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ELEC 341: Systems and Control

Lecture 14

Root locus: Lead-lag compensator design

Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - Electrical
  - Electromechanical
  - Mechanical
- Linearization, delay

Analysis

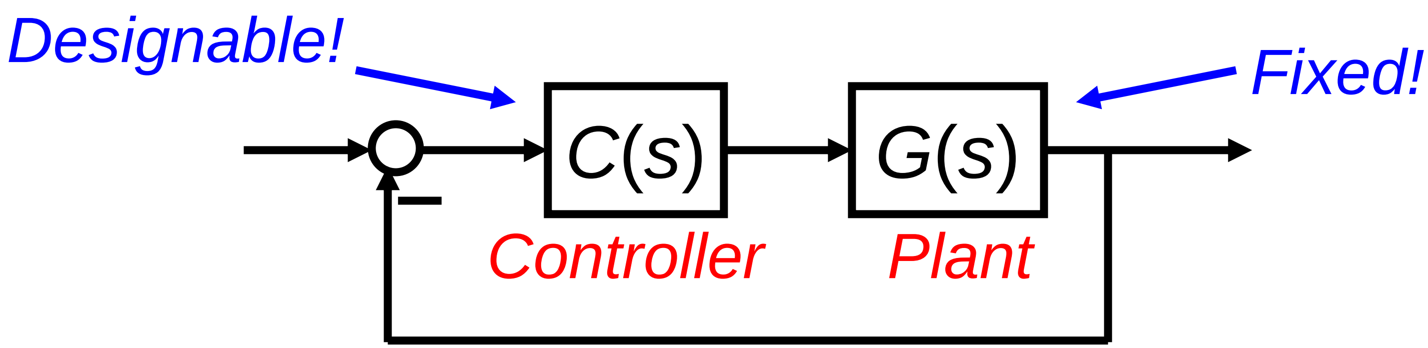
- Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

Controller design by root locus

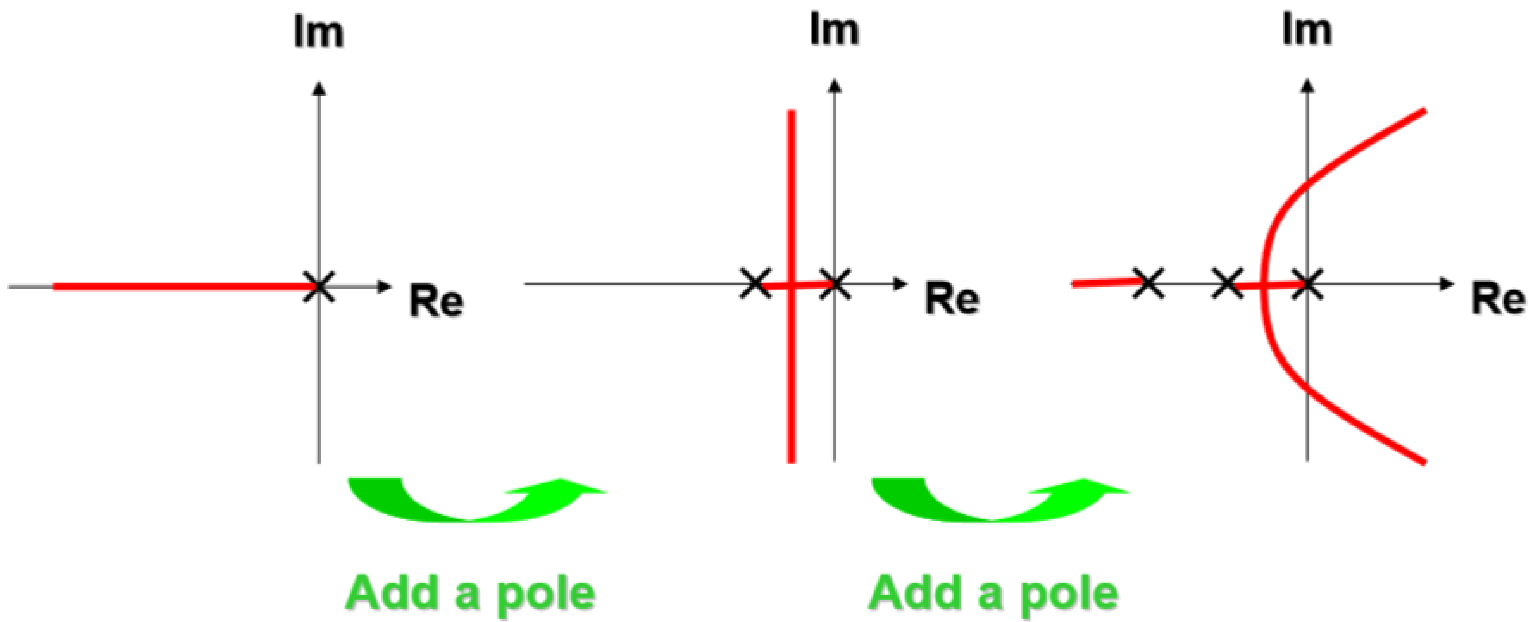


- Place closed-loop poles at desired locations
  - by tuning the gain $C(s) = K$. (for time domain specs)
- If root locus does not pass through the desired location, then **reshape** the root locus
  - by adding poles/zeros to $C(s)$. How?

  Compensation

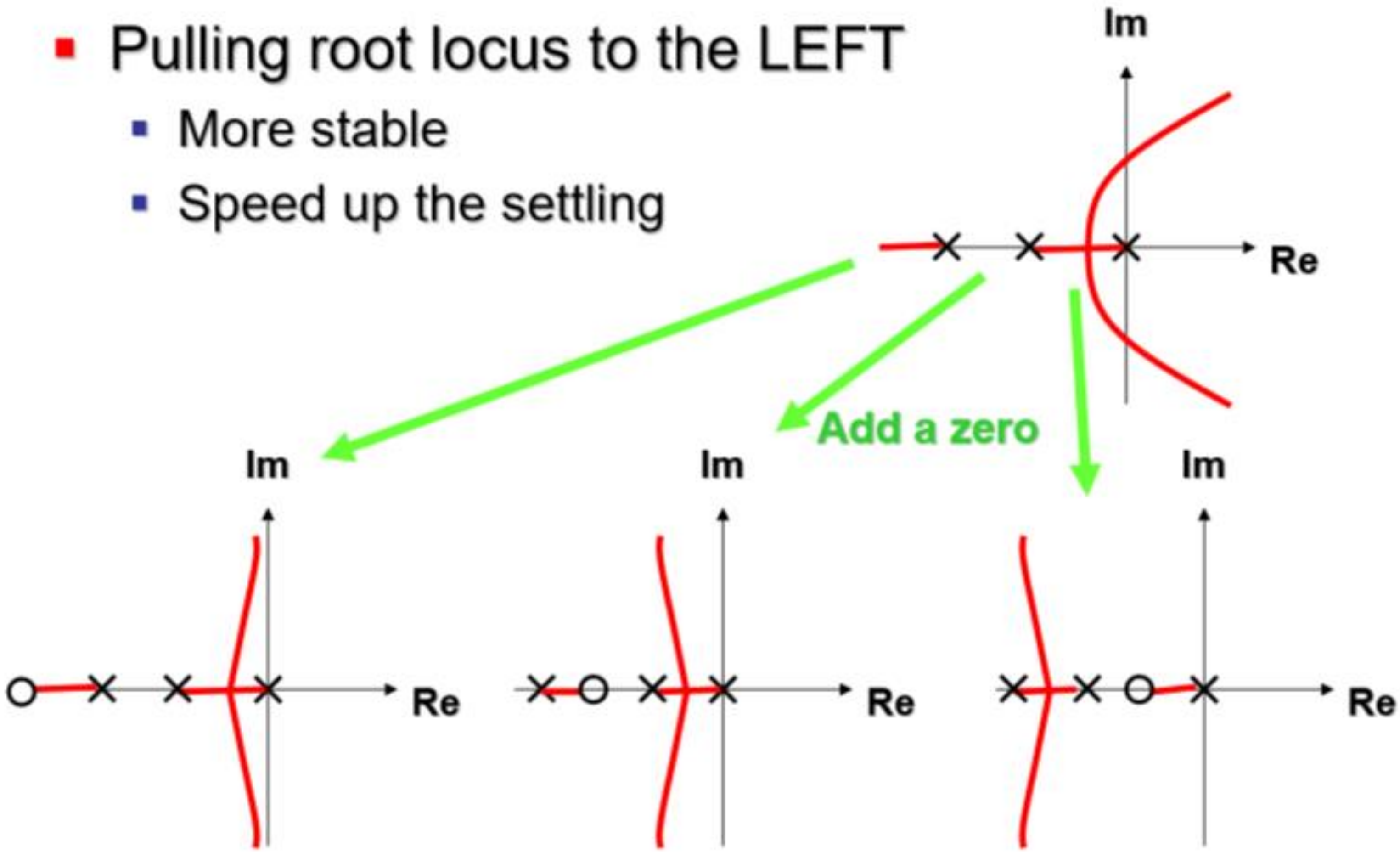
General effect of addition of poles

- Pulling root locus to the RIGHT
  - Less stable
  - Slow down the settling



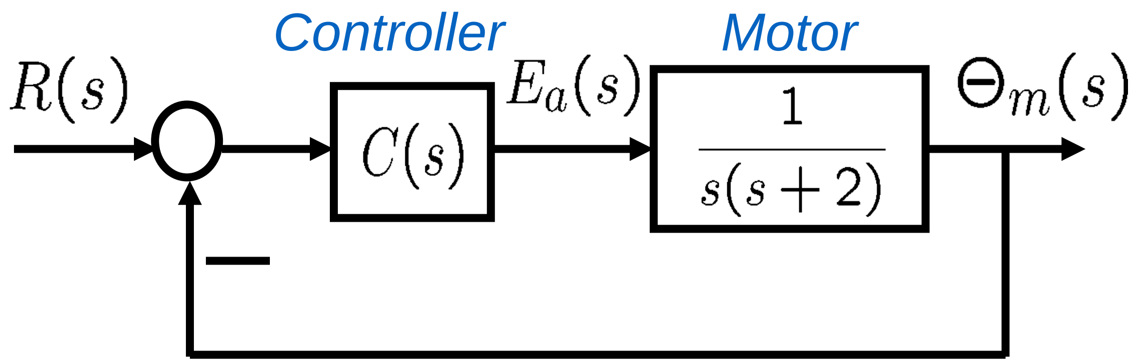
General effect of addition of zeros

- Pulling root locus to the LEFT
  - More stable
  - Speed up the settling



Example 1

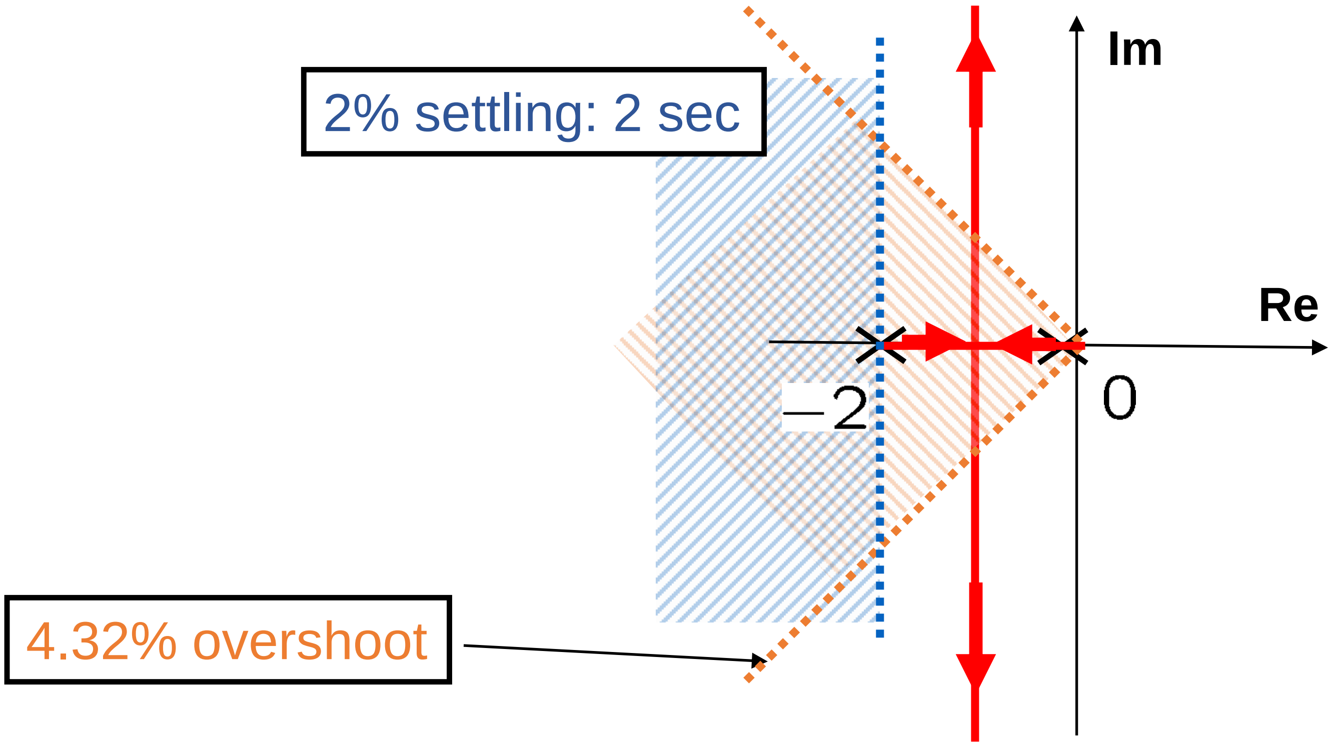
- A feedback system



- Design specifications
  - 2% settling time at most 2 seconds
  - Overshoot at most 4.32%
  - Steady state error
    - Zero for unit step $r(t) = u(t)$
    - At most 0.05 for unit ramp $r(t) = tu(t)$

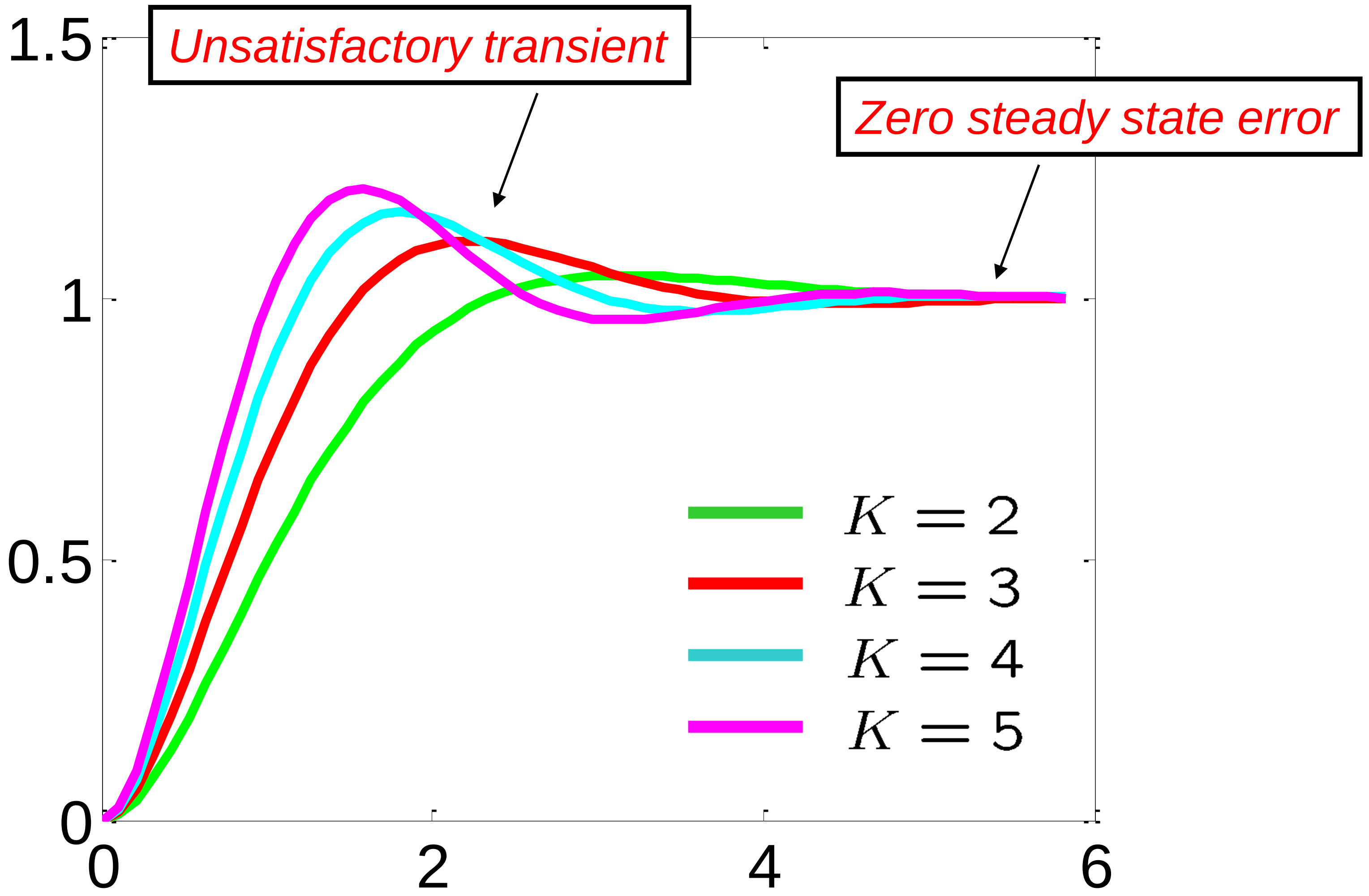
Example 1 (cont'd)

- Root locus for gain controller $C(s) = K > 0$



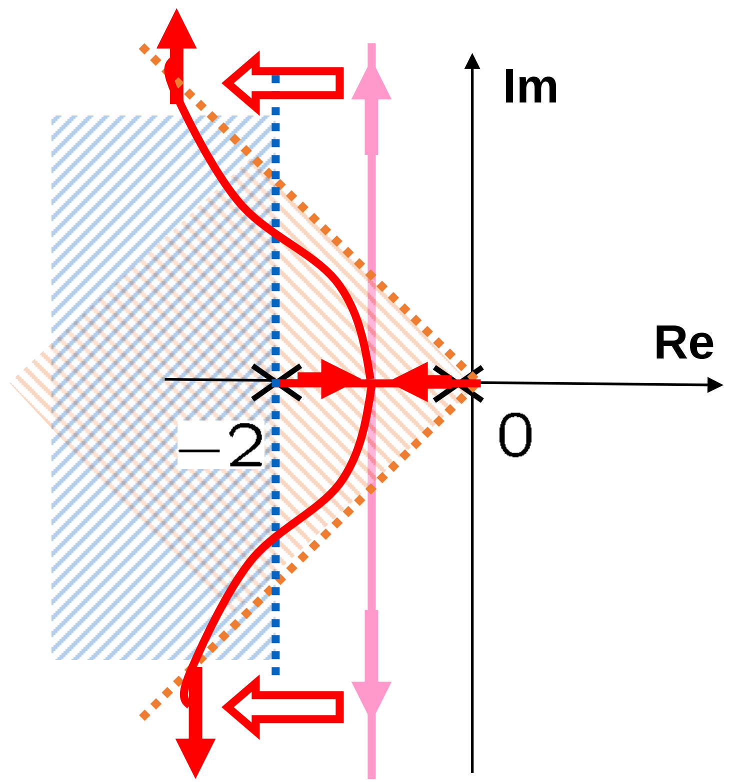
We cannot achieve the design specs with the gain feedback controller!

Example 1: Step responses for gain controllers $C(s) = K > 0$

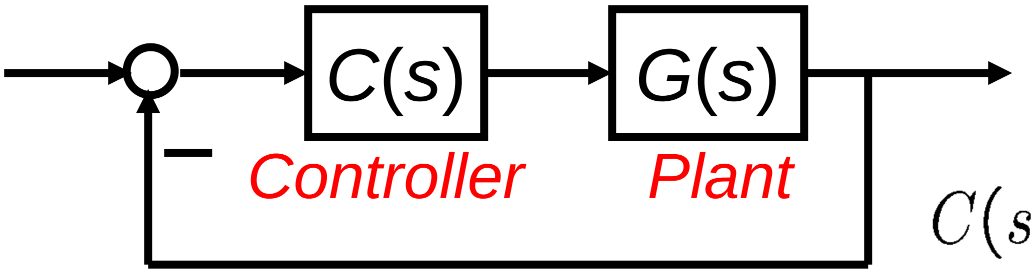


Example 1 (cont'd)

- We **reshape** the root locus so that it passes through the allowable region.

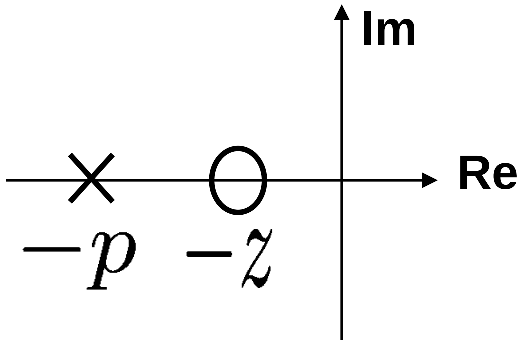


Lead and lag compensators

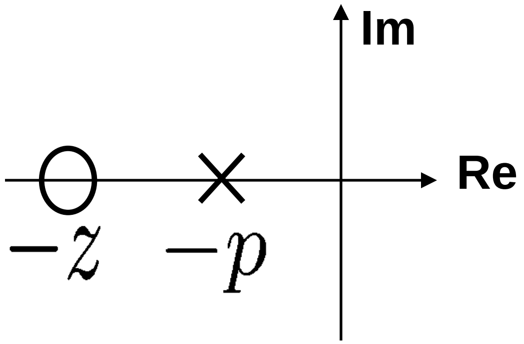


$$C(s) = K\frac{s+z}{s+p}, \quad (z > 0, p > 0)$$

- Lead compensator



- Lag compensator



The reason why these are called "lead" and "lag" will be explained in frequency response approach (later in this course).

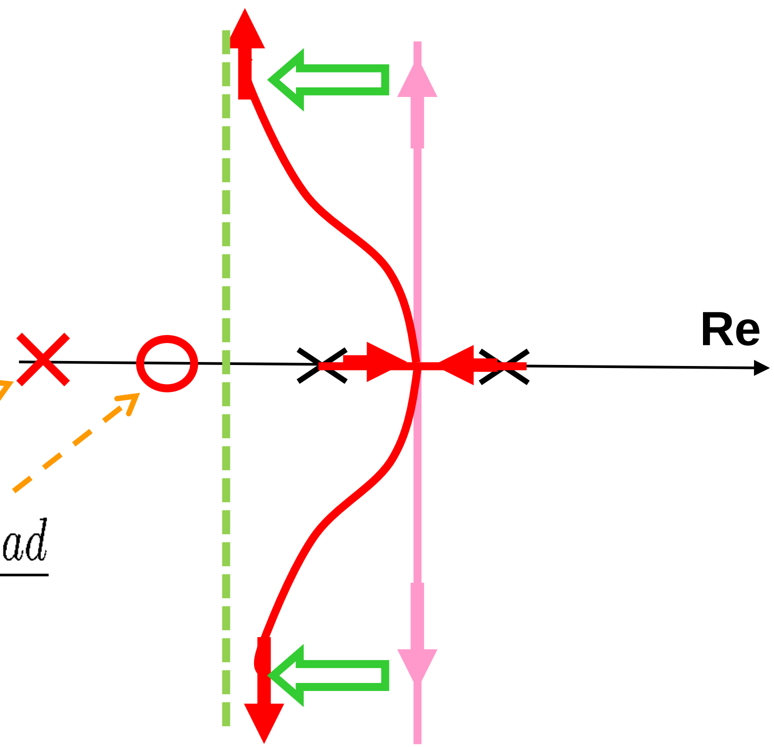
Reshaping root locus by lead compensators

- Lead compensators move the intersection of asymptotes (the centroid) to the left.
  - With plant only

$$\frac{\sum \text{pole} - \sum \text{zero}}{r}$$

  - With plant and lead

$$\frac{\sum \text{pole} - \sum \text{zero} + \text{pole}_{lead} - \text{zero}_{lead}}{r}$$



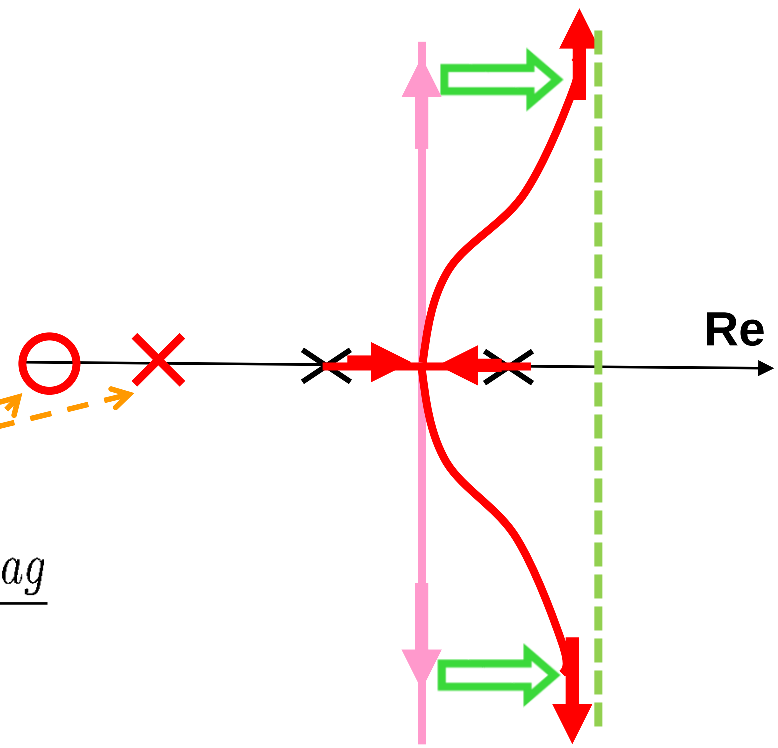
Reshaping root locus by lag compensators

- Lag compensators move the intersection of asymptotes (the centroid) to the right.
  - With plant only

$$\frac{\sum \text{pole} - \sum \text{zero}}{r}$$

  - With plant and lag

$$\frac{\sum \text{pole} - \sum \text{zero} + \text{pole}_{lag} - \text{zero}_{lag}}{r}$$



Roles of lead & lag compensators

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- Lead compensator
  - Improves transient response
  - Improves stability

$$C_{Lead}(s) = K\frac{s + z_{Lead}}{s + p_{Lead}}$$

- Lag compensator
  - Reduces steady state error

$$C_{Lag}(s) = \frac{s + z_{Lag}}{s + p_{Lag}}$$

- Lead-lag compensator
  - Takes into account both transient and steady state.

$$C_{LL}(s) = C_{Lead}(s)C_{Lag}(s)$$

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Analytical lead compensator design

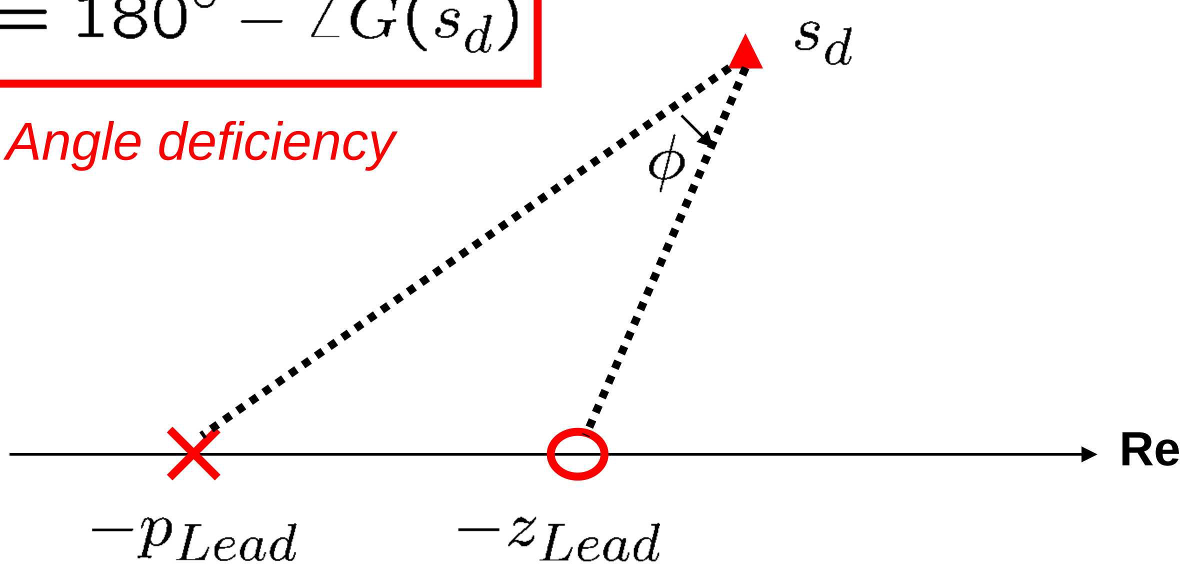
1. Select a desired pole in the allowable region. We aim at reshaping RL to pass through this pole.

Analytical lead compensator design

2. Select pole/zero in $C_{Lead}(s) = K\frac{s+z_{Lead}}{s+p_{Lead}}$ as

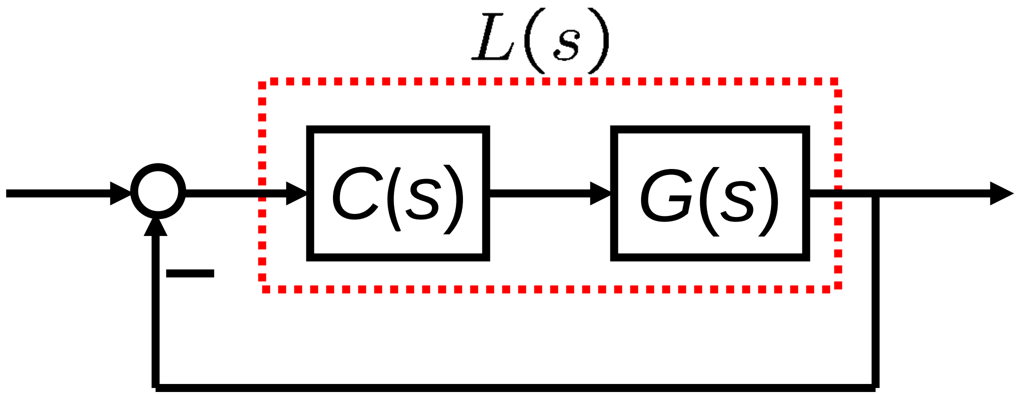
$$\boxed{\phi = 180^\circ - \angle G(s_d)}$$

Angle deficiency



Angle and magnitude conditions

- For an open-loop transfer function *L*(*s*),



- A point s to be on root locus ←→ it satisfies
  - **Angle condition**

Odd number

$$\angle L(s) = 180^0 \times (2k+1), \;\; k = 0, \pm 1, \pm 2, \ldots$$

- For a point on root locus, gain K is obtained by
  - **Magnitude condition**

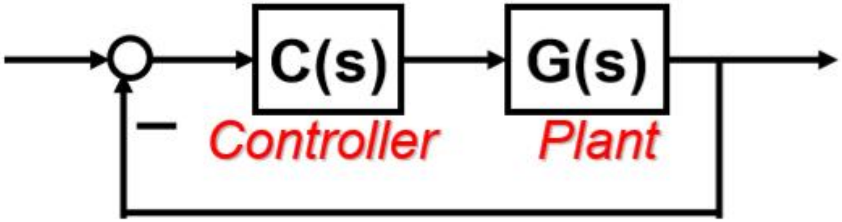
$$|L(s)| = \frac{1}{K}$$

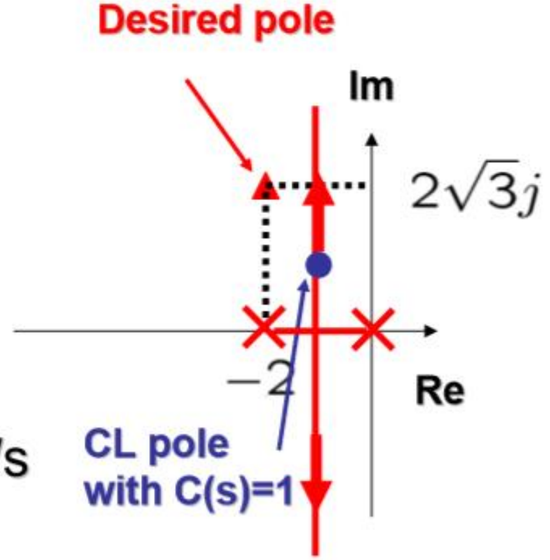
Example 2

Lead compensator design

- Consider a system

$$G(s) = \frac{4}{s(s+2)}$$





- Performance specification
  - Damping ratio ζ=0.5
  - Undamped natural freq. ω_n=4 rad/s

Example 2 (cont'd)

Lead compensator design (cont'd)

Evaluate G(s) at the desired pole. $s_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$

$$\underbrace{G(-2+2\sqrt{3}j)}_{s_d} = \frac{4}{(-2+2\sqrt{3}j)2\sqrt{3}j}$$

$$\angle G^*(s_d) = \angle 4 - \angle(-2+2\sqrt{3}j) - \angle 2\sqrt{3}j$$

$$= 0° - \left(\tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right)\right) - 90°$$

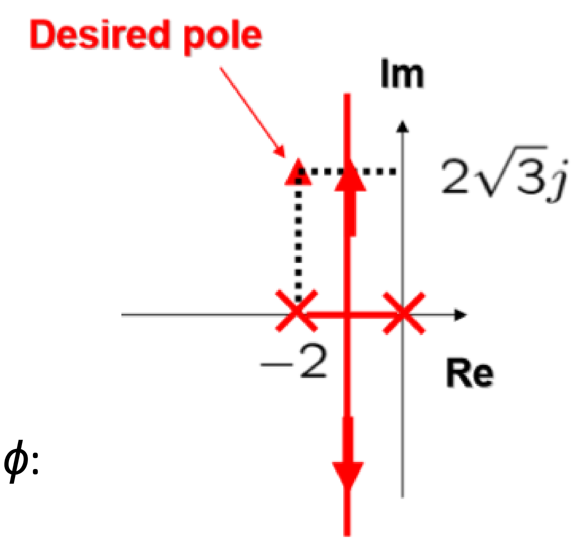
$$= -(+120°) - 90° = -210° \rightarrow$$

$\angle G^*(s_d) = -210°$

Convert "-210°" to an angle from +Re-axis:

$360° - 210° = +150° \rightarrow \angle G(s_d) = = +150°$

Now, use the **angle deficiency formula** to calculate ϕ:



$$\phi = 180° - \angle G(s_d) \rightarrow \phi = 180° - 150° \rightarrow \boldsymbol{\phi} = +\mathbf{30}°$$

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Example 2 (cont'd)

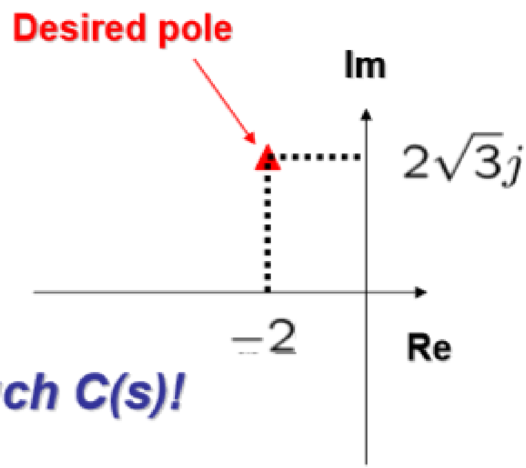
Lead compensator design (cont'd)

To compensate angle deficiency,
design a lead compensator C(s)

$$C(s) = K\frac{s+z}{s+p}$$

satisfying

$$\angle C(-2+2\sqrt{3}j) = 30 = \phi$$



There are many ways to design such C(s)!

Example 2 (cont'd)

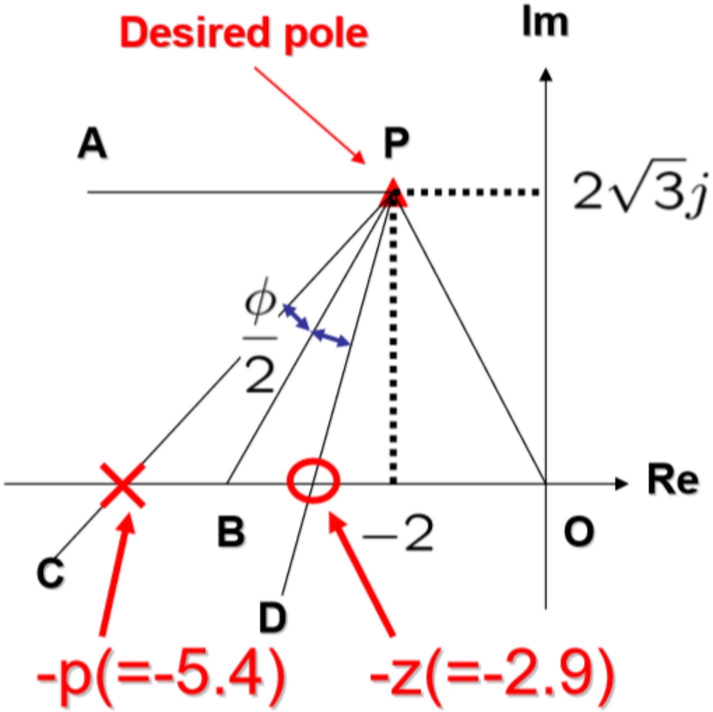
How to select pole and zero?

- Draw horizontal line PA
- Draw line PO
- Draw bisector PB

$$\angle APB = \angle BPO = \frac{1}{2}\angle APO$$

- Draw PC and PD

$$\angle CPB = \angle BPD = \frac{\phi}{2}$$

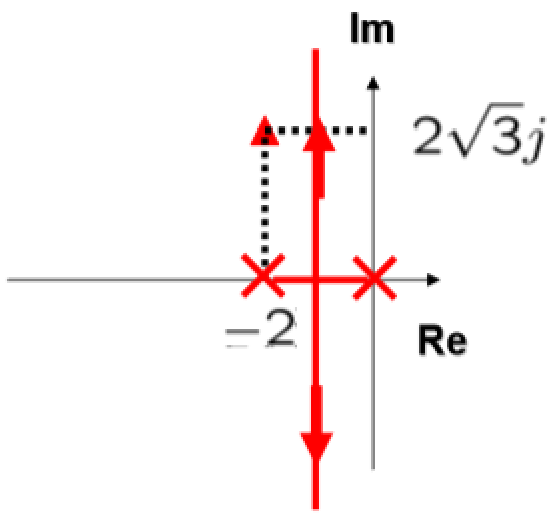


- Pole and zero of C(s) are shown in the figure.

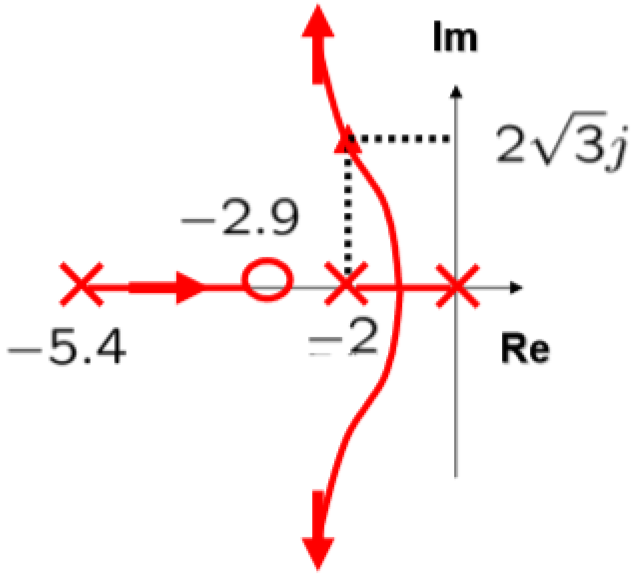
Example 2 (cont'd)

Comparison of root locus

- G(s)



- G(s)C(s)



Improved stability!

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Example 2 (cont'd)

How to design the gain K?

- Lead compensator

$$C(s) = K\frac{s+2.9}{s+5.4}$$

- Open loop transfer function

$$G(s)C(s) = K\frac{4(s+2.9)}{s(s+2)(s+5.4)}$$

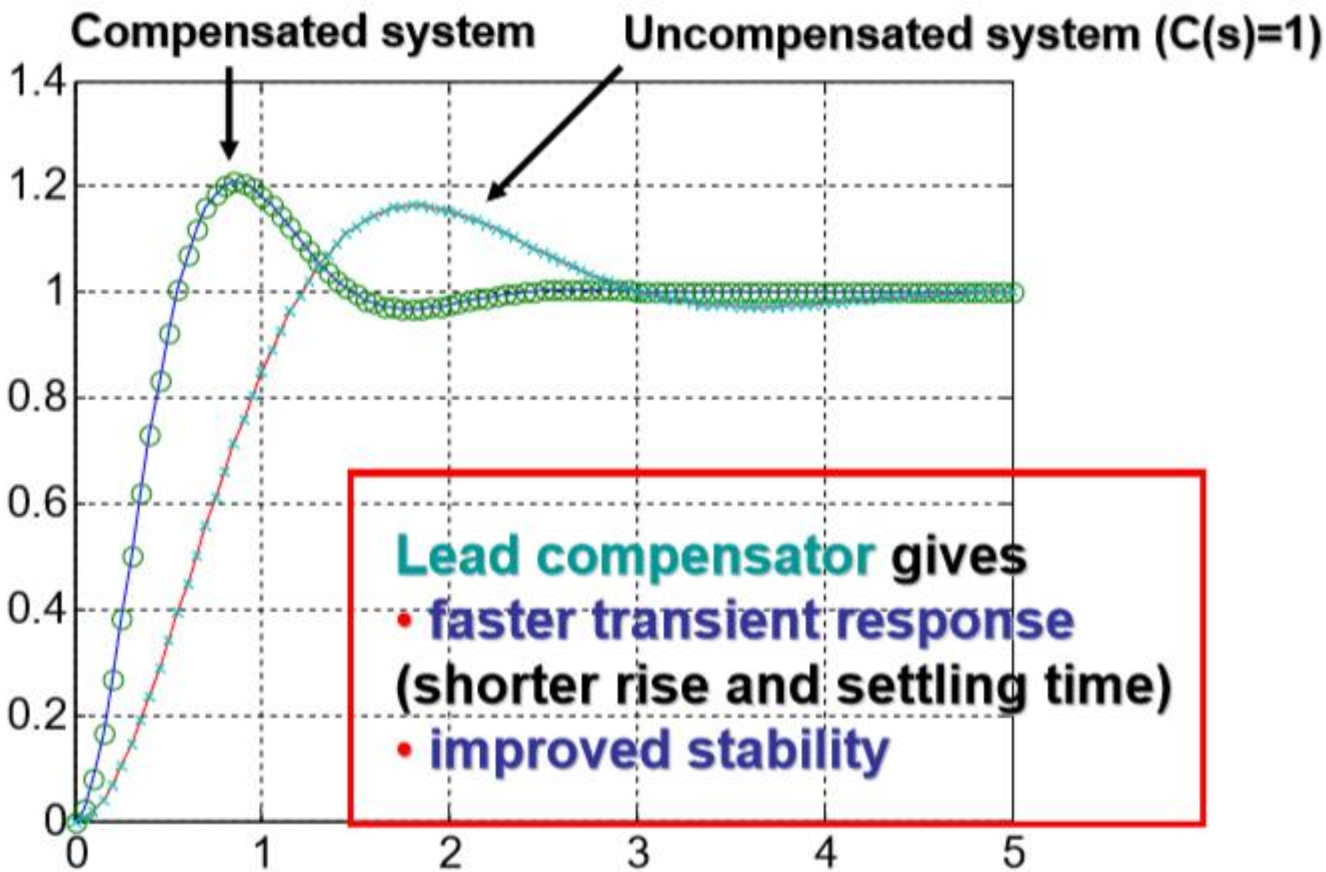
- Magnitude condition

$$K\left|\frac{4(s+2.9)}{s(s+2)(s+5.4)}\right|_{s=-2+2\sqrt{3}j} = 1 \quad \Rightarrow \quad \boxed{K = 4.675}$$

$$\Rightarrow \quad \boxed{G(s)C(s) = \frac{4}{s(s+2)} \cdot \frac{4.675(s+2.9)}{s+5.4}}$$

Example 2 (cont'd)

Comparison of step responses



Example 2 (cont'd)

Error constants

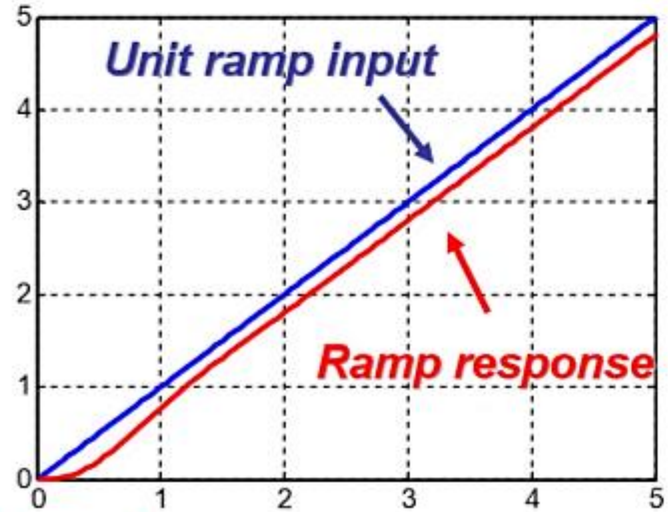
$$G(s)C(s) = \frac{4}{s(s+2)} \cdot \frac{4.675(s+2.9)}{s+5.4}$$

- Step-error constant

$$K_p = \lim_{s\to 0} G(s)C(s) = \infty$$

- Ramp-error constant

$$K_v = \lim_{s\to 0} sG(s)C(s) = 5.02$$



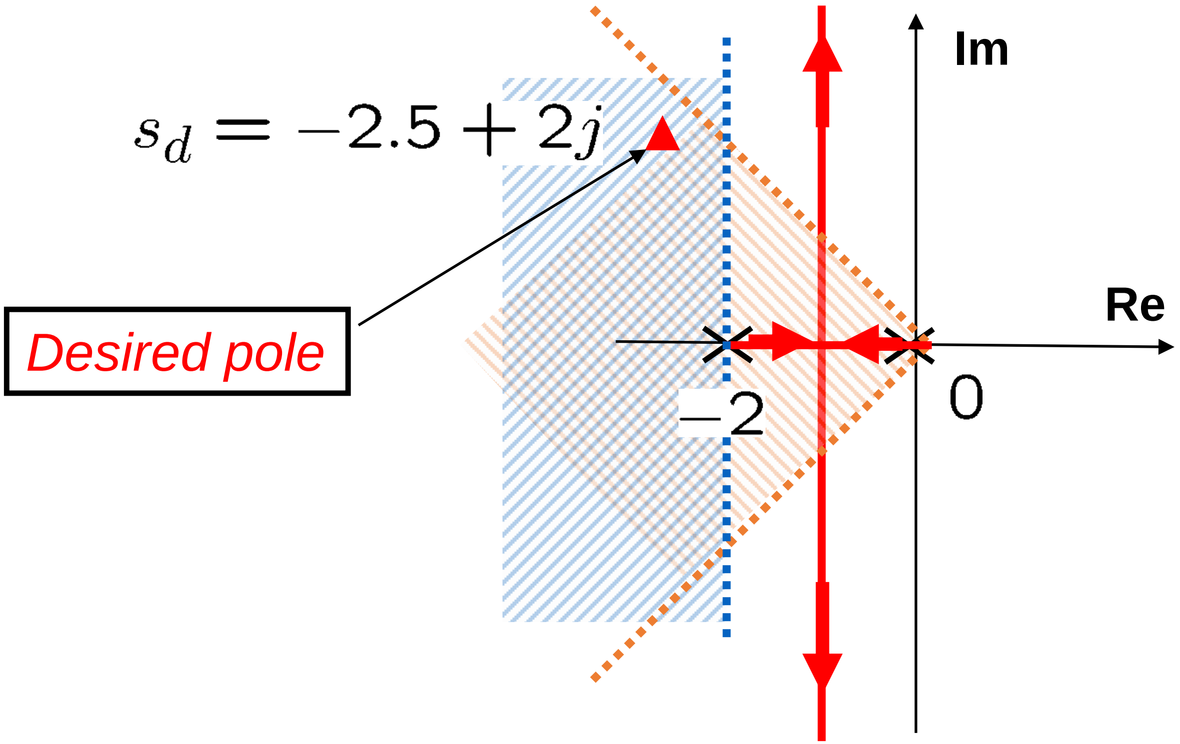
Lag compensator can be used to reduce steady-state error.

This will be shown soon.

Example 3

Design a lead compensator controller by using the given desired pole in the allowable region.

$$G(s) = \frac{1}{s(s+2)}$$



We aim at reshaping RL to pass through the desired pole.

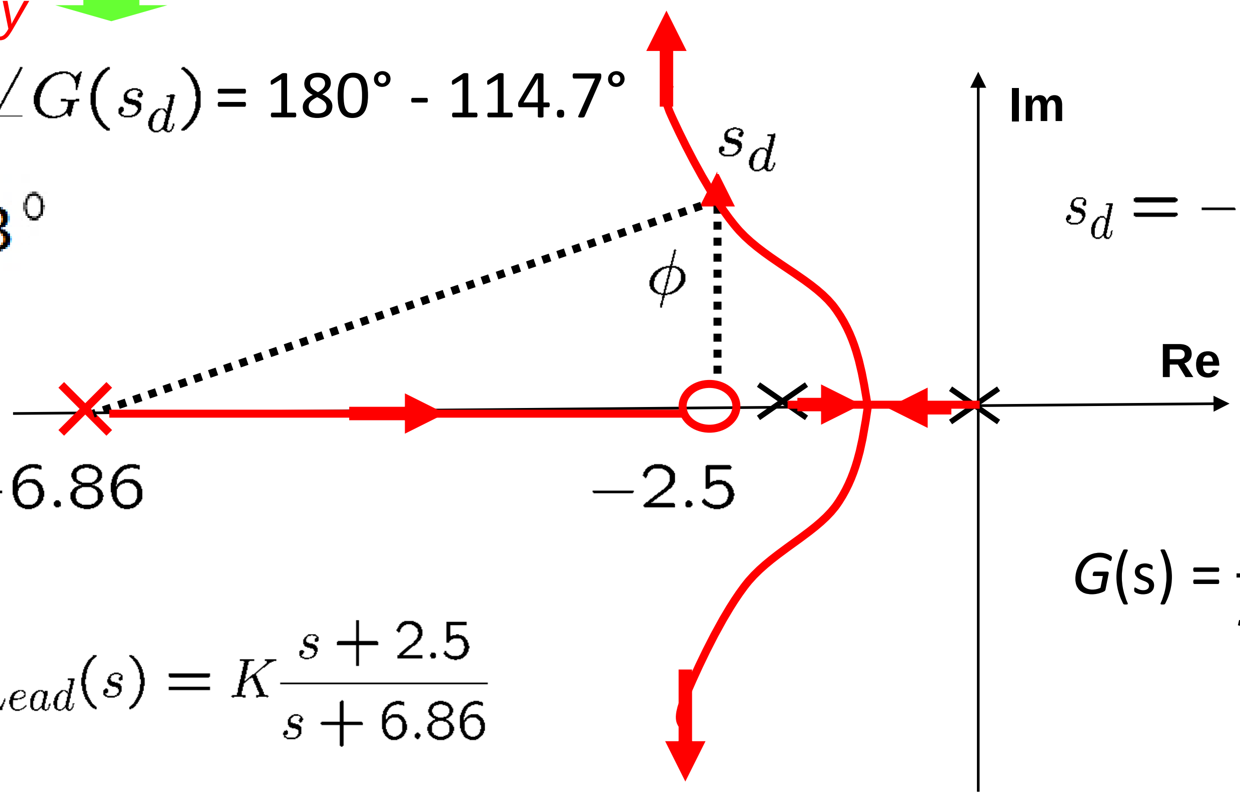
Example 3: One possible C_{Lead}

$$\angle \overset{*}{G}(s_d) = \angle \frac{1}{s_d(s_d+2)} = -245.3°$$ → 360° - 245.3° = 114.7° → $\angle G(s_d) = 114.7°$

Angle deficiency

$$\phi = 180° - \angle G(s_d) = 180° - 114.7°$$

$$\phi = 65.3°$$



$$s_d = -2.5 + 2j$$

$$G(s) = \frac{1}{s(s+2)}$$

$$C_{Lead}(s) = K \frac{s+2.5}{s+6.86}$$

Example 3: Design of pole/zero in C_{Lead}

- Lead compensator

$$C_{Lead}(s) = K\frac{s+2.5}{s+6.86}$$

- Open-loop transfer function

$$G(s)C_{Lead}(s) = K\underbrace{\frac{s+2.5}{s(s+2)(s+6.86)}}_{L_0(s)}$$

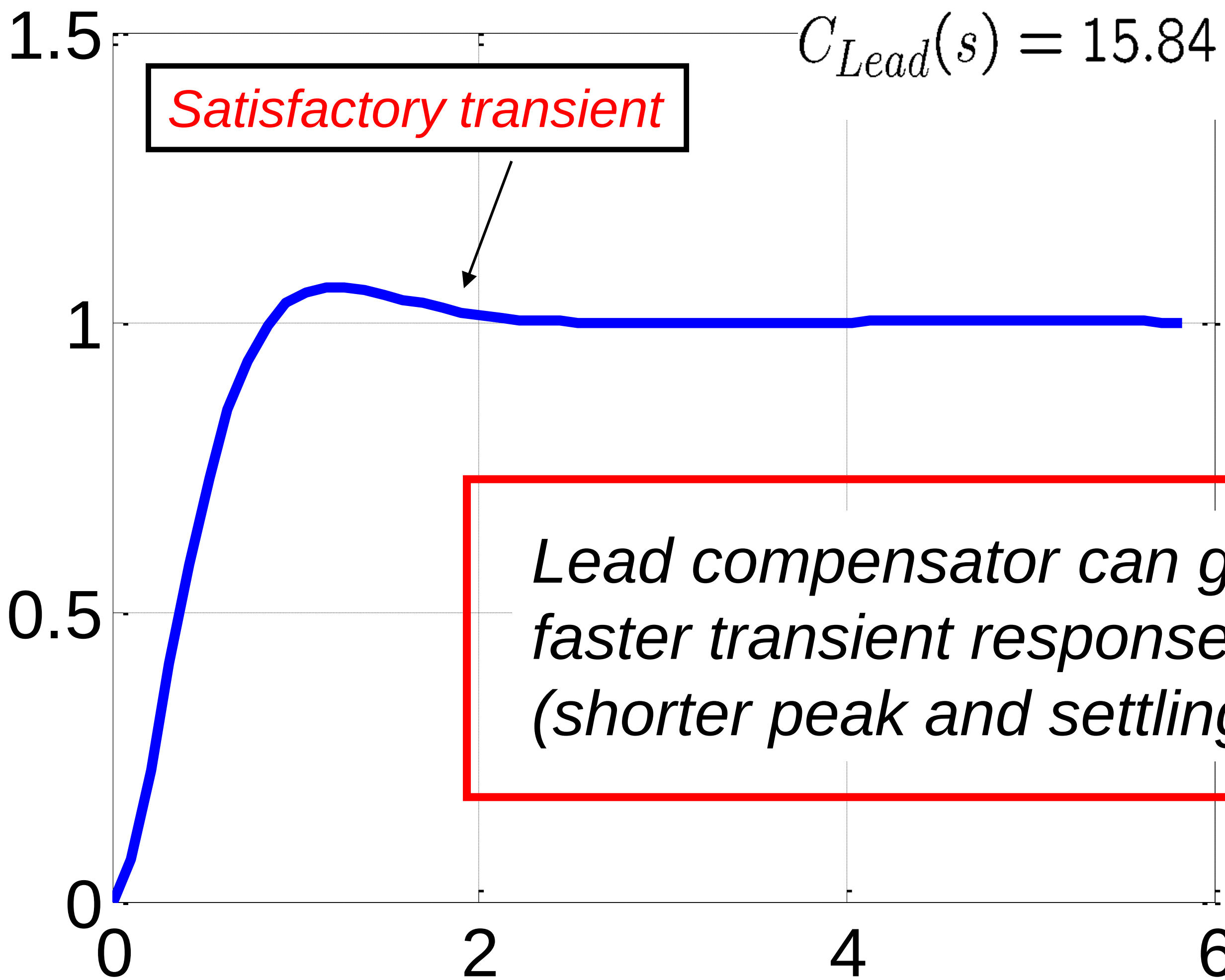
- Gain computation

$$K = -\frac{1}{L_0(s)}\bigg|_{s=s_d} = \cdots = 15.84$$

$$\boxed{G(s)C_{Lead}(s) = 15.84\,\frac{s+2.5}{s(s+2)(s+6.86)}}$$

Example 3: Step response with C_{Lead}

$$C_{Lead}(s) = 15.84 \times \frac{s+2.5}{s+6.86}$$



Lead compensator can give faster transient response (shorter peak and settling time)

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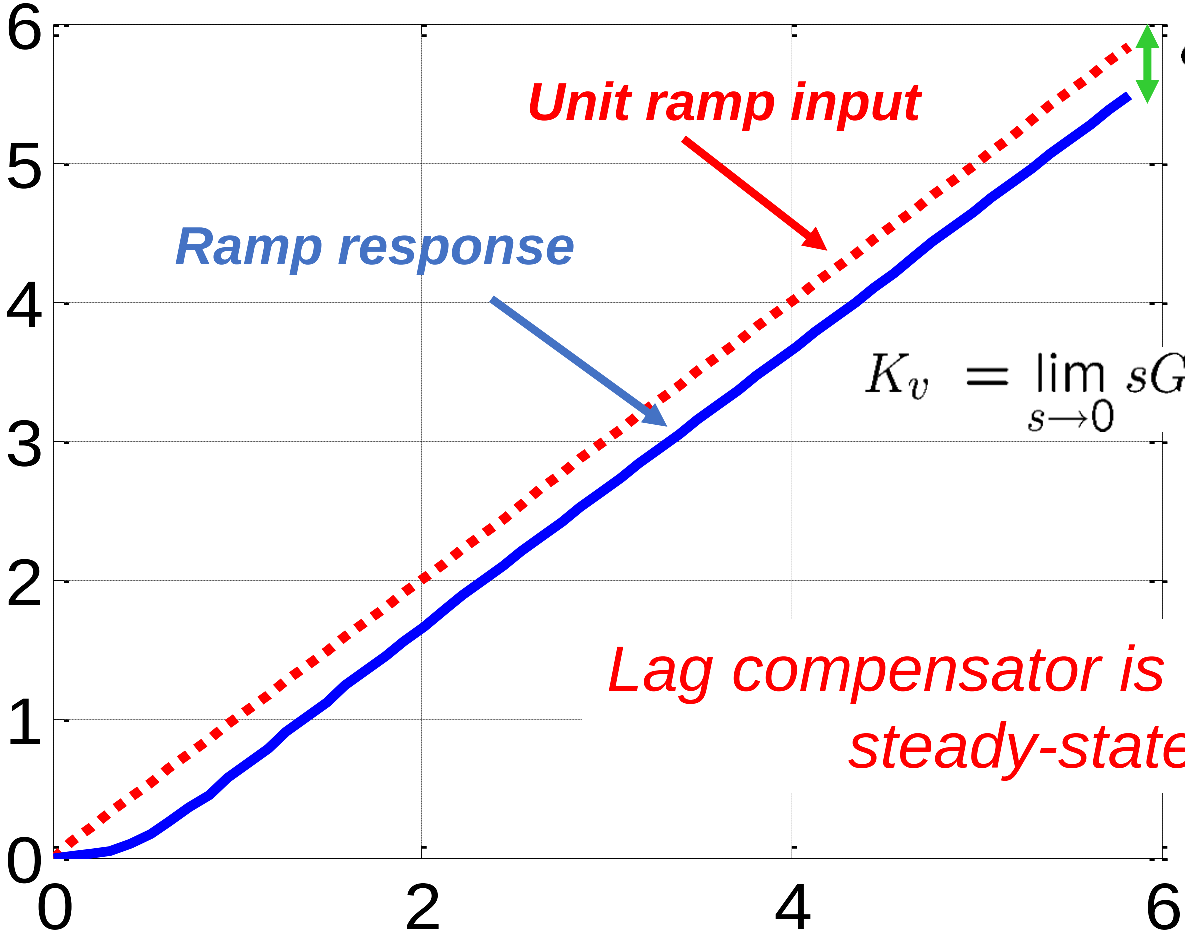
Example 3: Steady state error for $r(t) = tu(t)$



$$e_{ss} = \frac{1}{K_v} \approx 0.35$$

$$K_v = \lim_{s\to 0} sG(s)C_{Lead}(s) = 2.89$$

Lag compensator is used to reduce steady-state error.

Roles of lead & lag compensators

- Lead compensator
  - Improves transient response
  - Improves stability

$$C_{Lead}(s) = K\frac{s+z_{Lead}}{s+p_{Lead}}$$

- Lag compensator
  - Reduces steady state error

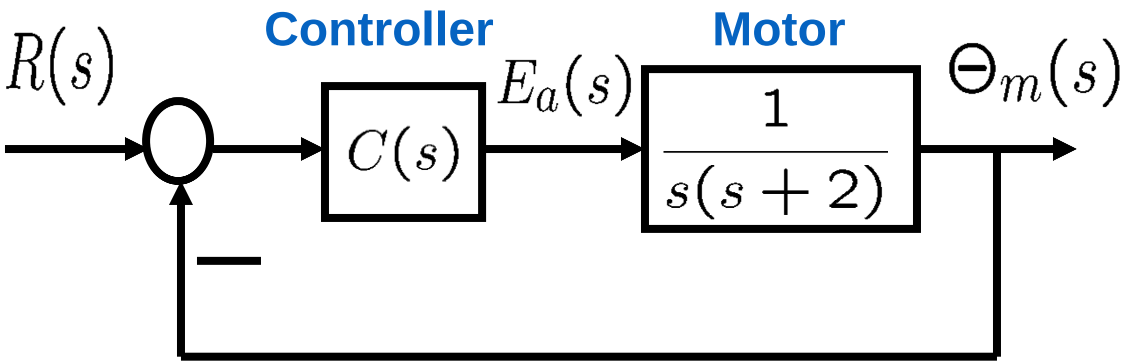
$$C_{Lag}(s) = \frac{s+z_{Lag}}{s+p_{Lag}}$$

- Lead-lag compensator
  - Takes into account both transient and steady state.

$$C_{LL}(s) = C_{Lead}(s)C_{Lag}(s)$$

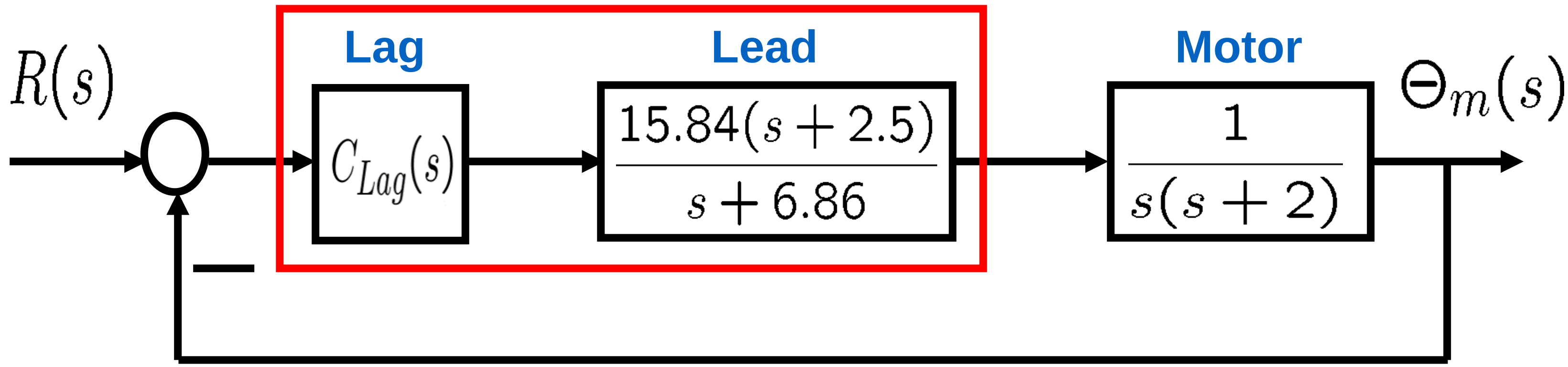
Example 4 (restated)

- A feedback system



- Design specifications
  - 2% settling time at most 2 seconds
  - Overshoot at most 4.32%
  - Steady state error
    - Zero for unit step $r(t) = u(t)$
    - At most 0.05 for unit ramp $r(t) = tu(t)$ ➡ $K_v > 20$

Example 4: After C_{Lead} design



- For a designed C_{Lead}, design C_{Lag} so that
  - Steady state error for unit ramp $r(t)$ is reduced (e_{ss} should be less than 0.02).
  - Transient is maintained. That is, we do not want to lose the satisfactory transient property achieved by the lead compensator.

Example 4: Design of $C_{Lag}(s) = \dfrac{s + z_{Lag}}{s + p_{Lag}}$

- We would like to reduce the steady state error, i.e., to increase ramp-error constant

$$K_v = \lim_{s \to 0} sG(s)C_{Lead}(s)C_{Lag}(s) = 2.89 \times \frac{z_{Lag}}{p_{Lag}} > 20$$

$$\frac{z_{Lag}}{p_{Lag}} > 6.92$$

Take, for example, $\boxed{z_{Lag} = 10 p_{Lag}}$

- Also, we want to maintain the transient property,

$$\boxed{|p_{Lag}| : \text{small}}$$

- We still want our RL to pass through s_d, despite modifying our $L(s)$.

$$\left.\begin{aligned} 1 + G(s_d)C_{Lead}(s_d) = 0 \\ C_{Lag}(s_d) \approx 1 \end{aligned}\right\} \Rightarrow 1 + G(s_d)C_{Lead}(s_d)C_{Lag}(s_d) \approx 0$$

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Why should p_{Lag} be small?

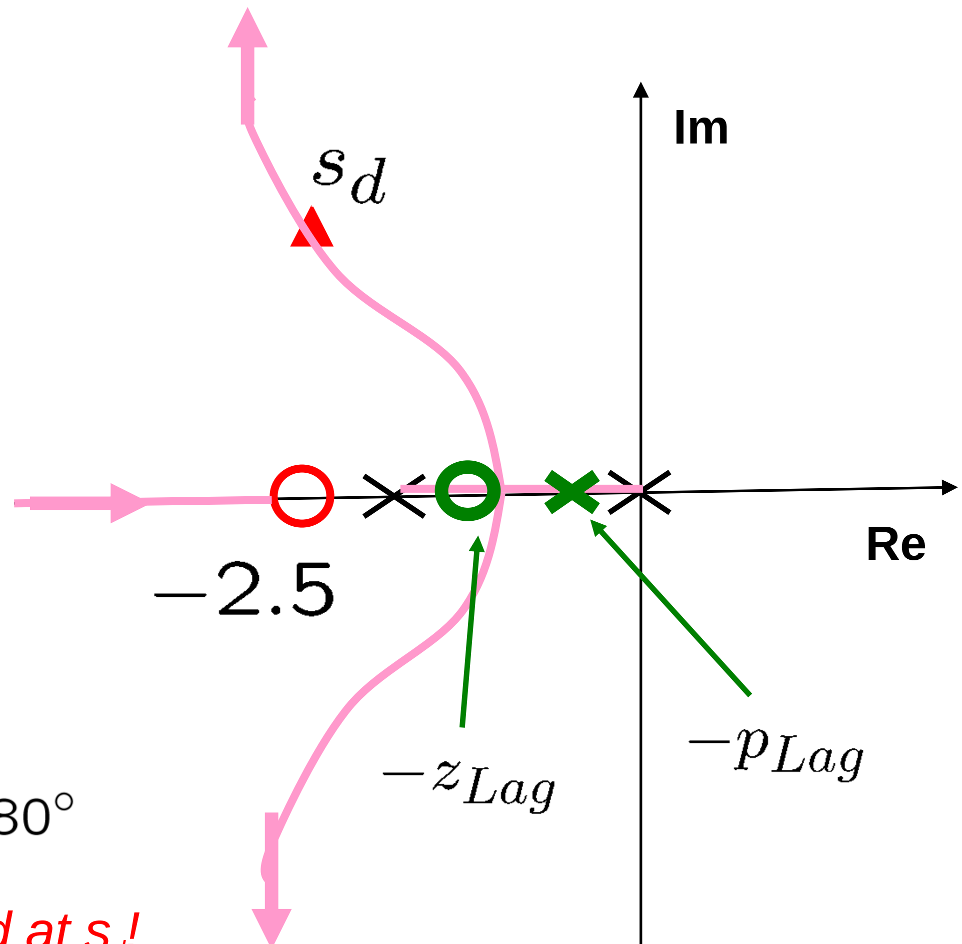
p_{Lag} : small

$$z_{Lag} = 10p_{Lag}$$

$$C_{Lag}(s_d) = \frac{s_d + z_{Lag}}{s_d + p_{Lag}} \approx 1$$

$$\angle G(s_d)C_{Lead}(s_d)C_{Lag}(s_d) \approx 180^\circ$$

Angle condition: almost satisfied at s_d!



Example 4: C_{Lag} design

- In design projects, we assume a small p_{Lag} and use trial-and-error! Here, p_{Lag} will be given to you.

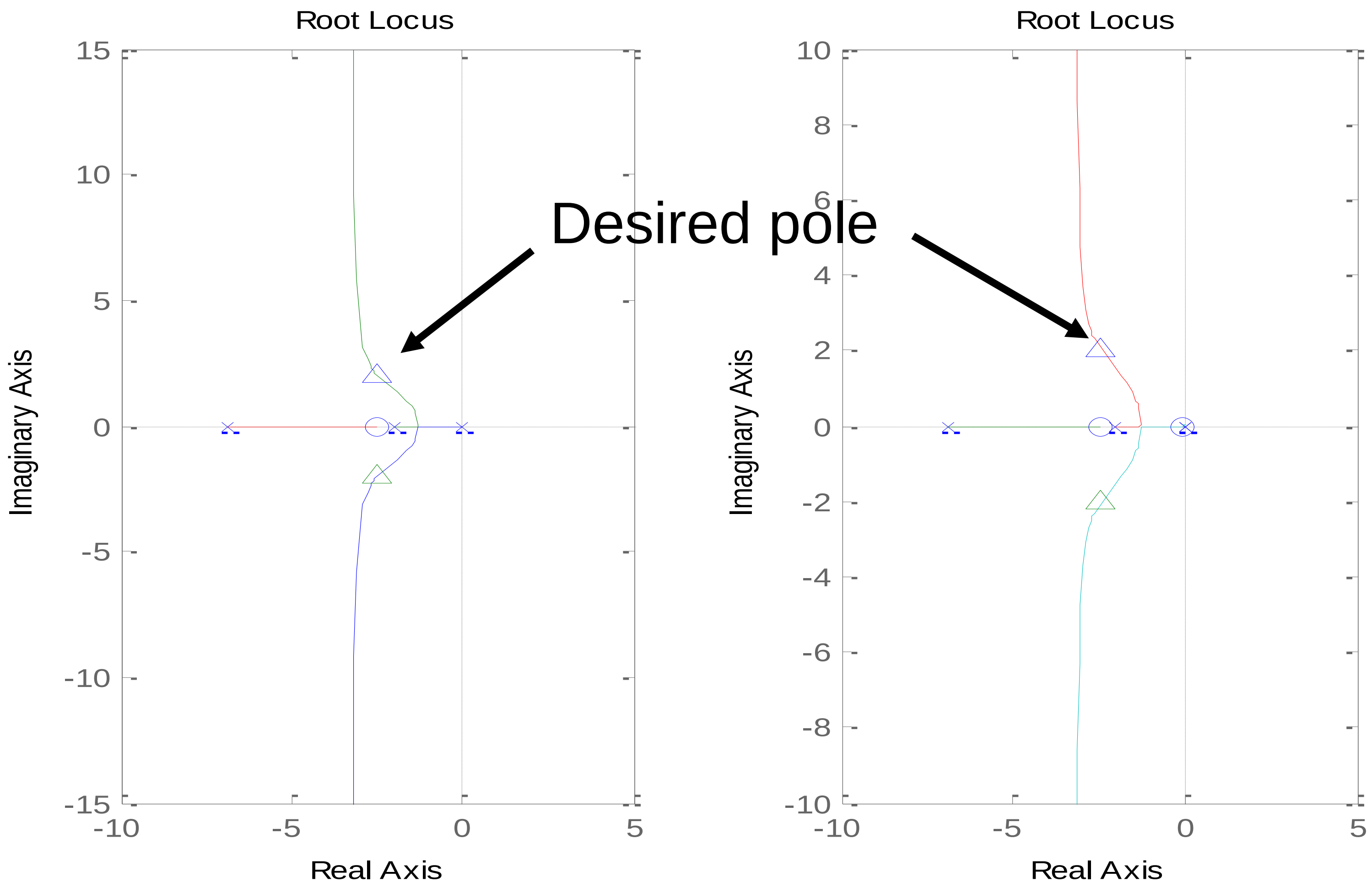
$$p_{Lag} = 0.005 \Rightarrow z_{Lag} = 0.05$$

$$\Rightarrow C_{Lag}(s) = \frac{s + 0.05}{s + 0.005}$$

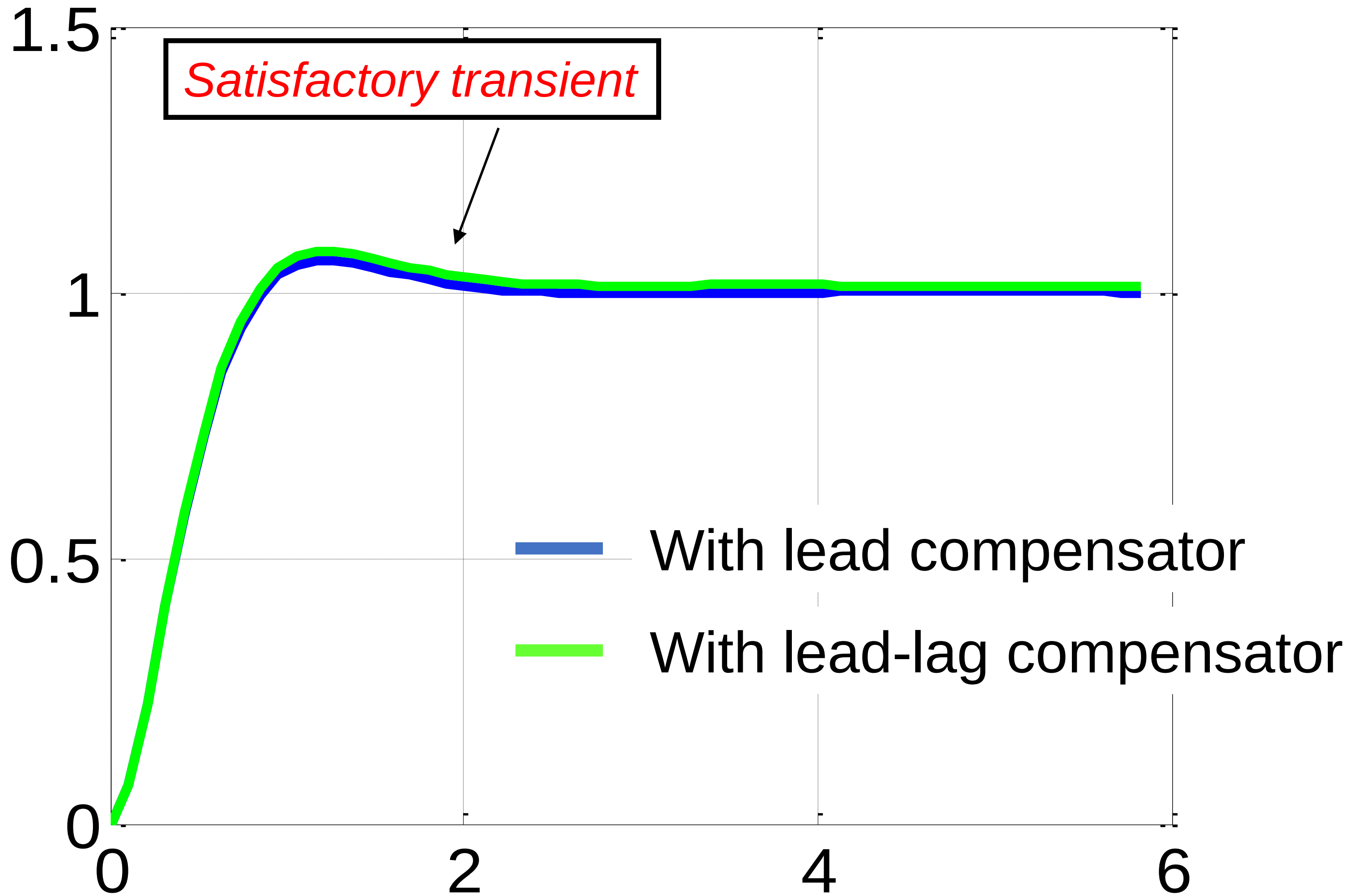
$$\Rightarrow C_{LL}(s) = \underbrace{\frac{15.84(s+2.5)}{s+6.86}}_{C_{Lead}(s)} \times \underbrace{\frac{s+0.05}{s+0.005}}_{C_{Lag}(s)}$$

Example 4: Root locus

With lead compensator | With lead-lag compensator



Example 4: Comparison of step responses



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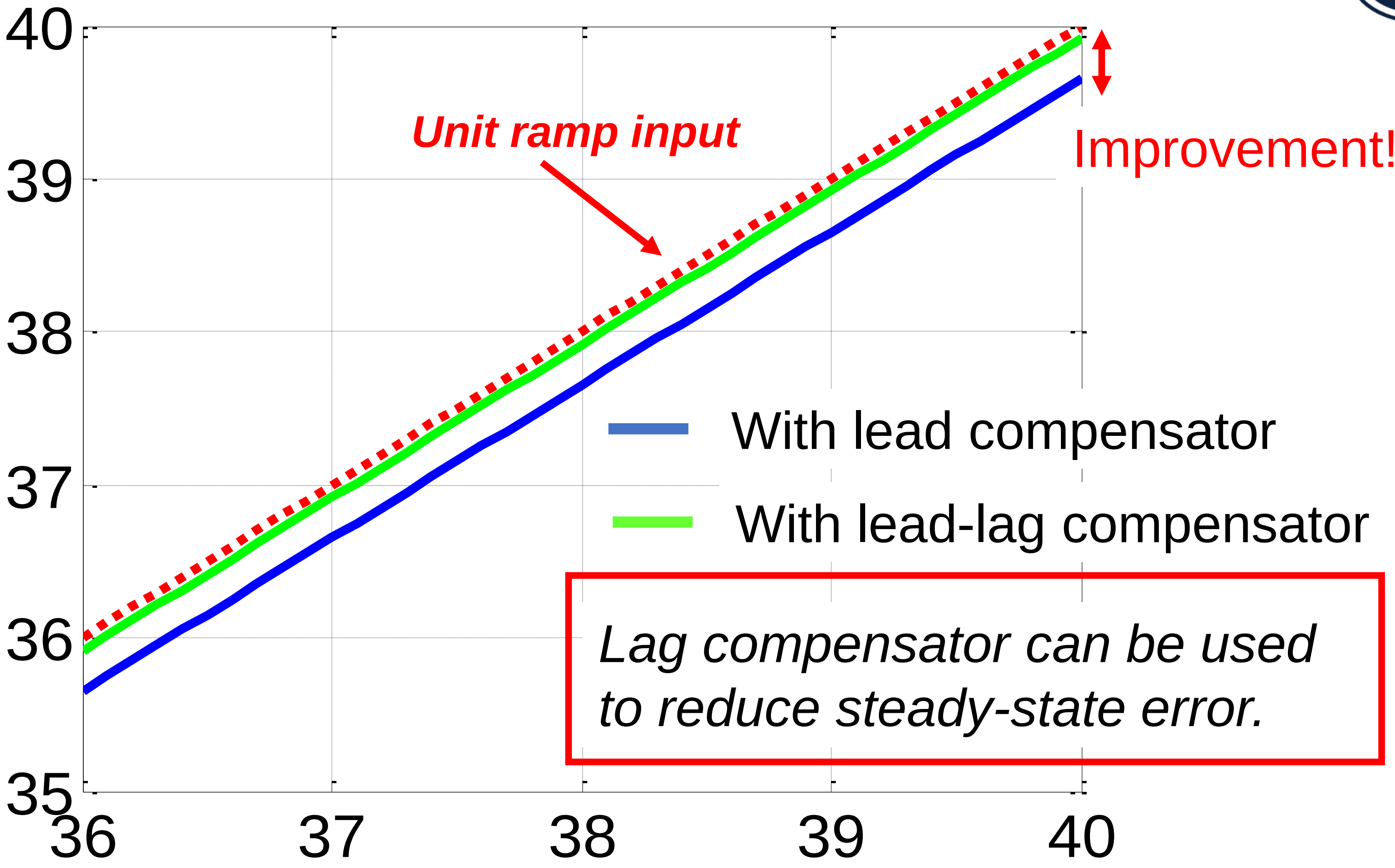
Example 4: Comparison of ramp responses



Example 2 (revisited)

If we only use a lead compensator.

Error constants

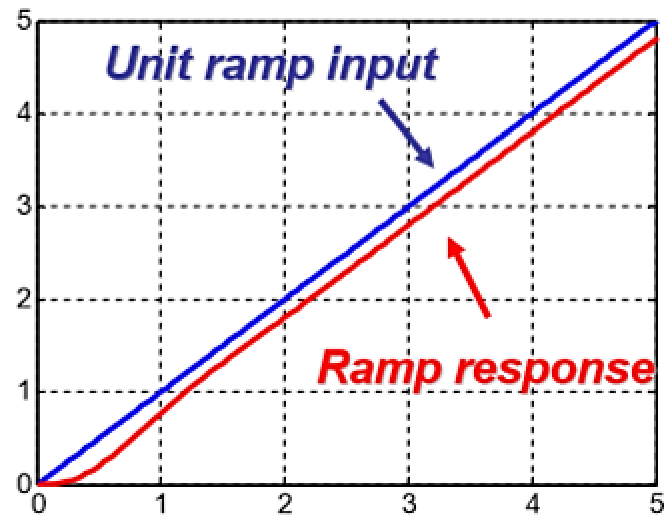
$$G(s)C(s) = \frac{4}{s(s+2)} \cdot \frac{4.675(s+2.9)}{s+5.4}$$

- Step-error constant

$$K_p = \lim_{s\to 0} G(s)C(s) = \infty$$

- Ramp-error constant

$$K_v = \lim_{s\to 0} sG(s)C(s) = 5.02$$



Lag compensator can be used to reduce steady-state error.

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Example 2 (revisited)

How to design lag compensator

- Lag compensator $C_{Lag}(s) = \dfrac{s+z}{s+p}$
- We want to increase ramp-error constant

$$e_{ss} < 0.02; \quad e_{ss} = \frac{R}{K_v}$$

$$\frac{1}{K_v} < 0.02 \rightarrow \boldsymbol{K_v > 50}$$

$$K_v = \lim_{s \to 0} sG(s)C_{Lead}(s)C_{Lag}(s) = 5.02 \cdot \frac{z}{p} > 50$$

$$\frac{z_{Lag}}{p_{Lag}} > 9.96 \rightarrow \boldsymbol{z_{Lag} = 10 p_{Lag}}$$

- We still want our RL to pass through s_d, despite modifying our $L(s)$.

$$\left.\begin{aligned} 1 + G(s_d)C_{Lead}(s_d) = 0 \\ C_{Lag}(s_d) \approx 1 \end{aligned}\right\} \Rightarrow 1 + G(s_d)C_{Lead}(s_d)C_{Lag}(s_d) \approx 0$$

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Example 2 (revisited)

- For the desired CL pole $s_d = -2 + 2\sqrt{3}j$

$$\boxed{C_{Lag}(s_d) \approx 1 \Leftrightarrow \left|\frac{s_d + 10p}{s_d + p}\right| \approx 1 \quad \angle\left(\frac{s_d + 10p}{s_d + p}\right) \approx 0}$$

- Let us say p_{Lag} is given as 0.025:

$z_{Lag} = 10p_{Lag}$
= (10)(0.025) = 0.25

$$\Rightarrow \left|\frac{s_d + 10p}{s_d + p}\right| = 0.97 \quad \angle\left(\frac{s_d + 10p}{s_d + p}\right) \approx -2.88^o$$

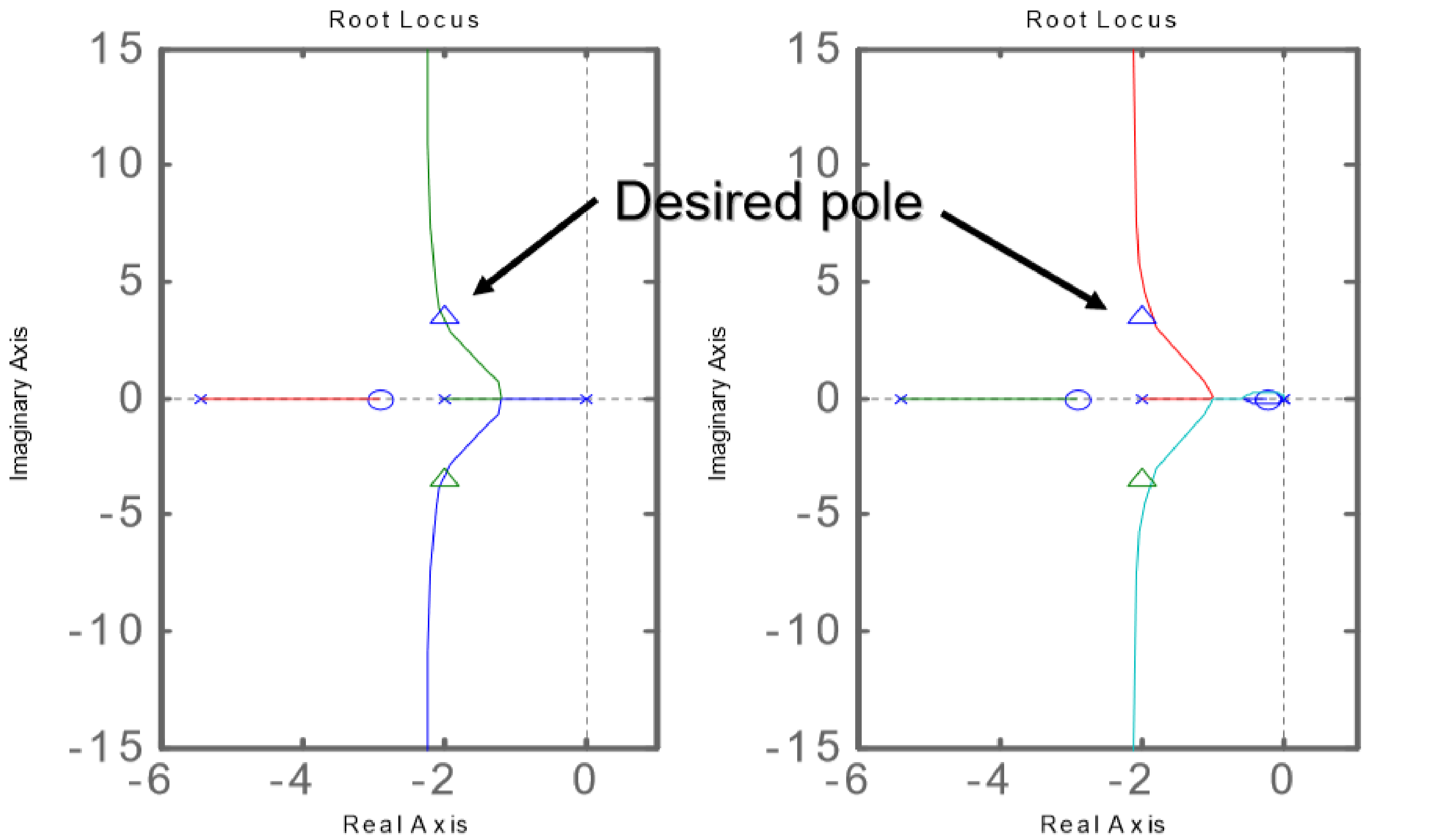
- Lead-lag controller

$$\boxed{C_{LL}(s) = 4.675\frac{s + 2.9}{s + 5.4} \cdot \frac{s + 0.25}{s + 0.025}}$$

Example 2 (revisited)

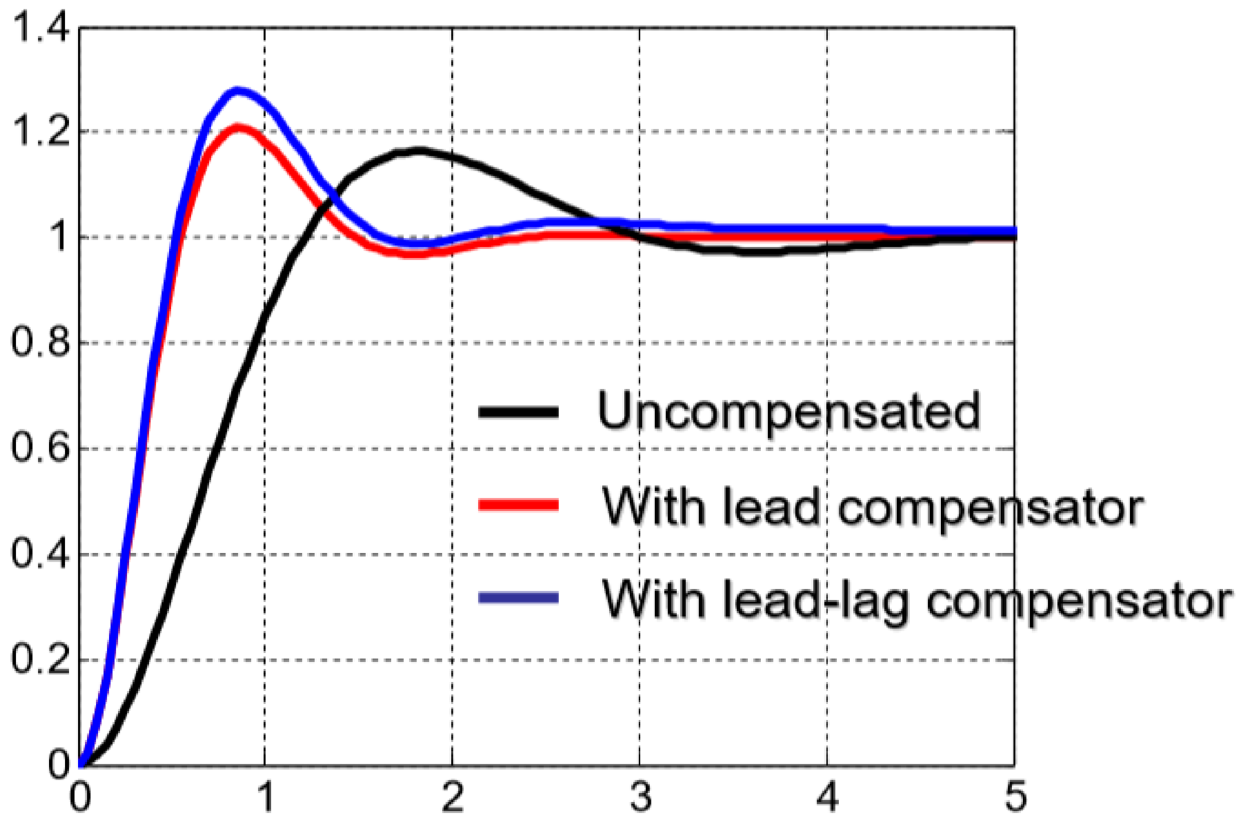
Root locus

With lead compensator With lead-lag compensator



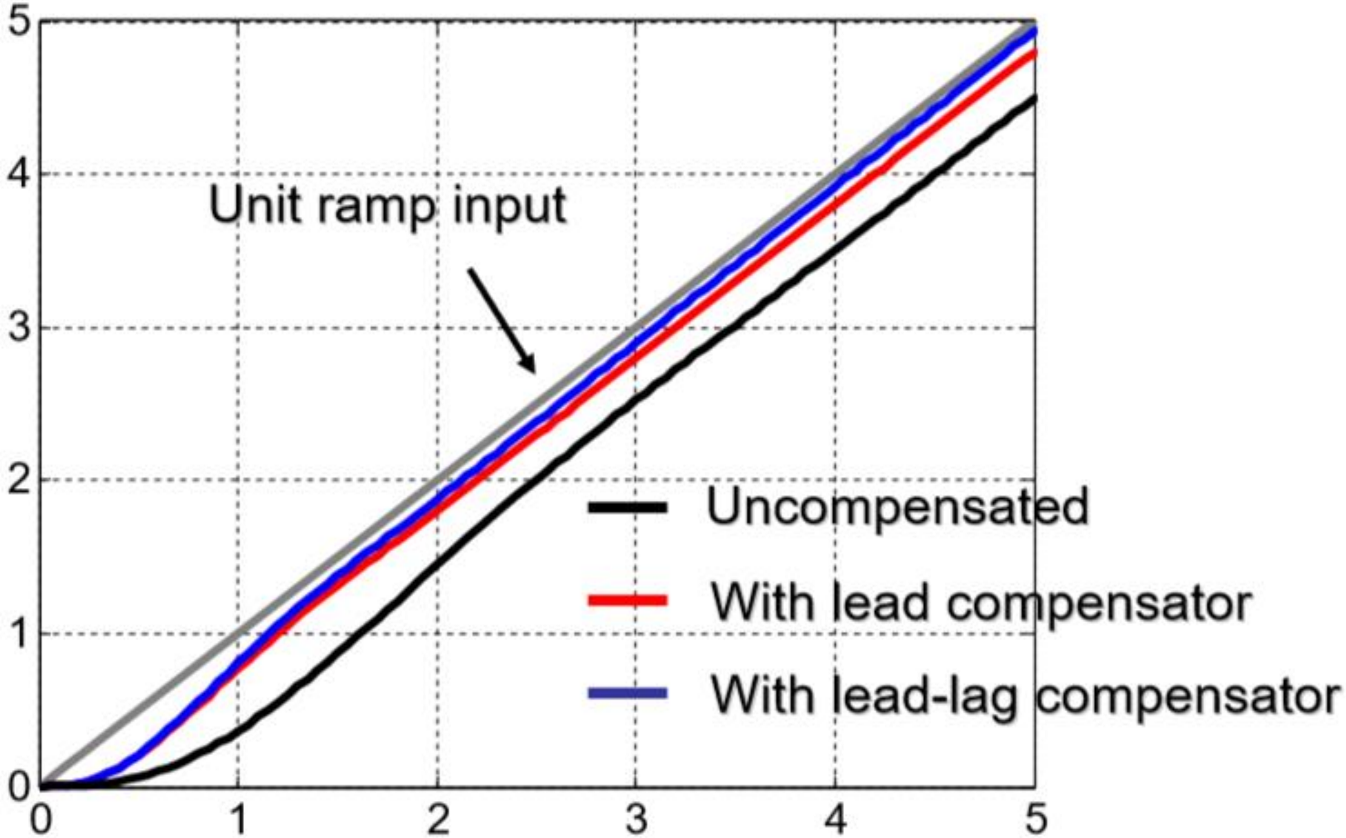
Example 2 (revisited)

Comparison of step responses



Example 2 (revisited)

Comparison of ramp responses



Remarks on controller design

- Existence of a satisfactory controller is generally unknown before the design.
- When a satisfactory controller exists, such controller is not unique.
- Controller design methods are not unique either.
  - Different references explain controller design in a different way.
  - Different control engineers design controllers in a different way.

a place of mind UBC

Summary

- Controller design based on root locus
  - Lead compensator **to improve transient response.**
  - Lag compensator **to improve steady state error.**
  - Lead-lag compensator **to improve both transient and steady state responses.**
- Next
  - Lead-lag compensator in Matlab and PID Controller design