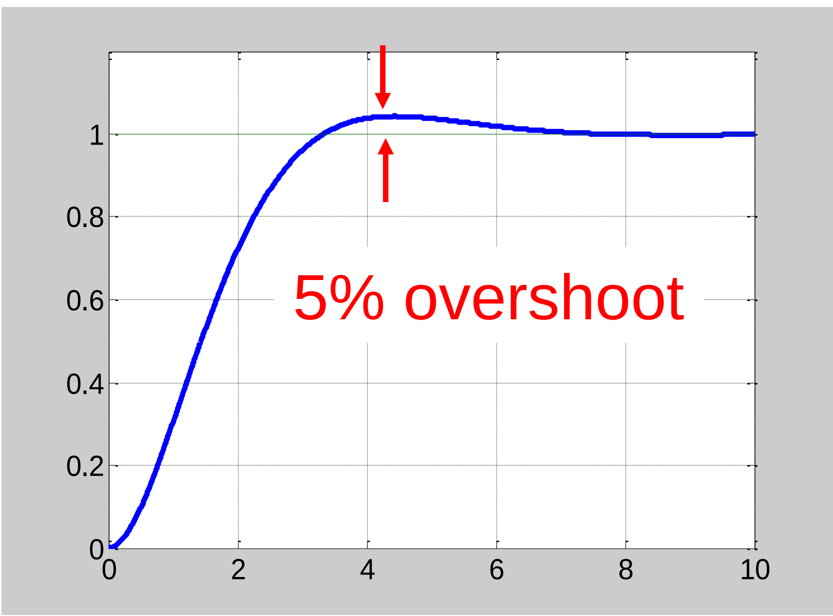


$$\zeta = \cos\theta = 0.5$$

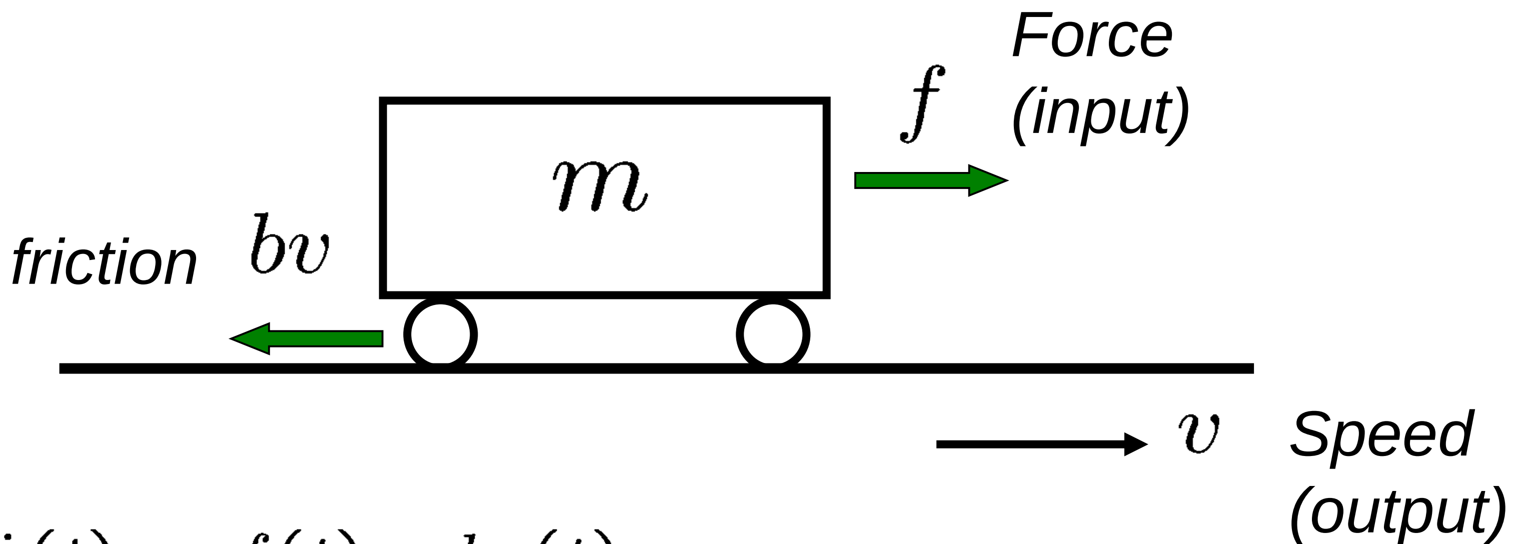


$$\zeta = \cos\theta = \frac{1}{\sqrt{2}} \approx 0.707$$





Example 1: Cruise control



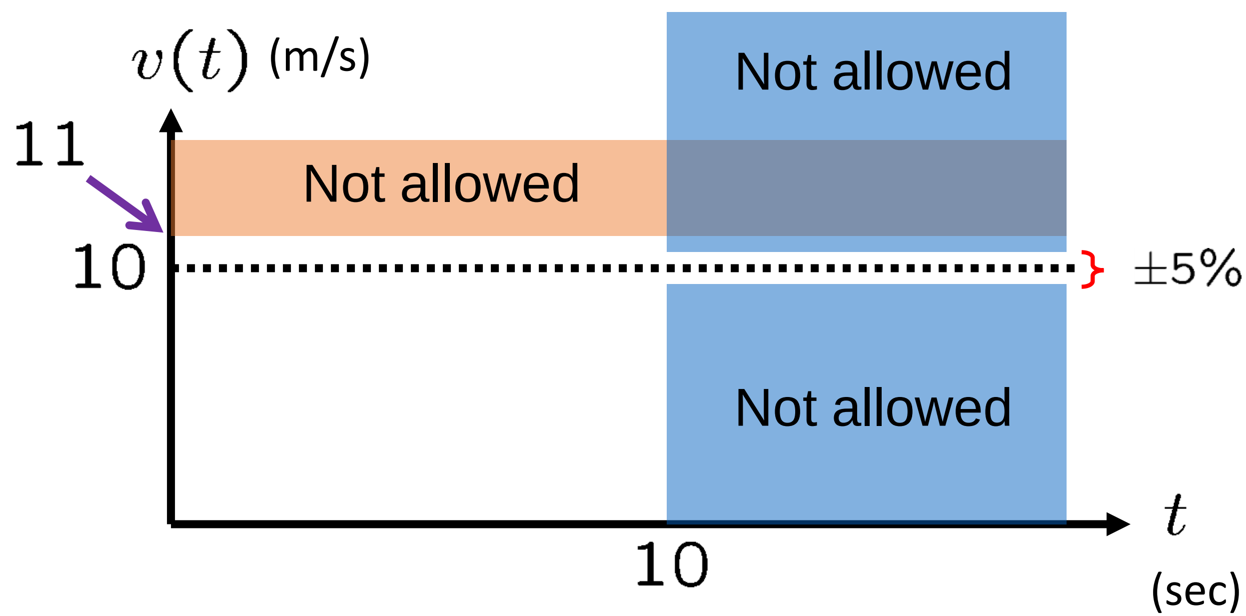
$$m\dot{v}(t) = f(t) - bv(t)$$

$$\Rightarrow \quad \frac{V(s)}{F(s)} = \frac{1}{ms+b}$$

$$\begin{aligned} m &= 1000 \text{ (kg)} \\ b &= 50 \text{ (N}\cdot\text{sec/m)} \end{aligned}$$

Example 1 (cont'd)

- Control goal: SS (steady-state) speed 10 m/s (= 36 km/h).
  - Stable
  - Zero steady-state error
  - Percent overshoot of 10 %
  - 5% T_s < 10 sec



Example 1: Open-loop control (cont'd)

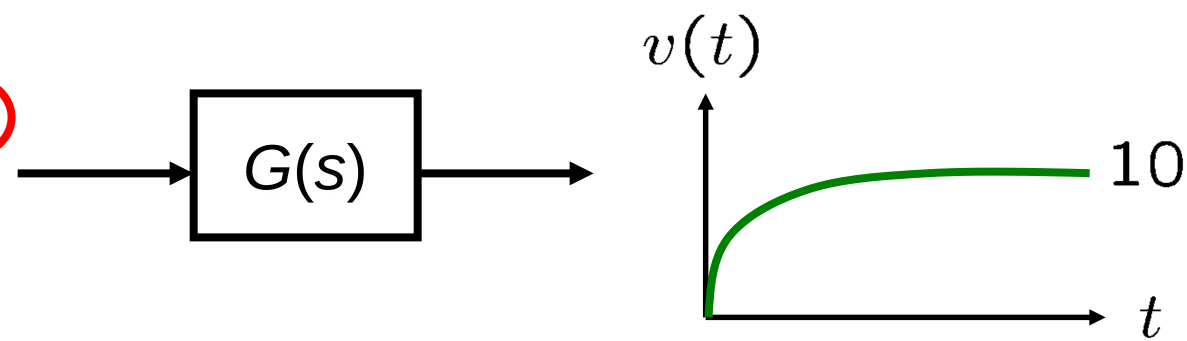
Solution:

$$G(s) = \frac{V(s)}{F(s)} = \frac{1}{1000s + 50} = \frac{1/50}{20s + 1}$$

T = 20

- Obtain the amplitude of the step force necessary to get the steady-state speed of 10 m/s. This is shown by *A* (or *R*).

$$F(s) = \frac{A}{s}$$



$$AG(0) = 10 \quad \Rightarrow \quad A = \frac{10}{G(0)} = 500(N)$$

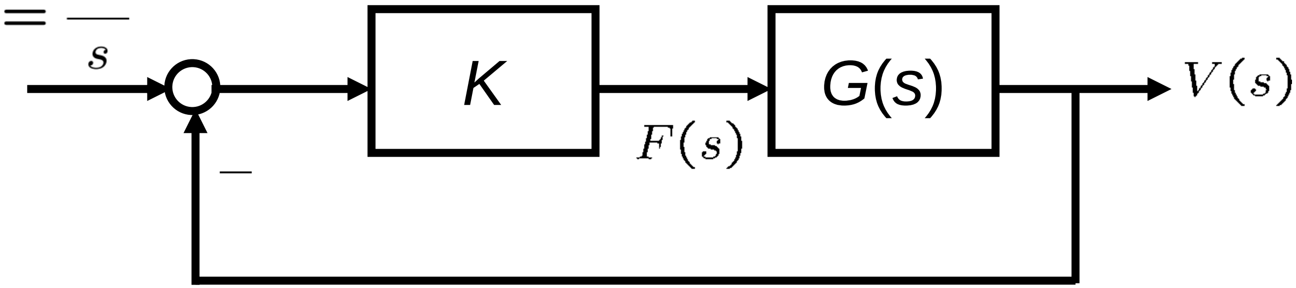
- However, by open-loop control, 5% T_s = 60 sec! and thus Too Slow!
- Note that 5% T_s = 3T or 3×20 = 60 sec, which we need it to be less 10 sec (which obviously is not).

Example 1: P control (cont'd)

- Block diagram

$$G(s) = \frac{1}{1000s + 50}$$

$$V_{desired}(s) = \frac{10}{s}$$



$$\frac{V(s)}{V_{desired}(s)} = \frac{KG(s)}{1 + KG(s)} = \frac{K}{1000s + 50 + K} = \frac{K/(50+K)}{\underbrace{[1000/(50+K)]}_{T}\, s + 1}$$

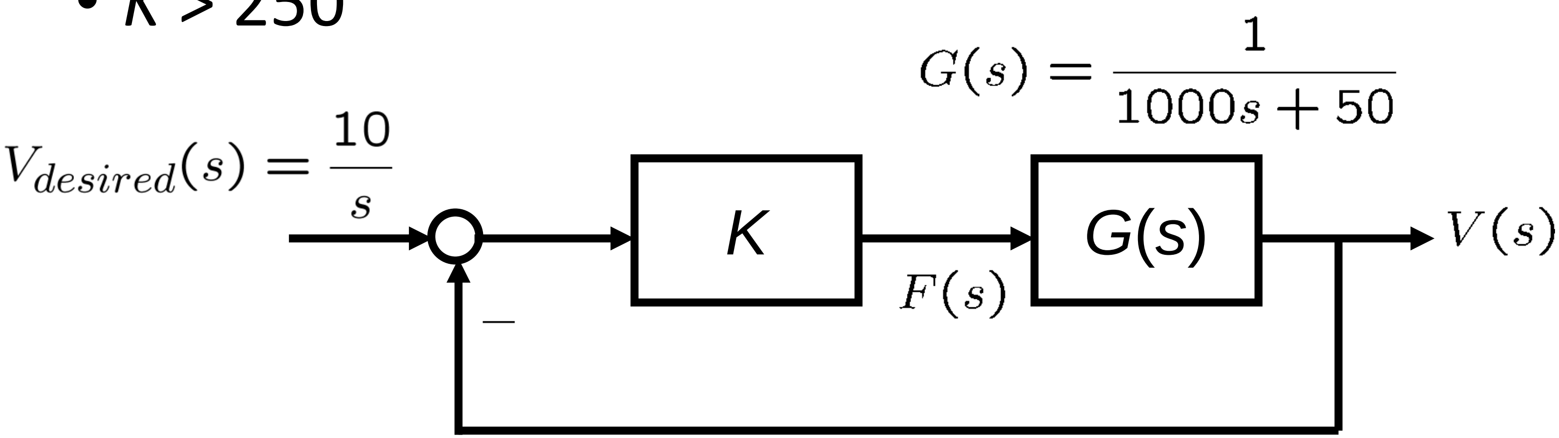
No overshoot

- 5% T_s < 10 sec ➡ $3T$ < 10 sec

➡ $3 \times \frac{1000}{50+K} < 10$ ➡ $300 < 50 + K$ ➡ $K > 250$

Example 1: P control (cont'd)

- $K > 250$



- To have zero SS error:

$$e_{ss} = \frac{10}{1 + K_p} = \frac{10}{1 + L(0)} = \frac{10}{1 + KG(0)} = \frac{10}{1 + K/50}$$

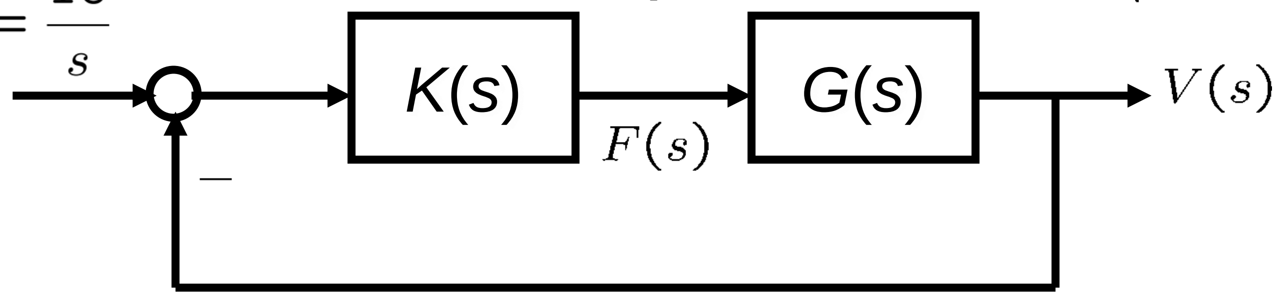
$$e_{ss} = 0 \quad \Rightarrow \quad K = \infty$$

Example 1: PI control (cont'd)

- Block diagram

$$K(s) = c_p + \frac{c_i}{s} \qquad G(s) = \frac{1}{1000s + 50}$$

$$V_{desired}(s) = \frac{10}{s}$$



$$\frac{V(s)}{V_{desired}(s)} = \frac{K(s)G(s)}{1 + K(s)G(s)} = \frac{c_p s + c_i}{1000s^2 + (c_p + 50)s + c_i}$$

- Stability: **OK**
- Zero SS error: **OK** $K_p = L(0) = K(0)G(0) = \infty$

Example 1: PI control (cont'd)

- How to satisfy transient requirements?
  - **Tune controller parameters** to place poles of the closed-loop transfer function at the "right places".

$$\frac{V(s)}{V_{desired}(s)} = \frac{c_p s + c_i}{1000s^2 + (c_p + 50)s + c_i}$$

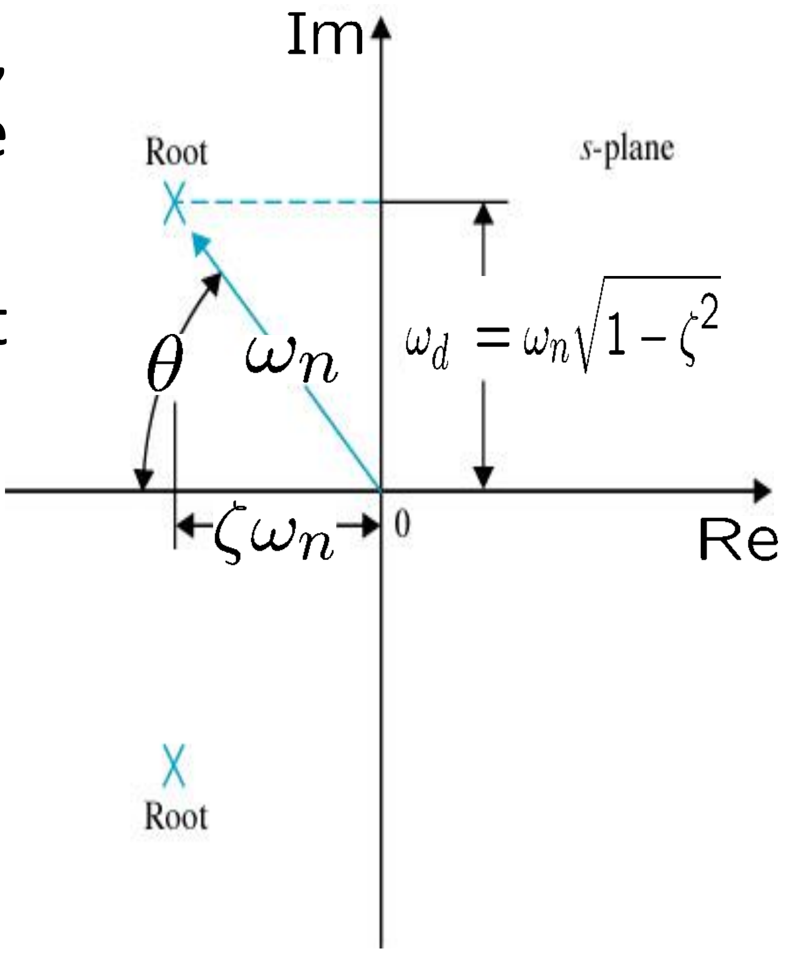
- We will learn
  - where the "right places" are (today: Example 2)
  - how to tune controller parameters (from next lecture)
- We can show (by trial and error and computer simulation) that the following controller is one of the controllers that can satisfy all the required conditions:

$$K(s) = 1050 + \frac{274}{s}$$

Example 2

- For the following design specs, find the allowable region of the second-order poles:

  PO at most 5% and T_s (5%) at most T_{smax} sec.



Example 2 (cont'd)

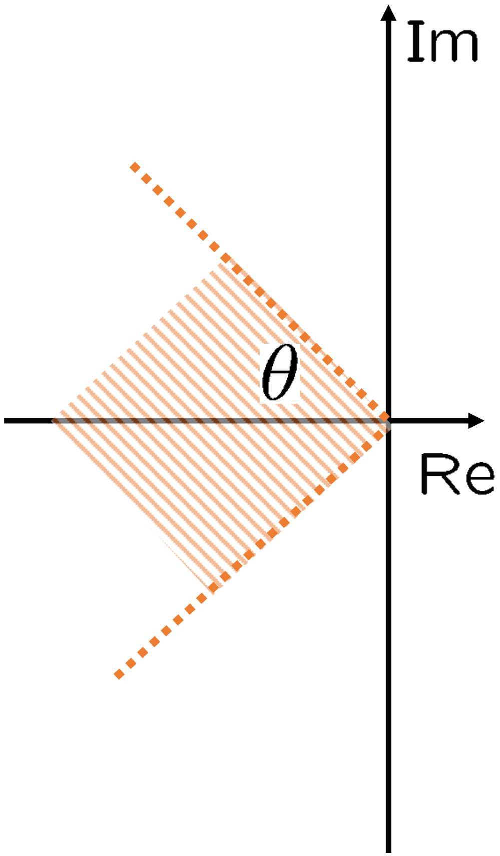
Solution:

- From the given *PO* constraint, we can obtain allowable angles θ.
- From the given settling time T_s constraint, we can obtain allowable real part of poles:

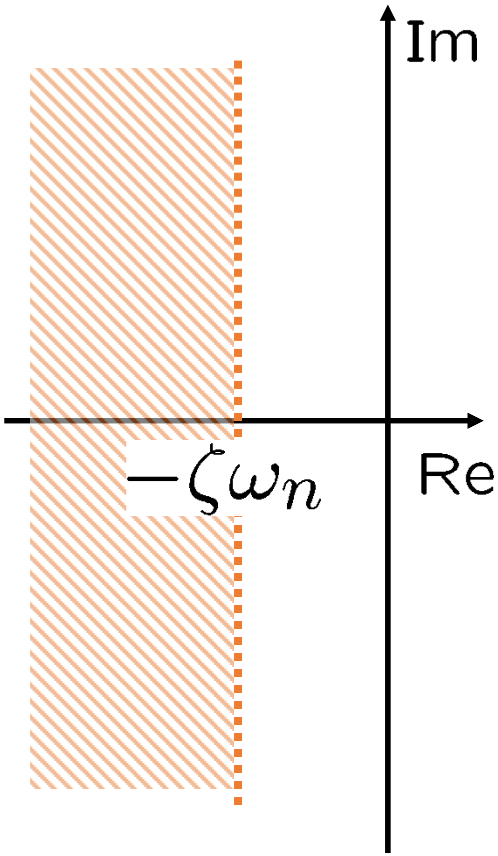
$$T_s = \frac{3}{\zeta\omega_n} \quad ; \qquad \frac{3}{\zeta\omega_n} < (\text{given } T_{smax}) \Leftrightarrow \zeta\omega_n > \frac{3}{T_{smax}}$$

Example 2 (cont'd)

PO constraint



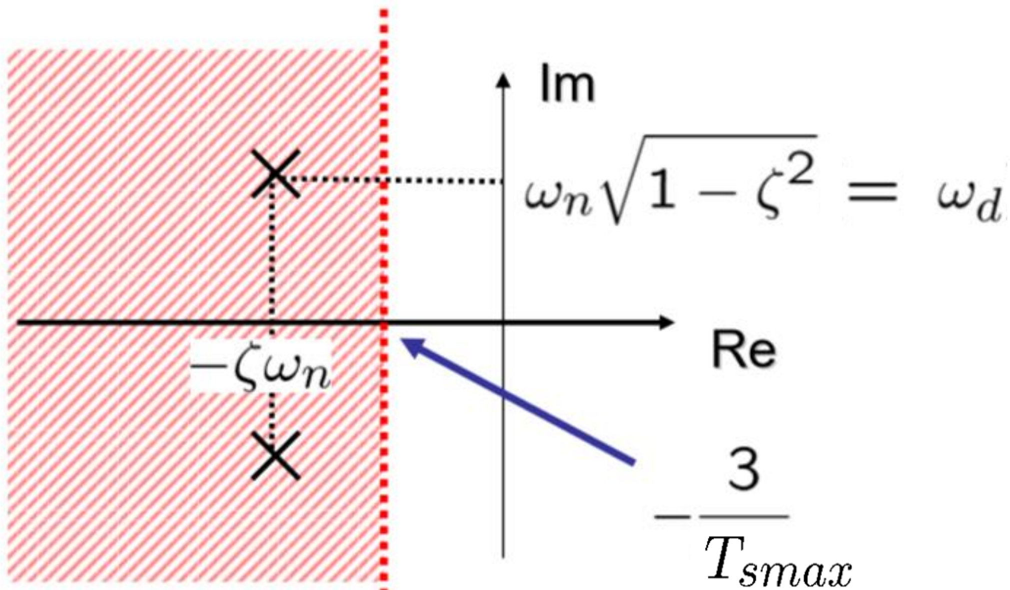
T_s constraint



Example 2 (cont'd)

- Require 5% settling time $T_s < T_{smax}$ (given):

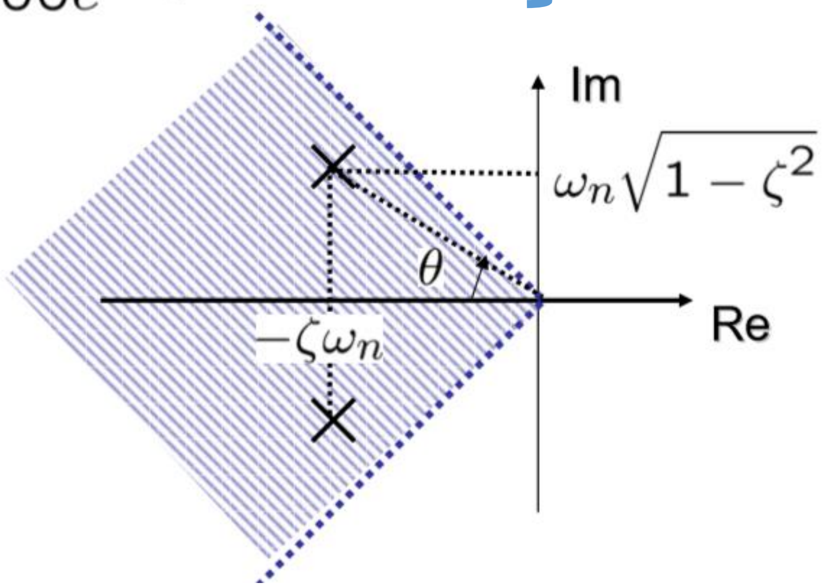
$$\left(T_s = \frac{3}{\zeta\omega_n} \right) < T_{smax} \Rightarrow \boxed{\zeta\omega_n > \frac{3}{T_{smax}}}$$



Example 2 (cont'd)

- Require $PO < PO_{max}$ (given):

$$\left.\begin{aligned} PO &= 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} < PO_{max} \\ &= 100e^{-\pi/\tan\theta} \end{aligned}\right\} \rightarrow \boxed{\theta < \theta_{max}}$$

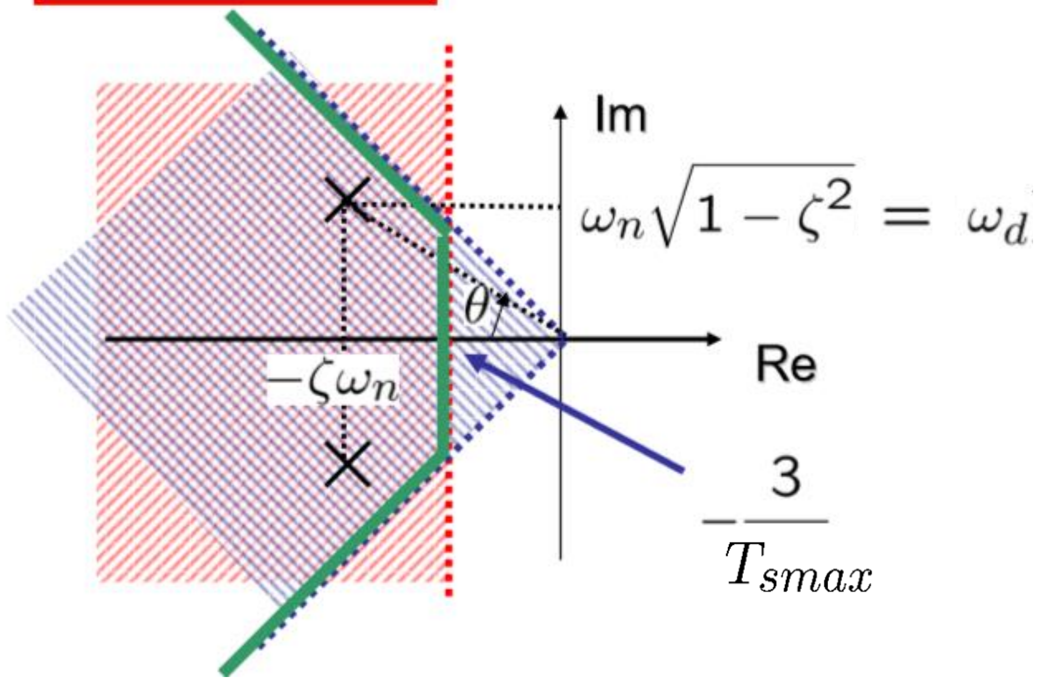


PO_{max}	θ_{max}
5%	46.3°

Example 2 (cont'd)

- Combination of two requirements

$$\boxed{\zeta\omega_n > \frac{3}{T_{smax}}} \quad \& \quad \boxed{\theta < \theta_{max}}$$



Example 3

- Consider a 2nd order overdamped system

$$G(s) = \frac{10}{s^2 + 11s + 10} = \frac{10}{(s+1)(s+10)}$$

For unit step input, obtain (or estimate)

- steady state value,
- 2% settling time, and
- percent overshoot.

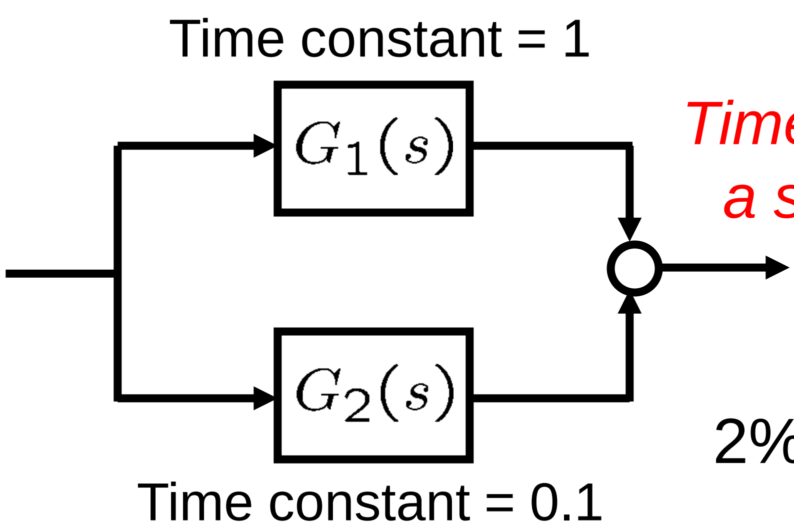
Solution:

- Steady state value is the DC gain $G(0) = 1$.
- No overshoot (overdamped!)

Example 3 (cont'd)

- Poles are -1 and -10.

$$G(s) = \frac{10}{s^2 + 11s + 10} = \underbrace{\frac{A}{s+1}}_{G_1(s)} + \underbrace{\frac{B}{s+10}}_{G_2(s)}$$

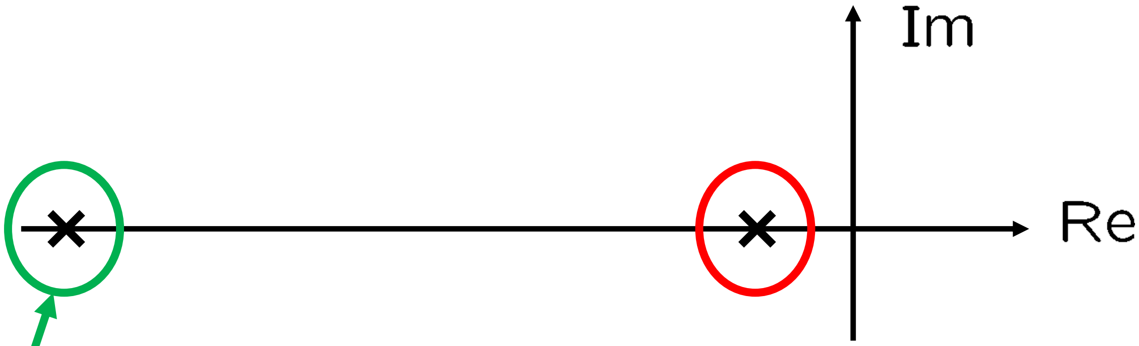


Time constant of G(s) is estimated by a slower subsystem $G_1(s)$, i.e., the larger T.

2% settling time: about 4 seconds

Example 3 (cont'd)

- Dominant poles: Poles closest to the imaginary axis and far away from remaining poles dominate the behavior of responses.

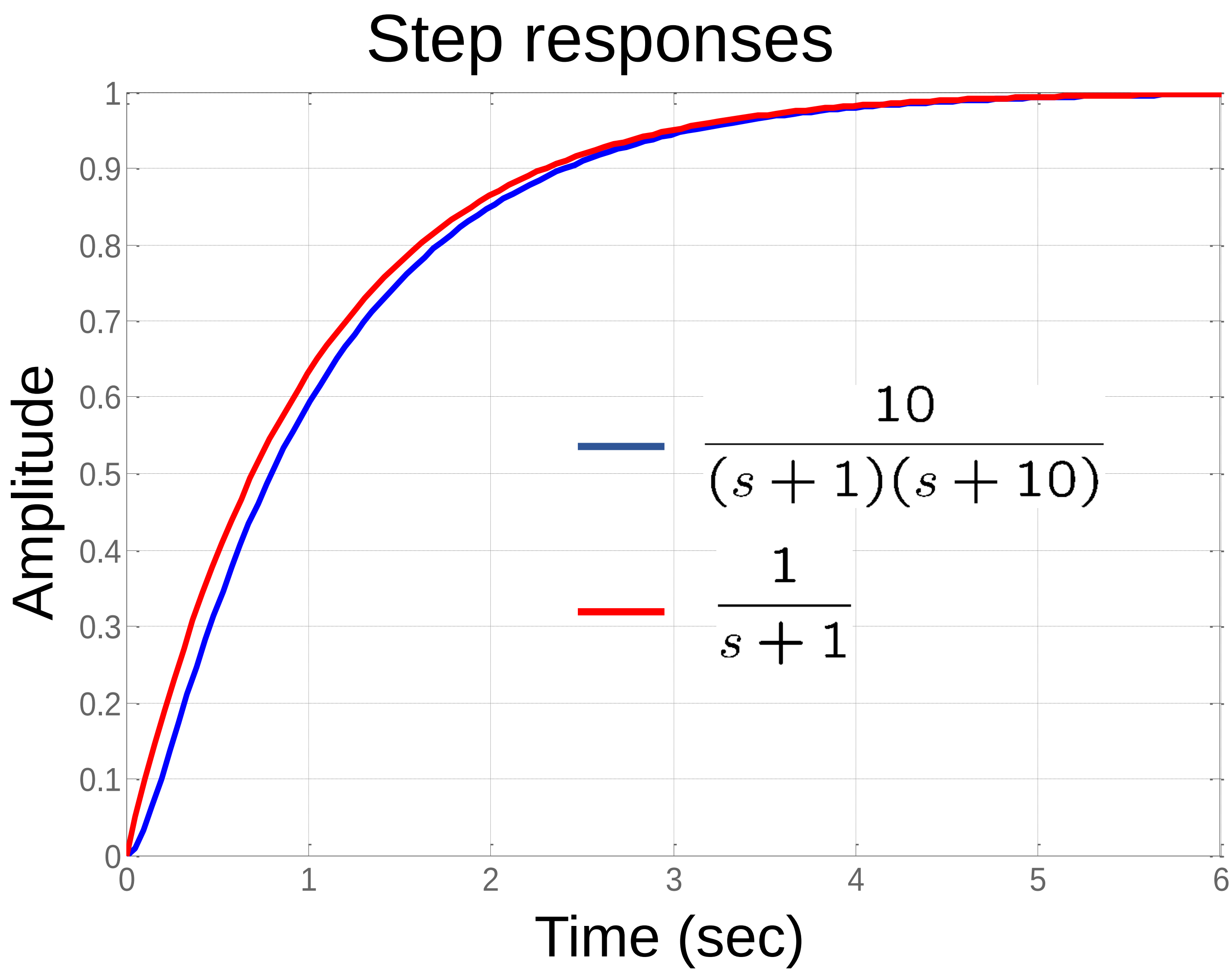


Poles far left (5-10 times) from dominant poles may be ignored.

$$G(s) = \frac{10}{(s+1)(s+10)} \approx \frac{1}{s+1}$$

Same DC gain

Example 3 (cont'd)



Example 4

- For the unit step response of a stable 3rd-order system,

$$G(s) = \frac{8}{(s+2.5)(s^2+2s+4)}$$

  obtain (or estimate)
  - steady state value,
  - 2% settling time, and
  - percent overshoot.

Solution:

- Steady state value is the DC gain $G(0) = 0.8$.

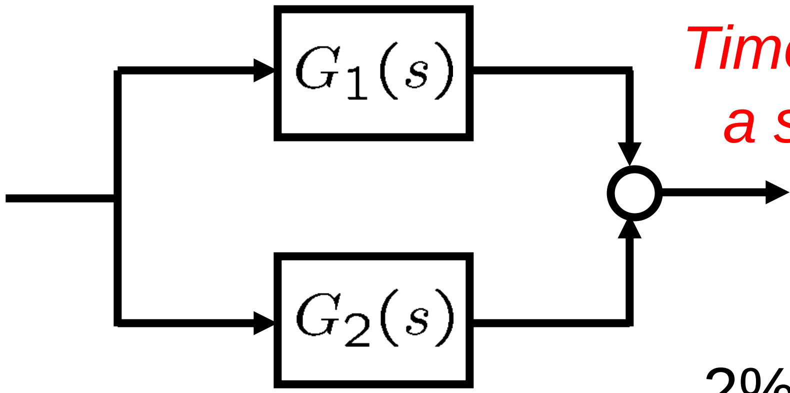
Example 4 (cont'd)

- Partial fraction expansion

$$G(s) = \frac{8}{(s+2.5)(s^2+2s+4)} = \underbrace{\frac{A}{s+2.5}}_{G_1(s)} + \underbrace{\frac{Bs+C}{s^2+2s+4}}_{G_2(s)}$$

$$\omega_n = 2, \ \zeta = 0.5$$

Time constant = 1/2.5 = 0.4 s



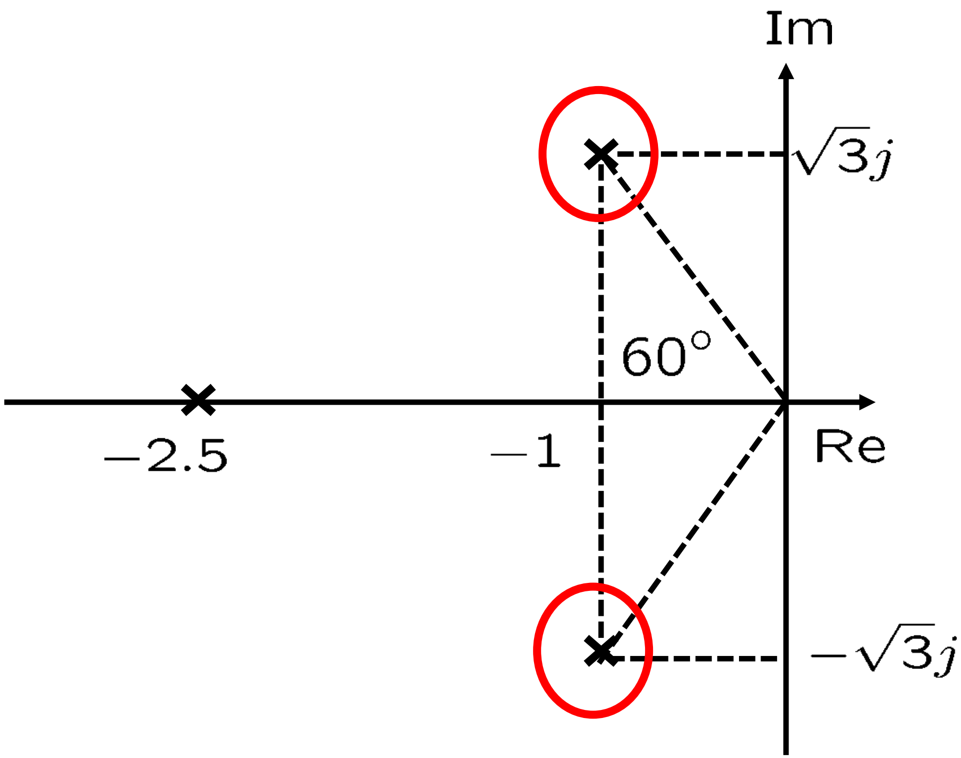
Time constant = $1/(\zeta\omega_n)$ = 1 s

Time constant of G(s) is estimated by a slower subsystem $G_2(s)$, i.e., the larger T.

2% settling time: about 4 seconds

Example 4 (cont'd)

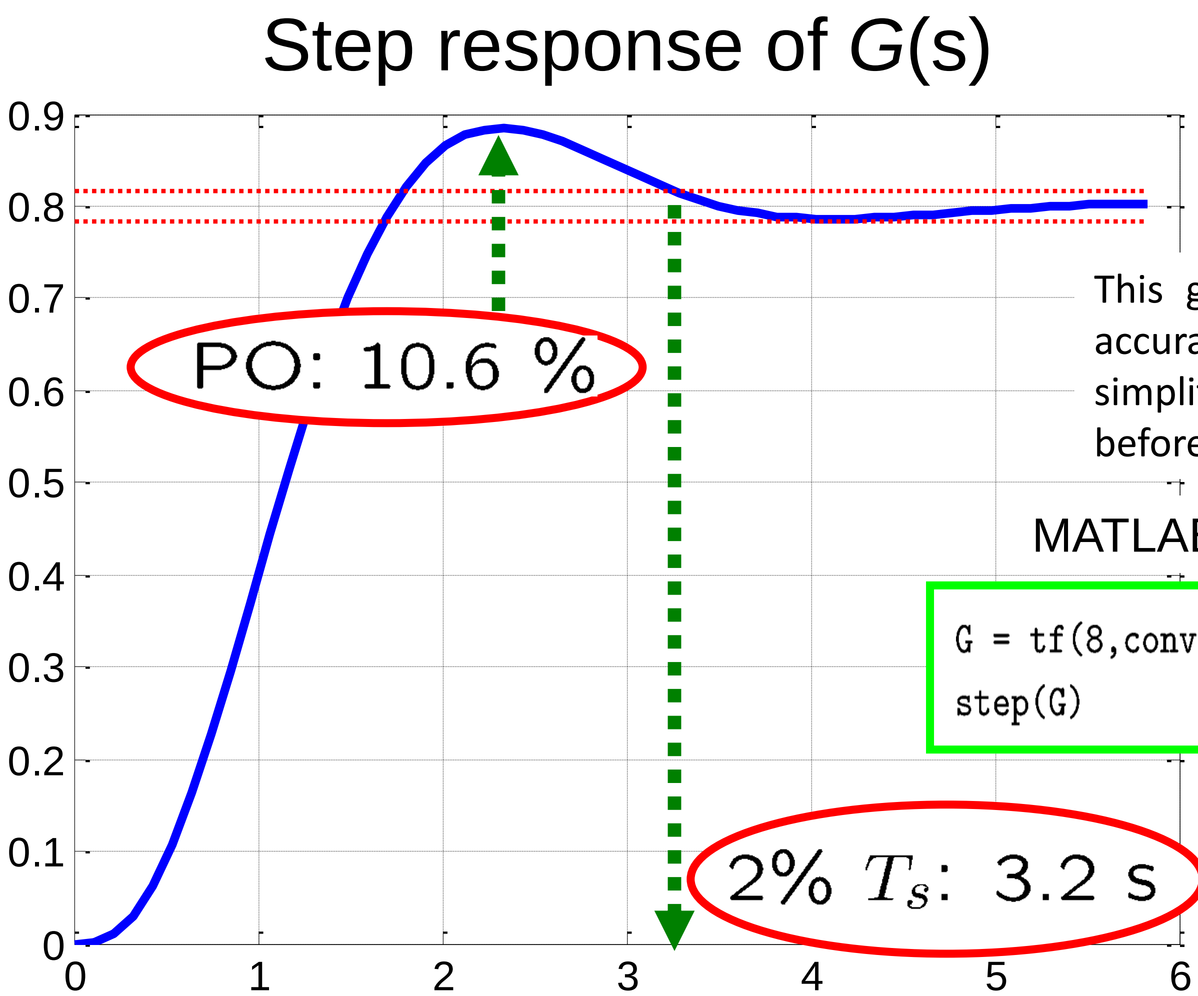
- Poles $s = -2.5,\ -1 \pm \sqrt{3}j$



These poles are not "sufficiently dominant"!

Estimation based on these poles may be inaccurate.

Example 4 (cont'd)



This graph was obtained accurately without the simplification mentioned before.

MATLAB command

```
G = tf(8,conv([1 2.5],[1 2 4]));
step(G)
```

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Example 5

- For the standard 2nd order systems, derive the relationship for damping ratio as a function of percent overshoot, from

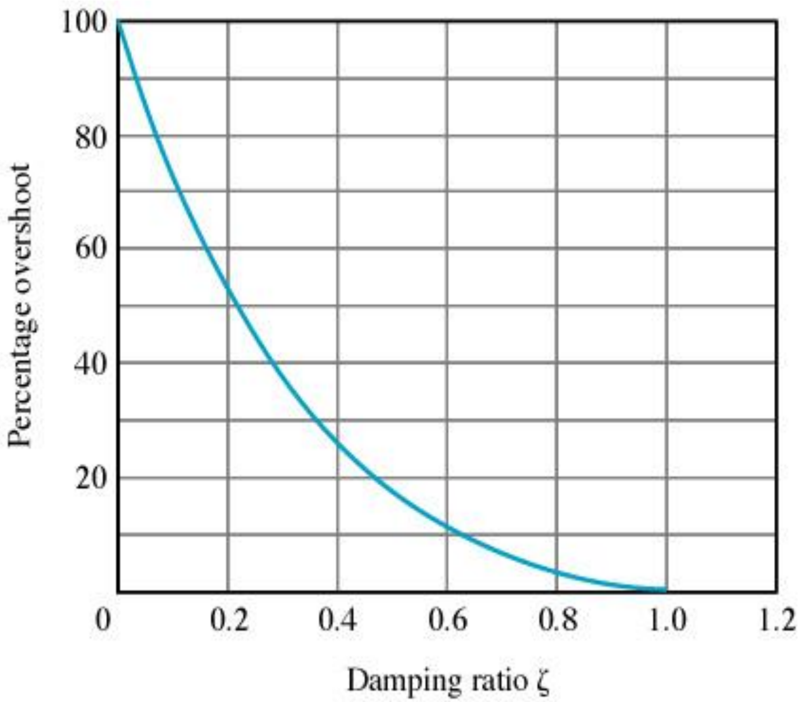
$$PO = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Solution:

$$\left(\ln\frac{PO}{100}\right)^2 = \frac{\zeta^2\pi^2}{1-\zeta^2}$$

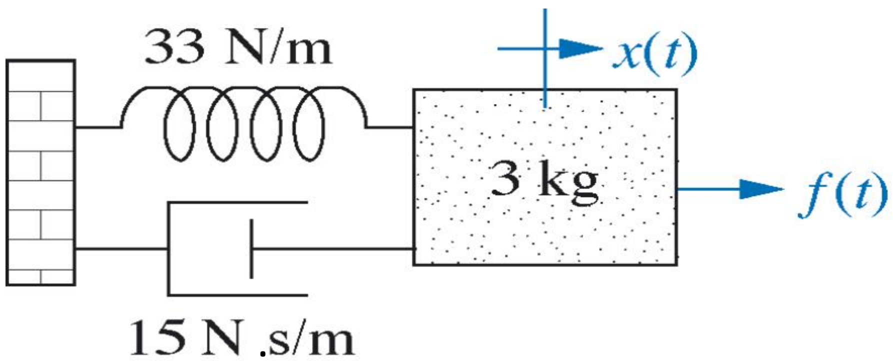
$$\Leftrightarrow \zeta^2 = \frac{\left(\ln\frac{PO}{100}\right)^2}{\pi^2 + \left(\ln\frac{PO}{100}\right)^2}$$

$$\Leftrightarrow \boxed{\zeta = \frac{\left|\ln\frac{PO}{100}\right|}{\sqrt{\pi^2 + \left(\ln\frac{PO}{100}\right)^2}}}$$



Example 6

- For the system below,
  - Find the transfer function from $F(s)$ to $X(s)$
  - Find DC gain, ζ, ω_n, T_s (2%), T_p, PO



Solution:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{3s^2 + 15s + 33}$$

Example 6 (cont'd)

- Find DC gain, ζ, ω_n, T_s (2%), T_p, PO

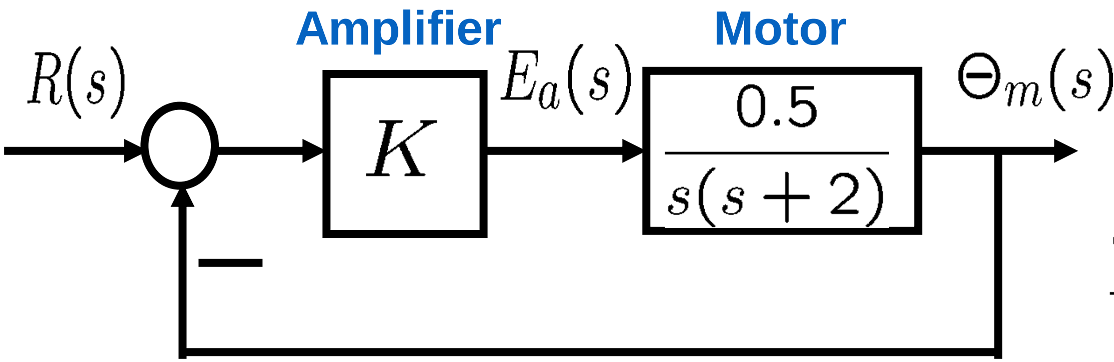
$$G(s) = \frac{1}{3s^2 + 15s + 33} = \frac{1}{33} \cdot \frac{11}{s^2 + 5s + 11}$$

$$2\zeta\omega_n = 5, \;\; \omega_n^2 = 11$$

- DC gain: $G(0) = 1/33$
- $\omega_n = \sqrt{11}$; $\zeta = \dfrac{5}{2\sqrt{11}}$; $\omega_d = \omega_n\sqrt{1-\zeta^2} = \dfrac{\sqrt{19}}{2}$
- $T_s = \dfrac{4}{\zeta\omega_n} = 1.6$; $T_p = \dfrac{\pi}{\omega_d} = \dfrac{2\pi}{\sqrt{19}}$
- $PO = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 2.72$

Example 7

- DC motor position control



Closed-loop TF

$$\frac{\Theta_m(s)}{R(s)} = \frac{0.5K}{s^2+2s+0.5K}$$

- Design the amplifier gain K so that
  - Percent overshoot is 5%.

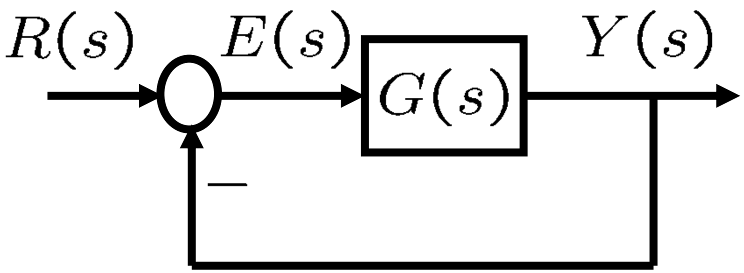
$$\begin{cases} 2\zeta\omega_n = 2 \\ \omega_n^2 = 0.5K \end{cases}$$

Solution:

$$\zeta = \frac{\left|\ln \frac{PO}{100}\right|}{\sqrt{\pi^2 + \left(\ln \frac{PO}{100}\right)^2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \omega_n = \sqrt{2} \quad \Rightarrow \quad \boxed{K = 4}$$

Example 8

- For the unity feedback system, check and compute



$$G(s) = \frac{4900}{s(s+70)}$$

- Stability
- 2% settling time for $r(t) = u(t)$ (unit step)
- Steady-state error for $r(t) = 5u(t)$ (step)
- Steady-state error for $r(t) = 5tu(t)$ (ramp)
- Steady-state error for $r(t) = 5t^2u(t)$ (parabolic)

Example 8 (cont'd)

Solution:

- **Stability**
  - Characteristic equation

$$1 + G(s) = 0 \Leftrightarrow s^2 + 70s + 4900 = 0$$

$$\zeta = \frac{1}{2}, \ \omega_n = 70$$

Since all the coefficients in the first column of Routh array have the same sign, the CL system is stable.

- **2% settling time** $(\zeta < 1 : \text{underdamped case})$
  - Time constant is $T = \frac{1}{|\mathrm{Re(pole)}|} = \frac{1}{\zeta\omega_n} = \frac{1}{35}$
  - 2% settling time is $4T = \frac{4}{35}$

Example 8 (cont'd)

- **Steady-state errors**

$$G(s) = \frac{4900}{s(s+70)}$$

  - For 5*u*(*t*) (step)

$$K_p = \lim_{s \to 0} G(s) = \infty \quad \Rightarrow \quad e_{ss} = \frac{5}{1+K_p} = 0$$

  - For 5*tu*(*t*) (ramp)

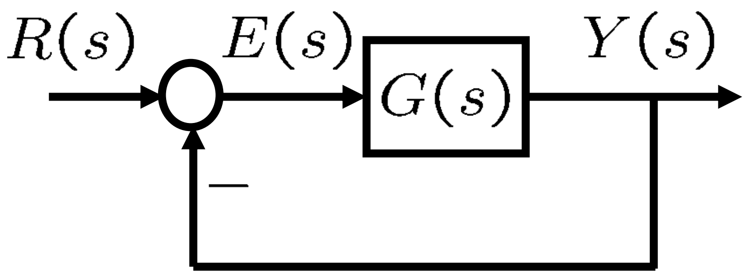
$$K_v = \lim_{s \to 0} sG(s) = 70 \quad \Rightarrow \quad e_{ss} = \frac{5}{K_v} = \frac{1}{14}$$

  - For 5t^2*u*(*t*) (parabolic)

$$K_a = \lim_{s \to 0} s^2 G(s) = 0 \quad \Rightarrow \quad e_{ss} = \frac{10}{K_a} = \infty$$

Example 9

- Design the unity feedback system such that:



$$G(s) = \frac{K(s+\alpha)}{s(s+\beta)}$$

  - (a) Steady-state error for unit ramp input is 1/10 and
  - (b) Closed-loop poles are $-1 \pm j$

Solution:

  - (a) $K_v = \lim_{s \to 0} sG(s) = \dfrac{K\alpha}{\beta} = 10$

  - (b) Closed-loop poles are at $-1 \pm j$

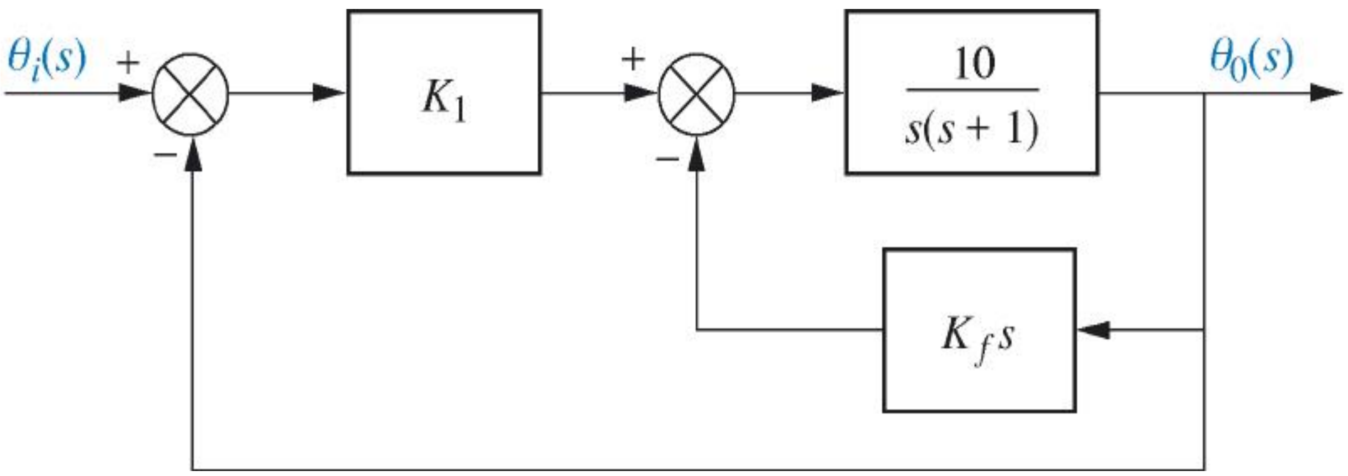
characteristic equation = $s(s+\beta) + K(s+\alpha) = s^2 + 2\zeta\omega_n s + \omega_n^2$

$$\boxed{\alpha = 1.111, \beta = 0.2, K = 1.8}$$

Example 10

- Design the parameters K_1 & K_f so that:

$$K_v = 10,\ \zeta = 0.5$$



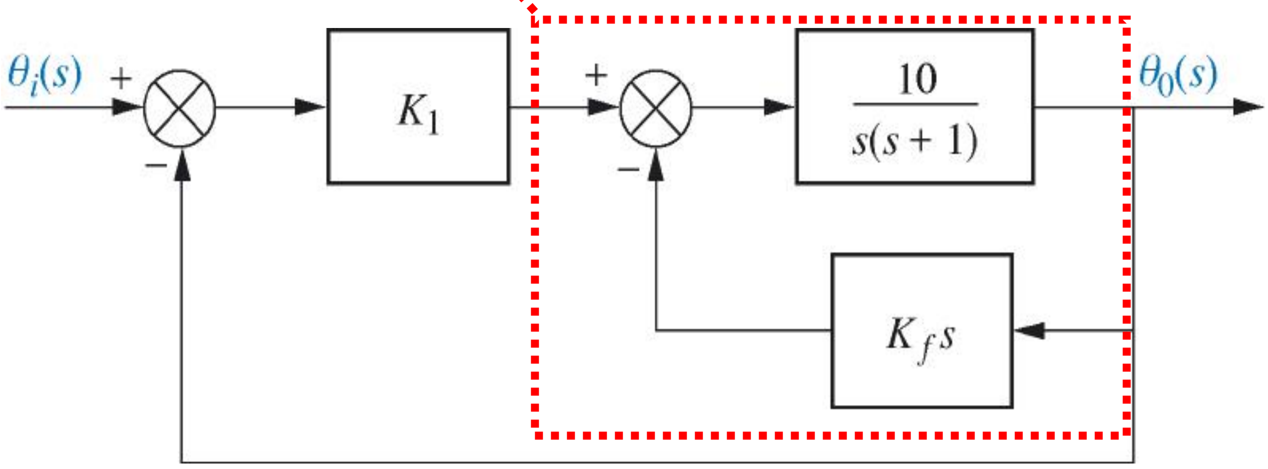
Example 10 (cont'd)

Solution:

- Closed-loop transfer function

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{K_1 F(s)}{1+K_1 F(s)} = \frac{10K_1}{s^2+(1+10K_f)s+10K_1}$$

$$\boxed{F(s)} = \frac{\frac{10}{s(s+1)}}{1+\frac{10}{s(s+1)}K_f s} = \frac{10}{s(s+1+10K_f)}$$



Example 10 (cont'd)

- From the two requirements,

$$K_v = \lim_{s\to 0} sK_1F(s) = \frac{10K_1}{1+10K_f} = 10$$

$$\Rightarrow \quad K_1 = 1 + 10K_f$$

$$\begin{cases} \zeta = 0.5 \\ 2\zeta\omega_n = K_1 \\ \omega_n^2 = 10K_1 \end{cases} \quad \Rightarrow \quad K_1^2 - 10K_1 = 0$$

$$\Rightarrow \quad \boxed{K_1 = 10, \;\; K_f = 0.9}$$

(For CL stability, K_1 has to be positive.)

Summary

- Cruise control example
- Allowable pole locations
- Step responses
  - 2^{nd} order overdamped systems
  - Higher order systems
- Design of some feedback systems
- Next
  - Root locus