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ELEC 341: Systems and Control

Lecture 13

Root locus: Controller design

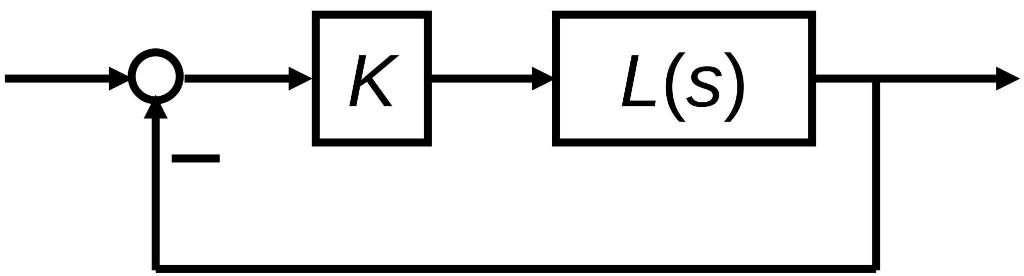
Course roadmap

Modeling	*Analysis*	*Design*
✓ Laplace transform	✓ Stability	Design specs
✓ Transfer function	✓ • Routh-Hurwitz	Root locus
Models for systems	• Nyquist	Frequency domain
✓ • Electrical	✓ Time response	PID & Lead-lag
✓ • Electromechanical	✓ • Transient	Design examples
✓ • Mechanical	✓ • Steady state	
✓ Linearization, delay	Frequency response	
	• Bode plot	

Matlab simulations

What is Root Locus? (review)

- *Pole locations* of the system characterizes stability and transient properties.
- Consider a feedback system that has one parameter (gain) $K > 0$ to be designed.



$K.L(s)$: open-loop TF

- *Root locus* graphically shows how poles of CL system varies as K varies from 0 to infinity.

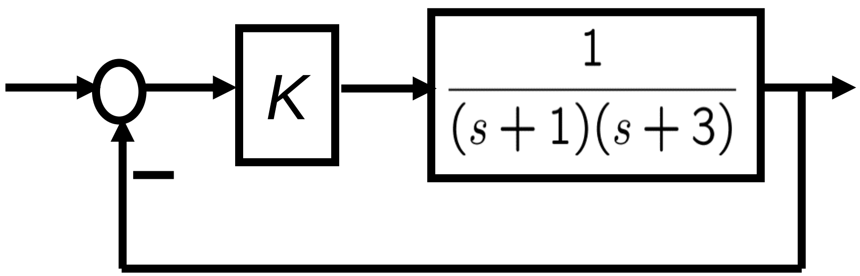
Today's topics

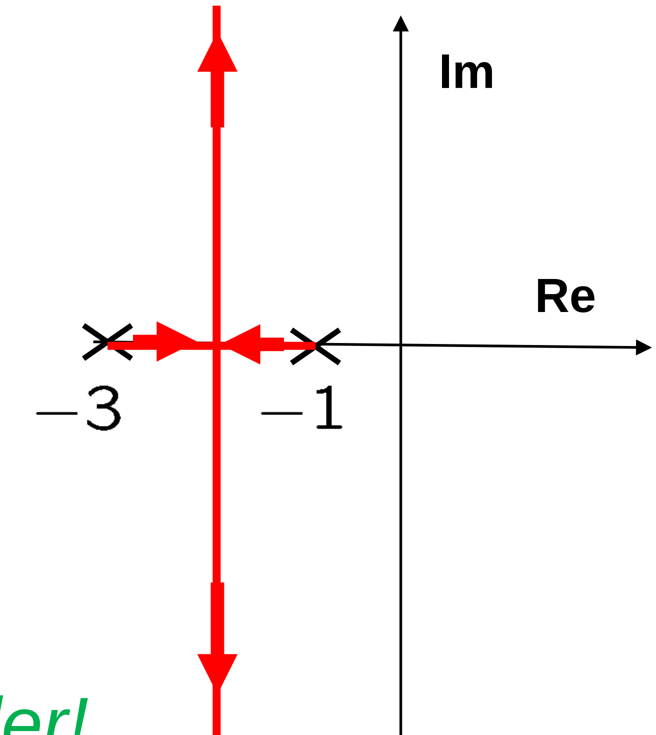
- How to use the root locus
  - Example 1: Gain design to meet design specifications
  - Example 2: Pole location design
  - Example 3: Multiple parameter design

  - In following lectures ....
    - Lead and lag compensator design
    - PID controller design

Example 1

- Design the gain K so that
  - Overshoot at most 4.32 %
  - 2% settling time at most 2 sec
  - Error constant $K_p > 1$
- Find minimum SS error for unit step





$$(\text{CL TF}) = \frac{K}{(s+1)(s+3)+K}$$

2nd order!

Example 1 (cont'd)

- Allowable region
  - Overshoot at most 4.32%

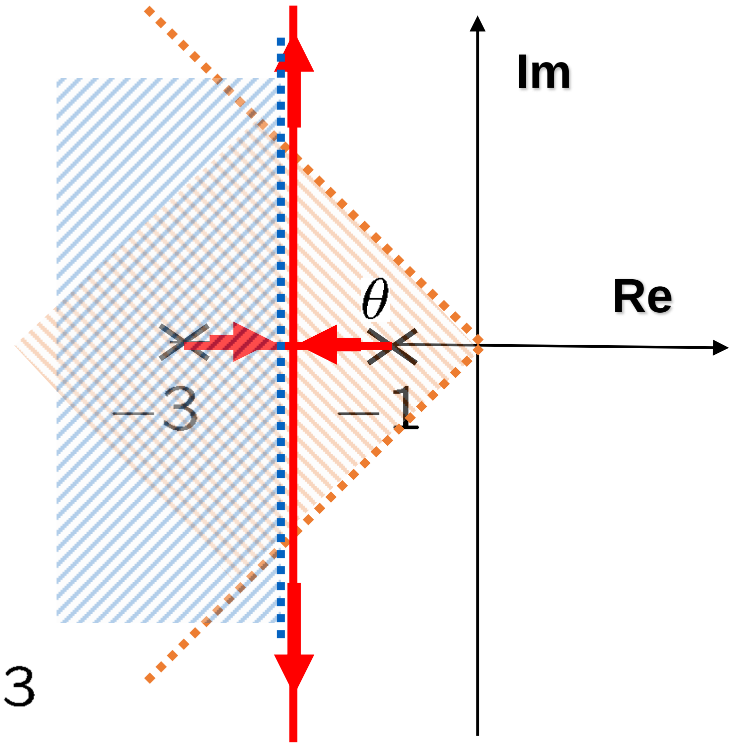
$$\theta < 45°$$

  - 2% settling time at most 2 sec

$$T_s = \frac{4}{|\mathrm{Re}|} \leq 2 \;\Leftrightarrow\; |\mathrm{Re}| \geq 2$$

- Error constant $K_p > 1$

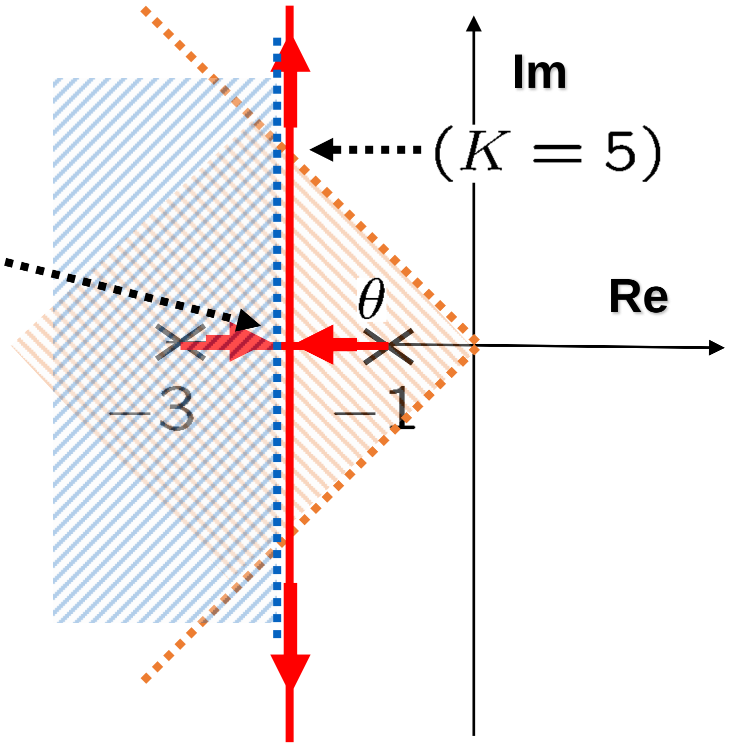
$$K_p = KL(0) = \frac{K}{3} > 1 \;\Leftrightarrow\; K > 3$$



Example 1 (cont'd)

- Gain value computations

$$\boxed{K = -\frac{1}{L(s)}} \quad (\Leftrightarrow 1 + KL(s) = 0)$$



- s = -2

$$K = -\frac{1}{L(-2)} = 1$$

- s = -2 + 2j

$$K = -\frac{1}{L(-2+2j)} = 5$$

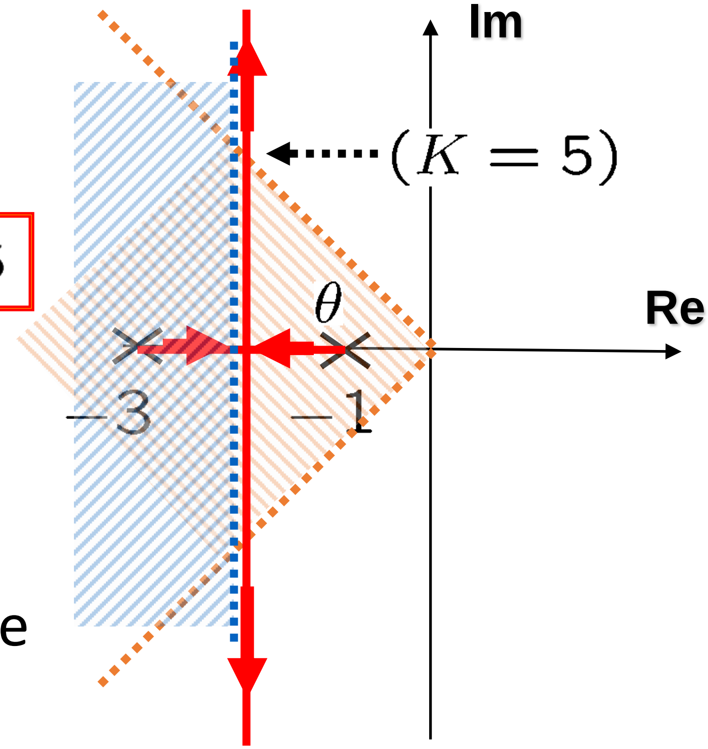
Example 1 (cont'd)

- Acceptable gain

$$\boxed{3 < K \leq 5}$$

- Minimum SS error for unit step

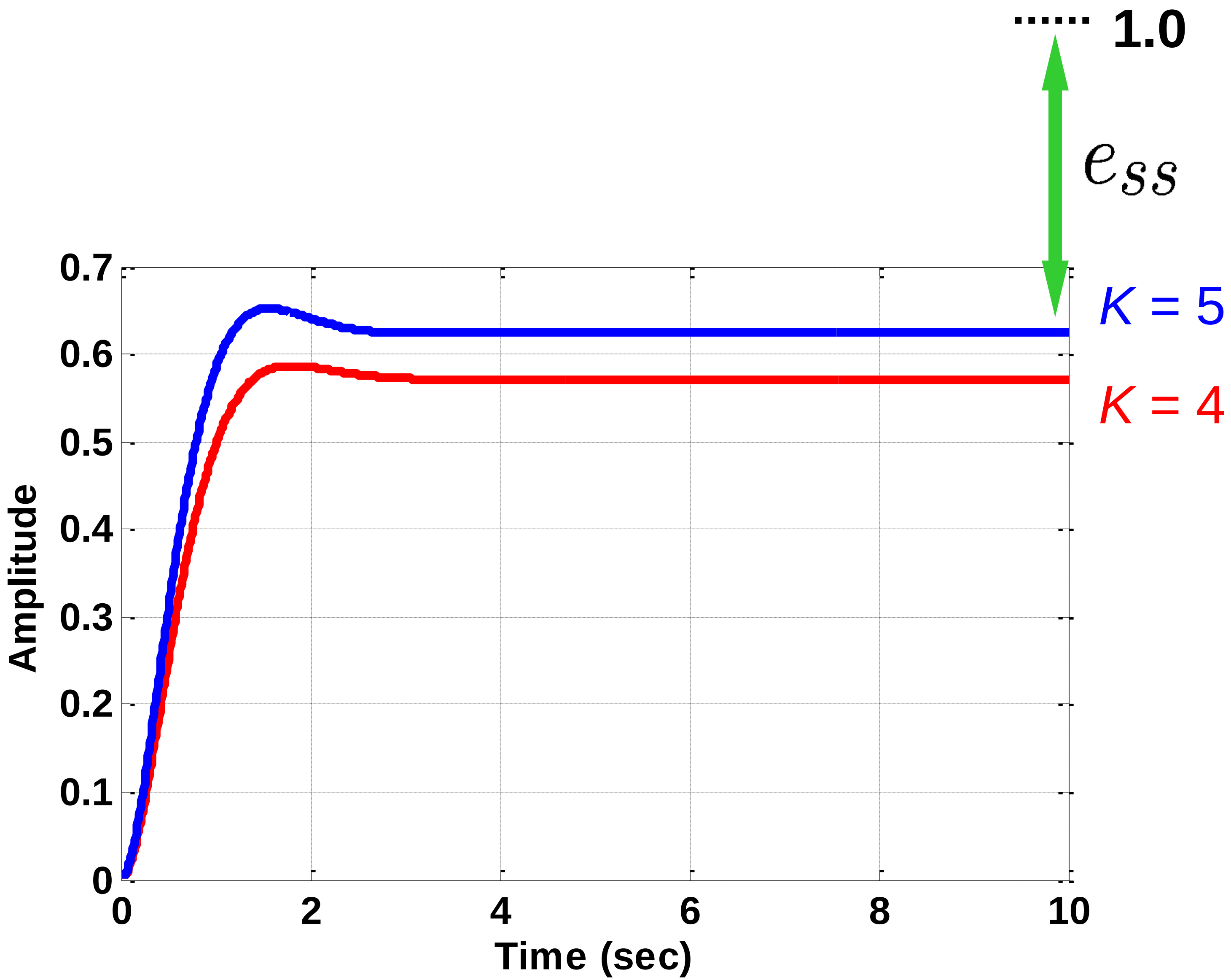
$$e_{ss} = \frac{1}{1+5/3} = \frac{3}{8} = 0.375 \Rightarrow \boxed{e_{ss} = 0.375}$$



- Limitations of gain controller
  - T_s cannot be less than 2 sec
  - Overshoot and SS error cannot be improved simultaneously.

➡ *Lead-lag or PID compensator design!*

Example 1 (cont'd): Step responses



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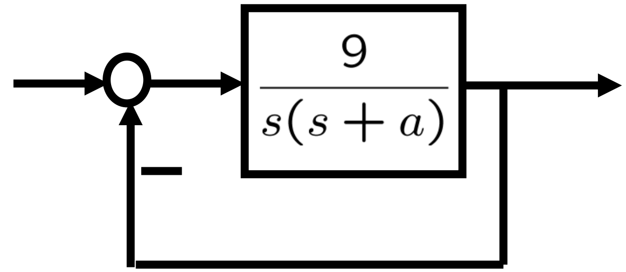
Example 2

Q: Draw root locus for $a > 0$



Characteristic eq.

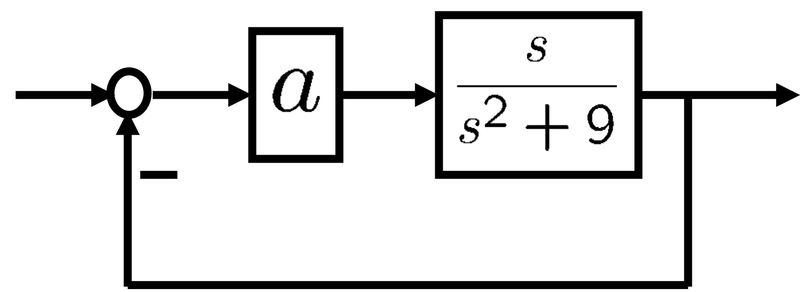
$$1 + \frac{9}{s(s+a)} = 0$$

$$\Leftrightarrow \quad \underbrace{s^2 + 9}_{\text{Term without } a} + \underbrace{sa}_{\text{Term with } a} = 0$$

$$\Leftrightarrow \quad 1 + a \underbrace{\frac{s}{s^2+9}}_{L(s)} = 0$$

Fictitious OL system

Two CL systems have the same characteristic eq.



Example 2 (cont'd)

- If a tuning parameter appears in the characteristic equation at a "non-standard location", to draw root locus with respect to the tuning parameter, we transform the equation into our "standard" form:

$$1 + KL(s) = 0$$

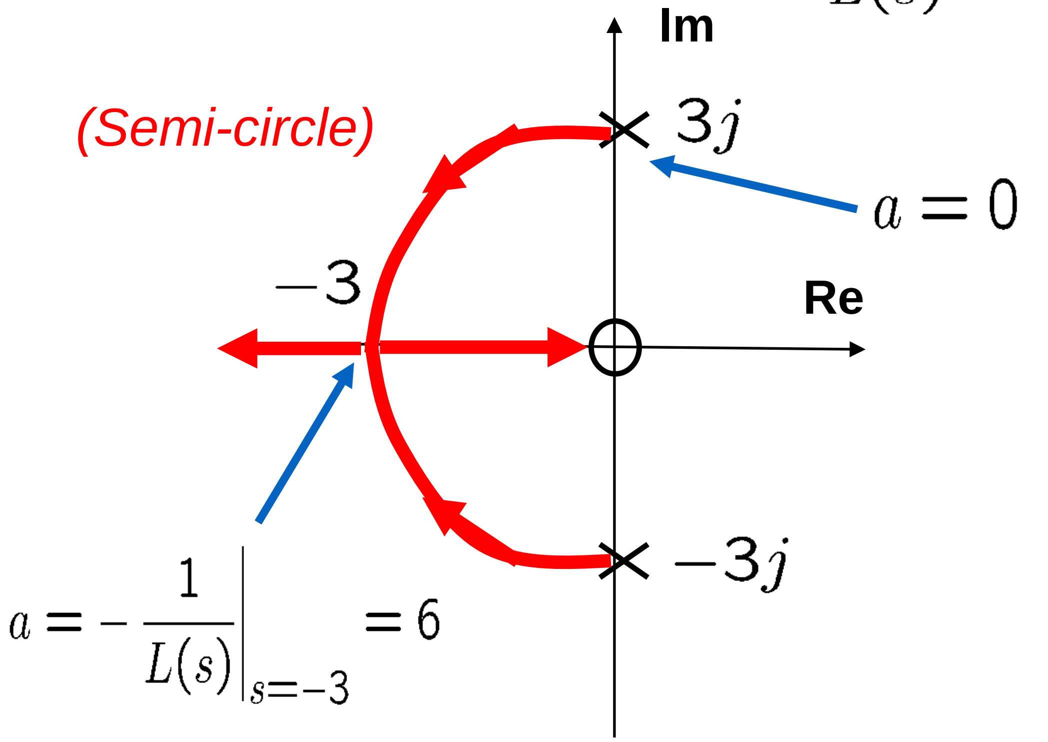
- Two examples:

$$\boxed{1 + \frac{1}{s(s+p)} = 0} \Rightarrow s(s+p) + 1 = 0 \Rightarrow s^2 + 1 + ps = 0 \Rightarrow \boxed{1 + p\frac{s}{s^2+1} = 0}$$

$$\boxed{1 + \frac{s+z}{s^2} = 0} \Rightarrow s^2 + s + z = 0 \Rightarrow \boxed{1 + z\frac{1}{s(s+1)} = 0}$$

Example 2 (cont'd)

Root locus for $1 + a \underbrace{\frac{s}{s^2+9}}_{L(s)} = 0$

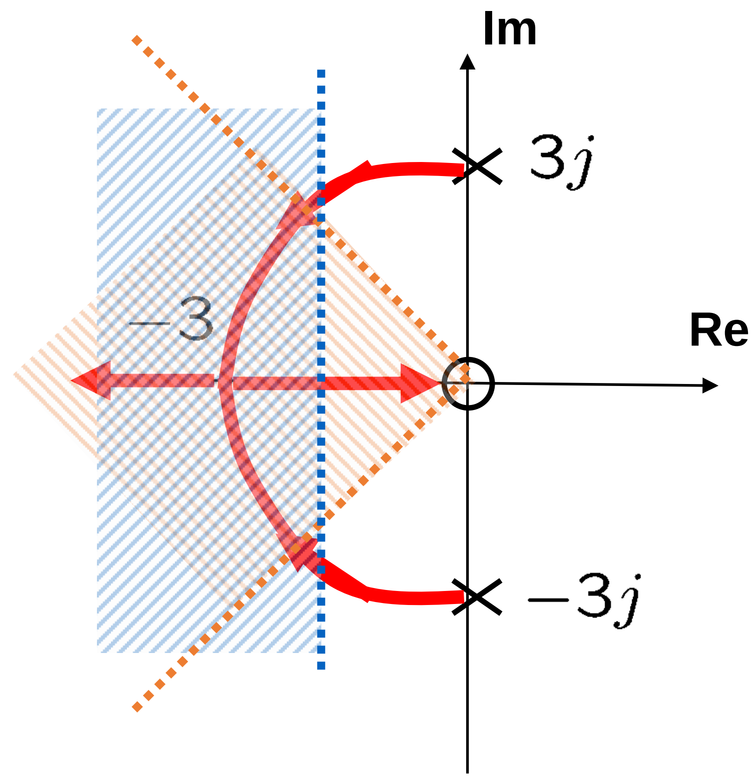


Example 2 (cont'd)

- Design the pole "*a*" satisfying
  - Overshoot at most 4.32%
  - 2% settling time at most 2 sec
  - Error constant $K_v > 2$

$$K_v = \lim_{s \to 0} s \frac{9}{s(s+a)} = \frac{9}{a} > 2$$

$$\boxed{a < 4.5}$$



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Today's topics

- How to use the root locus
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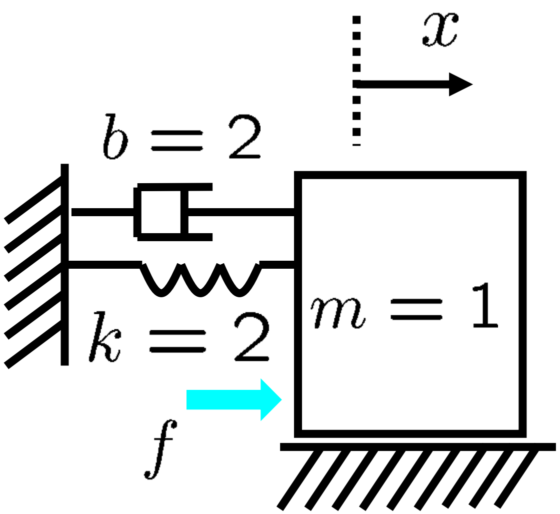
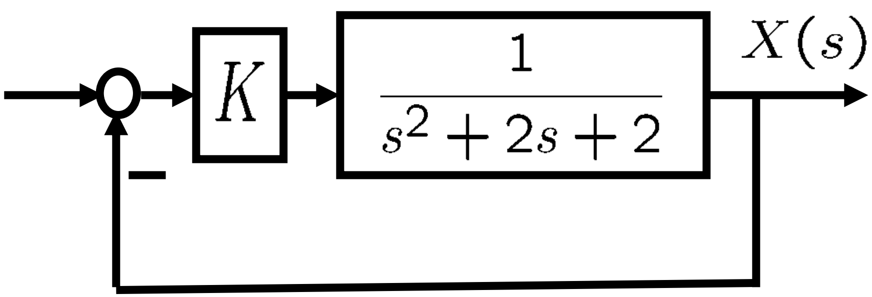
Example 3

- Mass-spring-damper system
  - Shock absorber (car, train)
  - Seismic isolator (building)
  - Accelerometer (Lec 3)
- Design a controller so that
  - 2% settling at most 1 sec
  - No overshoot
  - e_{ss} for unit step < 0.05



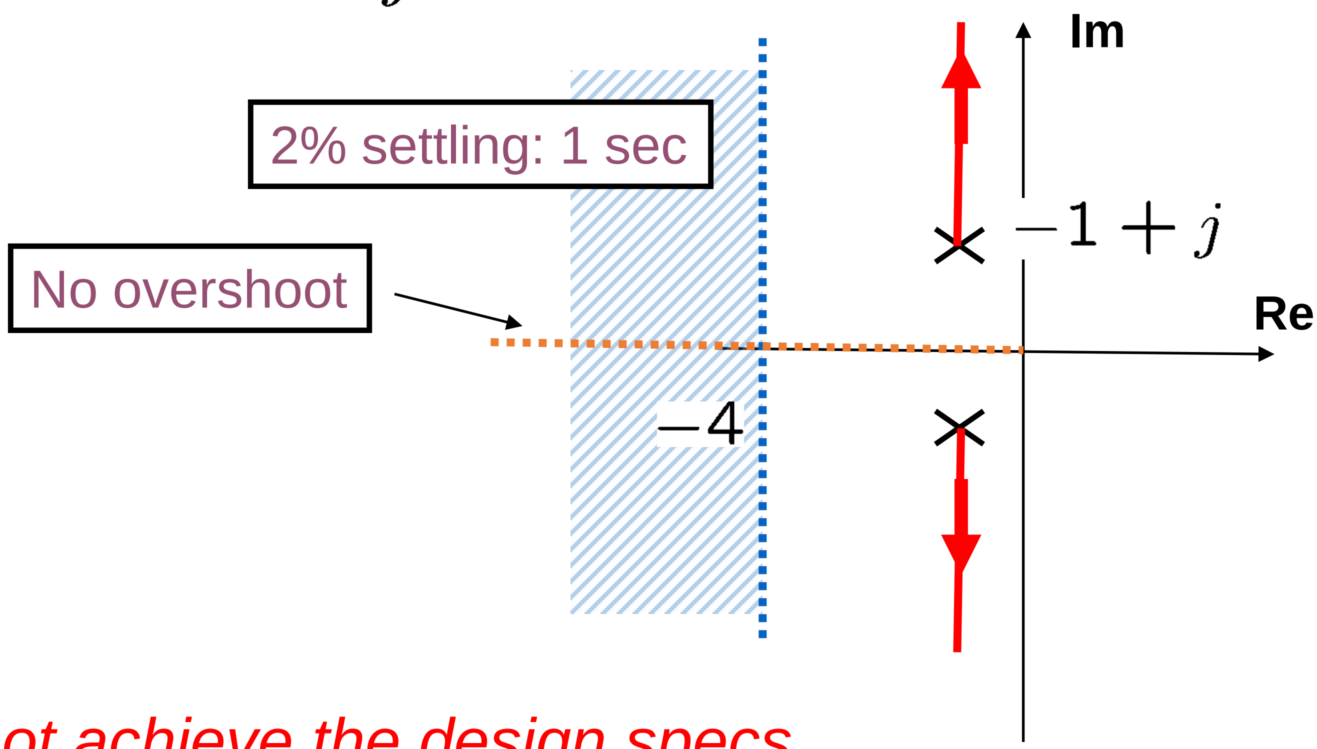
$$\Rightarrow \quad e_{ss} = \frac{1}{1+K_p} < 0.05$$

$$\Rightarrow \quad K_p > 19 \quad \Rightarrow \quad K > 38$$



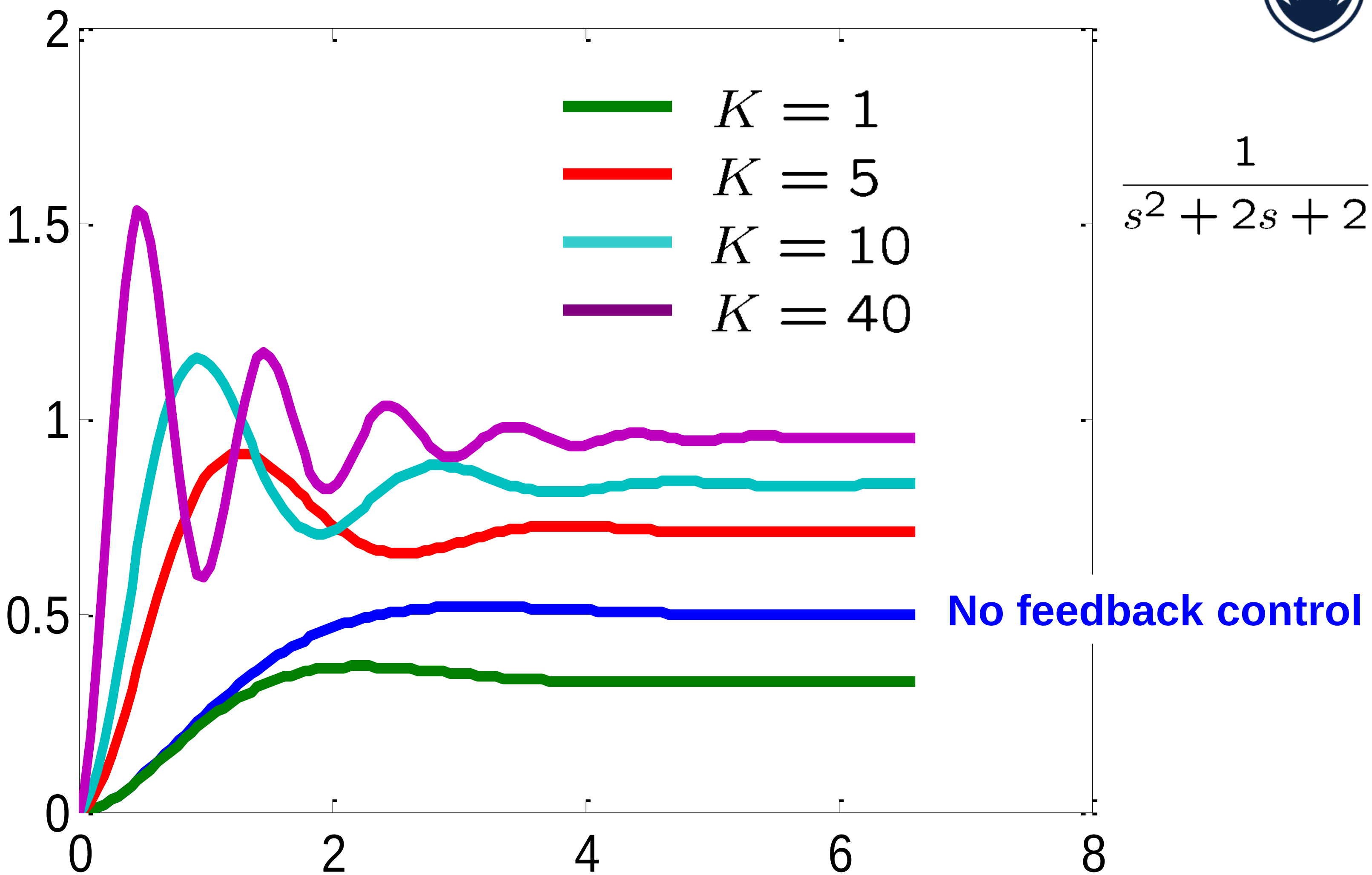
Example 3 (cont'd)

- Root locus
  - Poles: $s = -1 \pm j$



We cannot achieve the design specs with the position feedback gain controller!

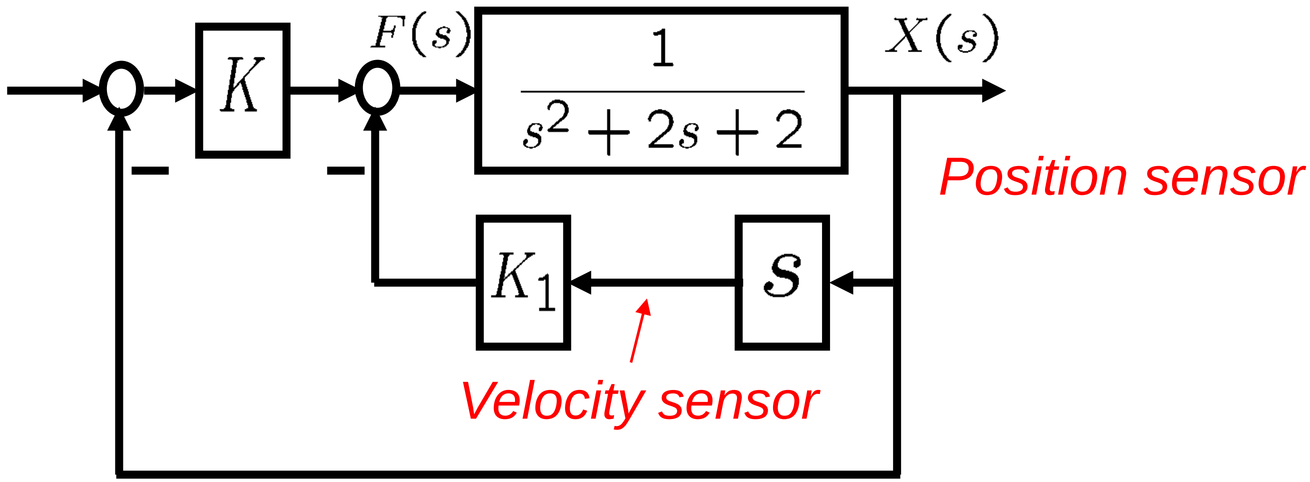
Example 3 (cont'd): Step responses



$$\frac{1}{s^2 + 2s + 2}$$

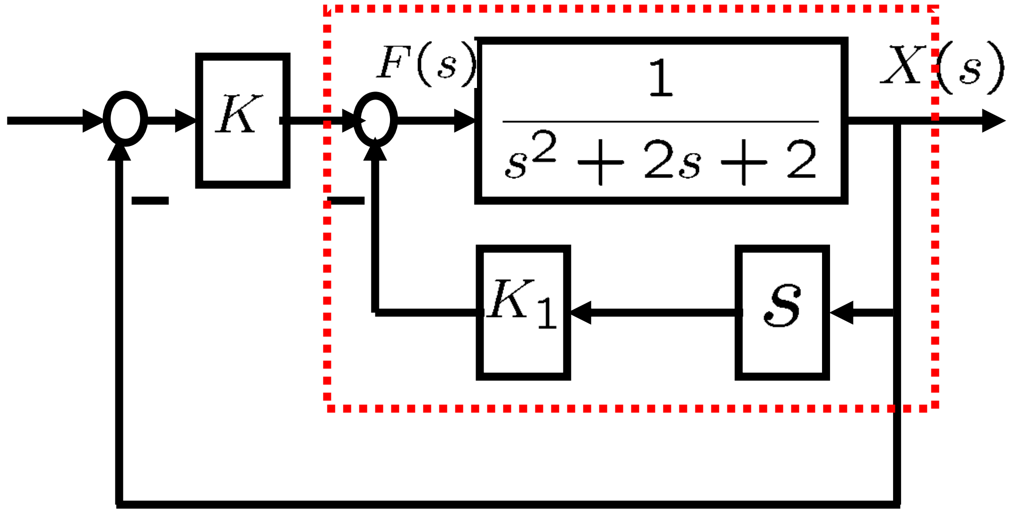
Example 3 (cont'd)

- Position & velocity (rate) feedback



- We have two design parameters to be tuned to satisfy the design specs, i.e., K & K_1.
- How to use root locus technique?

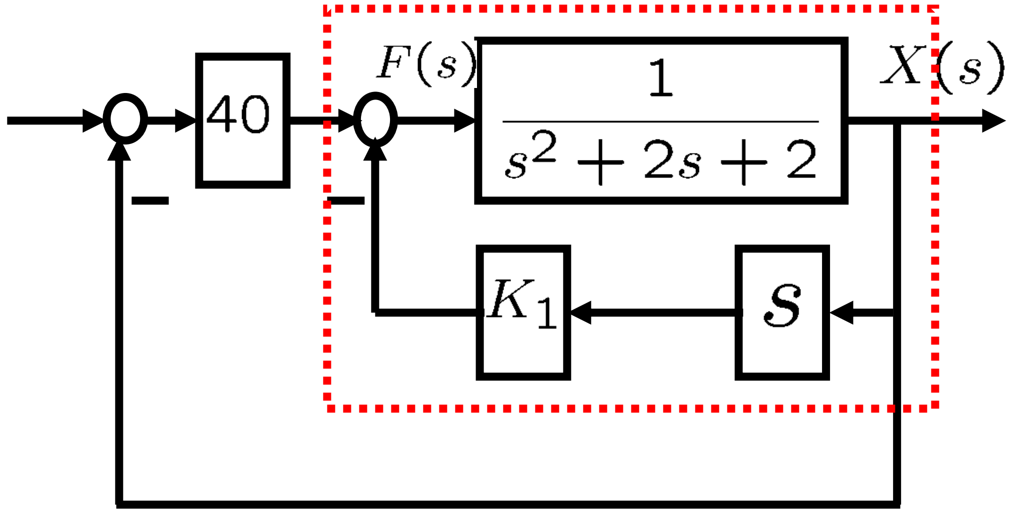
Example 3 (cont'd)



$$= \frac{\frac{1}{s^2+2s+2}}{1+\frac{K_1 s}{s^2+2s+2}} = \frac{1}{s^2+(K_1+2)s+2}$$

- SS error spec $\quad K_p = \frac{K}{2} > 19 \Rightarrow K > 38$

Example 3 (cont'd): Set $K = 40$



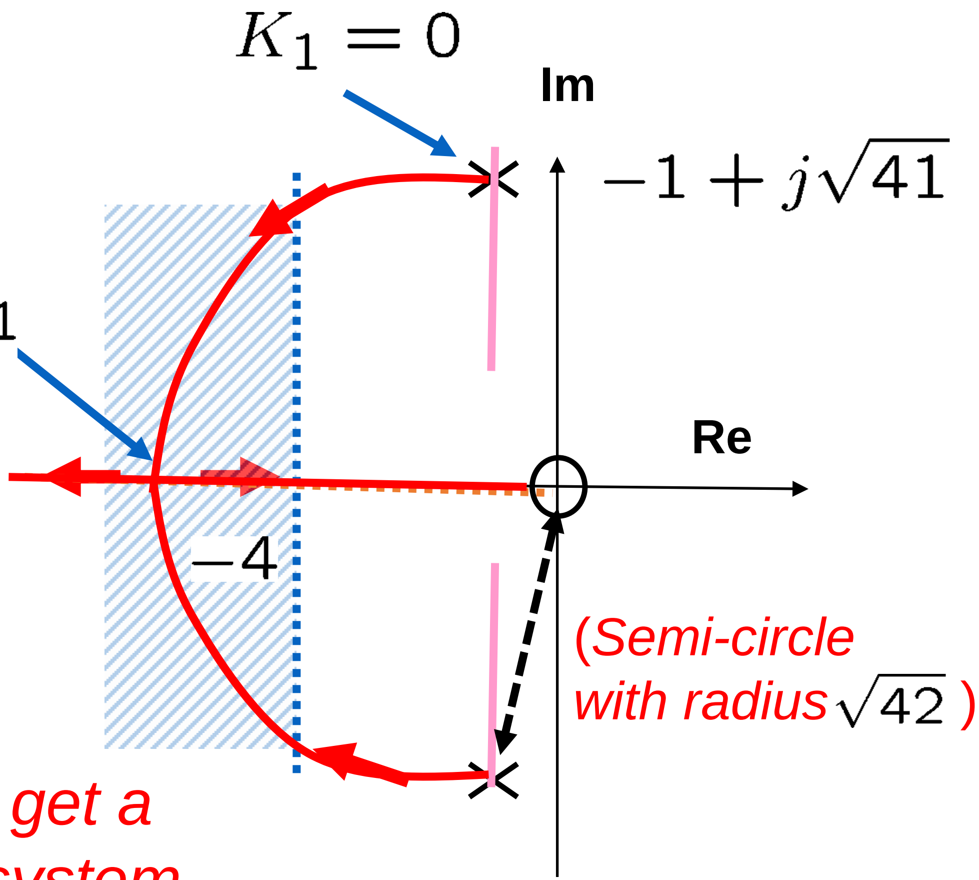
Characteristic eq. $1 + 40 \times \boxed{\frac{1}{s^2+(K_1+2)s+2}} = 0$

$$\Rightarrow \quad s^2 + 2s + 42 + K_1 s = 0 \quad \Rightarrow \quad 1 + K_1 \underbrace{\frac{s}{s^2+2s+42}}_{L(s)} = 0$$

Example 3 (cont'd): $K = 40$

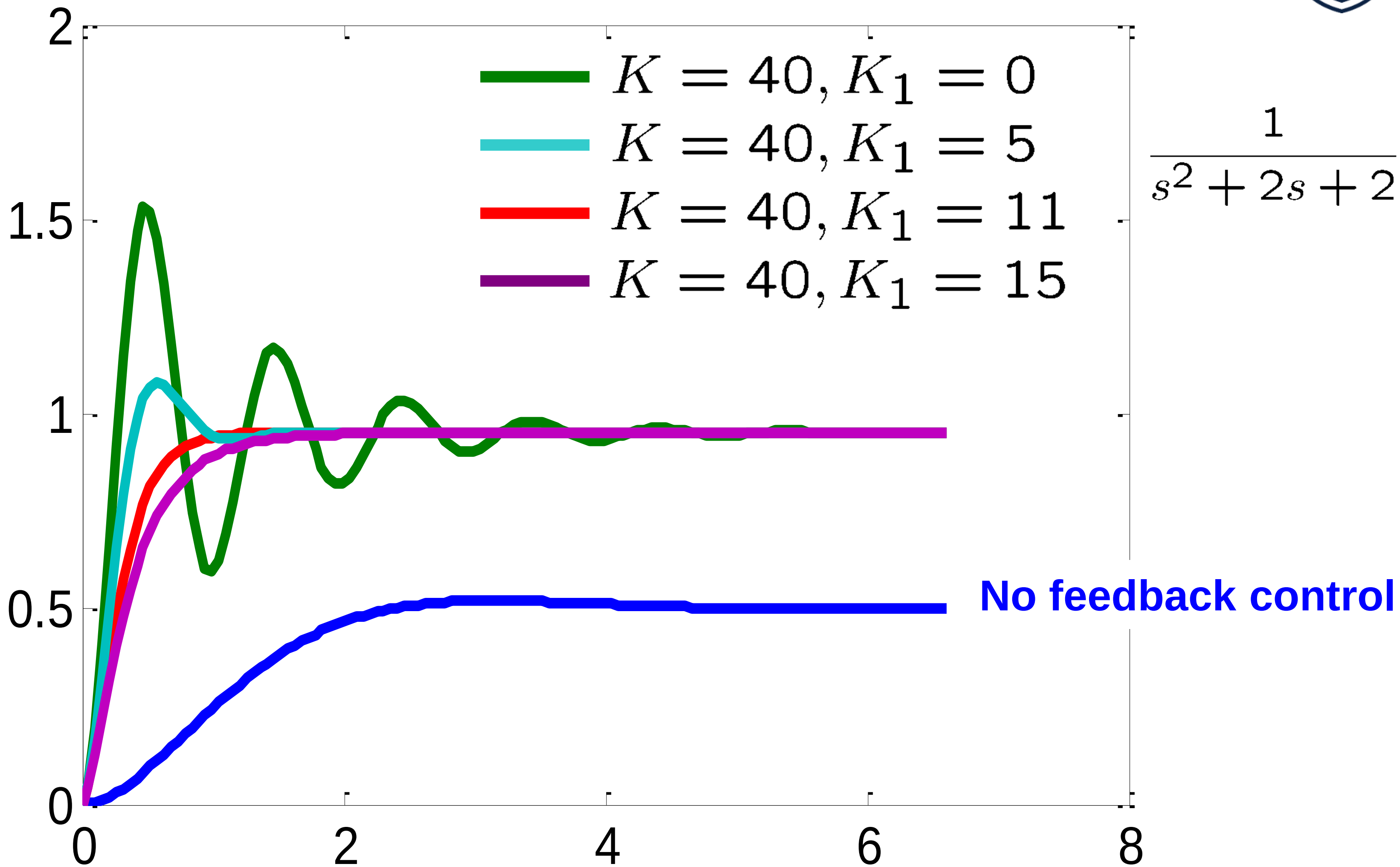
$$L(s) = \frac{s}{s^2 + 2s + 42}$$

$$K_1 = -\left.\frac{1}{L(s)}\right|_{s=-\sqrt{42}} \approx 11$$



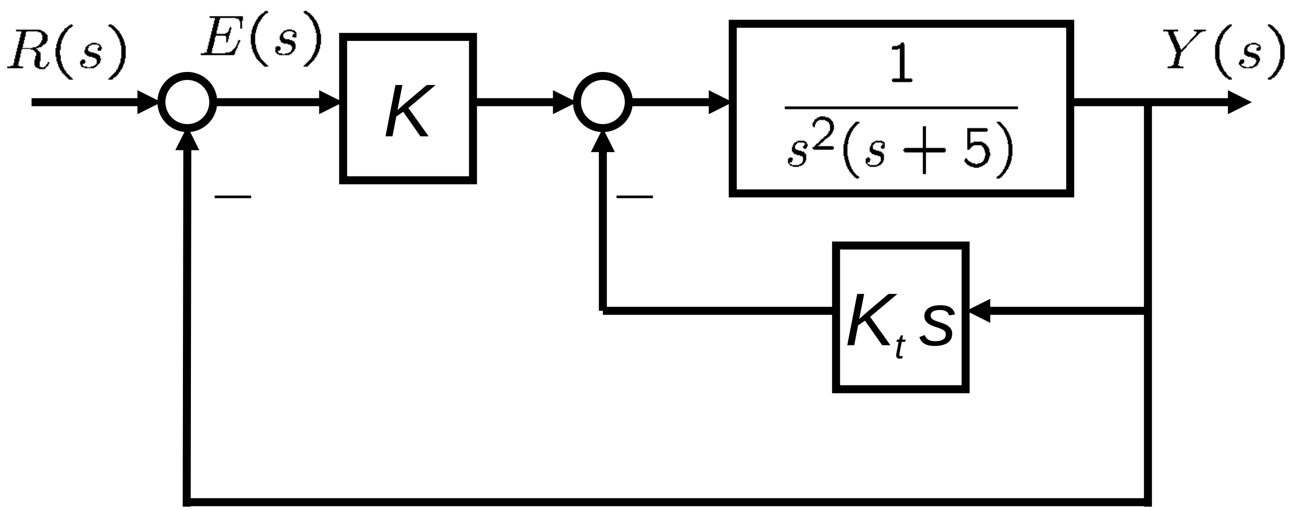
By increasing K_1, we can get a satisfactory closed-loop system.

Example 3 (cont'd): Step responses



$$\frac{1}{s^2 + 2s + 2}$$

Example 4



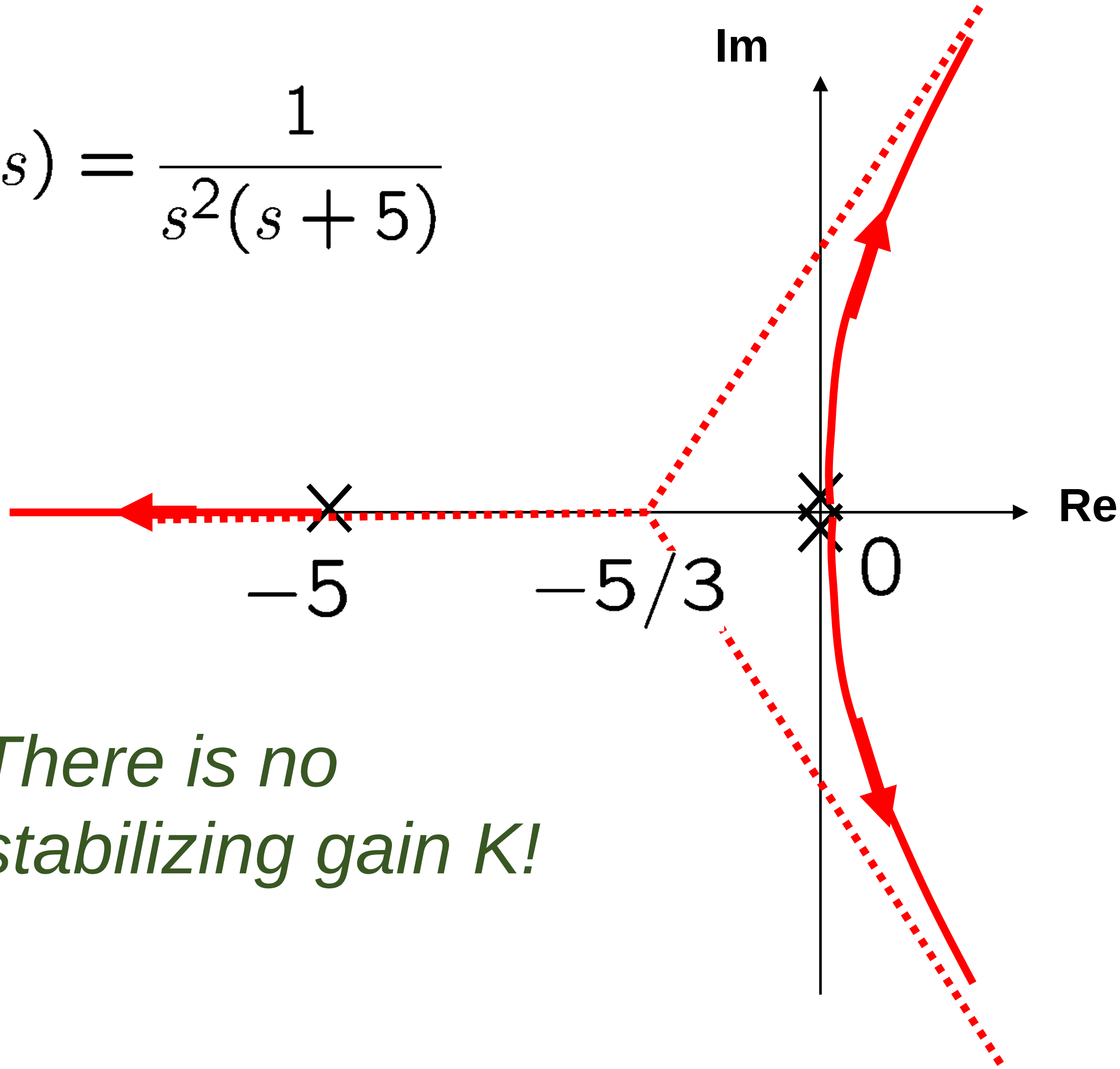
a) Set $K_t = 0$. Draw root locus for $K > 0$.

b) Set $K = 10$. Draw root locus for $K_t > 0$.

c) Set $K = 5$. Draw root locus for $K_t > 0$.

Example 4 (cont'd) (a): $K_t = 0$

$$L(s) = \frac{1}{s^2(s+5)}$$



There is no stabilizing gain K!

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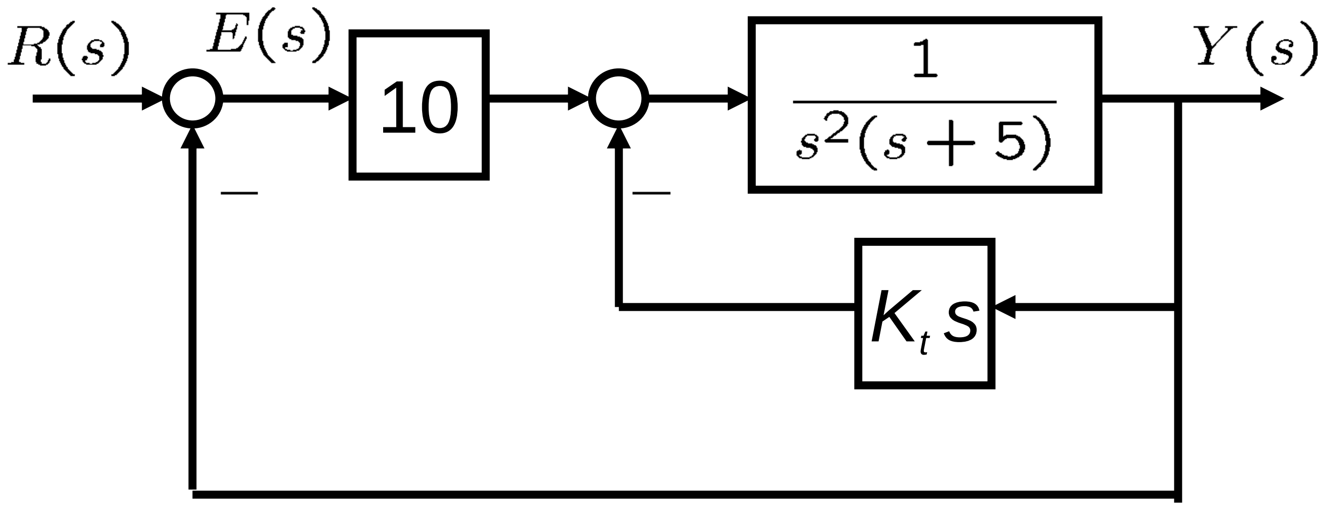
Example 4 (cont'd) (b): $K = 10$

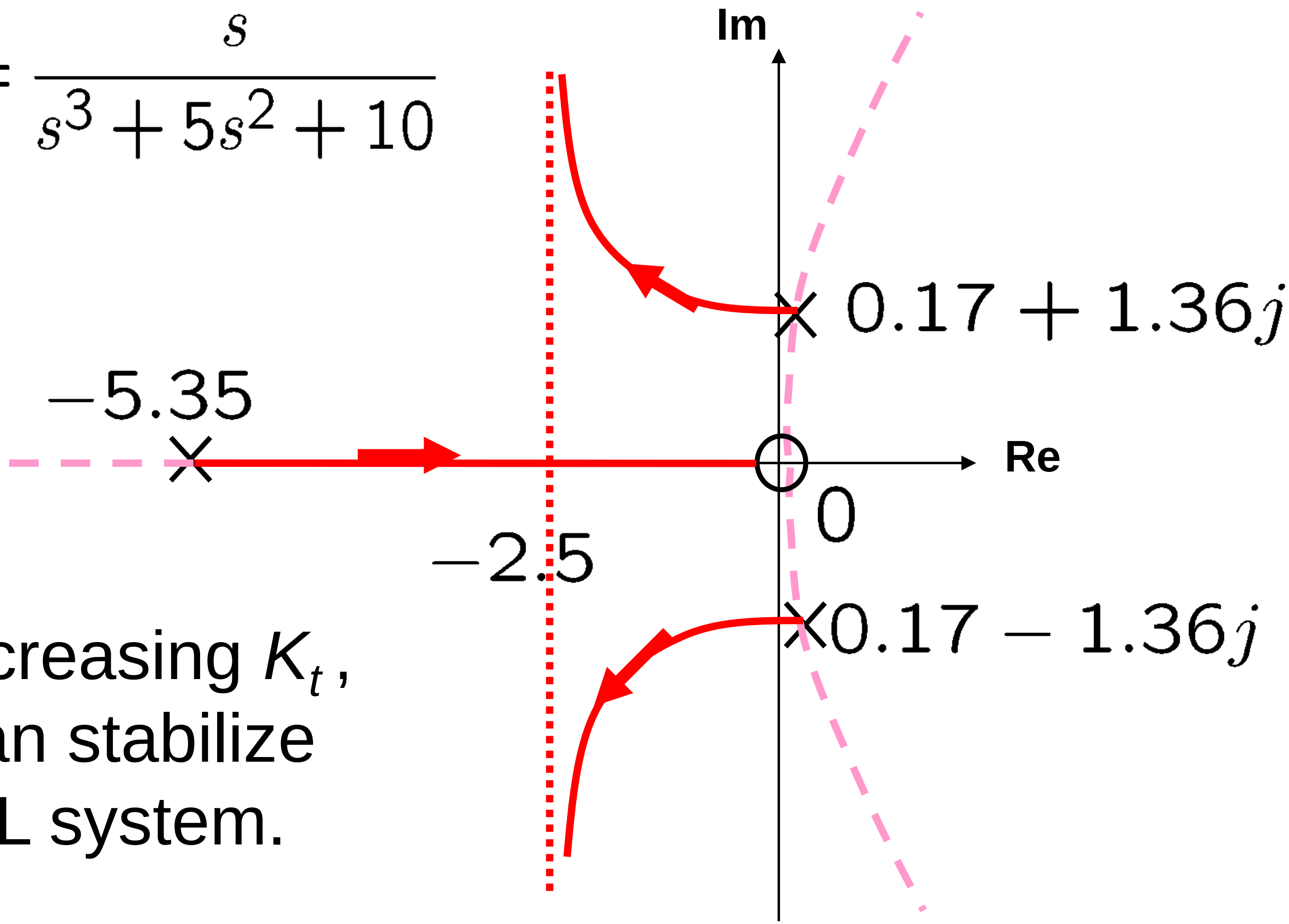


Characteristic eq. $$1 + 10\left(\frac{\frac{1}{s^2(s+5)}}{1 + \frac{K_t s}{s^2(s+5)}}\right) = 0$$

$$\Rightarrow \quad s^2(s+5) + K_t s + 10 = 0 \quad \Rightarrow \quad 1 + K_t \underbrace{\frac{s}{s^3 + 5s^2 + 10}}_{L(s)} = 0$$

Example 4 (cont'd) (b): $K = 10$

$$L(s) = \frac{s}{s^3 + 5s^2 + 10}$$



By increasing K_t, we can stabilize the CL system.

Example 4 (cont'd)
Finding K_t for marginal stability

- Characteristic equation

$$1+\frac{K_t s}{s^3+5s^2+10}=0 \Leftrightarrow s^3+5s^2+K_t s+10=0$$

- Routh array

$$\begin{array}{c|ll} s^3 & 1 & K_t \\ s^2 & 5 & 10 \\ s^1 & \frac{5K_t-10}{5} & \\ s^0 & 10 & \end{array}$$

Stability condition

$K_t > 2$

- When $K_t = 2$

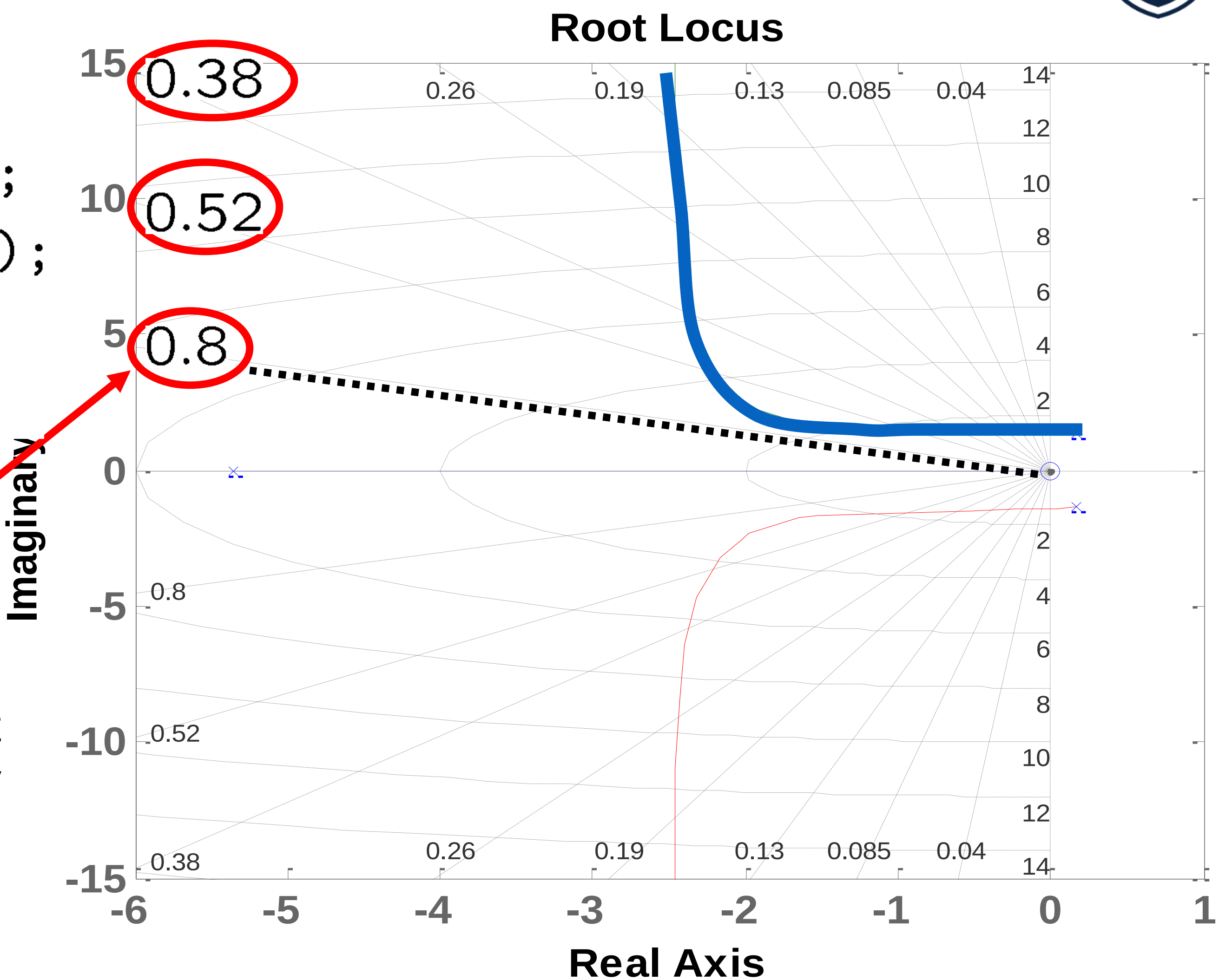
$$5s^2+10=0 \Rightarrow s=\pm\sqrt{2}j$$

Example 4 (cont'd)

Matlab command

```
num=[1 0];
den=[1 5 0 10];
sys=tf(num,den);
rlocus(sys)
grid on
```

Damping ratio



If K = 10, we cannot achieve $\zeta = 0.8$ for any $K_t > 0$.

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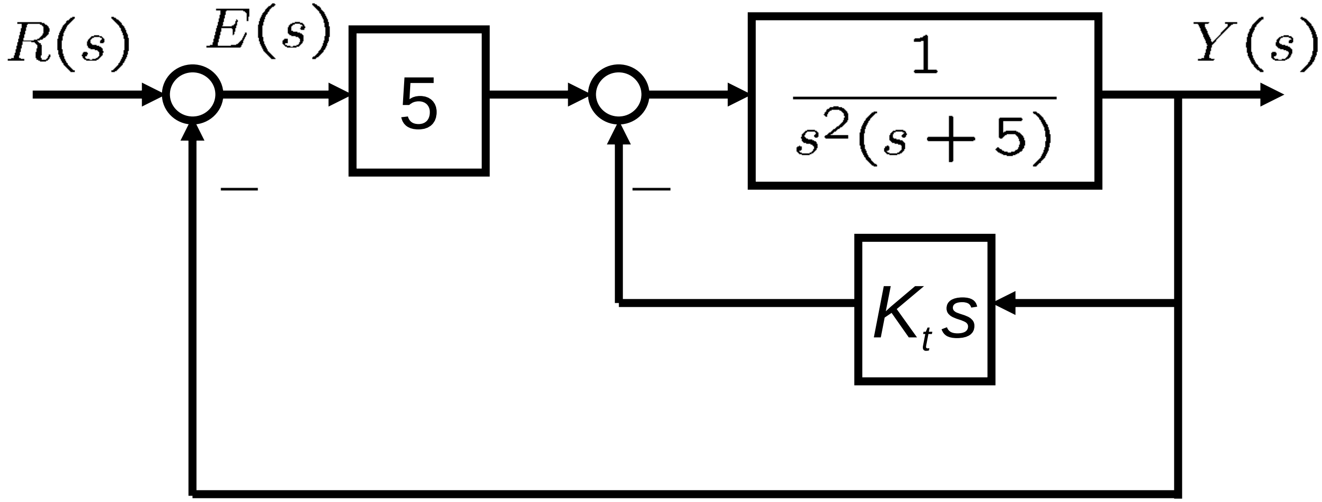
Example 4 (cont'd) (c): $K = 5$

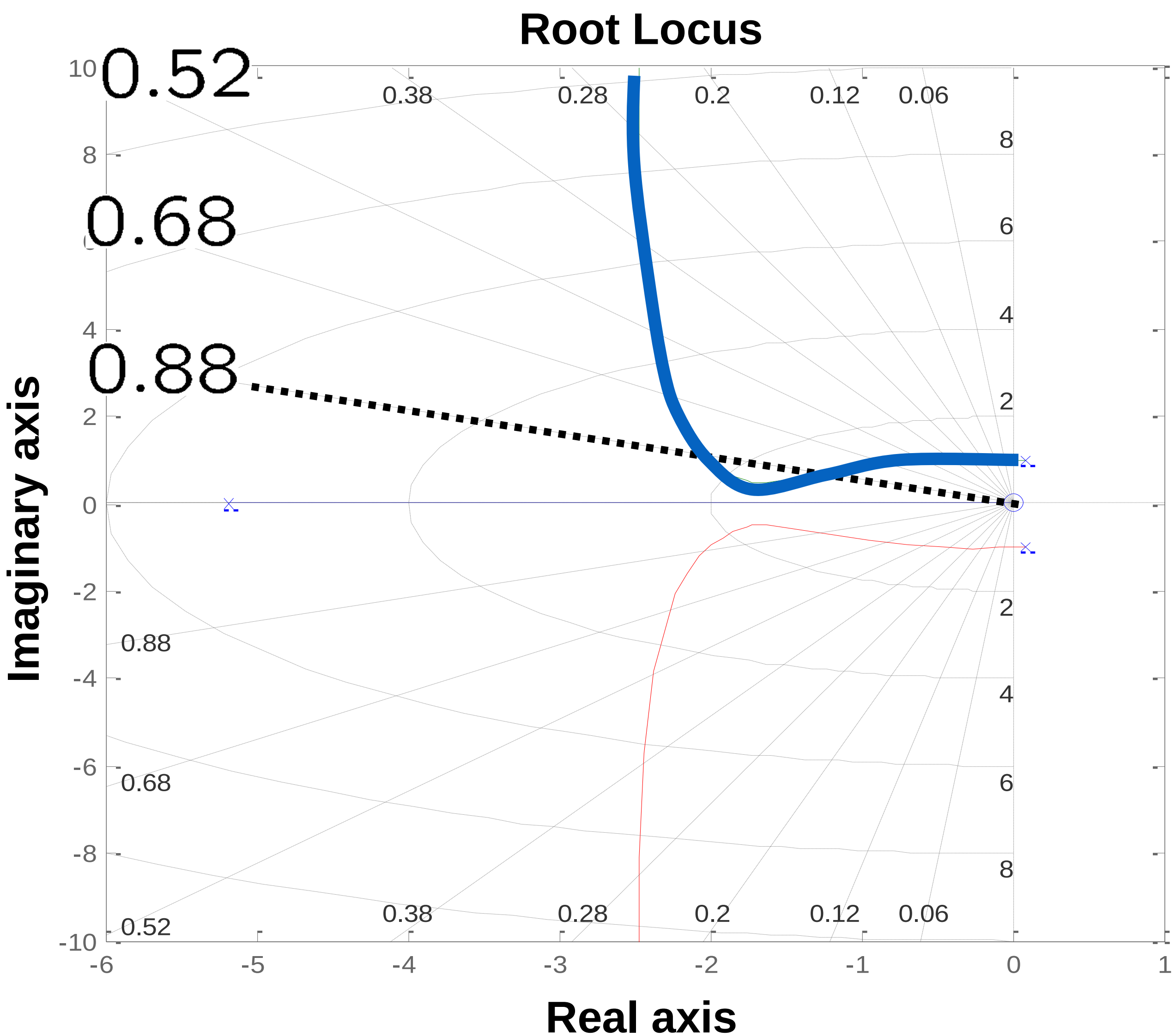


Characteristic eq.

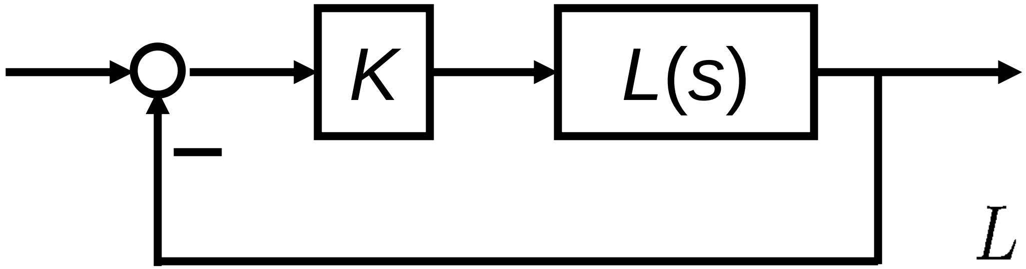
$$1 + 5\left(\frac{\frac{1}{s^2(s+5)}}{1 + \frac{K_t s}{s^2(s+5)}}\right) = 0$$

$$\Rightarrow \quad s^2(s+5) + K_t s + 5 = 0 \quad \Rightarrow \quad 1 + K_t \underbrace{\frac{s}{s^3 + 5s^2 + 5}}_{L(s)} = 0$$

Example 4 (cont'd) (c): Root locus plot



Example 5



$$L(s) = \frac{1+Ts}{s(s+1)(s+2)}$$

a) Set $T = 0$. Draw root locus for $K > 0$.

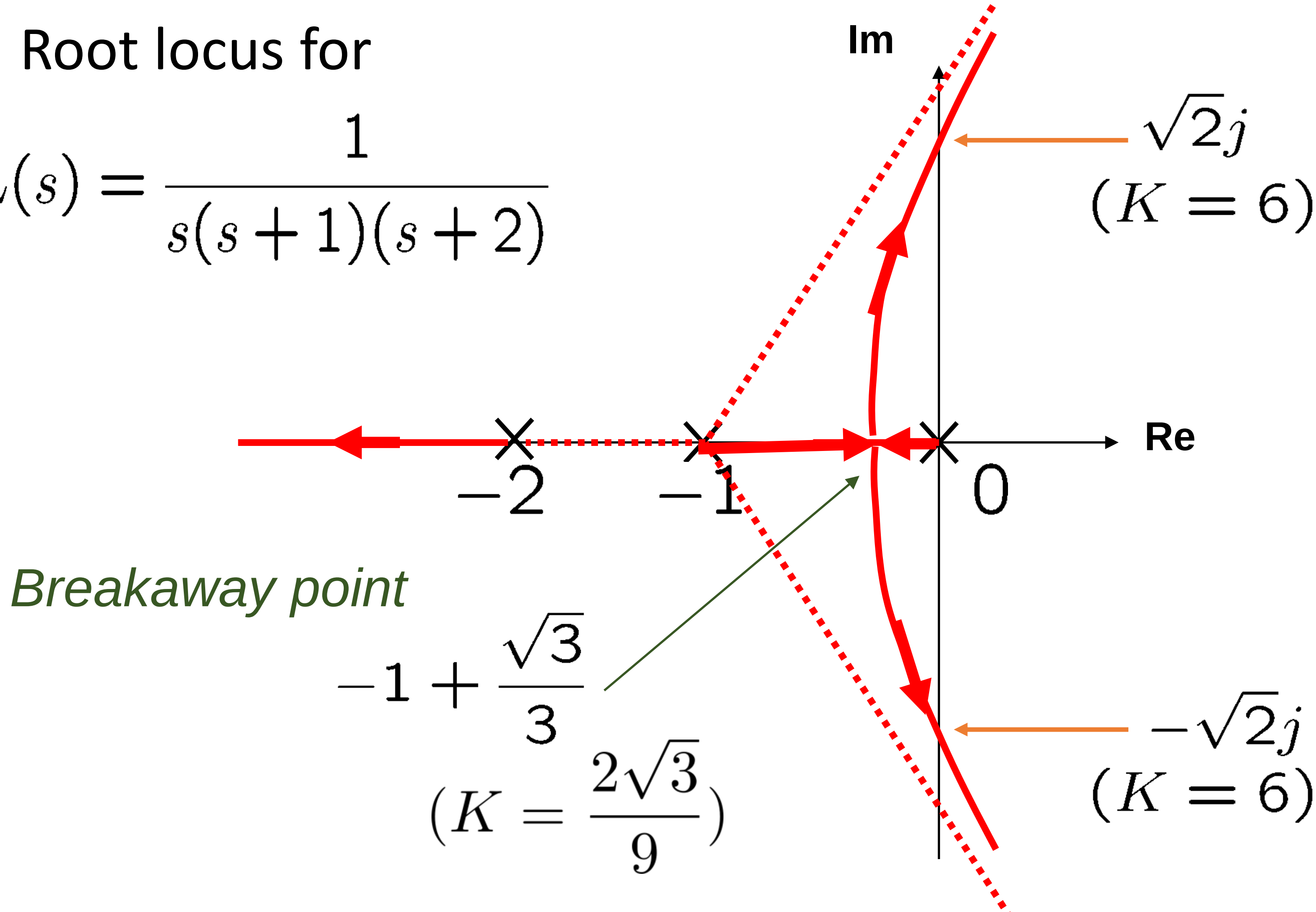
$$L(s) = \frac{1}{s(s+1)(s+2)}$$

b) Vary T to see the effect of a zero on root locus.

Example 5 (a) (cont'd)

- Root locus for

$$L(s) = \frac{1}{s(s+1)(s+2)}$$



Example 5 (b) (cont'd)

- When K is fixed and T is a positive parameter, the characteristic equation can be written as

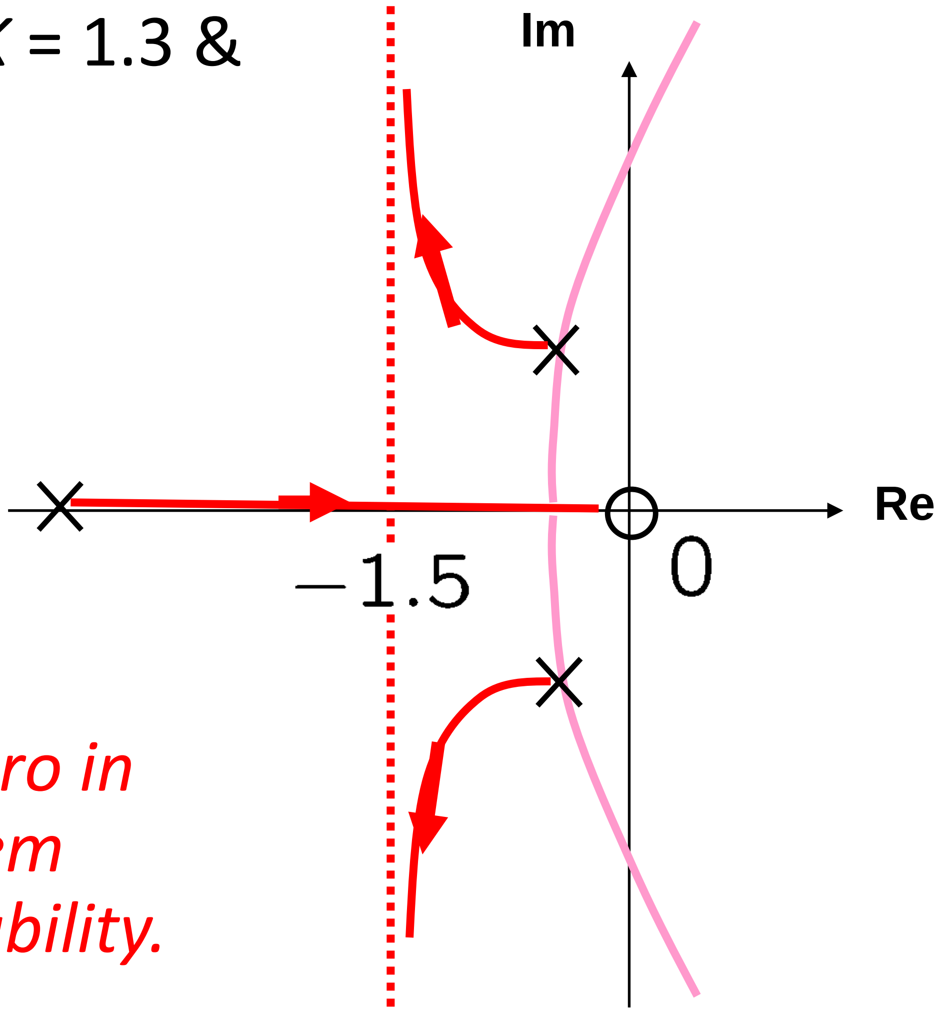
$$1+K\frac{1+Ts}{s(s+1)(s+2)}=0$$

$$\Rightarrow \quad \underbrace{s(s+1)(s+2)+K}_{\text{Term without } T}+\underbrace{TKs}_{\text{Term with } T}=0$$

$$\Rightarrow \quad 1+T\frac{Ks}{s(s+1)(s+2)+K}=0$$

Example 5 (b) (cont'd)

- Root locus for K = 1.3 & various T



- Generally,

  addition of a zero in open-loop system improves CL stability.

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Summary

- How to use the root locus
  - Example 1: Gain design to meet design specifications
  - Example 2: Pole location design
  - Example 3: Multiple parameter design
    - Velocity (rate) feedback can be used to improve the transient property.
- Next
  - We will study how to use the root locus to design lead-lag compensators.