Consider individual i and subpopulation k and let the estimated ancestry proportion of subpopulation k in individual i on chromosome c to be denoted as a^c_{ik}. We assume that for all c and c', $w_{i,cc'} = w_i$, that is, that the covariance between ancestry at any two pairs of chromosomes within an individual is the same, regardless of which pair of chromosomes we are considering. Denote

$$cov(a^c_{ik}, a^{c'}_{ik}) = w_{i,cc'} = w_i \tag{1}$$

$$var(a^c_{ik}) = \sigma^2_{ik} \tag{2}$$

First, we find $cov(\sum_{m \notin \{c\}} a^m_{ik}, \sum_{m \notin \{c'\}} a^m_{ik})$. In a small example where $c = 2$ and $c' = 3$ with $m = 4$ we see that

$$\sum_{m \notin \{c\}} a^m_{ik} = a_1 + a_3 + a_4 \tag{3}$$

$$\sum_{m \notin \{c'\}} a^m_{ik} = a_1 + a_2 + a_3 \tag{4}$$

Then it follows that

$$\begin{aligned}
cov(\sum_{m \notin \{c\}} a^m_{ik}, \sum_{m \notin \{c'\}} a^m_{ik}) &= cov(a_1 + a_3 + a_4, a_1 + a_2 + a_3) && (5)\\
&= cov(a_1, a_1) + cov(a_1, a_2) + cov(a_1, a_3) && (6)\\
&+ cov(a_3, a_1) + cov(a_3, a_2) + cov(a_3, a_3) && (7)\\
&+ cov(a_4, a_1) + cov(a_4, a_2) + cov(a_4, a_3) && (8)\\
&= 2\sigma^2_{ik} + 7w_i && (9)\\
&= (m-2)\sigma^2_{ik} + (m-2)^2 w_i + (m-1)w_i && (10)
\end{aligned}$$

We next find $cov(\sum_{m \notin \{c\}} a^m_{ik}, a^{c'}_{ik})$ under the same example where $c = 2$, $c' = 3$ and $m = 4$

$$\begin{aligned}
cov(\sum_{m \notin \{c\}} a^m_{ik}, a^{c'}_{ik}) &= cov(a_1 + a_3 + a_4, a_3) && (11)\\
&= cov(a_1, a_3) + cov(a_3, a_3) + cov(a_4, a_3) && (12)\\
&= 2w_i + \sigma^2_{ik} && (13)\\
&= (m-2)w_i + \sigma^2_{ik} && (14)
\end{aligned}$$

Now we have all the components to find $cov(D_{ik}^{c}, D_{ik}^{c'})$.

$$\begin{aligned}
cov(D_{ik}^{c}, D_{ik}^{c'}) &= cov(a_{ik}^{-c} - a_{ik}^{c}, a_{ik}^{-c'} - a_{ik}^{c'}) && (15)\\
&= cov(a_{ik}^{-c}, a_{ik}^{-c'}) - cov(a_{ik}^{-c}, a_{ik}^{c'}) - cov(a_{ik}^{c}, a_{ik}^{-c'}) + cov(a_{ik}^{c}, a_{ik}^{c'}) && (16)\\
&= cov(a_{ik}^{-c}, a_{ik}^{-c'}) - 2cov(a_{ik}^{-c}, a_{ik}^{c'}) + cov(a_{ik}^{c}, a_{ik}^{c'}) && (17)\\
&= cov(\frac{1}{22}\sum_{m\notin\{c\}} a_{ik}^{m}, \frac{1}{22}\sum_{m\notin\{c'\}} a_{ik}^{m}) - 2cov(\frac{1}{22}\sum_{m\notin\{c\}} a_{ik}^{m}, a_{ik}^{c'}) + w_i && (18)\\
&= \left(\frac{1}{22}\right)^2 cov(\sum_{m\notin\{c\}} a_{ik}^{m}, \sum_{m\notin\{c'\}} a_{ik}^{m}) - \frac{2}{22}cov(\sum_{m\notin\{c\}} a_{ik}^{m}, a_{ik}^{c'}) + w_i && (19)\\
&= \left(\frac{1}{22}\right)^2 \{21(\sigma_{ik}^2 + 21w_i) + 22w_i\} - \frac{2}{22}\{21w_i + \sigma_{ik}^2\} + w_i && (20)\\
&= \frac{1}{(m-1)^2}\{(m-2)\sigma_{ik}^2 + (m-2)^2 w_i + (m-1)w_i\} && (21)\\
&- \frac{2}{(m-1)}\{(m-2)w_i + \sigma_{ik}^2\} + w_i && (22)
\end{aligned}$$

where the first two terms are as found in Equations 10 and 14, respectively and $m = 23$. With some simplification and algebra, this equation becomes

$$cov(D_{ik}^{c}, D_{ik}^{c'}) = \frac{m}{(m-1)^2}(w_i - \sigma_{ik}^2) \tag{23}$$

Now, we are able to combine these results to calculate the entries in the variance-covariance matrix for our test statistics T_k^c.

$$\begin{aligned}
cov(T_k^c, T_k^{c'}) &= cov(\frac{\overline{D_k^c}}{\sigma_{ck}}, \frac{\overline{D_k^{c'}}}{\sigma_{c'k}}) && (24)\\
&= \frac{1}{\sigma_{ck}\sigma_{c'k}} cov(\overline{D_k^c}, \overline{D_k^{c'}}) && (25)\\
&= \frac{1}{n^2\sigma_{ck}\sigma_{c'k}} cov(D_{1k}^{c} + \cdots + D_{nk}^{c}, D_{1k}^{c'} + \cdots + D_{nk}^{c'}) && (26)\\
&= \frac{1}{n^2\sigma_{ck}\sigma_{c'k}} \sum_{i=1}^{n} \frac{m}{(m-1)^2}(w_i - \sigma_i^2) && (27)
\end{aligned}$$

The diagonal entries correspond to $var(T_k^c) = 1$.