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ELEC 341: Systems and Control

Lecture 16

Frequency response

Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - Electrical
  - Electromechanical
  - Mechanical
- Linearization, delay

Analysis

- Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

Design

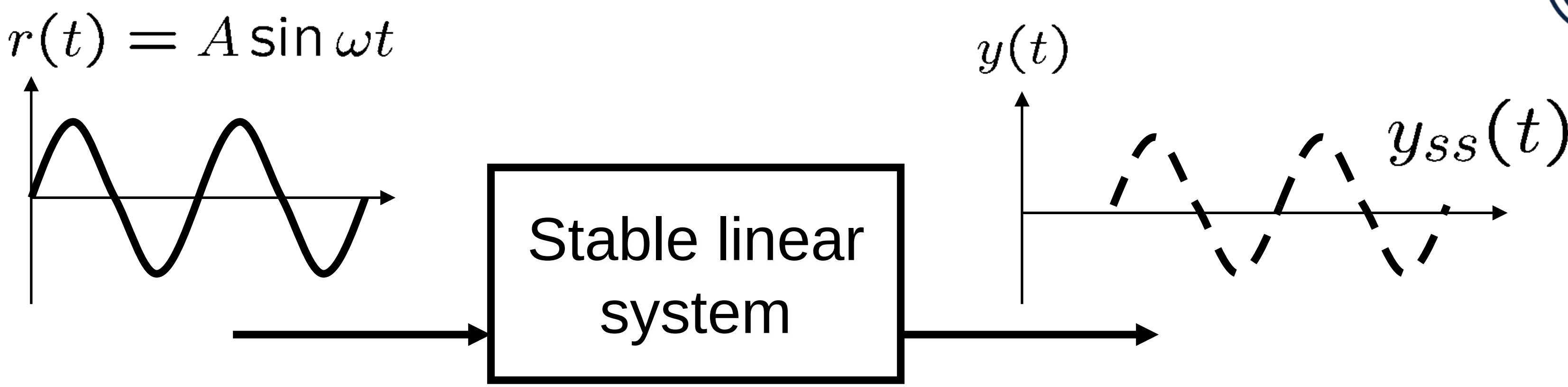
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

Summary up to now & *Topics from now on*

- **Modeling:** How to represent systems with transfer functions (s-domain).
- **Analysis:** How to extract time-response information from s-domain.
  - Steady-state error depends on TF evaluated at $s = 0$.
  - Stability and transient depends on pole locations.
  - ***Frequency responses contain all this information.***
- **Design:** How to obtain "nice" closed-loop system.
  - Poles can be placed by the root-locus technique.
  - ***System's frequency responses can be shaped in Bode plot.***

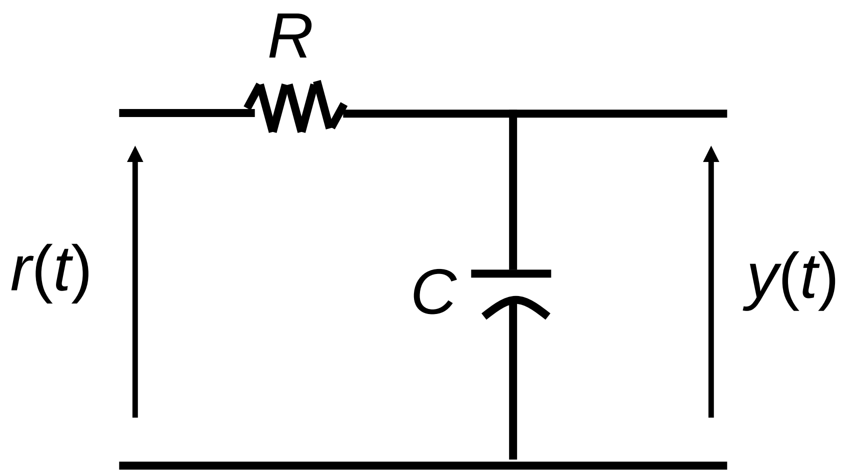
What is frequency response?



- We would like to analyze a system property by applying a *sinusoidal input* $r(t)$ and observing a response $y(t)$.
- Steady state response $y_{ss}(t)$ (after transient dies out) of a system to sinusoidal inputs is called ***frequency response***.

Example 1

- RC circuit



- Input a sinusoidal voltage $r(t)$
- What is the output voltage $y(t)$?

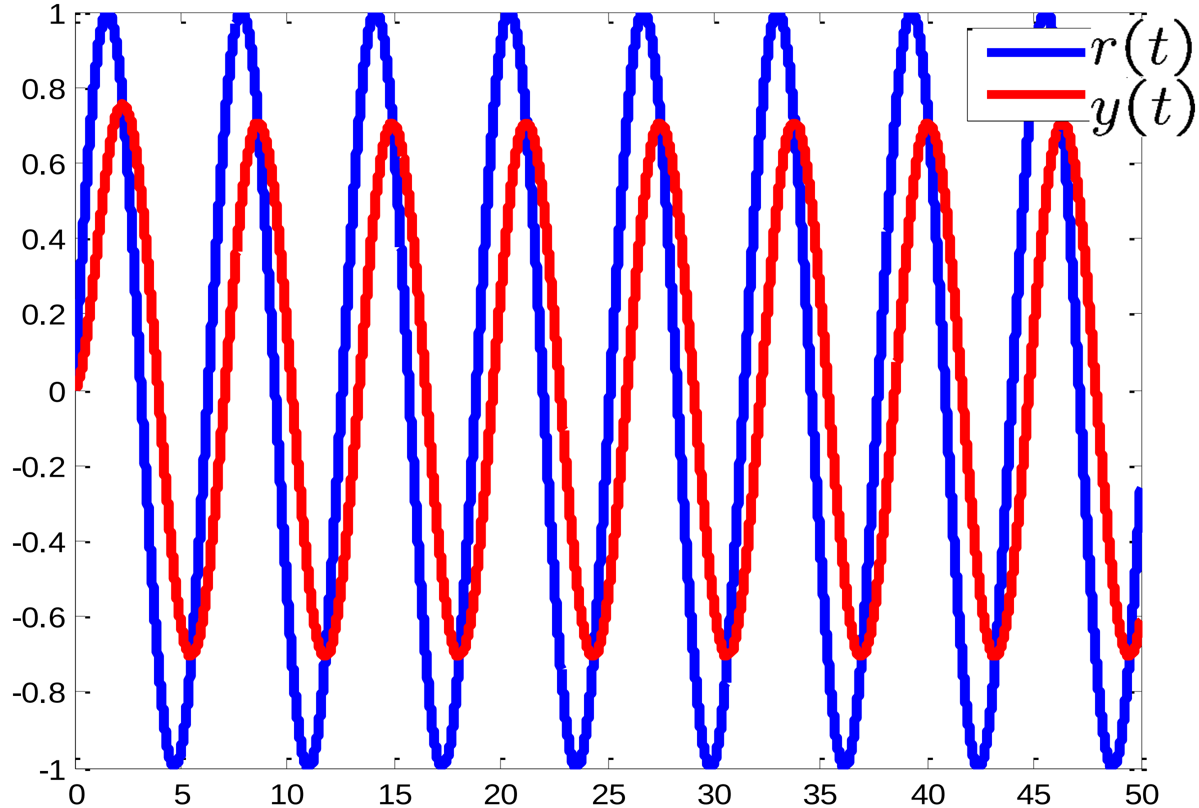
$$G(s) = \frac{Y(s)}{R(s)} \quad \Rightarrow \quad G(s) = \frac{1}{RCs+1}$$

Example 1 (cont'd)

- TF ($R = C = 1$)

$$G(s) = \frac{1}{s+1}$$

- $r(t) = \sin(t)$



At steady-state, $r(t)$ and $y(t)$ have the same frequency, but different amplitude and phase!

Example 1 (cont'd)

- Derivation of $y(t)$

$$Y(s) = G(s)R(s) = \frac{1}{s+1}\cdot\frac{1}{s^2+1} = \frac{1}{2}\left(\frac{1}{s+1} + \frac{-s+1}{s^2+1}\right)$$

Partial fraction expansion

- Inverse Laplace

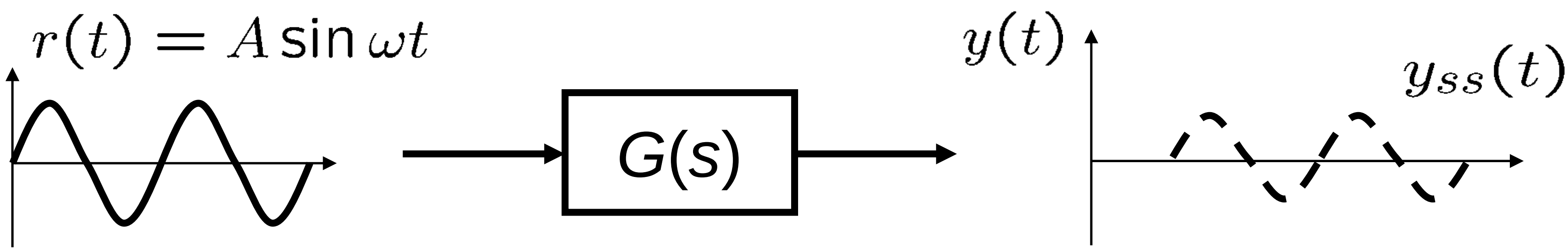
$$y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

0 as *t* goes to infinity.

$$\Rightarrow \quad y_{ss}(t) = \frac{1}{2}(-\cos t + \sin t) = \frac{1}{\sqrt{2}}\sin(t - 45^\circ)$$

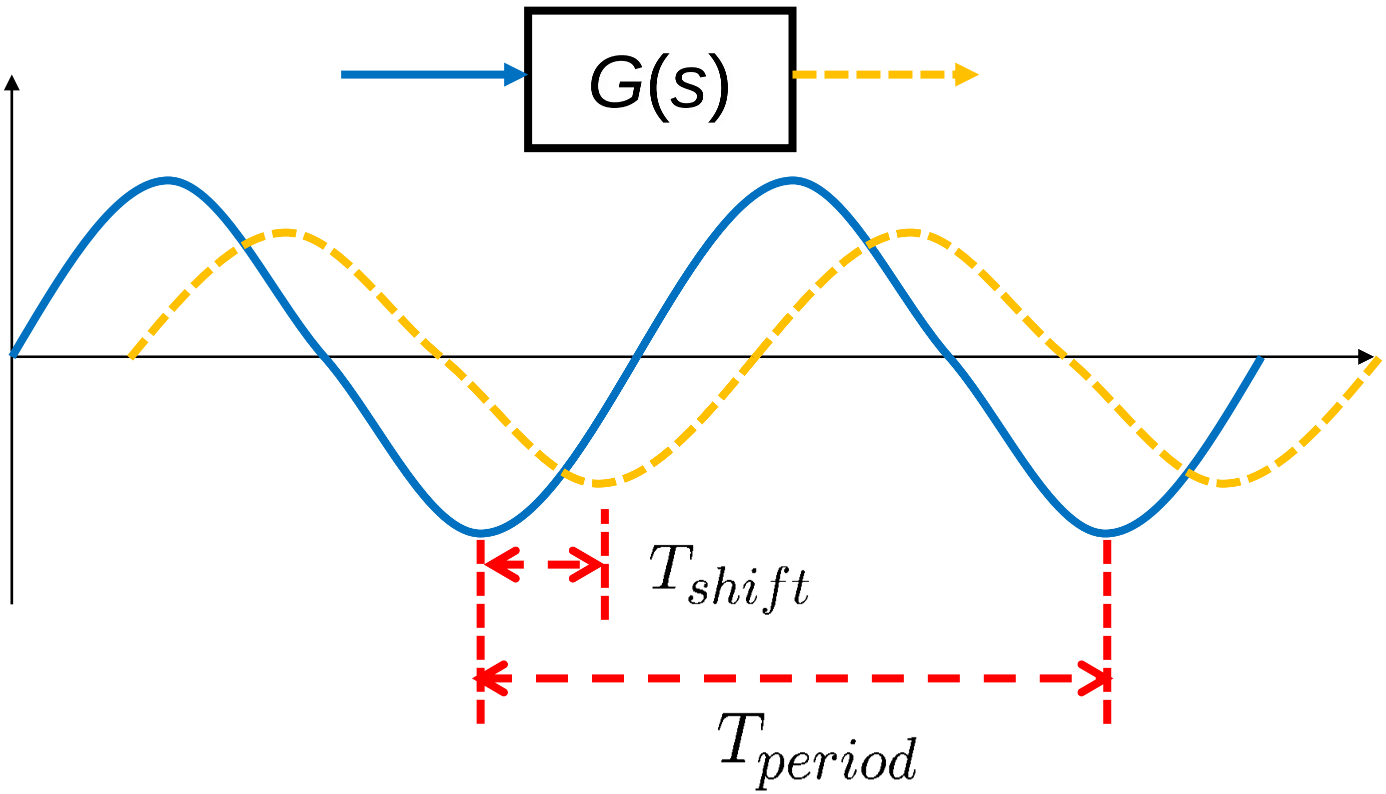
Response to sinusoidal input

- What is the steady state output of a stable linear system when the input is sinusoidal?



- Steady state output is $y_{ss}(t) = A\,|G(j\omega)|\,\sin(\omega t + \angle G(j\omega))$
  - Frequency is the same as the input frequency ω
  - Amplitude is that of input (A) multiplied by $\underbrace{|G(j\omega)|}_{\text{Gain}}$
  - Phase shifts is $\angle G(j\omega)$

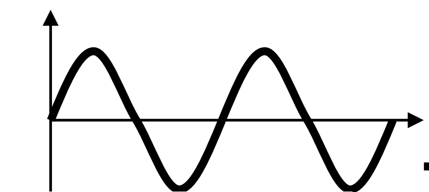
Phase shift



$$\frac{T_{shift}}{T_{period}} = \frac{-\angle G(j\omega)}{360^\circ} \quad \Rightarrow \quad \angle G(j\omega) = -\frac{T_{shift}}{T_{period}} \times 360^\circ$$

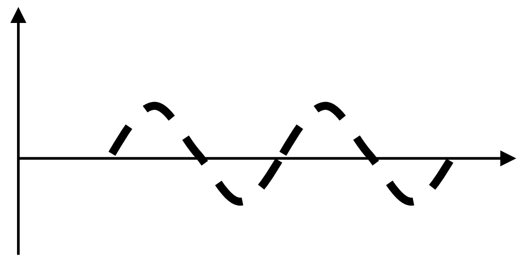
Example 1 (revisited)

- What is the steady state output of a stable linear system when the input is sinusoidal?



$r(t) = \sin t$

$$G(s) = \frac{1}{s+1}$$



$A = 1$; $\omega = 1$

$$y_{ss}(t) = 1 \cdot \underbrace{\frac{1}{\sqrt{2}}}_{|G(j \cdot 1)|} \sin(t \underbrace{-45^\circ}_{\angle G(j \cdot 1)})$$

Gain:

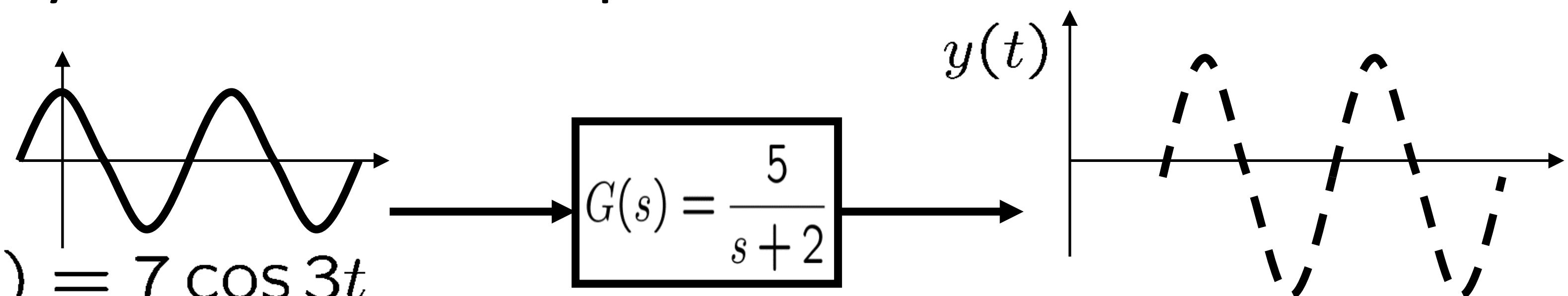
$$|G(j \cdot 1)| = \left|\frac{1}{j+1}\right| = \frac{1}{|j+1|} = \frac{1}{\sqrt{2}}$$

Phase:

$$\angle G(j \cdot 1) = \angle \frac{1}{j+1} = \underbrace{\angle 1}_{=0^\circ} - \underbrace{\angle (j+1)}_{=45^\circ}$$

Example 2

- What is the steady state output of a stable linear system when the input is sinusoidal?



$r(t) = 7\cos 3t$

$A = 7 \;;\; \omega = 3$

$$y_{ss}(t) = 7 \cdot \underbrace{\frac{5}{\sqrt{13}}}_{|G(j\cdot 3)|} \cos(3t + \underbrace{\theta}_{\angle G(j\cdot 3)})$$

Gain

Phase

Phase:

$$\angle G(j\cdot 3) = \angle \frac{5}{3j+2}$$
$$= \underbrace{\angle 5}_{=0^\circ} - \underbrace{\angle(3j+2)}_{=\tan^{-1}\frac{3}{2}}$$

θ = -56.3°

$$\boxed{y_{ss}(t) = 9.7\cos(3t - 56.3^\circ)}$$

Example 3: Frequency response function

- For a stable system $G(s)$, $G(j\omega)$ (ω is positive) is called *frequency response function (FRF)*.
- For each ω, FRF represents a complex number $G(j\omega)$, which has a gain and a phase.
- First order example

$$G(s) = \frac{1}{s+1} \Rightarrow G(j\omega) = \frac{1}{j\omega + 1}$$

$$\Rightarrow \begin{cases} |G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}} \\ \angle G(j\omega) = \angle(1) - \angle(j\omega + 1) = -\tan^{-1}\omega \end{cases}$$

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Example 3: First order FRF (cont'd)

- FRF $G(j\omega) = \dfrac{1}{j\omega + 1}$

frequency ω	gain $\lvert G(j\omega)\rvert$	phase $\angle G(j\omega)$
0	1	0°
0.5	0.894	-26.6°
1.0	0.707	-45°
$\vdots$	$\vdots$	$\vdots$
∞	0	-90°

- Two ways to represent a complex number (amplitude-phase or real-imaginary parts), therefore, two graphs representing FRF:
  - **Bode diagram (Bode plot)** (today's and next lecture)
  - **Nyquist diagram (Nyquist plot)** (in two lectures)

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Example 4: Second order FRF

- Second order system

$$G(s) = \frac{2}{s^2 + 3s + 2}$$

$$\Rightarrow \quad G(j\omega) = \frac{2}{(j\omega)^2 + 3(j\omega) + 2} = \frac{2}{2 - \omega^2 + j \cdot 3\omega}$$

$$\Rightarrow \quad \begin{cases} |G(j\omega)| = \dfrac{2}{\sqrt{(2-\omega^2)^2 + 9\omega^2}} \\ \angle G(j\omega) = \angle(2) - \angle(2 - \omega^2 + j \cdot 3\omega) \\ \qquad\quad\; = -\tan^{-1} \dfrac{3\omega}{2 - \omega^2} \end{cases}$$

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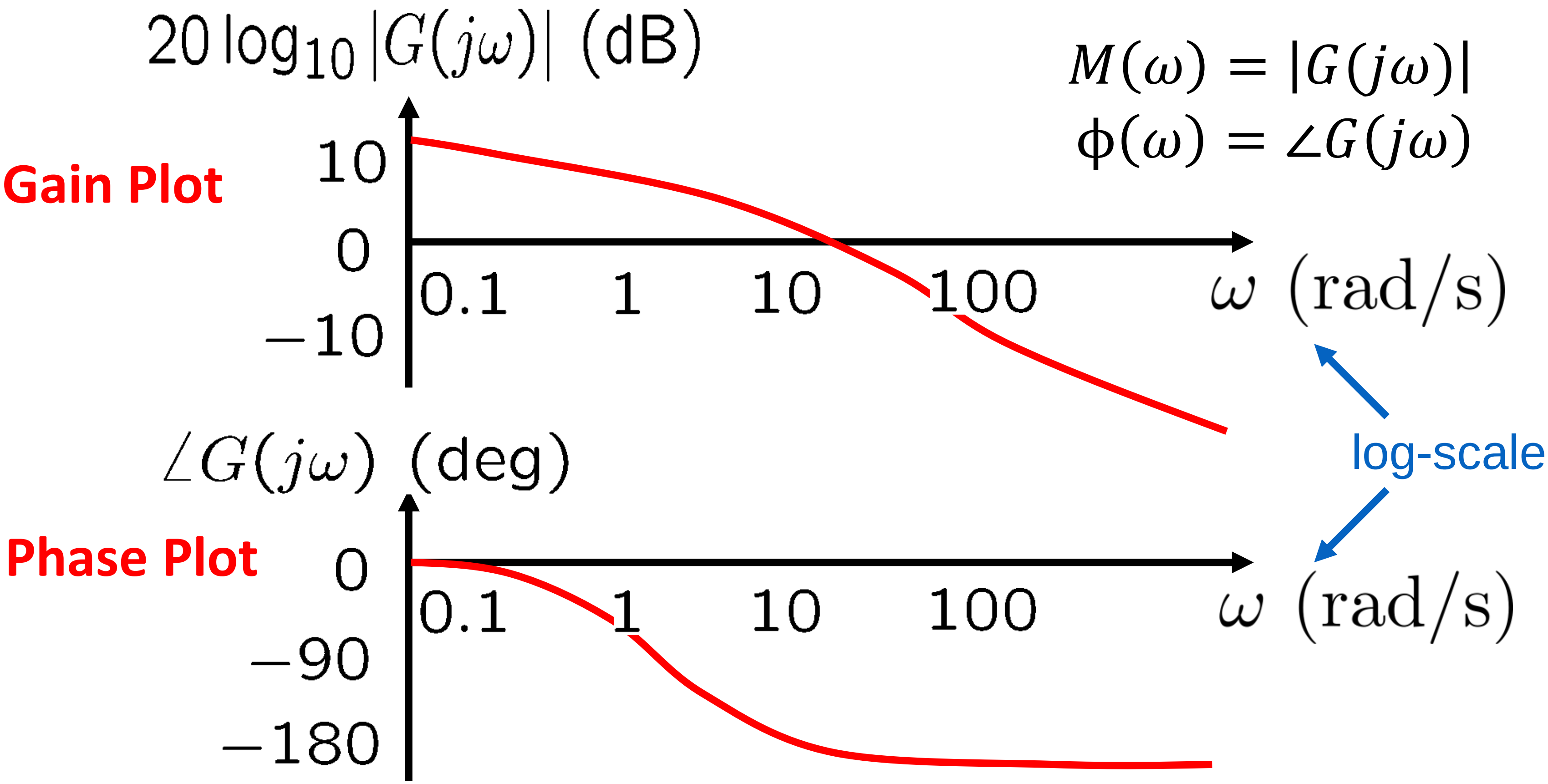
Matlab simulations

Bode plot (Bode diagram) of $G(j\omega)$

- Bode diagram consists of gain plot & phase plot

$$M(\omega) = |G(j\omega)|$$
$$\phi(\omega) = \angle G(j\omega)$$



Bode plot of a 1st order system

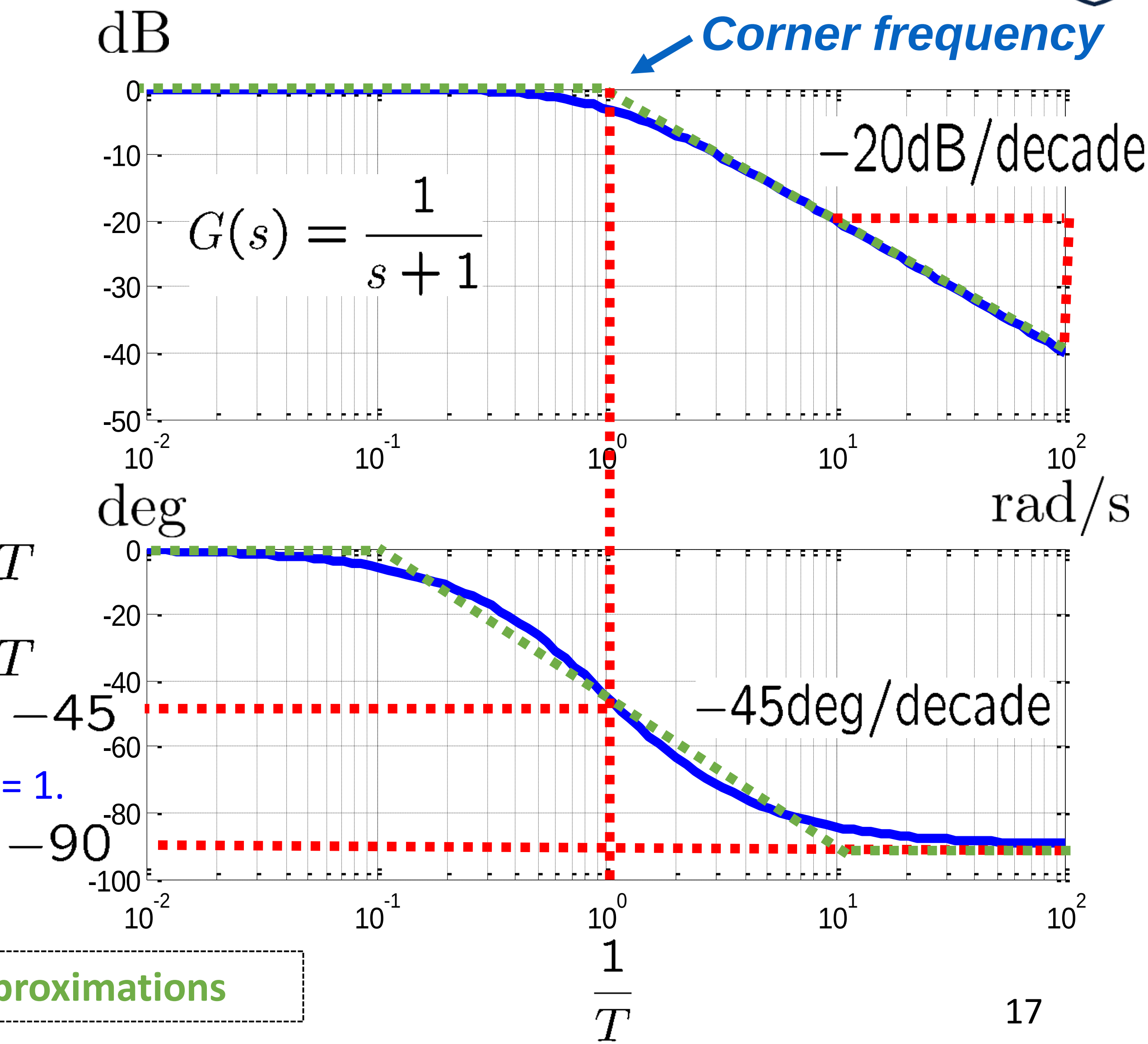
- TF

$$G(s) = \frac{1}{Ts+1}$$

$$G(j\omega) = \frac{1}{j\omega T + 1}$$

$$\approx \begin{cases} 1 & \text{if } 1 \gg \omega T \\ \frac{1}{j\omega T} & \text{if } 1 \ll \omega T \end{cases}$$

The shown Bode plot is for $T = 1$.



”-----” are straight line approximations

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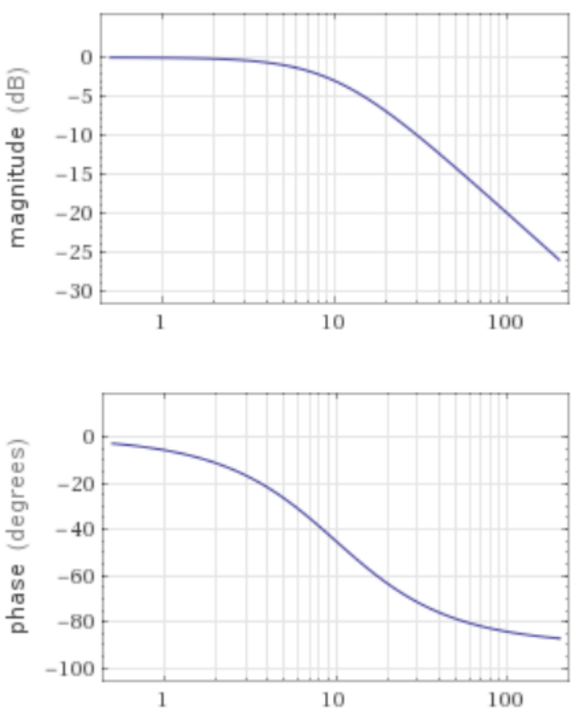
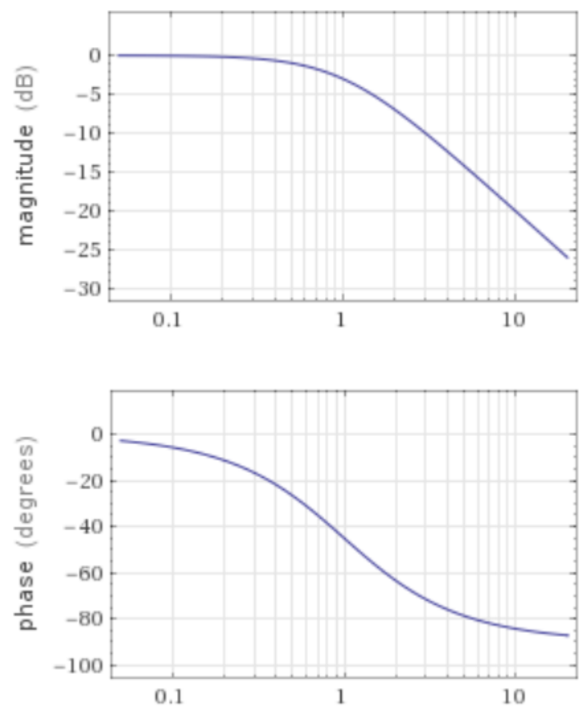
UBC

Effect of T on Bode plot (first order system)

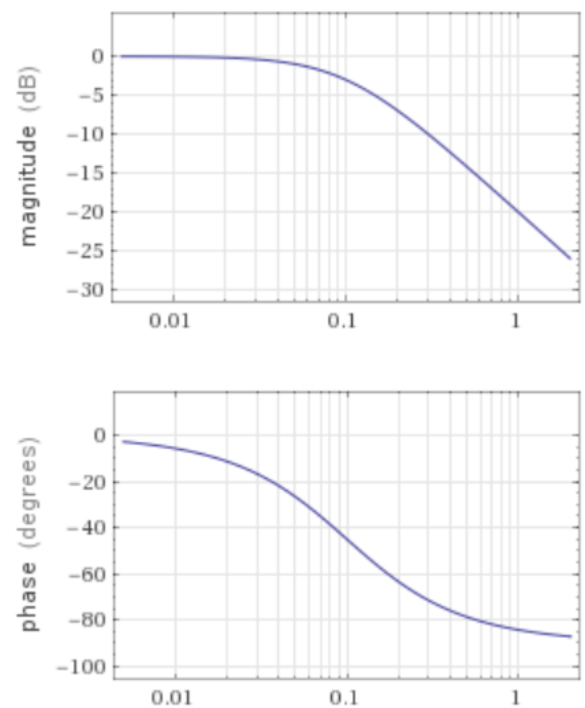
$$G(s) = \frac{1}{0.1s + 1}$$



$$G(s) = \frac{1}{s + 1}$$



$$G(s) = \frac{1}{10s + 1}$$



Remarks on Bode diagram

- Bode diagram shows **gain** and **phase shift** of a system output for sinusoidal inputs with various frequencies.
- Bode diagram is very useful and important in analysis and design of control systems.
- The shape of Bode plot contains information of CL stability, time responses, and much more!
- It can also be used for **system identification**.
  - Given FRF experimental data, obtain a transfer function that matches the data.
- MATLAB command for bode plot is “bode(sys)”.

System identification

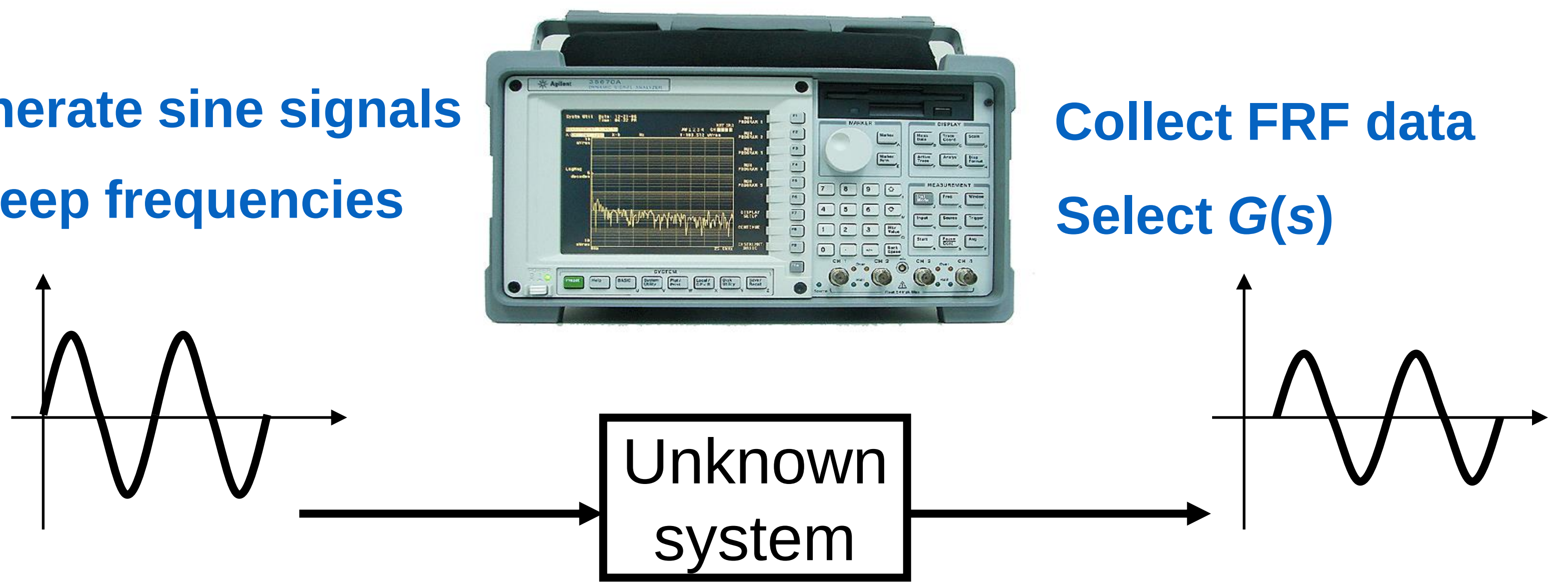
- Sweep frequencies of sinusoidal signals and obtain FRF data (i.e., gain and phase).
- Select $G(s)$ so that $G(j\omega)$ fits the FRF data.

Generate sine signals

Sweep frequencies

Collect FRF data

Select $G(s)$



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Why deg(den) ≥ deg(num)?

- All the transfer functions we encountered so far have the property $\deg(\mathrm{den}) \geq \deg(\mathrm{num})$

  Ex: $\dfrac{1}{Ms^2 + Bs + K}$ $\quad \dfrac{K}{Ts+1}$ $\quad K\dfrac{s+z}{s+p}$

- What if deg(num) is larger than deg(den)?
  - Then, $|G(j\omega)| \to \infty \text{ as } \omega \to \infty$
  - However, there is no such system in reality that has increasing gain as input frequency increases to infinity.
- That is why all the transfer function needs to meet

$$\deg(\mathrm{den}) \geq \deg(\mathrm{num})$$

Summary

- Frequency response
  - Steady state response to a sinusoidal input
  - For a linear stable system, a sinusoidal input generates a sinusoidal output with same frequency but different amplitude and phase.
- Bode plot is a graphical representation of frequency response function.
- Next
  - How to sketch Bode plots