THE UNIVERSITY OF BRITISH COLUMBIA

Department of Electrical and Computer Engineering

ELEC341 – Systems and Control, Midterm 1 – Feb 11, 2019

Answer all problems.
Time: 100min.

This examination consists of 12 pages. Please check that you have a complete copy. You may use both sides of each sheet if needed.

Solutions

Surname First

Student Number

#	MAX	GRADE
1	20	
2	15	
3	20	
4	15	
5	15	
6	15	
TOTAL	100	

READ THIS

IMPORTANT NOTE: The announcement "stop writing" will be made at the end of the examination. Anyone writing after this announcement will receive a score of 0. No exceptions, no excuses.

All writings must be on this booklet. The blank sides on the reverse of each page may also be used.

Each candidate should be prepared to produce, upon request, his/her student ID card.

Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of 10min, or to leave during the first 10min of the examination.

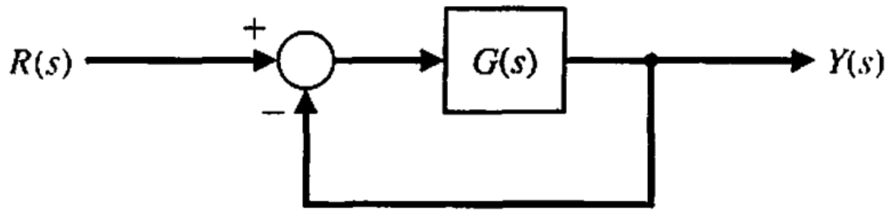
Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination-questions.

Caution *- Candidates guilty of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:*

- *Making use of any books, papers or memoranda, calculators, audio or visual cassette players or other memory aid devices, other than as authorized by the examiners.*
- *Speaking or communicating with other candidates.*
- *Purposely exposing written papers to the view of other candidates.*

The plea of accident or forgetfulness shall not be received.

1) [20 marks] A system is shown in the following figure



(a) Find the closed-loop transfer function Y(s)/R(s) when $G(s)=\dfrac{7}{s^2+2s+10}$

(b) Determine Y(s) when the input r(t) is a unit step, $r(t)=u(t)$, and when the input is a ramp, $r(t)=t\cdot u(t)$

(c) Compute the time response y(t) for a unit step input

(d) What is the value of the steady-state error signal, $\varepsilon_{SS}=\lim\limits_{t\to\infty}\varepsilon(t)=\lim\limits_{t\to\infty}\left(r(t)-y(t)\right)$, for an applied ramp input $r(t)=t\cdot u(t)$?

Solution:

(a) The closed-loop TF:

$$H(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\dfrac{7}{s^2+2s+10}}{1+\dfrac{7}{s^2+2s+10}}=\frac{7}{s^2+2s+17}$$

(b) The unit step response:

$$r(t)=u(t)\xrightarrow{L}R(s)=\frac{1}{s}\Rightarrow Y(s)=R(s)H(s)=\frac{7}{s\left(s^2+2s+17\right)}$$

The ramp response:

$$r(t)=tu(t)\xrightarrow{L}R(s)=\frac{1}{s^2}\Rightarrow Y(s)=R(s)H(s)=\frac{7}{s^2\left(s^2+2s+17\right)}$$

(c) The unit step response in time domain - decompose the response in simple fractions:

$$Y(s)\Big|_{R(s)=\frac{1}{s}}=\frac{7}{s\left(s^2+2s+17\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+2s+17}$$

$$s^2+2s+17=0\Leftrightarrow s_{1,2}=-1\pm\sqrt{1-17}=-1\pm j4$$

Determine A: $\dfrac{7}{s^2+2s+17}=A+s\dfrac{Bs+C}{s^2+2s+17}\xrightarrow{s\to 0}\dfrac{7}{17}=A$

Determine B and C by giving two extra values to s:

$$\frac{7}{s\left(s^2+2s+17\right)}=\frac{7}{17s}+\frac{Bs+C}{s^2+2s+17}$$

$$\begin{cases} s=1 \Rightarrow \dfrac{7}{20}=\dfrac{7}{17}+\dfrac{B+C}{20} \\ s=-1 \Rightarrow -\dfrac{7}{16}=-\dfrac{7}{17}+\dfrac{C-B}{16} \end{cases} \Rightarrow B=-\frac{7}{17}, C=-\frac{14}{17}$$

So the decomposition becomes: $\dfrac{7}{s\left(s^2+2s+17\right)}=\dfrac{7}{17s}-\dfrac{7s+14}{17\left(s^2+2s+17\right)}$

$$e^{-at}\sin(\omega t)u(t) \xleftrightarrow{L} \frac{\omega}{(s+\alpha)^2+\omega^2}$$

$$e^{-at}\cos(\omega t)u(t) \xleftrightarrow{L} \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$$

$$\frac{7s+14}{s^2+2s+17}=\frac{7s+14}{(s+1)^2+16}=\frac{7(s+1)}{(s+1)^2+16}+\frac{\frac{7}{4}4}{(s+1)^2+16}$$

$$Y(s)\Big|_{R(s)=\frac{1}{s}} \frac{7}{s\left(s^2+2s+17\right)}=\frac{7}{17s}-\frac{7}{17}\frac{s+1}{(s+1)^2+4^2}-\frac{7}{68}\frac{4}{(s+1)^2+4^2}$$

$$y(t)=\frac{7}{17}u(t)-\frac{7}{17}e^{-t}\cos(4t)u(t)-\frac{7}{68}e^{-t}\sin(4t)u(t)$$

(d) The steady state error can be found by applying the limit value theorem:

$$\varepsilon(s)=R(s)-Y(s)=R(s)\big(1-H(s)\big)=\frac{1}{s^2}\left(1-\frac{7}{s^2+2s+17}\right)=\frac{s^2+2s+10}{s^2\left(s^2+2s+17\right)}$$

$$\varepsilon_{SS}=\lim_{t\to\infty}\varepsilon(t)=\lim_{s\to 0}s\varepsilon(s)=\lim_{s\to 0}\frac{s^2+2s+10}{s\left(s^2+2s+17\right)}=\infty$$

The output cannot track a ramp input (but it can track, with a steady error, an input step)

2) [15 marks] A system has the following transfer function:

$$\frac{X(s)}{F(s)} = \frac{15}{(s+10)(s+11)}$$

a) Write the corresponding differential equation characterizing the system
b) Write an equivalent state-space model for this system
c) What initial conditions are being used in defining the transfer function?

Solution:

(a) Simply go from Laplace domain back into the time domain:

$$X(s) = \frac{15}{(s+10)(s+11)} F(s) \Leftrightarrow \left(s^2 + 10s + 11s + 110\right) X(s) = 15F(s)$$

$$s^2 X(s) + 21sX(s) + 110X(s) = 15F(s)$$

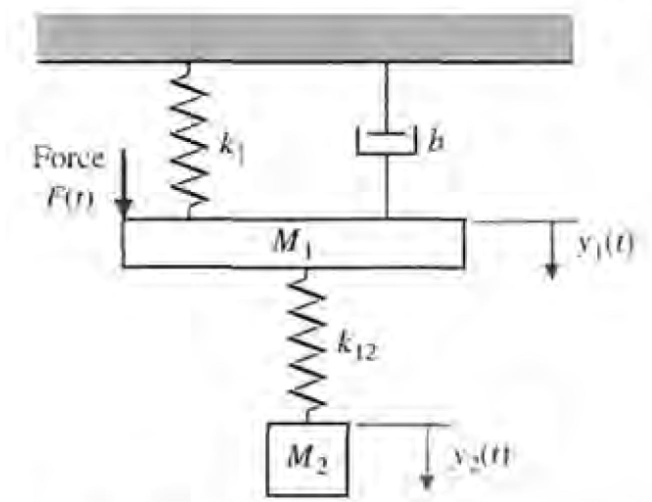
$$\frac{d^2x}{dt^2}(t) + 21\frac{dx}{dt}(t) + 110x(t) = 15f(t)$$

(b) We can build a simple state-space model by using as state variables x(t) and dx/dt(t):

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -110x_1 - 21x_2 + 15f \\ y = x_1 \end{cases} \Leftrightarrow \begin{cases} \dfrac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -110 & -21 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{pmatrix} 0 \\ 15 \end{pmatrix} f \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

(c) The transfer function of a system is defined for zero initial conditions

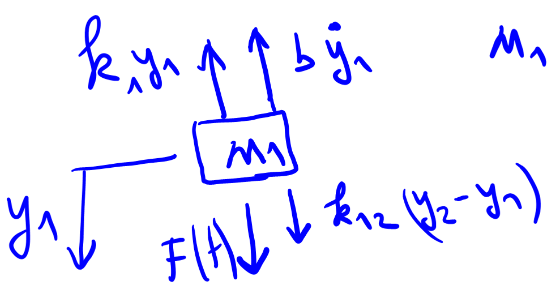
3) [20 marks]A dynamic vibration absorber is shown below. The parameters M2 and k12 may be chosen so that the mass M1 does not vibrate in the steady state when $F(t) = a\sin(\omega_0 t)$.



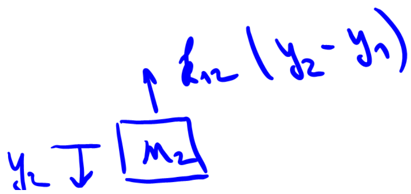
(a) Obtain the differential equations describing the system
(b) Draw the equivalent circuit diagram
(c) What is the order of a state-space model?
(d) Identify the A,B,C,D matrices in an equivalent state-space model, with F(t) as input and y2(t) as output.

Solution:

a. We can write the equations of motions for the two masses in the system:



$$M_1 \ddot{y}_1 = F(t) + k_{12}(y_2 - y_1) - k_1 y_1 - b\dot{y}_1$$

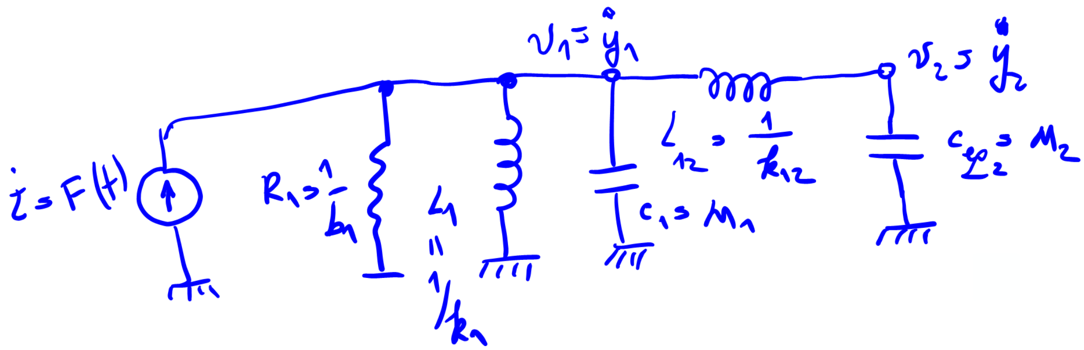


$$M_2 \ddot{y}_2 = -k_{12}(y_2 - y_1)$$

So the system of differential equations is:

$$\begin{cases} \dot{v}_1 = \dfrac{1}{M_1}\left(F(t) - k_1 y_1 - b v_1 - k_{12}(y_1 - y_2)\right) \\ \dot{v}_2 = -\dfrac{k_{12}}{M_2}(y_2 - y_1) \end{cases}$$

b. The equivalent generalized circuit for this system is given below:



(c) The order of the state-space model: we have two 2nd order coupled ordinary differential equations, which can be reduced to 4 1st order differential equations. We need 4 state variables in a variable state model (state space model of order 4). This can be also seen by checking the number of lumped components storing energy in the generalized circuit representation (4 storing elements, 2 inductors and 2 capacitors)

(d) A state-space model with F(t) as input and y2(t) as output can be obtained from the equations of motion:

$$\begin{cases} \dot{v}_1 = \dfrac{1}{M_1}\left(F(t) - k_1 y_1 - b v_1 - k_{12}(y_1 - y_2)\right) \\ \dot{v}_2 = -\dfrac{k_{12}}{M_2}(y_2 - y_1) \end{cases}$$

We take as state variables: $x_1 = y_1, x_2 = \dfrac{dy_1}{dt}, x_3 = y_2, x_4 = \dfrac{dy_2}{dt}$

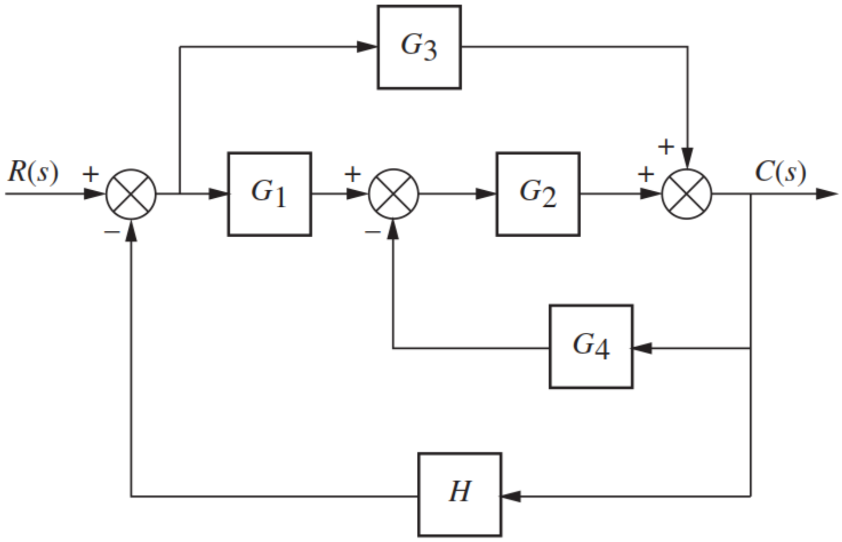
The state space model can be therefore written as:

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = \dfrac{1}{M_1}\left(-k_1 x_1 - b x_2 - k_{12} x_1 + k_{12} x_3\right) + \dfrac{1}{M_1} F \\ \dfrac{dx_3}{dt} = x_4 \\ \dfrac{dx_4}{dt} = -\dfrac{k_{12}}{M_2}(x_3 - x_1) \\ y = x_3 \end{cases}$$

The matrices A, B, C, D are as follows:

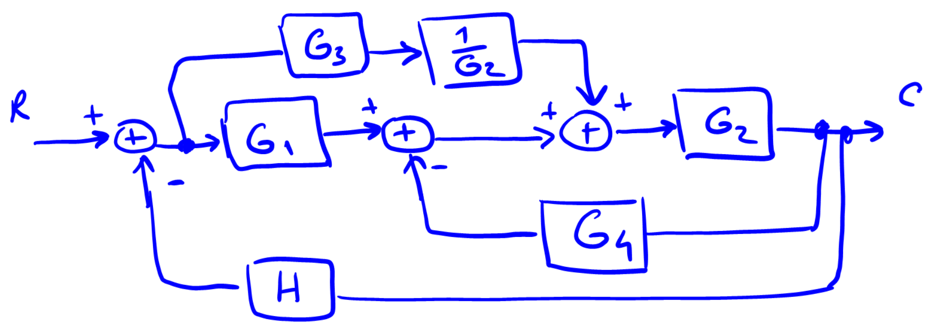
$$
\begin{cases}
\dfrac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \overbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_1+k_{12}}{M_1} & -\dfrac{b}{M_1} & \dfrac{k_{12}}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_{12}}{M_2} & 0 & -\dfrac{k_{12}}{M_2} & 0 \end{pmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \overbrace{\begin{pmatrix} 0 \\ \dfrac{1}{M_1} \\ 0 \\ 0 \end{pmatrix}}^{B} F \\
y = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{pmatrix} 0 \end{pmatrix}}_{D} F
\end{cases}
$$

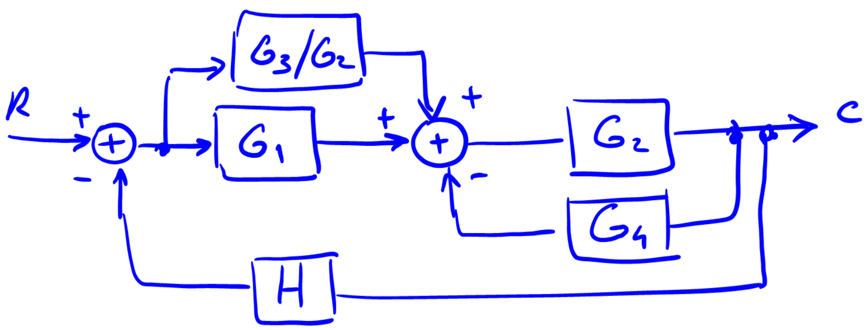
4) [15 marks]Find the equivalent transfer function, T(s)=C(s)/R(s), for the system shown below

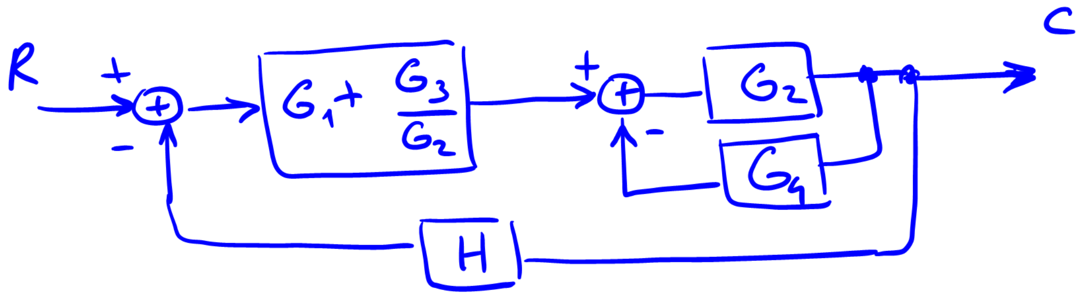


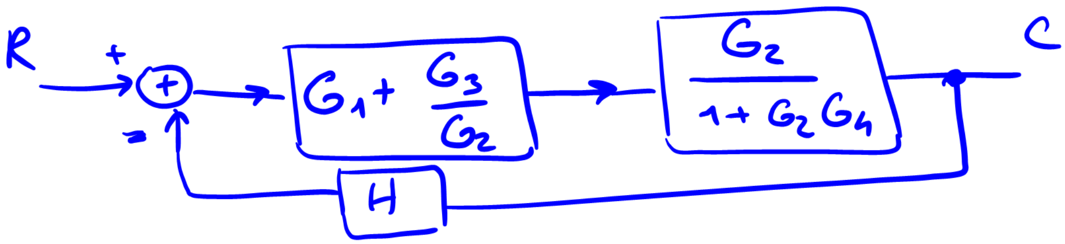
Solution:

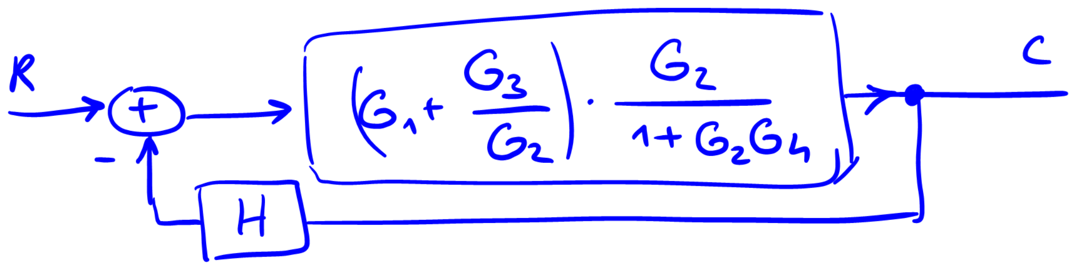
Sol.:

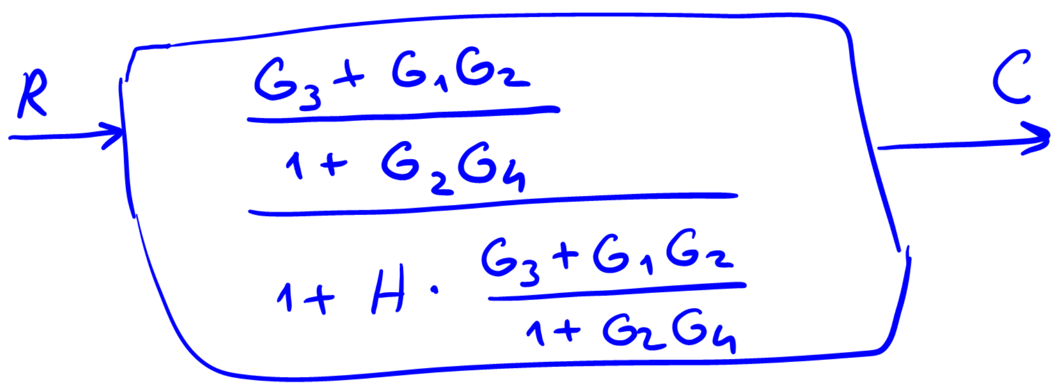










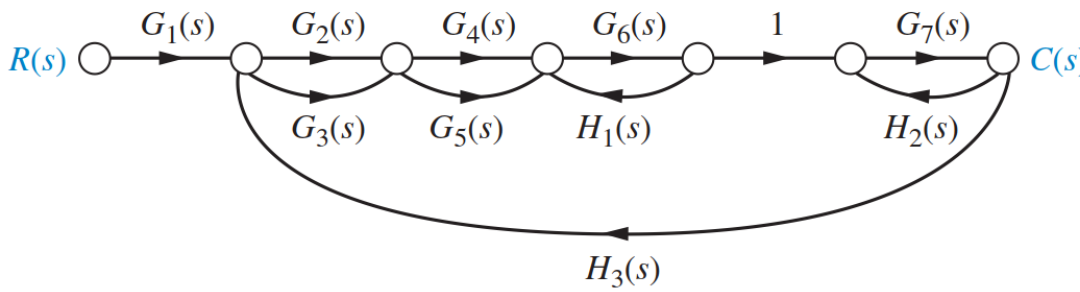


$$\frac{\dfrac{G_3 + G_1G_2}{1 + G_2G_4}}{1 + H \cdot \dfrac{G_3 + G_1G_2}{1 + G_2G_4}}$$

$$=$$

$$\frac{G_3 + G_1G_2}{1 + G_2G_4 + H(G_3 + G_1G_2)}$$

5) [15 marks] A system is represented by the following signal flow graph:



a) Find the transfer function, T(s)=C(s)/R(s), using Mason's rule
b) Map the SFG to an equivalent block diagram representation, and show that you can obtain the same transfer function by successive simplifications of the block diagram.

Solution:
a) Apply Mason's gain formula to the given SFG:
1. list of all forward paths and their gains: 4 possible forward paths - they can be reduced to a single one, if one considers G2+G3 and G4+G5 (you may separately consider all 4 distinct forward paths - it is correct)

$$G = G_1\left(G_2 + G_3\right)\left(G_4 + G_5\right)G_6 G_7$$

2. Make a list of all loops and their loop gains: only 3 loops (if we group together G2+G3 and G4+G5): $L_1 = G_6 H_1, L_2 = G_7 H_2, L_3 = \left(G_2 + G_3\right)\left(G_4 + G_5\right)G_6 G_7 H_3$

3. All pairs of non-touching loops: only L1 and L2: $L_1 L_2 = G_6 H_1 G_7 H_2$

4. Non-touching loops taken 3 at a time - there are none

5. Compute the graph determinant:

$$\Delta = 1 - \left(L_1 + L_2 + L_3\right) + L_1 L_2 =$$

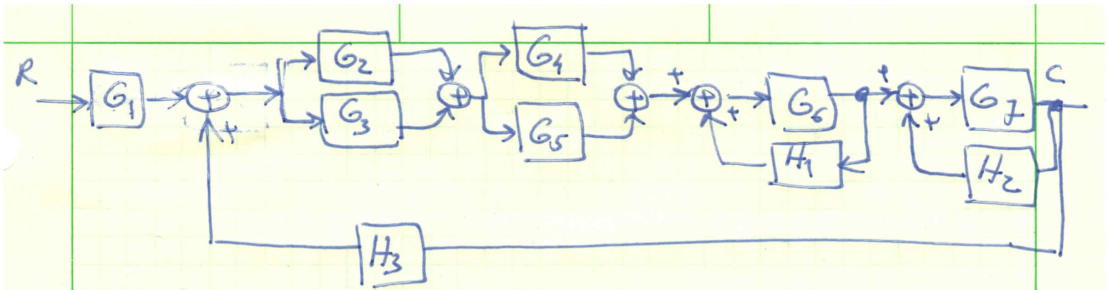
$$\Delta = 1 - \left(G_6 H + G_7 H_2 + \left(G_2 + G_3\right)\left(G_4 + G_5\right)G_6 G_7 H_3\right) + G_6 H_1 G_7 H_2$$

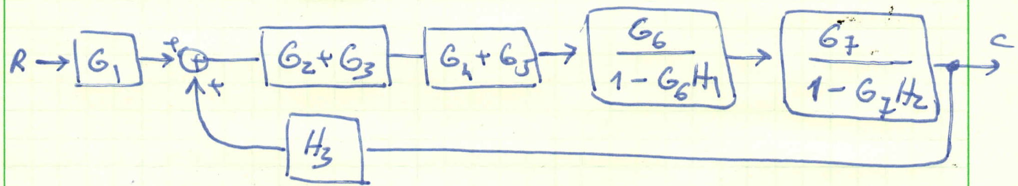
The co-factor associated with the forward path: $\Delta_1 = 1 - 0 = 1$

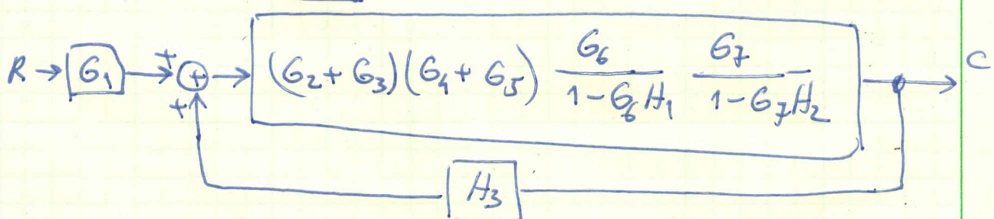
6. Apply Mason's gain formula:

$$\frac{C(s)}{R(s)} = \frac{G\Delta_1}{\Delta} = \frac{G_1\left(G_2 + G_3\right)\left(G_4 + G_5\right)G_6 G_7}{1 - \left(G_6 H + G_7 H_2 + \left(G_2 + G_3\right)\left(G_4 + G_5\right)G_6 G_7 H_3\right) + G_6 H_1 G_7 H_2}$$

b) Solve the problem by successive reductions of the SFG or block diagram - the result should be the same







$$\frac{C(s)}{R(s)} = G_1 \cdot \frac{(G_2+G_3)(G_4+G_5)\dfrac{G_6 \cdot G_7}{(1-G_6H_1)(1-G_7H_2)}}{1 - H_3\dfrac{(G_2+G_3)(G_4+G_5)G_6G_7}{(1-G_6H_1)(1-G_7H_2)}}$$

$$\frac{C(s)}{R(s)} = \frac{G_1 \cdot (G_2+G_3)(G_4+G_5)G_6 \cdot G_7}{(1-G_6H_1)(1-G_7H_2) - H_3(G_2+G_3)(G_4+G_5)G_6G_7}$$

$$\frac{C(s)}{R(s)} = \frac{G_1(G_2+G_3)(G_4+G_5)G_6G_7}{1-G_6H_1-G_7H_2+G_6H_1G_7H_2-H_3(G_2+G_3)(G_4+G_5)G_6G_7}$$

$$\frac{C(s)}{R(s)} = \frac{G_1(G_2+G_3)(G_4+G_5)G_6G_7}{1-G_6H_1-G_7H_2-(G_2+G_3)(G_4+G_5)G_6G_7H_3+G_2G_7H_1H_2}$$

6) [15 marks]The unit-step response of a system is

$$y(t)=\left[0.5-e^{-t}+0.5e^{-2t}\right]u(t)$$

a) Find the transfer function function H(s) of the system. (This is a system identification problem, when we apply a convenient test input, measure the output and compute H(s))
b) How could you use the given y(t) (unit step response) to find the impulse response h(t) and the ramp response (the response to t*u(t)) of the system?

Sol:
a) In time domain:

$$u(t)\to h(t)\to y(t)=h_u(t)=\left[0.5-e^{-t}+0.5e^{-2t}\right]u(t)$$

In Laplace domain:

$$\frac{1}{s}\to H(s)\to Y(s)\Big|_u=H_u(s)=\frac{1}{2s}-\frac{1}{s+1}+\frac{1}{2(s+2)}=H(s)\frac{1}{s}$$

It results:

$$H_u(s)=\frac{1}{2s}-\frac{1}{s+1}+\frac{1}{2(s+2)}=\frac{(s+1)(s+2)-2s(s+2)+s(s+1)}{2s(s+1)(s+2)}=\frac{s^2+s+2s+2-2s^2-4s+s^2+s}{2s(s+1)(s+2)}$$

$$H_u(s)=\frac{1}{s(s+1)(s+2)}$$

$$H(s)=sH_u(s)=\frac{1}{(s+1)(s+2)}$$

b) The signals are all related:

$$\delta(t)=\frac{d}{dt}u(t),tu(t)=\int_{0-}^{t}u(\tau)d\tau$$

The relation between the responses are therefore:

$$H(s)=sH_u(s)=\frac{1}{(s+1)(s+2)}$$

$$H_r(s)=\frac{1}{s}H_u(s)=\frac{1}{s^2(s+1)(s+2)}$$