THE SPACE OF STATES AKA THE HILBERT SPACE

$|\psi\rangle$ LIVES THERE IN THE "SPACE OF STATES"

KET

WHATEVER THIS SPACE OF STATES IS

THERE'S THE IDEA OF CALCULATING

AN "OVERLAP" BETWEEN STATES

$(|\phi\rangle, |\psi\rangle)$

(IDEA OF)

AN INNER PRODUCT

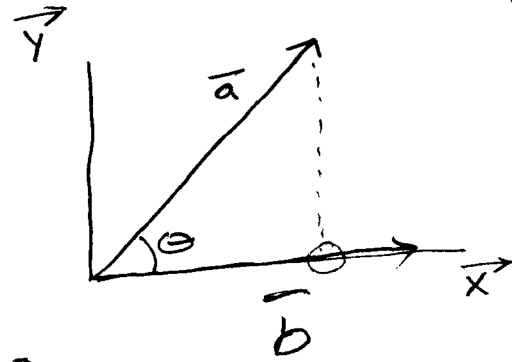
FAMILIAR INNER PRODUCT ; IN A VECTOR SPACE

VECTORS $\bar{a}, \bar{b}$

INNER PRODUCT $= \bar{a} \cdot \bar{b}$ "THE DOT PRODUCT"

WHAT IS IT?

WHAT IS THIS INNER PRODUCT



$\bar{a} \cdot \bar{b}$ = A REAL NUMBER

SCALAR

≥ 0

$$\bar{a} \cdot \bar{b} = |\bar{a}|\,|\bar{b}| \cos\theta$$

= LENGTH OF $\bar{a}$ * LENGTH OF b * PORTION PROJECTED

$(|\phi\rangle, |\psi\rangle)$ CORRESPONDS TO

$$\langle \phi | \psi \rangle$$

"THE AMOUNT TO WHICH $|\phi\rangle$ AND $|\psi\rangle$ OVERLAP"

$$(|\phi\rangle, |\psi\rangle) = \langle \phi | \psi \rangle$$

$|\psi\rangle \in \mathcal{E}$ CORRESPONDS TO $\psi(x) \in \mathcal{F}$

HILBERT SPACE

SPACE OF FUNCTIONS

$\psi(x)$ IN THE CONTINUOUS X BASIS

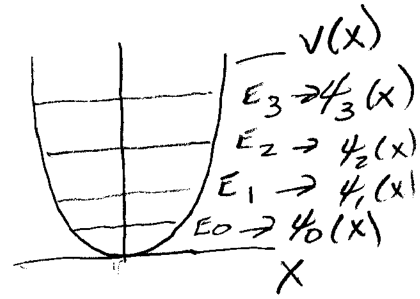
$$\langle \phi | \psi \rangle \triangleq \int_{-\infty}^{\infty} \phi^*(x)\, \psi(x)\, dx$$

$$\langle\phi|\psi\rangle = \int_{-\infty}^{\infty} \phi^*(x)\,\psi(x)\,dx$$

$$= \int_{-\infty}^{\infty} \phi^*(p)\,\psi(p)\,dp$$

SUPPOSE

$|\psi\rangle$ LIVED IN A DISCRETE SET OF STATES



$$|\psi\rangle = \sum_{\substack{\text{ALL}\\ n}} a_n\,|E_n\rangle$$

$$|\phi\rangle = \sum_{\substack{\text{ALL}\\ m}} b_m\,|E_m\rangle$$

$$\langle\phi|\psi\rangle = \sum_m b_m^* \langle E_m| \sum_n a_n |E_n\rangle$$

BUT THE ENERGY STATES ARE EIGENSTATES AND ORTHOGONAL SO

$$\langle E_m|E_n\rangle = 0 \quad \text{IF } m \neq n \quad = \delta_{mn}$$
$$= 1 \quad \text{IF } m = n$$

SO

$$\langle\phi|\psi\rangle = \sum_m b_m^* a_m = \text{A COMPLEX NUMBER}$$

WITHOUT HAVING TO CALCULATE

$$\langle\phi|\psi\rangle \longrightarrow \int \left(\left(\frac{\pi\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega^2 x^2}{2\hbar}}\right)\left(x\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{x}{\sqrt{2}}\, e^{-\frac{m\omega}{2\hbar}x^2}\right)dx$$

TERMINOLOGY

$|\psi\rangle$

↑

KET

$\langle\phi|$

↑

BRA

$$\underbrace{\langle \phi | \psi \rangle}_{\text{BRA(C)KET}}$$

WHAT IS THIS

$\langle\phi|$?

DOES IT LIVE IN THE HILBERT SPACE ?

NO, IT HANGS OUT IN SOMETHING CALLED THE DUAL SPACE $\mathcal{E}^*$ ("A SLIGHTLY LARGER NEIGHBORHOOD")

BUT WHAT IS $\langle\phi|$?

$\langle\phi|$ IS A $\boxed{\boxed{\text{LINEAR FUNCTIONAL}}}$ ON $|\psi\rangle$

SECTION 1.3 IN Q.F.T. FOR GIFTED AMATEUR

A FUNCTION $f(x) = x^2$ TAKES IN A NUMBER (x) AND RETURNS A NUMBER

A FUNCTIONAL TAKES IN A FUNCTION

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$ FOR EXAMPLE

AND RETURNS A NUMBER (MAY BE COMPLEX)

2nd POSTULATE WANT TO MEASURE $\mathcal{A}$ (OBSERVABLE QUANTITY)

IS DESCRIBED BY $A|\psi\rangle$ ← KET IN $\mathcal{E}$

OPERATOR = OBSERVABLE

BUT TO BE AN OBSERVABLE $\boxed{A \text{ MUST BE HERMITIAN}}$

REVIEW

CONJUGATE TRANSPOSE = $\dagger$ (DAGGER)

$$7^\dagger = 7$$

$$(a+ib) = a - ib$$

$$\hat{M} = \begin{pmatrix} \frac{\hbar\omega}{2} & i \\ -i & -\frac{\hbar\omega}{2} \end{pmatrix}$$

$$\hat{M}^\dagger = \left(\ \swarrow\nearrow \ \right)^* = \hat{M}^{T*}$$

$$= \begin{pmatrix} \frac{\hbar\omega}{2} & -i \\ i & -\frac{\hbar\omega}{2} \end{pmatrix}^* = \begin{pmatrix} \frac{\hbar\omega}{2} & i \\ -i & -\frac{\hbar\omega}{2} \end{pmatrix}$$

$$(|\psi\rangle)^\dagger \rightarrow \langle\psi|$$

$\dagger$ = CONJUGATE TRANSPOSE = ADJOINT

PROPERTIES

$$\left(\hat{M}^{+}\right)^{+} = \hat{M}$$

$$\left(\hat{M}\hat{N}\right)^{+} = \hat{N}^{+}\hat{M}^{+}$$

OPERATOR $\hat{M}$ IS SAID TO BE HERMITIAN

$$\boxed{\hat{M}^{+} = \hat{M}}$$

THE OPERATOR (MATRIX) = IT'S ADJOINT

IN PARTICULAR WE'LL LOOK FOR

$$\hat{H}^{+} = \hat{H}$$

$\hat{O}|\psi\rangle$ CHANGES STATE

COULD EQUIVALENTLY WRITE

$$\underbrace{\hat{O}|\psi\rangle}_{\text{OLD STATE}} = \underbrace{|\hat{O}\psi\rangle}_{\text{NEW STATE}}$$

$$\left(|\hat{O}\psi\rangle\right)^{+} = \boxed{\langle\hat{O}\psi| \triangleq \langle\psi|\hat{O}^{+}}$$

TO MOVE THE $\hat{O}$, MUST TAKE THE ADJOINT

$$\langle \hat{O}\psi | \hat{O}\psi \rangle = \text{A NUMBER} > 0$$

$$\langle \hat{O}\psi | \hat{O} | \psi \rangle$$

$$\langle \psi | \underbrace{\hat{O}^{+}\hat{O}}_{\hat{H}} | \psi \rangle$$

SO IF WE CAN WRITE

$$\boxed{\hat{H} = \hat{O}^{+}\hat{O}} = \text{A HERMITIAN OPERATOR}$$

BECAUSE

$$(\hat{M}\hat{N})^{+} = \hat{N}^{+}\hat{M}^{+}$$

$$\hat{M} = \hat{O}^{+} \quad \hat{N} = \hat{O}$$

$$\underbrace{(\hat{O}^{+}\hat{O})^{+}}_{\hat{H}^{+}} = \hat{O}^{+}(\hat{O}^{+})^{+} = \underbrace{\boxed{\hat{O}^{+}\hat{O}}}_{\hat{H}}$$

$$\hat{H}^{+} \stackrel{\Delta}{=} \hat{H}$$