SIGHTS ALONG THE WAY

$$\boxed{\begin{aligned} \psi(x) &= \frac{1}{\sqrt{2\pi\hbar}} \int \psi_p(p)\, e^{i(p/\hbar)x}\, dp \quad (1) \\ \psi_p(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x)\, e^{-i(p/\hbar)x}\, dx \end{aligned}}$$

← HI OLD FRIENDS

$$\Delta x^2 = \sigma_x^2 = \langle \psi | (\hat{x} - \langle \hat{x} \rangle)^2 | \psi \rangle$$

$$\boxed{\begin{aligned} \sigma_x^2 &= \int_x \psi^* (x - x_0)^2 \psi\, dx \quad (2) \\ \sigma_p^2 &= \int_p \psi^*(p)\, (p - p_0)^2\, \psi(p)\, dp \end{aligned}}$$

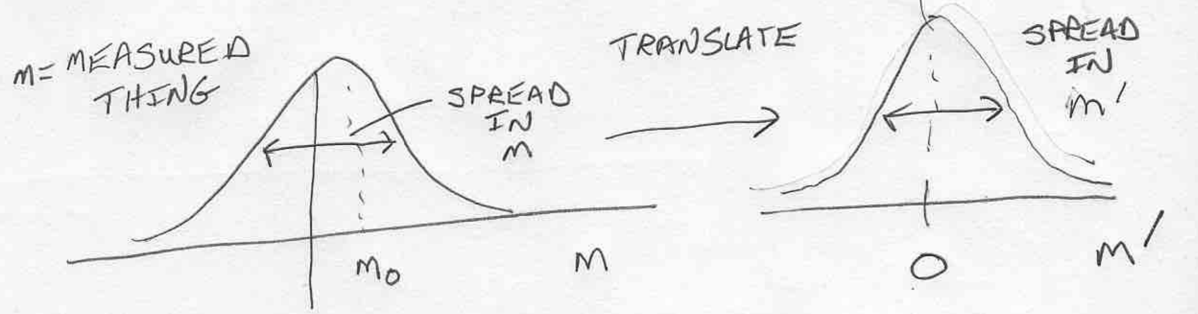
$\sigma_x^2 \sigma_p^2$ IS WHAT EXACTLY?

FIRST STOP = SAVE WORK AS POSSIBLE

Q: ASK DOES IT REALLY MATTER WHAT x_0, p_0 ARE?

A1: WELL, I COULD SEE MOVING MY RULER SO THAT I MEASURE $x_0 = 0$ MAYBE EVEN SO THAT THIS MOMENTUM THING'S AVERAGE $p_0 = 0$

REALLY WORRIED ABOUT THE DEVIATION



$$\sigma_x^2 = \int_x \psi^* (x - \cancel{x_0})^2 \psi \, dx$$

$$\boxed{\sigma_x^2 \sigma_p^2 = \int_x \psi^* x^2 \psi \, dx \int_p \psi^*(p)\, p^2 \psi(p)\, dp \quad (A)}$$

I'VE NOTICED WHEN I MISS THE SIMPLYING
⇒ APPLY BRUTE FORCE OR LEARN SOMETHING NEW
OR PROPERTIES OF F.T.

JUST MATH

$S = \text{DATA} = \{s_1, s_2, \ldots s_n\}$

$\bar{S} = \frac{1}{n} \sum s_i$

variance = σ^2

$\sigma_S = \sqrt{\sum (s_i - \bar{s})^2 /n}$

TRANSLATE THE DATA BY a

$T = \{t_1, t_2 \ldots t_n\} = \{s_1 + a, s_2 + a, \ldots\}$

$\bar{T} = \frac{1}{n} \sum (s_i + a) = \bar{s} + a$

$\sigma_T = \sqrt{\sum (t_i - \bar{T})^2 /n}$

$\sigma_T = \sqrt{\sum [s_i + a - (\bar{s} + a)]^2 /n}$

$\sigma_T = \sqrt{\sum (s_i - \bar{s})^2 /n} = \sigma_S$

$\sigma_T = \sigma_S$

$$\psi_p(p) = \mathcal{F}(\psi(x)) = \frac{1}{\sqrt{2\pi\hbar}} \int_x \psi(x) e^{-\frac{ip}{\hbar}x} dx$$

HOW ABOUT A TRANSLATION

$$\psi_p(p - p_0) = \frac{1}{\sqrt{2\pi\hbar}} \int_x \psi(x) e^{-\frac{i(p-p_0)}{\hbar}x} dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_x \underbrace{\psi(x) e^{+\frac{ip_0 x}{\hbar}}} e^{-\frac{ip}{\hbar}x} dx$$

$$\psi_p(p - p_0) = \mathcal{F}\left(\psi(x) e^{+\frac{ip_0}{\hbar}x}\right)$$

$$\psi(x \to x - x_0) \longrightarrow e^{-\frac{ip}{\hbar}x_0} \psi(x)$$

$$\psi(p \to p - p_0) \longrightarrow e^{+\frac{ip_0}{\hbar}} \psi(p)$$

TRANSLATION → PHASE SHIFT

$$\underbrace{\int \psi^* (x - x_0)^2 \psi \, dx} \quad \underbrace{\int \psi_p^*(p) (p - p_0)^2 \psi_p(p) \, dp}$$

GET MORE TRICKY DEPARTING FROM A ...

$$\int \psi^* \psi \, dx = 1 \quad \text{NORMALIZATION}$$

ONE OF SIGHTS!

INTEGRATION BY PARTS, ALWAYS!

$u = x$
$v = |\psi|^2$

$$d(uv) = u\,dv + v\,du$$

$$d(x\psi^2) =$$

$$= x \frac{d\,|\psi(x)|^2}{dx} + \psi^2$$

$$\int_{-\infty}^{\infty} d(x\psi^2)\,dx = \int x \frac{d\,|\psi(x)|^2}{dx}\,dx + \int \psi^2\,dx$$

$$x\psi^2 \Big|_{-\infty}^{\infty} = \int x \frac{d}{dx} |\psi|^2\,dx + 1$$

$x \to \infty$, $\psi^2 \to 0$ (FASTER)

DON'T MISS IT

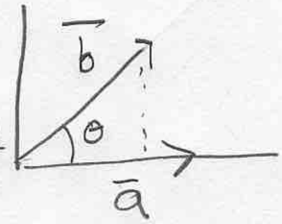
$$0 = \int x \frac{d}{dx} |\psi|^2\,dx + 1$$

$$\boxed{\left| \int x \frac{d}{dx} |\psi|^2\,dx \right| = 1}$$

B

$$\left|\frac{d|\psi|^2}{dx}\right| = \left|\psi^{*\prime}\psi + \psi^{*}\psi'\right|$$

$$\bar{a}\cdot\bar{b} = |a||b|\cos\theta$$



$$\left|\bar{a}\cdot\bar{b}\right| \le \left||a||b|\cos\theta\right|$$

MAX = 1

$$\left|\bar{a}\cdot\bar{b}\right| \le |a||b|$$

"CAUCHY-SCHWARZ"
THE BUDGET VERSION

A POPULAR VISTA ON THE ROAD TO $\sigma_x\sigma_p$

$$\bar{a} \sim \psi^{*\prime} \qquad \bar{b} \sim \psi$$

$$|\bar{a}| = |\psi^{*\prime}| \qquad |b| = |\psi|$$

$$|\psi^{*\prime}| = |\psi'| \qquad |\psi| = |\psi^{*}|$$

$$\boxed{\left|\frac{d|\psi|^2}{dx}\right| = \left|\psi^{*\prime}\psi + \psi^{*}\psi'\right| \le 2|\psi'||\psi|} \quad (C)$$

$$1 = \left|\int x\frac{d}{dx}|\psi|^2\,dx\right| \le 2\int |x||\psi'||\psi|\,dx$$

THE RETURN OF (B)

SUBING IN (C)

$$2 \int \underbrace{|x||\psi|}_{a} \underbrace{|\psi'|}_{b} dx \geq 1$$

APPLY CAUCHY SCHWARZ AGAIN (NORMS)

$$\boxed{\left(\left(\int |x|^2 |\psi|^2 dx\right) \int |\psi'|^2 dx\right)^{1/2} \geq \frac{1}{2}} \quad \text{D}$$

WELL THAT DOESN'T EVEN LOOK CLOSE TO WHAT WE STARTED WITH... WHICH WAS

ARE WE LOST?

$$\sigma_x^2 \sigma_p^2 = \int \psi^* x^2 \psi \, dx \int \psi_p^* p^2 \psi_p \, dp \quad \text{A}$$

BUT F.T. PROPERTIES TO THE RESCUE

MORE TRICKERY

$$\psi(x) = \mathcal{F}^{-1}(\psi(p))$$

$$\psi'(x) = \mathcal{F}^{-1}\left(\frac{ip}{\hbar} \psi_p\right)$$

DERIVATIVE → MULTIPLICATION

↑ IN D ↑ IN A

"ENERGY" CONSERVATION = PLANCHERAL / PARSEVALS RELATION

"UMPF"

$$\int |\psi(x)|^2 dx = \int |\psi(p)|^2 dp$$

6

$\psi(x) \to \psi'(x)$ $\quad \psi(x) = \mathcal{F}^{-1}(\psi(p))$

$$\psi'(x) = \mathcal{F}^{-1}\left(\frac{ip}{\hbar}\psi(p)\right)$$

$$\int |\psi'(x)|^2 dx = \text{EQUAL UMPH} = \left|\int \frac{ip}{\hbar}\psi(p)\right|^2 dp$$

$$\boxed{\int |\psi'(x)|^2 dx = \frac{1}{\hbar^2}\int p^2 \psi_p^2 \, dp}$$

E

D FS

$$\sqrt{\int_x |x|^2 |\psi|^2 dx \int |\psi'|^2 dx} \geq \frac{1}{2}$$

$$\frac{1}{\hbar}\sqrt{\int |x|^2 |\psi|^2 dx \int p^2 |\psi_p|^2 dp} \geq \frac{1}{2}$$

$$\sigma_x \quad \sigma_p \geq \frac{\hbar}{2}$$

Q. $\cancel{E}$. D.