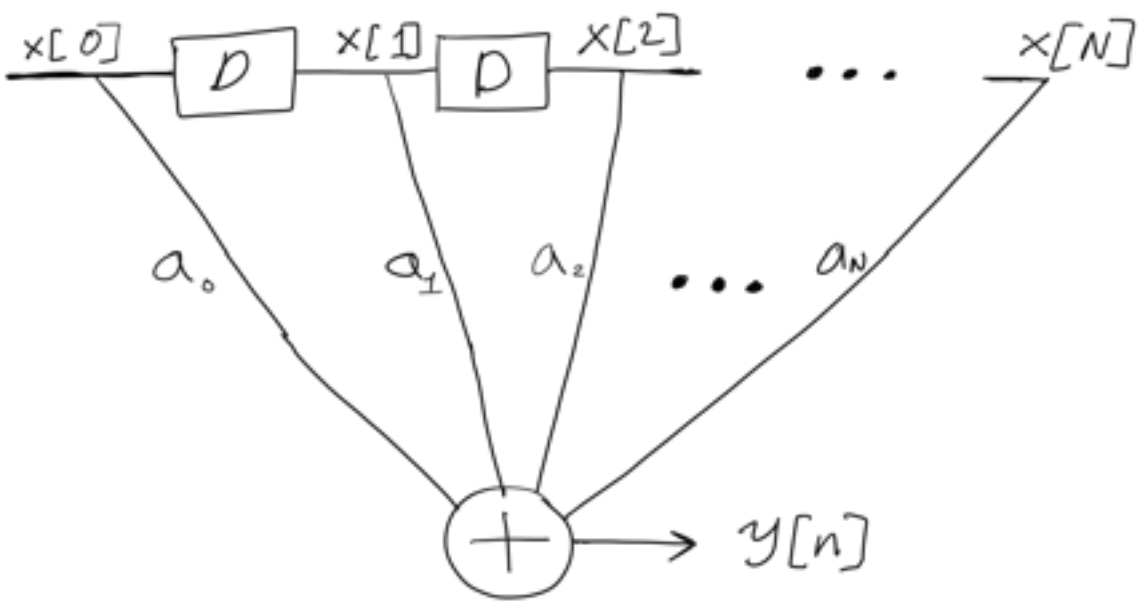
Question 2

Work to derive answers for questions a, b, c:

$$X[n] = \underbrace{S}_{\text{signal}} + \underbrace{Z[n]}_{\text{noise}}, \quad n = 0, 1, \dots N \text{ (or } M)$$

FIR Filter:



$$y[n] = \sum_{n=1}^{N} a_n x_n$$

$$= \sum_{n=1}^{N} a_n (S + Z[n])$$

$$= \sum_{n=1}^{N} a_n \cdot S + \sum_{n=1}^{N} a_n Z_n$$

$$= \underbrace{S\left(\sum_{n=1}^{N} a_n\right)}_{\text{signal}} + \underbrace{\sum_{n=1}^{N} a_n Z_n}_{\text{noise}}$$

$$SNR \triangleq \frac{P_{signal}}{P_{noise}}$$

$$= \frac{S^2 \left(\sum_{n=1}^{N} a_i\right)^2}{\left(\sum_{n=1}^{N} a_n Z_n\right)^2}$$

Solving for the denominator:

$$= \left(\sum_{n=1}^{N} a_n Z_n\right)\left(\sum_{n=1}^{N} a_n Z_n\right)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \overline{a_i a_j Z_i Z_j}$$

$$= a_1^2 \overline{Z_1^2} + a_1 a_2 \overline{Z_1 Z_2} + \dots + a_1 a_N \overline{Z_1 Z_N}$$
$$+ a_2 a_1 \overline{Z_1 Z_1} + a_2^2 \overline{Z_2^2} + \dots + a_2 a_N \overline{Z_2 Z_N}$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$+ a_N a_1 \overline{Z_N Z_1} + a_N a_2 \overline{Z_N Z_2} + \dots + a_N^2 \overline{Z_N^2}$$

$$= a_1^2 \overline{z_1^2} + a_2^2 \overline{z_2^2} + \ldots + a_N^2 \overline{z_N^2}$$

$$= \sigma^2 \left(\sum_{n=1}^{N} a_n^2 \right)$$

Assumption:
$z \sim \mathcal{N}(0, \sigma^2)$
$\therefore \bar{z} = 0, \overline{z^2} = \sigma^2$

$$\therefore \text{SNR} = \frac{S^2 \left(\sum_{n=1}^{N} a_n \right)^2}{\sigma^2 \left(\sum_{n=1}^{N} a_n^2 \right)}$$

$$\text{SNR}_{\text{uniform}} = \frac{S^2 \left(\sum_{n=1}^{N} \frac{1}{N} \right)^2}{\sigma^2 \left(\sum_{n=1}^{N} \left(\frac{1}{N} \right)^2 \right)}$$

$$= \frac{S^2 \left(N \cdot \frac{1}{N} \right)^2}{\sigma^2 \left(N \cdot \frac{1}{N^2} \right)}$$

$$= N \left(\frac{S^2}{\sigma^2} \right)$$

$$\text{SNR}_{\text{Harmonic}} = \frac{S^2 \left(\sum_{n=1}^{M} n \right)^2}{\sigma^2 \left(\sum_{n=1}^{M} n^2 \right)}$$

$$= \frac{S^2 \left(\frac{M(M+1)}{2} \right)^2}{\sigma^2 \left(\frac{M(M+1)(2M+1)}{6} \right)}$$

$$= \frac{S^2 \left(M^2 (M+1)^2 \cdot 6 \cdot 3 \right)}{\sigma^2 \left(M (M+1)(2M+1) \cdot 4 \cdot 2 \right)}$$

$$= \frac{S^2 \left(3M^2 + 3M \right)}{\sigma^2 \left(4M + 2 \right)}$$

$$\approx \frac{S^2 \left(3M^2 \right)}{\sigma^2 \left(4M \right)}$$

$$= \frac{3}{4} M \left(\frac{S^2}{\sigma^2} \right)$$

Boost SNR by factor of 5:

Uniform:

$$N \left(\frac{S^2}{\sigma^2} \right) = 5 \left(\frac{S^2}{\sigma^2} \right)$$

$$N = 5$$

Harmonic:

$$\frac{3}{4}M\left(\frac{5}{\sigma^2}\right) = 5\left(\frac{}{\sigma^2}\right)$$

$$M = \frac{20}{3}$$

M must be an integer ∴

$$M = \left\lceil \frac{20}{3} \right\rceil$$

$$M = 7$$

My Answers

a) $N = 5, M = 7$

b) $N < M$ ∴ Uniform

c) Uniform ∵ better SNR & delay