THIS BRINGS US TO THE S.H.O.

$$\boxed{\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2}$$

WANT TO WRITE AS A PRODUCT OF 2 OPERATORS

IF WE HAVE SOMETHING OF THE FORM

$$a^2 + b^2 = (a - ib)(a + ib)$$
$$= a^2 \underbrace{- iba + iab}_{\text{SCALARS} \to 0} + b^2$$

SO TRY IT

$$\boxed{\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2 = \frac{m\omega^2}{2}\left(\frac{P^2}{m^2\omega^2} + X^2\right)} \qquad (1)$$

LET'S CALCULATE THE PRODUCT

$$\frac{m\omega^2}{2}\left(X - \frac{iP}{m\omega}\right)\left(X + \frac{iP}{m\omega}\right) = \frac{m\omega^2}{2}\left(X^2 + \frac{i}{m\omega}(XP - PX) + \frac{P^2}{m^2\omega^2}\right)$$

$$= \underbrace{\frac{m\omega^2}{2}\left(\frac{P^2}{m^2\omega^2} + X^2\right)}_{\hat{H}} + \underbrace{\frac{m\omega^2}{2m\omega} i [X, P]}_{\text{AN EXTRA TERM}}$$

$$= \hat{H} + i\omega\left(\frac{i\hbar}{2}\right)$$

$$= \hat{H} - \frac{\hbar\omega}{2}$$

$$\hat{H} = \frac{m\omega^2}{2}\left(X - \frac{iP}{m\omega}\right)\left(X + \frac{iP}{m\omega}\right) + \frac{\hbar\omega}{2}$$

WANTED

$$\hat{H} = \hat{O}^{\dagger} \hat{O}$$

HAVE

$$\hat{H} = \frac{m\omega^2}{2}\left(X - \frac{iP}{m\omega}\right)\left(X + \frac{iP}{m\omega}\right) + \frac{\hbar\omega}{2}$$

X IS HERMITIAN

P IS HERMITIAN ... NOT AS OBVIOUS

TO SHOW COMPARE $\langle \phi | P | \psi \rangle^{\dagger}$ TO $\langle \phi | P | \psi \rangle$

HINT INTEGRATE BY PARTS **PROBLEM #1**

IF WE SAY

$$\hat{O} = X + \frac{iP}{m\omega}$$

$$\hat{O}^{\dagger} = X^{\dagger} + \left(\frac{iP}{m\omega}\right)^{\dagger} = X - \frac{iP}{m\omega}$$

$$\hat{H} = \frac{m\omega^2}{2}\left(O^{\dagger} O\right) + \frac{\hbar\omega}{2}$$

THIS EXTRA CONSTANT IS OK

$$\langle \psi | \frac{\hbar\omega}{2} | \psi \rangle = \frac{\hbar\omega}{2}$$

RECOGNIZE AS THE GROUND STATE ENERGY

AS LAST TIME WANT TO MOVE TO A DIMENSIONLESS H (ENERGY)

TO DO THIS LET'S CALCULATE A COMMUTATOR

COMMUTATORS ARE USEFUL AS WE SHALL SEE

$$[o, o^{+}] = \left(X + \frac{iP}{m\omega}\right)\left(X - \frac{iP}{m\omega}\right) - \left(X - \frac{iP}{m\omega}\right)\left(X + \frac{iP}{m\omega}\right)$$

X^2, P^2 TERMS VANISH

$$= \frac{i}{m\omega} - [X, P] - \frac{i}{m\omega}[X, P]$$

$$= -\frac{2i}{m\omega}[X, P] = -2i\frac{i\hbar}{m\omega}$$

$$\boxed{[o, o^{+}] = \frac{2\hbar}{m\omega}}$$

SO IF WE DEFINE

$$\boxed{\hat{a} \stackrel{\Delta}{=} \sqrt{\frac{m\omega}{2\hbar}}\,\hat{o}} \Rightarrow \boxed{[\hat{a}, \hat{a}^{+}] = [a, a^{+}] = 1}$$

$$\boxed{\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + \frac{i\hat{P}}{m\omega}\right) \\ \hat{a}^{+} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} - \frac{i\hat{P}}{m\omega}\right) \end{aligned}}$$

$\hat{X}$ + $\hat{P}$ IN TERMS OF $\hat{a}, \hat{a}^{+}$

SHOW BOTH ARE HERMITIAN

PROBLEM #2

$$\hat{H} = \frac{m\omega^2}{2}\left(\hat{O}^+\hat{O}\right) + \frac{\hbar\omega}{2}$$

$$\hat{H} = \frac{m\omega^2}{2}\left(\sqrt{\frac{2\hbar}{m\omega}}\, a^+ \sqrt{\frac{2\hbar}{m\omega}}\, a\right) + \frac{\hbar\omega}{2}$$

$$= \frac{m\omega^2 (2\hbar)}{2m\omega}\left(a^+ a\right) + \frac{\hbar\omega}{2}$$

$$\boxed{\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)}$$

LET'S LOOK AT THE ENERGY OF A STATE

$$\langle \hat{H} \rangle_\psi = \langle \psi | \hat{H} | \psi \rangle = \langle \psi | \hbar\omega\left(\hat{a}^+\hat{a} + \tfrac{1}{2}\right) | \psi \rangle$$

$$= \hbar\omega\left(\langle \psi | \hat{a}^+\hat{a} | \psi \rangle + \tfrac{1}{2}\langle \psi | \psi \rangle\right)$$

$$= \hbar\omega \langle \psi | a^+ a | \psi \rangle + \frac{\hbar\omega}{2}$$

$$= \hbar\omega \langle \psi | a^+ | a\psi \rangle + \frac{\hbar\omega}{2}$$

$$= \hbar\omega \langle a\psi | a\psi \rangle + \frac{\hbar\omega}{2}$$

$$\langle \phi | \phi \rangle$$

$$\int_{-\infty}^{\infty} \phi^*(x)\, \phi(x)\, dx \geq 0$$

$$\boxed{\langle \hat{H} \rangle_\psi \geq \frac{\hbar\omega}{2}} \Longleftarrow$$

SO THINKING ABOUT ENERGY EIGENSTATES

ALL

$$\langle H \rangle_\psi \geq \frac{\hbar\omega}{2}$$

WHAT'S THE GROUND STATE

$$\langle H \rangle_{\psi_0} = \hbar\omega \langle \psi_0 | a^\dagger a | \psi_0 \rangle + \frac{\hbar\omega}{2}$$

$$= \hbar\omega \langle a\psi_0 | a\psi_0 \rangle + \frac{\hbar\omega}{2}$$

$$|\phi\rangle = a|\psi_0\rangle$$

$$\langle \phi | \phi \rangle = \int \phi^*(x)\, \phi(x)\, dx \geq 0$$

ONLY $= 0$ IF $\phi(x) = 0$

$$\Rightarrow \quad a|\psi_0\rangle = 0$$

$$\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\psi_0(x) = 0$$

$$\left(\hat{x} + \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\psi_0(x) = 0$$

$$x\psi_0(x) + \frac{\hbar}{m\omega}\frac{d\psi_0(x)}{dx} = 0$$

$$\frac{d\psi_0(x)}{dx} = -\frac{m\omega x}{\hbar}\psi_0(x)$$ 1st ORDER O.D.E.

$$\Rightarrow \quad \boxed{\psi_0(x) = C_0\, e^{-\frac{m\omega}{2\hbar}x^2}} \qquad C_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$a|\psi_0\rangle = 0$$

ANNIHILATION OR DESTRUCTION OR LOWERING OPERATOR

a^+ = CREATION OR RAISING OPERATOR

WHY ?

REMEMBER

$$[a, a^+] = 1 = aa^+ - a^+a$$

$$(aa^+ - a^+a)|\psi_0\rangle = 1|\psi_0\rangle$$

$$aa^+|\psi_0\rangle = |\psi_0\rangle \ni a^+|\psi_0\rangle$$ MUST NOT BE THE GROUND STATE

$$\hat{H} = \hbar\omega\left(\hat{a}\hat{a}^+ + \tfrac{1}{2}\right)$$

$$= \hbar\omega\left(\hat{N} + \tfrac{1}{2}\right)$$

THE NUMBER OPERATOR

RECALL $E_n = \hbar\omega(n + \tfrac{1}{2})$ FROM SCHRODINGER'S EQUATION

$$\boxed{\hat{N} = \hat{a}\hat{a}^+}$$

ASK WHAT THE COMMUTATOR IS WITH OPERATORS

$$[\hat{N},\hat{a}] = [a^+a, a]$$

$$= a^+aa - a\,a^+a$$

$$= (a^+a - aa^+)a$$

$$= [a^+, a]\,a$$

$$\boxed{[\hat{N},\hat{a}] = -\hat{a}}$$

$$[\hat{N},\hat{a}^+] = [a^+a, a^+]$$

$$= a^+[a, a^+]$$

$$\boxed{[\hat{N},\hat{a}^+] = a^+}$$

$$[\hat{a}, a^{+K}] = a\,a^{+K} - a^{+K}a$$

MOVE THIS a ACROSS a^{+K}

REMEMBER $[a, a^+] = 1$ $\quad aa^+ - a^+a = 1$

$$\underbrace{a\,a^+}_{} = 1 + a^+a$$

$$a\,a^{+K} = (1 + a^+a)\,a^{+K-1}$$

$$= (1 + a^+a)\,a^{+K-1} - a^{+K}a$$

$$= a^{+K-1} + a^+(1 + a^+a)\,a^{+K-2} - a^{+K}a$$

$$= 2\,a^{+K-1} + a^{+2}a\,a^{+K-2} - a^{+K}a$$

$$\vdots$$

$$\boxed{[\hat{a}, a^{+K}] = K\,a^{+K-1}}$$

OTHERS

$$\boxed{[\hat{a}^+, \hat{a}^K] = ?}$$

$$\boxed{\begin{aligned}[\hat{N}, \hat{a}^K] &= -K\hat{a}^K \\ [\hat{N}, \hat{a}^{+K}] &= K\hat{a}^{+K}\end{aligned}}$$

N GIVES THE NUMBER OF CREATION OR DESTRUCTION OPERATORS

$$|\psi_1'\rangle \triangleq a^+|\psi_0\rangle$$

IF $|\psi\rangle$ IS AN EIGENSTATE OF $\hat{N}$ AN EIGENSTATE OF $\hat{H}$

$$\hat{N}|\psi_1'\rangle = \hat{N}a^+|\psi_0\rangle$$

$$= [a^+a\,a^+ - a^+a^+a]\,|\psi_0\rangle$$

$$= [\hat{N}, a^+]\,|\psi_0\rangle$$

$$\hat{N}|\psi_1'\rangle = \hat{a}^+|\psi_0\rangle = (1)|\psi_1'\rangle$$

NORMALIZED?

$$\langle\psi_1'|\psi_1'\rangle = \langle a^+\psi_0|a^+\psi_0\rangle$$

$$= \langle\psi_0|a\,a^+|\psi_0\rangle$$

$$= \langle\psi_0|[a, a^+]\,|\psi_0\rangle = \langle\psi_0|\psi_0\rangle$$

$$\Rightarrow |\psi_1\rangle = |\psi_1'\rangle$$

$$|\psi_2'\rangle = a^+ a^+ |\psi_0\rangle$$

$$N.|\psi_2'\rangle = \hat{N}\, a^{+2} |\psi_0\rangle$$

$$\downarrow$$

$$N|\psi_2'\rangle = [\hat{N}, \hat{a}^{+2}]\, |\psi_0\rangle$$

$$= 2\, \hat{a}^{+2} |\psi_0\rangle$$

$$= 2\, |\psi_2'\rangle$$

IS $|\psi_2'\rangle$ NORMALIZED

$$\langle \psi_2' | \psi_2' \rangle = \langle a^+ a^+ \psi_0 \,|\, a^+ a^+ \psi_0 \rangle$$

$$= \langle \psi_0 | \, a\, \underbrace{a\, a^+ a^+}_{K \hat{a}^{+K-1}} |\psi_0\rangle$$

$$= \langle \psi_0 | \, a\, 2\, a^+ |\psi_0\rangle$$

$$\langle \psi_2' | \psi_2' \rangle = 2 \langle \psi_0 | [a, a^+] | \psi_0 \rangle = 2$$

$$\Rightarrow |\psi_2\rangle = \frac{1}{\sqrt{2}}\, a^{+2} |\psi_0\rangle$$

$$\Rightarrow \boxed{|\psi_n\rangle = \frac{1}{\sqrt{n!}}\, a^{+n} |\psi_0\rangle} \quad 2.34$$