a place of mind

ELEC 341: Systems and Control

Lecture 6

Stability:
Routh-Hurwitz stability criterion

a place of mind UBC

Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - Electrical
  - Electromechanical
  - Mechanical
- Linearization, delay

Analysis

- Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

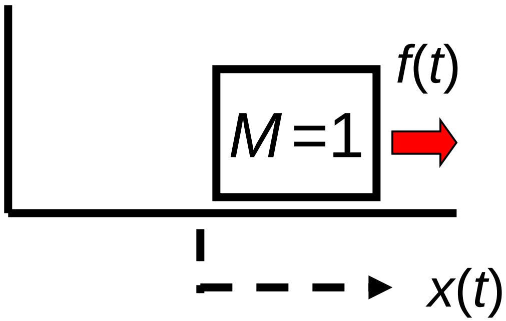
Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

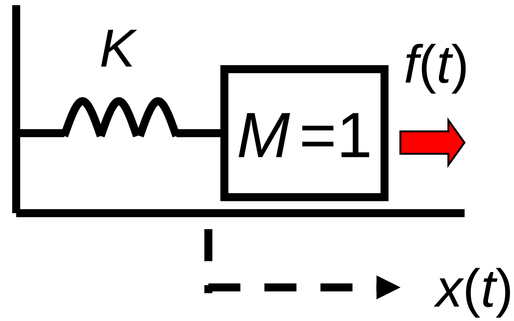
Matlab simulations

Some examples

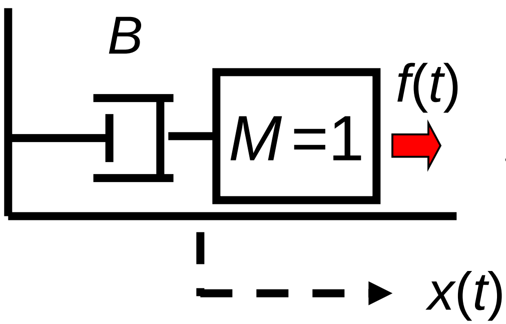
- We want the mass to stay at x = 0, but wind causes the mass to move. What will happen?



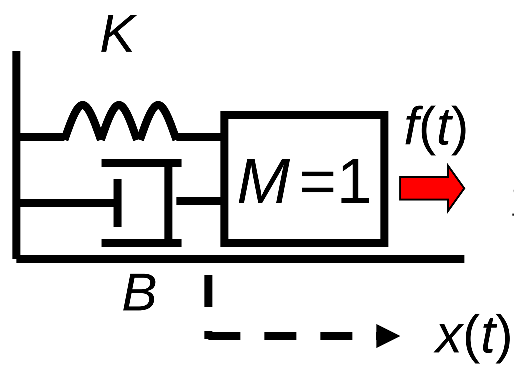
$$\frac{X(s)}{F(s)} = \frac{1}{s^2}$$



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + K}$$



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs}$$



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs + K}$$

- How to characterize different behaviors with TF?

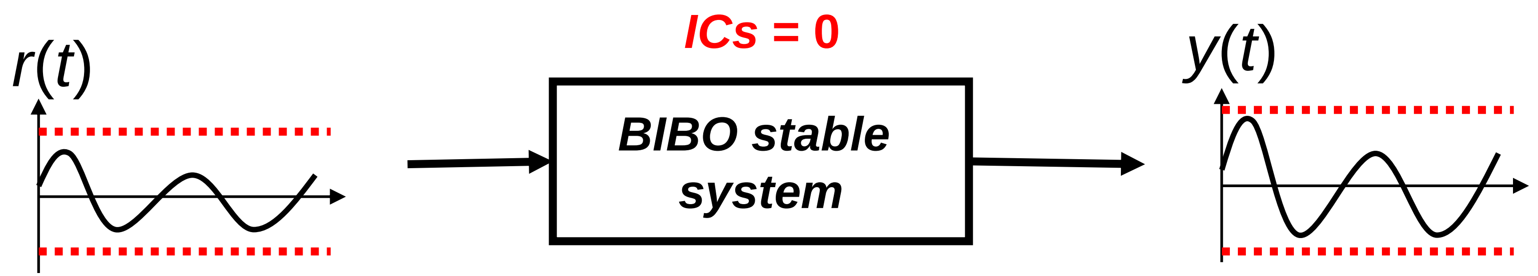
Stability

- Utmost important specification in control design!
- Unstable systems have to be stabilized by feedback.
- Unstable closed-loop systems are useless.
- What if a system is unstable? (“out-of-control”)
  - It may hit electrical/mechanical “stops” (saturation).
  - It may break down or burn out.
  - Signals diverge.
- Examples of unstable systems
  - Tacoma Narrows Bridge collapse in 1940
  - SAAB Gripen JAS-39 prototype accident in 1989
  - Wind turbine explosion in Denmark in 2008

Definitions of stability

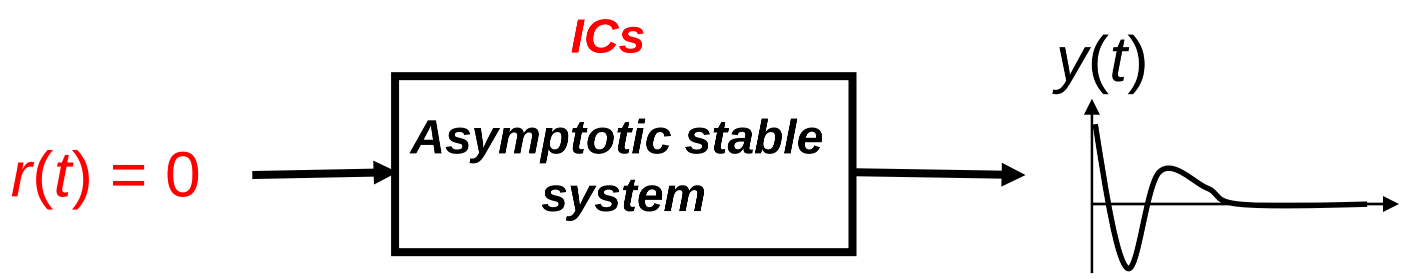
- **BIBO (Bounded-Input-Bounded-Output) stability**

  Any bounded input generates a bounded output.



- **Asymptotic stability**

  Any ICs generates y(t) converging to zero.



Some terminologies

$$G(s) = \frac{n(s)}{d(s)}$$

Ex. $$G(s) = \frac{(s-1)(s+1)}{(s+2)(s^2+1)}$$

- **Zero:** roots of $n(s)$ — (Zeros of G) $= \pm 1$
- **Pole:** roots of $d(s)$ — (Poles of G) $= -2, \pm j$
- **Characteristic polynomial:** $d(s)$
- **Characteristic equation:** $d(s) = 0$

Stability condition in *s*-domain

- For a system represented by transfer function $G(s)$,

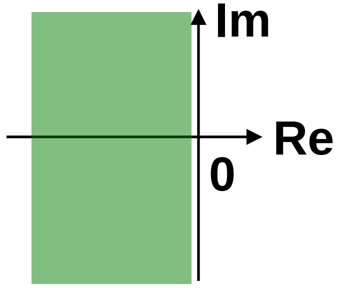
System is BIBO stable

$\Updownarrow$

All the poles of G(s) are in the open left half of the complex plane.

$\Updownarrow$

System is asymptotically stable



Idea of stability condition

- Example $y'(t) + \alpha y(t) = r(t)$

$$\Rightarrow \quad sY(s) - y(0) + \alpha Y(s) = R(s)$$

$$\Rightarrow \quad Y(s) = \frac{1}{s+\alpha}(R(s) + y(0))$$

Asymptotic Stability: $(r(t) = R(s) = 0)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}y(0)\right\} = e^{-\alpha t}y(0) \to 0 \Leftrightarrow \quad (-\alpha) < 0$$

BIBO Stability: $(y(0) = 0)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)R(s)\} = \int_0^t g(\tau)r(t-\tau)d\tau = \int_0^t e^{-\alpha\tau}r(t-\tau)d\tau$$

$$|y(t)| \le \int_0^t |e^{-\alpha\tau}||r(t-\tau)|d\tau \le \underline{\int_0^t |e^{-\alpha\tau}|d\tau} \cdot r_{max}$$

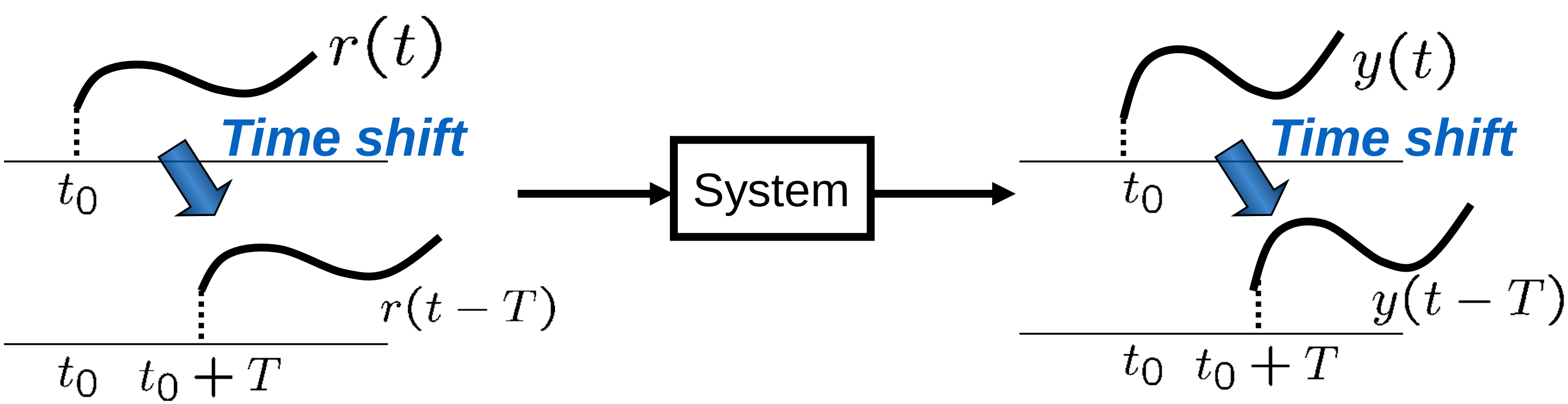
Bounded if $(-\alpha) < 0$

Remarks on stability

- For general systems (nonlinear, time-varying), BIBO stability condition and asymptotic stability condition are different.
- For linear time-invariant (LTI) systems (for which we can use Laplace transform and we can obtain transfer functions), these two conditions happen to be the same.
- In this course, since we are interested in only LTI systems, we simply use “stable” to mean both BIBO and asymptotic stability.

Time-invariant & time-varying

- A system is called *time-invariant* if system parameters do not change in time. If they do, it is called *time-varying*.
- Example: $Mx''(t) = f(t)$ is time-invariant while $M(t)x''(t) = f(t)$ is time-varying.
- For time-invariant systems:



- This course deals with time-invariant systems.

Remarks on stability (cont'd)

- Marginally stable if
  - $G(s)$ has no pole in the open RHP (Right Half Plane), and
  - $G(s)$ has at least one simple pole on $j\omega$-axis, and
  - $G(s)$ has no multiple pole on $j\omega$-axis.

$$G(s) = \frac{1}{s(s^2+4)(s+1)^2}$$

Marginally stable

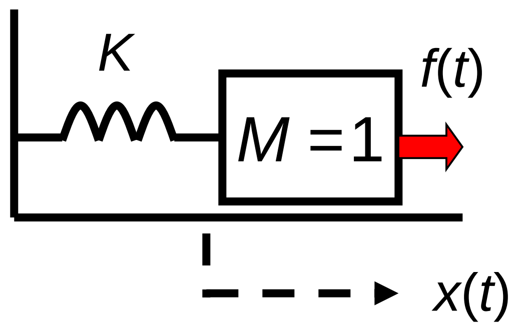
$$G(s) = \frac{1}{s(s^2+4)^2(s+1)^2}$$

NOT marginally stable

- Unstable if a system is neither stable nor marginally stable.

Note: A simple pole is a pole of order one (i.e., a non-repeating pole).

"Marginally stable" in *t*-domain



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + K}$$

- For any bounded input, except only special sinusoidal (bounded) inputs, the output is bounded.
  - In the example above, the special inputs are in the form of:

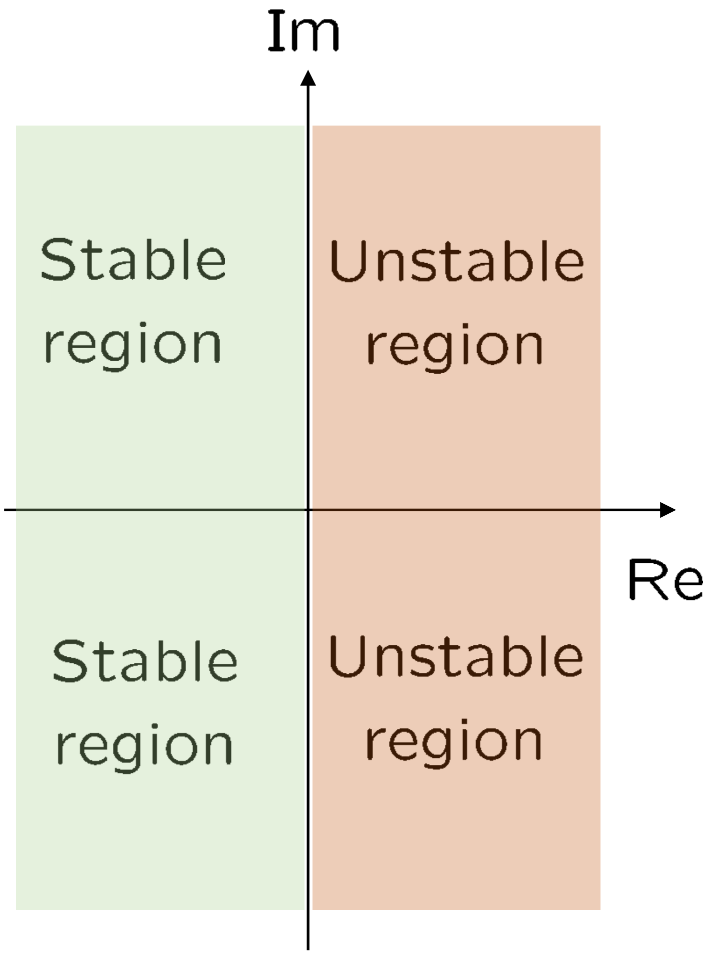
$$f(t) = \alpha \sin \sqrt{K} t + \beta \cos \sqrt{K} t \quad \Rightarrow \quad x(t) \to \pm\infty$$

Stability summary

Let s_i be poles of $G(s)$.
Then, $G(s)$ is …

- (BIBO, asymptotically) stable if

  $Re(s_i) < 0$ for all i.
- marginally stable if
  - $Re(s_i) \leq 0$ for all i, and
  - at least one simple pole for $Re(s_i) = 0$
- unstable if it is neither stable nor marginally stable.

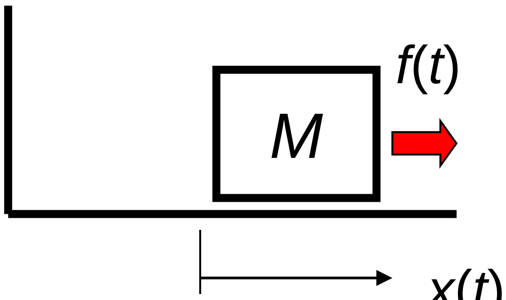


Re-axis is sometimes shown by $\boldsymbol{\sigma}$ or $\boldsymbol{\delta}$ while **Im-axis** is shown by $\boldsymbol{j\omega}$.

Examples: revisited

(Let *B* = 2 and *K* = 3)

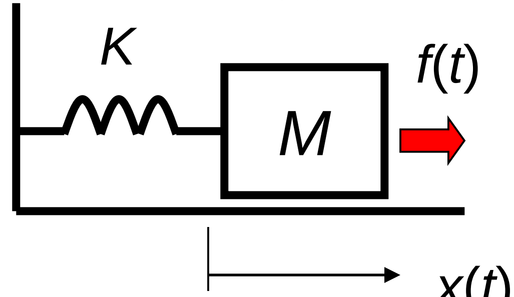
(a)



$$\frac{X(s)}{F(s)} = \frac{1}{s^2}$$

Poles ?
Stable ?

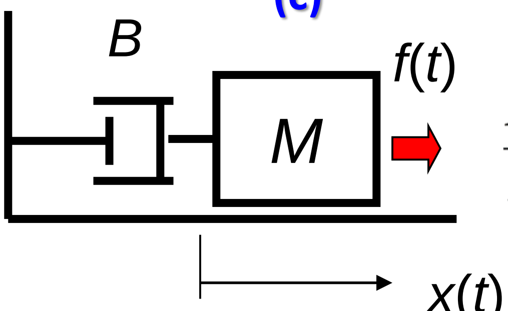
(b)



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + K}$$

Poles ?
Stable ?

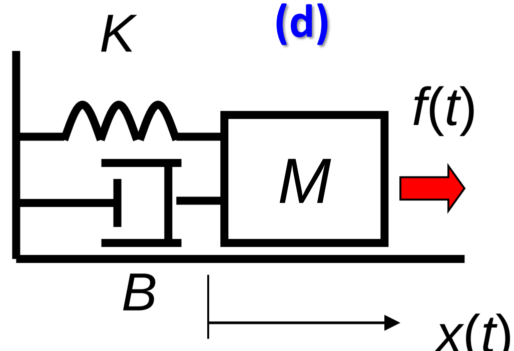
(c)



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs}$$

Poles ?
Stable ?

(d)



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs + K}$$

Poles ?
Stable ?

Examples

	$G(s)$	Stable/Marginally Stable/Unstable
(1)	$\dfrac{20}{(s+1)(s+2)(s+3)}$	**?**
(2)	$\dfrac{20(s+1)}{(s-1)(s^2+2s+3)}$	**?**
(3)	$\dfrac{1}{(s+5)(s^2+2)^2}$	**?**
(4)	$\dfrac{1}{s^4+5s^3+10s^2+3s+1}$	**?**

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

- ✓ Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

Routh-Hurwitz criterion

- This is for LTI systems with a *polynomial* denominator (without sine, cosine, exponential, etc.)
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does NOT explicitly compute the roots numerical values.
- Consider a polynomial

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Routh array

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

From the given polynomial

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Routh array (How to compute the third row)

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

$$
\begin{aligned}
b_1 &= \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}} \\
b_2 &= \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}} \\
&\vdots
\end{aligned}
$$

Routh array (How to compute the fourth row)

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

$$
\begin{aligned}
c_1 &= \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1} \\
c_2 &= \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1} \\
&\vdots
\end{aligned}
$$

Routh-Hurwitz criterion

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

The number of roots
in the open right half-plane
is equal to
the number of sign changes
in the first column of Routh array.

a place of mind

Example 1

$$Q(s) = s^3 + s^2 + 2s + 8 \;= (s+2)(s^2 - s + 4)$$

Routh array

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 8 \\ s^1 & -6 & \\ s^0 & 8 & \end{array}$$

$$\frac{2-8}{1}$$

$$\frac{8\times(-6)-1\times 0}{-6}$$

Two sign changes
in the first column
$1 \rightarrow -6 \rightarrow 8$

Two roots in RHP

$$\frac{1}{2} \pm \frac{j\sqrt{15}}{2}$$

Example 2

$$Q(s) = s^3 + 3s^2 + 6s + 8 = (s+2)(s^2+s+4)$$

Routh array

$$\begin{array}{c|cc} s^3 & 1 & 6 \\ s^2 & 3 & 8 \\ s^1 & 10/3 & \\ s^0 & 8 & \end{array} \qquad \frac{6\times 3 - 8}{3}$$

Always same!

No sign changes
in the first column
$1 \rightarrow 3 \rightarrow 10/3 \rightarrow 8$

No roots in RHP

$$-2, -\frac{1}{2} \pm \frac{j\sqrt{15}}{2}$$

Example 3 (from slide 15)

$$Q(s) = s^4 + 5s^3 + 10s^2 + 3s + 1$$

Routh array

$$\begin{array}{c|ccc} s^4 & 1 & 10 & 1 \\ s^3 & 5 & 3 & \\ s^2 & 9.4 & 1 & \\ s^1 & 2.468 & & \\ s^0 & 1 & & \end{array}$$

Always same!

No sign changes
in the first column → No roots in RHP

a place of mind

Simple important criteria for stability

- 1st order polynomial $Q(s) = a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ a_1 and a_0 have the same sign

- 2nd order polynomial $Q(s) = a_2 s^2 + a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ a_2, a_1 and a_0 have the same sign

- Higher order polynomial $Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$

  All roots are in LHP $\Rightarrow$ All a_k have the same sign

a place of mind

Examples

	$Q(s)$	All roots in open LHP?
(1)	$3s+5$	Yes / No
(2)	$-2s^2-5s-100$	Yes / No
(3)	$523s^2-57s+189$	Yes / No
(4)	$s^4+2s^3+s^2-1$	Yes / No
(5)	$s^3+5s^2+10s-3$	Yes / No

Summary

- Stability for LTI systems
  - (BIBO, asymptotically) stable, marginally stable, unstable
  - Stability for $G(s)$ is determined by poles of $G(s)$.
- Routh-Hurwitz stability criterion
  - To determine stability without explicitly computing the poles of a system.
- Next
  - Examples of Routh-Hurwitz criterion.