a place of mind

ELEC 341: Systems and Control

Lecture 7

Routh-Hurwitz stability criterion: Examples

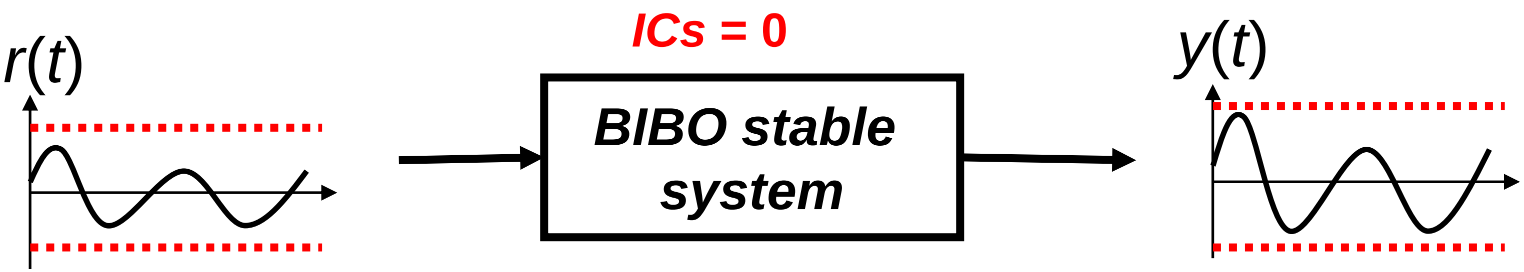
Course roadmap

Modeling	*Analysis*	*Design*
✓ Laplace transform	✓ Stability	Design specs
	• Routh-Hurwitz	
	• Nyquist	
✓ Transfer function		Root locus
Models for systems	Time response	Frequency domain
✓ • Electrical	• Transient	
✓ • Electromechanical	• Steady state	PID & Lead-lag
✓ • Mechanical		
✓ Linearization, delay	Frequency response	Design examples
	• Bode plot	

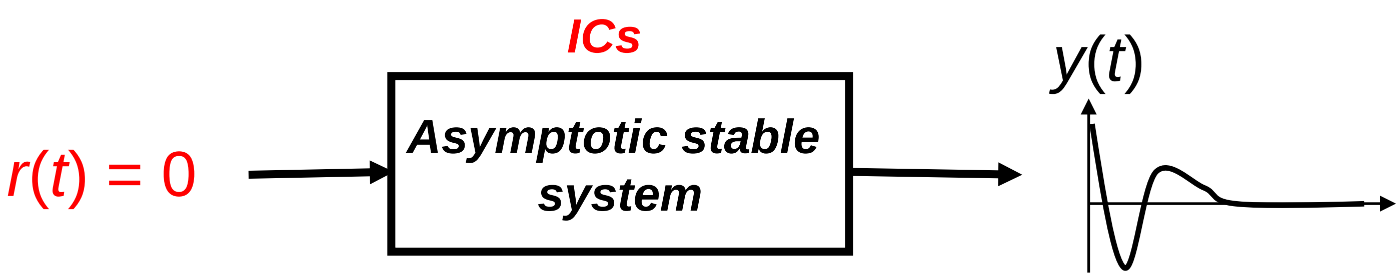
Matlab simulations

Definitions of stability (review)

- **BIBO (Bounded-Input-Bounded-Output) stability**
  Any bounded input generates a bounded output.



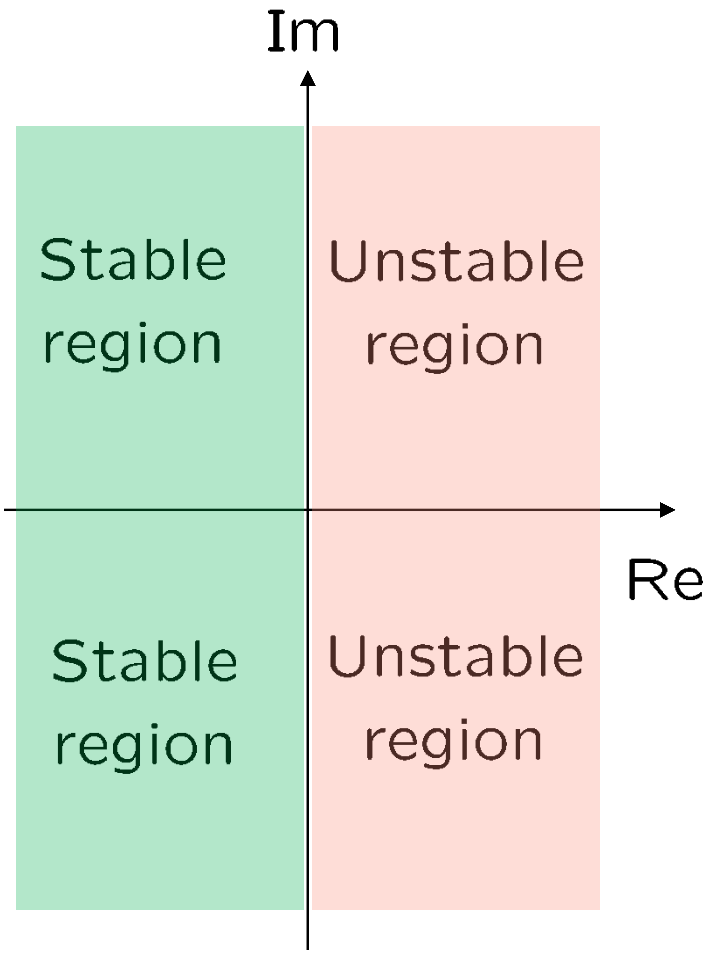
- **Asymptotic stability**
  Any ICs generates y(t) converging to zero.



Stability summary (review)

Let s_i be poles of $G(s)$. Then, $G(s)$ is …

- (BIBO, asymptotically) stable if $Re(s_i) < 0$ for all i.
- marginally stable if
  - $Re(s_i) \leq 0$ for all i, and
  - simple pole for $Re(s_i) = 0$
- unstable if it is neither stable nor marginally stable.



Routh-Hurwitz criterion (review)

- This is for LTI systems with a *polynomial* denominator (without sine, cosine, exponential, etc.)
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does NOT explicitly compute the roots numerical values.

Routh array (review)

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

From the given polynomial

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Routh array
(How to compute the third row)

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

$$
\begin{aligned}
b_1 &= \frac{a_{n-2}a_{n-1}-a_n a_{n-3}}{a_{n-1}} \\
b_2 &= \frac{a_{n-4}a_{n-1}-a_n a_{n-5}}{a_{n-1}} \\
&\vdots
\end{aligned}
$$

Routh array
(How to compute the fourth row)

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

$$
\begin{aligned}
c_1 &= \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1} \\
c_2 &= \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1} \\
&\vdots
\end{aligned}
$$

Routh-Hurwitz criterion

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

The number of roots in the open right half-plane is equal to the number of sign changes in the first column of Routh array.

Example 1

$$Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Routh array

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & \cancel{0}\ \varepsilon & 6 & \\ s^2 & \frac{4\varepsilon - 12}{\varepsilon} & 10 & \\ s^1 & \approx 6 & & \\ s^0 & 10 & & \end{array}$$

If 0 appears in the first column of a nonzero row in Routh array, replace it with a small positive number (ε).

Two sign changes in the first column ➡ Two roots in RHP

$$\varepsilon \to \underbrace{\frac{4\varepsilon - 12}{\varepsilon}}_{<0} \to 6$$

In this case, Q has some roots in RHP.

We can replace ε with 0^+ and $(4\varepsilon - 12)/\varepsilon$ with $-\infty$.

Example 2

$$Q(s) = s^4 + s^3 + 3s^2 + 2s + 2$$

Routh array

$$\begin{array}{c|ccc} s^4 & 1 & 3 & 2 \\ s^3 & 1 & 2 & \\ s^2 & 1 & 2 & \\ s^1 & \cancel{0}\ \mathbf{2} & & \\ s^0 & 2 & & \end{array}$$

If a zero row appears in Routh array (here, s^1 row), Q has roots either on the imaginary axis or in RHP.

No sign changes in the first column. No roots in RHP.

But some roots are on imag. axis.

S^1 is a ROZ = row of zero

How to write $P(s)$: Go one row above the zero row and begin with the power of s that starts the row and then skip every other power until you reach the end of the row.

$P(s) = (1)s^2 + (2)s^0 \longrightarrow P(s) = s^2 + 2$ **(auxiliary polynomial)**

Take the derivative of the **auxiliary polynomial** (which is a factor of $Q(s)$):

$\longrightarrow P' = \frac{dP}{ds} = \mathbf{2}s \longrightarrow$ Replace the zero row with the coefficients of the polynomial in the dP/ds equation.

Example 3

$$Q(s) = s^3 + s^2 + s + 1 \quad = (s+1)(s^2+1)$$

Routh array

$P(s) = s^2 + 1$ (auxiliary polynomial)

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 & 1 \\ s^1 & \cancel{0}\ 2 & \\ s^0 & 1 & \end{array}$$

ROZ → s^1

Derivative of auxiliary poly.

$$(s^2+1)' = 2s$$

(Auxiliary poly. is a factor of $Q(s)$.)

No sign changes
in the first column

No roots in OPEN RHP.

Example 4

$$Q(s) = s^5+s^4+2s^3+2s^2+s+1 \;= (s+1)(s^2+1)^2$$

Routh array

$$\begin{array}{r|lll} s^5 & 1 & 2 & 1 \\ s^4 & 1 & 2 & 1 \\ \text{ROZ} \rightarrow s^3 & \cancel{0}\ 4 & \cancel{0}\ 4 & \\ s^2 & 1 & 1 & \\ \text{ROZ} \rightarrow s^1 & \cancel{0}\ 2 & & \\ s^0 & 1 & & \end{array}$$

Derivative of auxiliary poly.

$$(s^4+2s^2+1)' = 4s^3+4s$$

$$(s^2+1)' = 2s$$

No sign changes
in the first column

$\Rightarrow$ No roots in OPEN RHP

Example 5

$$Q(s) = s^4 - 1 \quad = (s+1)(s-1)(s^2+1)$$

Routh array

Derivative of auxiliary poly.

$$(s^4 - 1)' = 4s^3$$

ROZ $\longrightarrow$

$$\begin{array}{c|ccc} s^4 & 1 & 0 & -1 \\ s^3 & \cancel{0}\ 4 & \cancel{0}\ 0 & \\ s^2 & \cancel{0}\ \varepsilon & -1 & \\ s^1 & 4/\varepsilon & & \\ s^0 & -1 & & \end{array}$$

One sign changes
in the first column $\Longrightarrow$ One root in OPEN RHP.

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Summary for stability when ROZ appears

Summary of procedure for determining if ROZ will lead to unstable or marginally stable systems:

- If there is a sign change in the first column, then we have poles in the RHP and the system is **unstable**.
- If there is no sign change in the first column, there is no poles in the RHP, but we will have poles on Im-axis and we should see whether the system is marginally stable or unstable.
  - ❑ If all the poles on the Im-axis are simple poles (i.e., with the multiplicity of 1), then the system is **marginally stable**.
  - ❑ If the poles on the Im-axis have multiplicity of more than 1, then the system is **unstable**.
    - ❖ Remember that sometimes we may have more than one ROZ, which means we need to look at the total multiplicity of the roots in the various obtained auxiliary polynomials.
      - ➢ For instance, we may have two similar roots in the two separately obtained auxiliary polynomials. In this case, the multiplicity is 2 and the system is **unstable**.

Notes on Routh-Hurwitz criterion

- Advantages
  - No need to explicitly compute roots of the polynomial.
    - High order $Q(s)$ can be handled by hand calculations.
  - Polynomials including undetermined parameters (plant and/or controller parameters in feedback systems) can be dealt with.
    - Root computation does not work in such cases!
- Disadvantage
  - Exponential functions (delay) cannot be dealt with.
    - Example:

$$Q(s) = e^{-s} + s^2 + s + 1$$

We will study Nyquist stability criterion later to deal with these cases!

Example 6

$$Q(s) = s^3 + 3Ks^2 + (K+2)s + 4$$

Find the range of K so that $Q(s)$ has all roots in the left half plane. Here, K is a design parameter.

Routh array

$$\begin{array}{c|ll} s^3 & 1 & K+2 \\ s^2 & 3K & 4 \\ s^1 & \frac{3K(K+2)-4}{3K} & \\ s^0 & 4 & \end{array}$$

In order to have no sign changes in the first column:

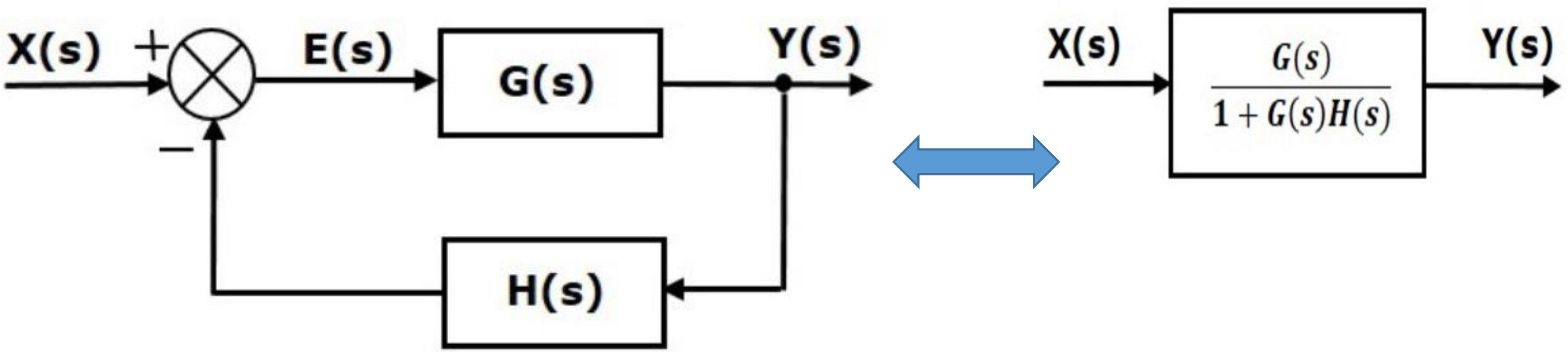
$$\Rightarrow \begin{cases} 3K > 0 \\ 3K(K+2) - 4 > 0 \end{cases}$$

$$\Rightarrow K > -1 + \frac{\sqrt{21}}{3}$$

$$\Rightarrow \boxed{K > 0.527}$$

Characteristic Equation

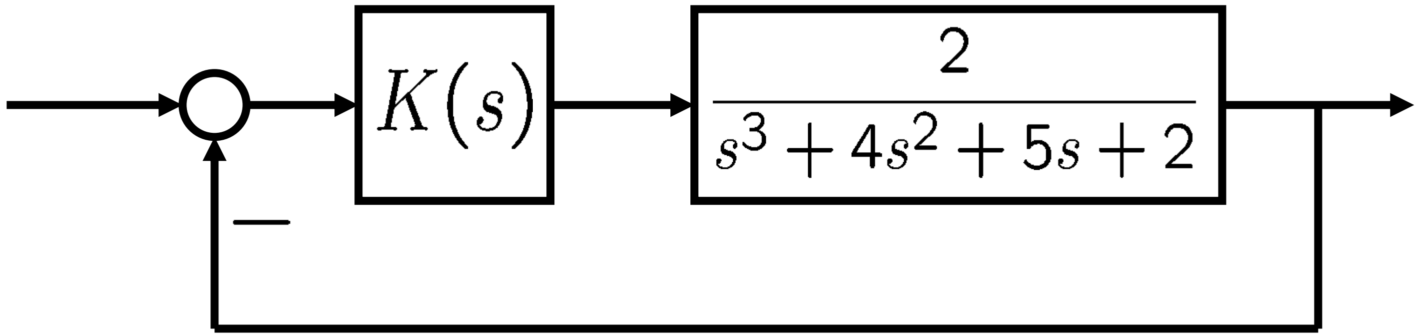
- The following figure shows a negative feedback control system. Here, two blocks having transfer functions $G(s)$ and $H(s)$ form a closed loop.



$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$1 + G(s)H(s) = 0$ is called the **Characteristic Equation**.

Example 7



- Design $K(s)$ that stabilizes the closed-loop system for the following cases.
  - $K(s) = K$ (constant, P or Proportional controller)
  - $K(s) = K_P + K_I/s$ (PI or Proportional-Integral controller)

Example 7 (cont'd): $K(s) = K$

- Characteristic equation

$$1 + K\frac{2}{s^3 + 4s^2 + 5s + 2} = 0$$

$$\Rightarrow \quad s^3 + 4s^2 + 5s + 2 + 2K = 0$$

- Routh array

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 4 & 2+2K \\ s^1 & \frac{18-2K}{4} & \\ s^0 & 2+2K & \end{array} \quad \Rightarrow \quad \boxed{-1 < K < 9}$$

Example 7 (cont'd): $K(s) = K_P + K_I/s$

- Characteristic equation

$$1 + \left(K_P + \frac{K_I}{s}\right)\frac{2}{s^3 + 4s^2 + 5s + 2} = 0$$

$$\Rightarrow \quad s^4 + 4s^3 + 5s^2 + (2 + 2K_P)s + 2K_I = 0$$

- Routh array

$$\begin{array}{c|ccc}
s^4 & 1 & 5 & 2K_I \\
s^3 & 4 & 2 + 2K_P & \\
s^2 & \frac{18-2K_P}{4} & 2K_I & \\
s^1 & (*) & & \\
s^0 & 2K_I & &
\end{array}$$

$$(*) = \frac{(2 + 2K_P)\left(\frac{18 - 2K_P}{4}\right) - 8K_I}{\left(\frac{18 - 2K_P}{4}\right)}$$

$\rightarrow K_P < 9$

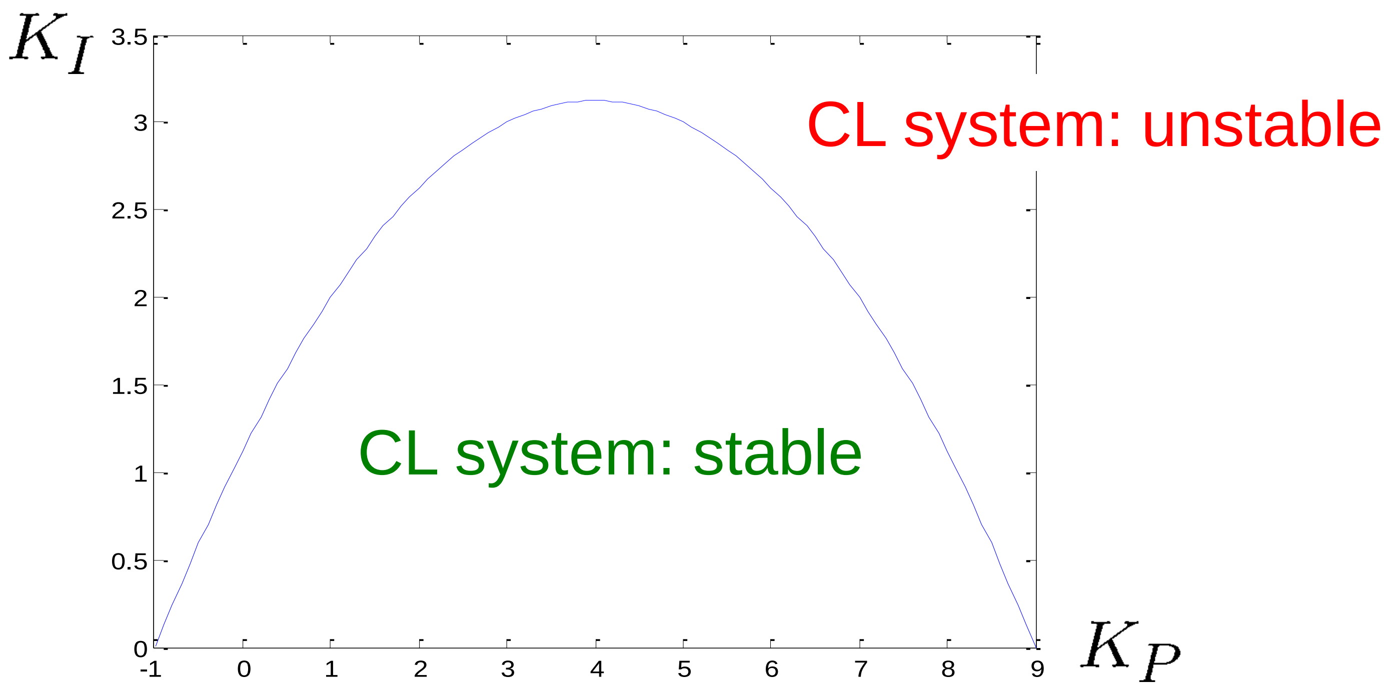
$\rightarrow K_I > 0$

Example 7 (cont'd): Range of (K_P, K_I)

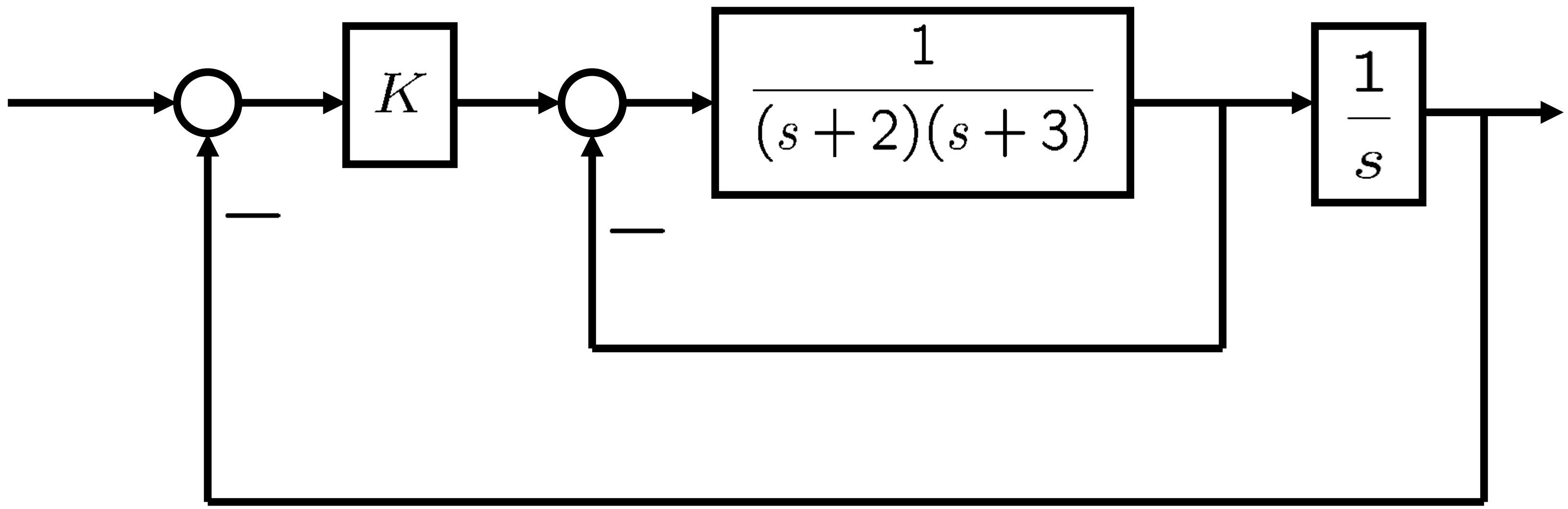
- From Routh array, $K_P < 9$

$$K_I > 0$$

$$(*) \Leftrightarrow (1 + K_P)(9 - K_P) - 8K_I > 0$$

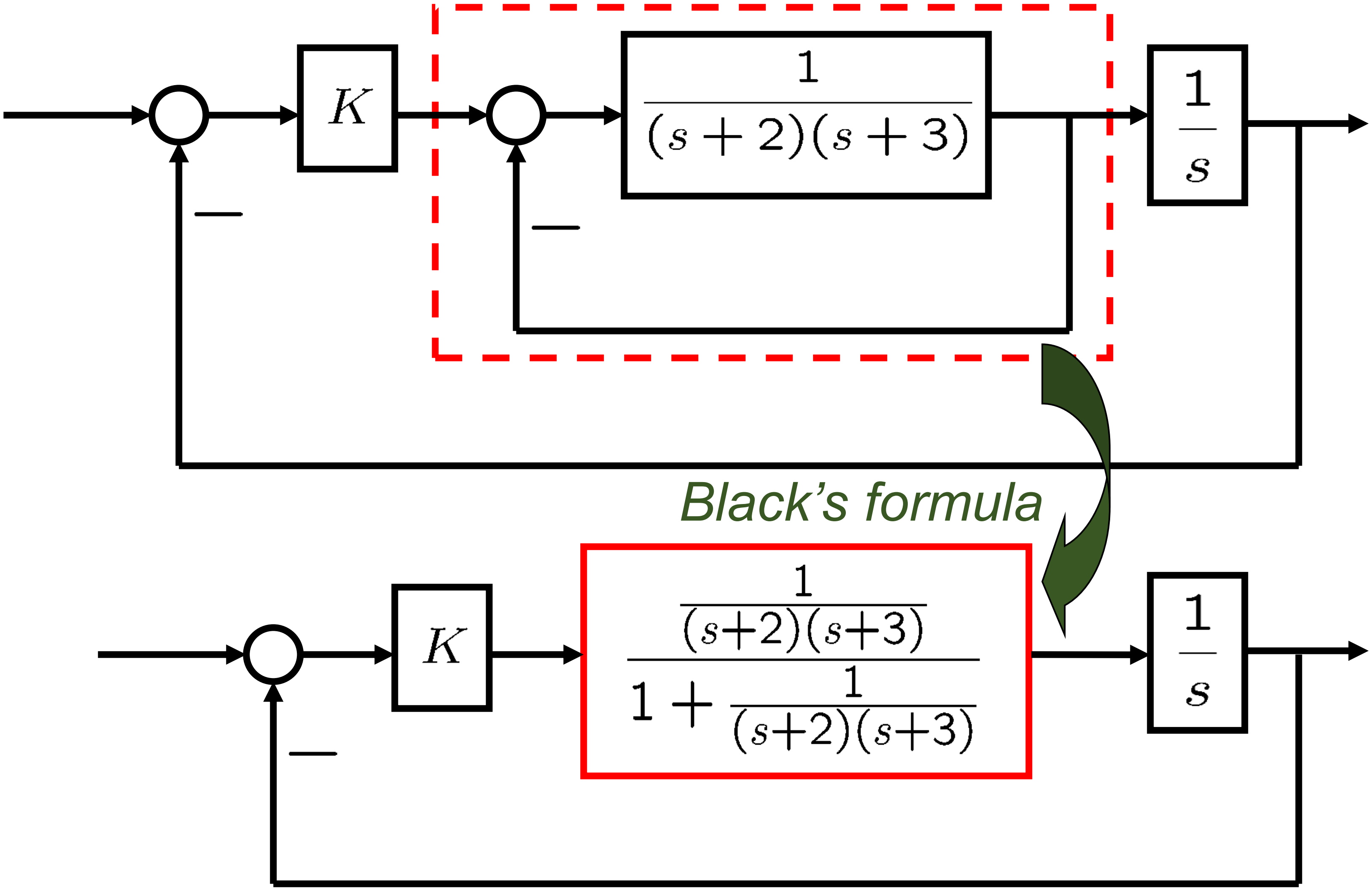


Example 8



- Determine the range of K that stabilizes the closed-loop system.

Example 8 (cont'd)



Example 8 (cont'd)

- Characteristic equation

$$1+K\frac{\frac{1}{(s+2)(s+3)}}{1+\frac{1}{(s+2)(s+3)}}\cdot\frac{1}{s}=0$$

$$\Rightarrow \quad 1+K\cdot\frac{1}{s(s+2)(s+3)+s}=0$$

$$\Rightarrow \quad s(s+2)(s+3)+s+K=0$$

$$\Rightarrow \quad s^3+5s^2+7s+K=0$$

Example 8 (cont'd)

- Routh array for $s^3 + 5s^2 + 7s + K = 0$:

$$
\begin{array}{c|cc}
s^3 & 1 & 7 \\
s^2 & 5 & K \\
s^1 & \frac{35-K}{5} & \\
s^0 & K &
\end{array}
\longrightarrow \boxed{0 < K < 35}
$$

- If K = 35, the closed-loop system is marginally stable.

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Summary

- Examples for Routh-Hurwitz criterion
  - Cases when zeros appear in Routh array
  - P controller gain range for stability
  - PI controller gain range for stability
- Next
  - Time responses and steady-state errors