DMD

$$X' \equiv X(t+\Delta t) = \hat{A}\,X(t) \qquad \hat{A} \equiv \text{Koopman Operator}$$

$$\frac{dx}{dt} = \hat{A}x \qquad \text{and} \qquad \hat{A}x \Rightarrow Ax = \lambda x$$

$$\frac{dx}{dt} = \lambda x \qquad \boxed{x(t) = c_1 e^{+\lambda \Delta t}}$$

$$\frac{dx}{x} = \lambda dt$$

So DMD solve $\underset{\sim}{A}\,\underset{\sim}{\Phi} = \underset{\sim}{\Phi}\,\underset{\sim}{\Lambda}$

$\Phi \equiv$ Dynamic eigen modes

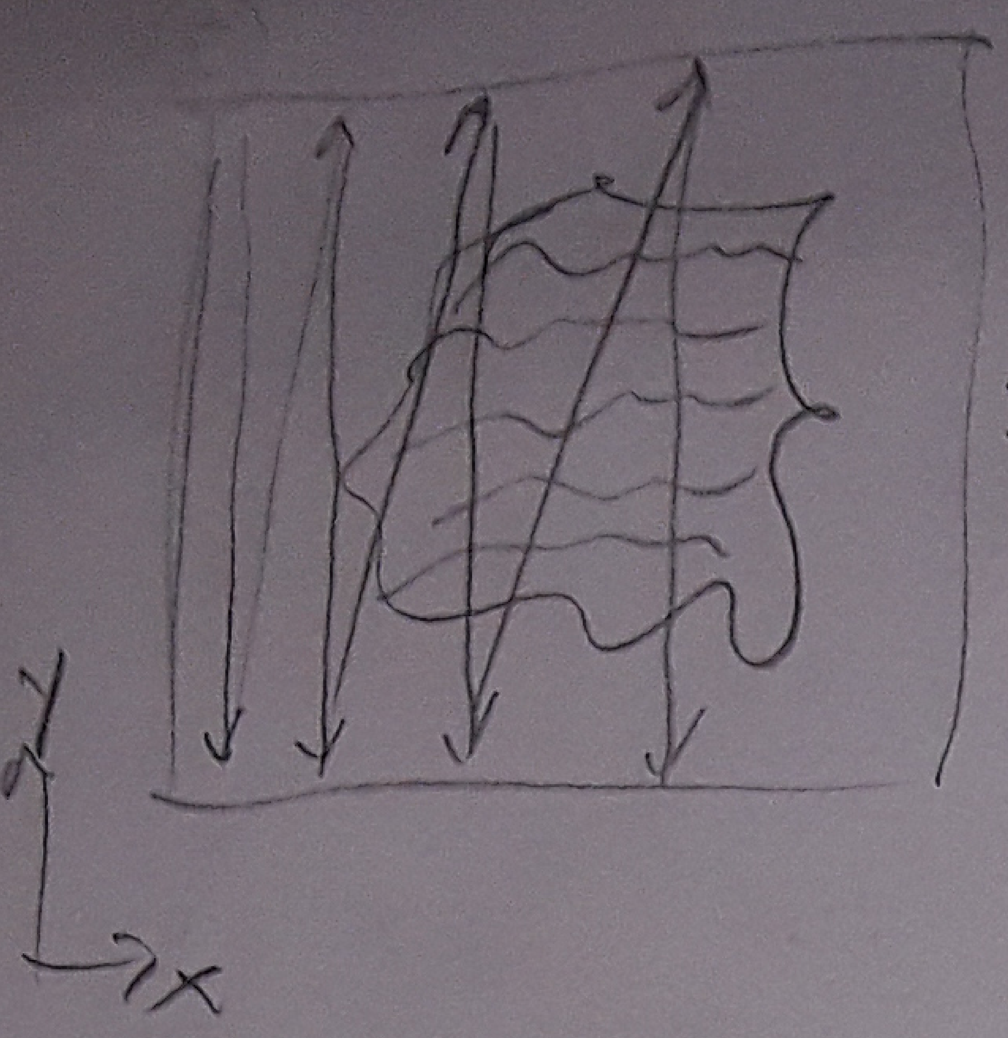
$\underset{\sim}{X}$ and $\underset{\sim}{X}'$ are matrices which represent time-series of data whose column vectors are data at a single time step. Each subsequent column are the next time steps. $\underset{\sim}{X}$ are time steps $1 \ldots N_{t-1}$ and $\underset{\sim}{X}'$ are the time steps advanced by one, $2 \ldots N_t$

Φ are DMD modes found

To "replay" the data as an animation, project the first animation frame onto Φ.

$$\vec{\underset{\sim}{X}}(t=1) = \Phi\,\vec{c} \qquad \text{so} \qquad \boxed{\vec{c} = \Phi^{-1}\vec{X}_1}$$

Then $\underset{\sim}{X}(t) = \left[e^{\Phi t}\right]\vec{c}$ ⇒ provides a new animation in continuous time with the ability to predict times past the end of the data ≡ prediction!



$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \\ x_{N_p} \end{pmatrix} \equiv \vec{x}$$

@ one time

$$\underset{\sim}{X} = \underbrace{\begin{pmatrix} \vec{x}(t_1) & \vec{x}(t_2) & \vec{x}(t_3) & \cdots & \vec{x}(t_{N_t-1}) \end{pmatrix}}_{N_p \times (N_t - 1)}$$

$\Rightarrow$ animation of time frames from $t = 1 \dots (N_t - 1)$

$$\underset{(\sim 10^3)}{N_t} \ll \underset{(\sim 10^5 - 10^7)}{N_p}$$

$$\underset{\sim}{X}' = \underset{\sim}{A}\,\underset{\sim}{X}$$

$\underset{\sim}{X}' = N_p \times (N_t - 1)$ for time frames $t = 2 \dots N_t$

= one time frame in the future

$\underset{\sim}{A}$ acts as a time-evolution operator

where $\hat{A} \equiv$ Koopman Operator

$$\underset{\sim}{A} = \underbrace{N_p \times N_p}_{\text{large!}}$$

so

$$A = X' X^{-1}$$

using SVD of X

$$X = U \Sigma V^{+}$$

where $U \equiv$ orthogonal matrix

$N_p \times N_p$

$\Sigma \equiv$ Diagonal matrix

$N_p \times N_{t-1}$

$V \equiv$ orthogonal matrix

$N_{t-1} \times N_{t-1}$

and $X X^{+} U = U \Sigma^{2}$ — U are eigen vectors of $X X^{+}$

$X^{+} X V = V \Sigma^{2}$ — V " " " $X^{+} X$

$$A = X' V \Sigma^{-1} U^{+}$$

$$\underbrace{A U}_{\substack{N_p \times N_{t-1} \\ \text{in econ mode}}} = X' V \Sigma^{-1}$$

$U = N_p \times N_p$ $\quad \Sigma = N_p \times N_{t-1}$

in economy mode

$U = N_p \times N_{t-1}$ $\quad \Sigma = N_{t-1} \times N_{t-1}$

$\Rightarrow$ so Solve the eigenmode problem in the lower dimensional space

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Define $\tilde{A} = U^{+} A U$ $\quad (N_{t-1} \times N_{t-1})$

solve eigvecs $\tilde{A} W = W \Lambda$

where $\Phi = X' V \Sigma^{-1} W$

$$\tilde{A} W = U^{+} X' V \Sigma^{-1} W = U^{+} \Phi \Rightarrow U \tilde{A} W = \cancel{U U^{+}} \Phi = \Phi$$

$$U \tilde{A} W = \cancel{U U^{+}} A U W = A U W = \Phi$$

we want to show $A \Phi = \Phi \Lambda$

$$\Lambda = W^{+} \tilde{A} W$$

$$A U W \Lambda = \Phi \Lambda$$

$$A U \cancel{W W^{+}} \tilde{A} W = \Phi \Lambda$$

$$A U \tilde{A} W = \Phi \Lambda$$

$$A \Phi = \Phi \Lambda \quad \checkmark$$

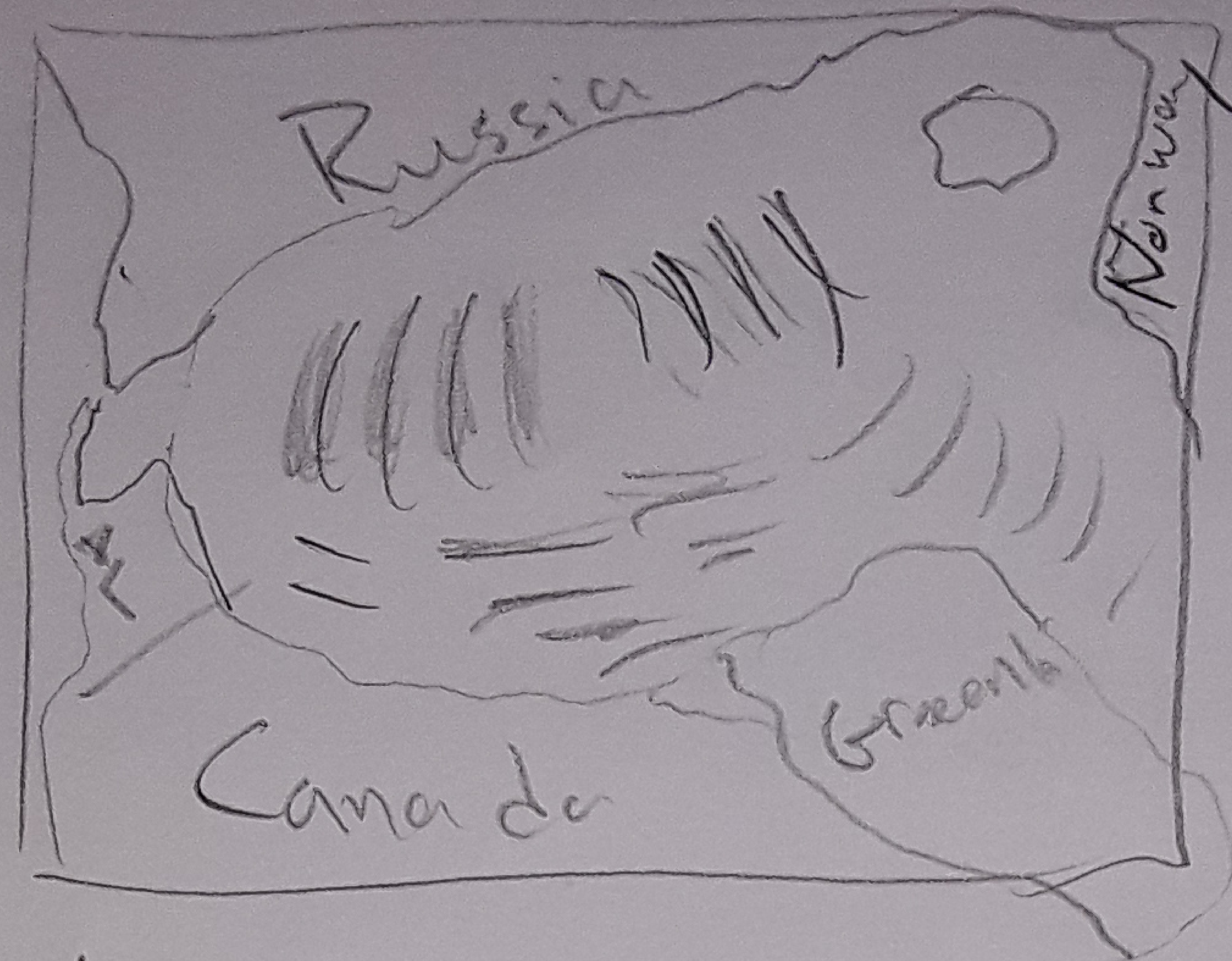
$\tilde{A} = N_{t-1} \times N_{t-1}$

$W = N_{t-1} \times N_{t-1}$

$\Lambda = N_{t-1} \times N_{t-1}$ = eigenvalues

$\Phi = N_p \times N_{t-1}$ = DMD modes

$\Phi_1(x,y)$



Kevin McIlhany

Arctic Ocean

(4)

DMD modes appear to have spatial structure but, in truth, Φ are modes whose information per location is the local λ-values $\equiv$ time-like info.

What happens if Russia blockades part of the arctic? This is called a data-denied region?

* Use neural networks trained based on previously computed DMD, $\underset{\sim}{\Phi}$

NN can provide data within the denied-zone via interpolation $\Rightarrow$ recover your prediction capability.