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ELEC 341: Systems and Control

Lecture 9

Step responses of 1st and 2nd order systems

Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - Electrical
  - Electromechanical
  - Mechanical
- Linearization, delay

Analysis

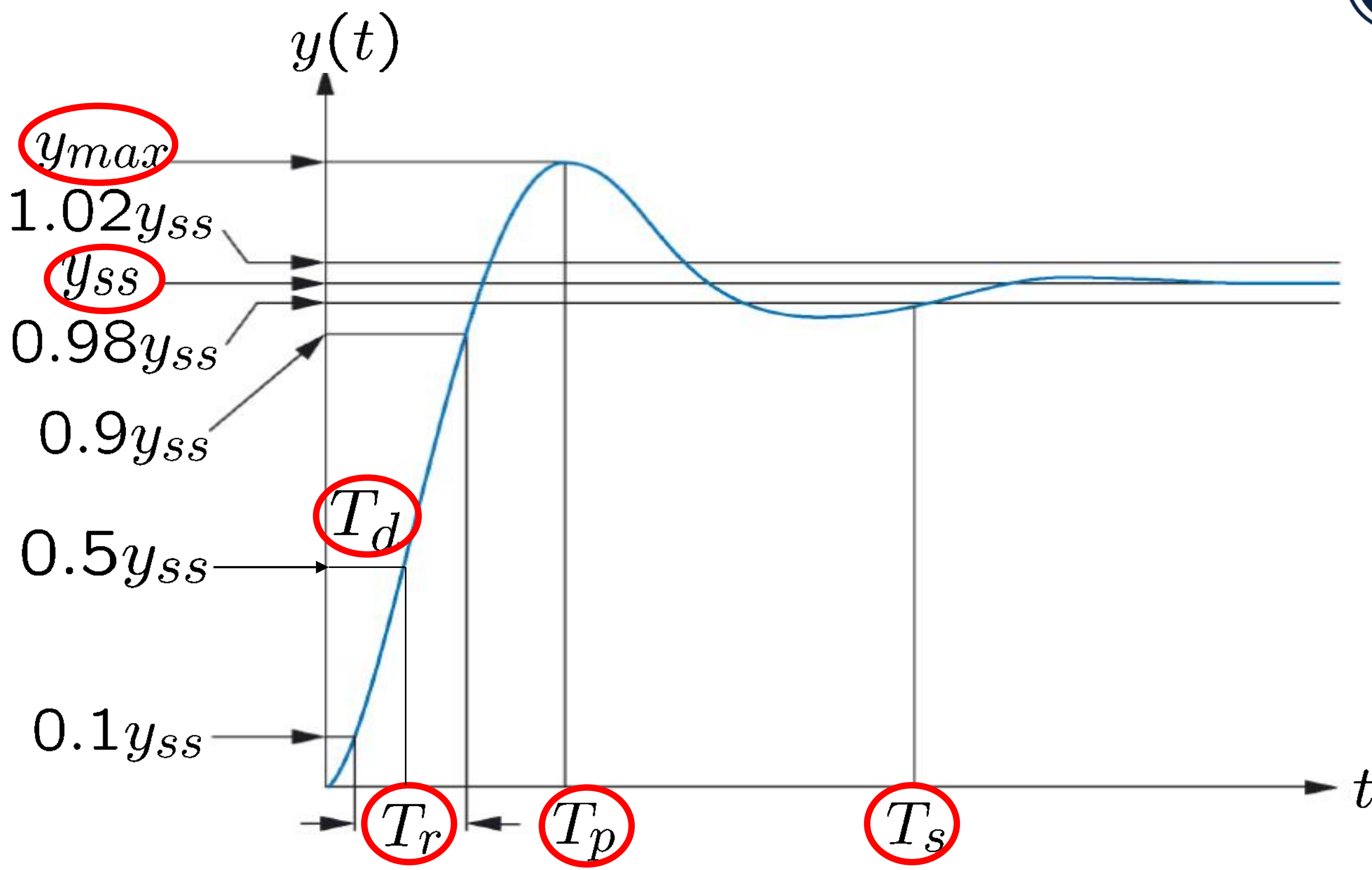
- Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

Typical step response (review)



Performance measures

- Transient response
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time
- Steady state response
  - Steady state error

(Today's lecture)

Next, we will connect these performance measures with s-domain.

(Done)

Today's topics

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- Characterization of step responses (**performance measures**) for
  - 1st-order system

$$G(s) = \frac{K}{Ts+1}$$

  - 2nd-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

in terms of

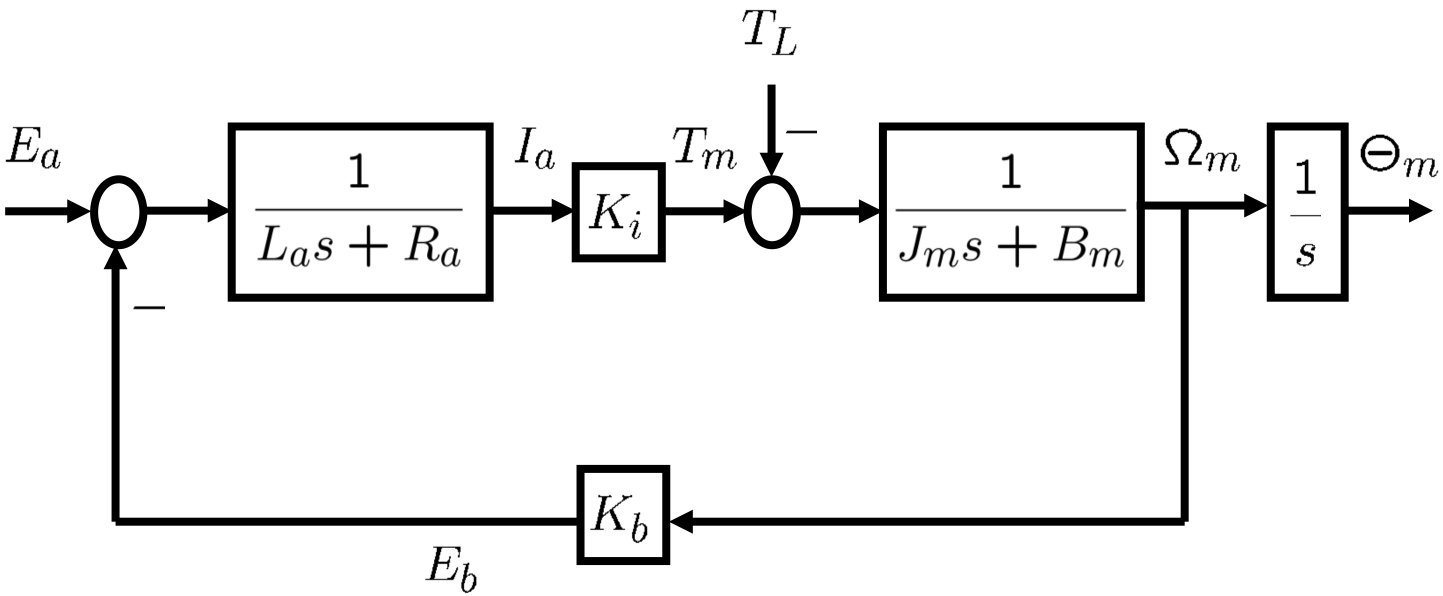
  - System parameters (K, T), (ζ, ω_n)
  - Pole locations

First-order system

- A standard form of the first-order system:

$$G(s) = \frac{K}{Ts+1}$$

- DC motor example (See L5)



DC motor example (cont'd)

- If $L_a << R_a$, we can obtain a 1st-order system

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{K_i}{(L_a s + R_a)(J_m s + B_m) + K_b K_i} \approx \frac{K_i}{R_a(J_m s + B_m) + K_b K_i}$$

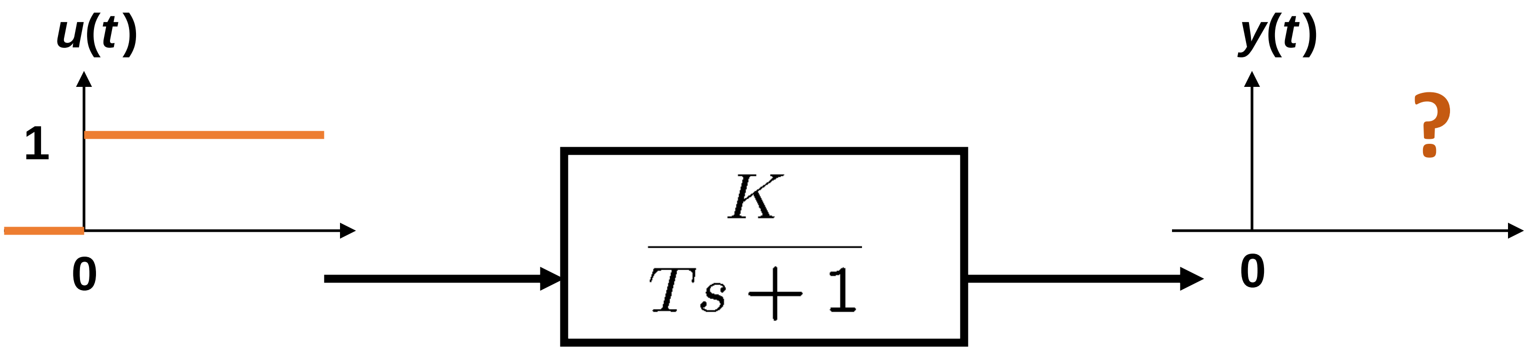
$$= \frac{K}{Ts+1} \quad \left(K = \frac{K_i}{R_a B_m + K_b K_i}, \; T = \frac{R_a J_m}{R_a B_m + K_b K_i} \right)$$

2nd order system → *1st order system*

- Remember that TF from motor voltage to
  - motor speed is 1st-order
  - motor position is 2nd-order

Step response of 1st-order system

- Input a unit step function to a first-order system. Then, what is the output?



$$
\begin{aligned}
Y(s) &= G(s)U(s) \\
&= \frac{K/T}{s+1/T} \cdot \frac{1}{s} \\
&= \frac{K}{s} + \frac{-K}{s+1/T}
\end{aligned}
$$

(Partial fraction expansion)

$$\xrightarrow{\mathcal{L}^{-1}}$$

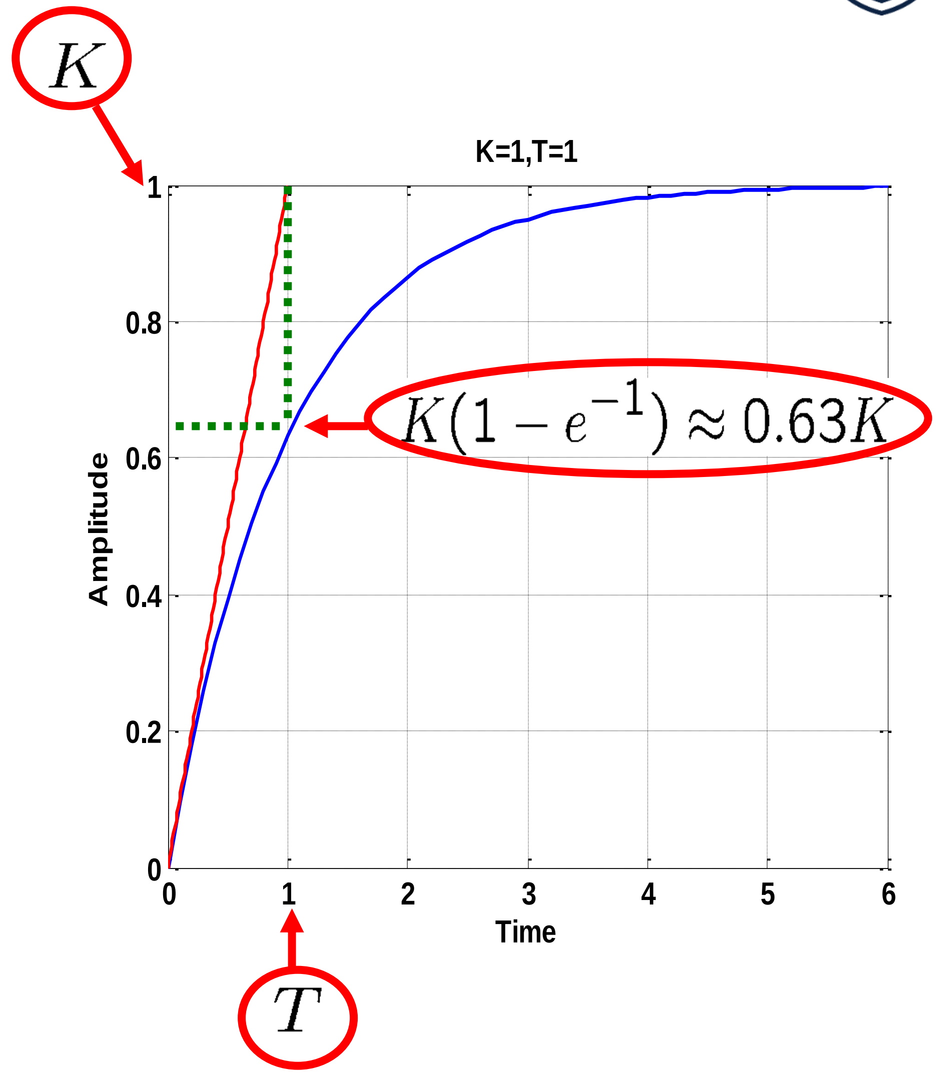
$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \underline{K(1-e^{-t/T})} \\
&\quad (t > 0)
\end{aligned}
$$

Meaning of K and T

- K : DC gain (next slide)
  - Final (steady-state) value

$$\lim_{t\to\infty} y(t) = K$$

- T : Time constant
  - Time when response rises to 63% of final value
  - Indication of speed of response (convergence)
  - Response is faster as T (also shown by τ) becomes smaller.



DC gain for a stable system

- DC gain: Final value of a unit step response
  - For a stable system G, DC gain is $G(0)$.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0)$$

Final value theorem

- Examples:

$$G(s) = \frac{3}{2s+5} \longrightarrow G(0) = \frac{3}{5}$$

$$G(s) = \frac{7}{s^2+2s+3} \longrightarrow G(0) = \frac{7}{3}$$

Settling time of 1st-order systems

$$y(t) = K(1 - e^{-t/T})$$

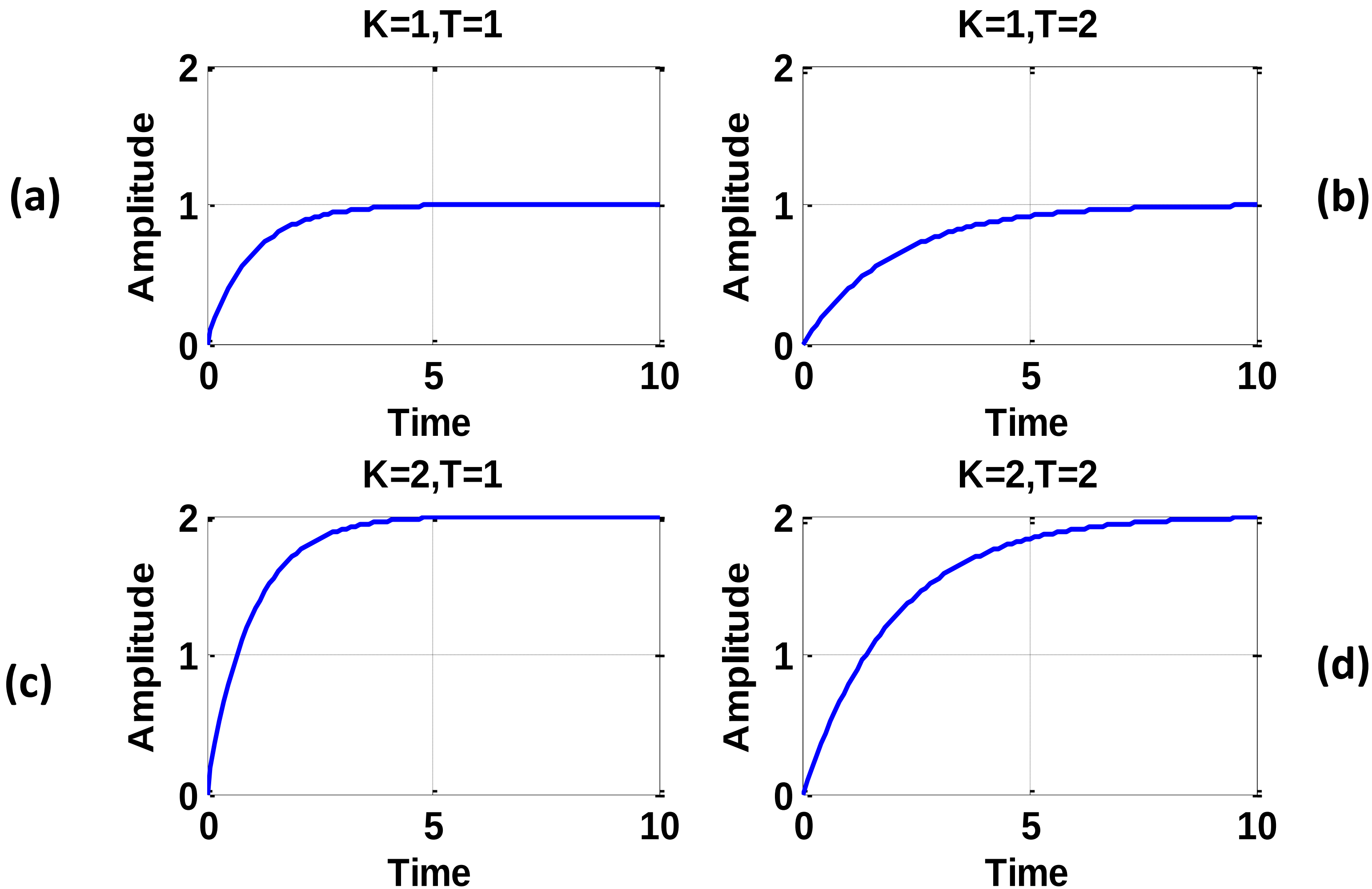
- Relation between time and exponential decay

t	$e^{-t/T}$
0	1
T	0.3679
$2T$	0.1353
$\boxed{3T}$	$\boxed{0.0498}$
$\boxed{4T}$	$\boxed{0.0183}$
$5T$	0.0067

5% settling time is about 3T

2% settling time is about 4T

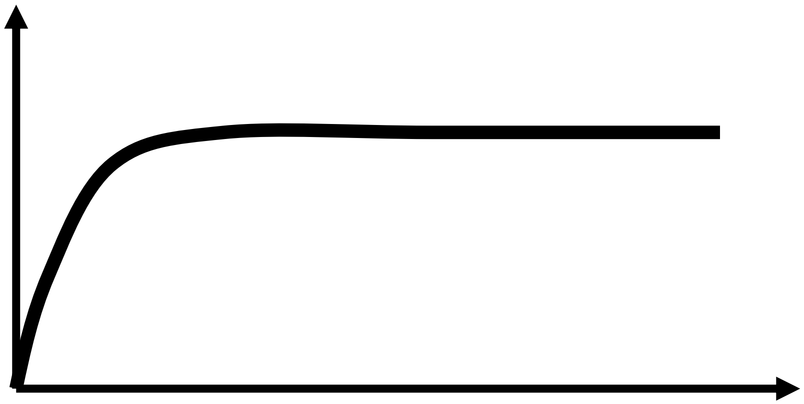
Step response for some K & T



System identification

- Suppose that we have a “black-box” system:

- Obtain step response:

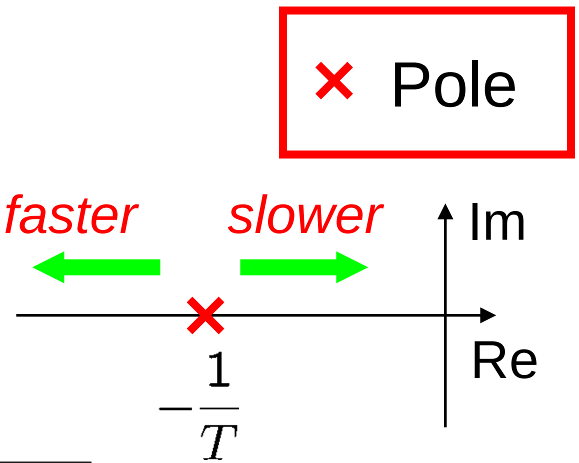


- Can you obtain a transfer function? How?

Summary:
Step response of 1st order systems

$$G(s) = \frac{K}{Ts+1}$$

- For 1st order systems, step responses have:
  - Steady-state value: K
  - Peak value, peak time, percent overshoot: undefined
  - Delay time: 0.7T
  - Rise time: 2.2T
  - Settling time
    - 2%: 4T
    - 5%: 3T
  - Characterization in terms of poles



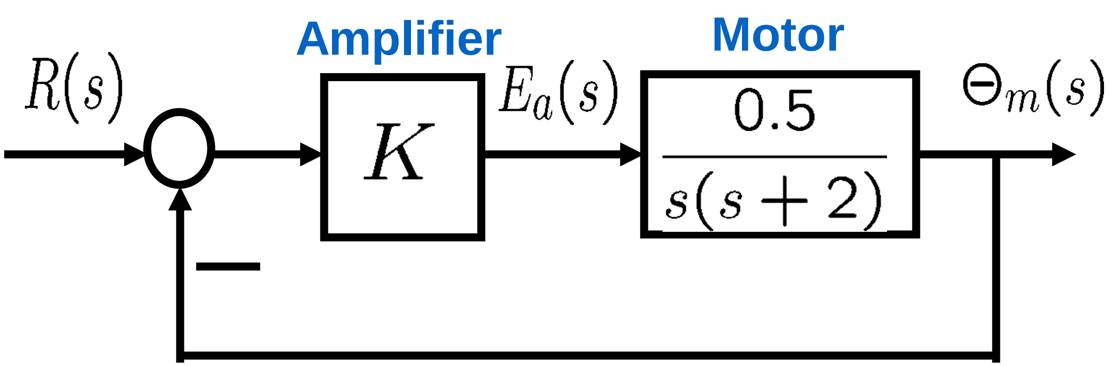
$$|\text{Real part of the pole}| = \frac{1}{T} \longrightarrow T = \frac{1}{|\text{Real part of the pole}|}$$

Second-order systems

- A standard form of the second-order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \begin{cases} \zeta : & \text{damping ratio} \\ \omega_n : & \text{undamped natural frequency} \end{cases}$$

- DC motor position control example:



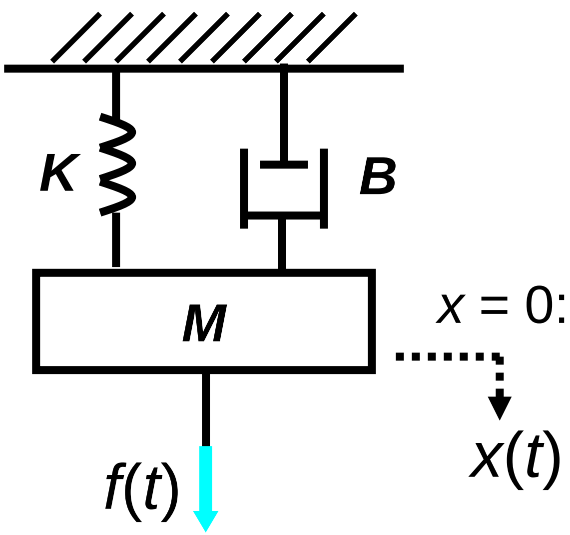
Closed-loop TF (CLTF):

$$\frac{\Theta_m(s)}{R(s)} = \frac{0.5K}{s^2 + 2s + 0.5K}$$

Second-order system (Mechanical Example)

- Mass spring damper system (L4):

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$
$$= \frac{1}{K} \cdot \frac{1}{(M/K)s^2 + (B/K)s + 1}$$
$$= \frac{1}{K} \cdot \frac{(K/M)}{s^2 + (B/M)s + (K/M)}$$
$$= \frac{1}{K} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

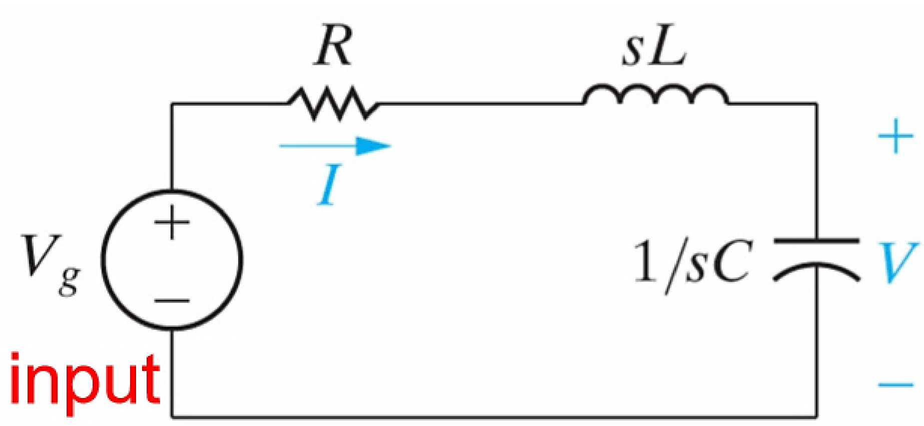


x = 0: Static equilibrium

$$\zeta = \frac{B}{2\sqrt{KM}}, \quad \omega_n = \sqrt{\frac{K}{M}}$$

Second-order system (Electrical Example)

- Series RLC circuit system:

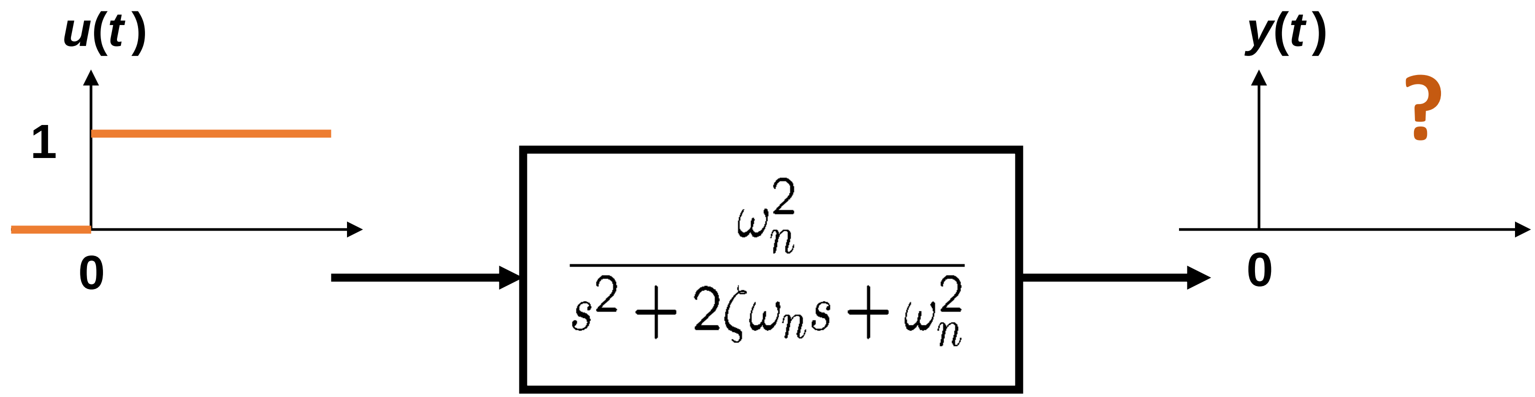


- If the output is the capacitor voltage (V):

$$\frac{V}{V_g} = \frac{(sC)^{-1}}{R + sL + (sC)^{-1}} = \frac{1}{s^2LC + sRC + 1}$$

Step response of 2nd-order system

- Input a unit step function to a 2nd-order system. What is the output?



$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0)$$

Final value theorem

DC gain

$$\lim_{t\to\infty} y(t) = G(0) = 1 \text{ if } G \text{ is stable}$$

Step response of 2nd-order system for various damping ratios

- Undamped

  $$\zeta = 0$$

- Underdamped

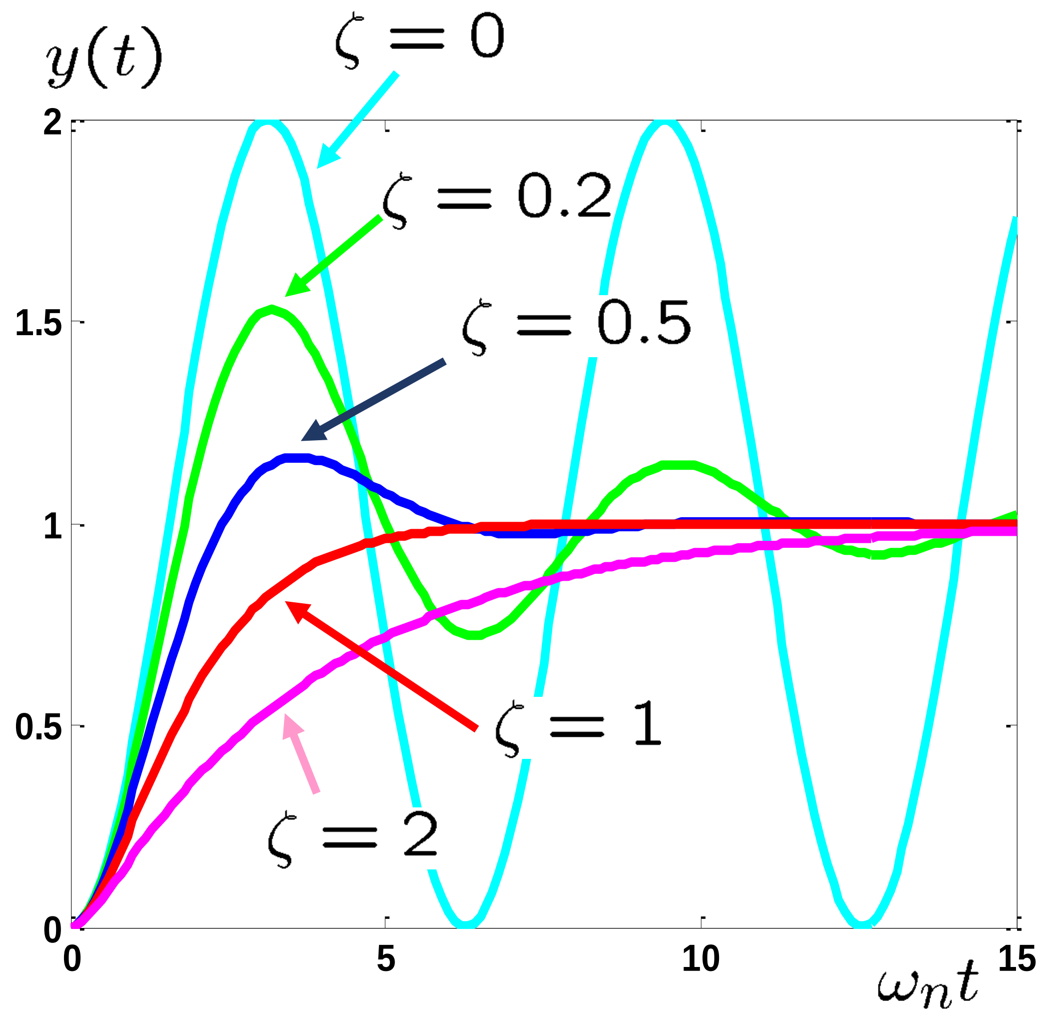
  $$0 < \zeta < 1$$

- Critically damped

  $$\zeta = 1$$

- Overdamped

  $$\zeta > 1$$



Step response of 2nd-order system: Underdamped case

- Math expression of $y(t)$ for underdamped case

$$0 < \zeta < 1$$

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

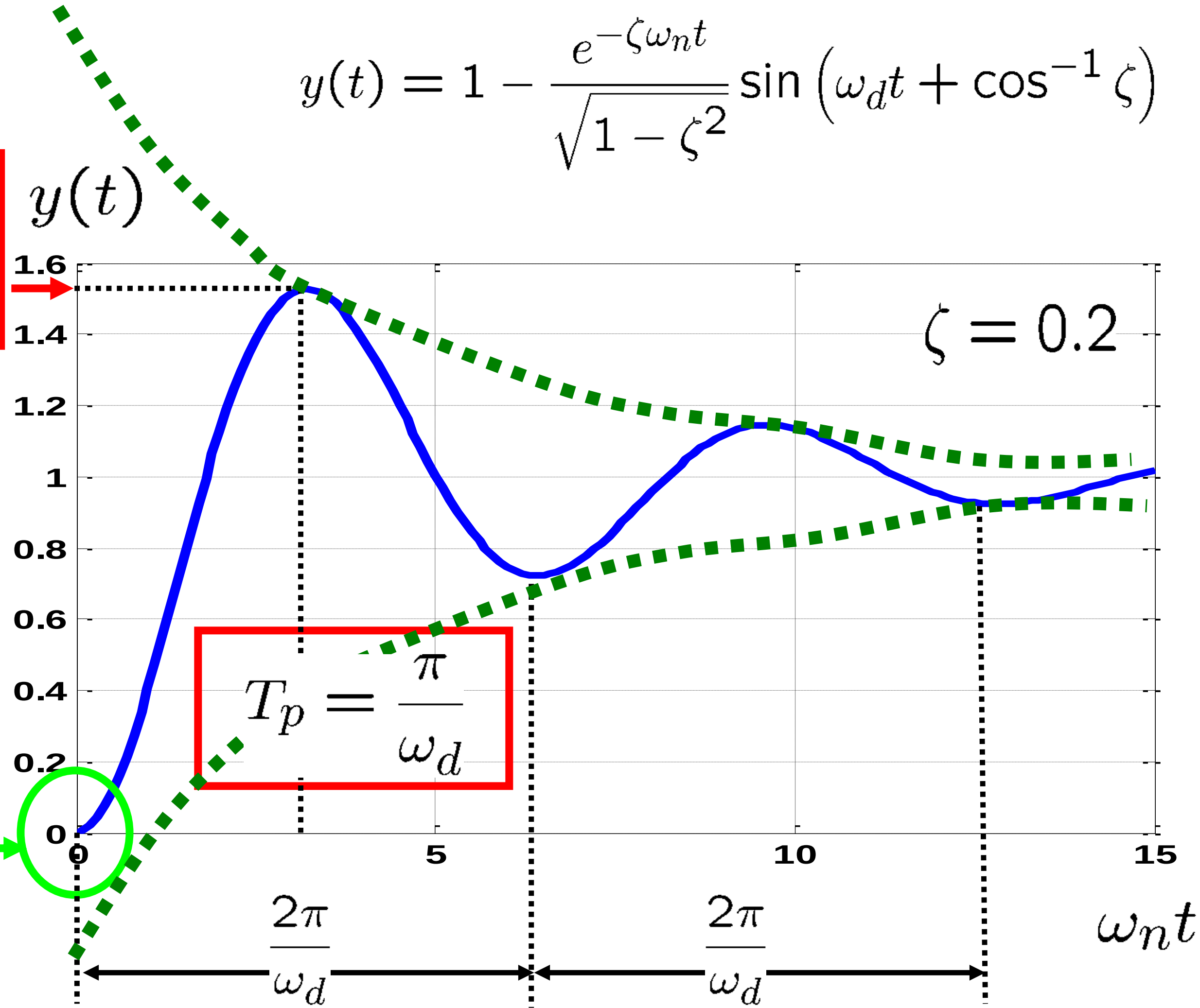
$$\xrightarrow{\mathcal{L}^{-1}} \boxed{y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)}$$

Damped natural frequency $\longrightarrow$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Peak value and peak time: Underdamped case

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \cos^{-1}\zeta\right)$$

$$y_{max} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$



$$T_p = \frac{\pi}{\omega_d}$$

$$y(0) = 0$$
$$y'(0) = 0$$

Properties of underdamped 2nd-order system in terms of ζ and ω_n

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 0 < \zeta < 1$$

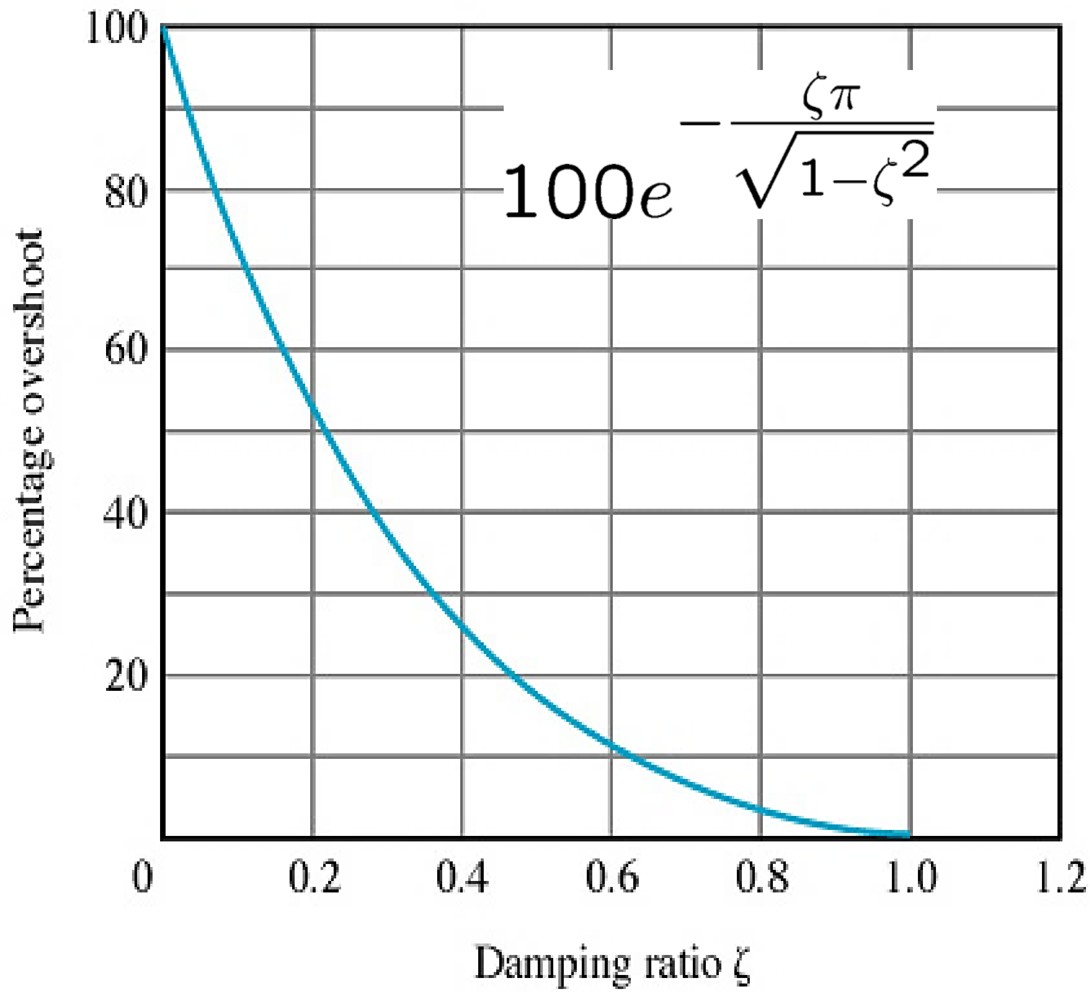
	(5%) (2%)
Settling time	$\approx \dfrac{3}{\zeta\omega_n}$ or $\dfrac{4}{\zeta\omega_n}$
Peak time	$\dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
Peak value	$1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$
Percent overshoot	$100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Remarks for underdamped cases

- Time constant is $1/(\zeta\omega_n)$, indicating convergence speed.
- Percent overshoot depends on ζ, but NOT ω_n. (See the next slide.)
- For ζ >1 (overdamped case), we cannot define peak time, peak value, and percent overshoot (PO).
- For the 2nd-order transfer function, analytic expressions of delay & rise time are harder to obtain. You can use the following formula for rise time:

$$\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$$

PO vs. damping ratio

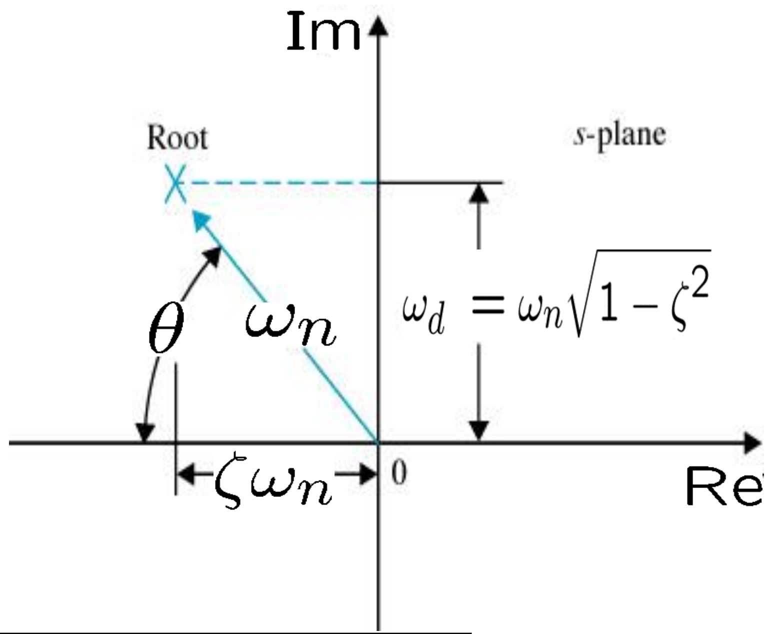


Step response properties of underdamped 2nd order system in terms of pole locations

- Poles ($0 < \zeta < 1$)

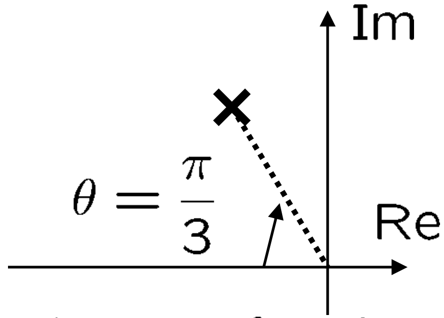
$$s = -\zeta\omega_n \pm \underbrace{j\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

$$\zeta = \cos\theta$$

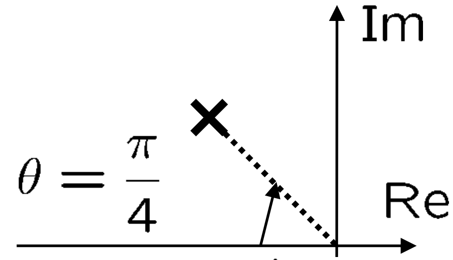


Pole			Performance
Real part	$\zeta\omega_n$	determines	$T_s = \frac{3}{\zeta\omega_n}, \frac{4}{\zeta\omega_n}$ $= \frac{3}{\lvert\mathrm{Re}\rvert}, \frac{4}{\lvert\mathrm{Re}\rvert}$
Imag. part	ω_d	determines	$T_p = \frac{\pi}{\omega_d}$ $= \frac{\pi}{\lvert\mathrm{Im}\rvert}$
Angle	θ	determines	overshoot

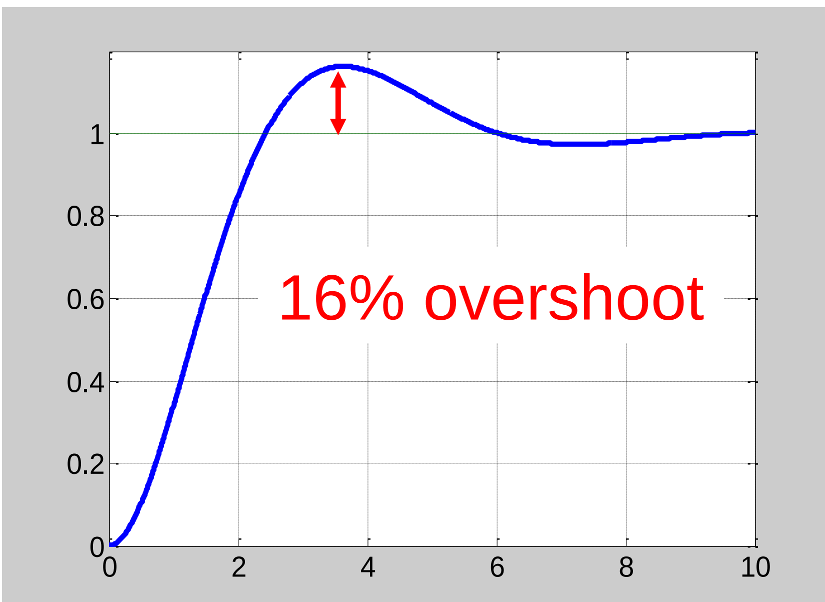
Angle θ and the overshoot

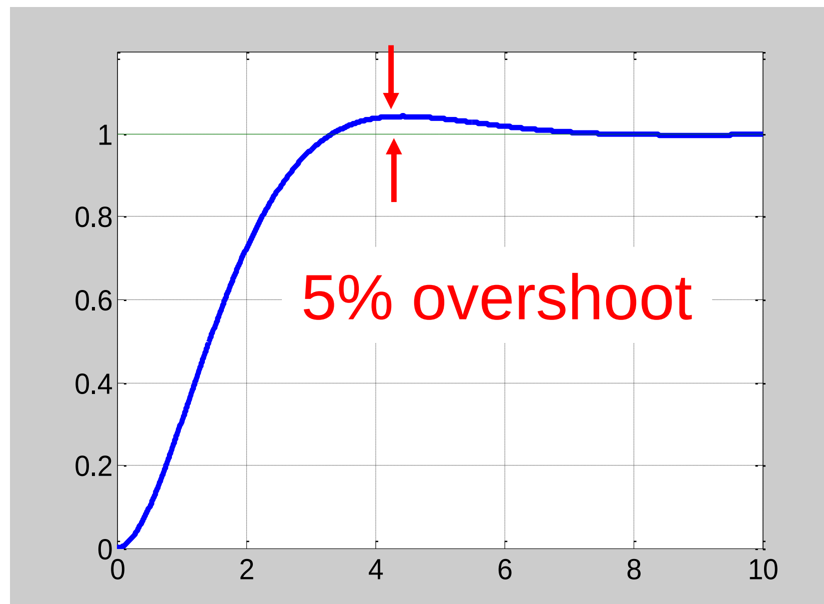


$$\zeta = \cos\theta = 0.5$$



$$\zeta = \cos\theta = \frac{1}{\sqrt{2}} \approx 0.707$$

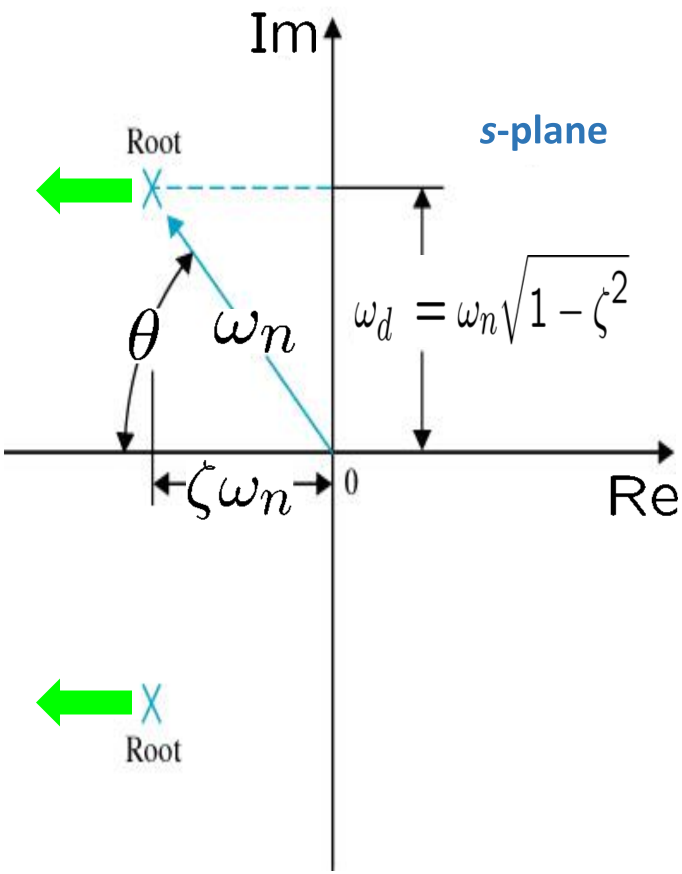


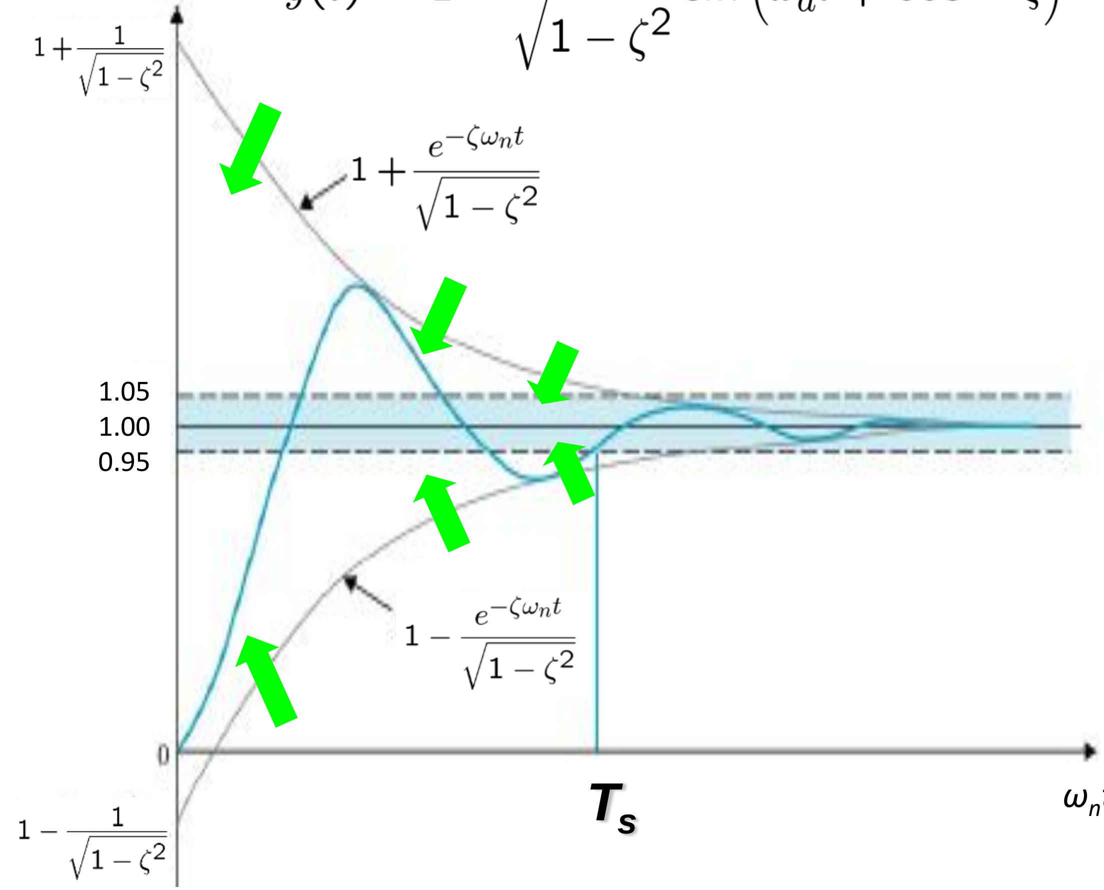


Influence of real part of poles

- Settling time T_s decreases.

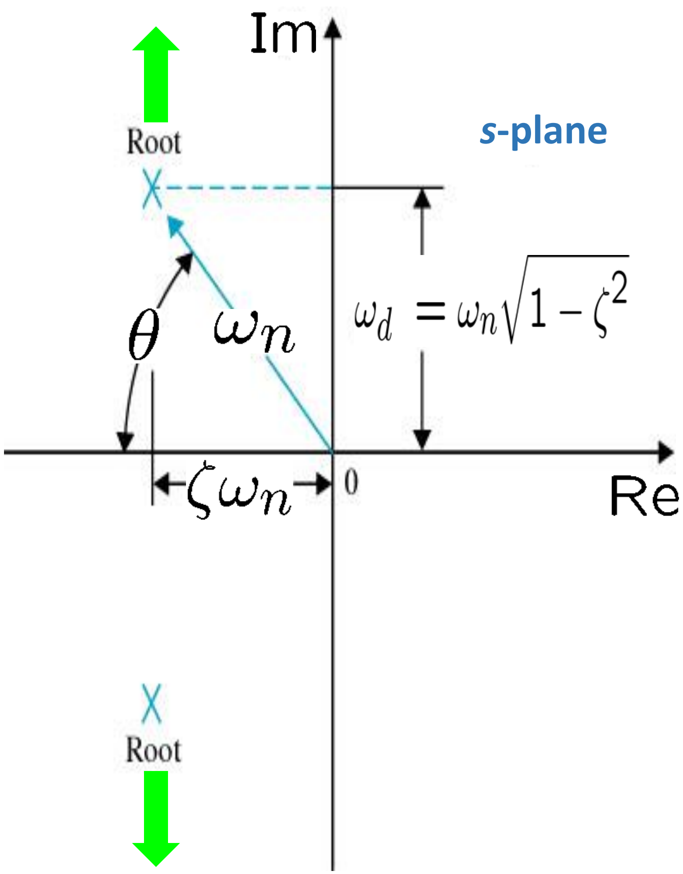
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$

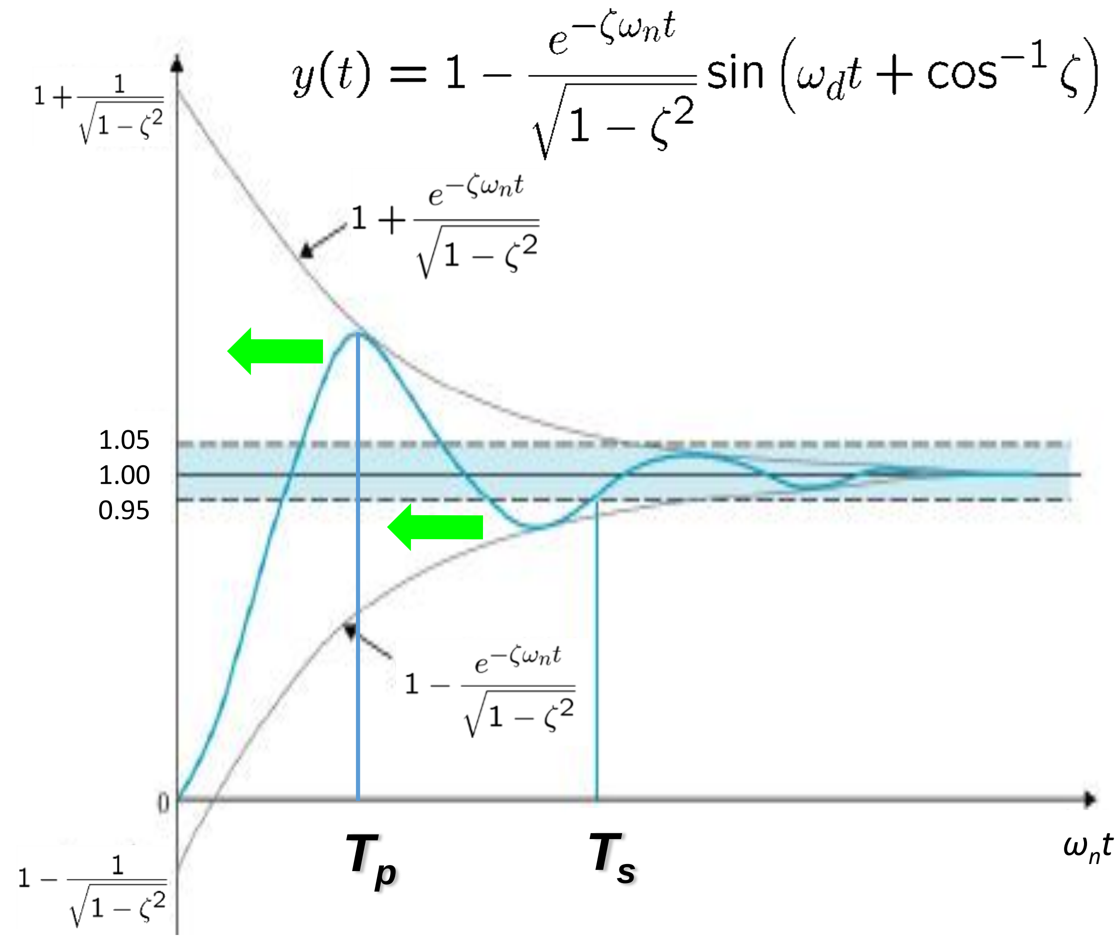




Influence of imag. part of poles

- As oscillation frequency (ω_d) increases, T_P decreases.

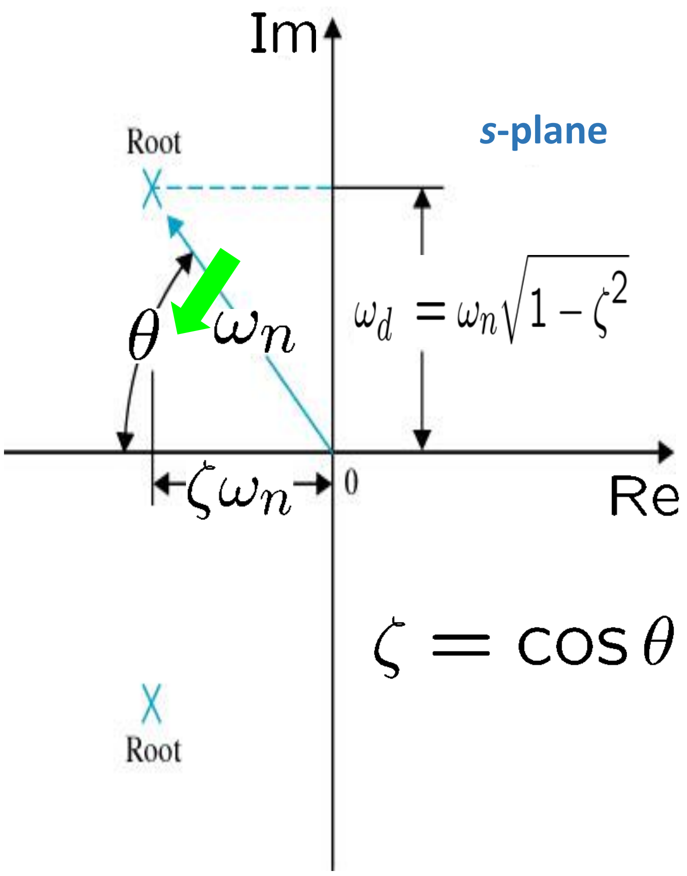


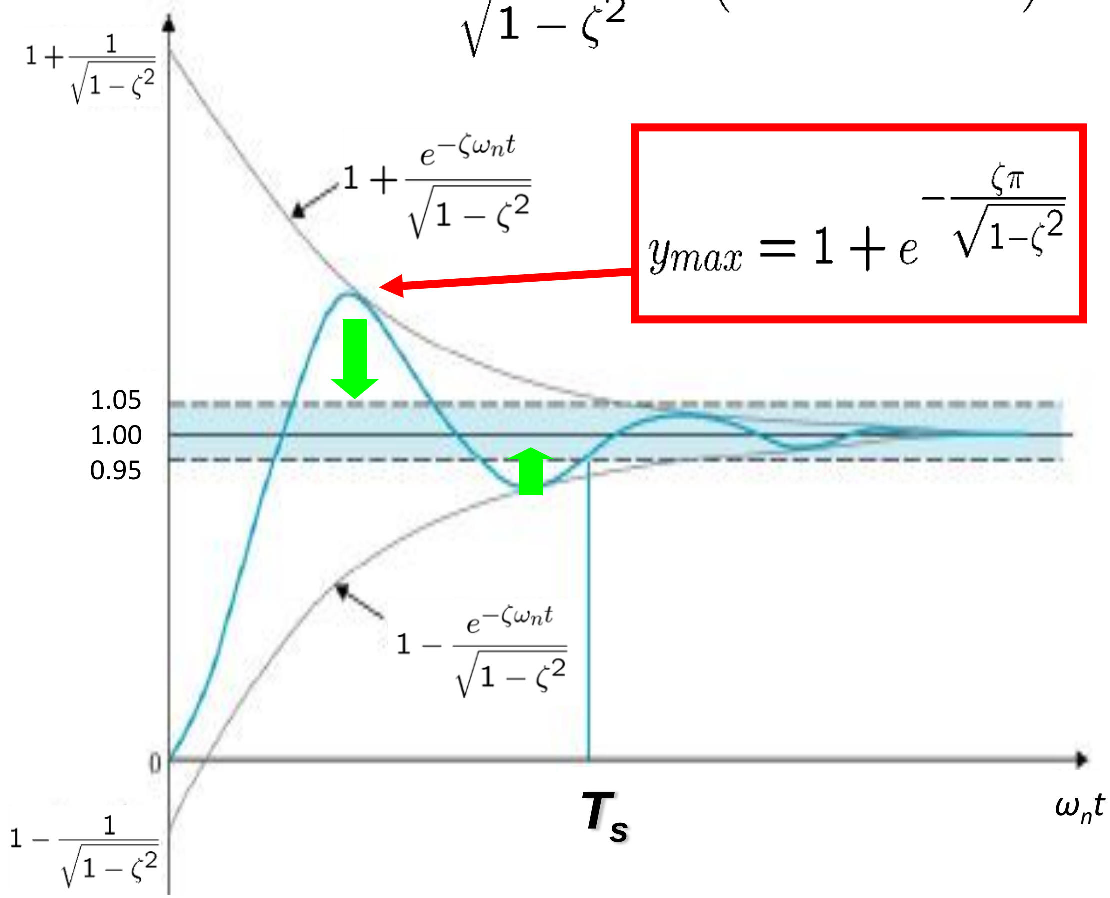


Influence of angle of poles

- As θ decreases from 90° to 0°, ζ increases from 0 to 1 and over/under-shoot decreases.

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$





Summary

- Step responses of
  - 1st-order system is characterized by
    - Time constant, T, and DC gain, K
    - Pole location
  - 2nd-order system is characterized by
    - Damping ratio ζ & undamped natural frequency ω_n
    - Pole location
- Next
  - Time response examples