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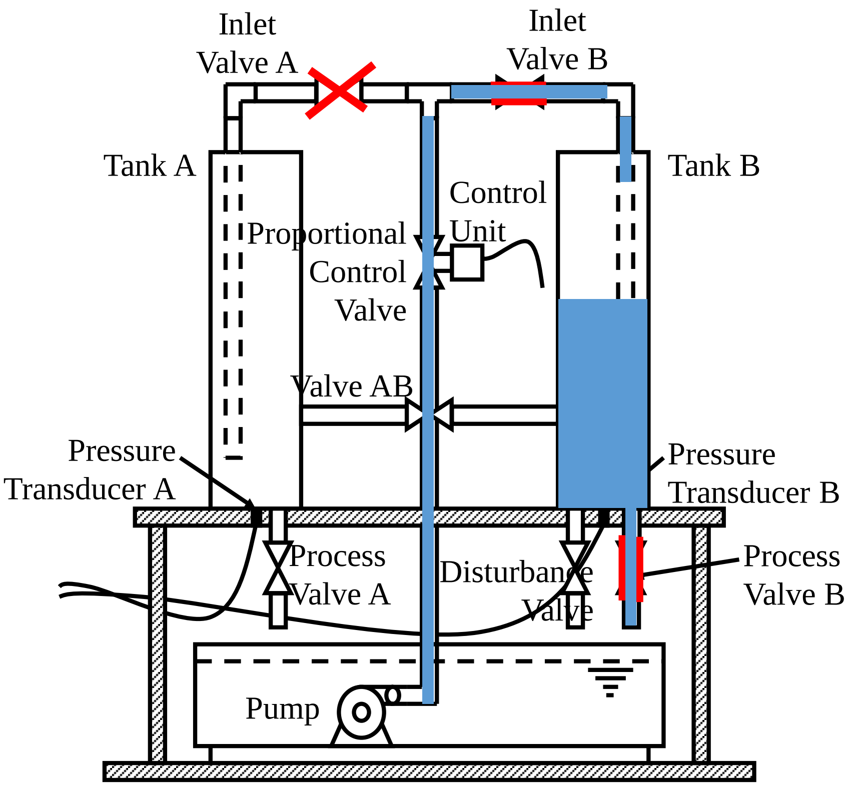
ELEC 341: Systems and Control

Lecture 8

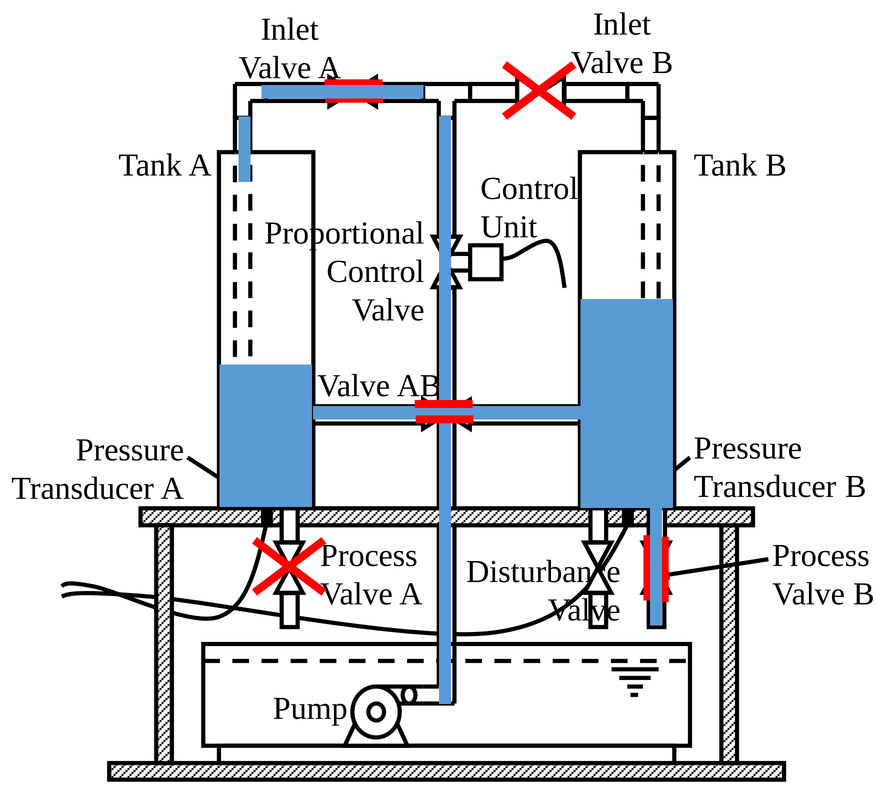
Time-domain specifications and steady-state error

Water tank level control

- Requirement: Maintain the level of Tank B at a desired level by controlling the control valve.

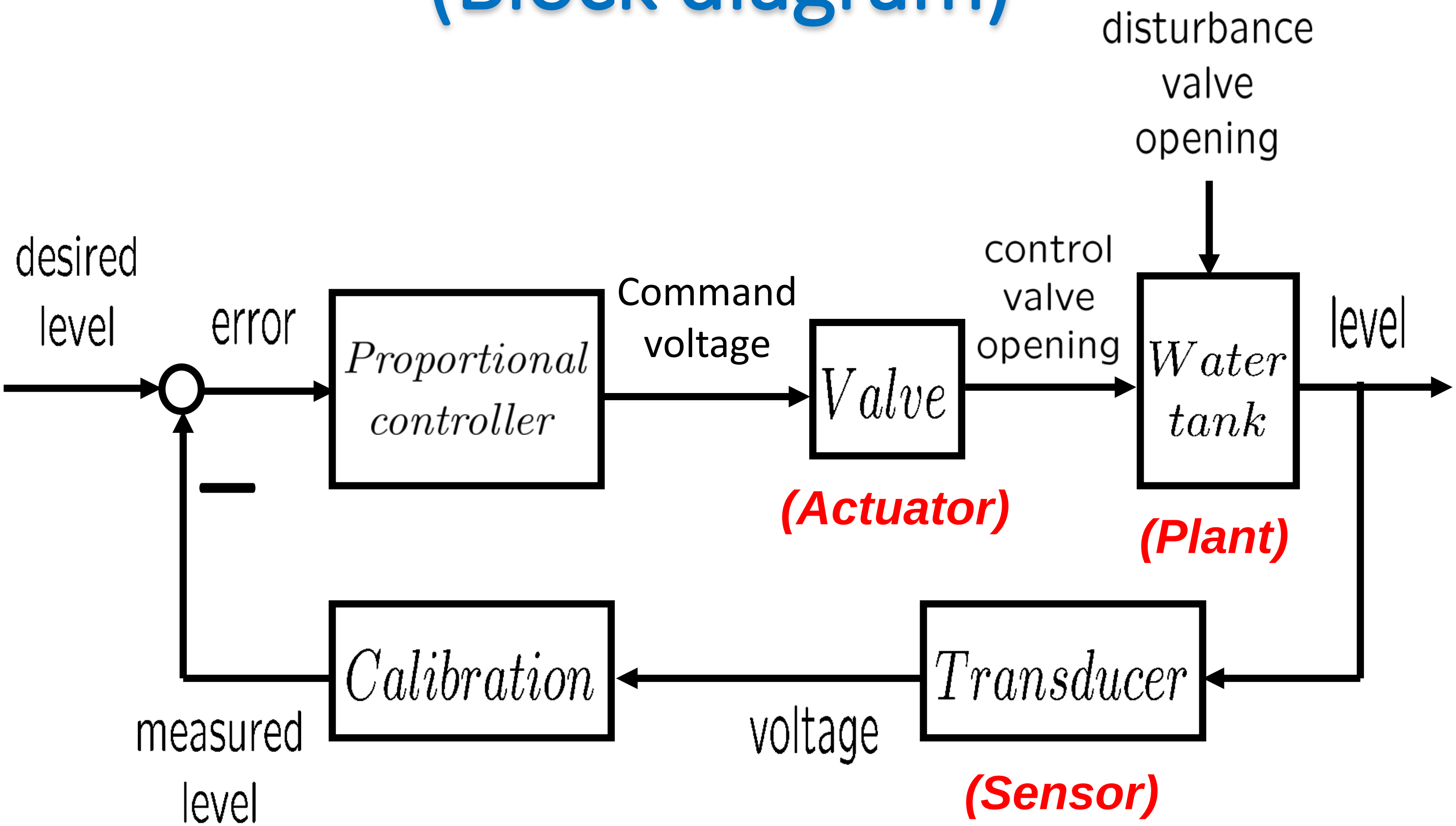


Single tank case



Two tank case

Water tank level control (Block diagram)

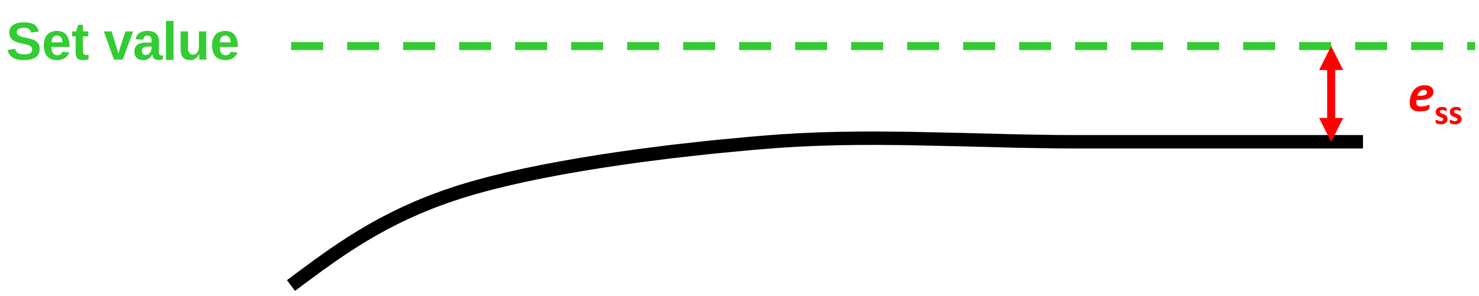


Proportional control (P-control)

- Set value (desired level)
- Controller gain K_P (design parameter)
- Command voltage $V(t)$ ($e(t)$ is an error)

$$V(t) = \begin{cases} V_{max} & \text{if } K_P e(t) > V_{max} \\ K_P e(t) & \text{if } K_P e(t) \in [V_{min}, V_{max}] \\ V_{min} & \text{if } K_P e(t) < V_{min} \end{cases}$$

- Note that there is a steady state error.



Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - Electrical
  - Electromechanical
  - Mechanical
- Linearization, delay

Analysis

- Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

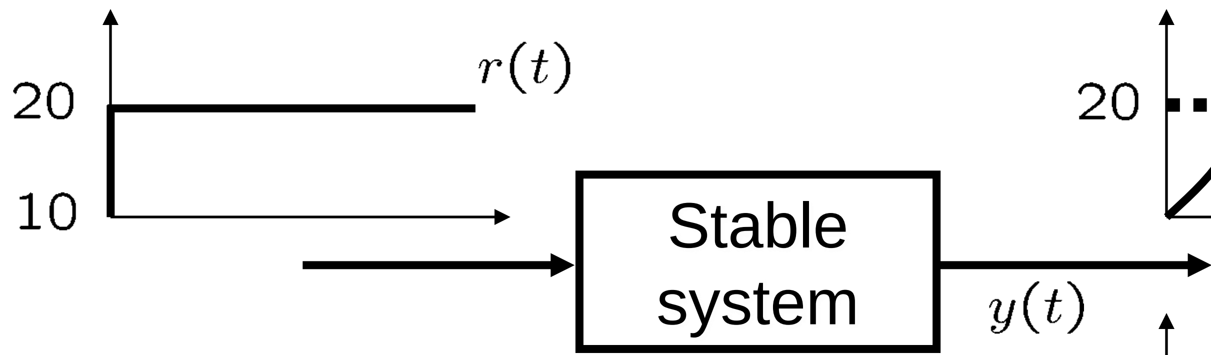
Matlab simulations

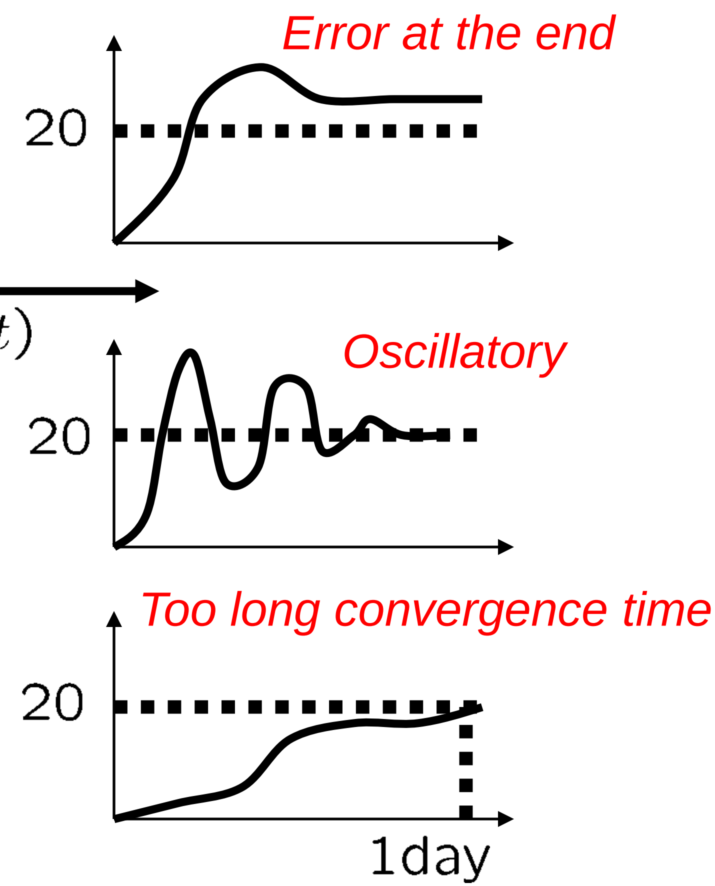
Review and next topics

- We have learned about stability.
  - Definition in time domain
  - Condition in *s*-domain
  - Routh-Hurwitz criterion to check the condition
- Stability is a necessary requirement, but not sufficient in most control systems. (next slide)
- Specifications other than stability
  - How to evaluate a system quantitatively in *t*-domain?
  - How to give **design specifications** in *t*-domain?
  - What are the corresponding conditions in *s*-domain?

Temperature control example

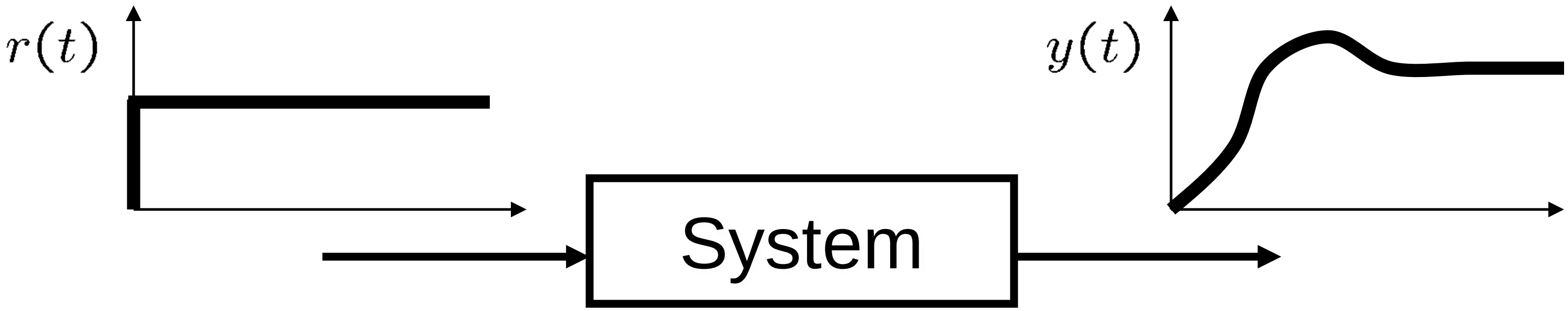
- We want to change the room temperature from 10 °C to 20 °C.





Are we happy with these responses?

Time response



- We would like to **analyze** our stable system's property by applying a *test input* $r(t)$ and observing its time response $y(t)$.
- Time response is divided into two components as below:

$$y(t) = \underbrace{y_t(t)}_{\text{Transient (natural) response}} + \underbrace{y_{ss}(t)}_{\text{Steady-state (forced) response}}$$

Transient (natural) response

$$\lim_{t \to \infty} y_t(t) = 0$$

Steady-state (forced) response

(after y_t dies out)

Example:
Transient & steady-state responses

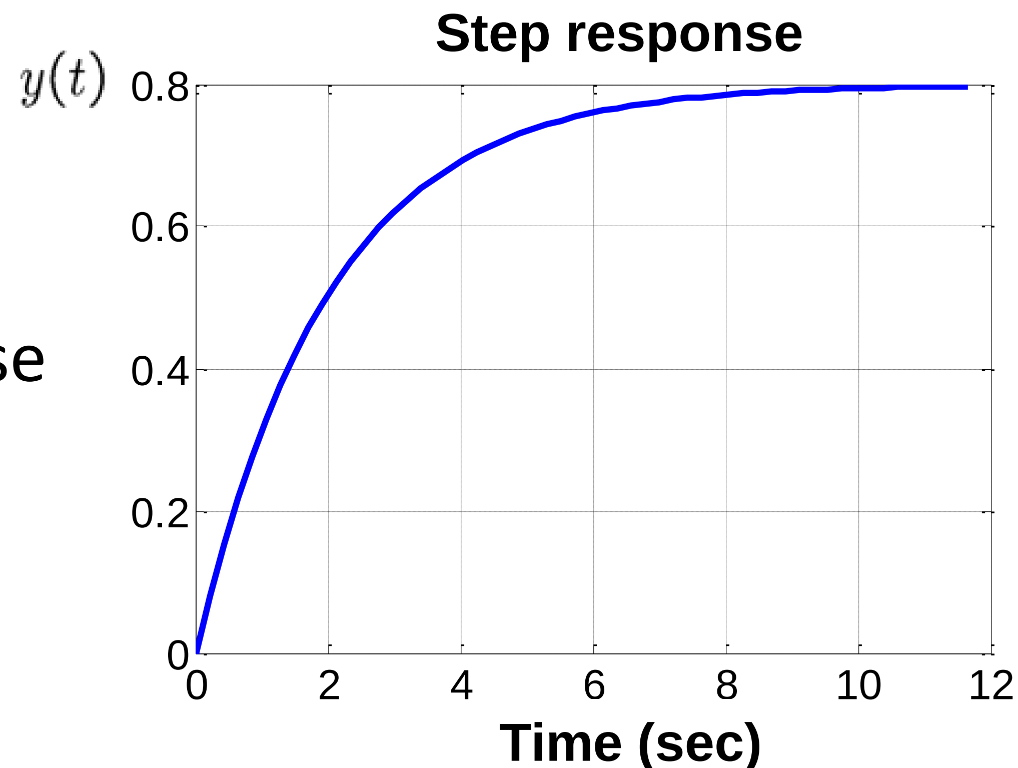
$$r(t) = u(t) \longrightarrow \boxed{\frac{0.8}{2s+1}} \longrightarrow y(t) = 0.8 - 0.8e^{-\frac{t}{2}}$$

- Transient response

$$y_t(t) = -0.8e^{-\frac{t}{2}}$$

- Steady-state response

$$y_{ss}(t) = 0.8$$



Usage of time responses

- **Modeling**
  - Some parameters in the system may be estimated by time responses.
- **Analysis**
  - A system can be evaluated by seeing transient and steady-state responses. (Satisfactory or not?)
- **Design**
  - Given design specs in terms of transient and steady-state responses, controllers are designed to satisfy all the design specs.

Typical test inputs

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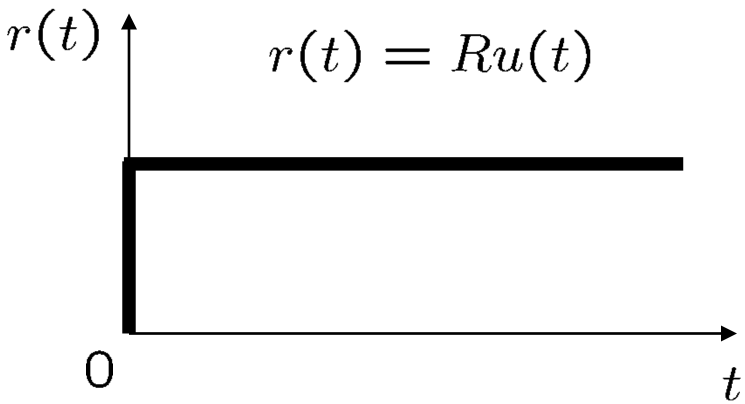
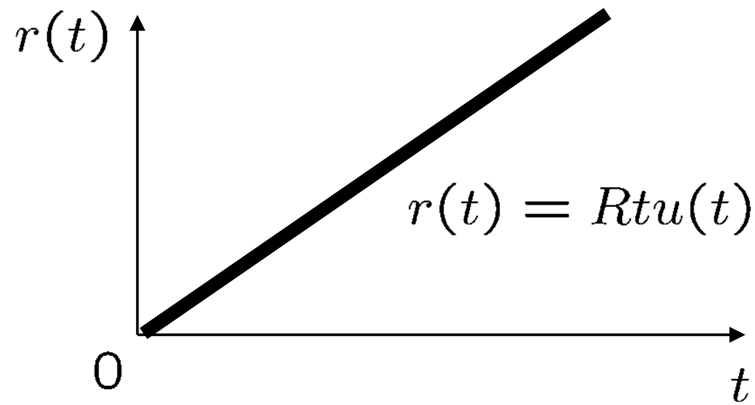
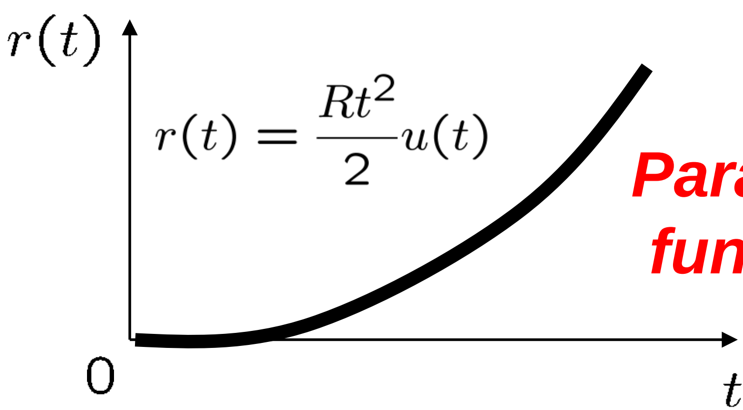


Step function

(Most popular)



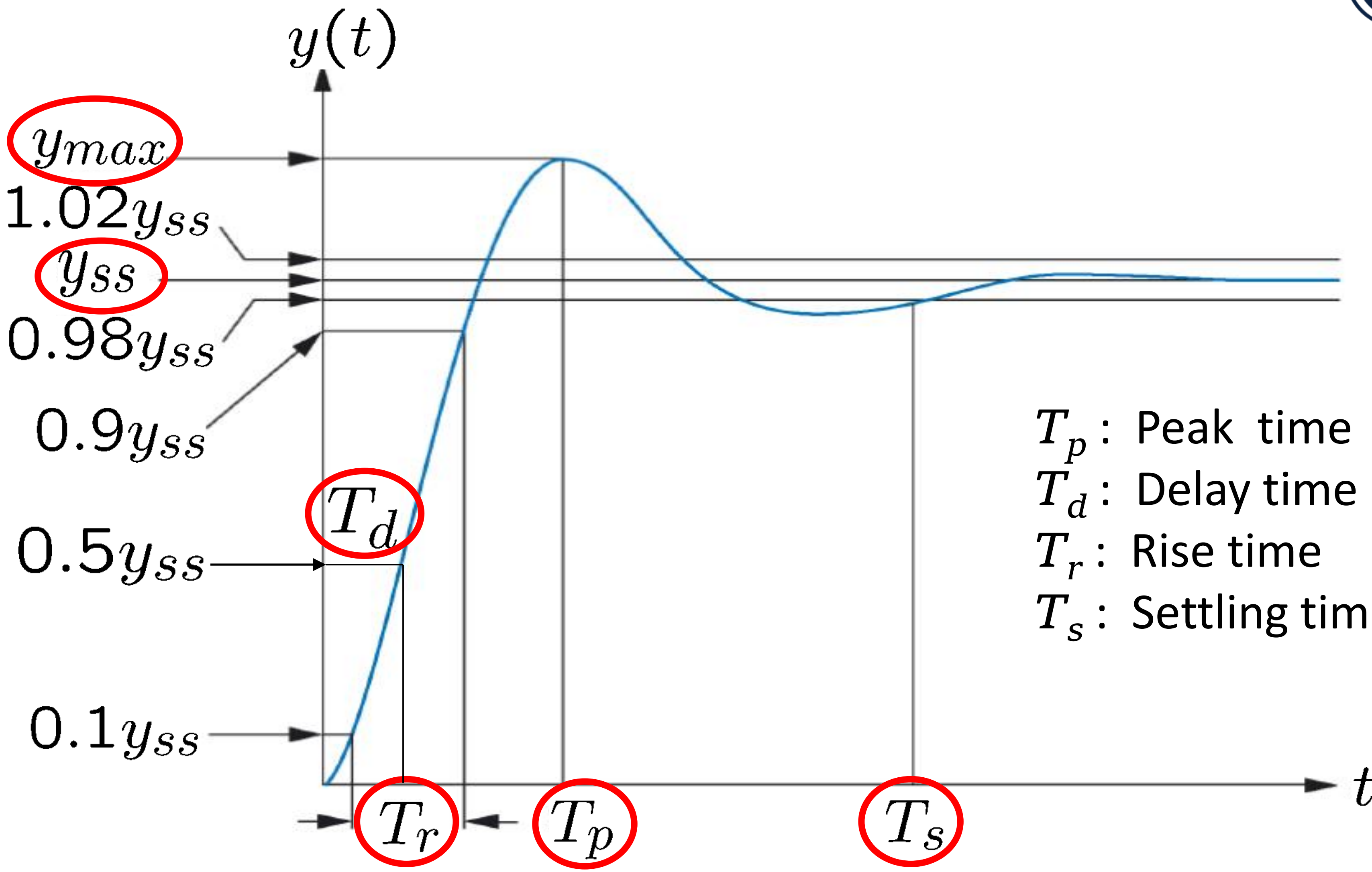
Ramp function



Parabolic function

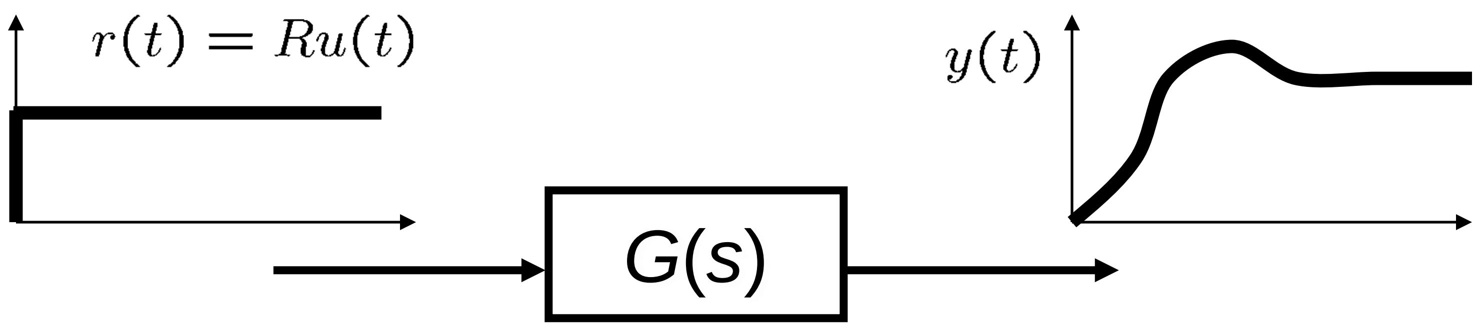
Sinusoidal input
will be dealt with later.

Typical step response



T_p : Peak time
T_d : Delay time
T_r : Rise time
T_s : Settling time

Steady-state value for step input



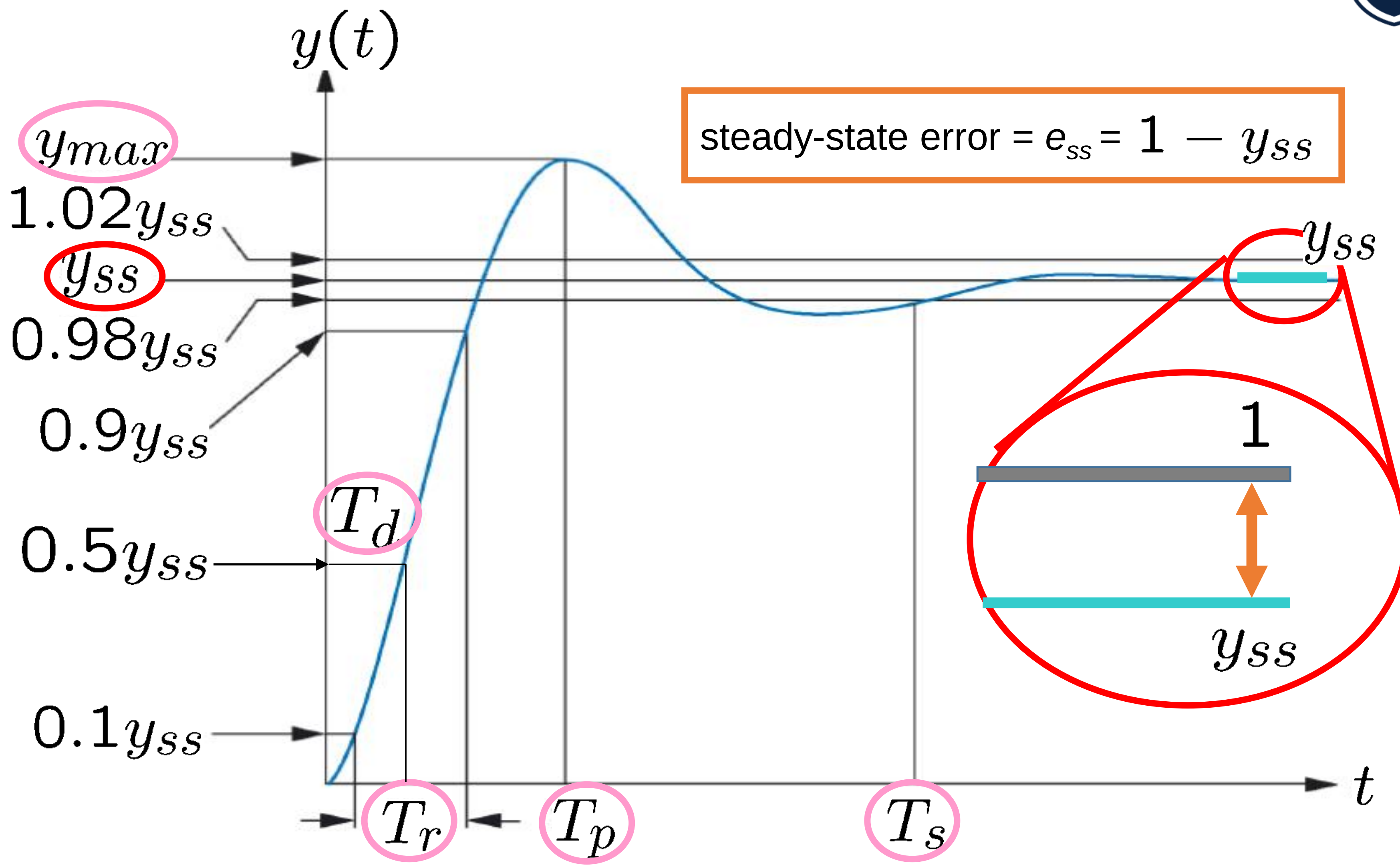
- Suppose that *G(s)* is stable.
- By the final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot Y(s) = \lim_{s\to 0} s\cdot G(s)\cdot R(s) = \lim_{s\to 0} s\cdot G(s)\cdot U(s) = \lim_{s\to 0} s\cdot G(s)\cdot \frac{R}{s}$$

$$\longrightarrow \quad \lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)\frac{R}{s} = RG(0)$$

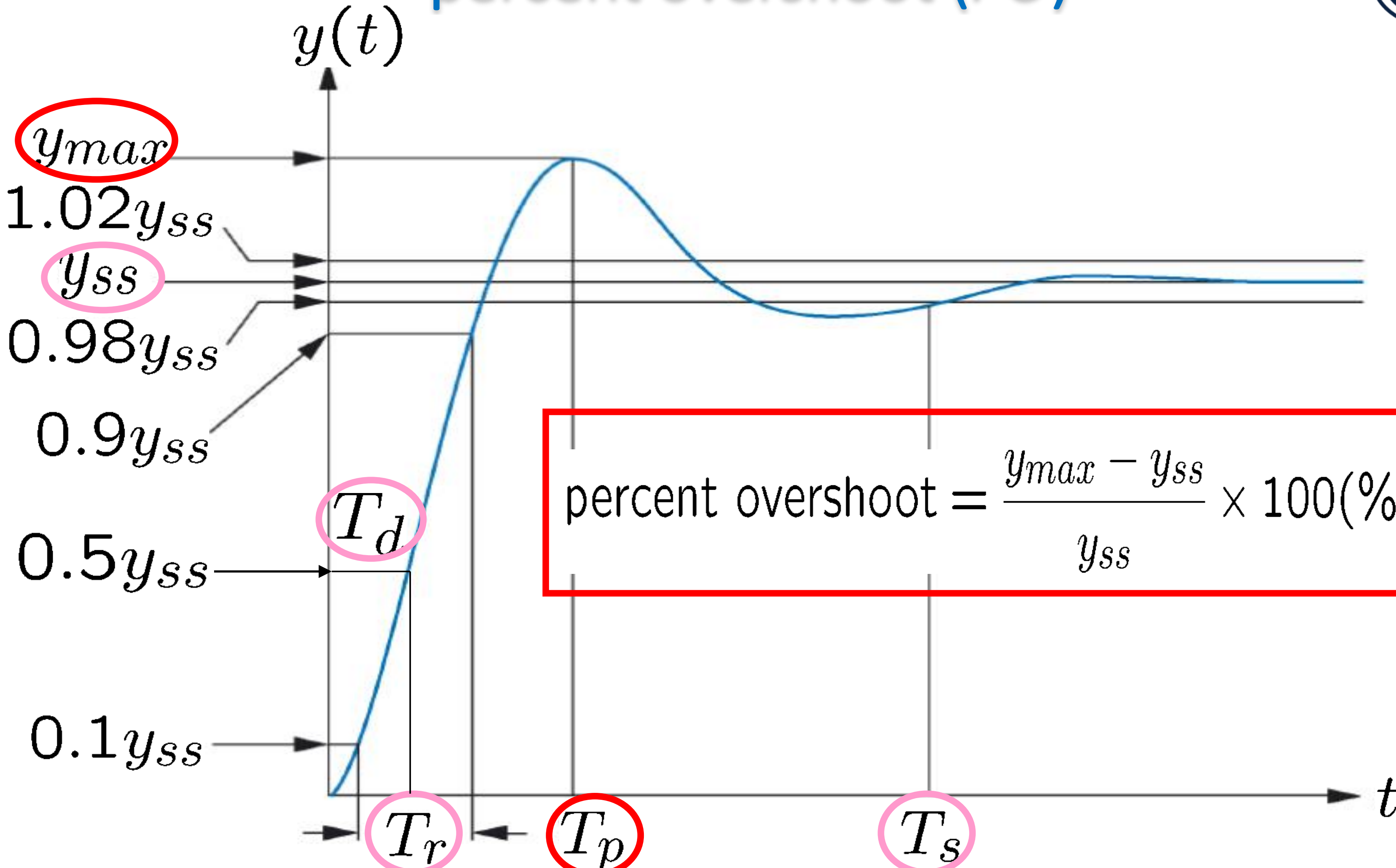
- Step response converges to some finite value, called *steady-state value* y_{ss}.

Steady-state error for input $u(t)$



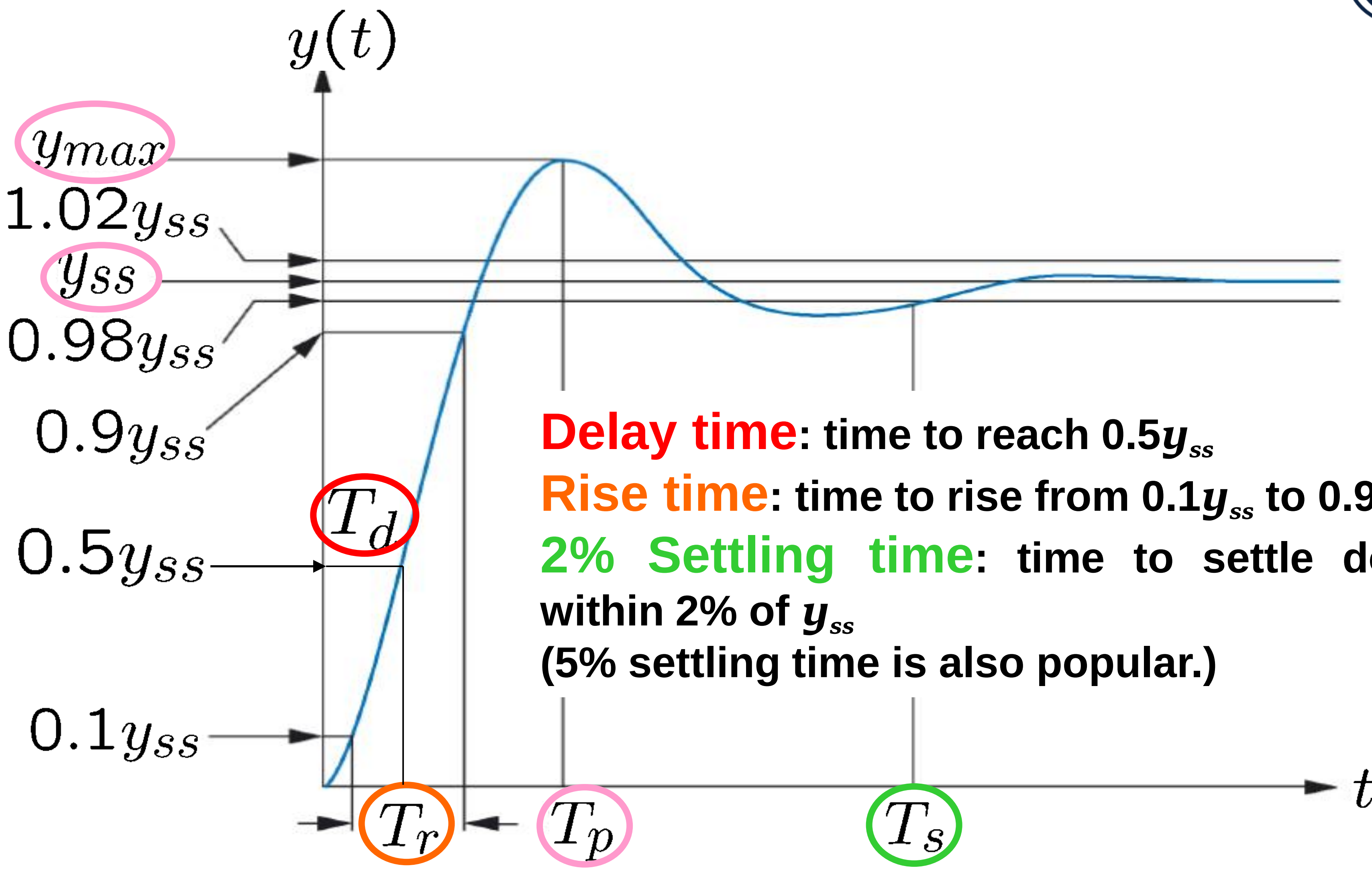
steady-state error = e_{ss} = $1 - y_{ss}$

Peak value (y_{max}), peak time (T_p), and percent overshoot (PO)



$$\text{percent overshoot} = \frac{y_{max} - y_{ss}}{y_{ss}} \times 100(\%)$$

Delay, rise, and settling times



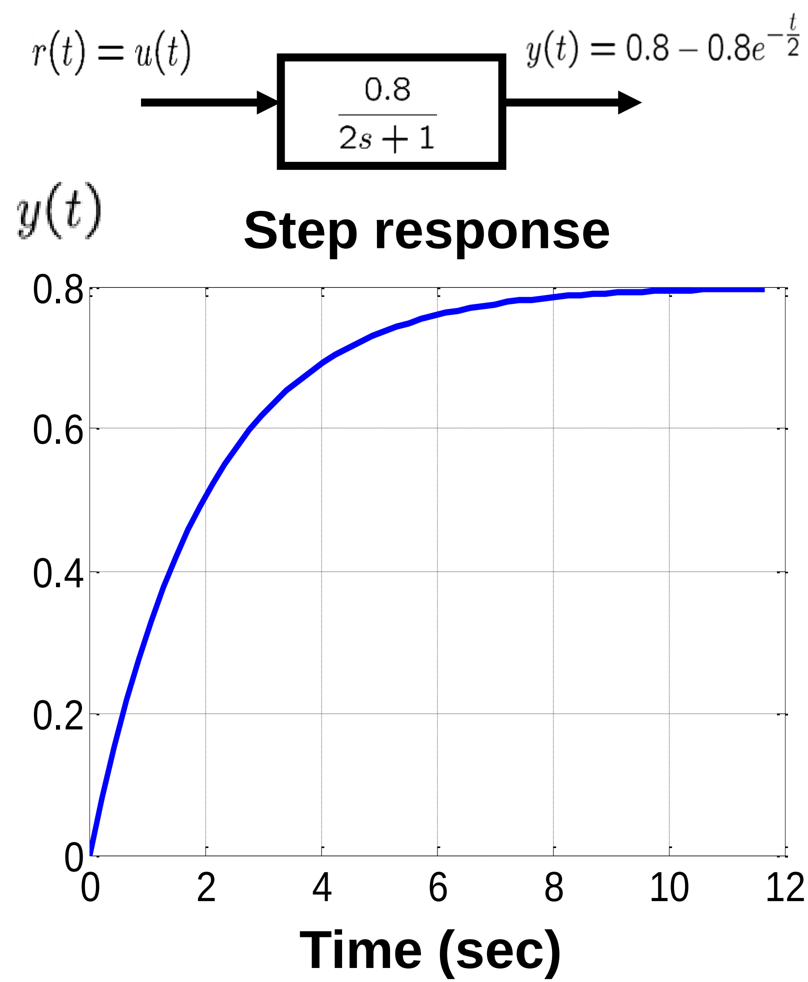
Delay time: time to reach $0.5y_{ss}$

Rise time: time to rise from $0.1y_{ss}$ to $0.9y_{ss}$

2% Settling time: time to settle down within 2% of y_{ss}
(5% settling time is also popular.)

An example revisited

- For the previous example shown here again:
  - Steady-state error: 1 - 0.8 = 0.2
  - Delay time around 1.4 sec
  - Rise time around 5 sec
  - Settling time around 8 sec
- Remark: There is no peak in this case, so the following are undefined:
  - peak value
  - peak time
  - percent overshoot



Remarks on time responses

- Speed of response is measured by
  - Rise time, delay time, peak time, and settling time
- Relative stability is measured by
  - Percent overshoot
- Typically …
  - Fast response (short rise time, short peak time)

  → Large percent overshoot → Small stability margin
- In controller design, we normally face a trade-off between response speed and stability.

  (“***No-free-lunch theorem***” in Control Engineering!)

Performance measures

- Transient response
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time
- Steady state response
  - Steady state error

(Next lecture)

Next, we will connect these performance measures with s-domain.

(Today's lecture)

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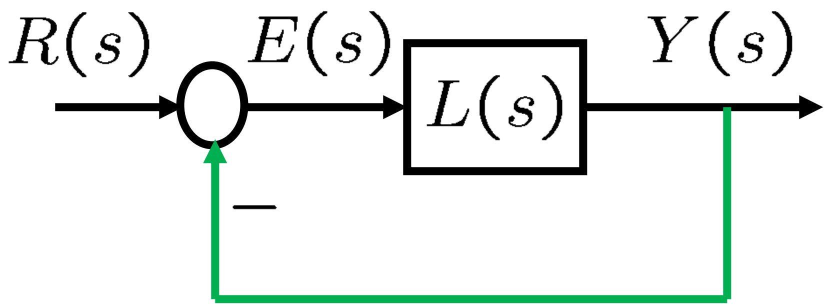
Matlab simulations

Steady-state error of feedback system



Assumptions

- $L(s)$ = Controller(s)×Plant(s)
- Unity feedback (no block on feedback path)
- CL system is stable

$L(s) = K.G(s)$

- Suppose that we want output $y(t)$ to track $r(t)$.
- Error $e(t) = r(t) - y(t)$
- Steady-state error

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+L(s)}R(s)$$

Final value theorem
(Suppose CL system is stable!)

Error constants

- CL system's ability to reduce steady-state error e_{ss}
  - "Large error constant" means "large ability".
- Three error constants (also called static error constants) are:
  - Step-error (**p**osition-error) constant

$$K_p = \lim_{s \to 0} L(s)$$

  - Ramp-error (**v**elocity-error) constant

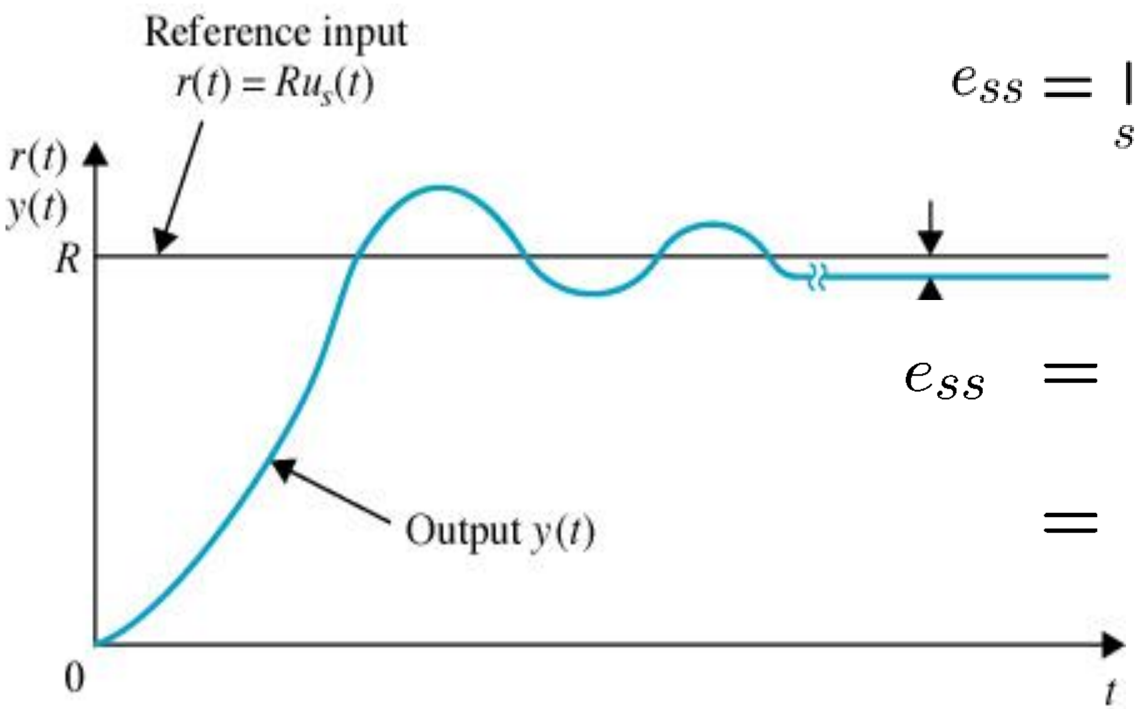
$$K_v = \lim_{s \to 0} sL(s)$$

  - Parabolic-error (**a**cceleration-error) constant

$$K_a = \lim_{s \to 0} s^2 L(s)$$

Steady-state error for step *r*(*t*)

$$\boxed{r(t) = Ru(t) \Rightarrow e_{ss} = \frac{R}{1+K_p}}$$



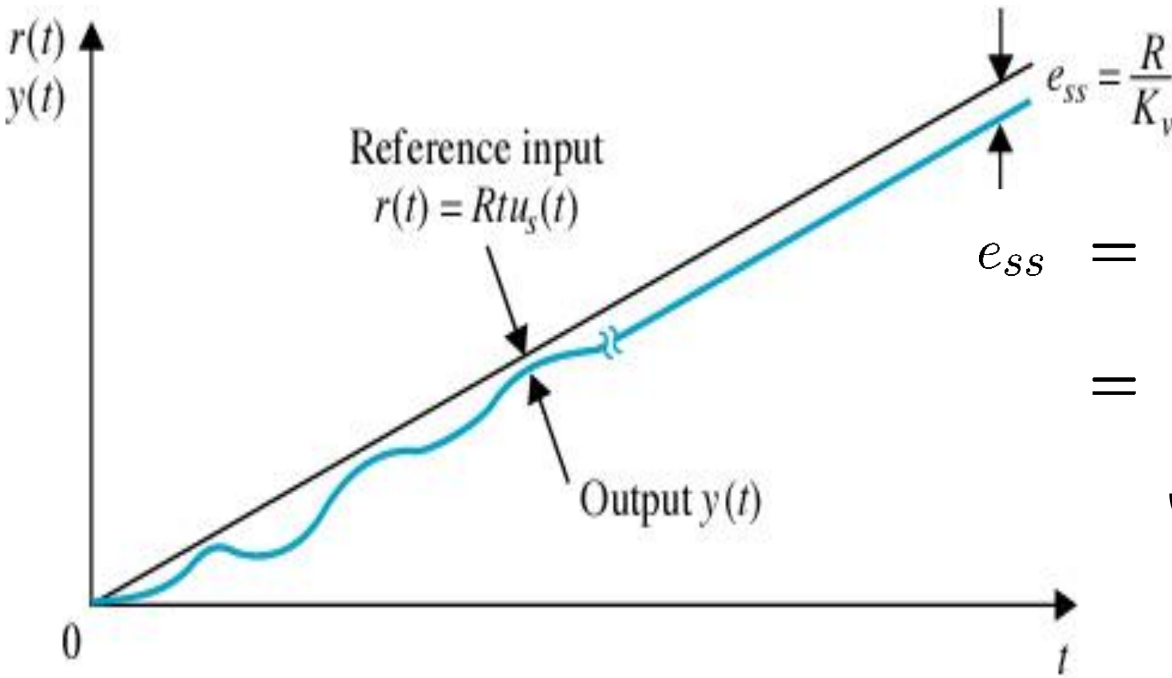
$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+L(s)}R(s)$$

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+L(s)}\cdot\frac{R}{s}$$

$$= \frac{R}{1+\underbrace{L(0)}_{K_p}}$$

Steady-state error for ramp *r*(*t*)

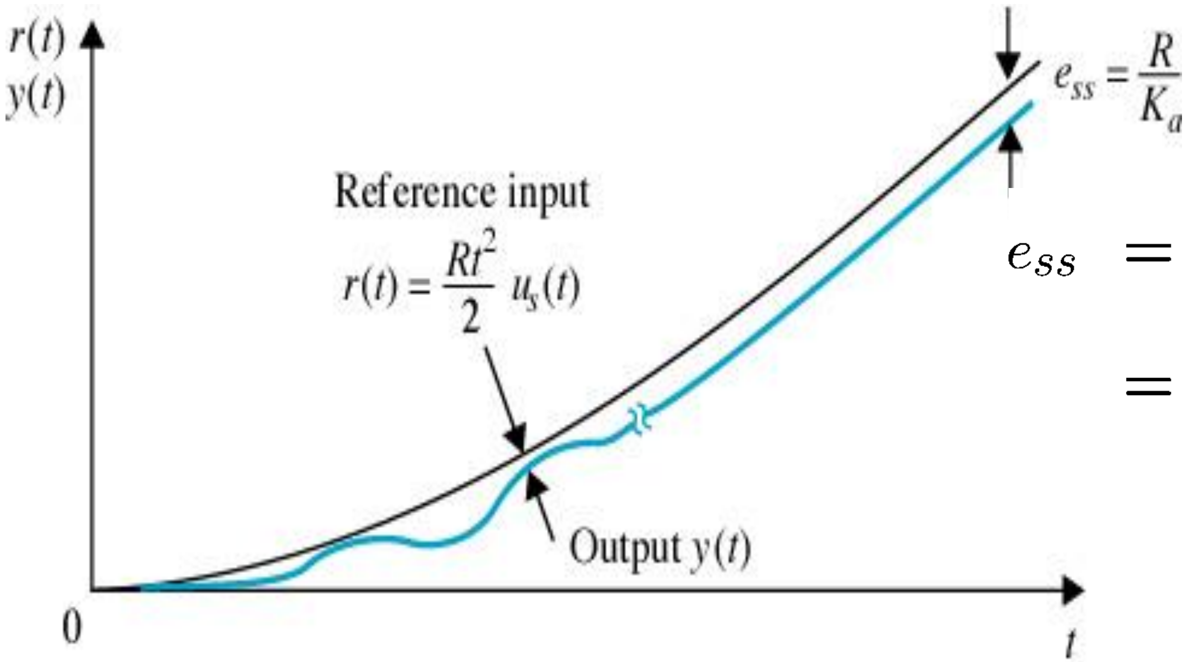
$$\boxed{r(t) = Rtu(t) \Rightarrow e_{ss} = \frac{R}{K_v}}$$



$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s \frac{1}{1 + L(s)} \cdot \frac{R}{s^2} \\ &= \frac{R}{\underbrace{\lim_{s \to 0} sL(s)}_{K_v}} \end{aligned}$$

Steady-state error for parabolic *r*(*t*)

$$\boxed{r(t) = \frac{Rt^2}{2}u(t) \Rightarrow e_{ss} = \frac{R}{K_a}}$$



$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\frac{1}{1+L(s)} \cdot \frac{R}{s^3} \\ &= \frac{R}{\underbrace{\lim_{s\to 0} s^2 L(s)}_{K_a}} \end{aligned}$$

Zero steady-state error

- When does steady-state error become zero? (i.e., **accurate tracking**!)
- Infinite error constant!
  - For step $r(t)$ $K_p = \lim_{s\to 0} L(s) = \infty$

➡ $L(s)$ must have at least 1-integrator. (Type 1 system)

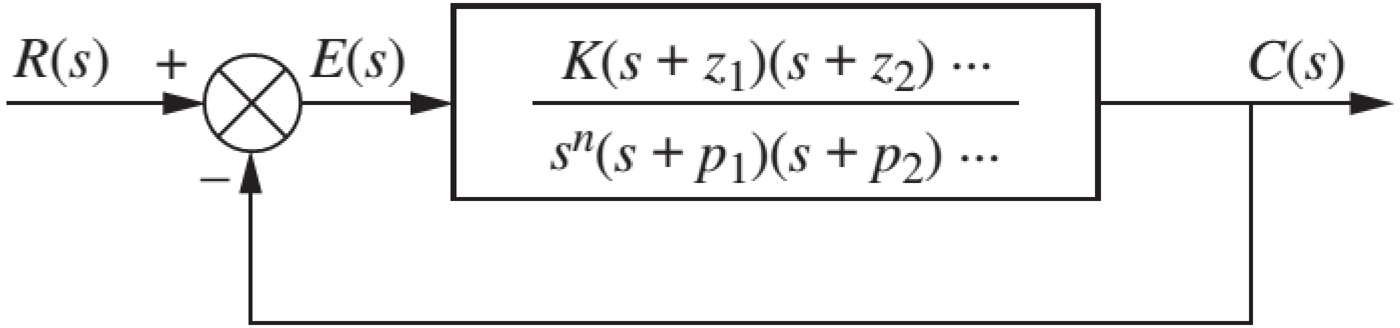
  - For ramp $r(t)$ $K_v = \lim_{s\to 0} sL(s) = \infty$

➡ $L(s)$ must have at least 2-integrators. (Type 2 system)

  - For parabolic $r(t)$ $K_a = \lim_{s\to 0} s^2L(s) = \infty$

➡ $L(s)$ must have at least 3-integrators. (Type 3 system)

System Type

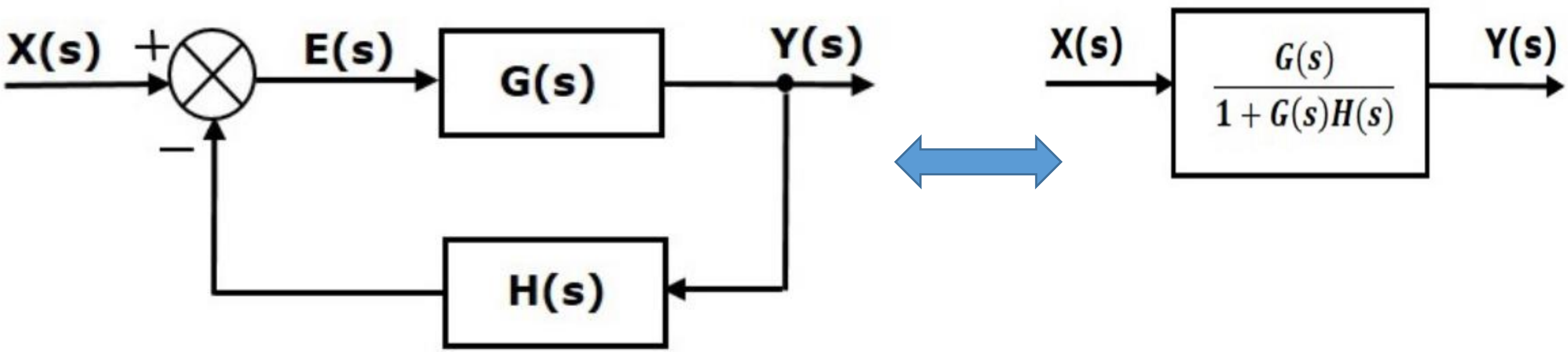


Feedback control system for defining system type

- We define system type to be the value of n in the denominator or, equivalently, the number of pure integrations in the forward path. Therefore, a system with ***n* = 0** is a **Type 0** system. If ***n* = 1** or ***n* = 2**, the corresponding system is a **Type 1** or **Type 2** system, respectively.

Characteristic Equation (review)

- The following figure shows a negative feedback control system. Here, two blocks having transfer functions $G(s)$ and $H(s)$ form a closed loop.



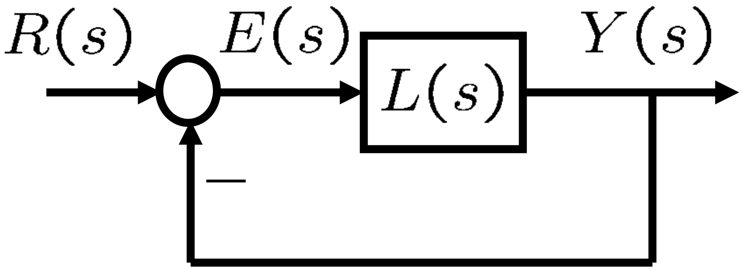
$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$1 + G(s)H(s) = 0$ is called the **Characteristic Equation**.

Example 1

- $L(s)$ has 2-integrators.

$$L(s) = \frac{K}{s^2(s+12)}$$



- Characteristic equation:

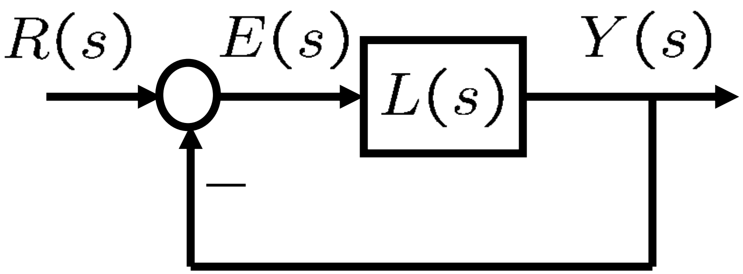
$$1+L(s)=0 \Leftrightarrow s^2(s+12)+K=0 \Leftrightarrow s^3+12s^2+K=0$$

- CL system is NOT stable for any *K*.
- *e*(*t*) will not converge. (Do not use today's results if CL system is not stable!)

Example 2

- $L(s)$ has 1-integrator.

$$L(s) = \frac{K(s+3.15)}{s(s+1.5)(s+0.5)}$$

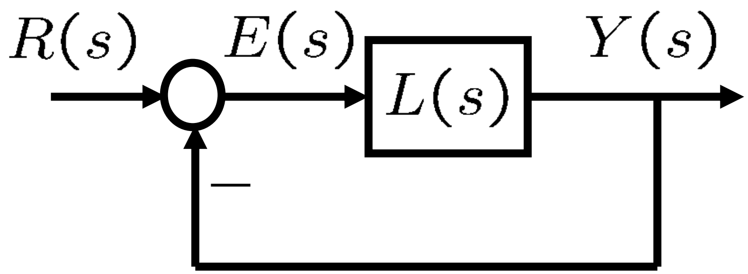


- By Routh-Hurwitz criterion, CL is stable if $0 < K < 1.304$
- Step $r(t)$ $\quad K_p = \lim_{s\to 0} L(s) = \infty \Rightarrow e_{ss} = \frac{R}{1+K_p} = 0$
- Ramp $r(t)$ $\quad K_v = \lim_{s\to 0} sL(s) = \frac{3.15K}{0.75} = 4.2K \Rightarrow e_{ss} = \frac{R}{K_v} = \frac{R}{4.2K}$
- Parabolic $r(t)$ $\quad K_a = \lim_{s\to 0} s^2 L(s) = 0 \Rightarrow e_{ss} = \frac{R}{K_a} = \infty$

Example 3

- $L(s)$ has 2-integrators.

$$L(s) = \frac{5(s+1)}{s^2(s+12)(s+5)}$$



- By Routh-Hurwitz criterion, we can show that CL system is stable.
- Step $r(t)$ $\quad K_p = \lim_{s\to 0} L(s) = \infty \quad \Rightarrow \quad e_{ss} = \frac{R}{1+K_p} = 0$
- Ramp $r(t)$ $\quad K_v = \lim_{s\to 0} sL(s) = \infty \quad \Rightarrow \quad e_{ss} = \frac{R}{K_v} = 0$
- Parabolic $r(t)$ $\quad K_a = \lim_{s\to 0} s^2 L(s) = \frac{1}{12} \quad \Rightarrow \quad e_{ss} = \frac{R}{K_a} = 12R$

Integrators in $L(s)$

- Integrators in $L(s)$ (i.e., plant and controller) are very powerful to eliminate the steady-state errors.
  - Examples 2 & 3
- However, integrators in $L(s)$ try to destabilize the feedback system.
  - Example 1
  - To be explained later in "Nyquist stability criterion".

("***No-free-lunch theorem***" again.)

Summary

- Time response and time domain specifications.
- Steady-state error
  - For stable unity feedback systems and for a specific type of input (step, ramp, parabolic, etc.), the number of integrators determines if the steady-state error is zero.
  - The key tool is the final value theorem.
- Next
  - Step responses of 1st and 2nd order systems.