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ELEC 341: Systems and Control

Lecture 11

Root locus: Introduction

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Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist
- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- Frequency response
  - Bode plot

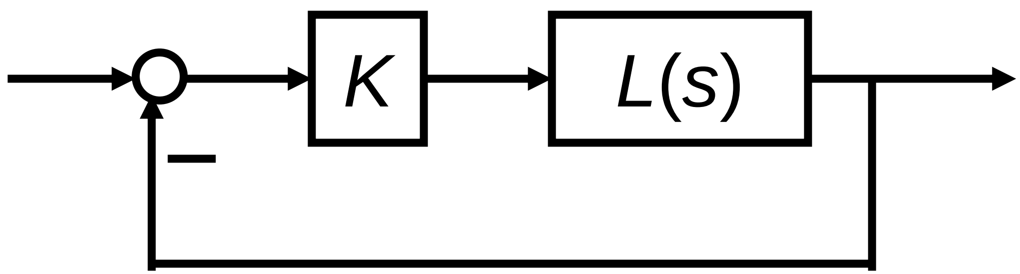
Design

- Design specs
- ➤ Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

What is Root Locus?

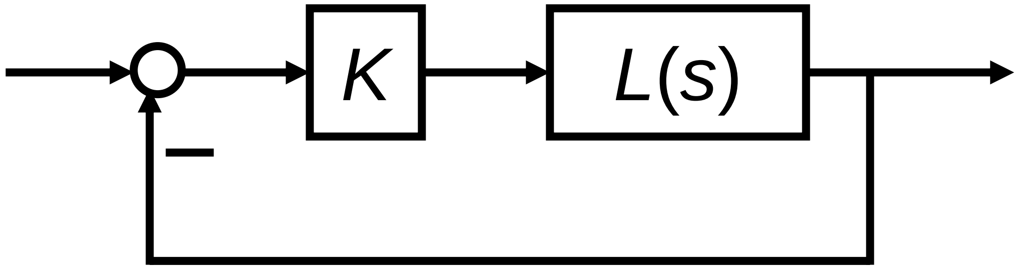
- *Pole locations* of the system characterizes stability and transient properties.
- Consider a feedback system that has one parameter (gain) $K > 0$ to be designed:



$K.L(s)$: open-loop TF

- *Root locus* (RL) graphically shows how poles of CL system vary as K varies from 0 to infinity.

Example 1



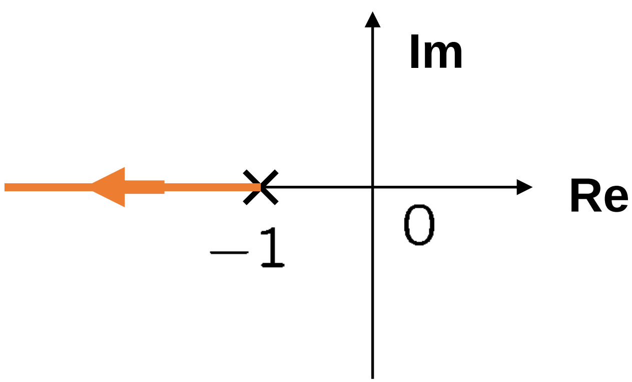
$$L(s) = \frac{1}{s+1}$$

- Characteristic eq. $1 + K\frac{1}{s+1} = 0$

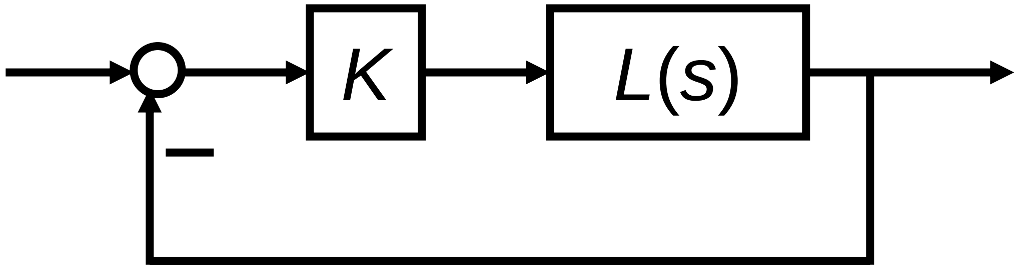
Closed-loop poles

$$\Rightarrow \quad s + 1 + K = 0 \quad \Rightarrow \quad s = -1 - K$$

- K = 0: s = -1
- K = 1: s = -2
- K = 2: s = -3, etc.



Example 2



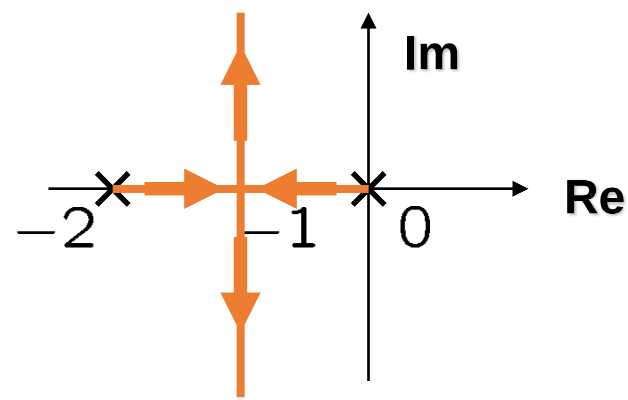
$$L(s) = \frac{1}{s(s+2)}$$

- Characteristic eq. $1 + K\frac{1}{s(s+2)} = 0$

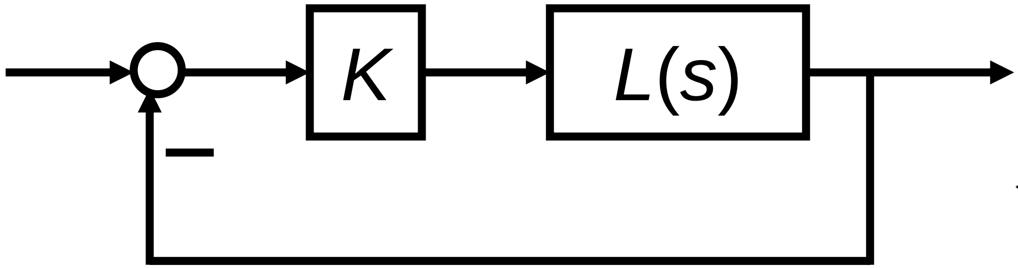
Closed-loop poles

$$\Rightarrow s^2 + 2s + K = 0 \Rightarrow s = -1 \pm \sqrt{1-K}$$

- $K = 0$: $s = 0,-2$
- $K = 1$: $s = -1,-1$
- $K > 1$: complex numbers



Example 3



$$L(s) = \frac{s+1}{s(s+2)(s+3)}$$

- Characteristic eq. $1 + K\frac{s+1}{s(s+2)(s+3)} = 0$

$\Rightarrow$ $s(s+2)(s+3) + K(s+1) = 0$ $\Rightarrow$ $s = ???$

- It is hard to solve this analytically for each *K*.
- Is there some way to sketch roughly and quickly root locus by hand?

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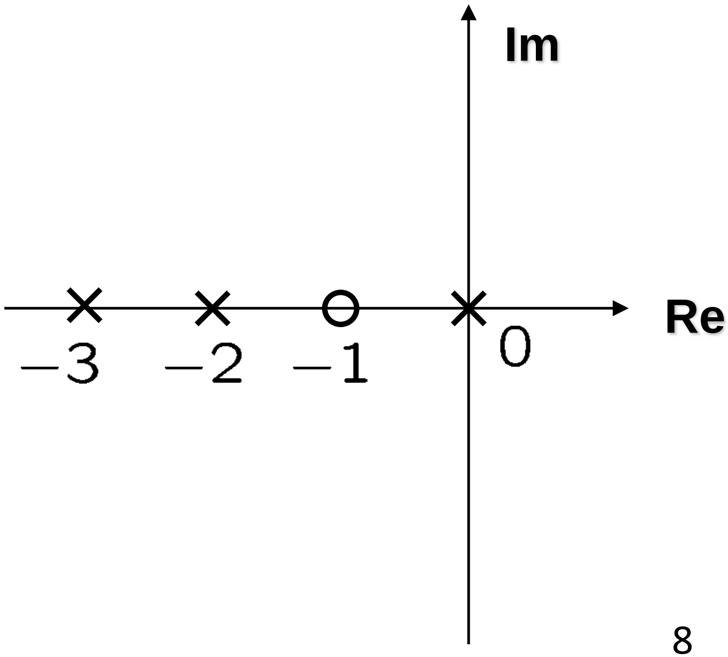
Example 3 (cont'd): Root locus sketching

- Step 0: Mark open-loop poles and zeros
- Step 1: On the real axis
- Step 2: Asymptotes
- Step 3: Breakaway points
- Step 4: Angles of departures and arrivals

Example 3 (cont'd)
Root locus: Step 0

- *Root locus is symmetric w.r.t. the real axis.*
- *Mark poles of L(s) with "x" and zeros of L(s) with "o".*
- *The number of branches that go to infinity = # of poles - # of zeros*
- *The number of branches = order of L(s)*

$$L(s) = \frac{s+1}{s(s+2)(s+3)}$$

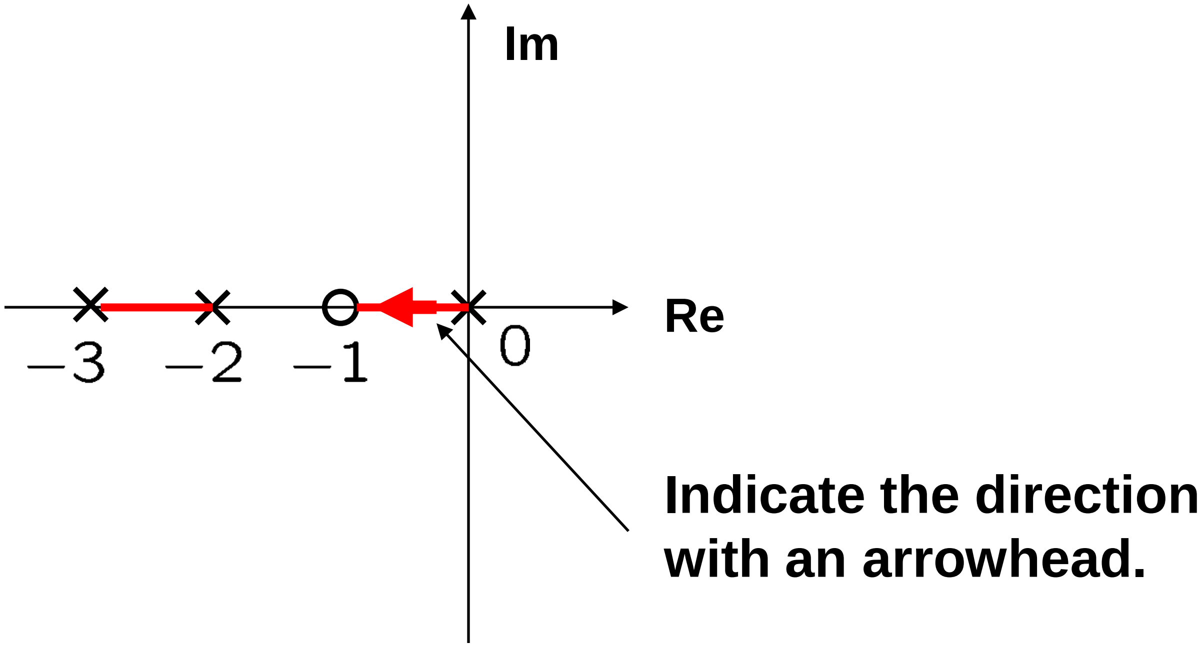


Example 3 (cont'd): Root locus sketching

- Step 0: Mark open-loop poles and zeros
- Step 1: On the real axis (also called real-axis segments)
- Step 2: Asymptotes
- Step 3: Breakaway points
- Step 4: Angles of departures and arrivals

Example 3 (cont'd)
Root locus: Step 1 (On the real axis)

- *RL includes all points on real axis to the* ***left of an odd number*** *of real poles/zeros.*
- *RL originates from the poles of L(s) and terminates at the zeros of L(s), including infinity zeros.*



Example 3 (cont'd): Root locus sketching

- Step 0: Mark open-loop poles and zeros
- Step 1: On the real axis
- Step 2: Asymptotes (i.e., lines to which root locus converges)
- Step 3: Breakaway points
- Step 4: Angles of departures and arrivals

Example 3 (cont'd)
Root locus: Step 2 (Asymptotes)

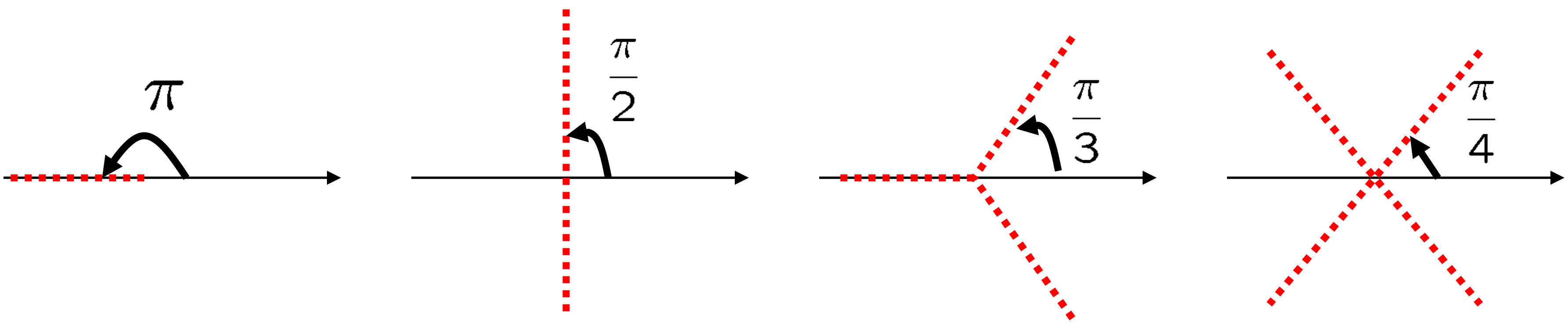
- *Number of asymptotes = relative degree (r) of L(s):*

$$r = \underbrace{n}_{\text{deg (den)}} - \underbrace{m}_{\text{deg (num)}}$$

- *Angles of asymptotes are*

$$\frac{\pi}{r} \times \underbrace{(2k+1)}_{\text{Odd number}}, \ k = 0, 1, \ldots, (r-1)$$

$r = 1$ $\quad r = 2$ $\quad r = 3$ $\quad r = 4$

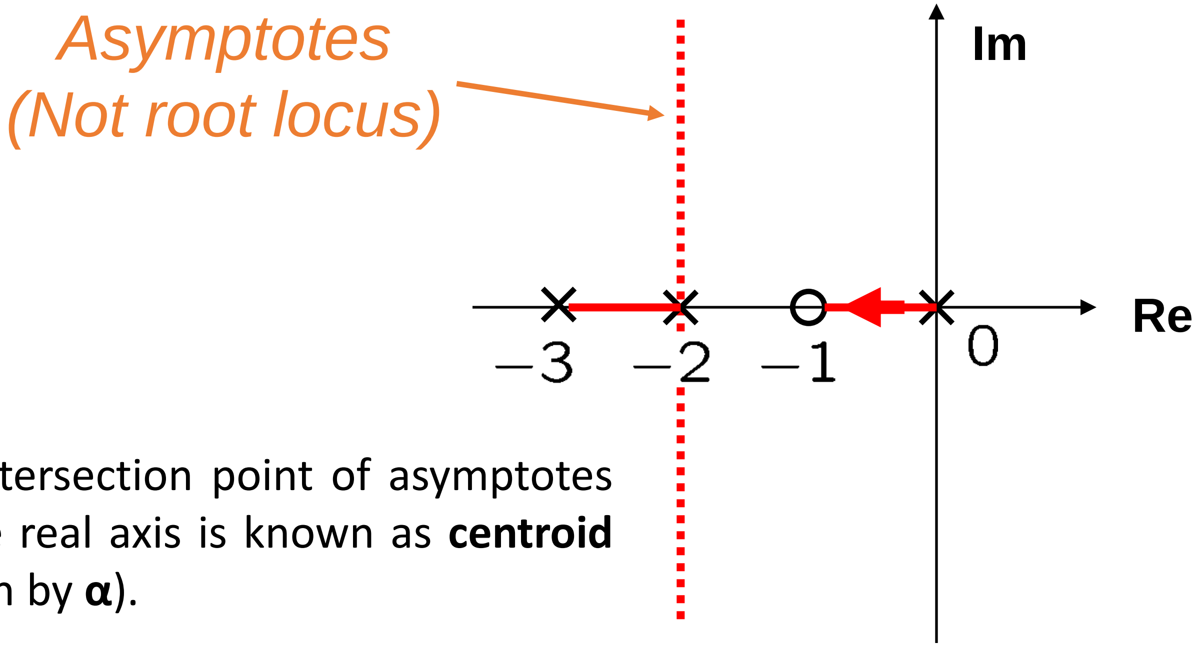


Example 3 (cont'd)
Root locus: Step 2 (Asymptotes)

- *Intersections of asymptotes* $\alpha = \dfrac{\sum \text{pole} - \sum \text{zero}}{r}$

$$L(s) = \frac{s+1}{s(s+2)(s+3)} \Rightarrow \frac{\sum \text{pole} - \sum \text{zero}}{r} = \frac{(0+(-2)+(-3))-(-1)}{2} = -2$$

$\alpha = -2$



The intersection point of asymptotes on the real axis is known as **centroid** (shown by **α**).

Example 3 (cont'd): Root locus sketching

- Step 0: Mark open-loop poles and zeros
- Step 1: On the real axis
- Step 2: Asymptotes
- Step 3: Breakaway points
- Step 4: Angles of departures and arrivals

Example 3 (cont'd)
Root locus: Step 3

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- *Breakaway points are among roots of* $\boxed{\dfrac{dL(s)}{ds} = 0}$

Points where two or more branches meet and break away.

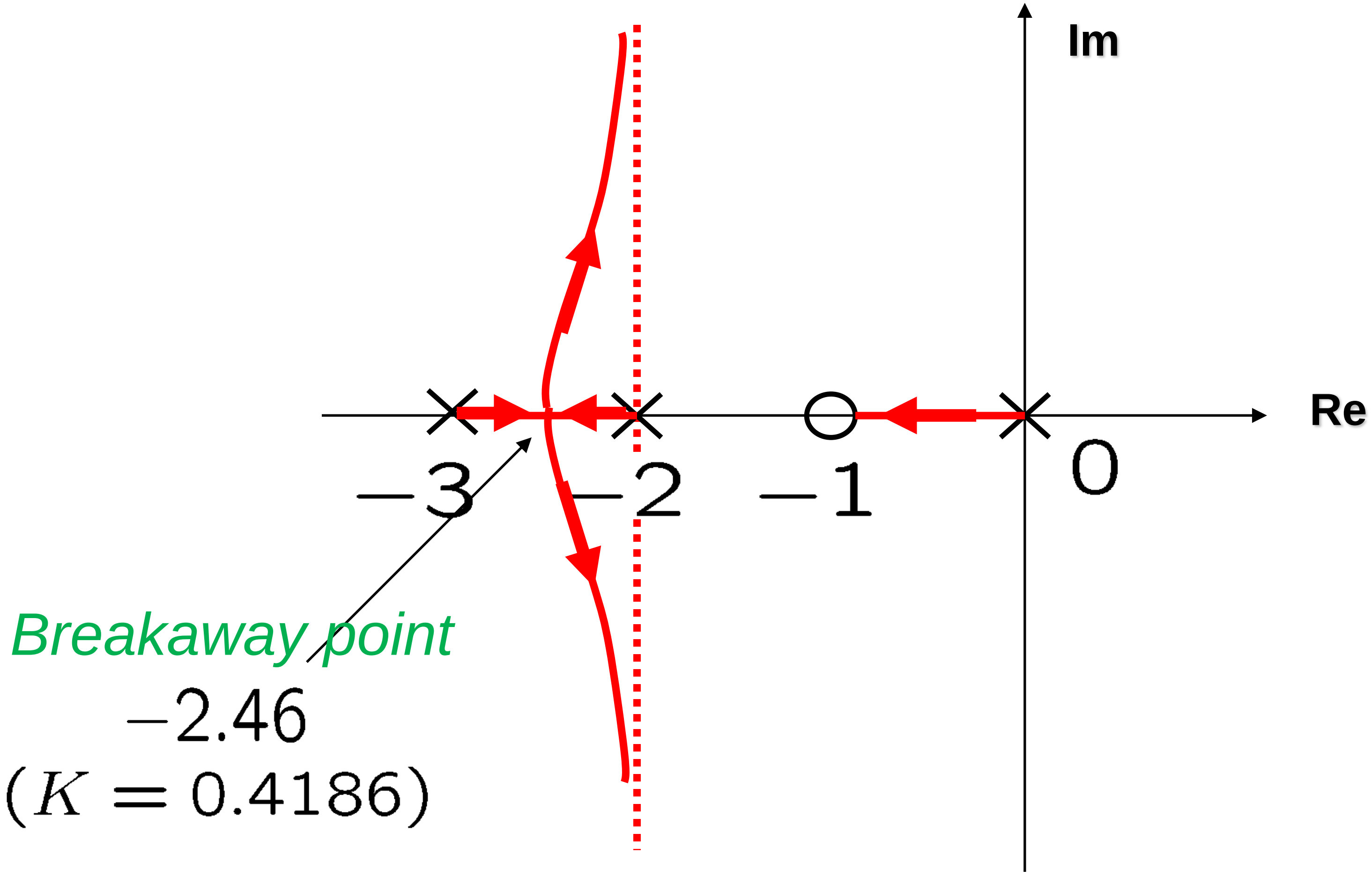
$$L(s) = \frac{s+1}{s(s+2)(s+3)} \Rightarrow \frac{dL(s)}{ds} = -2\frac{s^3+4s^2+5s+3}{(*)} = 0$$

$$\Rightarrow \quad s = -2.4656, \; -0.7672 \pm 0.7926i$$

For each candidate s, check the positivity of $K = -\dfrac{1}{L(s)}$

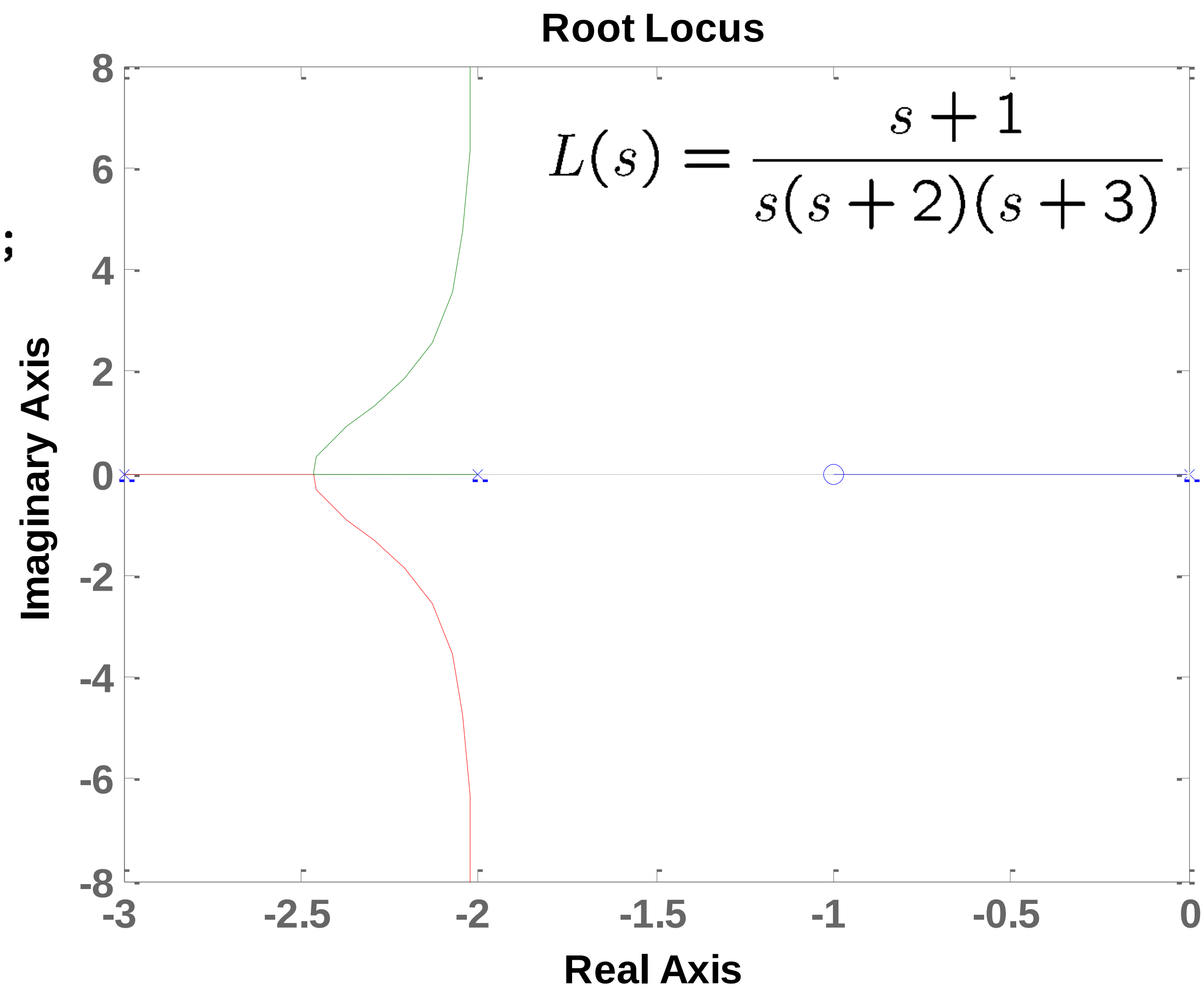
$$\Rightarrow \quad K = 0.4186, \; 1.7907 \mp 4.2772i$$

Example 3 (cont'd)
Root locus: Step 3



Example 3 (cont'd)
Matlab command "rlocus"

```
num=[1 1];
den=[1 5 6 0];
sys=tf(num,den);
rlocus(sys)
```



Example 3 (cont'd)
Root locus: Step 3

Find the number of root locus branches (as a prediction):

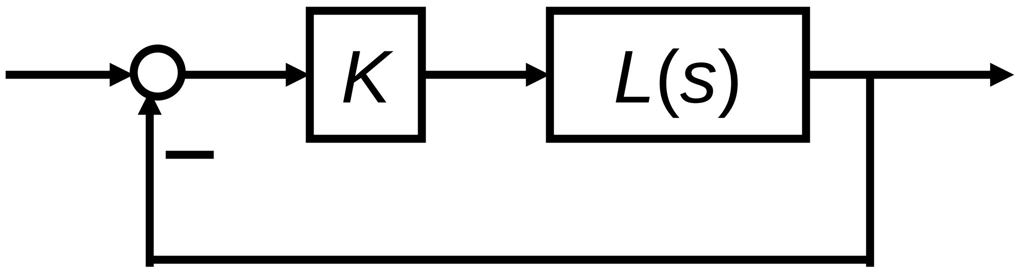
- We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches ***N*** is equal to the number of finite open loop poles ***P*** or the number of finite open loop zeros ***Z***, whichever is greater.
- Mathematically, we can write the number of root locus branches ***N*** as

$$N = P \text{ if } P \geq Z$$

$$N = Z \text{ if } P < Z$$

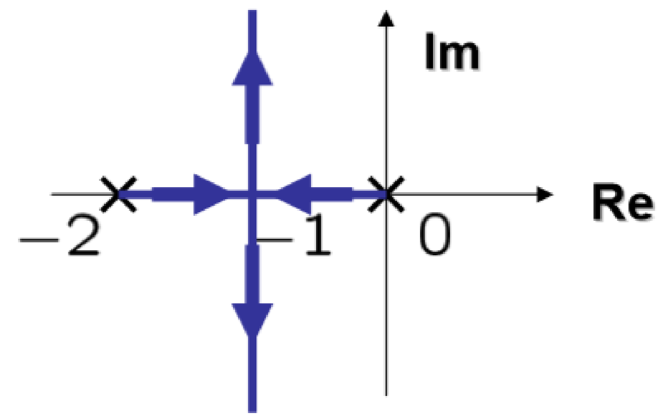
- In this example, ***P*** = 3, ***Z*** = 1.
- So, ***P*** ≥ ***Z*** → ***N*** = ***P*** = 3 = Number of branches.

Example 4



$$L(s) = \frac{1}{s(s+2)}$$

- Asymptotes
  - Relative degree: $r = 2$
  - Intersection: $\dfrac{0+(-2)}{2} = -1$



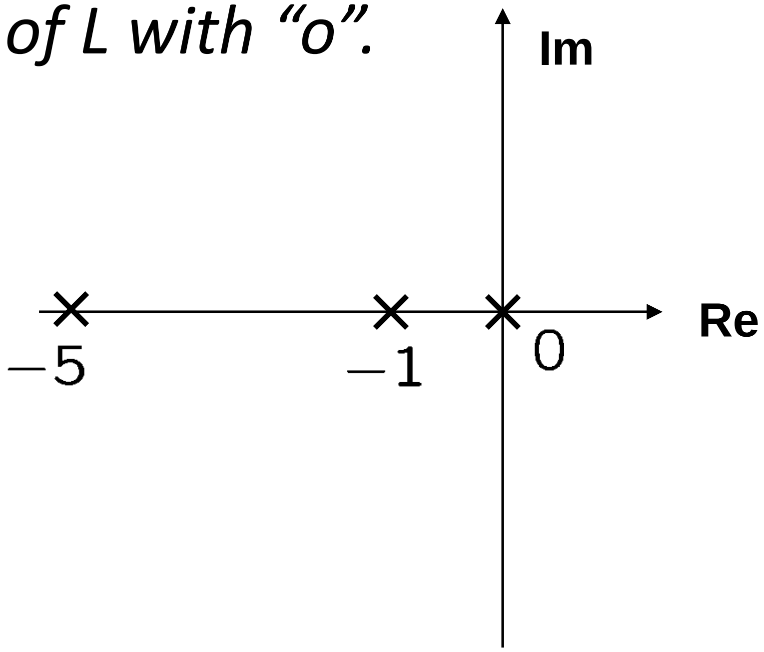
- Breakaway point

$$L'(s) = \frac{-(2s+2)}{*} = 0 \;\Rightarrow\; s = -1$$

Example 5
Root locus: Step 0

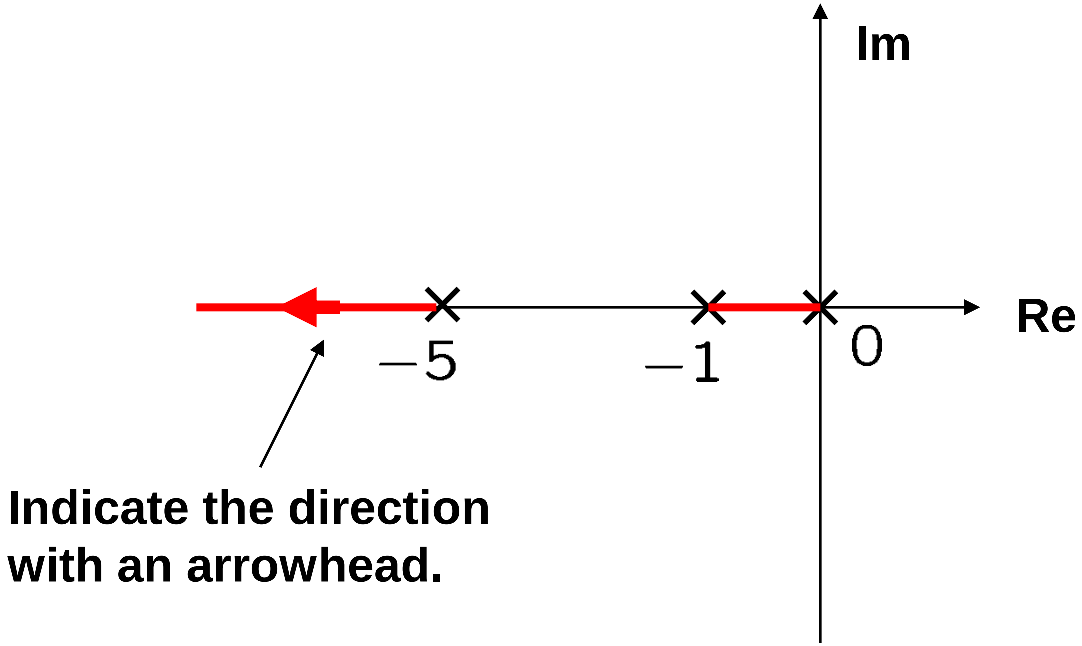
$$L(s) = \frac{1}{s(s+1)(s+5)}$$

- *Root locus is symmetric w.r.t. the real axis.*
- *The number of branches = order of L(s)*
- *Mark poles of L with "x" and zeros of L with "o".*



Example 5 (cont'd)
Root locus: Step 1 (Real axis)

- *RL includes all points on real axis to the left of an odd number of real poles/zeros.*
- *RL originates from the poles of L and terminates at the zeros of L, including infinity zeros.*



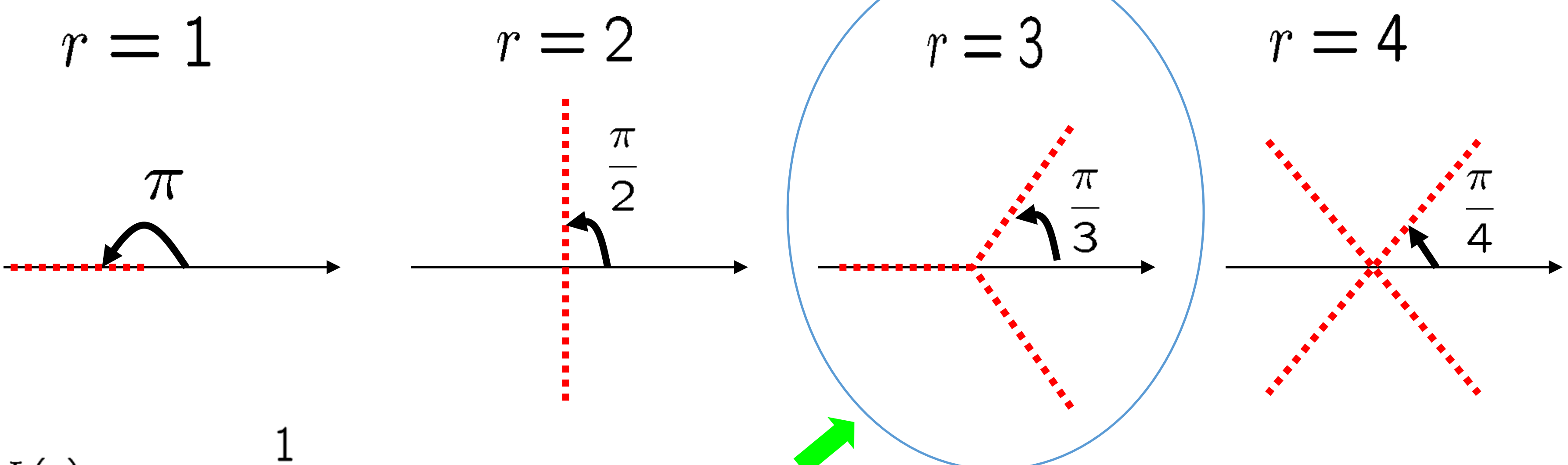
Example 5 (cont'd)
Root locus: Step 2 (Asymptotes)

- *Number of asymptotes = relative degree (r) of L:*

$$r = \text{deg}(\text{den}) - \text{deg}(\text{num})$$

- *Angles of asymptotes are*

$$\frac{\pi}{r} \times (2k+1), \;\; k = 0, 1, \ldots, (r-1)$$



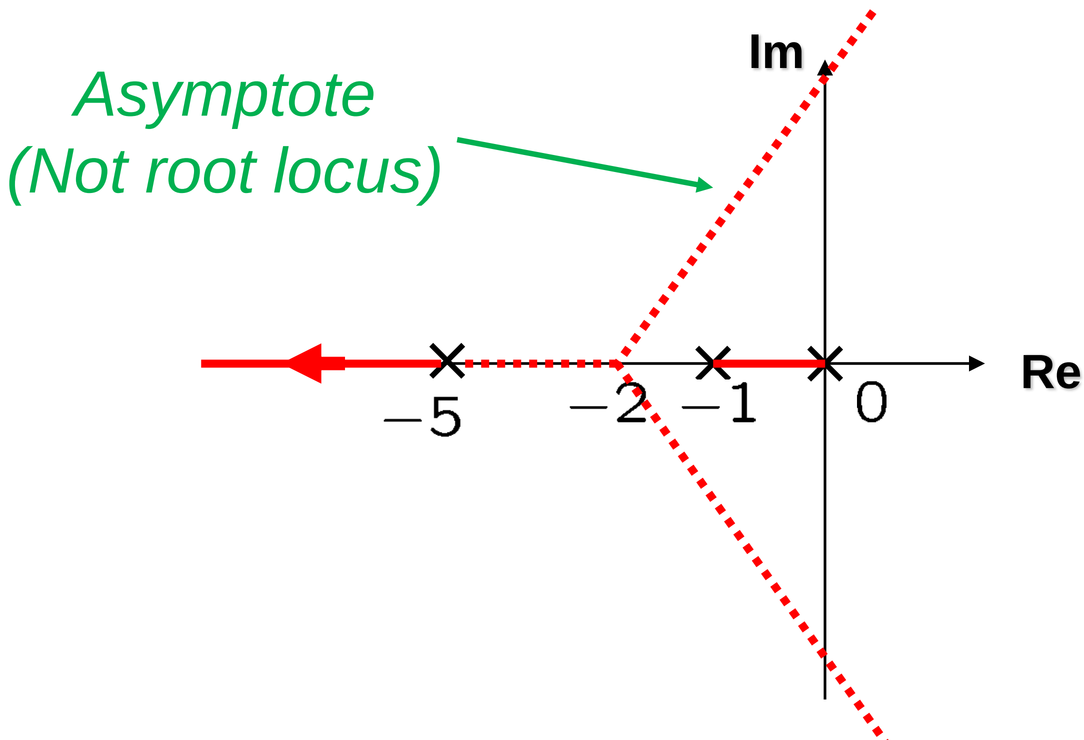
$$L(s) = \frac{1}{s(s+1)(s+5)}$$

$r = 3 - 0 = 3$

Example 5 (cont'd)
Root locus: Step 2 (Asymptotes)

- *Intersections of asymptotes* $\alpha = \dfrac{\sum \text{pole} - \sum \text{zero}}{r}$

$$L(s) = \frac{1}{s(s+1)(s+5)} \Rightarrow \alpha = \frac{\sum \text{pole} - \sum \text{zero}}{r} = \frac{0+(-1)+(-5)}{3} = -2$$



Example 5 (cont'd)
Root locus: Step 3 (Breakaway)

- *Breakaway points are among roots of* $\boxed{\frac{dL(s)}{ds} = 0}$

$$L(s) = \frac{1}{s(s+1)(s+5)} \Rightarrow \frac{dL(s)}{ds} = -\frac{3s^2 + 12s + 5}{(*)} = 0$$

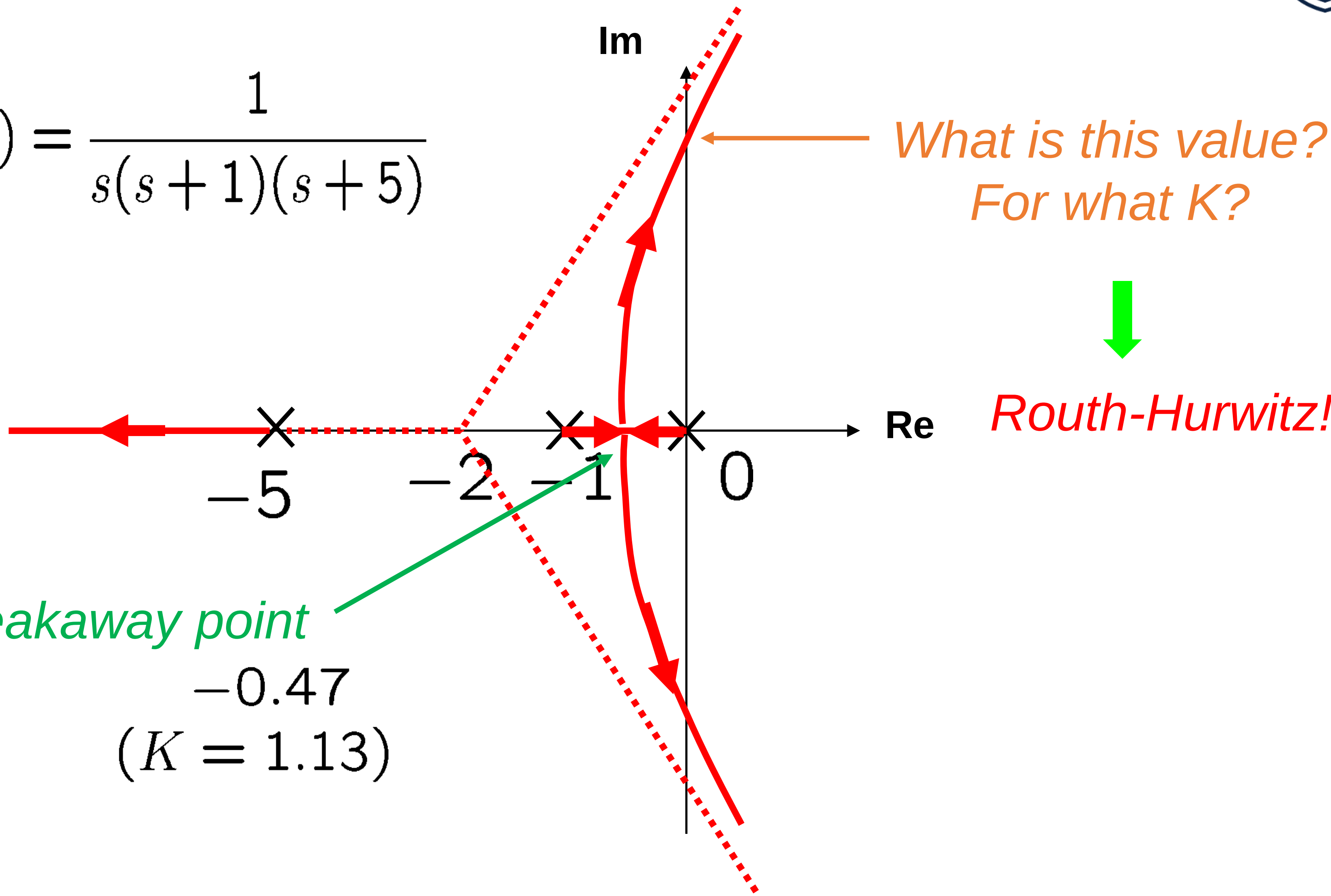
$$\Rightarrow s = -2 \pm \frac{\sqrt{21}}{3}$$

For each candidate s, check the positivity of $K = -\frac{1}{L(s)}$

$$\Rightarrow \begin{cases} s = -2 + \frac{\sqrt{21}}{3} \approx -0.47 & K \approx 1.13 \\ s = -2 - \frac{\sqrt{21}}{3} \approx -3.52 & K \approx -13.1 \end{cases}$$

Example 5 (cont'd)
Root locus: Step 3 (Breakaway)

$$L(s) = \frac{1}{s(s+1)(s+5)}$$



Example 5 (cont'd)
Finding K for marginal/critical stability

- Characteristic equation

$$1+\frac{K}{s(s+1)(s+5)}=0 \Leftrightarrow s^3+6s^2+5s+K=0$$

- Routh array

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 6 & K \\ s^1 & \frac{30-K}{6} & \\ s^0 & K & \end{array}$$

Stability condition

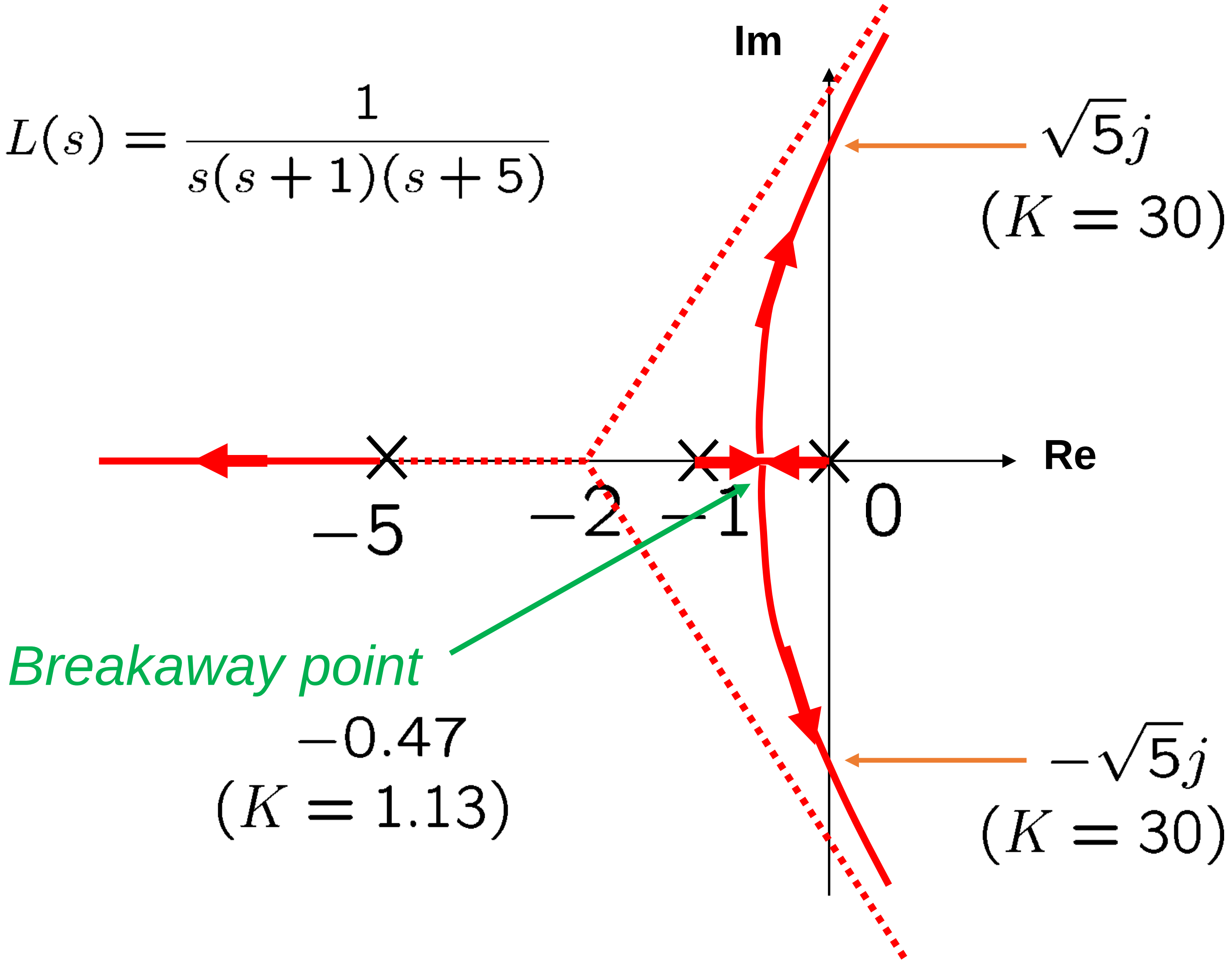
$$0 < K < 30$$

- When K = 30

$$6s^2+30=0 \Rightarrow s=\pm\sqrt{5}j$$

Example 5 (cont'd)
Root locus

$$L(s) = \frac{1}{s(s+1)(s+5)}$$



Summary

- Root locus
  - What is root locus
  - How to roughly and quickly sketch root locus
- Sketching root locus relies heavily on experience. Please practice!
- To accurately draw root locus, use Matlab.
- Next
  - Root locus examples