$\boxed{1}$ Mass on a cart

1. $$ms^2Y(s) = -kY(s) + kU(s) - bsY(s) + bsU(s)$$

$$Y(s)(ms^2+bs+k) = U(s)(bs+k)$$

$$\frac{Y(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k} = \frac{s\cdot b/m + k/m}{s^2 + b/m\,s + k/m}$$

2. $$A = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} k/m & b/m \end{bmatrix} \qquad D = 0$$

3. $$\Phi(s) = (sI-A)^{-1} = \begin{bmatrix} s & -1 \\ k/m & s+b/m \end{bmatrix}^{-1}$$

$$= \frac{1}{s^2 + b/m\,s + k/m}\begin{bmatrix} s+b/m & +1 \\ -k/m & s \end{bmatrix}$$

$$= \frac{1}{s^2+4}\begin{bmatrix} s & 1 \\ -4 & s \end{bmatrix}, \qquad b=0, \quad k=\cancel{3}\,4, \quad M=1$$

4. Forced response

$$X(s) = \Phi(s) B U(s)$$

$$= \frac{1}{s^2+4}\begin{bmatrix} s & 1 \\ -4 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} U(s) \quad = \frac{1}{s^2+4}\begin{bmatrix} 1 \\ s \end{bmatrix} \cdot U(s)$$

For $U(s) = \frac{1}{s}$ $\quad (u(t) = \underline{1}(t))$,

$$X(s) = \frac{1}{s^2+4} \cdot \begin{bmatrix} 1 \\ s \end{bmatrix} \cdot \frac{1}{s} = \begin{bmatrix} \frac{1}{s(s^2+4)} \\ \frac{1}{s^2+4} \end{bmatrix}$$

For $U(s) = \frac{1}{s+3}$ $\quad (u(t) = e^{-3t})$,

$$X(s) = \frac{1}{s^2+4}\begin{bmatrix} 1 \\ s \end{bmatrix} \cdot \frac{1}{s+3} = \begin{bmatrix} \frac{1}{(s^2+4)(s+3)} \\ \frac{s}{(s^2+4)(s+3)} \end{bmatrix}$$

5. Natural response

$$X(s) = \Phi(s) X_0$$

$$= \frac{1}{s^2+4}\begin{bmatrix} s & 1 \\ -4 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X(s) = \begin{bmatrix} \frac{s+1}{s^2+4} \\ \frac{s-4}{s^2+4} \end{bmatrix} \qquad \left(= \begin{bmatrix} \frac{s}{s^2+2^2} + \frac{1}{2}\cdot\frac{2}{s^2+2^2} \\ \frac{s}{s^2+2^2} - 2\,\frac{2}{s^2+2^2} \end{bmatrix} \right.$$

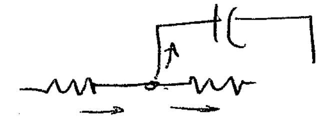
$$\therefore \quad x(t) = \begin{bmatrix} \cos(2t) + \frac{1}{2}\sin(2t) \\ \cos(2t) - 2\sin(2t) \end{bmatrix}$$

using $\begin{cases} \mathcal{L}(\sin wt) = \frac{w}{s^2+w^2} \\ \mathcal{L}(\cos wt) = \frac{s}{s^2+w^2} \end{cases}$

[2]. RLC Op-Amp Circuit

1. Integro-diff. equations.

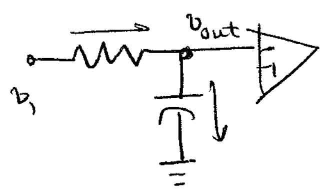
Use KCL at node w/ voltage v_1:



$$\frac{v_{in} - v_1}{R} = \frac{d(v_1 - v_{out})}{dt} \cdot C + \frac{v_1 - v_{out}}{R}$$

$$(1) \qquad \frac{v_{in}}{R} = \dot{v}_c \cdot C + \frac{v_1}{R} + \frac{v_c}{R}$$

and at node near "−" ~~...~~ terminal of op-amp



$$\frac{v_1 - v_{out}}{R} = \frac{dv_{out}}{dt} \cdot C$$

$$(2) \qquad \frac{v_1}{R} - \frac{v_{out}}{R} = \dot{v}_{out} \cdot C$$

2. state-space: $x = \begin{bmatrix} v_{out} \\ v_c \end{bmatrix}$

$$v_1 = v_c + v_{out}$$

$$\dot{x} = \begin{bmatrix} \dot{v}_{out} \\ \dot{v}_c \end{bmatrix} \begin{matrix} \leftarrow \text{from (2)} \rightarrow \\ \leftarrow \text{from (1)} \end{matrix} \qquad \dot{v}_{out} = \frac{1}{RC} \cdot v_1 - \frac{1}{RC} v_{out}$$

$$= \frac{1}{RC} v_c + \frac{1}{RC} v_{out} - \frac{1}{RC} v_{out}$$

$$\dot{v}_c = \frac{1}{RC} v_{in} - \frac{1}{RC} v_1 - \frac{1}{RC} v_c \qquad \boxed{\dot{v}_{out} = \frac{1}{RC} \cdot v_c}$$

$$= \frac{1}{RC} v_{in} - \frac{1}{RC}(v_c + v_{out}) - \frac{1}{RC} v_c$$

$$\boxed{\dot{v}_c = -\frac{1}{RC} v_{out} - \frac{2}{RC} v_c + \frac{1}{RC} v_{in}}$$

$$\therefore \quad \dot{x} = \begin{bmatrix} 0 & 1/RC \\ -1/RC & -2/RC \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/RC \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

3) $G(s) = C(sI - A)^{-1} B$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} (sI - A)^{-1} \cancel{B} \begin{bmatrix} 0 \\ 1/RC \end{bmatrix}$$

where $$(sI - A)^{-1} = \begin{bmatrix} s & -1/RC \\ +1/RC & s + \frac{2}{RC} \end{bmatrix}^{-1}$$

$$= \frac{1}{s(s+\frac{2}{RC}) + (\frac{1}{RC})^2} \begin{bmatrix} s + 2/RC & +1/RC \\ -1/RC & s \end{bmatrix}$$

$$= \frac{1}{s^2 + s\frac{2}{RC} + (\frac{1}{RC})^2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + 2/RC & 1/RC \\ -1/RC & s \end{bmatrix} \begin{bmatrix} 0 \\ 1/RC \end{bmatrix}$$

$$= \frac{1}{s^2 + s\frac{2}{RC} + (\frac{1}{RC})^2} \begin{bmatrix} s + \frac{2}{RC} & 1/RC \end{bmatrix} \begin{bmatrix} 0 \\ 1/RC \end{bmatrix}$$

$$= \frac{(1/RC)^2}{s^2 + \frac{2s}{RC} + (\frac{1}{RC})^2}$$

$$= \frac{(1/RC)^2}{(s + 1/RC)^2}$$

4) ~~A~~ eig(A): $|sI - A| = 0 = s^2 + \frac{2}{RC}s + \left(\frac{1}{RC}\right)^2$

$$= \left(s + \frac{1}{RC}\right)^2$$

$$s = -1/RC, -1/RC.$$

~~A~~ $D(s) = 0$: $(s + 1/RC)^2 = 0 \Rightarrow s = -1/RC, -1/RC$

5) Only 1 characteristic ~~eq~~ eqn exists.
All possible state-space reali~~zations~~ will have the same eigenvalues.