Synchronizing Emission Halvings in Dynamic TAO: A Mathematical Analysis of Asymmetric Halving Effects

Maciej Kula

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Abstract

Dynamic TAO (DTAO) allocates emissions via subnet AMMs rather than validator weights, but independent halving schedules for TAO and subnet ALPHA create an asynchronous gap between the TAO halving index k and the ALPHA halving index n. This paper derives closed-form relationships that expose how this gap drives cohort asymmetries across four dimensions: compression of ALPHA-halving intervals, liquidity impact in constant-product AMMs, the pace of root-proportion decline, and subnet liquidation. The results show that later TAO cohorts face progressively stronger effects under identical conditions due to the exponential factor 2^{k-n}. A synchronized-halving variant is examined that ties both ALPHA components to the global TAO schedule, thereby removing the $k-n$ gap. In this variant, ALPHA-halving intervals cease to compress, liquidity impact depends only on the sell fraction (not cohort timing), the emissions–injections ratio equals one (eliminating liquidation discounts), and the root-proportion trajectory depends only on global time. ALPHA supply implications under synchronization are outlined, with Zeno-Halvings proposed to prevent zero tails when on-chain precision would otherwise truncate injections and thus emissions. A simple scaling of the TAO-weight, $\gamma_k = \gamma_0/2^k$, stabilizes the root proportion across halvings. Overall, the analysis identifies 2^{k-n} as the common driver of cohort asymmetries in the asynchronous design and clarifies how synchronization can neutralize protocol-level bias.

1 Introduction

Dynamic TAO (DTAO) introduces market-driven emission allocation through subnet-specific Automated Market Makers (AMMs), where subnet tokens (ALPHA, α) trade against the universal TAO token (τ). This mechanism replaces validator-weighted allocations with price-discovery-based distribution, fundamentally altering how subnet rewards are determined and distributed.

A critical design choice in DTAO is the use of independent halving schedules for TAO and ALPHA tokens. While both follow issuance-based thresholds, TAO halvings occur globally based on total TAO issuance, whereas each subnet's ALPHA halvings depend on that subnet's individual issuance. This creates an asynchronous relationship where subnet cohorts experience different combinations of TAO halving index k and ALPHA halving index n based purely on their registration timing. The mathematical consequences of this asynchronous design have not been systematically analyzed. This paper demonstrates that the (k, n) halving gap creates exponential asymmetries across multiple dimensions. These asymmetries compound over time, producing systematically different outcomes for economically identical subnet behavior based solely on registration timing relative to TAO halvings.

Our analysis reveals that all phenomena share the same underlying mathematical structure driven by the exponential scaling factor 2^{n-k}, where larger gaps between ALPHA and TAO halving indices amplify cohort disparities, demonstrating how the asynchronous halving design embeds mathematical inequality.

Scope. This paper analyzes current cohort asymmetries arising from the asynchronous design and explores a synchronized-halving variant as a hypothetical alternative that removes the $(n-k)$ gap; the synchronized variant is presented to isolate the effect of gap removal and is not claimed to be uniquely optimal.

2 Notation and Assumptions

Global indices and schedules. k: global TAO halving index; n: subnet ALPHA halving index. $\Delta\tau_k$: global TAO block emission at halving index k (halves with k). B: blocks per day.

Prices and reserves. For subnet i, $p_i := \tau_i/\alpha_i$ is the AMM spot price (TAO per ALPHA) and $\tilde{p}_i$ is its exponentially weighted moving average (EMA) used for TERP.

Injections and emissions. $\Delta\tau_i = \dfrac{\tilde{p}_i}{\sum_j \tilde{p}_j} \cdot \Delta\tau_k$ is the TAO injection share to subnet i. The ALPHA injection per block is

$$\Delta\alpha_i = \min\left(\frac{\tilde{p}_i}{\sum_j \tilde{p}_j} \cdot \frac{\Delta\tau_k}{p_i},\ \Delta\overline{\alpha}_i(k)\right), \quad \text{with} \quad \Delta\overline{\alpha}_i(k) := \Delta\tau_k,$$

a per-block cap tied to the TAO schedule. The per-block ALPHA emission to participants is $\Delta\alpha'_i$ following the subnet's own ALPHA schedule.

Cap (upper bound). $\Delta\overline{\alpha}_i(k)$: per-block upper bound on ALPHA injection for subnet i, set by the TAO schedule as $\Delta\overline{\alpha}_i(k) := \Delta\tau_k$.

Outstanding and root. α_i^{o}: ALPHA outstanding on subnet i (held by participants, not in the AMM reserve). τ_0: TAO staked in the root. $\gamma > 0$: TAO-weight parameter for dividend splits.

For a complete list of symbols used throughout, see Appendix B (Notation Index).

For all examples below, take the baseline $\Delta\alpha(0)_i = \Delta\alpha'(0)_i = 1$. Equivalently, assume $\Delta\tau_i/p_i = 1$ so that $\Delta\alpha(0)_i = \Delta\overline{\alpha}_i(0)$.

3 Mathematical Framework

3.1 Injection Mechanism and TERP

The injection mechanism allocates the per-block TAO emission across subnets using the TAO Emission Ratio Property (TERP) with EMA prices. Let $\Delta\tau_k$ denote the global TAO emission per block at TAO halving index k. TERP assigns subnet i a TAO share

$$\Delta\tau_i \;=\; \frac{\tilde{p}_i}{\sum_j \tilde{p}_j} \cdot \Delta\tau_k \tag{1}$$

where $\tilde{p}_i$ is the EMA price (TAO per ALPHA) for subnet i. Given a spot price p_i (TAO per ALPHA), the protocol mints ALPHA into the AMM reserve so that the TAO value of the ALPHA injection does not exceed the assigned TAO injection, subject to a per-block cap tied to the TAO schedule:

$$\Delta\alpha_i \;=\; \min\left(\frac{\tilde{p}_i}{\sum_j \tilde{p}_j} \cdot \frac{\Delta\tau_k}{p_i},\; \Delta\overline{\alpha}_i(k)\right), \qquad \Delta\overline{\alpha}_i(k) := \Delta\tau_k. \tag{2}$$

The first term in (2) converts the TAO share in (1) into ALPHA units using the spot price; the cap enforces that, per block, the number of ALPHA injected for subnet i cannot exceed the global TAO emission scale at index k.

Normalization for analysis. To make closed forms tractable and cohort-comparable, the analysis proceeds under a neutral baseline where price normalization implies a common scale and the cap is not binding on the studied path. Concretely:

$$\frac{\tilde{p}_i}{\sum_j \tilde{p}_j} \cdot \frac{\Delta\tau_k}{p_i} \;\equiv\; \frac{\Delta\alpha(0)}{2^k}, \qquad \Delta\overline{\alpha}_i(k) \;\geq\; \frac{\Delta\alpha(0)}{2^k}, \tag{3}$$

so that the effective per-block ALPHA injected into the AMM for subnet i is

$$\Delta\alpha_i(k) \;=\; \frac{\Delta\alpha(0)}{2^k}. \tag{4}$$

Here $\Delta\alpha(0) > 0$ sets the pre-halving injection scale and halving index k applies the geometric decay inherited from $\Delta\tau_k$.

3.2 ALPHA Block Inflation

Beyond reserve injections, each subnet mints ALPHA directly to participants according to its own ALPHA halving schedule indexed by n, producing a per-block emission

$$\Delta\alpha'_i(n) \;=\; \frac{\Delta\alpha'(0)}{2^n}, \tag{5}$$

with $\Delta\alpha'(0) > 0$ the pre-halving emission scale. The total per-block ALPHA minted (to the AMM reserve plus to participants) on subnet i at halving indices (n, k) is the sum of (4) and (5):

$$\mu_{k,n} \;:=\; \Delta\alpha_i(k) \;+\; \Delta\alpha'_i(n) \;=\; \frac{\Delta\alpha(0)}{2^k} \;+\; \frac{\Delta\alpha'(0)}{2^n}\,, \tag{6}$$

where $\Delta\alpha(0)/2^k$ represents the injection component (scaled by TAO halving index k) and $\Delta\alpha'(0)/2^n$ represents the emission component (scaled by ALPHA halving index n).

4 ALPHA Halving Compression

4.1 Compression Formula

The asynchronous halving design creates an interaction between TAO-scaled injections and ALPHA emissions; as TAO halvings (index k) reduce the per-block ALPHA injection $\Delta\alpha_i$, the per-block ALPHA emission to participants $\Delta\alpha_i'$ (index n) becomes a different share of the total, changing the length of ALPHA-halving intervals for subnet i. With B blocks per day and per-block minting rate $\mu_{k,n}$ from (6), an ALPHA-halving interval ends when its issuance threshold is reached. Let $D(0) = 10{,}500{,}000$ be the issuance needed for the first halving; after n ALPHA halvings, the threshold is $D(0)/2^n$. When the per-block rate is constant (no TAO halving within the ALPHA interval), the interval duration is

$$T_{k,n} \;=\; \frac{D(0)/2^n}{\mu_{k,n}\,B} \;=\; \frac{D(0)/2^n}{\left(\dfrac{\Delta\alpha(0)}{2^k} + \dfrac{\Delta\alpha'(0)}{2^n}\right)B}. \tag{7}$$

To compare successive intervals, the compression ratio is defined as the next duration divided by the current duration and is derived directly from (7):

$$C_{k,n} := \frac{T_{k,n+1}}{T_{k,n}} = \frac{\dfrac{D(0)}{2^{n+1}}\Big/\big(\mu_{k,n+1}B\big)}{\dfrac{D(0)}{2^n}\Big/\big(\mu_{k,n}B\big)} = \frac{1}{2}\cdot\frac{\mu_{k,n}}{\mu_{k,n+1}} = \frac{1}{2}\cdot\frac{\dfrac{\Delta\alpha(0)}{2^k}+\dfrac{\Delta\alpha'(0)}{2^n}}{\dfrac{\Delta\alpha(0)}{2^k}+\dfrac{\Delta\alpha'(0)}{2^{n+1}}} \in \left(\tfrac{1}{2},\,1\right]. \tag{8}$$

The factor $\frac{1}{2}$ arises from the threshold ratio $\big(D(0)/2^{n+1}\big)/\big(D(0)/2^n\big)$; the remaining factor $\mu_{k,n}/\mu_{k,n+1}$ is the relative change in the per-block minting rate between the two intervals and determines the amount of compression.

Under the baseline $\Delta\alpha'(0) = \Delta\alpha(0)$, (8) becomes

$$C_{k,n} = \frac{1}{2}\cdot\frac{\dfrac{\Delta\alpha(0)}{2^k}+\dfrac{\Delta\alpha(0)}{2^n}}{\dfrac{\Delta\alpha(0)}{2^k}+\dfrac{\Delta\alpha(0)}{2^{n+1}}} = \frac{1}{2}\cdot\frac{\Delta\alpha(0)\left(\dfrac{1}{2^k}+\dfrac{1}{2^n}\right)}{\Delta\alpha(0)\left(\dfrac{1}{2^k}+\dfrac{1}{2^{n+1}}\right)} = \frac{1}{2}\cdot\frac{1+2^{k-n}}{1+2^{k-n-1}}. \tag{9}$$

For fixed k and increasing n, the emission term halves across intervals while the injection term is unchanged, driving $C_{k,n}$ monotonically toward $\frac{1}{2}$. The exponential gap 2^{k-n} that governs $C_{k,n}$ will reappear in the liquidity and price asymmetries below, providing a common driver across sections.

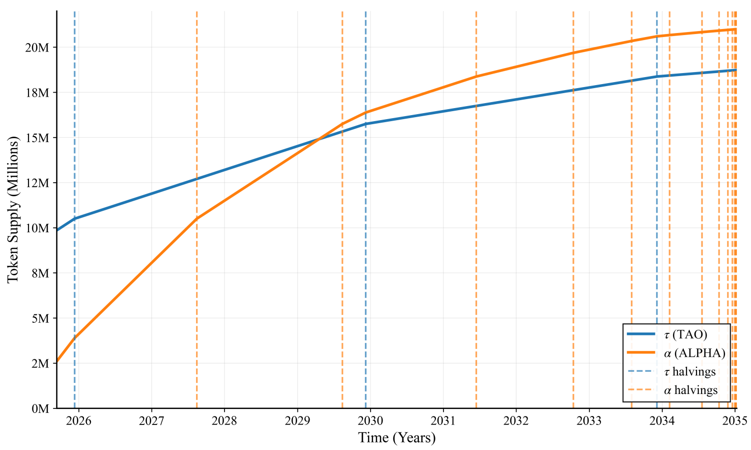

Figure 1: As $n-k$ grows, $C_{k,n}$ approaches $\frac{1}{2}$, so each subsequent ALPHA-halving interval tends toward half of the preceding one (intervals compress). The halving trajectories used to generate this figure are reproduced in Appendix C.

4.2 TAO Halving Within the Current Interval (Case A)

There can be a TAO halving inside the current ALPHA interval $n \to n{+}1$. In this case, (8) still applies, but the current interval runs at two rates because the TAO index changes from k to $k{+}1$ partway through. Let $\phi \in [0,1]$ denote the issuance fraction of the current threshold produced before the TAO halving (i.e., the proportion of $D(0)/2^{\,n}$ minted prior to $k \to k{+}1$). Using (7) segment-wise for the current interval and (7) at indices $(k{+}1, n{+}1)$ for the next interval,

$$T^{\mathrm{A}}_{k,n} = \frac{D(0)}{2^{\,n}B}\left(\frac{\phi}{\mu_{k,n}} + \frac{1-\phi}{\mu_{k+1,n}}\right), \qquad T^{\mathrm{A}}_{k,n+1} := T_{k+1,n+1} = \frac{D(0)/2^{\,n+1}}{\mu_{k+1,n+1}\,B}. \tag{10}$$

Taking the ratio and cancelling $D(0)$ and B,

$$C^{\mathrm{A}}_{k,n}(\phi) := \frac{T^{\mathrm{A}}_{k,n+1}}{T^{\mathrm{A}}_{k,n}} = \frac{\dfrac{D(0)}{2^{\,n+1}}\Big/\big(\mu_{k+1,n+1}B\big)}{\dfrac{D(0)}{2^{\,n}}\Big/(B)\left(\dfrac{\phi}{\mu_{k,n}} + \dfrac{1-\phi}{\mu_{k+1,n}}\right)} = \frac{1}{2}\cdot\frac{1}{\phi\,\dfrac{\mu_{k+1,n+1}}{\mu_{k,n}} + (1-\phi)\,\dfrac{\mu_{k+1,n+1}}{\mu_{k+1,n}}}. \tag{11}$$

This follows the same structure as (8): the $\frac{1}{2}$ threshold factor remains, and the effective current-interval rate becomes a weighted average of the two segment rates.

4.3 TAO Halving Within the Next Interval (Case B)

There can be a TAO halving inside the next ALPHA interval $n{+}1 \to n{+}2$. Here the current interval is constant-rate, while the next interval splits. Let $\psi \in [0,1]$ denote the issuance fraction of the next threshold produced before the TAO halving (i.e., the proportion of $D(0)/2^{\,n+1}$ minted prior to $k \to k{+}1$). Using (7) for each segment,

$$T^{\mathrm{B}}_{k,n} = \frac{D(0)/2^{\,n}}{\mu_{k,n}\,B}, \qquad T^{\mathrm{B}}_{k,n+1} = \frac{D(0)}{2^{\,n+1}B}\left(\frac{\psi}{\mu_{k,n+1}} + \frac{1-\psi}{\mu_{k+1,n+1}}\right). \tag{12}$$

Taking the ratio and cancelling $D(0)$ and B,

$$C^{\mathrm{B}}_{k,n}(\psi) := \frac{T^{\mathrm{B}}_{k,n+1}}{T^{\mathrm{B}}_{k,n}} = \frac{1}{2}\left(\psi\,\frac{\mu_{k,n}}{\mu_{k,n+1}} + (1-\psi)\,\frac{\mu_{k,n}}{\mu_{k+1,n+1}}\right). \tag{13}$$

Again, this follows (8) with the next-interval rate represented as a weighted average.

4.4 Behavior and Examples

Example 1 (Case A: TAO halving inside the current interval). Let $(k,n) = (0,0)$, $\phi = \frac{1}{2}$, and the rates:

$$\mu_{0,0} = 2^{-0} + 2^{-0} = 2, \quad \mu_{1,0} = 2^{-1} + 2^{-0} = 1.5, \quad \mu_{1,1} = 2^{-1} + 2^{-1} = 1.$$

Substitute into (11):

$$C^{\mathrm{A}}_{0,0}\big(\tfrac{1}{2}\big) = \frac{1}{2}\cdot\frac{1}{\frac{1}{2}\cdot\dfrac{1}{2} + \frac{1}{2}\cdot\dfrac{1}{1.5}} = \frac{1}{2}\cdot\frac{1}{0.25 + 0.333\overline{3}} \approx 0.8571.$$

Benchmark without a TAO halving in either interval from (9) is $C_{0,0} = \frac{1}{2}\cdot\frac{1+1}{1+1/2} = 2/3 \approx 0.6667$. The in-interval halving makes the next interval run entirely at the slower post-halving rate and lengthens the current interval via its slower segment, raising the ratio from $2/3$ to ≈ 0.857.

Example 2 (Case B: TAO halving inside the next interval). Let $(k, n) = (1, 1)$, $\psi = \frac{1}{2}$, and the rates:

$$\mu_{1,1} = 2^{-1} + 2^{-1} = 1, \quad \mu_{1,2} = 2^{-1} + 2^{-2} = 0.75, \quad \mu_{2,2} = 2^{-2} + 2^{-2} = 0.5.$$

Substitute into (13):

$$C^{\mathrm{B}}_{1,1}(\tfrac{1}{2}) = \frac{1}{2}\left(\tfrac{1}{2} \cdot \frac{1}{0.75} + \tfrac{1}{2} \cdot \frac{1}{0.5}\right) = \frac{1}{2}\big(\tfrac{2}{3} + 1\big) = \frac{5}{6} \approx 0.8333.$$

Because the next interval spends a portion at the faster pre-halving rate, it shortens less than the expected, so the ratio increases.

Example 3 (No TAO halving during current and next interval). Let $(k, n) = (2, 4)$ with constant rate:

$$\mu_{2,4} = 2^{-2} + 2^{-4} = 0.3125, \quad \mu_{2,5} = 2^{-2} + 2^{-5} = 0.28125.$$

Substitute into (8):

$$C_{2,4} = \frac{1}{2} \cdot \frac{0.3125}{0.28125} = \frac{1}{2} \cdot \frac{10}{9} \approx 0.5556.$$

Here k is fixed and only n advances by one, so the injection term is unchanged while the emission term halves; the resulting rate change yields $C \approx 0.5556$, illustrating compression toward $\frac{1}{2}$.

5 Liquidity Impact

5.1 Liquidity Impact Formula

Building on the compression analysis, the same gap in halving indices (k, n) that governs (9) also governs AMM liquidity. In a constant-product AMM with reserves (α_i, τ_i), selling an amount α^{swap} of ALPHA pays out TAO

$$\tau^{\mathrm{swap}} := \tau_i - \frac{\tau_i\, \alpha_i}{\alpha_i + \alpha^{\mathrm{swap}}} = \tau_i\left(1 - \frac{\alpha_i}{\alpha_i + \alpha^{\mathrm{swap}}}\right), \tag{14}$$

so the normalized impact on TAO reserves (fraction of TAO liquidity removed) is

$$\mathcal{R} := \frac{\tau^{\mathrm{swap}}}{\tau_i} = 1 - \frac{\alpha_i}{\alpha_i + \alpha^{\mathrm{swap}}} = \frac{\alpha^{\mathrm{swap}}}{\alpha_i + \alpha^{\mathrm{swap}}}\,. \tag{15}$$

5.2 Cohort-Impact Analysis

Over an interval with fixed (k, n), the per-block ALPHA emission minted to participants is $\Delta\alpha'(0)/2^{\,n}$ and the per-block ALPHA injection minted into the AMM reserve is $\Delta\alpha(0)/2^{\,k}$. For any common horizon $H > 0$ (no prior swaps), selling a fraction $s \in [0, 1]$ of the newly minted emissions yields

$$\alpha^{\mathrm{swap}}_{H,n} = s\, H\, \frac{\Delta\alpha'(0)}{2^{\,n}}, \qquad \alpha_{H,k} = H\, \frac{\Delta\alpha(0)}{2^{\,k}}.$$

Substituting into $\mathcal{R}$ and cancelling H gives

$$\mathcal{R}_{k,n}(s) = \frac{s\, \dfrac{\Delta\alpha'(0)}{2^{\,n}}}{\dfrac{\Delta\alpha(0)}{2^{\,k}} + s\, \dfrac{\Delta\alpha'(0)}{2^{\,n}}} = \frac{s\, \dfrac{\Delta\alpha'(0)}{\Delta\alpha(0)}}{2^{\,n-k} + s\, \dfrac{\Delta\alpha'(0)}{\Delta\alpha(0)}} \tag{16}$$

$$= \frac{s}{2^{\,n-k} + s} \qquad (\Delta\alpha'(0) = \Delta\alpha(0) \text{ baseline})\,. \tag{17}$$

For fixed s and n, increasing k reduces 2^{n-k} and increases $\mathcal{R}_{k,n}(s)$ (larger fraction of TAO liquidity removed); for fixed s and k, increasing n enlarges 2^{n-k} and decreases $\mathcal{R}_{k,n}(s)$. The same asymmetry in 2^{k-n} that compresses intervals therefore amplifies liquidity impact for later TAO cohorts.

5.3 Behavior and Examples

Example 1 (Baseline $k = 0$, $n = 0$). With $s = 0.41$ and $\Delta\alpha'(0) = \Delta\alpha(0)$, substitute in (16):

$$\mathcal{R}_{0,0}(0.41) = \frac{0.41}{2^{0-0} + 0.41} = \frac{0.41}{1 + 0.41} = \frac{0.41}{1.41} \approx 0.291 \ (29.1\%).$$

Example 2 (Later TAO cohort $k = 5$, $n = 0$). With $s = 0.41$ and $\Delta\alpha'(0) = \Delta\alpha(0)$,

$$\mathcal{R}_{5,0}(0.41) = \frac{0.41}{2^{0-5} + 0.41} = \frac{0.41}{1/32 + 0.41} \approx 0.929 \ (92.9\%).$$

A larger k reduces the AMM reserve minted per block while the sold emissions are unchanged at $n = 0$, so the same selling fraction drains a larger share of TAO liquidity.

Example 3 (Later ALPHA cohort $n = 1$, $k = 0$). With $s = 0.41$ and $\Delta\alpha'(0) = \Delta\alpha(0)$,

$$\mathcal{R}_{0,1}(0.41) = \frac{0.41}{2^{1-0} + 0.41} = \frac{0.41}{2 + 0.41} = \frac{0.41}{2.41} \approx 0.170 \ (17.0\%).$$

Advancing n halves emissions while injection is unchanged at $k = 0$, so the same selling fraction drains a smaller share of TAO liquidity.

These relationships formalize that identical selling pressure s creates larger liquidity impact for later TAO cohorts because AMM reserves scale with $\Delta\alpha(0)/2^k$ while sold emissions scale with $\Delta\alpha'(0)/2^n$. The next section shows how the same rates shape the pace of root-dominated dividends.

6 Root Proportion Slowdown

6.1 Definition and Formula

The root proportion r_i splits validator dividends between root stakers and subnet validators and is defined as

$$r_i \;=\; \frac{\gamma\,\tau_0}{\gamma\,\tau_0 + \alpha_i^{\mathrm{o}} + \alpha_i}\,, \tag{18}$$

where α_i^{o} is the ALPHA outstanding (held by participants, not AMM reserves), α_i is the ALPHA in the AMM reserve, τ_0 is the total TAO staked in the root subnet, and $\gamma > 0$ is the TAO-weight parameter.

6.2 Slowdown Analysis

Consider the case where the ALPHA halving index is $n = 0$ (before any ALPHA halving). Over a common horizon of $H > 0$ blocks, the total ALPHA minted satisfies

$$\alpha_{i,H,k} + \alpha_{i,H}^{\mathrm{o}} \;=\; A(0) \;+\; H\,\mu_{k,0} \;=\; A(0) \;+\; H\left(\frac{\Delta\alpha(0)}{2^k} + \Delta\alpha'(0)\right),$$

where $A(0)$ denotes the initial total ALPHA minted and $\mu_{k,0}$ is the per-block minting rate from (6) with $n = 0$. Substituting into (18) gives

$$r_{i,k}(H) = \frac{\gamma\,\tau_0}{\gamma\,\tau_0 + \left(\alpha_i^{\mathrm{o}} + \alpha_i\right)} = \frac{\gamma\,\tau_0}{\gamma\,\tau_0 + A(0) + H\,\mu_{k,0}} = \frac{\gamma\,\tau_0}{\gamma\,\tau_0 + A(0) + H\left(\dfrac{\Delta\alpha(0)}{2^k} + \Delta\alpha'(0)\right)}. \tag{19}$$

Under the baseline $\Delta\alpha'(0) = \Delta\alpha(0)$, (19) becomes

$$r_{i,k}(H) = \frac{\gamma\,\tau_0}{\gamma\,\tau_0 + A(0) + H\,\Delta\alpha(0)\left(\dfrac{1}{2^k} + 1\right)} = \frac{\gamma\,\tau_0}{\gamma\,\tau_0 + A(0) + H\,\Delta\alpha(0)(2^{-k} + 1)}. \tag{20}$$

6.3 Behavior and Examples

Example 1 (Baseline horizon at $k = 0$). Take $\tau_0 = 6{,}000{,}000$, $\gamma = 0.18$, $H = 1{,}296{,}000$, and the baseline $A(0) = 0$, $\Delta\alpha(0) = \Delta\alpha'(0) = 1$ (so $\mu_{0,0} = 2^{-0} + 1 = 2$). Using (20),

$$r_{i,0}(1{,}296{,}000) = \frac{6{,}000{,}000 \cdot 0.18}{6{,}000{,}000 \cdot 0.18 + 1{,}296{,}000 \cdot \big(1 \cdot (1+1)\big)} = \frac{1{,}080{,}000}{1{,}080{,}000 + 2{,}592{,}000} \approx 0.2941.$$

Example 2 (Baseline horizon at $k = 5$). With the same τ_0, γ, H, $A(0)$, and $\Delta\alpha(0) = \Delta\alpha'(0) = 1$ (so $\mu_{5,0} = 2^{-5} + 1 = 1.03125$), (20) gives

$$r_{i,5}(1{,}296{,}000) = \frac{6{,}000{,}000 \cdot 0.18}{6{,}000{,}000 \cdot 0.18 + 1{,}296{,}000 \cdot \big(1 \cdot (2^{-5}+1)\big)} = \frac{1{,}080{,}000}{1{,}080{,}000 + 1{,}336{,}500} \approx 0.4469.$$

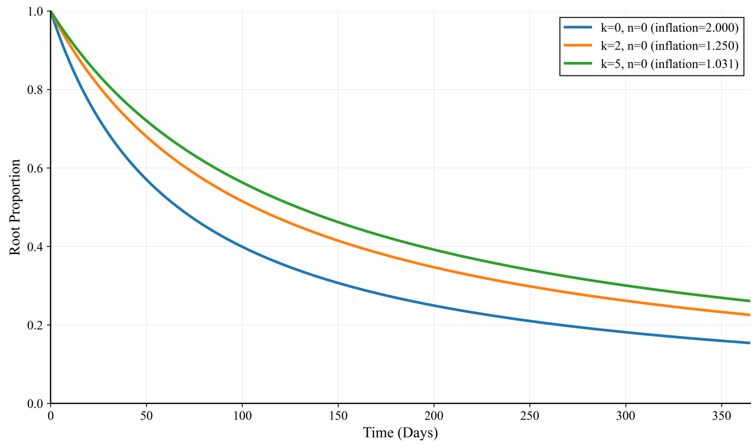

Figure 2: Root proportion trajectories for $n = 0$ across TAO halving indices k. Larger k reduces the injection term 2^{-k}, slowing the decline of $r_{i,k}(H)$ over a common horizon H.

This demonstrates how identical time periods produce markedly different outcomes based purely on subnet registration timing relative to TAO halvings, with later cohorts experiencing progressively slower transitions from root-dominated to ALPHA-dominated dividend distribution. The same gap 2^{k-n} also determines liquidation premia and discounts in the next section.

7 Liquidation Asymmetry

7.1 Definition and Formula

Asynchronous halvings create a systematic divergence between the price implied by AMM reserves and the price implied by outstanding supply, producing liquidation asymmetry. The spot price for subnet i is $p_i := \tau_i/\alpha_i$ (TAO per ALPHA using AMM reserves), and the liquidation price is $p_{L,i} := \tau_i/\alpha_i^{\mathrm{o}}$ (TAO per ALPHA using ALPHA outstanding).

$$\frac{p_i}{p_{L,i}} = \frac{\tau_i/\alpha_i}{\tau_i/\alpha_i^{\mathrm{o}}} = \frac{\alpha_i^{\mathrm{o}}}{\alpha_i} = \frac{\dfrac{\Delta\alpha'(0)}{2^{\,n}}}{\dfrac{\Delta\alpha(0)}{2^{\,k}}}\,. \tag{21}$$

Define the emissions-to-injection ratio at (k, n) by

$$\mathrm{ADR}_{k,n} := \frac{\dfrac{\Delta\alpha'(0)}{2^{\,n}}}{\dfrac{\Delta\alpha(0)}{2^{\,k}}} = 2^{\,k-n} \cdot \frac{\Delta\alpha'(0)}{\Delta\alpha(0)}\,, \tag{22}$$

so that

$$p_{L,i} = \frac{p_i}{\mathrm{ADR}_{k,n}} = p_i \cdot \frac{\dfrac{\Delta\alpha(0)}{2^{\,k}}}{\dfrac{\Delta\alpha'(0)}{2^{\,n}}}\,. \tag{23}$$

Under the baseline $\Delta\alpha'(0) = \Delta\alpha(0)$, (22) and (23) simplify to

$$\mathrm{ADR}_{k,n} = 2^{\,k-n}, \qquad\qquad p_{L,i} = p_i \cdot 2^{-(k-n)}\,. \tag{24}$$

7.2 Haircut

The liquidation haircut relative to spot is

$$h := 1 - \frac{p_{L,i}}{p_i} = 1 - \frac{1}{\mathrm{ADR}_{k,n}} \qquad \text{(general)} \tag{25}$$

$$= 1 - 2^{-(k-n)} \qquad (\Delta\alpha'(0) = \Delta\alpha(0))\,. \tag{26}$$

If $k > n$ then $h \in (0, 1)$ (a discount); if $k < n$ the expression is negative, indicating a premium relative to spot of magnitude $1/\mathrm{ADR}_{k,n} - 1$.

7.3 Behavior and Examples

Example 1 (Baseline $k = 0$, $n = 0$). With $p_i = 0.10$ TAO/ALPHA and $\Delta\alpha'(0) = \Delta\alpha(0)$,

$$\mathrm{ADR}_{0,0} = 2^{\,0-0} = 1, \qquad p_{L,i} = \frac{0.10}{1} = 0.10, \qquad h = 1 - \frac{1}{1} = 0.$$

Interpretation: when emissions and injections scale equally ($k = n$), liquidation occurs at spot (no discount or premium).

Example 2 (Later TAO cohort $k = 5$, $n = 0$). With $p_i = 0.10$,

$$\mathrm{ADR}_{5,0} = 2^{\,5-0} = 32, \qquad p_{L,i} = \frac{0.10}{32} = 0.003125, \qquad h = 1 - \frac{1}{32} \approx 0.96875.$$

Interpretation: larger k (smaller injections) raises ADR, lowering $p_{L,i}$ and producing a deeper liquidation discount.

Example 3 (Later ALPHA cohort $n = 4$, $k = 2$). With $p_i = 0.10$,

$$\mathrm{ADR}_{2,4} = 2^{\,2-4} = 0.25, \qquad p_{L,i} = \frac{0.10}{0.25} = 0.40, \qquad h = 1 - \frac{1}{0.25} = -3.$$

Across Sections 4–7, the same exponential gap $2^{\,k-n}$ drives interval compression (9), liquidity impact (16), root proportion trajectories (19)–(20), and liquidation premia/discounts (24), providing a single, coherent lens on cohort asymmetries under asynchronous halvings.

8 Synchronized Halving Design

8.1 Mechanism

Motivated by the prior sections' identification of the exponential gap $2^{\,k-n}$ as the common driver of interval compression (9), liquidity impact (16), root proportion trajectories (19), and liquidation asymmetries (24), this section explores a synchronized-halving variant that eliminates this gap. The idea is to align ALPHA emissions with the global TAO halving schedule so that both injection and emission scale at the same global index k, removing the subnet-specific ALPHA halving index n and thus the difference $(k - n)$.

Concretely, the synchronized variant sets

$$\Delta\alpha_k \;=\; \frac{\Delta\alpha(0)}{2^{\,k}}, \qquad \Delta\alpha'_k \;=\; \frac{\Delta\alpha'(0)}{2^{\,k}}, \tag{27}$$

so the total per-block minting rate becomes

$$\mu_k \;:=\; \Delta\alpha_k + \Delta\alpha'_k = \frac{\Delta\alpha(0) + \Delta\alpha'(0)}{2^{\,k}}\,. \tag{28}$$

Under the baseline $\Delta\alpha'(0) = \Delta\alpha(0)$, both components halve together at every global TAO halving, yielding $\mu_k = 2\,\Delta\alpha(0)/2^{\,k}$ and an emissions-to-injection ratio that remains constant. In effect, the factor $2^{\,k-n}$ collapses to unity throughout, so expressions that previously depended on $2^{\,k-n}$ lose their asymmetry and depend only on the global index k.

8.2 Properties

Starting from the compression definition (8) and setting $n = k$ (both components follow the same halving schedule),

$$C_k := \frac{1}{2}\cdot\frac{\mu_{k,k}}{\mu_{k,k+1}} = \frac{1}{2}\cdot\frac{\dfrac{\Delta\alpha(0)}{2^k} + \dfrac{\Delta\alpha'(0)}{2^k}}{\dfrac{\Delta\alpha(0)}{2^{k+1}} + \dfrac{\Delta\alpha'(0)}{2^{k+1}}} = \frac{1}{2}\cdot\frac{\dfrac{\Delta\alpha(0) + \Delta\alpha'(0)}{2^k}}{\dfrac{\Delta\alpha(0) + \Delta\alpha'(0)}{2^{k+1}}} = \frac{1}{2}\cdot 2 = 1\,. \tag{29}$$

Temporal compression vanishes because both injection and emission scale identically across intervals.

For liquidity impact, substitute $n = k$ into (16):

$$\mathcal{R}_k(s) = \frac{s}{2^{\,k-k} + s} = \frac{s}{1+s}\,. \qquad (\Delta\alpha'(0) = \Delta\alpha(0)\ \text{baseline})\,. \tag{30}$$

Under synchronization, liquidity impact depends only on the selling fraction s, not on cohort timing; all subnets face identical impact $\mathcal{R} = s/(1+s)$ regardless of their registration timing relative to TAO halvings.

For root proportion trajectories, substitute the synchronized rate μ_k from (28) into (19):

$$r_{i,k}(H) = \frac{\gamma\,\tau_0}{\gamma\,\tau_0 + A(0) + H\,\mu_k} = \frac{\gamma\,\tau_0}{\gamma\,\tau_0 + A(0) + H\,\dfrac{\Delta\alpha(0) + \Delta\alpha'(0)}{2^{\,k}}}\,. \tag{31}$$

At any fixed horizon H, all subnets with the same global k experience identical root proportion pace; cohort disparities based on registration timing disappear.

For liquidation asymmetry, apply $n = k$ to the ADR definition (22):

$$\mathrm{ADR}_k = 2^{\,k-k} \cdot \frac{\Delta\alpha'(0)}{\Delta\alpha(0)} = 1 \cdot \frac{\Delta\alpha'(0)}{\Delta\alpha(0)} = 1 \quad \text{(baseline)}\,. \tag{32}$$

From (23), this yields $p_{L,i} = p_i/\mathrm{ADR}_k = p_i/1 = p_i$, so liquidation occurs at spot price with zero haircut for all cohorts.

8.3 ALPHA Supply Trajectory

Within the synchronized variant, the maximal ALPHA minted by a subnet registering when TAO supply is S_τ becomes asymptotically bounded by twice the remaining TAO supply to the 21M cap, because both injection and emission track the same decaying TAO schedule:

$$S_i^{max} \;\lesssim\; 2\left(21\mathrm{M} - S_\tau\right).$$

However, after the last discrete TAO halving, a hard per-block emission floor could stall issuance unless the tail continues to decay.

8.4 Zeno-Halvings

The synchronized design still faces granularity constraints: at $k = 0$ the TAO block reward is $E_0 = 1{,}000{,}000{,}000$ RAO and halves each halving; once $E_k < 1$ RAO, it would round to 0. To avoid a zero tail, Zeno-Halvings maintain the exact halving law below the RAO unit so that $E_k = E_0/2^k > 0$ for all k and the total remains finite:

$$\sum_{k=0}^{\infty} \frac{E_0}{2^k} \;=\; 2E_0\,.$$

This approach increases token precision beyond RAO so that $E_k < 1$ RAO remains representable without rounding to zero. Since Zeno-Halvings preserve the per-index scaling, the synchronized conclusions hold at every k.

k	Exact (RAO)	Floor (RAO)	Zeno (RAO)
0	10^9	10^9	10^9
10	976,562	976,562	976,562
20	954	954	954
29	1.86	1	1.86
30	0.93	0	0.93
31	0.46	0	0.46
32	0.23	0	0.23
Sum	2×10^9	$< 2 \times 10^9$	2×10^9

Table 1: Under a 1-RAO floor, emission becomes 0 for $k \geq 31$ (tail truncation). Zeno-Halvings keep $E_k > 0$ for all k while preserving the same finite geometric total.

8.5 TAO-weight Calibration

In the synchronized variant, the root proportion (31) includes the issuance term $H(\Delta\alpha(0) + \Delta\alpha'(0))/2^k$; if γ is held fixed, this term shrinks with k and $r_{i,k}(H)$ drifts upward at a given horizon. To keep $r_{i,k}(H)$ approximately stable across halvings, the TAO-weight must scale with the same factor as issuance,

$$\gamma_k := \frac{\gamma_0}{2^k}.$$

This keeps $\gamma_k \tau_0$ aligned with the per-block ALPHA minting rate across halvings without reintroducing asymmetries.

9 Conclusion

This paper identifies the exponential gap 2^{k-n} between the global TAO halving index k and the subnet ALPHA halving index n as the common driver of cohort asymmetries across four dimensions of Dynamic TAO: interval compression $C_{k,n}$ in (9), normalized AMM liquidity impact $\mathcal{R}_{k,n}(s)$ in (16), the pace of the root proportion $r_{i,k}(H)$ in (19)–(20), and liquidation premia/discounts via $\mathrm{ADR}_{k,n}$ and $p_{L,i}$ in (24). By deriving closed-form expressions that expose the gap explicitly, the analysis shows that later TAO cohorts (larger k at fixed n) face progressively stronger effects under identical horizons and thresholds, while advancing n at fixed k compresses intervals toward $\frac{1}{2}$ through the emission term alone.

To isolate the source of asymmetry, the synchronized-halving variant sets $n = k$, removing the gap and collapsing 2^{k-n} to unity. Under synchronization, the identities become cohort-invariant on a rate basis: $C_k = 1$ by (29), $\mathcal{R}_k(s) = s/(1+s)$ by (30), $\mathrm{ADR}_k = 1$ by (32), and $p_{L,i} = p_i$, while $r_{i,k}(H)$ depends only on global time via (31). Zeno-Halvings preserve the tail without rounding to zero, so the synchronized conclusions hold at all k, and scaling the TAO-weight as $\gamma_k = \gamma_0/2^k$ keeps $r_{i,k}(H)$ approximately stable across halvings without reintroducing the asymmetry.

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A Additional Derivations

A.1 Deriving the compression ratio $C_{k,n}$ from $T_{k,n}$

Starting from the interval duration in (7),

$$T_{k,n} = \frac{D(0)/2^n}{\mu_{k,n} B} = \frac{D(0)/2^n}{\Big(\frac{\Delta\alpha(0)}{2^k} + \frac{\Delta\alpha'(0)}{2^n}\Big) B}, \tag{33}$$

$$T_{k,n+1} = \frac{D(0)/2^{n+1}}{\mu_{k,n+1} B} = \frac{D(0)/2^{n+1}}{\Big(\frac{\Delta\alpha(0)}{2^k} + \frac{\Delta\alpha'(0)}{2^{n+1}}\Big) B}. \tag{34}$$

Taking the ratio and cancelling $D(0)$ and B,

$$C_{k,n} := \frac{T_{k,n+1}}{T_{k,n}} = \frac{\frac{D(0)}{2^{n+1}} \Big/ \big(\mu_{k,n+1} B\big)}{\frac{D(0)}{2^n} \Big/ \big(\mu_{k,n} B\big)} = \frac{1}{2} \cdot \frac{\mu_{k,n}}{\mu_{k,n+1}} \tag{35}$$

$$= \frac{1}{2} \cdot \frac{\frac{\Delta\alpha(0)}{2^k} + \frac{\Delta\alpha'(0)}{2^n}}{\frac{\Delta\alpha(0)}{2^k} + \frac{\Delta\alpha'(0)}{2^{n+1}}} \qquad \text{(matches (8))}. \tag{36}$$

A.2 Bounds and monotonicity of $C_{k,n}$

Let $a := \dfrac{\Delta\alpha(0)}{2^k} > 0$ and $b_n := \dfrac{\Delta\alpha'(0)}{2^n} > 0$. Then

$$C_{k,n} = \frac{1}{2} \cdot \frac{a + b_n}{a + b_{n+1}} = \frac{1}{2} \cdot \frac{a + b_n}{a + \frac{b_n}{2}} = \frac{1}{2} \cdot \left(1 + \frac{\frac{b_n}{2}}{a + \frac{b_n}{2}}\right) \cdot 2 = \frac{a + b_n}{2a + b_n}.$$

Because $a, b_n > 0$,

$$\frac{1}{2} < \frac{a + b_n}{2a + b_n} \le 1,$$

with equality $= 1$ iff $b_n = 0$ (not attainable for finite n with $\Delta\alpha'(0) > 0$). Thus $C_{k,n} \in (\frac{1}{2}, 1]$. Moreover, since $b_{n+1} = \frac{b_n}{2}$,

$$C_{k,n+1} = \frac{a + b_{n+1}}{2a + b_{n+1}} = \frac{a + \frac{b_n}{2}}{2a + \frac{b_n}{2}} < \frac{a + b_n}{2a + b_n} = C_{k,n},$$

so $n \mapsto C_{k,n}$ is strictly decreasing and $C_{k,n} \to \frac{1}{2}$ as $n \to \infty$. Under the baseline $\Delta\alpha'(0) = \Delta\alpha(0)$, substituting $b_n = a \cdot 2^{k-n}$ gives (9).

A.3 Split-interval derivations (Case A and Case B)

Case A (TAO halving inside the current interval). Let $\phi \in [0, 1]$ be the issuance fraction of the current threshold produced before the TAO halving. The current interval splits into two segments with rates $\mu_{k,n}$ and $\mu_{k+1,n}$. Using (7) segment-wise,

$$T^{\mathrm{A}}_{k,n} = \frac{D(0)}{2^n B}\Big(\frac{\phi}{\mu_{k,n}} + \frac{1-\phi}{\mu_{k+1,n}}\Big), \tag{37}$$

$$T^{\mathrm{A}}_{k,n+1} := T_{k+1,n+1} = \frac{D(0)/2^{n+1}}{\mu_{k+1,n+1} B}. \tag{38}$$

Taking the ratio and cancelling $D(0)$ and B,

$$C^{\mathrm{A}}_{k,n}(\phi) = \frac{T^{\mathrm{A}}_{k,n+1}}{T^{\mathrm{A}}_{k,n}} = \frac{\dfrac{D(0)}{2^{\,n+1}} \cdot \dfrac{1}{\mu_{k+1,n+1}B}}{\dfrac{D(0)}{2^{\,n}B}\Big(\dfrac{\phi}{\mu_{k,n}} + \dfrac{1-\phi}{\mu_{k+1,n}}\Big)} = \frac{1}{2} \cdot \frac{1}{\phi\,\dfrac{\mu_{k+1,n+1}}{\mu_{k,n}} + (1-\phi)\,\dfrac{\mu_{k+1,n+1}}{\mu_{k+1,n}}} \tag{39}$$

which matches (11).

Case B (TAO halving inside the next interval). Here the current interval is single-rate and the next interval splits. With $\psi \in [0,1]$ the issuance fraction of the next threshold produced before the halving,

$$T^{\mathrm{B}}_{k,n} = \frac{D(0)}{2^{\,n}} \cdot \frac{1}{\mu_{k,n}B}, \tag{40}$$

$$T^{\mathrm{B}}_{k,n+1} = \frac{D(0)}{2^{\,n+1}B}\Big(\frac{\psi}{\mu_{k,n+1}} + \frac{1-\psi}{\mu_{k+1,n+1}}\Big). \tag{41}$$

Taking the ratio,

$$C^{\mathrm{B}}_{k,n}(\psi) = \frac{T^{\mathrm{B}}_{k,n+1}}{T^{\mathrm{B}}_{k,n}} = \frac{1}{2}\Big(\psi \cdot \frac{\mu_{k,n}}{\mu_{k,n+1}} + (1-\psi) \cdot \frac{\mu_{k,n}}{\mu_{k+1,n+1}}\Big), \tag{42}$$

which is (13).

A.4 Derivation of the liquidity impact $\mathcal{R}_{k,n}(s)$

By definition of the constant-product swap impact,

$$\mathcal{R} = \frac{\tau^{\mathrm{swap}}}{\tau_i} = \frac{\alpha^{\mathrm{swap}}}{\alpha_i + \alpha^{\mathrm{swap}}}.$$

Over a horizon $H > 0$ with no prior swaps, the sold emissions and the AMM reserve (at fixed (k,n)) satisfy

$$\alpha^{\mathrm{swap}}_{H,n} = s\,H\,\frac{\Delta\alpha'(0)}{2^{\,n}}, \qquad \alpha_{H,k} = H\,\frac{\Delta\alpha(0)}{2^{\,k}}.$$

Substitution and cancellation of H gives

$$\mathcal{R}_{k,n}(s) = \frac{s\,\dfrac{\Delta\alpha'(0)}{2^{\,n}}}{\dfrac{\Delta\alpha(0)}{2^{\,k}} + s\,\dfrac{\Delta\alpha'(0)}{2^{\,n}}} = \frac{s\,\dfrac{\Delta\alpha'(0)}{\Delta\alpha(0)}}{2^{\,n-k} + s\,\dfrac{\Delta\alpha'(0)}{\Delta\alpha(0)}} \qquad \text{(matches (16))}. \tag{43}$$

Under synchronization $n = k$ and the baseline $\Delta\alpha'(0) = \Delta\alpha(0)$, $\mathcal{R}_k(s) = \dfrac{s}{1+s}$ (cf. (30)).

A.5 Root proportion $r_{i,k}(H)$ and comparative statics

Starting from (18) and using

$$\alpha_{H,k} + \alpha^{\mathrm{o}}_{H} = A(0) + H\Big(\frac{\Delta\alpha(0)}{2^k} + \frac{\Delta\alpha'(0)}{2^0}\Big) = A(0) + H\,\mu_{k,0},$$

substitution yields (19):

$$r_{i,k}(H) = \frac{\gamma\,\tau_0}{\gamma\,\tau_0 + A(0) + H\,\mu_{k,0}}, \qquad \mu_{k,0} = \frac{\Delta\alpha(0)}{2^k} + \Delta\alpha'(0). \tag{44}$$

Discrete monotonicity in k at fixed H follows since $\mu_{k+1,0} < \mu_{k,0}$ and therefore the denominator decreases with k, increasing $r_{i,k}(H)$. Under synchronization, replace $\mu_{k,0}$ by μ_k from (28) to obtain (31).

A.6 Liquidation identities and haircut

By definition $p_i := \tau_i/\alpha_i$ and $p_{L,i} := \tau_i/\alpha_i^{\mathrm{o}}$. Over the horizon $H > 0$,

$$\alpha_i = H\,\frac{\Delta\alpha(0)}{2^{\,k}}, \qquad \alpha_i^{\mathrm{o}} = H\,\frac{\Delta\alpha'(0)}{2^{\,n}},$$

so

$$\frac{p_i}{p_{L,i}} = \frac{\tau_i/\alpha_i}{\tau_i/\alpha_i^{\mathrm{o}}} = \frac{\alpha_i^{\mathrm{o}}}{\alpha_i} = \frac{\dfrac{\Delta\alpha'(0)}{2^{\,n}}}{\dfrac{\Delta\alpha(0)}{2^{\,k}}} = \mathrm{ADR}_{k,n} \quad \Rightarrow \quad p_{L,i} = \frac{p_i}{\mathrm{ADR}_{k,n}}, \tag{45}$$

which matches (22) and (23). The haircut in (25) satisfies

$$h = 1 - \frac{p_{L,i}}{p_i} = 1 - \frac{1}{\mathrm{ADR}_{k,n}} \begin{cases} \in (0,1), & k > n, \\ = 0, & k = n, \\ < 0, & k < n, \end{cases}$$

and under synchronization with the baseline $\mathrm{ADR}_k = 1$ (cf. (32)) so $h = 0$.

A.7 Synchronized identities

Setting $n = k$ in (8) gives

$$C_k = \frac{1}{2} \cdot \frac{\dfrac{\Delta\alpha(0)}{2^k} + \dfrac{\Delta\alpha'(0)}{2^k}}{\dfrac{\Delta\alpha(0)}{2^{k+1}} + \dfrac{\Delta\alpha'(0)}{2^{k+1}}} = \frac{1}{2} \cdot \frac{\dfrac{\Delta\alpha(0) + \Delta\alpha'(0)}{2^k}}{\dfrac{\Delta\alpha(0) + \Delta\alpha'(0)}{2^{k+1}}} = 1 \quad \text{(matches (29))}.$$

Similarly, (43) with $n = k$ gives $\mathcal{R}_k(s) = \dfrac{s\,\Delta\alpha'(0)}{\Delta\alpha(0) + s\,\Delta\alpha'(0)}$, which reduces to $\dfrac{s}{1+s}$ under the baseline (cf. (30)). From (22), $\mathrm{ADR}_k = 2^{\,k-k} \cdot \dfrac{\Delta\alpha'(0)}{\Delta\alpha(0)} = \dfrac{\Delta\alpha'(0)}{\Delta\alpha(0)}$, equal to 1 under the baseline (cf. (32)).

B Notation Index

This index summarizes the symbols used throughout the paper for quick reference. Unless explicitly noted as "synchronized," definitions assume the baseline normalization $\Delta\alpha'(0) = \Delta\alpha(0)$ and the asynchronous setting with distinct halving indices k (TAO) and n (ALPHA).

- Global indices and schedules
  - k: global TAO halving index.
  - n: subnet ALPHA halving index.
  - B: blocks per day.
  - $D(0)$: ALPHA issuance threshold for the first halving.
  - E_0, E_k: TAO block reward at indices 0 and k (RAO units).
  - S_τ: current TAO supply.
  - $S_{\tau,\max}$: TAO supply cap (21,000,000 TAO).
- Subnet index
  - i: subnet identifier.
- Prices and reserves (subnet i)
  - α_i: ALPHA in the AMM reserve.
  - α_i^{o}: ALPHA outstanding (held by participants).
  - τ_i: TAO in the AMM reserve.
  - p_i: AMM spot price (TAO per ALPHA).
  - $\tilde{p}_i$: EMA price (TAO per ALPHA) used by TERP.
- Injections and emissions
  - $\Delta\tau_k$: global TAO emission per block at TAO index k.
  - $\Delta\tau_i$: TAO injection share to subnet i.
  - $\Delta\overline{\alpha}_i(k)$: per-block cap on ALPHA injection for subnet i (tied to the TAO schedule).
  - $\Delta\alpha(0)$, $\Delta\alpha'(0)$: pre-halving ALPHA injection and emission scales.
  - $\Delta\alpha_i(k)$: per-block ALPHA injection into the AMM for subnet i at TAO index k.
  - $\Delta\alpha_i'(n)$: per-block ALPHA emission to participants at ALPHA index n.
- Rates and durations
  - $\mu_{k,n}$: per-block ALPHA minting rate at indices (k, n).
  - $T_{k,n}$: ALPHA-halving interval duration at indices (k, n).
- Compression
  - $C_{k,n}$: compression ratio between consecutive ALPHA-halving intervals.
  - $\phi \in [0, 1]$: issuance fraction of the current threshold produced before a TAO halving (Case A).
  - $\psi \in [0, 1]$: issuance fraction of the next threshold produced before a TAO halving (Case B).

- Liquidity impact
  - α^{swap}: ALPHA sold into the AMM during a swap.
  - τ^{swap}: TAO paid out by the AMM during a swap.
  - $s \in [0, 1]$: fraction of newly minted emissions sold over the horizon.
  - $H > 0$: common evaluation horizon (blocks).
  - $\mathcal{R}_{k,n}(s)$: normalized TAO-reserve impact (fraction removed) at indices (k, n).
- Root proportion
  - r_i: root proportion for subnet i (dividend split parameter).
  - $r_{i,k}(H)$: root proportion trajectory over horizon H at TAO index k.
  - $A(0)$: initial total ALPHA minted (outstanding plus AMM reserve).
  - τ_0: TAO staked in the root subnet.
  - $\gamma > 0$: TAO-weight parameter.
  - γ_k: TAO-weight parameter scaled at TAO index k.
- Liquidation
  - $\mathrm{ADR}_{k,n}$: emissions-to-injection ratio at indices (k, n).
  - $p_{L,i}$: liquidation price (TAO per ALPHA using ALPHA outstanding).
  - h: liquidation haircut relative to spot.
- Synchronized halving (notations)
  - $\Delta\alpha_k$, $\Delta\alpha'_k$: ALPHA injection and emission per block at global index k.
  - μ_k: per-block ALPHA minting rate at global index k.
  - C_k: compression ratio between consecutive intervals in the (synchronized).
  - $\mathcal{R}_k(s)$: normalized TAO-reserve impact in the (synchronized).
  - ADR_k: emissions-to-injection ratio in the (synchronized).
  - $r_{i,k}(H)$: root proportion at horizon H with synchronized rate at index k.
  - $S_{i,\max}$: maximum ALPHA supply of subnet i.
- Miscellaneous
  - TERP: TAO Emission Ratio Property (EMA-based TAO allocation across subnets).
  - RAO: smallest TAO unit.

C Halving Trajectory Simulation Code

```
S_tau_max, S_tau_0, A_0 = 21_000_000, 9_860_998, 2_610_000

# Generate halving thresholds (issuance-based)
D = [
    S_tau_max*(1 - 2**-i)
    for i in range(1, 25)
    if S_tau_max - S_tau_max*(1 - 2**-i) > 1
]

# Initialize indices and per-block scales
k_0 = sum(S_tau_0 >= d for d in D)
n_0 = sum(A_0     >= d for d in D)
Delta_tau_k, Delta_alpha_n = 2**-k_0, 2**-n_0

# Simulate block-by-block issuance and halvings
S_tau, S_i, block, k, n = S_tau_0, A_0, 0, k_0, n_0
halvings = []

while max(S_tau, S_i) < S_tau_max and block < 50_000_000:
    # TAO halving: affects Delta_tau_k
    if k < len(D) and S_tau >= D[k]:
        Delta_tau_k, k = Delta_tau_k / 2, k + 1
        halvings.append(
            f"TAO HALVING #{k}\n"
            f"   Block: {block:,}\n"
            f"   Supply: {S_tau:,.0f}\n"
            f"   Delta_tau_k: {Delta_tau_k:.6f}\n"
        )

    # ALPHA halving: affects Delta_alpha_n
    if n < len(D) and S_i >= D[n]:
        Delta_alpha_n, n = Delta_alpha_n / 2, n + 1
        halvings.append(
            f"ALPHA HALVING #{n}\n"
            f"   Block: {block:,}\n"
            f"   Supply: {S_i:,.0f}\n"
            f"   Delta_alpha_n: {Delta_alpha_n:.6f}\n"
        )

    # Advance one block of issuance (AMM injection + participant emission)
    S_tau = min(S_tau + Delta_tau_k, S_tau_max)
    S_i   = min(S_i   + Delta_tau_k + Delta_alpha_n, S_tau_max)
    block += 1

# Output proof-of-trajectory
for h in halvings:
    print(h)
```

Note: This code simulates a subnet for reproducibility of the halving paths underlying Figure 1.

References

[1] Y. Rao, Dr. Nick, and 0xcacti, “Dynamic TAO (BIT001): A market-driven emission allocation mechanism,” Whitepaper, 2025.

[2] Dr. Nick and 0xcacti, “The dTAO Injection Mechanism: Derivation and design goals,” Technical note, 2025.