ELEC 341: Systems and Control

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Lecture 18

Nyquist stability criterion: Introduction

Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - Electrical
  - Electromechanical
  - Mechanical
- Linearization, delay

Analysis

- Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

Design

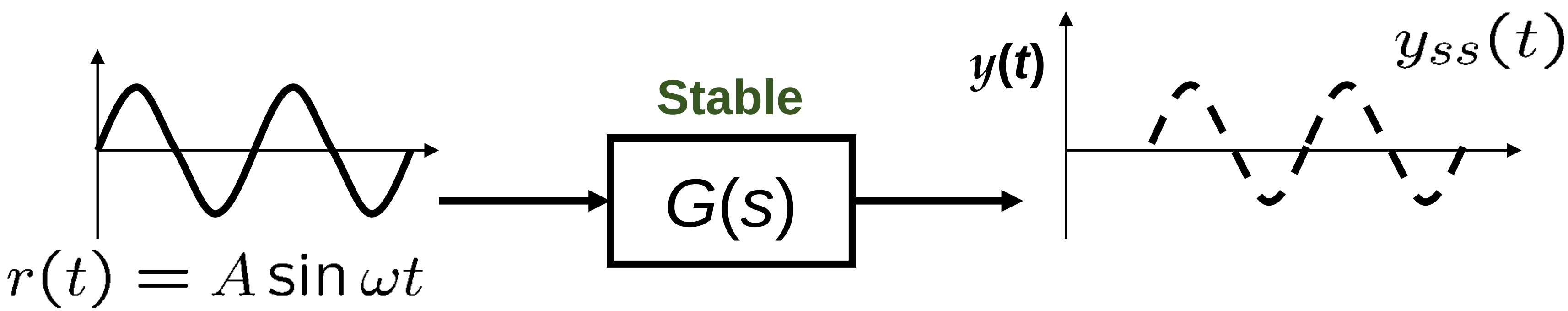
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

Frequency response (review)

- Steady state output $y_{ss}(t) = A\,|G(j\omega)|\,\text{sin}(\omega t + \angle G(j\omega))$
  - Frequency is same as the input frequency ω
  - Amplitude is that of input (*A*) multiplied by $|G(j\omega)|$
  - Phase shifts $\angle G(j\omega)$

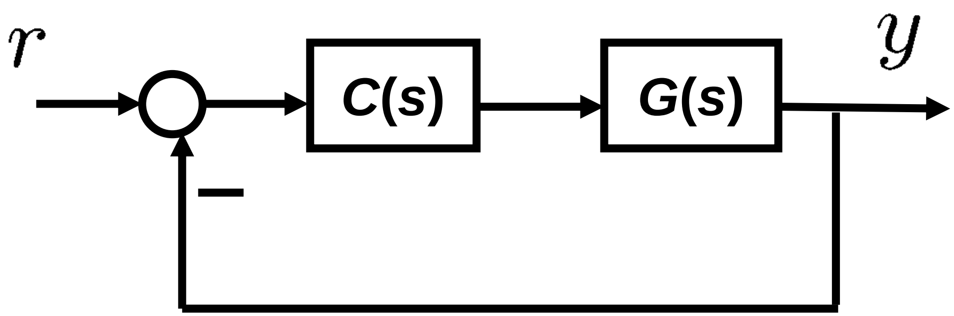
Gain



- Frequency response function (FRF): $G(j\omega)$
- Bode plot: Graphical representation of $G(j\omega)$

Stability of feedback system

- Consider the feedback system



- Fundamental questions
  - If *G* and *C* are stable, is the closed-loop system *always stable*?
  - If *G* and *C* are unstable, is the closed-loop system *always unstable*?

Closed-loop stability criterion

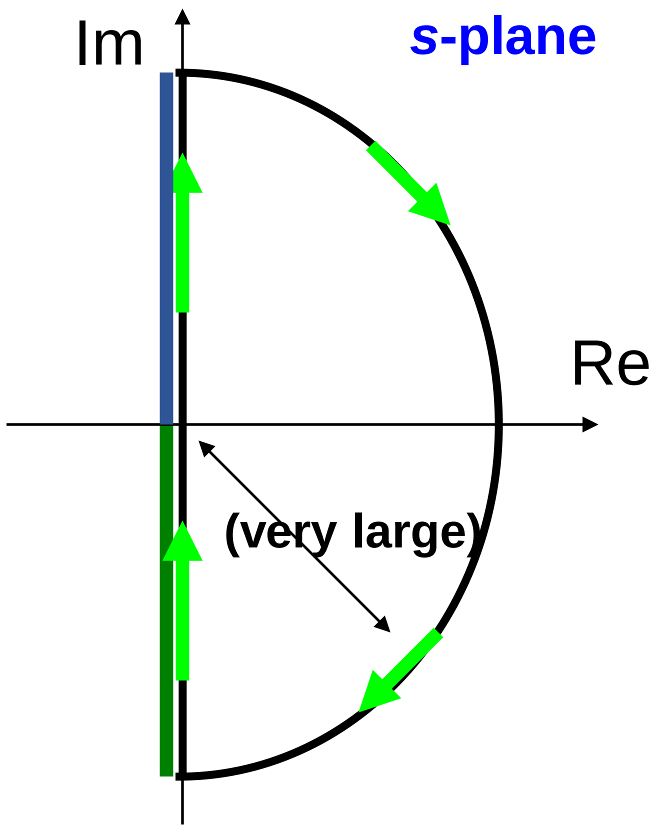
- Closed-loop stability can be determined by the roots of the characteristic equation

$$1 + L(s) = 0, \;\; L(s) = G(s)C(s)$$

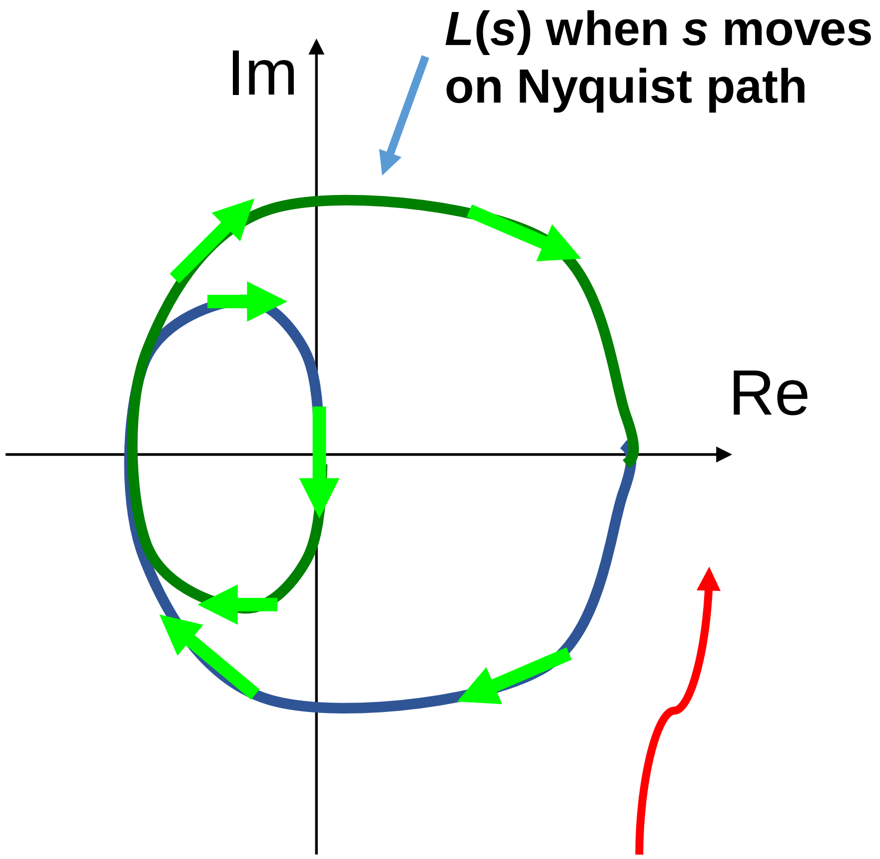
- Closed-loop system is stable if the Ch. Eq. has all roots in the open left half plane.
- How to check the closed-loop stability?
  - Computation of all the roots
  - Routh-Hurwitz stability criterion
  - *Nyquist stability criterion: Open-loop FRF L($j\omega$) contains information of closed-loop stability.*

Nyquist plot

- Nyquist path



- Nyquist plot



This plane is called ***L(jω)*-plane** or ***w*-plane**.

Nyquist Plot or Polar Plot

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- Nyquist Plots were invented by Nyquist - who worked at Bell Laboratories, the premiere technical organization in the U.S. at the time.
- Nyquist Plots are a way of showing frequency responses of linear systems.
- There are several ways of displaying frequency response data, including Bode plots and Nyquist plots.
- **Bode plots** use frequency as the horizontal axis and use ***two separate plots*** to display amplitude and phase of the frequency response. This was covered earlier.
- **Nyquist plots** display both amplitude and phase angle on ***a single plot***, using frequency as a parameter in the plot.
- Nyquist plots have properties that allow you to see whether a system is stable or unstable.

Nyquist Plot

- Simply put, a **Nyquist plot** is basically a **polar plot** of the frequency response function of a linear system.

- That means a Nyquist plot is a plot of the transfer function, $G(s)$ with $s = j\omega$. Here, you will be plotting $G(j\omega)$.

- $G(j\omega)$ is a complex number for any angular frequency, ω, so the plot is a plot of complex numbers.

- The complex number, $G(j\omega)$, depends upon frequency, so frequency will be a parameter if you plot the imaginary part of $G(j\omega)$ against the real part of $G(j\omega)$.

Sketch the Polar Plot of Frequency Response

- To sketch the polar plot of $G(j\omega)$ for the entire range of frequency ω, i.e., from 0 to infinity, there are four key points that usually need to be known:

  1) The start of plot where $\omega = 0$,
  2) The end of plot where $\omega = \infty$,
  3) Where the plot crosses the real axis, i.e., $\mathrm{Im}(G(j\omega)) = 0$, and
  4) Where the plot crosses the imaginary axis, i.e., $\mathrm{Re}(G(j\omega)) = 0$.

Example 1: Polar Plot of Integrator

Consider a first order system, $G(s) = \frac{1}{s}$

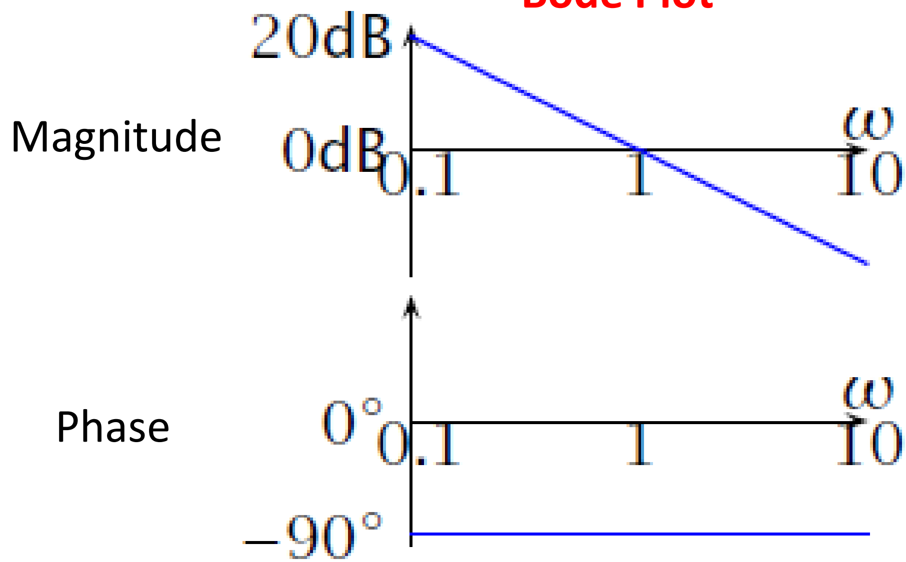
Representing *G*(*s*) in the frequency response form G(*jω*) by replacing *s* = *jω*:

$$G(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega}$$

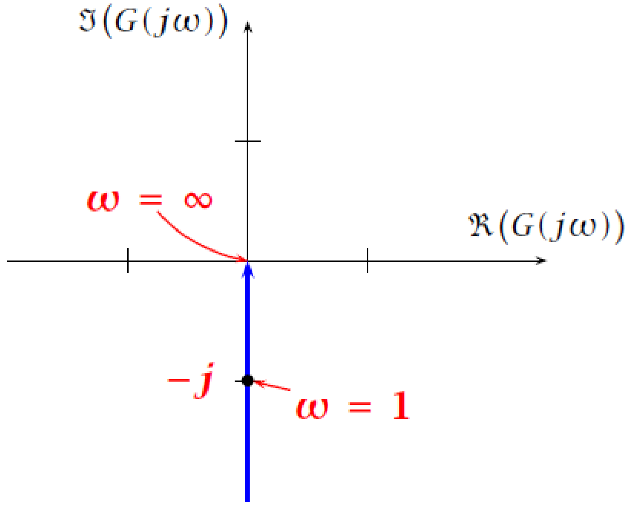
The **magnitude** of *G*(*jω*), i.e., |*G*(*jω*)|, is obtained as $|G(j\omega)| = 1/\omega$

The **phase** of *G*(*jω*), denoted by, ϕ , is obtained as $\angle G(j\omega) = -90^\circ$

Bode Plot



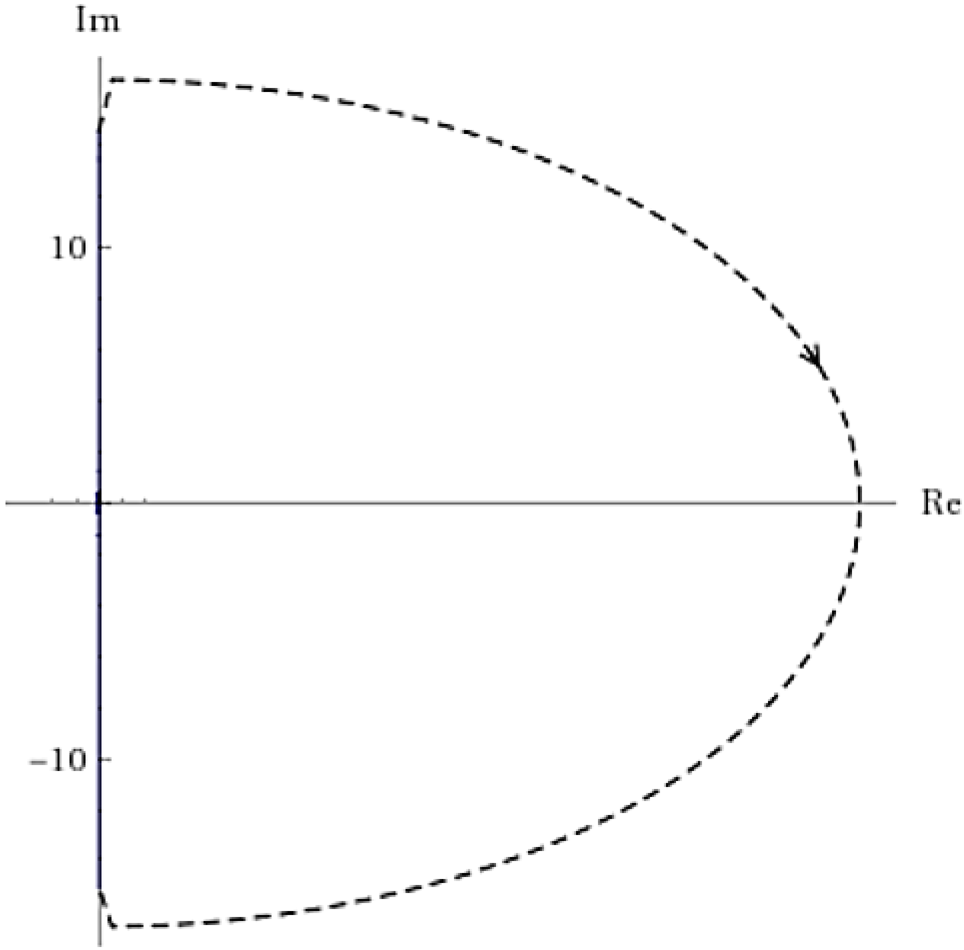
Polar Plot



Example 1 (cont'd): Polar Plot of Integrator

$$G(s) = \frac{1}{s}$$

Nyquist plot:



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Example 2: Polar Plot of First Order System

Consider a first order system $G(s) = \dfrac{1}{1+sT}$ where T is the time constant.

Representing $G(s)$ in the frequency response form $G(j\omega)$ by replacing $s = j\omega$:

$$G(j\omega) = \frac{1}{1 + j\omega T}$$

The **magnitude** of $G(j\omega)$, i.e., $|G(j\omega)|$, is obtained as;

$$|G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 T^2}}$$

The **phase** of $G(j\omega)$, denoted by, ϕ , is obtained as;

$$\angle G(j\omega) = -\arctan(\omega T)$$

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Example 2 (cont'd): Polar Plot of First Order System

The start of plot where $\omega = 0^+$

$$|G(j\omega)| = \frac{1}{\sqrt{1+0}} = 1,$$

For $\omega = 0^+$: $\angle G(j.0^+) = -\tan^{-1}(0^+.T) = 0°$

The end of plot where $\omega = +\infty$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\infty}} = 0,$$

For $\omega = +\infty$: $\angle G(j.(+\infty)) = -\tan^{-1}(+\infty.T) = -90°$

The mid part of plot where $\omega = 1/T$

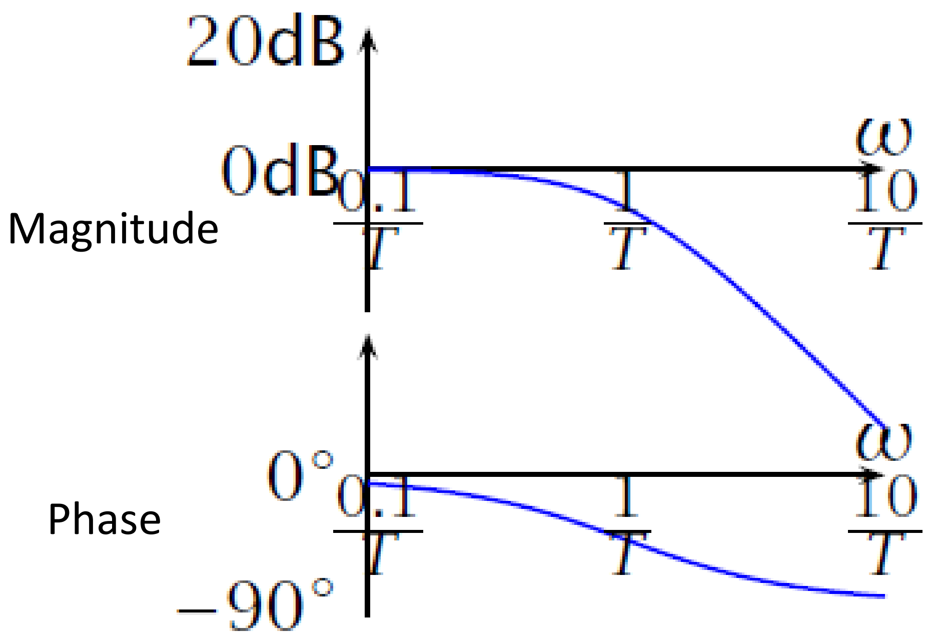
$$|G(j\omega)| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}},$$

For $\omega = \frac{1}{T}$: $\angle G\left(j.\frac{1}{T}\right) = -\tan^{-1}(\frac{1}{T}.T) = -\tan^{-1}(1) = -45°$

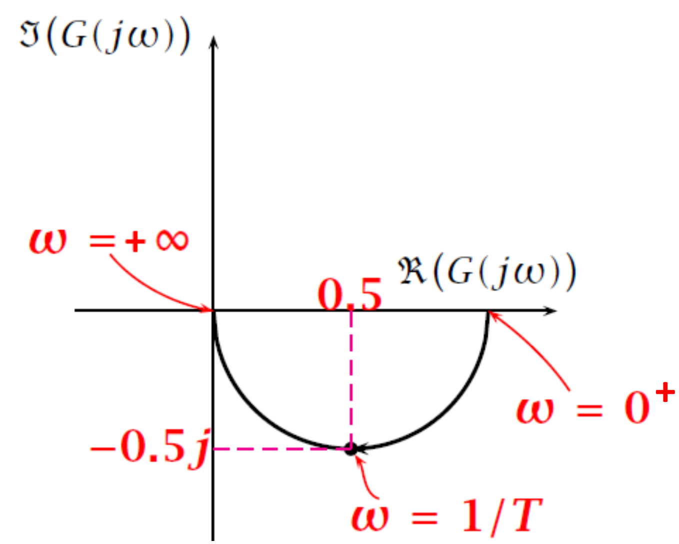
Example 2 (cont'd): Polar Plot of First Order System

	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
$\omega = 0^+$	1	0
$\omega = \dfrac{1}{T}$	$\dfrac{1}{\sqrt{2}}$	$-45°$
$\omega = +\infty$	0	$-90°$

Bode Plot



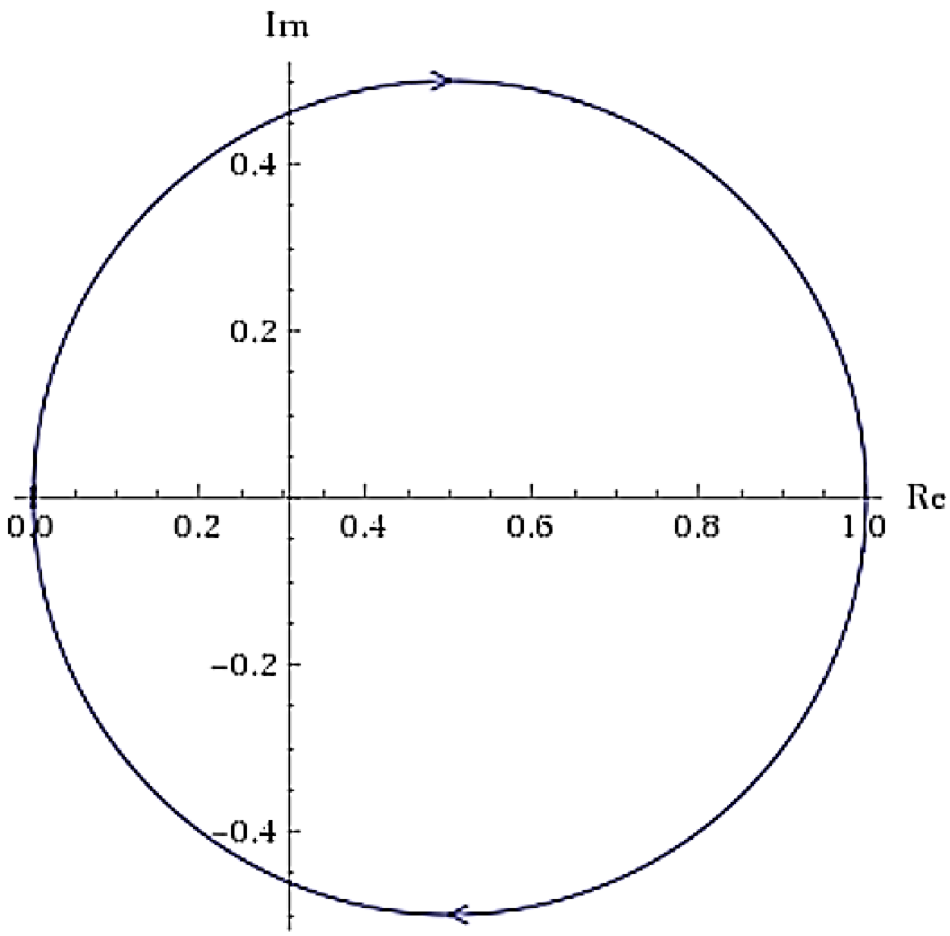
Polar Plot



Example 2 (cont'd): Polar Plot of First Order System

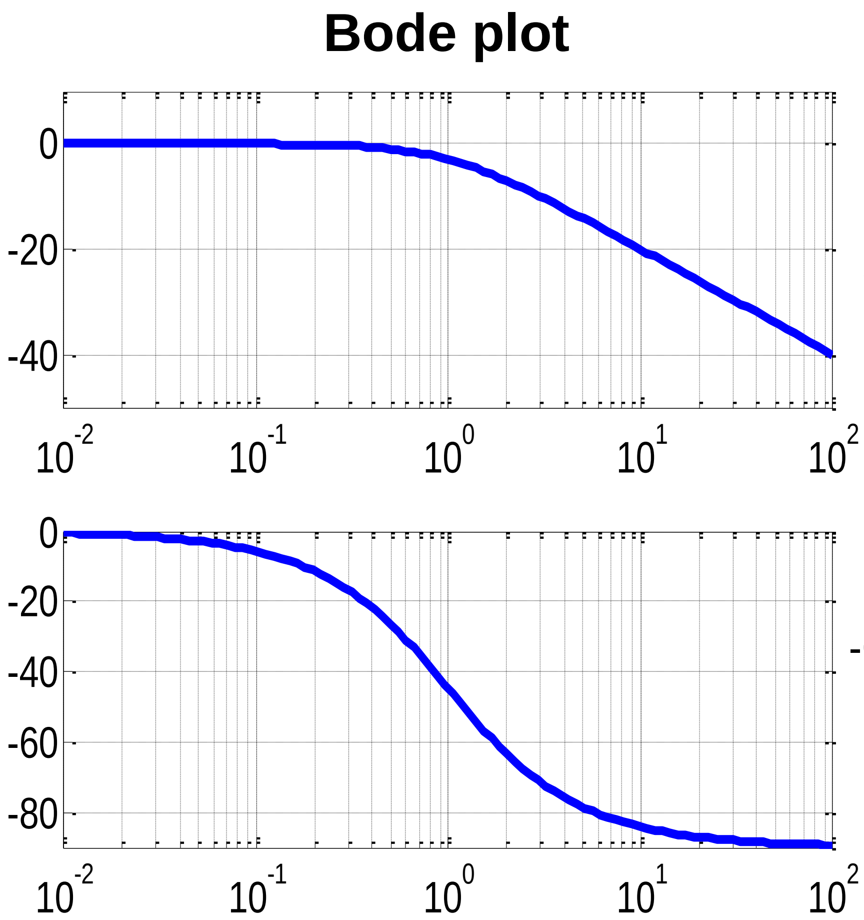
Nyquist plot:

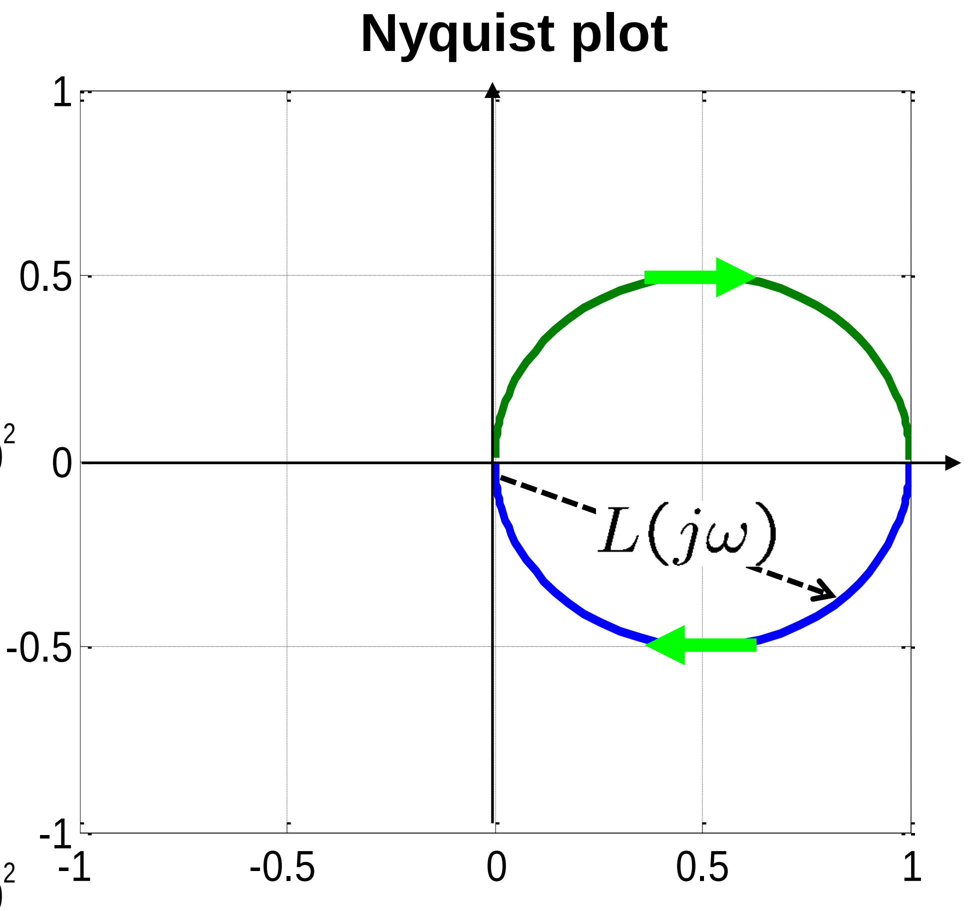
For $T = 2$.



Example 3: (Bode & Nyquist plots)

- First order system $L(s) = \dfrac{1}{s+1}$

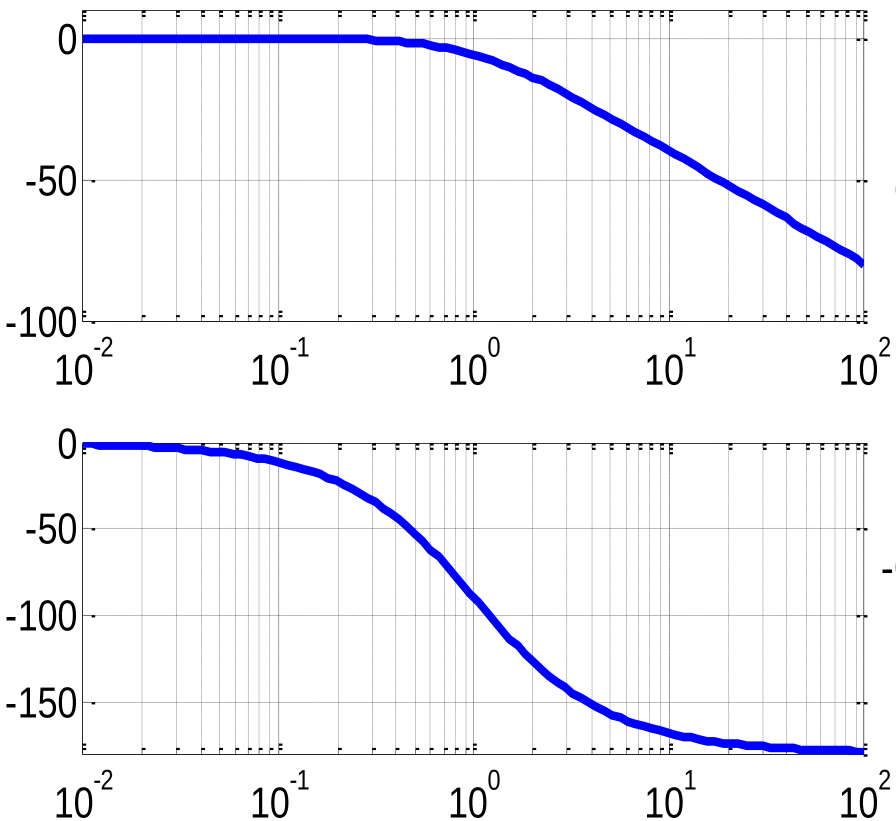




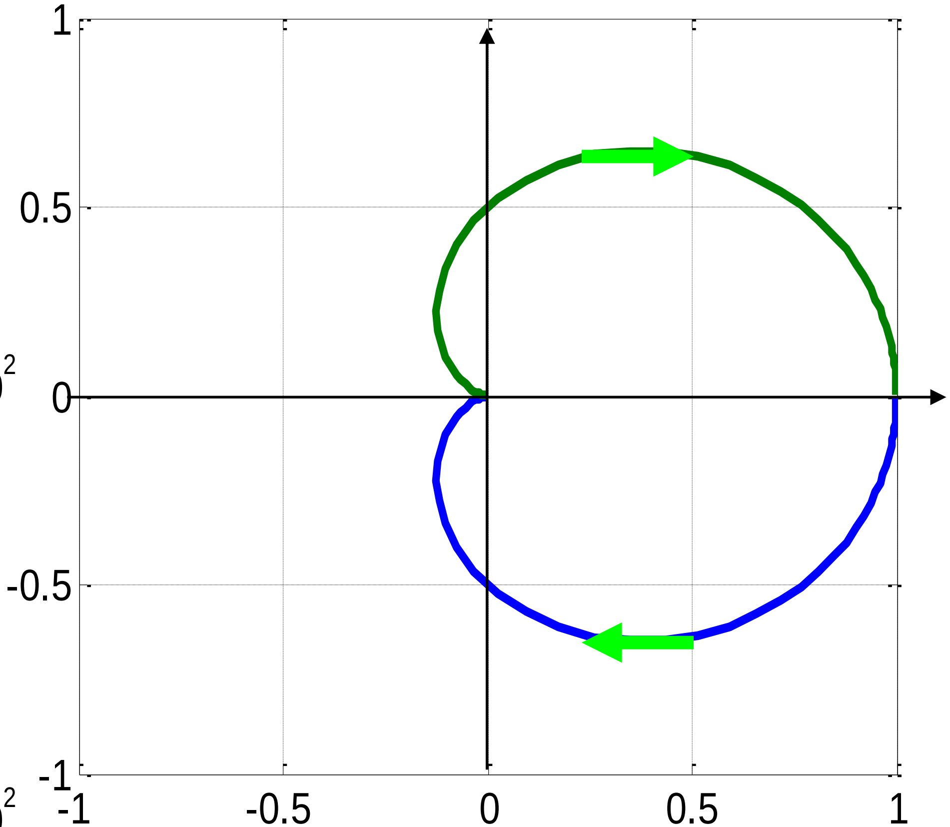
Example 4: (Bode & Nyquist plots)

- Second order system $L(s) = \dfrac{1}{(s+1)^2}$

Bode plot



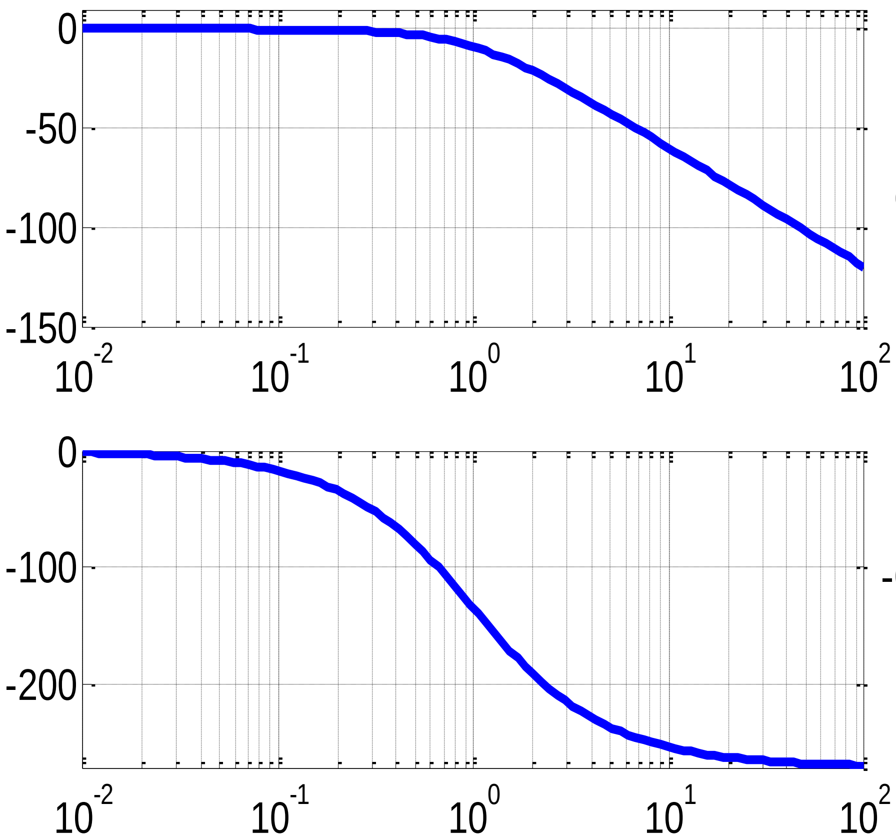
Nyquist plot



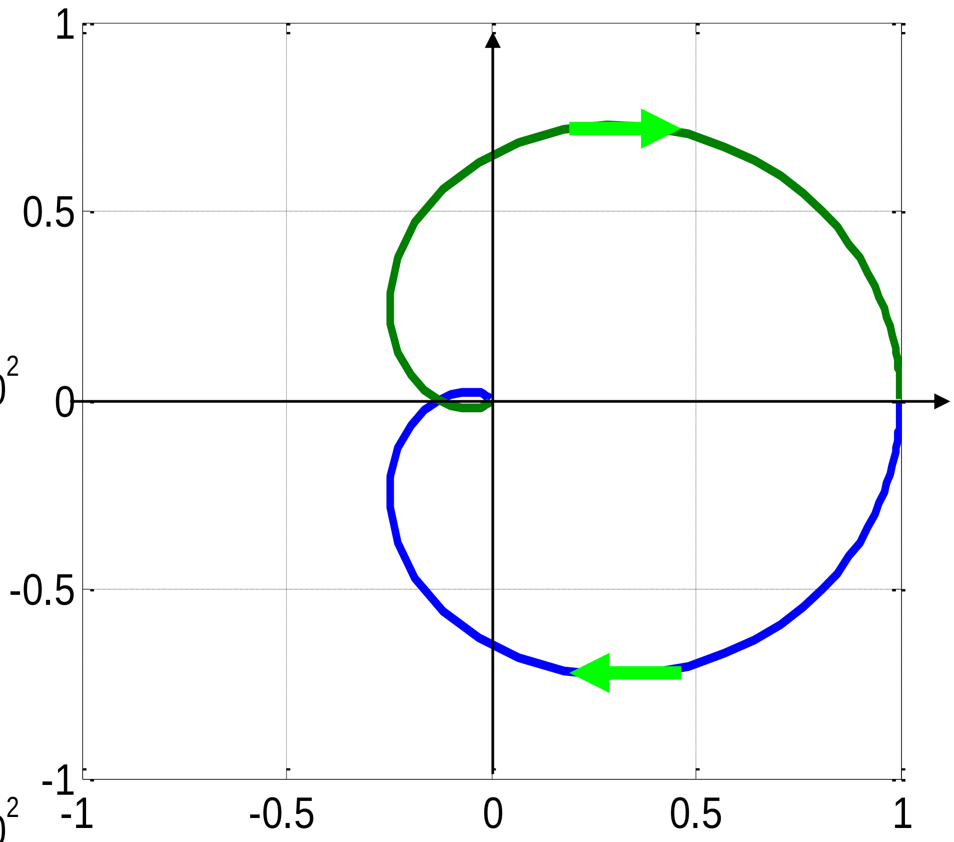
Example 5: (Bode & Nyquist plots)

- Third order system $L(s) = \dfrac{1}{(s+1)^3}$

Bode plot



Nyquist plot



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Nyquist stability criterion

$$\boxed{\text{CL system is stable} \Leftrightarrow Z = P + N = 0}$$

Z: # of CL poles in open RHP

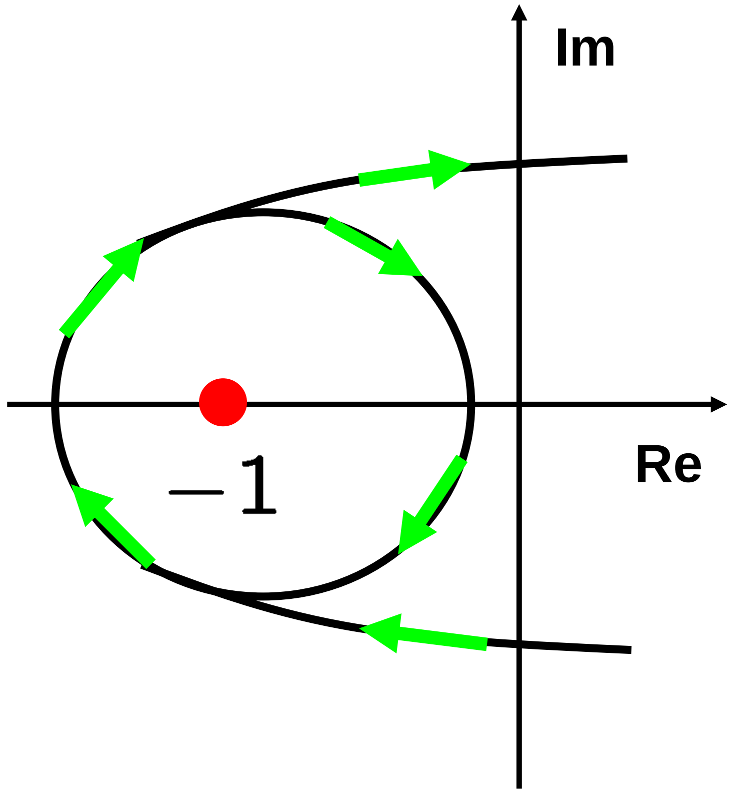
P: # of OL poles in open RHP (given)

N: # of clockwise/counterclockwise encirclement of -1 by Nyquist plot of OL transfer function $L(s)$ (counted by using Nyquist plot of $L(s)$)

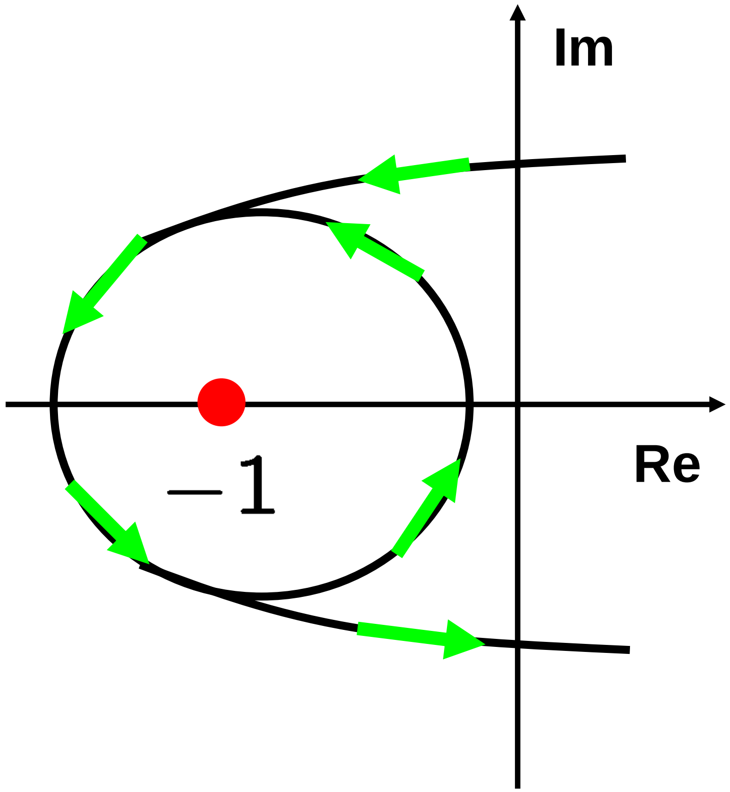
Remark: A negative value for *N* means a *counterclockwise* encirclement. For example, $N = -2$ means we encircle point -1 twice and in counterclockwise direction.

Encirclements in Nyquist plot

- Clockwise

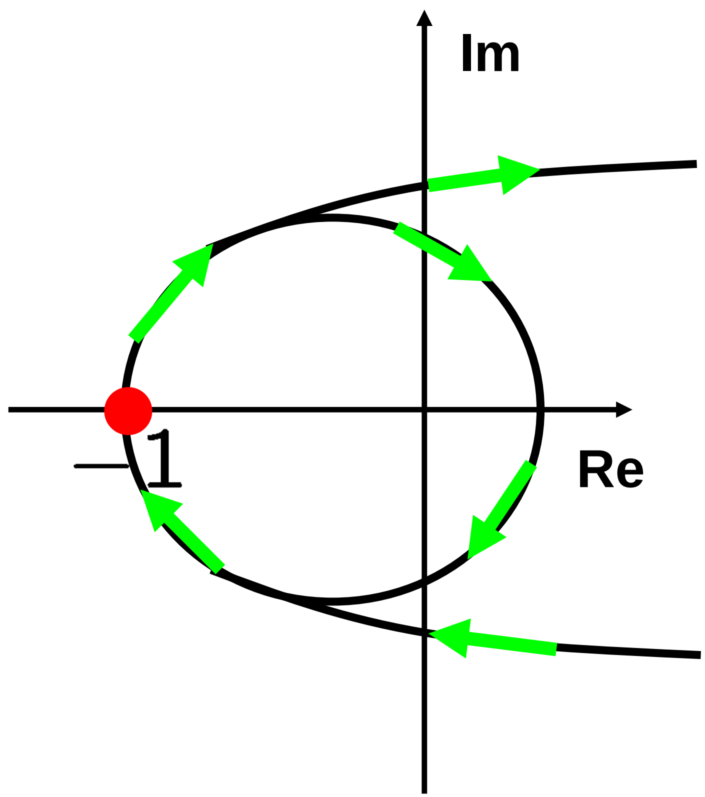


- Counter-clockwise

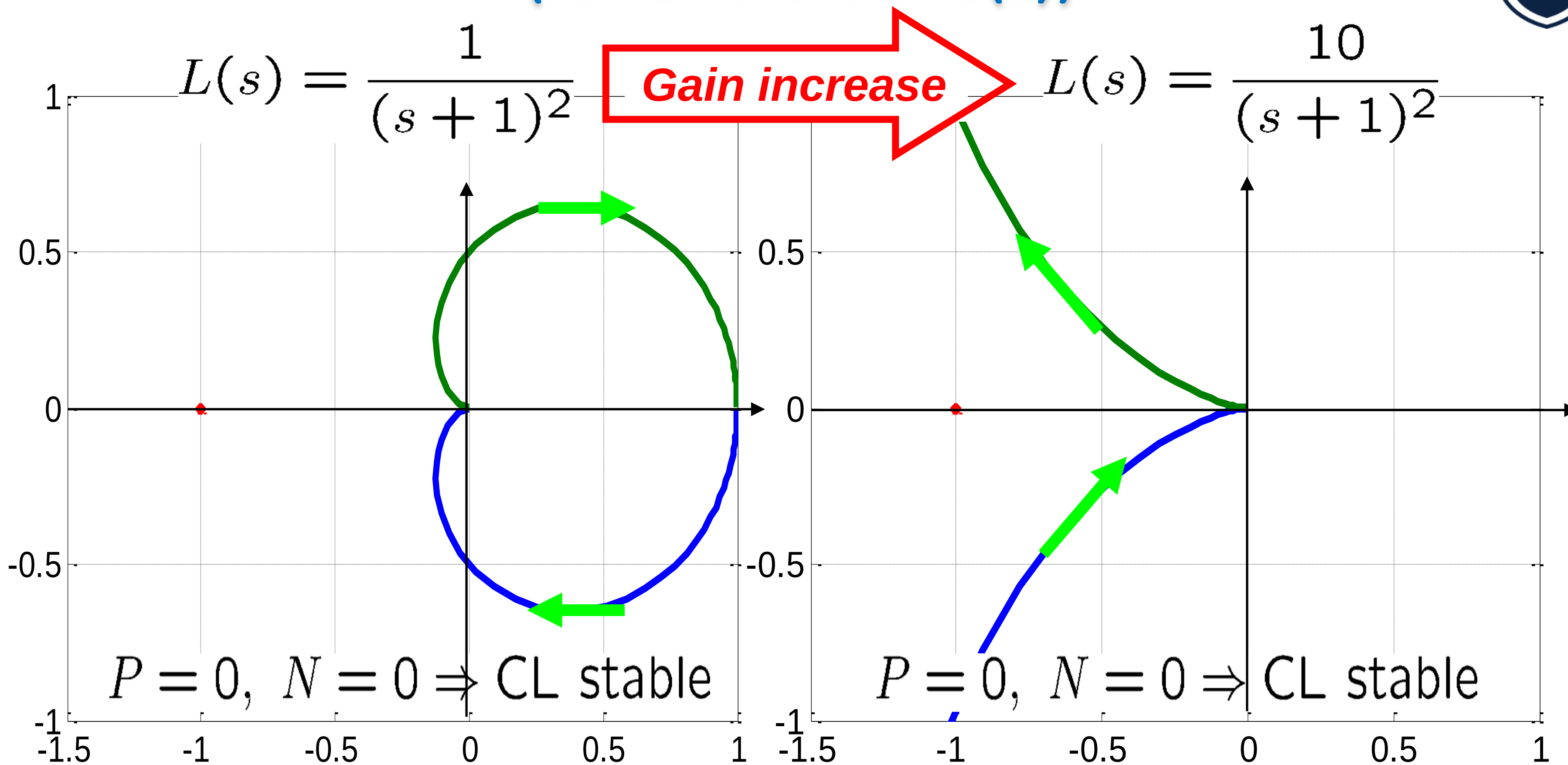


Remark

- If Nyquist plot passes the point -1, it means that the closed-loop system has a pole on the imaginary axis (and thus, marginally stable).

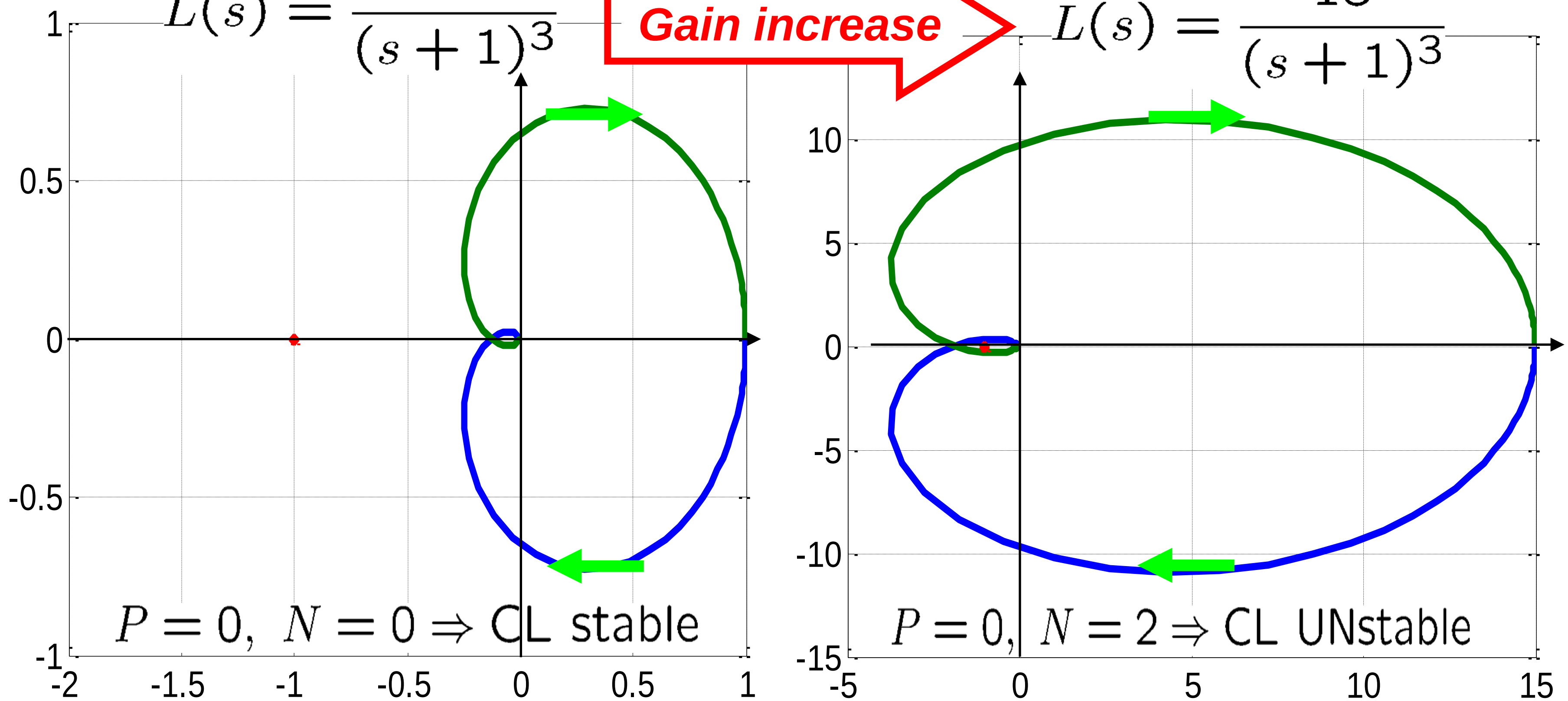


Example 6
(for 2nd order *L*(*s*))



Example 7
(for 3rd order $L(s)$)

$$L(s) = \frac{1}{(s+1)^3} \quad \xrightarrow{\text{Gain increase}} \quad L(s) = \frac{15}{(s+1)^3}$$



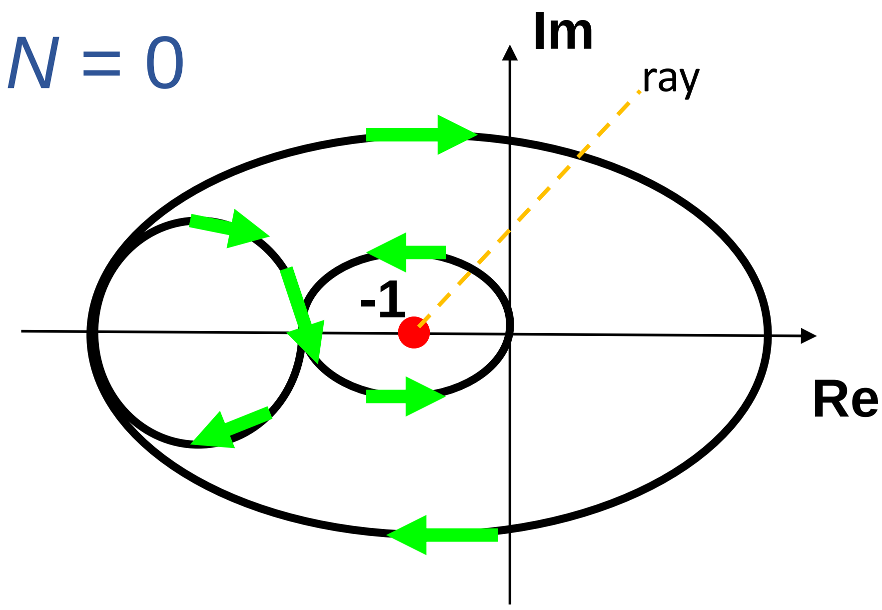
$P = 0,\ N = 0 \Rightarrow$ CL stable

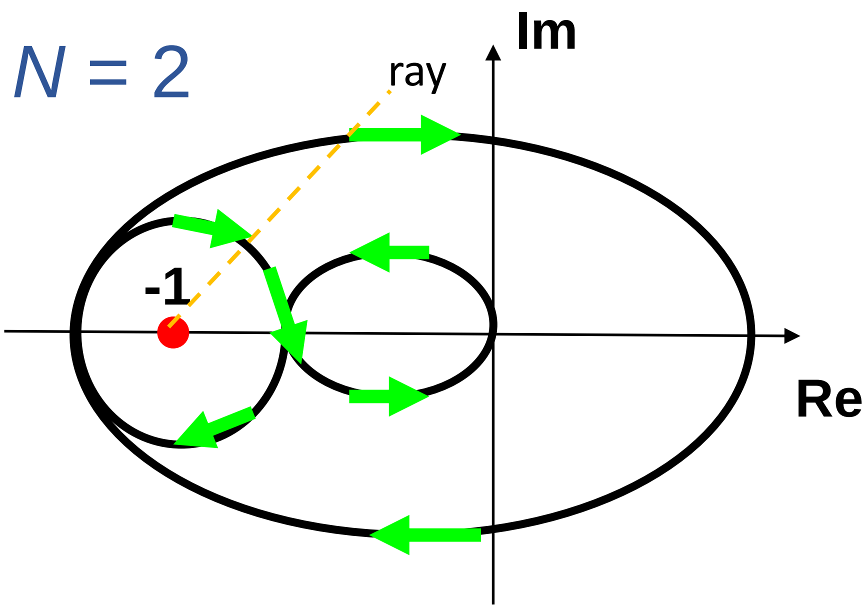
$P = 0,\ N = 2 \Rightarrow$ CL UNstable

How to count # of encirclement

- A ray is drawn from -1 point in any convenient direction. Then,

$$N = (\#\text{ of crossing of ray by Nyquist plot in clockwise direction}) \\ -(\#\text{ of crossing of ray by Nyquist plot in counterclockwise direction})$$



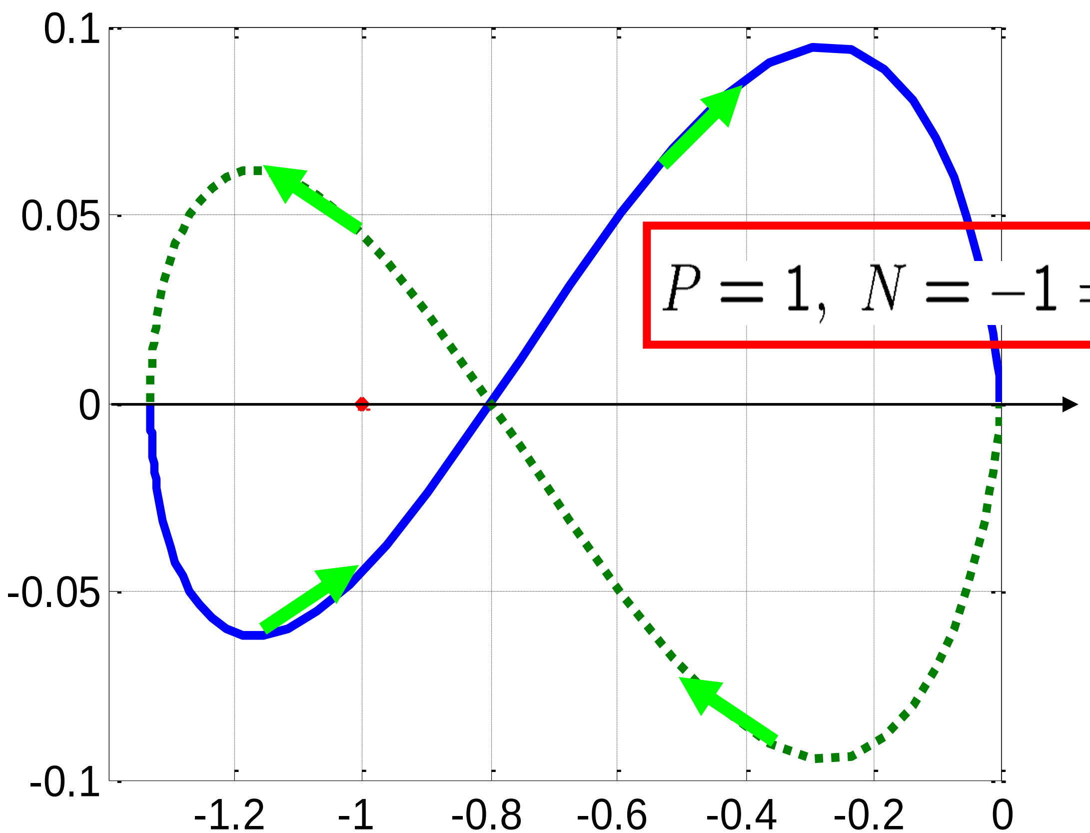


Notes on Nyquist stability criterion

- Nyquist stability criterion allows us to determine the stability of CL system from knowledge of the OL system.
- It can deal with time delay (next lecture), which Routh-Hurwitz criterion cannot.
- We often draw only half of Nyquist plot. The other half is mirror image with respect to real axis.
- Nyquist plot in MATLAB: nyquist(sys).

Example 8 (for unstable *L*(*s*))

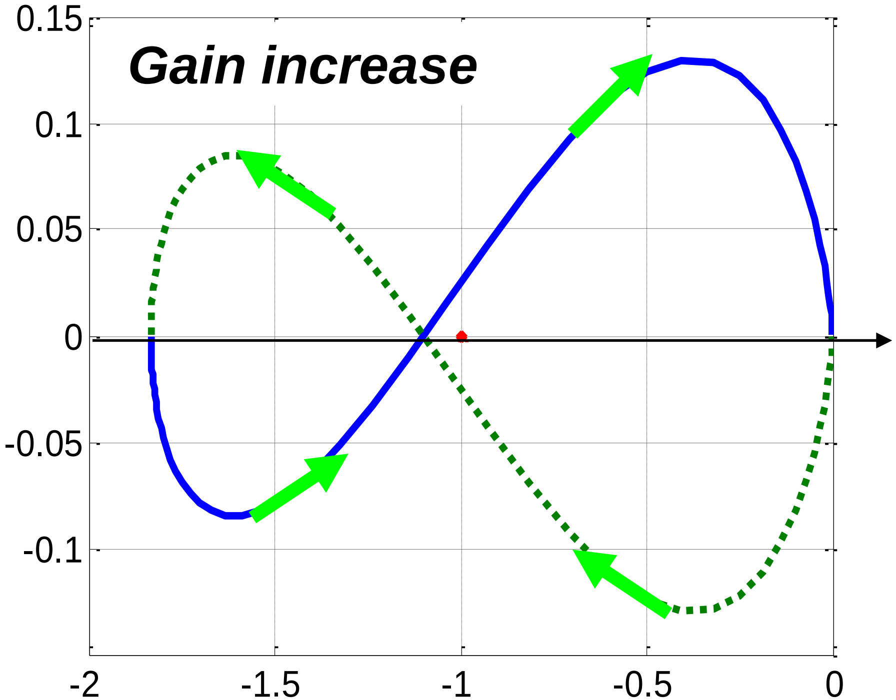
$$L(s) = \frac{8}{(s-1)(s+2)(s+3)}$$



$P = 1, \;\; N = -1 \Rightarrow \text{CL stable}$

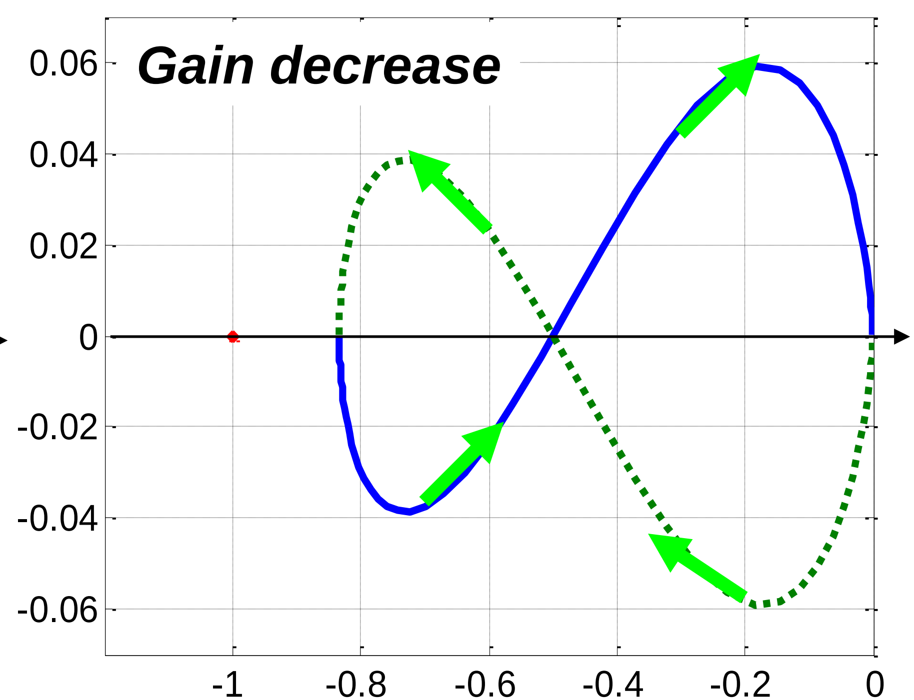
Example 8
(for unstable *L*(*s*), cont'd)

$$L(s) = \frac{11}{(s-1)(s+2)(s+3)}$$



$$\boxed{P = 1,\ \ N = 1 \Rightarrow \text{CL unstable}}$$

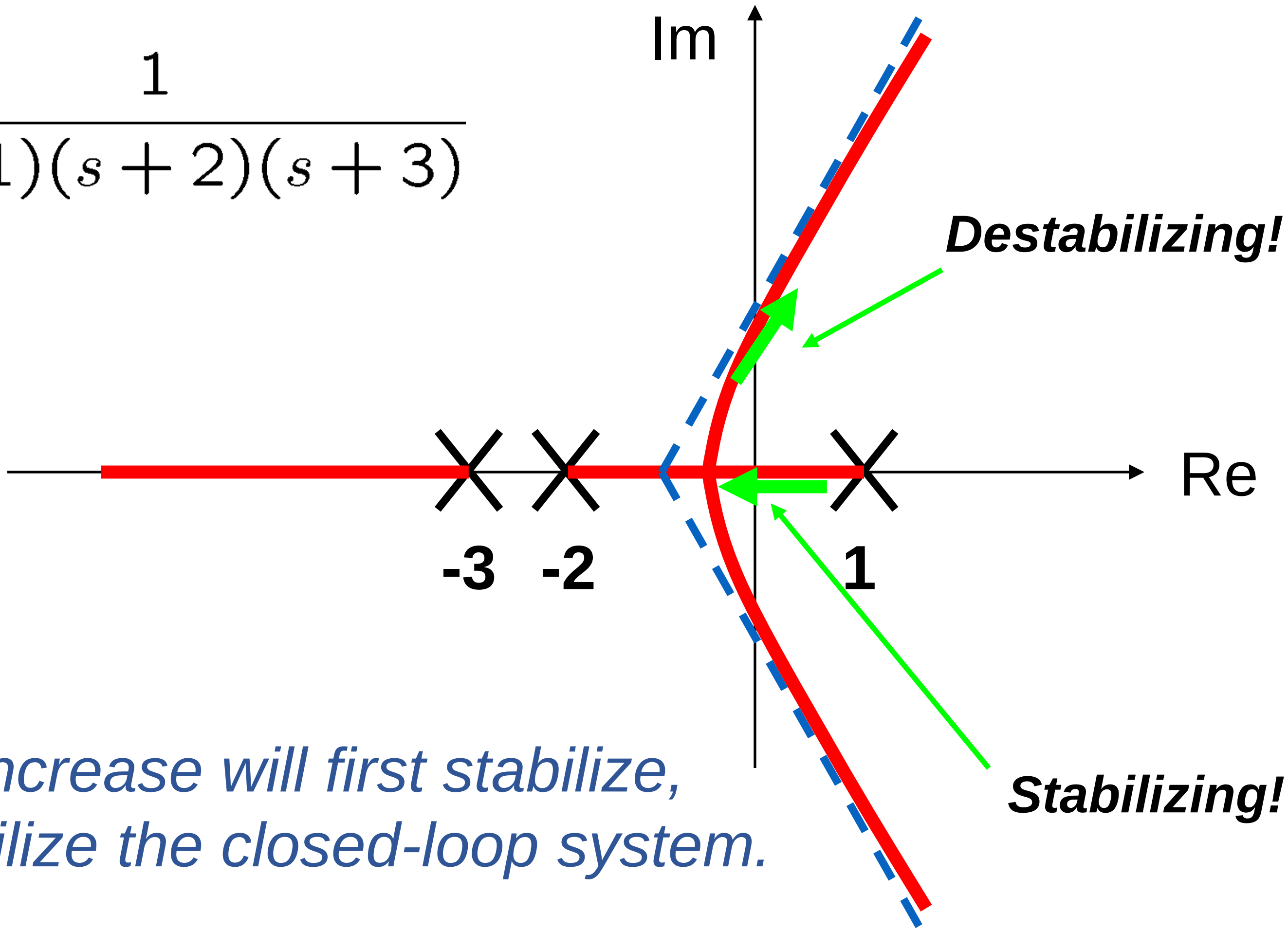
$$L(s) = \frac{5}{(s-1)(s+2)(s+3)}$$



$$\boxed{P = 1,\ \ N = 0 \Rightarrow \text{CL unstable}}$$

Example 8 (for unstable *L*(*s*), cont'd): Interpretation by root locus

$$L(s) = \frac{1}{(s-1)(s+2)(s+3)}$$



Open loop gain increase will first stabilize, and then, destabilize the closed-loop system.

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Nyquist criterion: A special case

$$\boxed{\text{CL system is stable} \Leftrightarrow Z = P + N = 0}$$

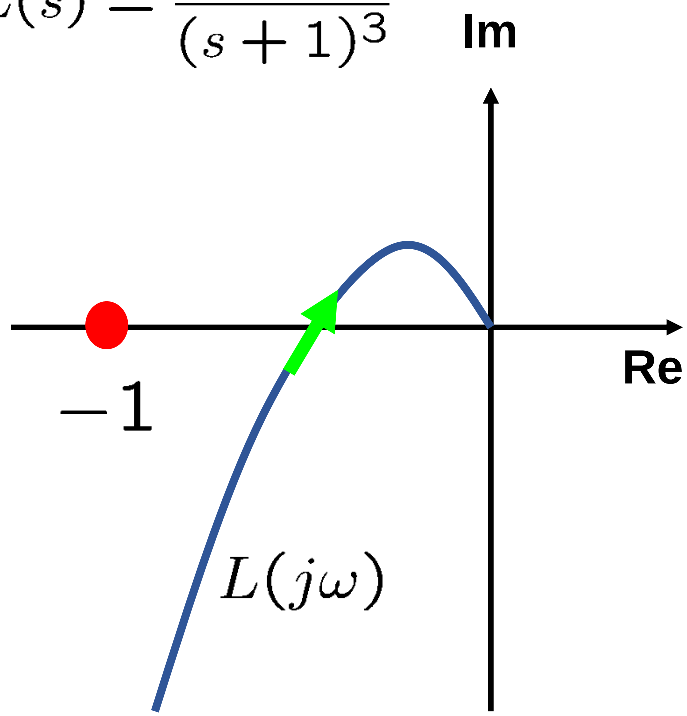
- If P = 0 (i.e., if $L(s)$ has no pole in open RHP)

$$\boxed{\text{CL system is stable} \Leftrightarrow N = 0}$$

This fact is very important since open-loop systems in many practical problems have no pole in open RHP!

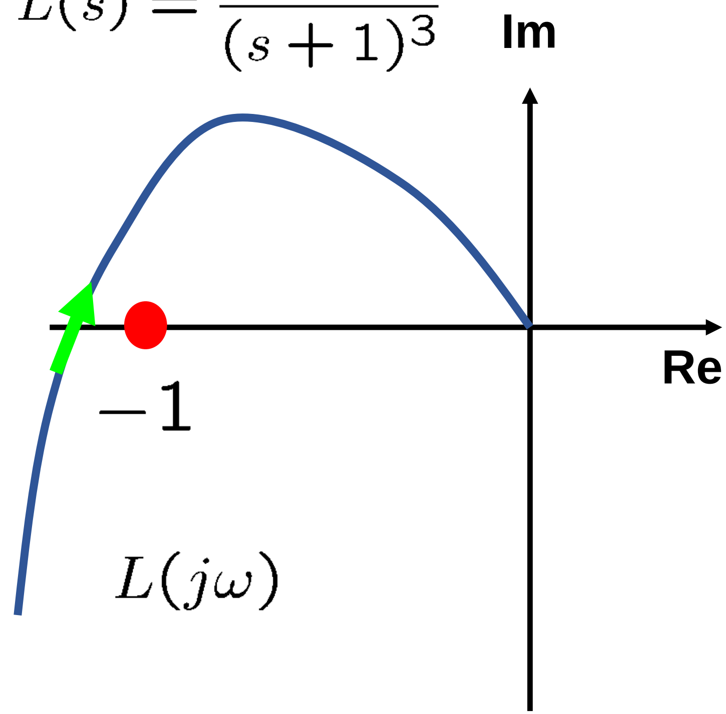
Example 9 (when $P = 0$)

$$L(s) = \frac{1}{(s+1)^3}$$



CL stable

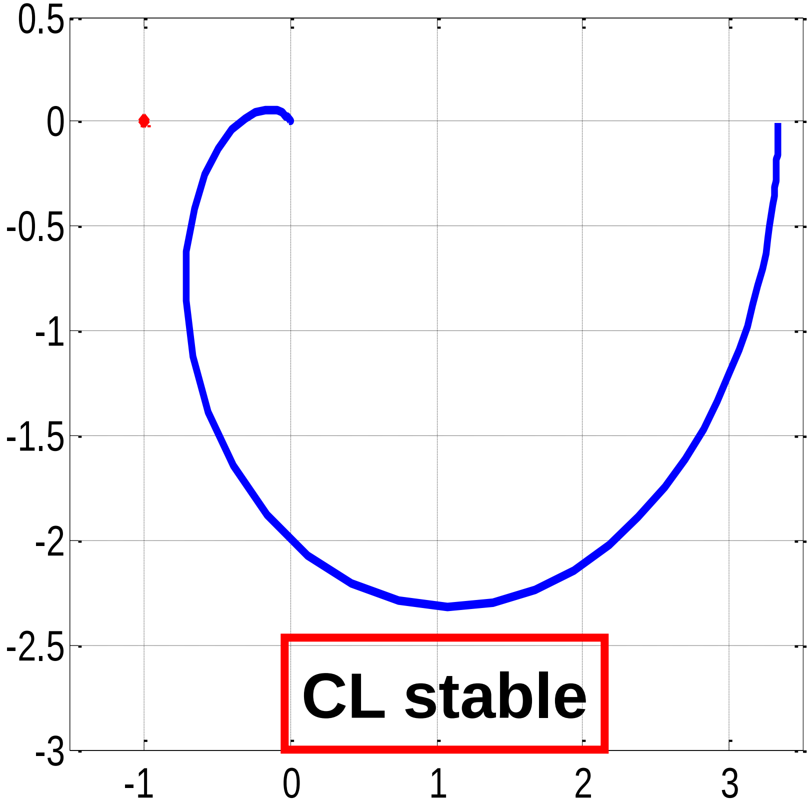
$$L(s) = \frac{15}{(s+1)^3}$$



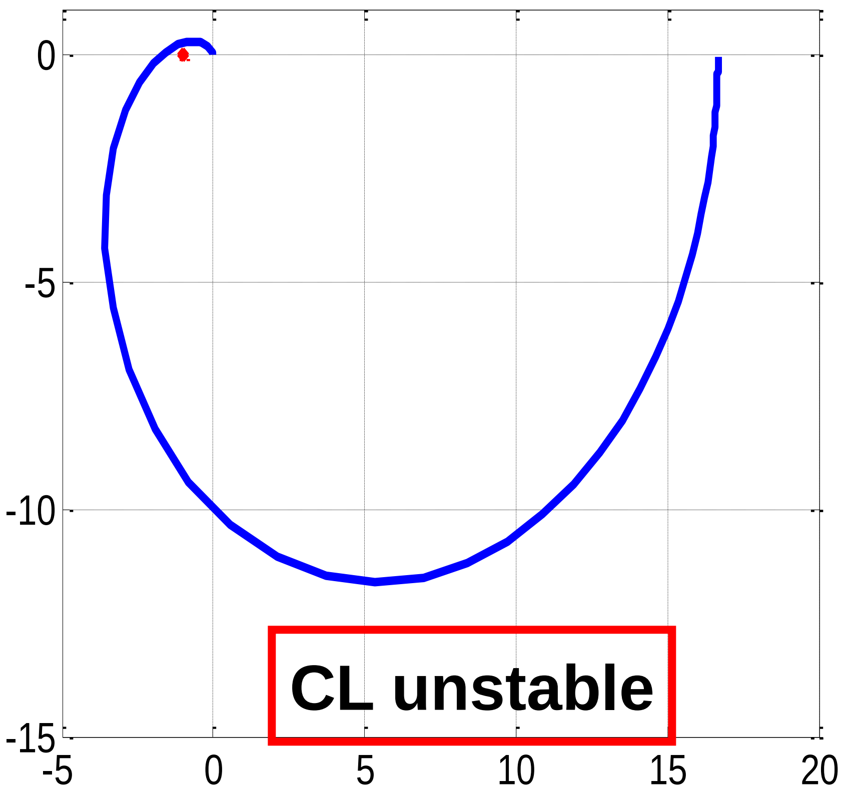
CL unstable

Example 10
(for stable *L*(*s*), cont'd)

$$L(s) = \frac{20}{(s+1)(s+2)(s+3)}$$

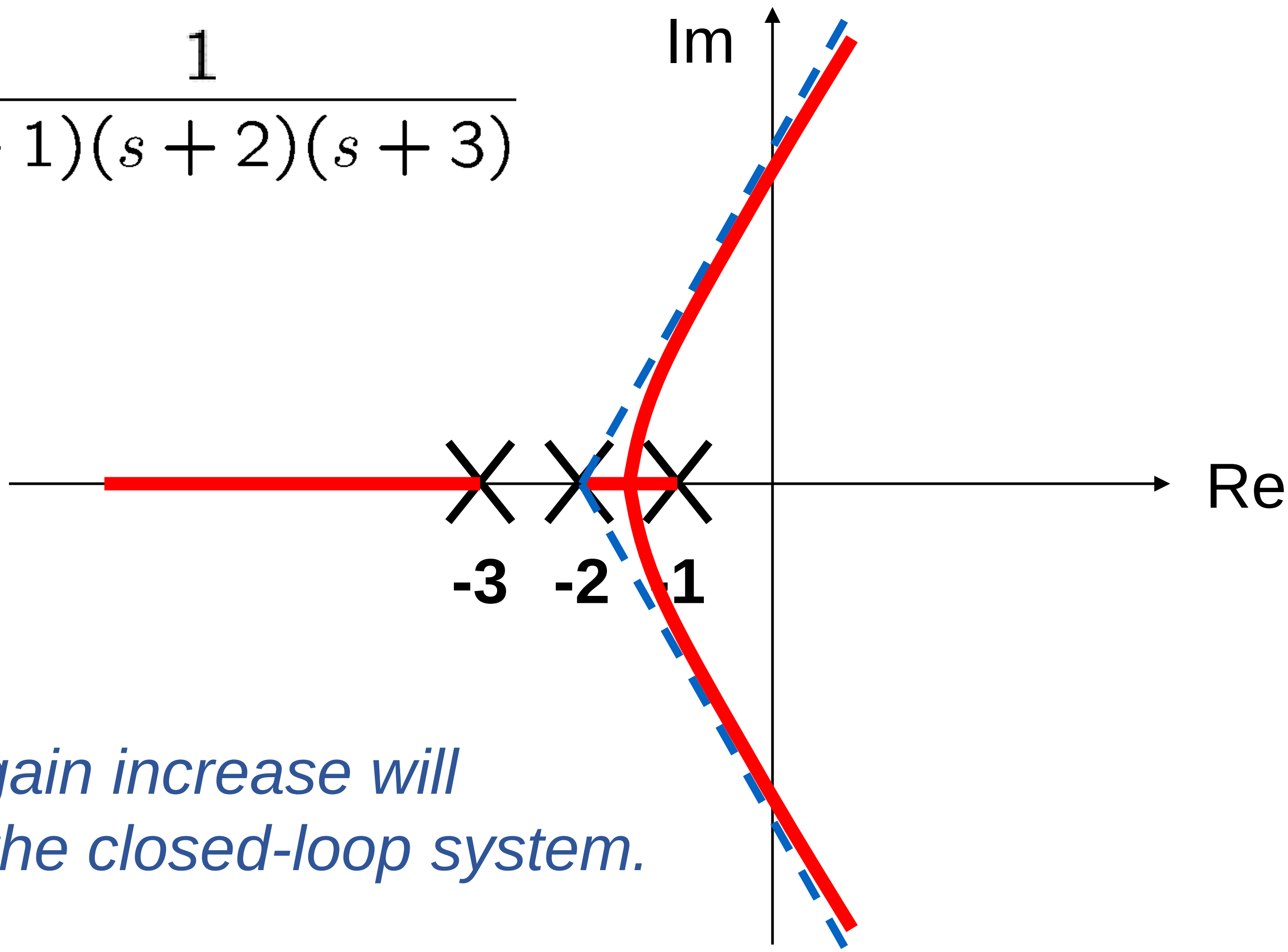


$$L(s) = \frac{100}{(s+1)(s+2)(s+3)}$$



Example 10 (Interpretation by root locus, cont'd)

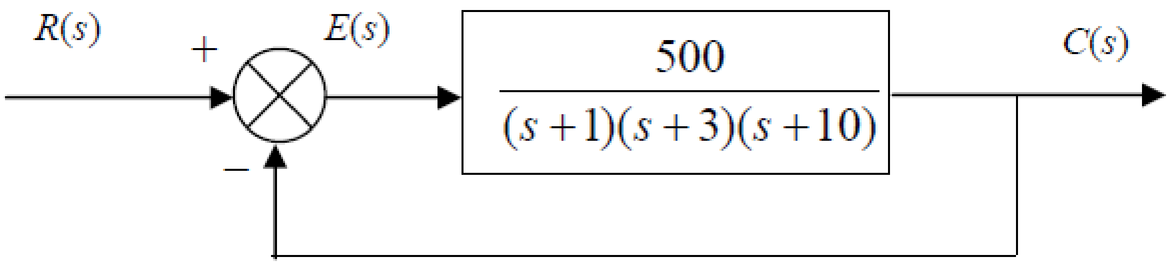
$$L(s) = \frac{1}{(s+1)(s+2)(s+3)}$$



Open loop gain increase will destabilize the closed-loop system.

Example 11

Sketch the Nyquist diagram for the system shown in the following figure, and then determine the system stability using the Nyquist criterion.



The open-loop transfer function: $G(s)H(s)=\dfrac{500}{(s+1)(s+3)(s+10)}$.

Replacing s with $j\omega$ yields the frequency response of $G(s)H(s)$, i.e.,

$$G(j\omega)H(j\omega)=\frac{500}{(j\omega+1)(j\omega+3)(j\omega+10)}=\frac{500}{(-14\omega^2+30)+j(43\omega-\omega^3)}$$

$$=500\frac{(-14\omega^2+30)-j(43\omega-\omega^3)}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2} \longrightarrow \textbf{(a)}$$

Example 11 (cont'd)

Magnitude response:

$$|G(j\omega)H(j\omega)| = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \frac{500}{\sqrt{(-14\omega^2+30)^2 + (43\omega-\omega^3)^2}}$$

Phase response:

$$\angle G(j\omega)H(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = \tan^{-1}\left(\frac{-\left(43\omega-\omega^3\right)}{-14\omega^2+30}\right)$$

Now that when we have expressions for the magnitude and phase of the frequency response, we can sketch the polar plot using the **4 key points**.

Point 1: The start of plot where $\omega = 0$

$$|G(0)H(0)| = \frac{500}{\sqrt{(30)^2}} = 16.67 \qquad \angle G(0)H(0) = \tan^{-1}\frac{0}{30} = 0^\circ$$

Example 11 (cont'd)

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Point 2: The end of plot where $\omega = \infty$

$$|G(\infty)H(\infty)| = \frac{500}{\sqrt{\infty}} = 0$$

$$\angle G(\infty)H(\infty) = \tan^{-1}\infty = -270°$$

Point 3: Where the plot crosses the real axis, i.e., **Im(G(jω)H(jω)) = 0**

Take the imaginary part of equation **(a)**, and set it equal to zero, to get the value of frequency ω at the interception of real axis:

$$\frac{-(43\omega-\omega^3)}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2} = 0 \Rightarrow \begin{cases} 43\omega-\omega^3 = 0 \quad \Rightarrow \quad \omega = 0 \quad \text{and} \quad \omega = 6.56\ \text{rad/s} \\ \omega = \infty \end{cases}$$

Example 11 (cont'd)

Point 4: Where the plot crosses the imaginary axis, Re(*G*(jω)H(jω)) = 0

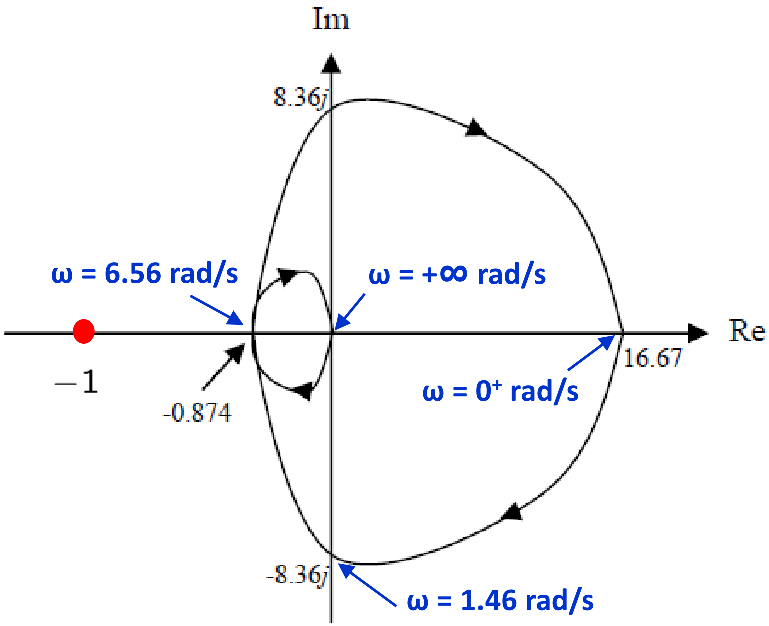
Take the real part of equation **(a)**, and set it equal to zero, to get the value of frequency ω at the interception of imaginary axis:

$$\frac{-14\omega^2+30}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2}=0 \Rightarrow \begin{cases} -14\omega^2+30=0 & \Rightarrow \quad \omega=1.46\ \mathrm{rad/s} \\ \omega=\infty \end{cases}$$

Key Points of the polar plot:

	$\lvert GH \rvert$	$\angle GH$
$\omega=0$	16.67	0°
$\omega=\infty$	0	-270°
Re Crossing:		
$\omega=0$	16.67	0°
$\omega=\infty$	0	-270°
$\omega=6.56$ rad/s	0.874	-180°
Im Crossing:		
$\omega=\infty$	0	-270°
$\omega=1.46$ rad/s	8.36	-90°

Nyquist diagram



P = 0, N = 0 (no encirclements of -1). So based on $Z = P + N$ = 0 (**CL is stable**).

Summary

- Nyquist stability criterion for feedback stability
- Examples for Nyquist stability criterion
- Next
    - Relative stability