The Quantum Harmonic Oscillator

Physics Group
April 2021

Agenda

Classical Simple Harmonic Oscillator Quick Review

Everywhere there's a SHO

Quantum SHO

- Method 1: … Math …
- Method 2: Ladder Operators

1st Formulation - The Differential Equation

- We already know that $a(t) = -\frac{k}{m}x(t)$.
- But acceleration is just the 2nd derivative of position (wrt time), so:

$$x''(t) = -\frac{k}{m}x(t).$$

- This is a *2nd degree* (two derivatives) *ordinary* (no partial derivatives) differential equation. Abbreviated "ODE."
- Solving the ODE

  Either sines or cosines could be solutions: $\sin\left(\sqrt{\frac{k}{m}}\,t\right), \cos\left(\sqrt{\frac{k}{m}}\,t\right)$

  But since we define $t = 0$ to be the time when we initially release the block (which we have pulled out to $x = x_0$), We must have: $x(0) = x_0$. So the solution we need is:

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

  To get the velocity we simply take the first derivative of the position:

$$v(t) = -\sqrt{\frac{k}{m}}\,x_0 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

(Aside) Angular Velocity

Angular velocity, usually denoted by ω (omega), is the velocity at which an angle changes. For example, picture the position of the oscillator as a vector on a circle of radius x_0. At time zero, when the oscillator is at position x_0 the vector has an angle of zero (it's pointing straight to the right). As the oscillator moves through one cycle, the vector rotates counter clockwise through the circle, the angle returning to zero as the oscillator returns to $x = x_0$.

The position of the oscillator is $x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right)$. The radius of the circle comes from the x_0 and the angle is t multiplied by $\sqrt{\frac{k}{m}}$. So, for each second that elapses, the angle moves by $\sqrt{\frac{k}{m}}$ radians. This means that $\omega = \sqrt{\frac{k}{m}}$.

Clearly the units of ω would be radians per second, but by a "definitional trick" the SI standard calls it dimensionless. iopscience.iop.org/article/10.1088/0026-1394/52/1/40

In any event, if you don't want to have $\sqrt{\frac{k}{m}}$ all over the place, then you can say, for example $x = x_0 \cos(\omega t)$, and (as long as we're using radians) our oscillator will go through one complete cycle in $\frac{2\pi}{\omega}$ seconds.

2nd Formulation - The Hamiltonian

- Momentum: $p = mv$.
- Momentum in kinetic energy: $K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$.
- The Hamiltonian: $H = \text{Total energy} = K + U = \frac{p^2}{2m} + U$.
- Hamiltonian for the oscillator: $H = \frac{p^2}{2m} + \frac{1}{2}kx^2$.
- Position and Momentum, from the Hamiltonian

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x} = -kx$$

In other words: $\frac{dx}{dt} = v, \quad \frac{dv}{dt} = a$ ☹

- Why do we care about the Hamiltonian formulation?

It may seem like this didn't accomplish anything. But these equations describe the oscillator in *phase space*. More important, the Hamiltonian will provide the starting point for the calculations in the *quantum* harmonic oscillator (and the Hamiltonian is almost always used in basic quantum mechanical systems).

Motivations: Everywhere there’s a S.H.O.

Every particle and every wave in the Universe is simply an excitation of a quantum field that is defined over all space and time.

That remarkable assertion is at the heart of **quantum field theory**.

Part I

The Universe as a set of harmonic oscillators

Quantum Field Theory for the Gifted Amateur

Tom Lancaster
Department of Physics, University of Durham

Stephen J. Blundell
Department of Physics, University of Oxford

One Experimental Example

Vol 464 | 1 April 2010 | doi:10.1038/nature08967 nature

ARTICLES

Quantum ground state and single-phonon control of a mechanical resonator

A. D. O'Connell[1], M. Hofheinz[1], M. Ansmann[1], Radoslaw C. Bialczak[1], M. Lenander[1], Erik Lucero[1], M. Neeley[1], D. Sank[1], H. Wang[1], M. Weides[1], J. Wenner[1], John M. Martinis[1] & A. N. Cleland[1]

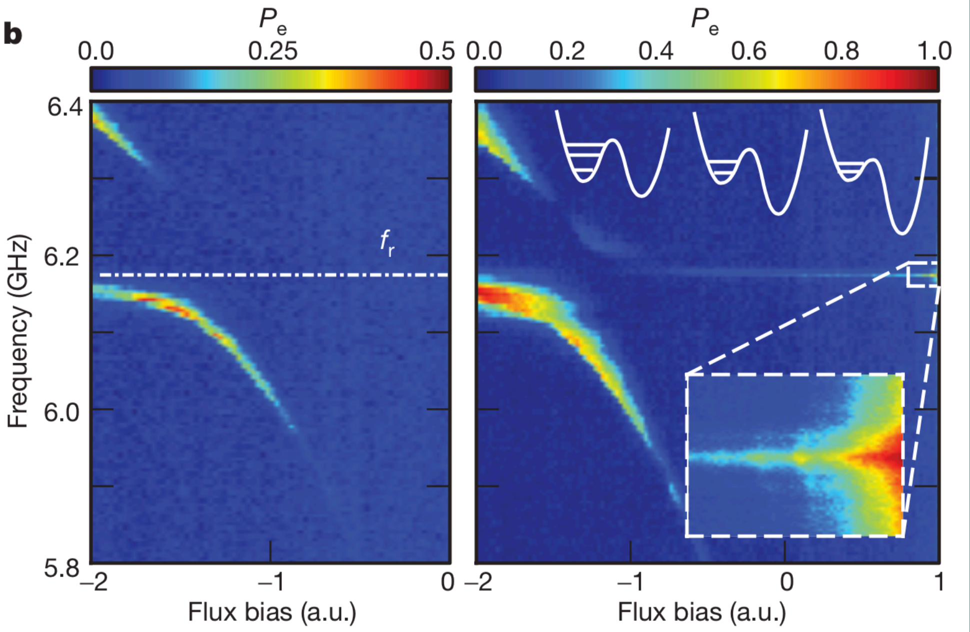
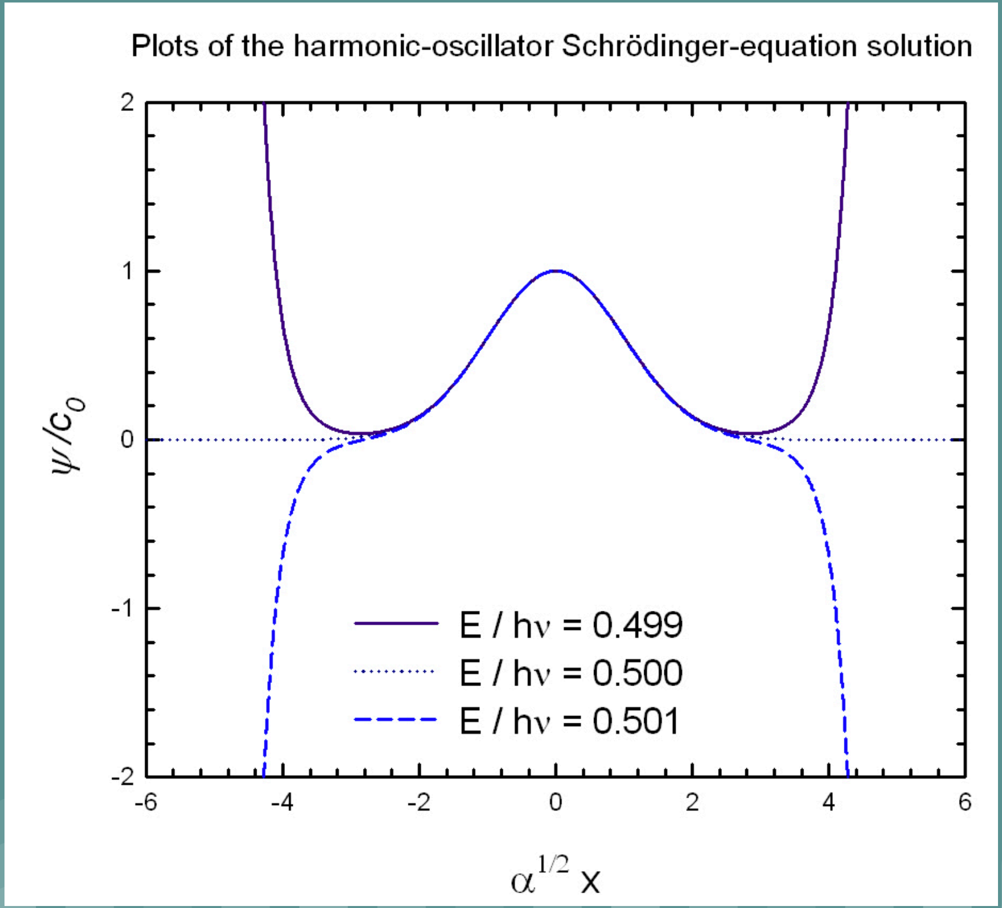


Figure 3 | Resonator spectroscopy. **a**, Pulse sequence applied to the qubit (blue) and the mechanical resonator (red). The qubit is tuned to within Δ of the resonator frequency and a 1-μs microwave tone is applied to the resonator through C_x. The resulting qubit P_e is then evaluated. **b**, Left: spectroscopic P_e as a function of qubit flux bias and applied microwave frequency. Right: same as left-hand panel but with higher microwave power, showing qubit state ejection due to interaction with a highly excited mechanical resonator. The structure of the qubit well levels is indicated schematically as a function of flux bias, showing marginal excited-state confinement at higher positive flux bias. Inset, detail at the highest flux bias, for which the left-hand well is very shallow, highlighting qubit ejection at the resonant frequency of the mechanical resonator.

Convergence or Not !



The quantization of energy in the harmonic-oscillator potential: Power series solution

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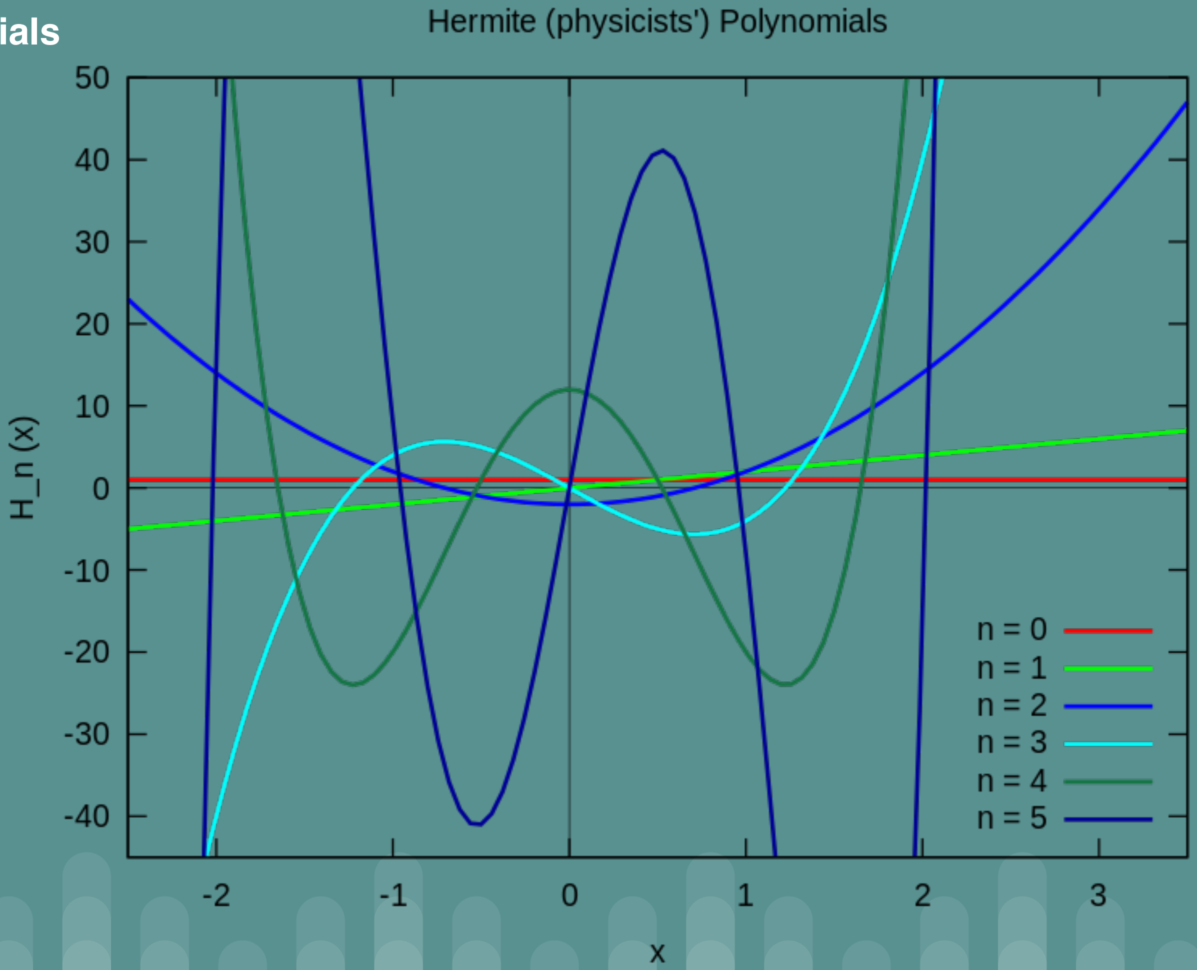
Hermite Polynomials



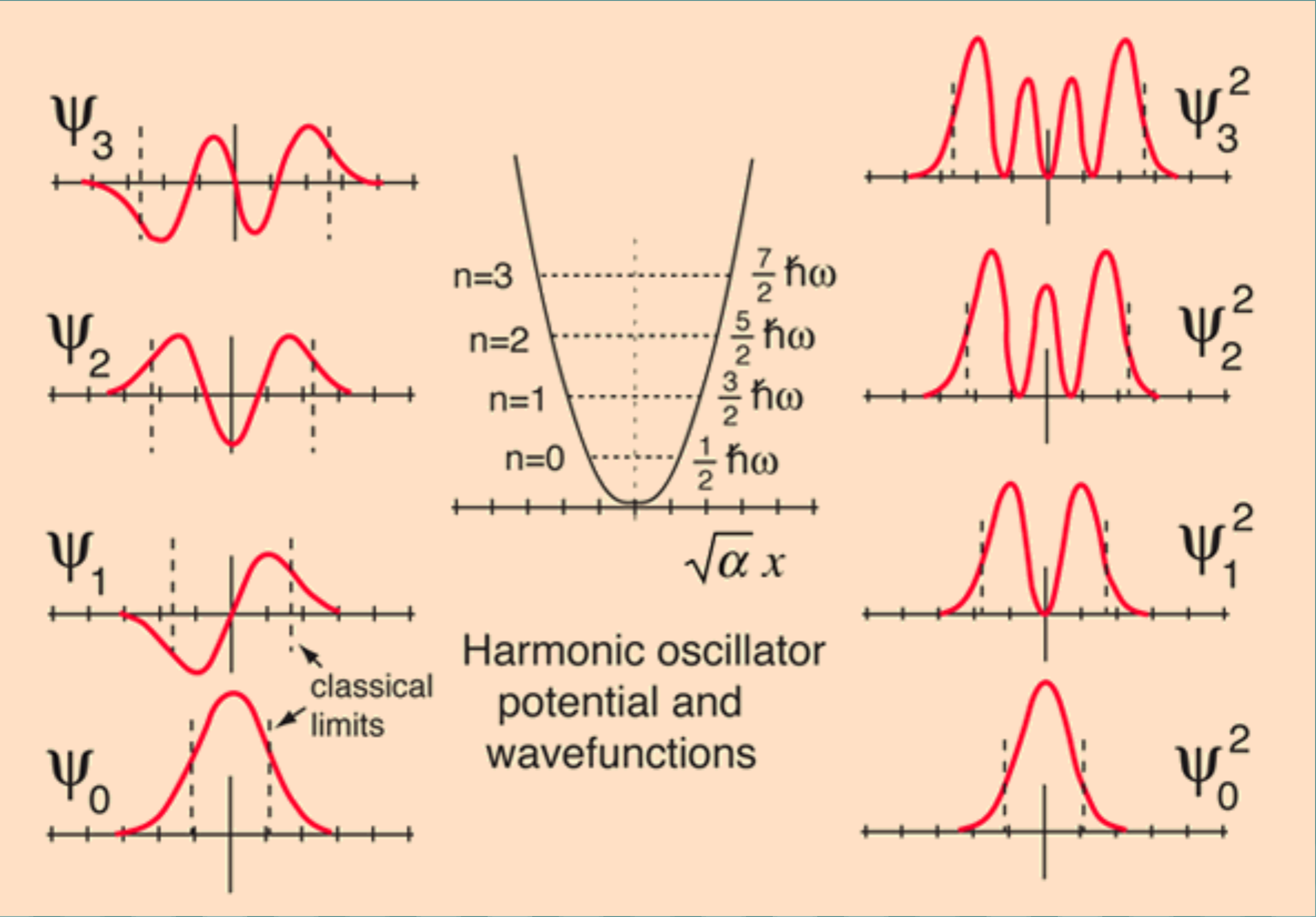
https://en.wikipedia.org/wiki/Hermite_polynomials

Hermite Polynomials

$$
\begin{aligned}
H_0(x) &= 1, \\
H_1(x) &= 2x, \\
H_2(x) &= 4x^2 - 2, \\
H_3(x) &= 8x^3 - 12x, \\
H_4(x) &= 16x^4 - 48x^2 + 12, \\
H_5(x) &= 32x^5 - 160x^3 + 120x, \\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120, \\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x, \\
H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680, \\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x, \\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240.
\end{aligned}
$$

https://en.wikipedia.org/wiki/Hermite_polynomials

Wavefunctions



http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/hosc.html