ELEC 341: Systems and Control

Lecture 5

Modeling of DC motor, linearization, and time delay

Course roadmap

Modeling

✓ Laplace transform

✓ Transfer function

Models for systems
- ✓ Electrical
- ► Electromechanical
- ✓ Mechanical

Linearization, delay

Analysis

Stability
- Routh-Hurwitz
- Nyquist

Time response
- Transient
- Steady state

Frequency response
- Bode plot

Design

Design specs

Root locus

Frequency domain

PID & Lead-lag

Design examples

Matlab simulations

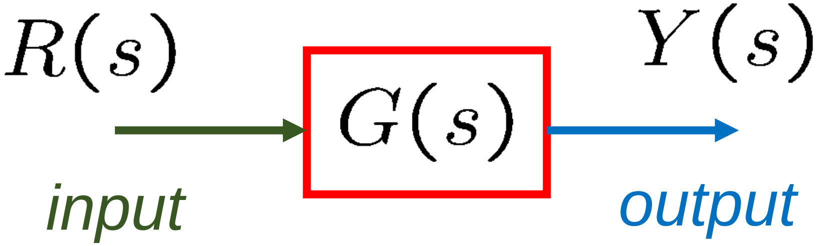
Transfer function (review)

- A transfer function is defined by

$$G(s) = \frac{Y(s)}{R(s)}$$

Laplace transform of system output

Laplace transform of system input



$$Y(s) = G(s)R(s)$$

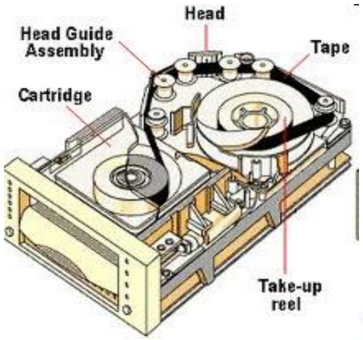
- Transfer function is a generalization of “gain” concept.

DC motor

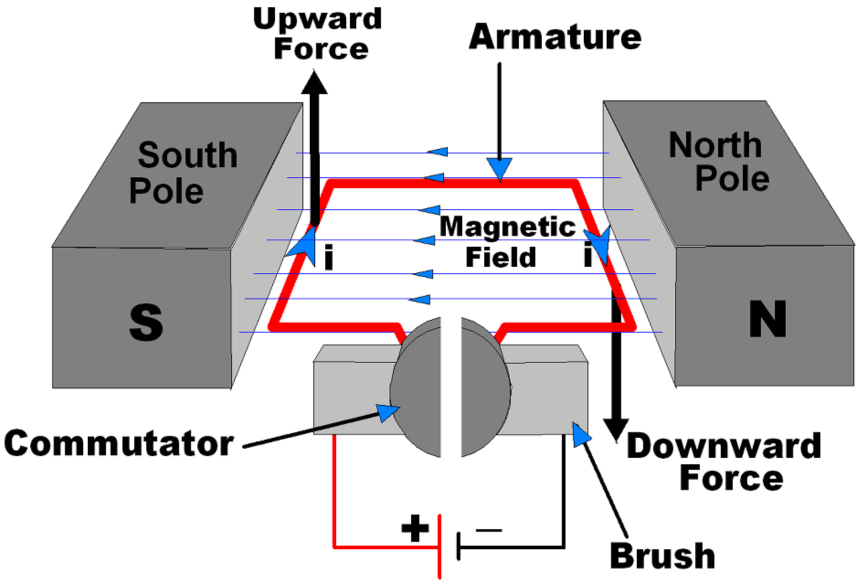
An actuator, converting electrical energy into rotational mechanical energy.

Why DC motor?

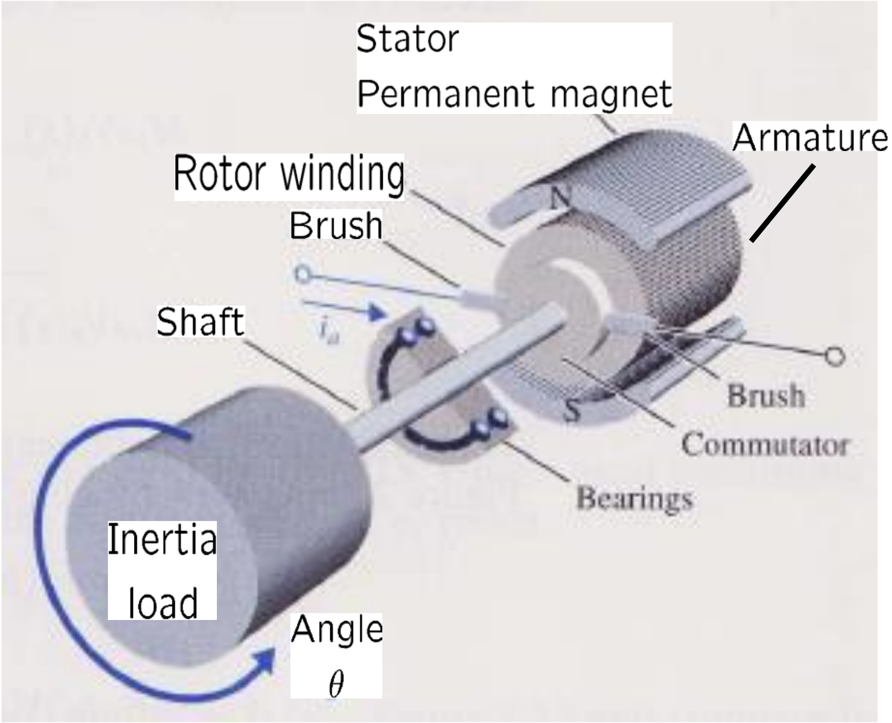
- Advantages
  - high torque
  - speed controllability
  - portability, etc.
- Widely used in control applications
  - Robots
  - Surgical tools
  - Tape drives
  - Printers
  - Machine tool industries
  - Radar tracking systems, etc.



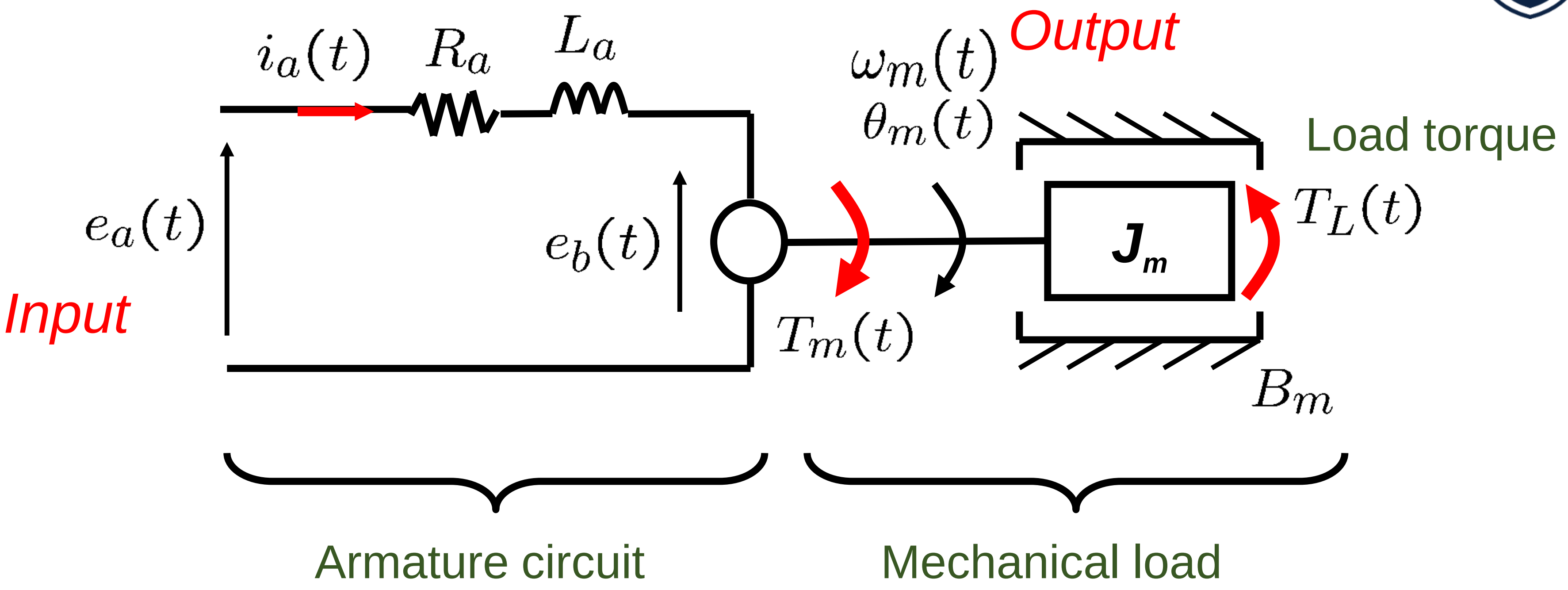
How does DC motor work?



DC Motor Conceptual Diagram



Model of DC motor



"a": armature
e_a : applied voltage
i_a : armature current

"b": back EMF

"m": mechanical
θ_m : angular position
ω_m : angular velocity
J_m : total inertia
B_m : viscous friction

Modeling of DC motor: *t*-domain

- Armature circuit $$e_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + e_b(t)$$
- Mechanical load $$J_m \dot{\omega}_m(t) = T_m(t) - B_m \omega_m(t) - T_L(t)$$

  Driving torque

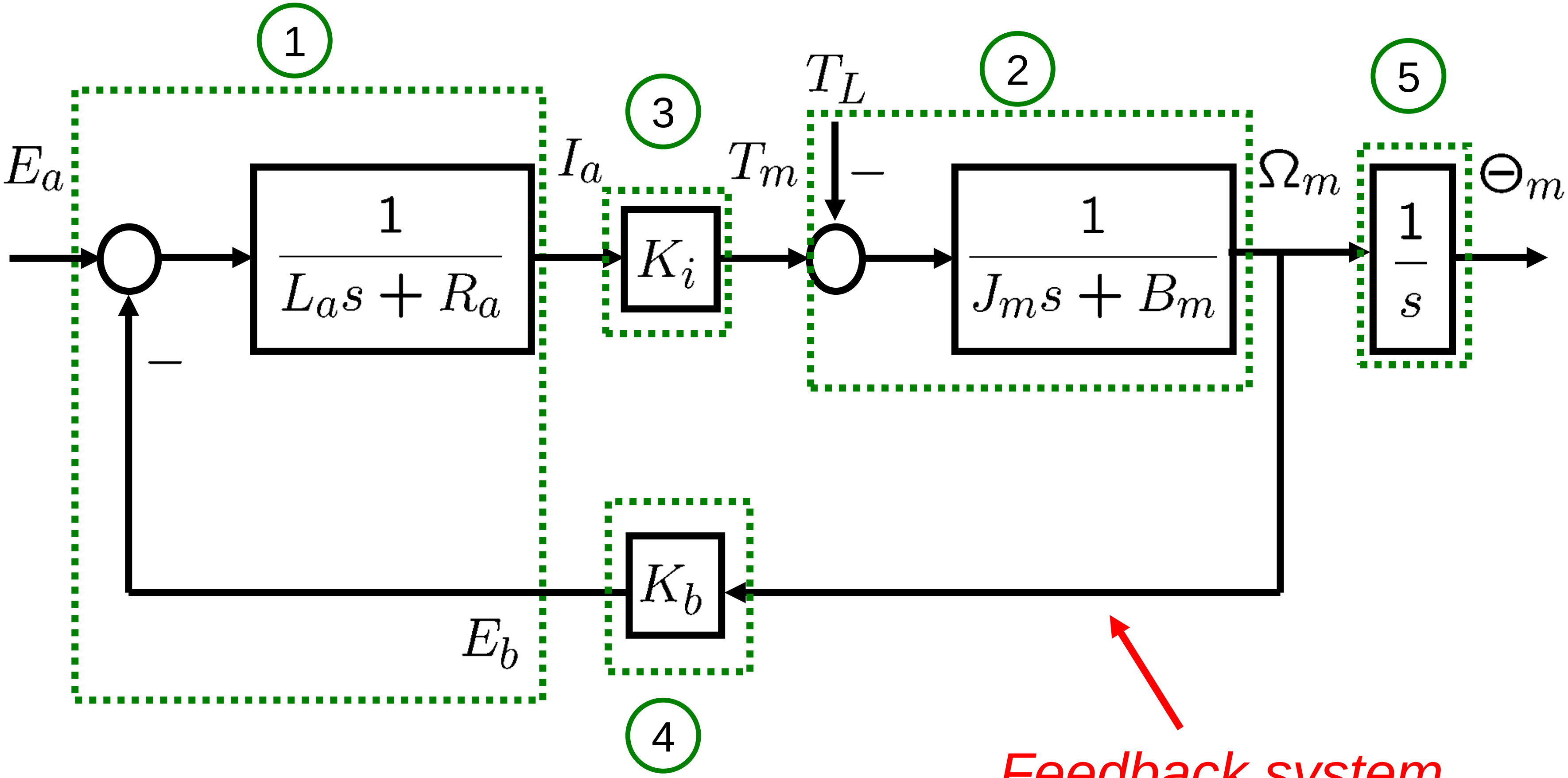
  Load torque
- Connection between mechanical/electrical parts
  - Motor torque $$T_m(t) = K_i i_a(t)$$
  - Back EMF $$e_b(t) = K_b \omega_m(t)$$
- Angular position $$\omega_m(t) = \dot{\theta}_m(t)$$

Modeling of DC motor: *s*-domain

- Armature circuit $$I_a(s) = \frac{1}{L_a s + R_a}\left(E_a(s) - E_b(s)\right)$$ ①
- Mechanical load $$\Omega_m(s) = \frac{1}{J_m s + B_m}\left(T_m(s) - T_L(s)\right)$$ ②

  Note that $\mathcal{L}\{\omega(t)\} = \Omega(s)$ and $\mathcal{L}\{\theta(t)\} = \Theta(s)$.
- Connection between mechanical/electrical parts
  - Motor torque $$T_m(s) = K_i I_a(s)$$ ③
  - Back EMF $$E_b(s) = K_b \Omega_m(s)$$ ④
- Angular position $$\Theta_m(s) = \frac{1}{s}\Omega_m(s)$$ ⑤

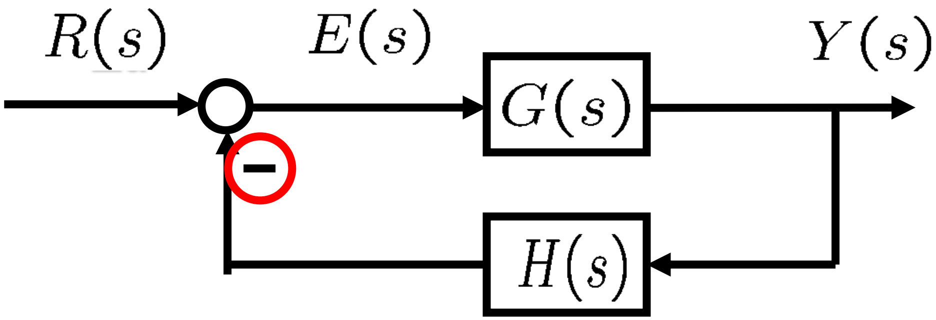
DC motor: Block diagram



Feedback system

Transfer function (TF) with feedback (Black's formula)

- Negative feedback system



$$E(s) = R(s) - H(s)G(s)E(s) \quad \Rightarrow \quad E(s) = \frac{1}{1+G(s)H(s)}R(s)$$

$$Y(s) = G(s)E(s) \quad \Rightarrow \quad \boxed{Y(s) = \frac{G(s)}{1+G(s)H(s)}R(s)}$$

OLTF = Open Loop Transfer Function
CLTF = Closed Loop Transfer Function

$$\left(\begin{array}{ll} G(s) & : \text{forward path TF} \\ G(s)H(s) & : \text{open-loop TF} \end{array} \right)$$

- **Black's Formula: Closed-loop transfer function is given by:** $\dfrac{\textbf{Forward Gain}}{\textbf{1+Loop Gain}} = \dfrac{Y(s)}{R(s)}$

DC motor: Transfer functions

If $T_L = 0$, then $$\frac{\Omega_m(s)}{E_a(s)} = \frac{\frac{K_i}{(L_a s + R_a)(J_m s + B_m)}}{1 + \frac{K_b K_i}{(L_a s + R_a)(J_m s + B_m)}} = \underbrace{\frac{K_i}{(L_a s + R_a)(J_m s + B_m) + K_b K_i}}_{G_1(s)}$$

If $E_a = 0$, then $$\frac{\Omega_m(s)}{T_L(s)} = -\frac{\frac{1}{J_m s + B_m}}{1 + \frac{K_b K_i}{(L_a s + R_a)(J_m s + B_m)}} = \underbrace{-\frac{L_a s + R_a}{(L_a s + R_a)(J_m s + B_m) + K_b K_i}}_{G_2(s)}$$

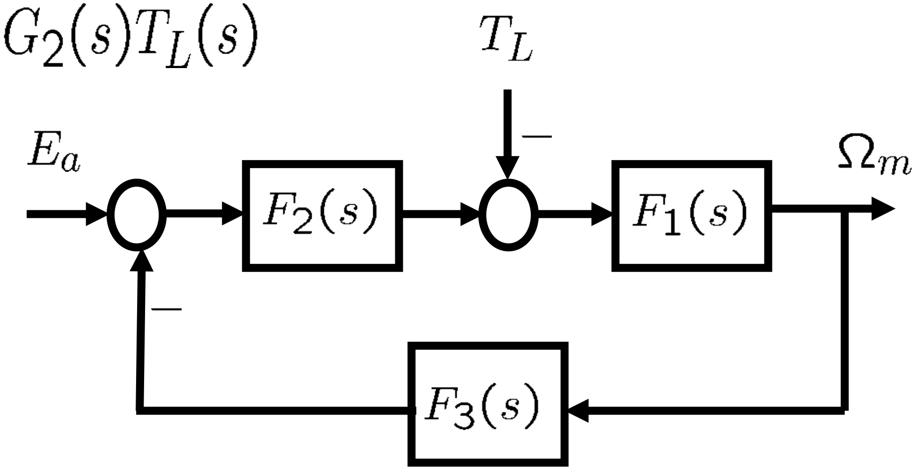
In general, when $T_L \neq 0$ and $E_a \neq 0$, we can prove the following equation:

➡ $$\Omega_m(s) = G_1(s)E_a(s) + G_2(s)T_L(s)$$ ➡

$$\boxed{\Theta_m(s) = \frac{1}{s}\Omega_m(s) = \frac{1}{s}\left(G_1(s)E_a(s) + G_2(s)T_L(s)\right)}$$

DC motor: Derivation of TFs

- Why $\Omega_m(s) = G_1(s)E_a(s) + G_2(s)T_L(s)$



$$\Omega_m(s) = F_1(s)\left[-T_L(s) + F_2(s)\left\{E_a(s) - F_3(s)\Omega_m(s)\right\}\right]$$

$$\Rightarrow \{1 + F_1(s)F_2(s)F_3(s)\}\,\Omega_m(s) = F_1(s)\left\{-T_L(s) + F_2(s)E_a(s)\right\}$$

$$\Rightarrow \boxed{\Omega_m(s) = \frac{F_1(s)F_2(s)}{1 + F_1(s)F_2(s)F_3(s)}E_a(s) - \frac{F_1(s)}{1 + F_1(s)F_2(s)F_3(s)}T_L(s)}$$

DC motor: TFs (cont'd)

- Note: For DC motors, $L_a << R_a$. Then, an approximated TF is obtained by setting $L_a = 0$.

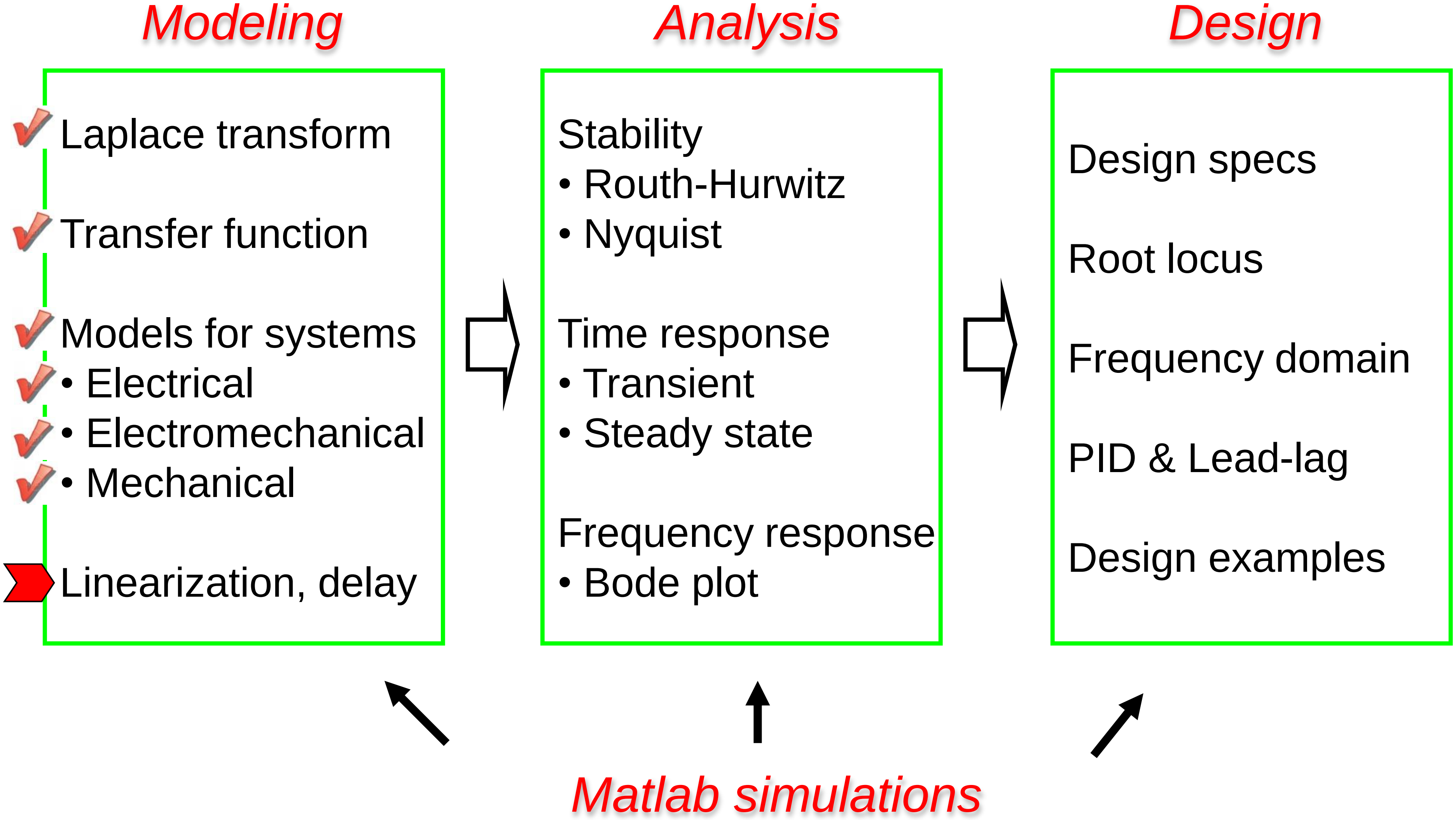
$$\frac{\Omega_m(s)}{E_a(s)} = \frac{K_i}{(L_a s + R_a)(J_m s + B_m) + K_b K_i} \approx \frac{K_i}{R_a(J_m s + B_m) + K_b K_i}$$

$$= \frac{K}{Ts+1} \qquad \left(K = \frac{K_i}{R_a B_m + K_b K_i}, \; T = \frac{R_a J_m}{R_a B_m + K_b K_i} \right)$$

2nd order system → *1st order system*

$$\Theta_m(s) = \frac{1}{s}\Omega_m(s) \Rightarrow \boxed{\frac{\Theta_m(s)}{E_a(s)} = \frac{K}{s(Ts+1)}}$$

Course roadmap



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  - ✓ Electrical
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- ➤ Linearization, delay

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Stability
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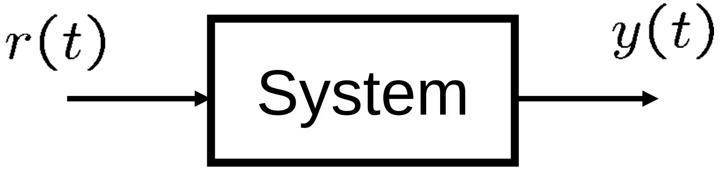
Linear system

- A linear system satisfies the *Principle of Superposition*



$$\left.\begin{array}{l} r_1(t) \rightarrow y_1(t) \\ r_2(t) \rightarrow y_2(t) \end{array}\right\} \Rightarrow \alpha_1 r_1(t) + \alpha_2 r_2(t) \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

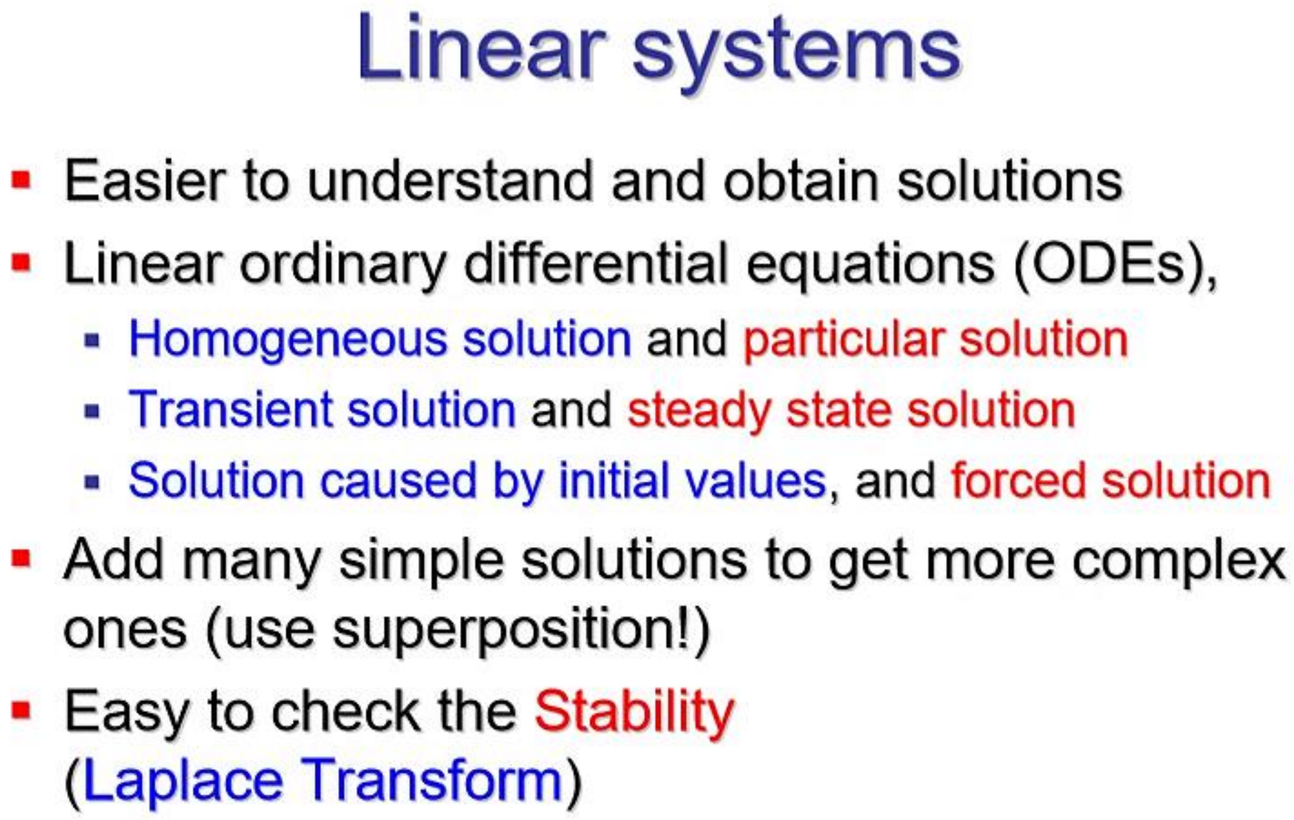
$$\forall \alpha_1, \alpha_2 \in \mathbb{R} \quad (\text{'}\forall\text{' means 'for all'})$$

A nonlinear system does not satisfy the principle of superposition.

Why linearization?

- Real systems are inherently nonlinear. (Linear systems do not exist!) Ex. $f(t) = K.x(t)$, $v(t) = R.i(t)$
- TF models are only for linear systems.
- Many control analysis/design techniques are available for linear systems.
- Nonlinear systems are difficult to deal with mathematically.
- Often we linearize nonlinear systems before analysis and design. How?

Linearization



Linear systems

- Easier to understand and obtain solutions
- Linear ordinary differential equations (ODEs),
    - Homogeneous solution and particular solution
    - Transient solution and steady state solution
    - Solution caused by initial values, and forced solution
- Add many simple solutions to get more complex ones (use superposition!)
- Easy to check the Stability (Laplace Transform)

- We will look at a systematic method called **Generalized Method for Linearization** for handling these types of control problems.

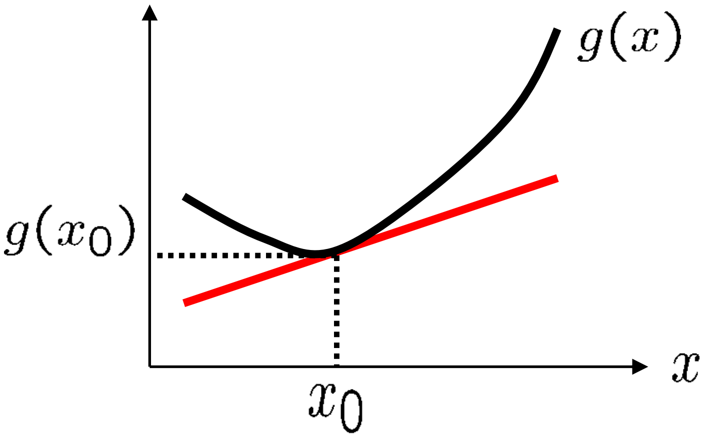
Taylor series expansion

- Taylor series expansion of a smooth (i.e., infinitely differentiable) function *g*(*x*) around $x = x_o$

$$g(x) = \underline{g(x_0) + \left.\frac{dg(x)}{dx}\right|_{x=x_0}(x - x_0)}$$

$$\underbrace{+ \left.\frac{d^2 g(x)}{dx^2}\right|_{x=x_0}\frac{(x - x_0)^2}{2} + \cdots}_{(\approx 0 \text{ if } x \approx x_0)}$$

$$g(x) \approx g(x_0) + \left.\frac{dg(x)}{dx}\right|_{x=x_0}(x - x_0)$$

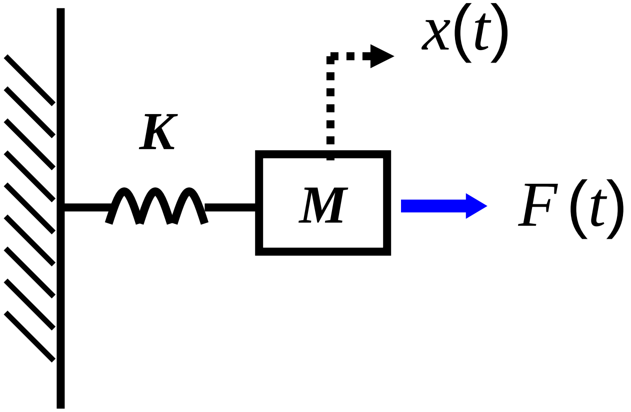


Example 1: Nonlinear spring

- Linear spring

$$M\ddot{x}(t) = F(t) - Kx(t)$$

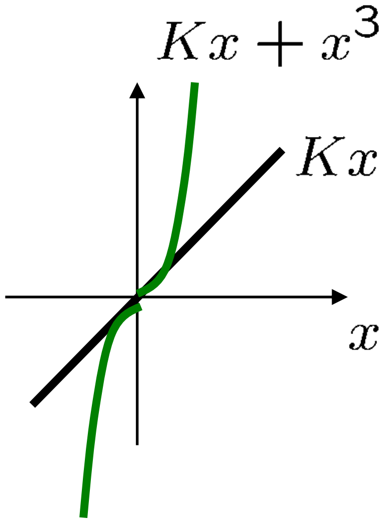
$$\Rightarrow \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + K}$$



- Example of nonlinear spring

Nonlinear term!

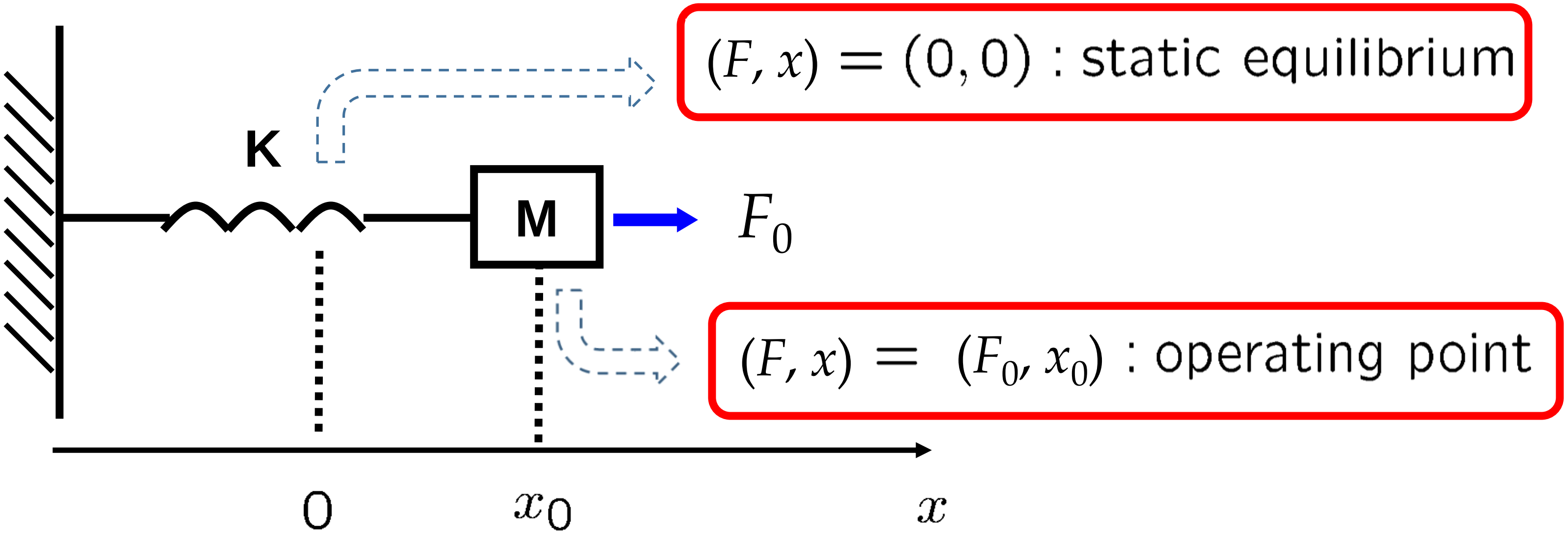
$$M\ddot{x}(t) = F(t) - Kx(t) - \underline{x^3(t)}$$



$$\boxed{\text{We cannot represent } \mathcal{L}\{x^3(t)\} \text{ in terms of } X(s).}$$

Example 1 (cont'd): Specifying an operating point

- Operating point: The point around which the system is assumed to be operating.



$(F, x) = (0, 0)$: static equilibrium

$(F, x) = (F_0, x_0)$: operating point

- We linearize the nonlinear system around a specific operating point.

Example 1 (cont'd): Linearization Procedure

Step 1: Identify input and output variables:

$$\begin{cases} F(t) = \text{input} \\ x(t) = \text{output} \end{cases}$$

Step 2: Express non-linear ODE in the form of $f(\ddot{x}, \dot{x}, x, F) = 0$.

Step 3: Find the Operating Point (OP) of (x_0, F_0). That is, find F_0 at the given x_0.

Step 4: Write the Taylor series expansion at (x_0, F_0), i.e., at the OP.

Step 5: Change variables to perturbation variables in the Taylor series expansion.

Step 6: Re-write the Taylor series expansion as a linear ODE.

The linearized model is valid only around the specified operating point!

Example 1 (cont’d)

Our aim is to linearize the ODE at $x = x_0$.

Step 1:

Identify input and output variables:

$$\boxed{\begin{cases} F(t) = \text{input} \\ x(t) = \text{output} \end{cases}}$$

Step 2:

Express non-linear ODE in the form of $f(\ddot{x}, \dot{x}, x, F) = 0$, (in this case, $f(\ddot{x}, x, F) = 0$):

$$M\ddot{x}(t) = F(t) - K \cdot x(t) - x^3(t) \Longrightarrow f(\ddot{x}, x, F) = M\ddot{x} + K \cdot x + x^3 - F(t) = 0$$

$$\Longrightarrow \boxed{M\ddot{x} + K \cdot x + x^3 - F(t) = 0}$$

Step 3:

Find the operating point (x_0, F_0). That is, find F_0, which is the operating value of F at $x = x_0$:

$$x(t) = x_0,\ \dot{x}(t) = 0,\ \ddot{x}(t) = 0 \Longrightarrow M\underset{0}{\cancel{\ddot{x}_0}} + K.x_0 + x_0^3 - F_0 = 0$$

$$\Longrightarrow \boxed{F_0 = K.x_0 + x_0^3}$$

Example 1 (cont'd)

Step 4:

Write the Taylor series expansion at (x_0, F_0), i.e., at the Operating Point (OP):

$$\boxed{f(\ddot{x}, x, F) = f(\ddot{x}_0, x_0, F_0) + \left.\frac{\partial f}{\partial \ddot{x}}\right|_{OP} (\ddot{x} - \ddot{x}_0) + \left.\frac{\partial f}{\partial x}\right|_{OP} (x - x_0) + \left.\frac{\partial f}{\partial F}\right|_{OP} (F - F_0)}$$

Step 5:

Change variables to perturbation variables in the Taylor series expansion:

$$\begin{cases} \delta x = x - x_0 \\ \delta F = F - F_0 \\ \delta \ddot{x} = \ddot{x} - \ddot{x}_0 \end{cases}$$

$$\boxed{f(\ddot{x}, x, F) = f(\ddot{x}, x_0, F_0) + \left.\frac{\partial f}{\partial \ddot{x}}\right|_{OP} \delta \ddot{x} + \left.\frac{\partial f}{\partial x}\right|_{OP} \delta x + \left.\frac{\partial f}{\partial F}\right|_{OP} \delta F}$$

Example 1 (cont'd)

Step 6:

Re-write the Taylor series expansion as a linear ODE:

$$f(\ddot{x}, x, F) = M\ddot{x} + K \cdot x + x^3 - F(t) \quad \longrightarrow$$

$$\left.\frac{\partial f}{\partial \ddot{x}}\right|_{OP} = M \qquad \left.\frac{\partial f}{\partial x}\right|_{OP} = K + 3x_0^2 = K^* \qquad \left.\frac{\partial f}{\partial F}\right|_{OP} = -1 \quad \longrightarrow$$

$$f(\ddot{x}, x, F) = \overset{0}{\cancel{f}}(\ddot{x}_0, x_0, F_0) + M\delta\ddot{x} + K^* \cdot \delta x + (-1)\delta F$$

We know that $f(\ddot{x}, x, F)$ is also equal to 0: → $\boxed{M\delta\ddot{x} + K^* \cdot \delta x = \delta F}$

Linear!

Note that if we take the Laplace transform of the above linear ODE, we obtain:

$$\boxed{\frac{\tilde{X}(s)}{\tilde{F}(s)} = \frac{1}{Ms^2 + K^*}}$$

where $\mathcal{L}\{\delta x(t)\} = \tilde{X}(s)$ and $\mathcal{L}\{\delta F(t)\} = \tilde{F}(s)$.

Linearization (General Method)

The Six Steps of Linearization

1) Identify the system model's input $r(t)$ and output $c(t)$.

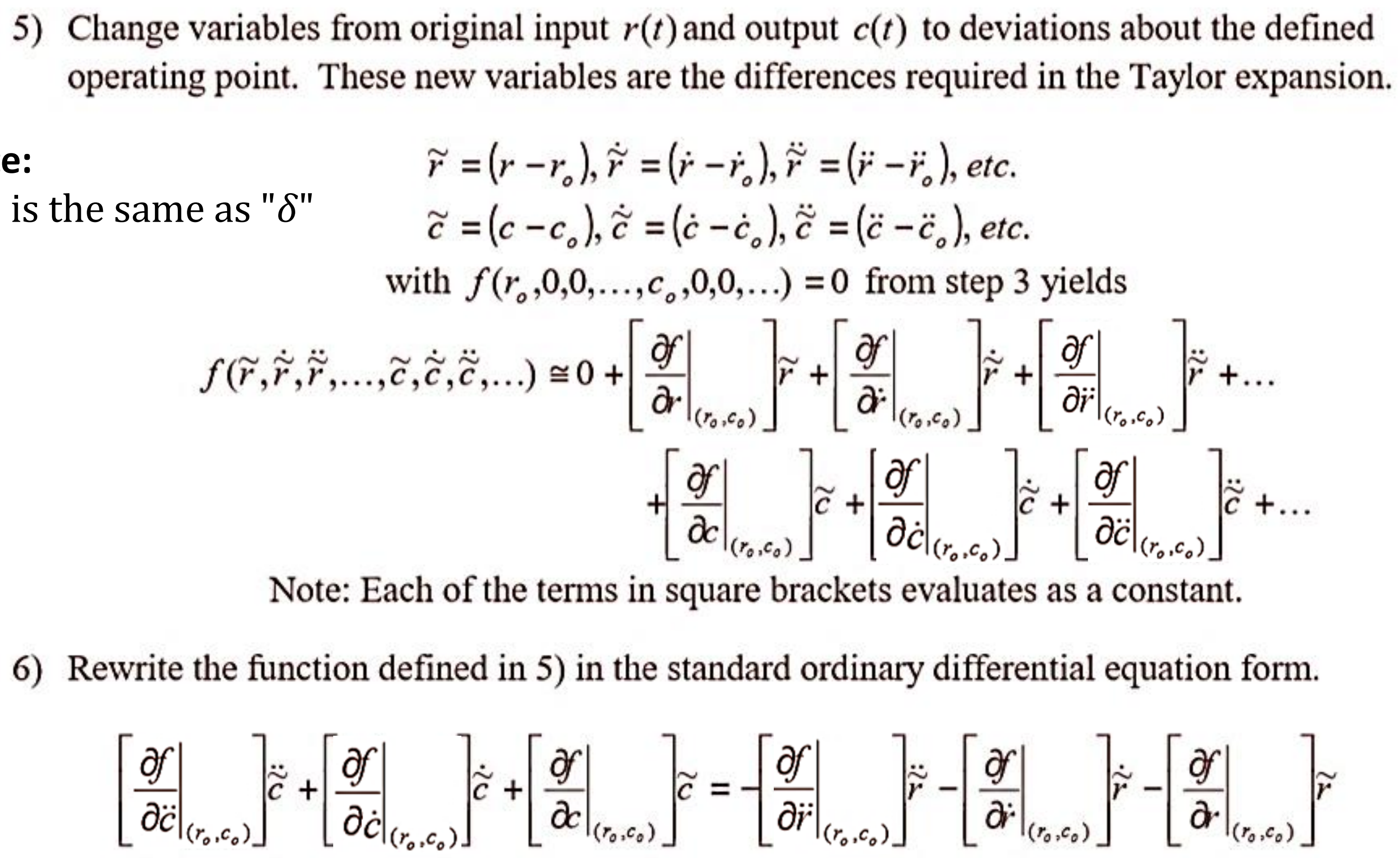
2) Express Model in the form $f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots)=0$.

3) Define an equilibrium operating point where all input and output derivatives are zero ($\dot{r}=\ddot{r}=\ldots=\dot{c}=\ddot{c}=\ldots=0$) and the operating point (r_o,c_o) satisfies the original model such that $f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots)=f(r_o,0,0,\ldots,c_o,0,0,\ldots)=0$ at the input/output values $(r,c)=(r_o,c_o)$.

4) Perform a Taylor Series expansion about the operating point (r_o,c_o) retaining only 1st derivative terms.

$$
\begin{aligned}
f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots) \cong{}& f(r_o,0,0,\ldots,c_o,0,0,\ldots)\\
&+\left.\frac{\partial f}{\partial r}\right|_{(r_o,c_o)}(r-r_o)+\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}(\dot{r}-\dot{r}_o)+\left.\frac{\partial f}{\partial \ddot{r}}\right|_{(r_o,c_o)}(\ddot{r}-\ddot{r}_o)+\ldots\\
&+\left.\frac{\partial f}{\partial c}\right|_{(r_o,c_o)}(c-c_o)+\left.\frac{\partial f}{\partial \dot{c}}\right|_{(r_o,c_o)}(\dot{c}-\dot{c}_o)+\left.\frac{\partial f}{\partial \ddot{c}}\right|_{(r_o,c_o)}(\ddot{c}-\ddot{c}_o)+\ldots
\end{aligned}
$$

Linearization (General Method)



5) Change variables from original input $r(t)$ and output $c(t)$ to deviations about the defined operating point. These new variables are the differences required in the Taylor expansion.

Note:
"$\sim$" is the same as "δ"

$$\tilde{r} = (r - r_o), \dot{\tilde{r}} = (\dot{r} - \dot{r}_o), \ddot{\tilde{r}} = (\ddot{r} - \ddot{r}_o), etc.$$

$$\tilde{c} = (c - c_o), \dot{\tilde{c}} = (\dot{c} - \dot{c}_o), \ddot{\tilde{c}} = (\ddot{c} - \ddot{c}_o), etc.$$

with $f(r_o,0,0,\ldots,c_o,0,0,\ldots) = 0$ from step 3 yields

$$f(\tilde{r},\dot{\tilde{r}},\ddot{\tilde{r}},\ldots,\tilde{c},\dot{\tilde{c}},\ddot{\tilde{c}},\ldots) \cong 0 + \left[\left.\frac{\partial f}{\partial r}\right|_{(r_o,c_o)}\right]\tilde{r} + \left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\dot{\tilde{r}} + \left[\left.\frac{\partial f}{\partial \ddot{r}}\right|_{(r_o,c_o)}\right]\ddot{\tilde{r}} + \ldots$$

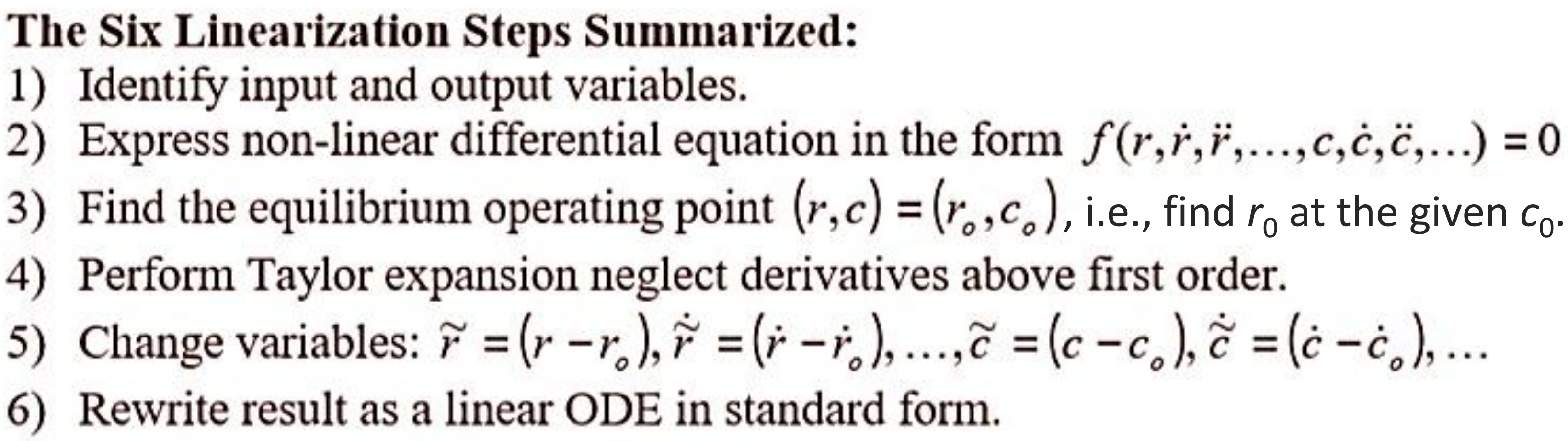
$$+\left[\left.\frac{\partial f}{\partial c}\right|_{(r_o,c_o)}\right]\tilde{c} + \left[\left.\frac{\partial f}{\partial \dot{c}}\right|_{(r_o,c_o)}\right]\dot{\tilde{c}} + \left[\left.\frac{\partial f}{\partial \ddot{c}}\right|_{(r_o,c_o)}\right]\ddot{\tilde{c}} + \ldots$$

Note: Each of the terms in square brackets evaluates as a constant.

6) Rewrite the function defined in 5) in the standard ordinary differential equation form.

$$\left[\left.\frac{\partial f}{\partial \ddot{c}}\right|_{(r_o,c_o)}\right]\ddot{\tilde{c}} + \left[\left.\frac{\partial f}{\partial \dot{c}}\right|_{(r_o,c_o)}\right]\dot{\tilde{c}} + \left[\left.\frac{\partial f}{\partial c}\right|_{(r_o,c_o)}\right]\tilde{c} = -\left[\left.\frac{\partial f}{\partial \ddot{r}}\right|_{(r_o,c_o)}\right]\ddot{\tilde{r}} - \left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\dot{\tilde{r}} - \left[\left.\frac{\partial f}{\partial r}\right|_{(r_o,c_o)}\right]\tilde{r}$$

Linearization (General Method)

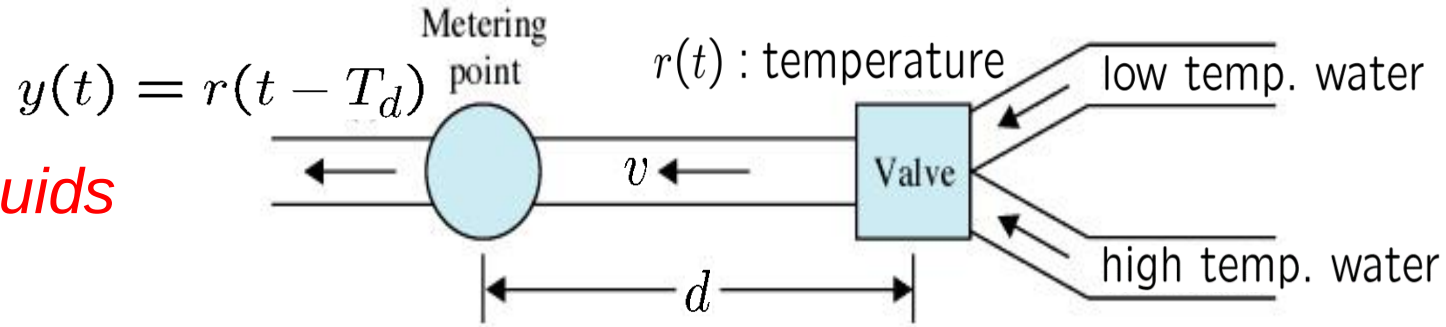


The Six Linearization Steps Summarized:

1) Identify input and output variables.
2) Express non-linear differential equation in the form $f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots)=0$
3) Find the equilibrium operating point $(r,c)=(r_o,c_o)$, i.e., find r_0 at the given c_0.
4) Perform Taylor expansion neglect derivatives above first order.
5) Change variables: $\tilde{r}=(r-r_o),\dot{\tilde{r}}=(\dot{r}-\dot{r}_o),\ldots,\tilde{c}=(c-c_o),\dot{\tilde{c}}=(\dot{c}-\dot{c}_o),\ldots$
6) Rewrite result as a linear ODE in standard form.

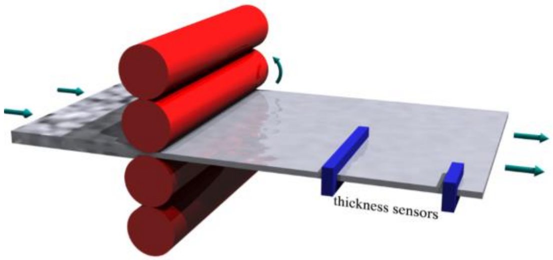
Time-delay example

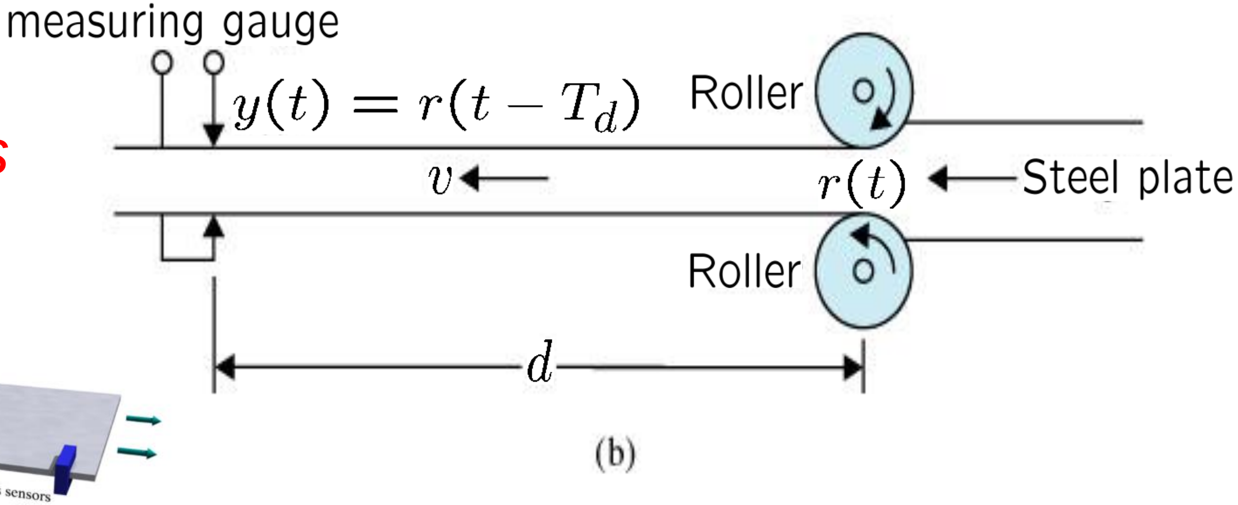
Mixing fluids



Delay time: $T_d = \frac{d}{v}$

Steel thickness control





Time-delay transfer function

- TF derivation

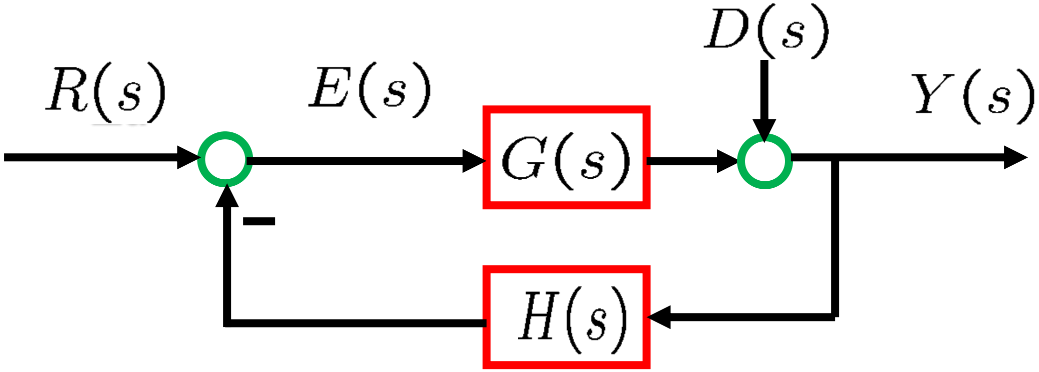
$$y(t) = r(t - T_d) \ (T_d\text{: delay time})$$

$$\xrightarrow{\mathcal{L}} \quad Y(s) = e^{-T_d s} R(s) \quad \Rightarrow \quad \frac{Y(s)}{R(s)} = e^{-T_d s}$$

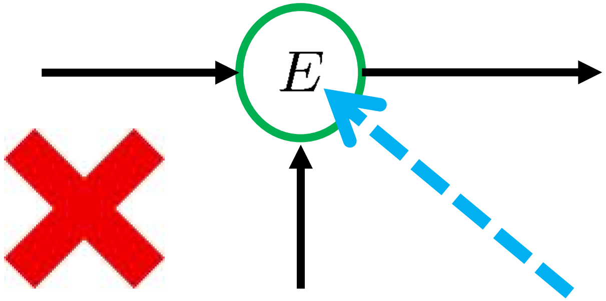
- The more the time delay is, the more difficult to control! (You will learn this fact theoretically later.)
- Imagine that you are controlling the temperature of your shower with a very long hose. You will either get **burned** or **frozen**!

Block diagram

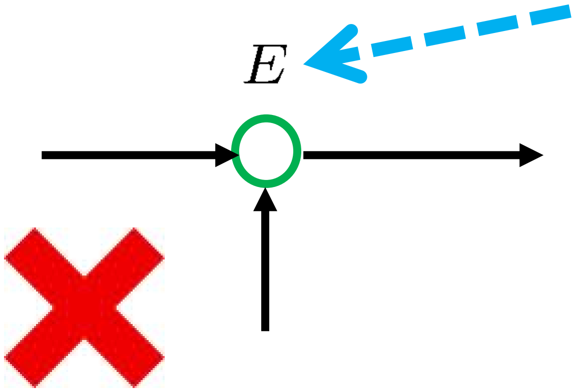
- Represents relations among signals and systems
- Very useful in representing control systems
- Also useful in computer simulations (Simulink)
- Elements
  - **Block**: transfer function ("gain" block)
  - **Arrow**: signal
  - **Node**: summation (or subtraction) of signals



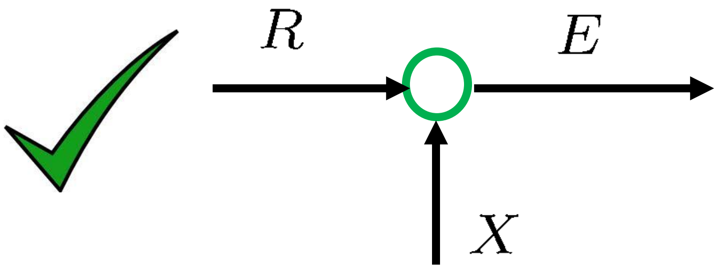
Typical mistakes



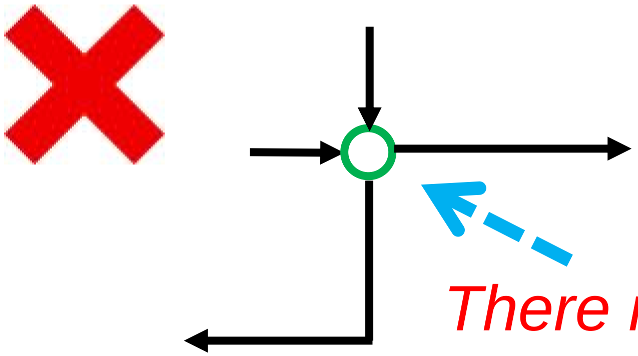
Unclear which signal is "E"



Signal must be indicated on an arrow.

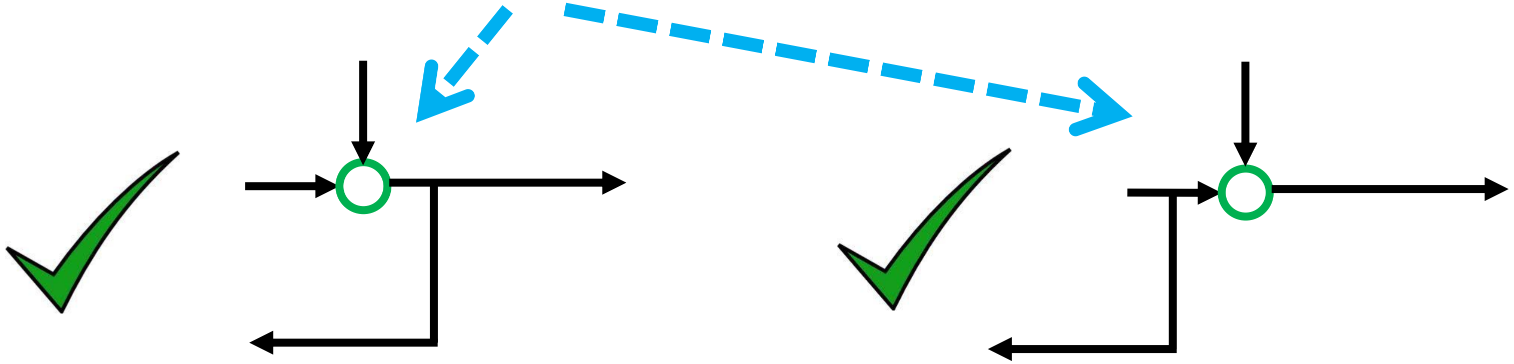


Typical mistakes (cont'd)



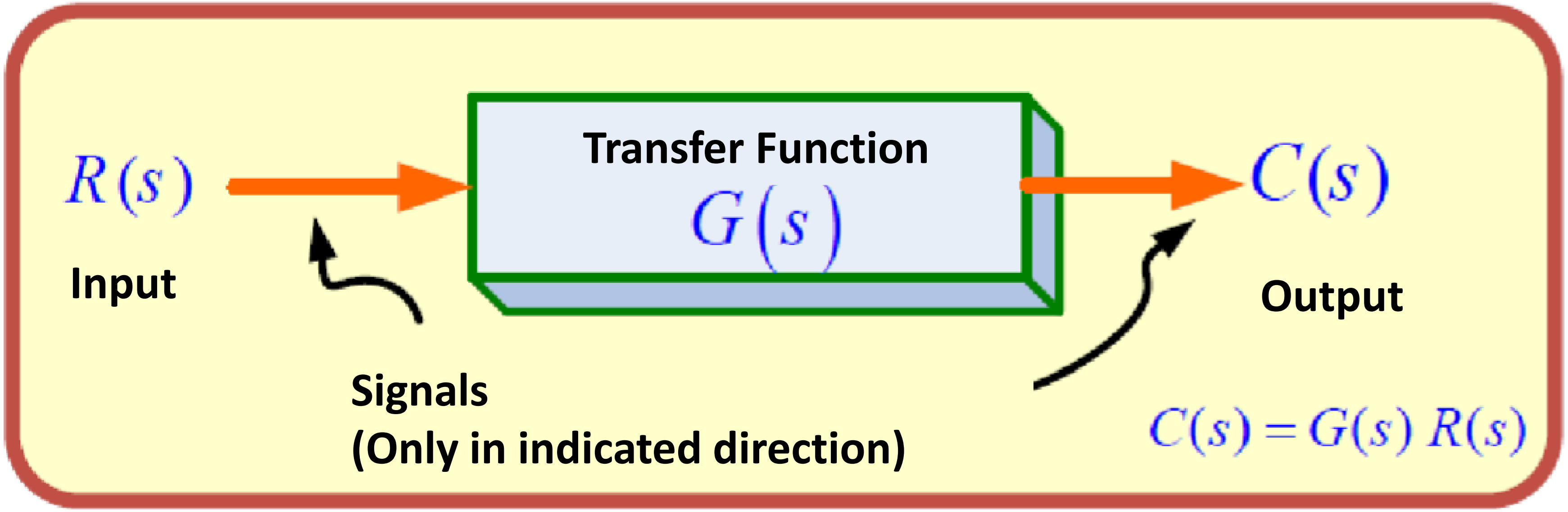
There must be only one output from a node.

Both are fine, but they have different meanings!



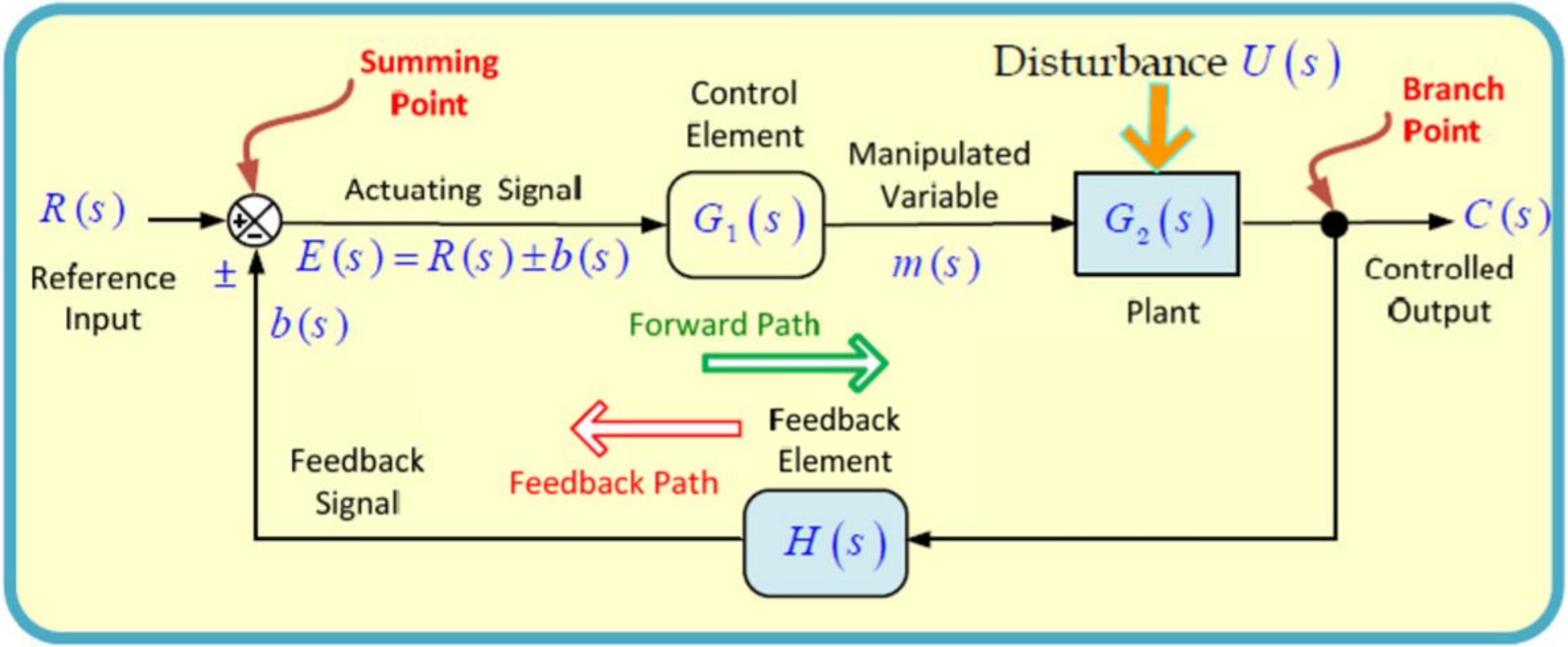
Block diagram

Block Diagram Reduction



$C(s) = G(s)\,R(s)$

Block diagram

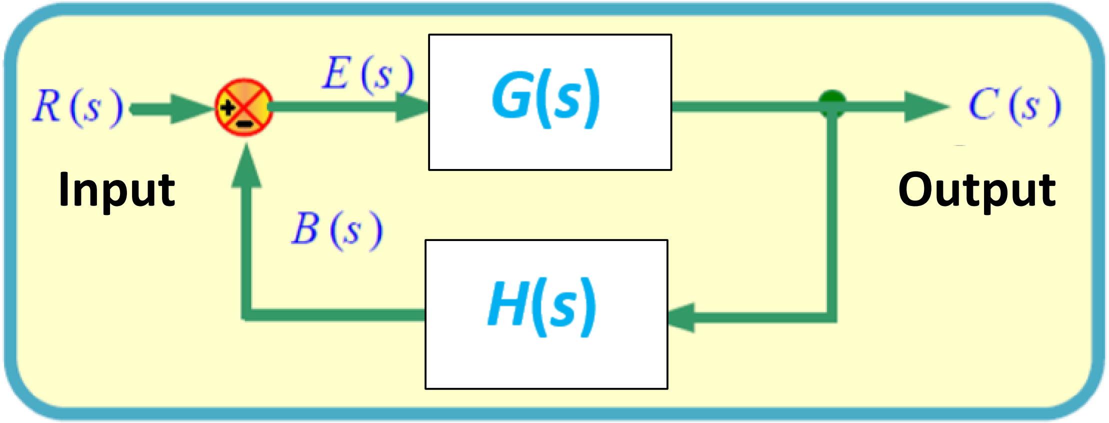


Block diagram

Definitions

- $G(s)$ ≡ Direct transfer function = Forward transfer function.
- $H(s)$ ≡ Feedback transfer function.
- $G(s)H(s)$ ≡ Open-loop transfer function.
- $C(s)/R(s)$ ≡ Closed-loop transfer function = Control ratio
- $C(s)/E(s)$ ≡ Feed-forward transfer function.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

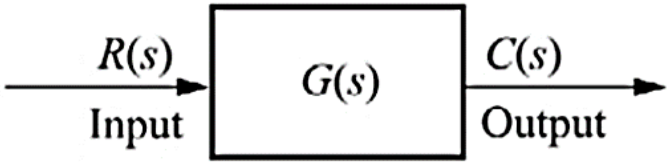


Block diagram of a closed-loop system with a feedback element

Block diagram

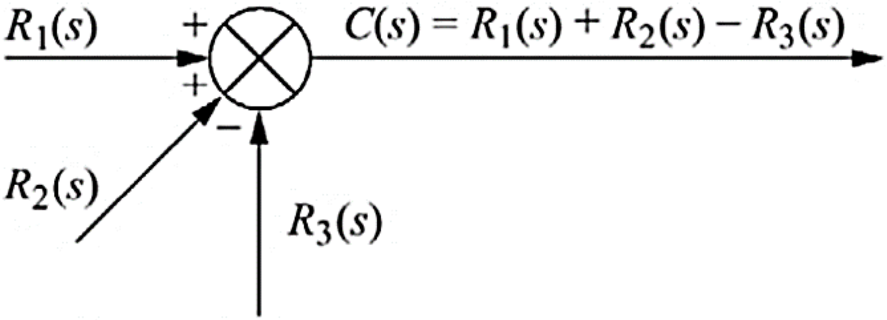
Signals

(*a*)



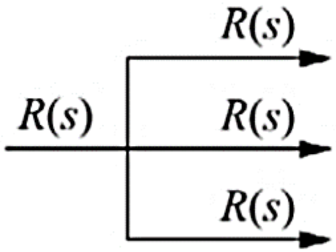
System

(*b*)



Summing junction

(*c*)

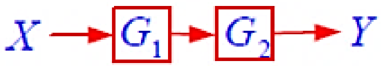
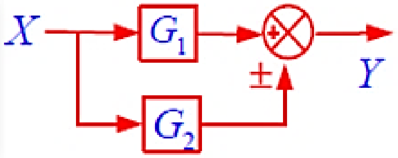
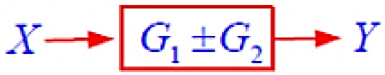
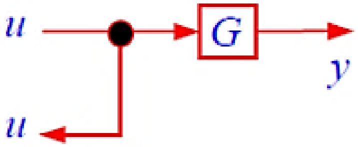
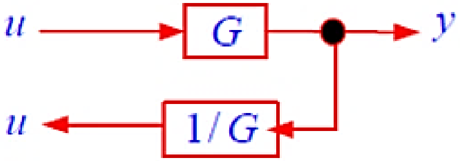


Pickoff point

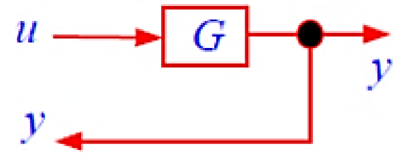
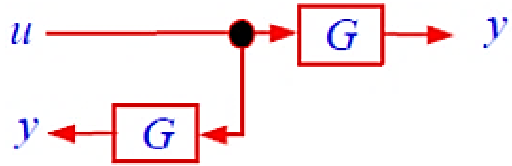
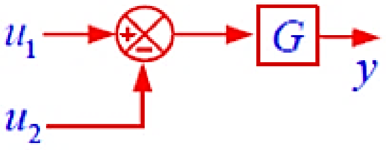
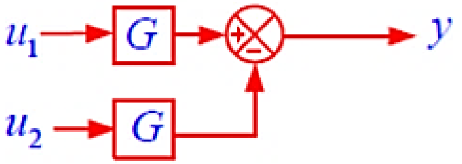
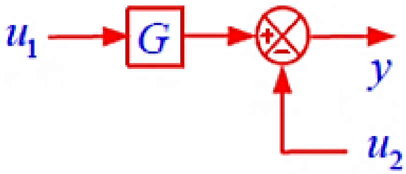
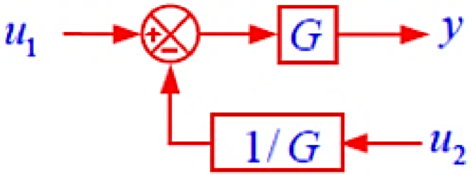
(*d*)

Block diagram

Basic rules with block diagram transformation

	Manipulation	Original Block Diagram	Equivalent Block Diagram	Equation
1	Combining blocks in cascade			$Y=(G_1G_2)X$
2	Combining blocks in parallel; or eliminating a forward loop			$Y=(G_1 \pm G_2)X$
3	Moving a pickoff point after a block			$y = Gu$ $u = \dfrac{1}{G}y$

Block diagram

	Manipulation	Original Block Diagram	Equivalent Block Diagram	Equation
4	Moving a pickoff point before a block			$y = Gu$
5	Moving a summing point after a block			$y = G(u_1 - u_2)$
6	Moving a summing point before a block			$y = Gu_1 - u_2$

Summary

- Modeling of DC motor, nonlinear systems, delay time.
- Main message up to this point: "*Many systems can be represented as transfer functions!*
- Next
  - *Stability* of linear control systems, which is one of the most important topics in feedback control.