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ELEC 341: Systems and Control

Lecture 3

ODE solution via Laplace transform

Course roadmap

Modeling

Laplace transform

Transfer function

Models for systems
- Electrical
- Electromechanical
- Mechanical

Linearization, delay

Analysis

Stability
- Routh-Hurwitz
- Nyquist

Time response
- Transient
- Steady state

Frequency response
- Bode plot

Design

Design specs

Root locus

Frequency domain

PID & Lead-lag

Design examples

Matlab simulations

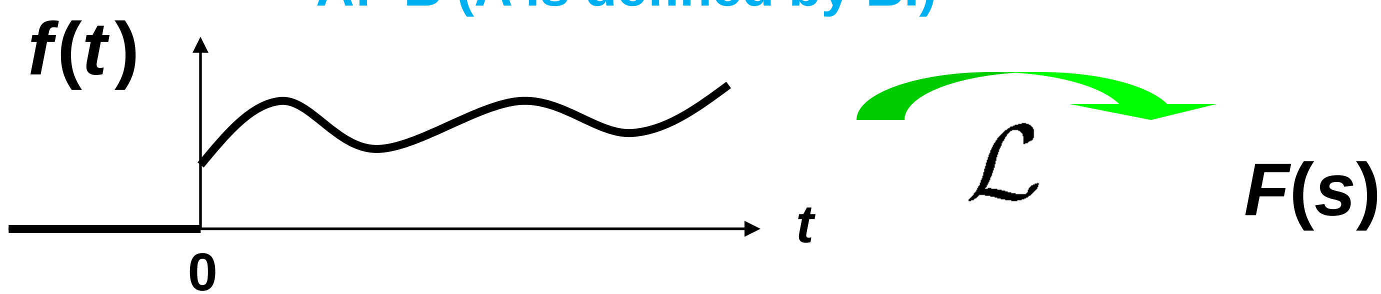
Laplace transform (review)

- One of most important math tools in the course!
- Definition: For a function *f*(*t*) (*f*(*t*) = 0 for *t* < 0),

$$\boxed{F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}dt}$$

A:=B (A is defined by B.)

(*s*: complex variable)



- We denote Laplace transform of *f*(*t*) by *F*(*s*).

Laplace transform table

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t \cdot u(t)$	$\frac{s}{s^2+\omega^2}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$

$\xrightarrow{\mathcal{L}}$

$\xleftarrow{\mathcal{L}^{-1}}$

Inverse Laplace Transform

(u(t) is often omitted.)

Advantages of *s*-domain (review)

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- We can transform an ordinary differential equation (ODE) into an algebraic equation which becomes easier to solve.

  (This lecture)

- It is easier to analyze and design interconnected (series, feedback etc.) systems.

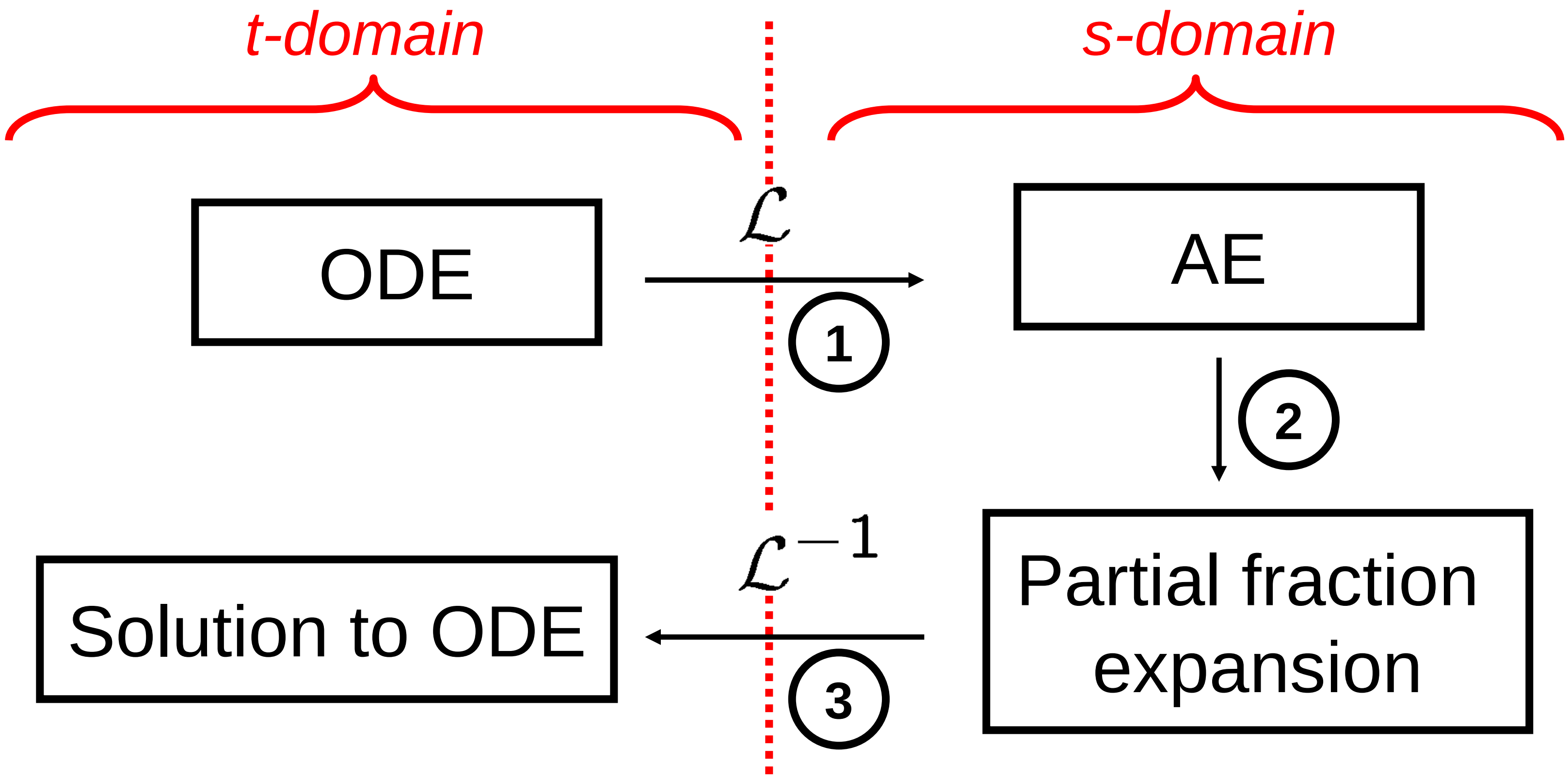
  (Throughout the course)

- Frequency domain information of signals can be dealt with.

  (Lectures for frequency responses)

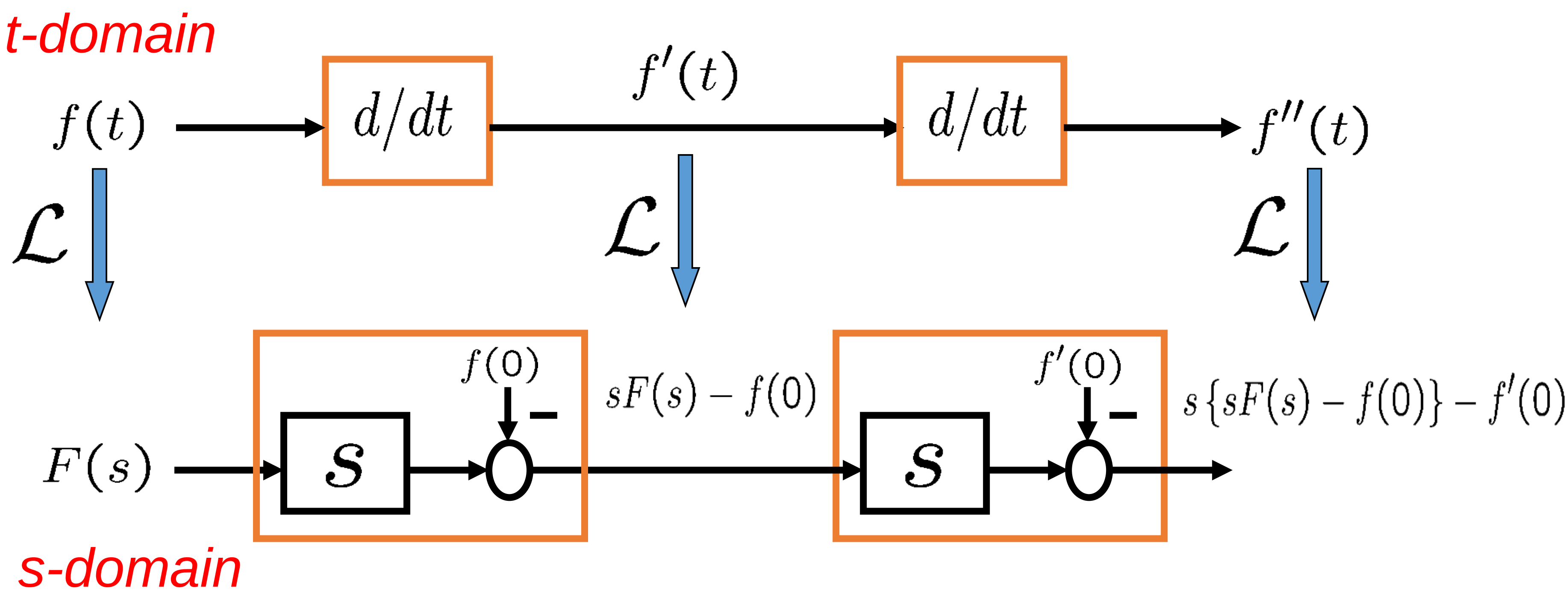
An advantage of Laplace transform

We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Properties of Laplace transform
Differentiation (review)

$$\boxed{\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)}$$



Or, in general: $\mathcal{L}\left\{f^n(t)\right\} = s^n\mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$

Example 1 (distinct roots)

ODE with initial conditions (ICs):

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=5u(t),\ y(0)=-1,\ y'(0)=2$$

1. Laplace transform

$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathcal{L}\left\{y''(t)\right\}}+3\underbrace{\{sY(s)-y(0)\}}_{\mathcal{L}\left\{y'(t)\right\}}+2Y(s)=\frac{5}{s}$$

$$\Rightarrow\quad Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}$$

distinct roots

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Example 1 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

Multiply both sides by *s*(*s*+1)(*s*+2) :

$$-s^2-s+5=A(s+1)(s+2)+Bs(s+2)+Cs(s+1)$$

Compare coefficients:

$$\begin{aligned} s^2\text{-term} &: \quad -1=A+B+C \\ s^1\text{-term} &: \quad -1=3A+2B+C \\ s^0\text{-term} &: \quad 5=2A \end{aligned} \quad \Rightarrow \quad \begin{cases} A=\dfrac{5}{2} \\ B=-5 \\ C=\dfrac{3}{2} \end{cases}$$

Example 1 (cont'd)

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3. Inverse Laplace transform

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

(*You may omit u(t)*.)

$$\Rightarrow \boxed{y(t) = \left(\underbrace{\frac{5}{2}}_{A} + \underbrace{(-5)}_{B} e^{-t} + \underbrace{\frac{3}{2}}_{C} e^{-2t} \right) u(t)}$$

If we are interested in only the final value of *y*(*t*), apply the Final Value Theorem, without explicitly computing *y*(*t*):

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{-s^2 - s + 5}{(s+1)(s+2)} = \frac{5}{2}$$

Example 2 (repeated roots)

ODE with zero initial conditions (ICs):

$$\frac{d^3y(t)}{dt^3}+5\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+4y(t)=2\delta(t),\ y(0)=y'(0)=y''(0)=0$$

1. Laplace transform

$$\begin{aligned}&s^3Y(s)-s^2y(0)-sy'(0)-y''(0) && \longleftarrow \mathcal{L}\left\{y'''(t)\right\}\\&+5\left\{s^2Y(s)-sy(0)-y'(0)\right\} && \longleftarrow 5\mathcal{L}\left\{y''(t)\right\}\\&+8\left\{sY(s)-y(0)\right\}+4Y(s)\\&=2\end{aligned}$$

$$\Rightarrow \quad Y(s)=\frac{2}{(s+1)(s+2)^2} \longleftarrow \textit{Repeated roots}$$

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Example 2 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

Multiply both sides by $(s+1)(s+2)^2$

$$2 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Compare coefficients:

$$\begin{aligned} s^2\text{-term} &: \quad 0 = A + B \\ s^1\text{-term} &: \quad 0 = 4A + 3B + C \\ s^0\text{-term} &: \quad 2 = 4A + 2B + C \end{aligned} \quad \Rightarrow \quad \begin{cases} A = 2 \\ B = -2 \\ C = -2 \end{cases}$$

Example 2 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

(u(t) omitted.)

$$\Rightarrow \boxed{y(t) = \underbrace{2}_{A} e^{-t} + \underbrace{(-2)}_{B} e^{-2t} + \underbrace{(-2)}_{C} te^{-2t}}$$

If we are interested in only the final value of *y*(*t*), apply the Final Value Theorem, without explicitly computing *y*(*t*):

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{2s}{(s+1)(s+2)^2} = 0$$

Properties of Laplace transform
Frequency shift theorem (review)

$$\boxed{\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)}$$

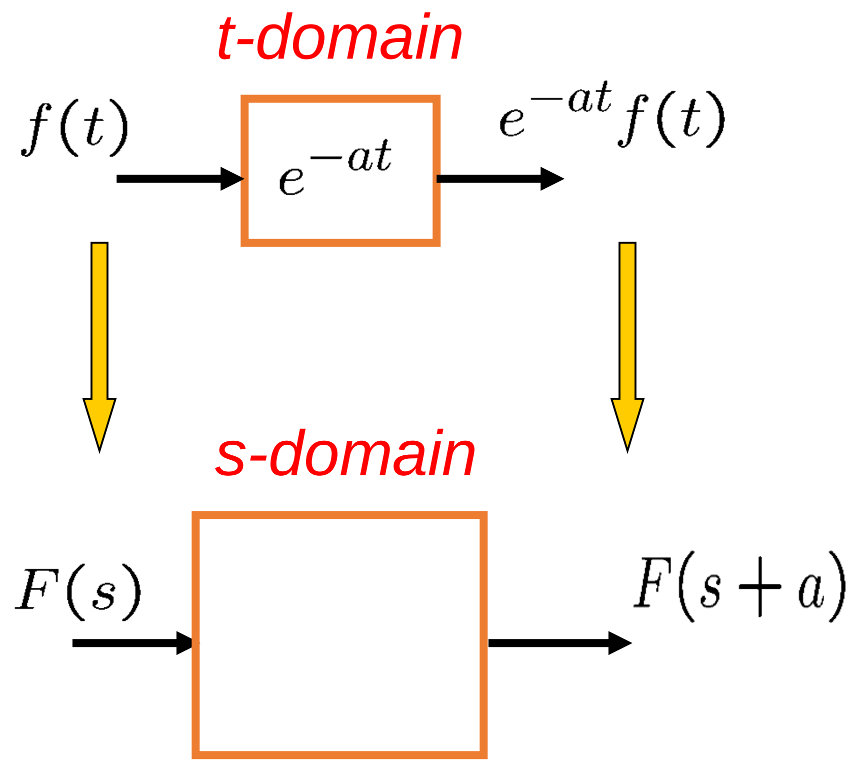
Proof.

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = \int_0^\infty e^{-at}f(t)e^{-st}dt$$

$$= \int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$$

Ex.

$$\mathcal{L}\left\{te^{-2t}\right\} = \frac{1}{(s+2)^2}$$



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Example 3 (complex roots)

ODE with zero initial conditions (ICs):

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+5y(t)=3u(t),\ \ y(0)=0,\ \ y'(0)=0$$

1. Laplace transform

$$s^2Y(s)+2sY(s)+5Y(s)=\frac{3}{s}$$

$$\Rightarrow\ Y(s)=\frac{3}{s(s^2+2s+5)}$$

Complex roots

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Example 3 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{3}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

Multiply both sides by $s(s^2+2s+5)$

$$3 = A(s^2+2s+5) + s(Bs+C)$$

Compare coefficients:

$$\begin{aligned} s^2\text{-term} &: \quad 0 = A + B \\ s^1\text{-term} &: \quad 0 = 2A + C \\ s^0\text{-term} &: \quad 3 = 5A \end{aligned} \quad \Rightarrow \quad \begin{cases} A = \frac{3}{5} \\ B = -\frac{3}{5} \\ C = -\frac{6}{5} \end{cases}$$

Example 3 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s} + \boxed{\frac{Bs + C}{s^2 + 2s + 5}}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{Bs+C}{s^2+2s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{B(s+1)+C-B}{(s+1)^2+4}\right\} \\
&= B\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} + \frac{C-B}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} \\
&= Be^{-t}\cos 2t + \frac{C-B}{2}e^{-t}\sin 2t
\end{aligned}$$

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)}$$

Laplace transform table for frequency shift theorem

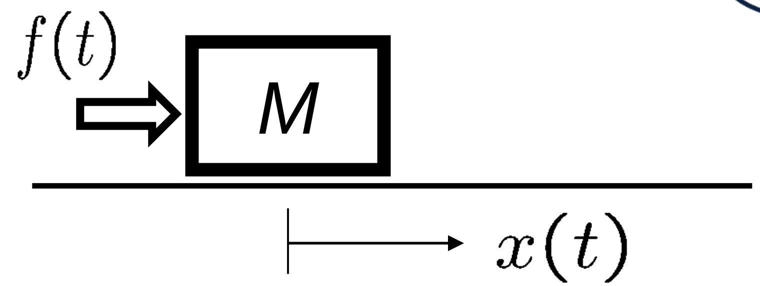
$f(t)$	$F(s)$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$e^{-\alpha t} \sin \omega t$	$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$e^{-\alpha t} \cos \omega t$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Frequency shift theorem

$$\boxed{\mathcal{L}\left\{e^{-at} f(t)\right\} = F(s+a)}$$

Example 4: Newton's law

$$M\frac{d^2x(t)}{dt^2} = f(t)$$



Want to know position *x*(*t*) when force *f*(*t*) is applied.

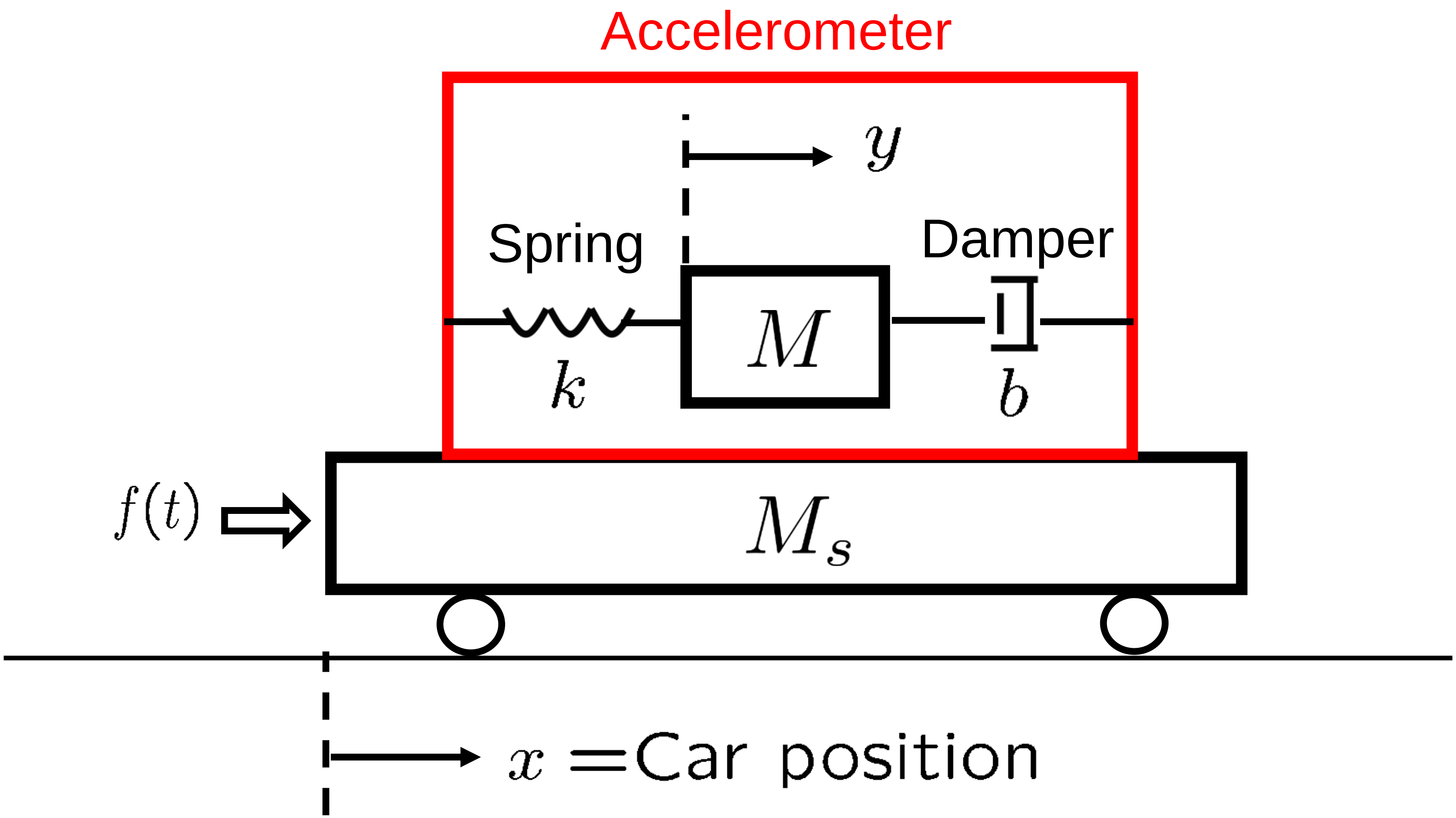
$$M\left(s^2X(s) - sx(0) - x'(0)\right) = F(s)$$

$$\Rightarrow \quad X(s) = \frac{1}{Ms^2}F(s) + \frac{x(0)}{s} + \frac{x'(0)}{s^2}$$

(Total Response) = (Forced Response) + (IC Response)

$$\Rightarrow \quad \boxed{x(t) = \mathcal{L}^{-1}\left\{\frac{1}{Ms^2}F(s)\right\} + x(0)u(t) + x'(0)tu(t)}$$

Example 5: Mechanical accelerometer



Example 5 (cont'd): Accelerometer

- Want to know how $y(t)$ moves when unit step $f(t)$ is applied with zero ICs.
- By Newton's law

$$\begin{cases} M\dfrac{d^2}{dt^2}(x(t)+y(t)) = -b\dfrac{dy(t)}{dt} - ky(t) \\ M_s\dfrac{d^2x(t)}{dt^2} = f(t) \end{cases}$$

$$\Rightarrow \quad My''(t) + by'(t) + ky(t) = -\frac{M}{M_s}f(t)$$

$$\xrightarrow{\mathcal{L}} \quad Y(s) = -\frac{M}{M_s}\cdot\frac{1}{Ms^2+bs+k}\cdot\frac{1}{s} = -\frac{1}{M_s}\cdot\frac{1}{s^2+(b/M)s+(k/M)}\cdot\frac{1}{s}$$

Example 5 (cont'd): Accelerometer

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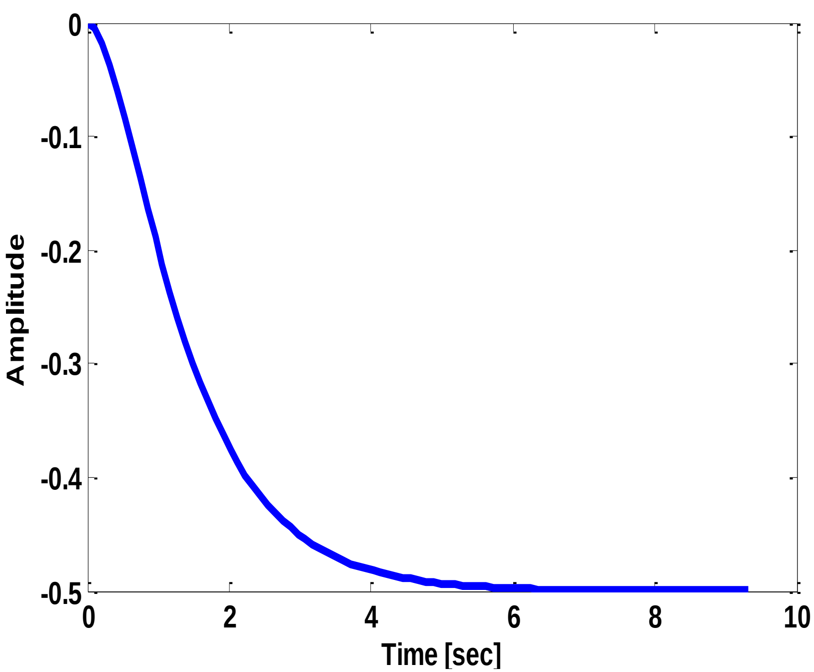
- Suppose that $b/M = 3$, $k/M = 2$ and $M_s = 1$.
- Partial fraction expansion

$$Y(s) = -\frac{1}{s^2+3s+2}\cdot\frac{1}{s} = -\frac{1}{2s}+\frac{1}{s+1}-\frac{1}{2(s+2)}$$

- Inverse Laplace transform

$$\boxed{y(t) = \left(-\frac{1}{2}+e^{-t}-\frac{1}{2}e^{-2t}\right)u(t)}$$



Summary

- Solution to ODE via Laplace transform
    1. Laplace transform
    2. Partial fraction expansion
    3. Inverse Laplace transform
- Next
    - Modeling of physical systems in *s*-domain