Curriculum Vitæ

Kevin L. McIlhany

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Education:	B.A., 1989	Northwestern University
	M.S., 1992	University of California, Riverside
	Ph.D., 1996	University of California, Riverside
Career:	1995–1996	Visiting Scientist, W.K. Kellogg Radiation Laboratory California Institute of Technology
	1996–1997	Post Doctoral Associate, W.K. Kellogg Radiation Laboratory California Institute of Technology
	1997–2000	Post Doctoral Associate, Laboratory for Nuclear Science Massachusetts Institute of Technology
	2000–2005	Assistant Professor of Physics, Physics Department United States Naval Academy
	2006–2014	Associate Professor of Physics, Physics Department United States Naval Academy
	2014–present	Professor of Physics, Physics Department United States Naval Academy

Personal Data: **Born:** August 1, 1967, Dallas, Texas
Citizenship: U.S.A.
Family: Married

Career Goal: Teaching and research in an academic environment. Long range goals include administration within the community.

Research Interests: Currently working on geo-physical flows and microfluidics studying the application of Eulerian Indicators to optimized mixing schemes. Application of the inverse problem to acoustic scattering for imaging hand positions for use with control systems. Recent interests have included computational analysis of the Chesapeake Bay flow through a Normal Mode Analysis. Previous work in this field has included neural network development as well as finite difference development of eigenmodes and time-series analysis. Throughout most of my career to date, I have worked as an experimental particle physicist with emphasis on nucleon structure. Past projects include study of electron scattering using polarized beams and internal targets.

Professional Societies: American Physical Society – Division of Nuclear Physics,
American Physical Society – Division of Fluid Dynamics

Publication List (Primary Authorship): Kevin L. McIlhany

1. K. McIlhany, S. Wiggins, “High Dimensional Cluster Analysis Using Path Lengths,” Jour. Data Analysis & Information Processing Vol.6 No.3, DOI:10.4236 (2018).
2. K. McIlhany, S. Wiggins, “High Dimensional Cluster Analysis Using Path Lengths,” arXiv:1710.04886, (2017).
3. K. McIlhany, S. Guth, S. Wiggins, “ Lagrangian and Eulerian Analysis of Transport and Mixing in the Three Dimensional, Time Dependent Hillś Spherical Vortex,” Phys. Fluids 27, 063603 (2015).
4. B. Hanlon, K. McIlhany, “Eulerian Study of the Kuroshio Current,” Bowman Scholar Report, USNA (2013).
5. K. McIlhany, S. Wiggins, “Eulerian Indicators under Continuously Varying Conditions,” Phys. Fluids 24, 073601 (2012).
6. K. McIlhany, D. Mott, E. Oran, S. Wiggins, “Optimizing mixing in lid-driven flow designs through predictions from Eulerian indicators,” Phys. Fluids 23, 082005 (2011).
7. K. McIlhany, S. Wiggins, “Optimizing Mixing in Channel Flows: Kinematic Aspects Associated with Secondary Flows in the Cross-Section”, Journal of Micro and Nano Fluidics, Vol. 10, Num. 2, pp. 249-262 (2010).
8. ... 15 primary publications in total as of March, 2018.

Talk List: Kevin L. McIlhany

1. *“SPace of Eulerian MeasureS (SPEMS) application to Oceanographic Flows”,* 3nd International Workshop on Non-Linear Ocean and Atmospheric Processes, Madrid, ES 7 July 2016.
2. *“SPace of Eulerian MeasureS (SPEMS) application to Chesapeake Bay Mouth”,* Chesapeake Modeling Symposium 2016, Williamsburg, VA 4 June 2016.
3. *“Normal Mode Analysis with the Chesapeake Bay Under Non-Optimal Conditions”,* 2nd International Workshop on Non-Linear Ocean and Atmospheric Processes, Madrid, ES 5 July 2012.
4. *“Use of Eulerian Indicators to Predict Best Mixing Configurations for a Blinking 2D Lid-Driven Flow”,* American Physical Society, 64th Annual Division of Fluid Dynamics Meeting, Baltimore, MD, 22 Nov. 2011.
5. ... 21 talks in total as of March, 2018.

Poster List: Kevin L. McIlhany

1. *“Optimizing Mixing in Channel Flows: Kinematic Aspects Associated with Secondary Flows in the Cross-Section”,* Dynamics Days 2010, Northwestern Univ. 7 Jan 2010.

1 Budget

Summer 2019 Proposed Budget			
Purpose	Quantity	Cost per item	Total
Labor	41 days	$875.00	$35,793.00
Labor (optional)	10 days	$875.00	$8,750.00
Travel (optional)	1 trip	$3,500.00	$3,500.00
Total (without options)			$37,082.00
Total (with options)			$48,043.00

Table 1: Proposed budget for the summer of 2019 funding.

Explanation The proposed budget covers the labor costs for the 41 work days of the summer of 2019 for Prof. McIlhany using an acceleration rate of 33% with an overhead of 5%. The optional 10 work days proposed would fund 10 additional days during the Fall semester to "buy out" Prof. McIlhany for one day per week for 10 weeks. Finally, the travel budget is proposed for a visit to University of Bristol to collaborate with Prof. Stephen Wiggins.

2 Introduction

ONR has funded the collaboration between Prof. Kevin Mcilhany and Prof. Stephen Wiggins from 2009-present in order to study Eulerian measures of fluid flow as a complementary approach to characterization of manifolds in comparison to Lagrangian techniques[13;14;17;18]. As this approach developed from toy models to analyses of ocean models[7], the size and scope of the project grew. From 2015-2018 the main focus was to understand the data space formed when a fluid flow is described by multiple measurements spanning many dimensions. The data size (approximately 1TB) and the dimension size (7 dimensions or higher) necessitated the development of new data clustering techniques that scale easily with size in both data and dimensions[15;16]. The clusters formed from the high dimensional space can lead to groupings within the data that are otherwise difficult to observe. Work on applying these data clustering techniques along with other recent "Data Science" approaches to oceanography are the goals of this proposal.

This proposal looks towards the data itself to provide clues as to the structure and dynamics hidden with it. Two relatively new techniques will be used to analyze data of Naval interest; deep learning of physics models as well as dynamic mode decomposition. The data set will be chosen in the Spring of 2019 in consultation with other members at USNA, however, the South China Seas will most likely be investigated, with Arctic Sea ice as another possible candidate. The main issue regarding which region of the ocean to consider is that varying degrees of data are currently available, and the data choice directly informs how the data is analyzed. Ideally, a combination of multiple data sources would be used, including satellite data for sea surface height (SSH), temperature and salinity, from which the surface velocity vector field can be derived[30]. When cloud cover permits, the satellite data is the closest to a full domain coverage of the region of interest, making the satellite data relatively easy to compare to models - which cover the domain. Unfortunately, satellites only provide information at the sea surface (2D); in order to validate information at some depth, Lagrangian drifters (gliders) can be utilized to provide a transect of positions versus time. Presently, thousands of glider transects have been collected covering large portions of the oceans of the world, but always at one location at one time. While this data provides information at depth, it fails to provide a "snapshot" over a large region, making data assimilation difficult to reconcile with large-scale models[11;25;27].

Although neural networks have a rich and long history[8;12], the idea of machine learning started in the late 50's with gradual growth and understanding from the 60's-80's, peaking during the 90's then receding in light of comparable techniques requiring less rigor starting in the 90's through the new millennia, such as genetic algorithms[9;19;24;31]. Recent studies have incorporated traditional neural networks in new configurations that attempt to extract the structure of a dynamical system from the data by solving multiple neural nets in tandem[22;23]. These *machine learning* techniques employ state of the art algorithms applied to massive data sets to extract the parameters of an equation system that best represents the data. In this regard, the machine learning techniques can be viewed as optimization processes for complicated non-linear and high dimensional systems. These "Physics Informed Neural Networks" (PINN) and similar "Deep Learning of Physics Models" attempt to utilize neural networks as a nonlinear fit to the data while simultaneously extracting the parameters from a loosely defined set of partial differential equations which model the data, reaching convergence when the two outputs agree within some tolerance. By optimizing all three data sets (from satellite, model and glider), a deep learning approach might reconcile this multi-scale problem.

A different approach to data driven science is the search for the dynamics of a system by seeking an appropriate modal decomposition based on the data given. Seemingly similar methods, Proper Orthogonal Decomposition (POD)[10] and Empirical Orthogonal Functions (EOFs)[2;6], in that eigenmodes are sought from a covariance matrix based on data, Dynamic Mode Decomposition (DMD)[26] distinguishes itself by seeking spatio-temporal modes. Differing from traditional modal decomposition, the modes found are not static, but evolve with the data which defines them. Further, no assumptions are made about the form taken by the system, in other words, no conditions are imposed on the modes sought either by a model or boundary conditions. By utilizing the fewest modes which account for the majority of the variances within the data, a much smaller (*compressed*) representation of the data is found [1;3;29].

3 Deep Learning of Physics Models from Data

Traditional neural networks employing backpropagation error correction require some mechanism to adjust values of the network. Through an iterative process, the network changes values in matrices and vectors until some form of “agreement” is reached. One approach is to provide a neural network with a training set of paired state vectors, an input state which maps to an target output state, $[\vec{\mathbb{I}}_j, \vec{\mathbb{T}}_j] \in (\mathbb{R}^N, \mathbb{R}^M)$, with $j \in 1...N_D$ as the index over the set of data. The neural net is a composition of nonlinear functions acting on matrices along with “bias” vectors, where the combination of these for a layer, whose output is a vector, $\vec{L}_k$, is of some dimension determined by the size of the weight matrices, W_k, representing the network values, where k represents the layers number:

$$\begin{aligned}
\vec{L}_1 &= f_1\left(W_1 \cdot \vec{\mathbb{I}} + \vec{b}_1\right) \\
\vec{L}_2 &= f_2\left(W_2 \cdot \vec{L}_1 + \vec{b}_2\right) \\
\vec{L}_3 &= f_3\left(W_3 \cdot \vec{L}_2 + \vec{b}_3\right) \\
&\dots \\
\vec{\mathbb{O}} &= f_n\left(W_n \cdot \vec{L}_{n-1} + \vec{b}_n\right)
\end{aligned}$$

The sigmoid function, $f(\vec{L}, \vec{b})_k$, can take many forms, however, a common form taken is $f(x) = tanh(\beta(x - x_0))$, which exhibits linear behavior near $x = x_0$ and is asymptotically bound. The bias vectors, $\vec{b}_k$, act as the term x_0 in the sigmoid, effectively shifting the origin. For two layers, back-to-back, using sigmoids of the form, a gaussian-like mapping can be affected. Sigmoids can perform as discreet maps, where $\beta \gg 1$, and with proper normalization can represent binary values as well.

The final vector produced, $\vec{\mathbb{O}}_j$, is the output for a given input, $\vec{\mathbb{I}}_j$ based on the current values of the weight matrices and bias vectors. *Training* a neural network can take many forms, but commonly, values within the network are adjusted iteratively until an error function, $\mathbb{E}$, has been minimized using any number of common optimization techniques. *Back propagation* refers to the process of evaluating the gradient of the error function with respect to the network values, then altering the network values such that the error is lowered in each step.

$$\begin{aligned}\vec{\mathbb{E}}_j &= \vec{\mathbb{O}}_j - \vec{\mathbb{T}}_j \\ \mathbb{E} &= \sum_{j=1}^{N_d} |\vec{\mathbb{E}}|_j^2\end{aligned}$$

The gradient of the error function with respect to a specific network weight, $\frac{\partial \mathbb{E}}{\partial W_{nm}}$ is easily calculated via chain rule. Inverting this terms allows each weight to be adjusted by setting a step size which reduces the error in each iteration of the network: $W'_{nm} = W_{nm} + \frac{\partial W_{nm}}{\partial \mathbb{E}}(\mu \Delta \mathbb{E})$, where μ is a *learning rate* term, which is scaled to prevent too large of a stepsize[5;9].

Other approaches to determining the error - or cost - function help during optimization to avoid stopping early at a local minima, avoiding root-finding loops, as well as annealing the system to find the most general non-linear mapping from $\vec{\mathbb{I}} \Rightarrow \vec{\mathbb{O}}$.

Key features of neural networks are that a well-trained network is effectively a complex non-linear function, whose interpretation based on matrix values is difficult. The network acts as a mapping for *whatever elements are used during training.* In this regard, neural networks are only capable of mapping what has been presented to the network during training. Any degrees of freedom present in the data may be accounted for in the network, provided enough terms were employed in the layers of the network to allow for the mapping to be robust enough to minimize the error function. If the network is too large in size, training is equivalent to over-fitting the data set, making the mapping susceptible to any noise present in the data. The ideal neural network is the smallest sized network, which finds a global minimum to the cost function chosen, leading to a robust non-linear mapping, insensitive to noise.

The form of a network presented above using arc-tangent sigmoids has the advantage of being easily differentiated. If data is represented by a function, $u(x,t)$, then a model of the data might take the form:

$$\begin{aligned}u_t &= \mathcal{N}(t, x, u, u_x, u_{xx}, ...) \\ &= a_0 + a_1 u + a_2 u_x + a_3 u_{xx} + a_4 u u_x + a_5 x u + ...\end{aligned}$$

with subscripts representing partial differentiation and the function, $\mathcal{N}(...)$, can be written as a linear combination of its arguements, representing a single differential equation or a set of equations representing a physical model, such as the primitive equations of the ocean[4;20;21;28]. Solving a *second* neural network which attempts to find a mapping from $[t, x, u, u_x, ...] \Rightarrow u_t$, the network formed becomes an ansatz to the physical model, $\mathcal{N}$.

The advantage of solving two networks is that the first network finds a substitution for the data which is effectively "denoised", with easily calculable derivatives needed as inputs to the second network. After training, the end product is two-fold; a non-linear fitted mapping from raw data to parts of the domain where data is not present, as well as physical model based on the denoised data, representing the physics, with predictive capabilities in u_t. The cost function minimizes takes the form:

$$\mathbb{E} = \sum_{j=1}^{N_d} |u(x_j, t_j) - u_j|^2 + |u_{t,j} - \mathcal{N}(t_j, x_j, u_j, u_x, j, u_{xx,j}, ...)|^2$$

where the first term is the error between training sets of the data and the non-linear representation of the data while the second term is the error between the denoised data and the physical model represented by the data[22;23].

Solving two networks in tandem in order to recreate a physical model based on data is burgeoning research, with some initial encouraging studies done. Full studies for complicated models have yet to be done, and the current proposal will apply this technique for representative data that the Navy uses applied in ocean settings. Adding a third network to represent Lagrangian data will be considered as well. In this manner, the first network represents Eulerian data (taken from an absolute coordinate system), the next network would represent the Lagrangian data (taken from gliders/drifters) and the last network representing the physical model. Separating the networks allows each to have its own scale and resolution in space and time.

An example of how naval data might be applied would be to take a satellite data set giving sea surface height, temperature and salinity. The first network maps the sea surface height as inputs to the temperature and salinity as output. This network is highly sensitive to the surface (2D), yet can only infer the 3D structure. The glider data, however, maps position and time to similarly recorded values from the drifter, temperature and salinity; this time *at depth*. Finally, the physical model is solved assuming a functional form that could represent a primitive set of equations yielding temperature and velocity field information.

4 Dynamic Mode Decomposition

A parallel effort in data science is the application of modal decomposition by creating a matrix based on data, usually with columns vectors represent positions, with each column being a single data point in the space. For a system with a high number of dimensions, the matrices formed may not be square. Several techniques proceed by forming a covariance matrix from the initial rectangular data matrix, where the eigenmodes of the covariance matrix found account for the degree of variance found in the data starting from the lowest modes (larger distance scales) to higher modes (smaller size scales).

This approach allows data to be represented in a compressed manner when only a select few eigenmodes are utilized from the set found. Typically, the fewest modes accounting for the most variation are retained, allowing the data to be reconstructed in a lower dimensional space. Further, the eigenvectors formed span the domain where data has been sampled, yet can also act as interpolating functions into regions where data has not been collected. The eigenvalues for each mode represent the variance for that mode with respect to the entire space. In this regard, arranging the modes according to highest contributions down to lowest, orders the mode to align along dimensions which span the data set from most to least. This approach goes by many names depending on the field, where it is commonly known as Principle Component Analysis (PCA) for statistics, Empirical Orthogonal Functions (EOF) in the mechanical engineering sector, and a host other names (see `https://en.wikipedia.org/wiki/Principal_component_analysis`).

A dynamical systems varaition has recently been explored, where the columns represent a scanned 2D or 3D collection of positions, with subsequent columns for the spatial information at differing times. Often the data come in varying resolutions, where the spatial resolution may be large, while the temporal collection may be limited, leading to tall and thin matrices representing the data. A propagation operator (called a Koopman operator, A) is sought, taking the first (1...N-1) columns to the next (2...N) columns, by writing the data as:

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_{n-1} \\ | & | & & | \end{bmatrix} \qquad X' = \begin{bmatrix} | & | & & | \\ x_2 & x_2 & \dots & x_n \\ | & | & & | \end{bmatrix} \qquad X' \; = \; AX$$

Applying singular value decomposition (SVD) to data formed matrix, ($A = X'X\dagger$), the lowest modes from U can be used to rebuild the propagator, A', such that a lower dimensional representation of the data is formed, effectively denoised, where the modes found are spatio-temporal modes, which evolve over time as the data set evolves.

5 Statement of Work

Prof. Mcilhany will work at USNA with a group recently formed to address applying new techniques to oceanographic data, namely, PINN and DMD. Further collaboration with Prof. Stephen Wiggins will continue focusing on the understanding of the data manifold formed from the use of Eulerian measures developed in past years. The combination of the Eulerian view (satellite, model based) with Lagrangian (gliders, drifters) will be applied as simply "data" where the effort will be to interpret the utility of the information provided by these new methods. Ultimately, this is an attempt to answer whether simple questions about the nature of ocean flow can be addressed given the extensive data set the Navy has collected. For example, "Is a gyre 2km or 10km in front of a vessel at a depth of 200m?", which affects how sonar returns are viewed.

Work done over the Spring of 2019 to determine an optimal data set to illustrate these techniques is a precursor to the application. Most likely, the Arctic Sea ice lifetime data will be analyzed by MIDM Smith as part of her overall research program. Should enough data be acquired of useful content, the South China Seas will also be analyzed. MATLAB will be used as the IDE of choice, although initially, TensorFlow (via PYTHON) will be used to transition from previous work[22] to the present effort.

Applying these techniques to gain operational understanding is the goal for the Summer of 2019. The techniques have been established previously, however, the attempt to combine both Eulerian and Lagrangian data sets within a common data framework is novel. Working with faculty as well as MIDN Smith on a rich data set will encompass the summer intercession.

6 Naval Impact

MIDN Alexandra Smith is currently studying dynamic mode decomposition as her thesis for a proposed Bowman scholar program. At USNA, a small number of gifted midshipmen are advanced enough in their coursework such that they can pursue an enhanced research program in their senior year. Further, Bowman scholars are automatically entered into the submarine fleets pipeline *after* completing a mandatory masters degree program - which can be at the Naval Postgraduate School (NPS), but may also take place at other compatible institutions, such as MIT. Recent effort has been made to begin such a collaboration with UCSD's Scripps Institute as well.

With attention paid to the Task Force Ocean (N00014-19-S-SN04), MIDN Smith's education is an attempt at aligning advanced techniques culled from the physics and mathematics world with applications to oceanography. Further, the faculty at USNA need to remain in touch with the most recent advances in the field, necessitating our commitment to both teaching our students as well as ourselves to avoid complacency.

References

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