Numeration Systems

A numeration system is a method for writing and expressing numbers.

The decimal system is a positional numeration system that uses the digits from the set {0,1,2,3,4,5,6,7,8,9} and their relative position to express numbers.

We read the decimal number 2099, for example, as two thousand ninety-nine. This number uses valid decimal digits in four different positions.

The digit 2 is in the thousands place. The digit 0 is in the hundreds place. There is a 9 in the tens place and a 9 in the ones place.

2	0	9	9
1000s (or 10^3)	100s (or 10^2)	10s (or 10^1)	1s (or 10^0)

So, the expanded version of this number is $2 \times 1000 + 0 \times 100 + 9 \times 10 + 9 \times 1$, or equivalently, $2 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0$.

Decimal is a "base 10" positional system where each position in a number is a power of 10.

We can use this same basic structure to express numbers in any "base b" positional system where b is an integer greater than 1.

Historically, civilizations have developed a variety numeration systems, both positional and non-positional (e.g. Roman numerals).

In this course will concentrate solely on positional numeration systems – primarily binary, octal, and hexadecimal (and decimal, of course). To avoid confusion in this document, we will write all non-decimal numbers with the base as a subscript.

Binary

Binary is a base 2 positional numeration system where the two valid digits are 0 and 1. Binary applies naturally to situations where the base unit involves two possible value – e.g. logic (true/false), electronics (on/off), etc.

Below we show how to write the first twenty natural numbers in binary and decimal.

decimal	binary
0	0_2
1	1_2
2	10_2
3	11_2
4	100_2
5	101_2
6	110_2
7	111_2

8	1000_2
9	1001_2
10	1010_2
11	1011_2
12	1100_2
13	1101_2
14	1110_2
15	1111_2
16	10000_2
17	10001_2
18	10010_2
19	10011_2

As with decimal, the unit for each position in the expression of a binary number is a power of the base. So the last number written above, 10011_2, contains digits in the 1s, 2s, 4s, 8s, and 16s place as shown below.

1	0	0	1	1
16s (or 2^4)	8s (or 2^3)	4s (or 2^2)	2s (or 2^1)	1s (or 2^0)

Thus, the expanded version of this number is $1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$, which is 16 + 2 + 1 = 19.

Work the following examples and then go to the next page to check your answers.

Example 1

Write the decimal equivalent to the binary number 11100_2.

Example 2

Write the decimal equivalent to the binary number 110101_2.

Example 1

Write the decimal equivalent to the binary number 11100_2.

1	1	1	0	0
16s (or 2^4)	8s (or 2^3)	4s (or 2^2)	2s (or 2^1)	1s (or 2^0)

The expanded version of this number is $1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$, which is 16 + 8 + 4 = 28.

Example 2

Write the decimal equivalent to the binary number 110101_2.

1	1	0	1	0	1
32s (or 2^5)	16s (or 2^4)	8s (or 2^3)	4s (or 2^2)	2s (or 2^1)	1s (or 2^0)

The expanded version is $1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$, which is 32 + 16 + 4 + 1 = 53.

Octal

Octal is a base 8 positional numeration system where the valid digits are 0, 1, 2, 3, 4, 5, 6, 7.

Below we add the first twenty natural numbers written in octal to our binary and decimal list.

decimal	binary	octal
0	0_2	0_8
1	1_2	1_8
2	10_2	2_8
3	11_2	3_8
4	100_2	4_8
5	101_2	5_8
6	110_2	6_8
7	111_2	7_8
8	1000_2	10_8
9	1001_2	11_8
10	1010_2	12_8
11	1011_2	13_8
12	1100_2	14_8
13	1101_2	15_8
14	1110_2	16_8
15	1111_2	17_8

16	10000_2	20_8
17	10001_2	21_8
18	10010_2	22_8
19	10011_2	23_8

The unit for each position in the expression of a octal number is a power of 8. So the last number written above, 23_8, contains digits in the 1s and 8s place as shown below.

2	3
8s (or 8^1)	1s (or 8^0)

Thus, the expanded version of this number is $2 \times 8^1 + 3 \times 8^0$, which is $2 \times 8 + 3 \times 1 = 19$.

Work the following examples and then check your answers on the next page.

Example 3

Write the decimal equivalent to the octal number 150_8.

Example 4

Write the decimal equivalent to the octal number 2010_8.

Example 3

Write the decimal equivalent to the octal number 150_8.

1	5	0
64s (or 8^2)	8s (or 8^1)	1s (or 8^0)

$$1 \times 8^2 + 5 \times 8^1 + 0 \times 8^0 = 64 + 40 = 104.$$

Example 4

Write the decimal equivalent to the octal number 2010_8.

2	0	1	0
512s (or 8^3)	64s (or 8^2)	8s (or 8^1)	1s (or 8^0)

$$2 \times 8^3 + 0 \times 8^2 + 1 \times 8^1 + 0 \times 8^0 = 1024 + 8 = 1032.$$

Hexadecimal

Hexadecimal is a base 16 positional numeration system where the valid digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.

Below we add the first twenty natural numbers written in octal to our binary and decimal list.

decimal	binary	octal	hexadecimal
0	0_2	0_8	0_{16}
1	1_2	1_8	1_{16}
2	10_2	2_8	2_{16}
3	11_2	3_8	3_{16}
4	100_2	4_8	4_{16}
5	101_2	5_8	5_{16}
6	110_2	6_8	6_{16}
7	111_2	7_8	7_{16}
8	1000_2	10_8	8_{16}
9	1001_2	11_8	9_{16}
10	1010_2	12_8	A_{16}
11	1011_2	13_8	B_{16}
12	1100_2	14_8	C_{16}
13	1101_2	15_8	D_{16}
14	1110_2	16_8	E_{16}
15	1111_2	17_8	F_{16}

16	10000_2	20_8	10_{16}
17	10001_2	21_8	11_{16}
18	10010_2	22_8	12_{16}
19	10011_2	23_8	13_{16}

The unit for each position in the expression of a octal number is a power of 16. So the last number written above, 13_8, contains digits in the 1s and 16s place as shown below.

1	3
16s (or 16^1)	1s (or 16^0)

Thus, the expanded version of this number is $1 \times 16^1 + 3 \times 16^0$, which is $1 \times 16 + 3 \times 1 = 19$.

Work the following examples and then check your answers on the next page.

Example 5

Write the decimal equivalent to the hexadecimal number $3D_{16}$.

Example 6

Write the decimal equivalent to the hexadecimal number $A07_{16}$.

Example 5

Write the decimal equivalent to the hexadecimal number $3D_{16}$.

3	D
16s (or 16^1)	1s (or 16^0)

$$3 \times 16^1 + 13 \times 16^0 = 48 + 13 = 61.$$

Example 6

Write the decimal equivalent to the binary number $A07_{16}$.

A	0	7
256s (or 16^2)	16s (or 16^1)	1s (or 16^0)

$$10 \times 16^2 + 0 \times 16^1 + 7 \times 16^0 = 2560 + 7 = 2567.$$