THE QUANTUM S.H.O.

$H\,\psi(x) = E\,\psi(x)$ TIME INDEPENDENT SCHRODINGER'S

$\hat{H}\,|\psi_n\rangle = E_n\,|\psi_n\rangle$ IN THE n^{th} EIGENSTATE

(1) $\boxed{H = \frac{p^2}{2m} + \frac{1}{2}Kx^2}$ $\quad \omega = \sqrt{\frac{K}{m}} \Rightarrow \boxed{K = m\omega^2}$ (2)

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$$

$\hat{p} = -i\hbar\frac{\partial}{\partial x}$ $\quad \hat{x}$ WELL IT'S JUST x

$$p^2 = -i\hbar\frac{\partial}{\partial x}\left(-i\hbar\frac{\partial}{\partial x}\right) = -\hbar^2\frac{\partial^2}{\partial x^2}$$

(3) $$\boxed{\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2\right)\psi(x) = E\,\psi(x)}$$

TRYING TO SOLVE FOR $\psi(x)$

BUT DON'T KNOW E

KNOW WE WANT TO FIND BOUND STATES

I.E. $\psi(x) \to 0 \quad x \to \pm\infty$

NORMALIZABLE

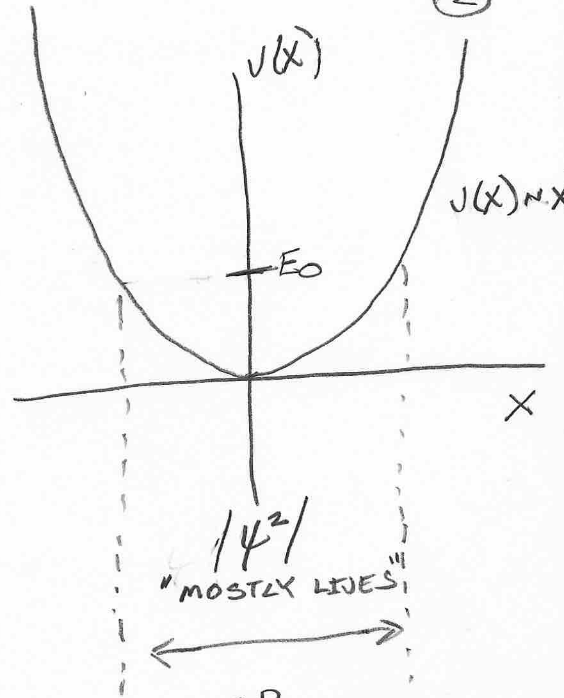
$$\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x) = 1$$

BOUNDED TO WHAT ?
BY

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x)$$

KINETIC ENERGY

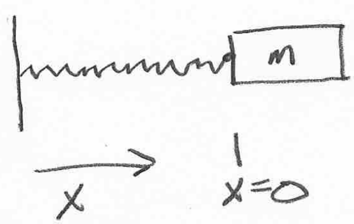
POTENTIAL ENERGY $V(x)$

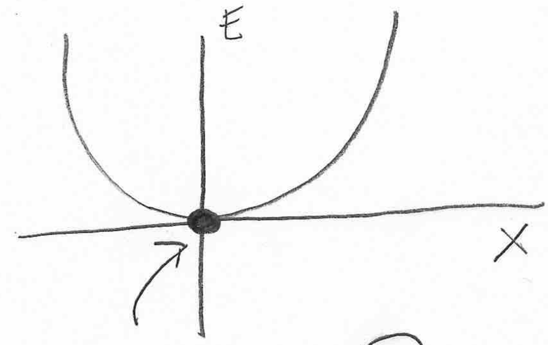


OR PERHAPS BETTER SAID WE EXPECT $\langle \hat{x} \rangle$ MOSTLY HERE

$\psi(x)$, E ARE BOTH UNKNOWN

CLASSICALLY





WHAT IS THE LOWEST ENERGY OF THE SYSTEM ?

THE BLOCK JUST SITS THERE

$x = 0$ (ALWAYS)

$p = m\dot{x} = 0$ (ALWAYS)

$\Rightarrow$ $\boxed{E_{\text{CLASSICAL S.H.O.}} = 0}$

$\Delta x = 0$

$\Delta p_x = 0$

MIGHT WE EXPECT SOMETHING DIFFERENT IN THE QUANTUM CASE ?

$\sigma_x \sigma_{p_x} \geq \hbar/2$

SO WHATEVER X IS

$x \rightsquigarrow \langle x \rangle = \langle \psi | x | \psi \rangle$

THE PRODUCT OF ITS UNCERTAINY OR STANDARD DEVIATION TIMES THE UNCERTAINTY OF p_x ≥ 0

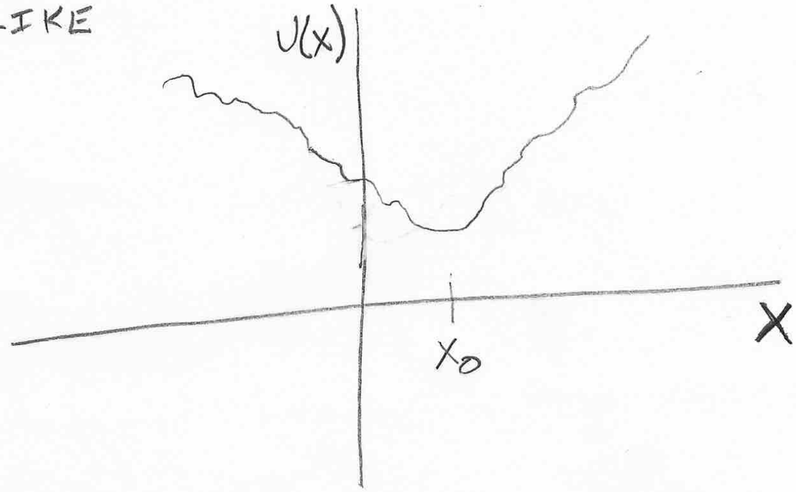
AS THE ENERGY (IN A EIGENSTATE) IS

$\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \Rightarrow$

GROUND STATE ENERGY $\neq 0$

WHY SO MUCH S.H.O. ?

ANY POTENTIAL LIKE



$V(x)$ "SMOOTH"

NEAR x_0

$$V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{V''(x_0)(x-x_0)^2}{2!} + \dots$$

AT
NEAR x_0 $V'(x_0) = 0$ THE SLOPE IS 0

SO NEAR x_0 $\approx$

$$V(x) \approx V(x_0) + \underbrace{V''(x_0)\frac{(x-x_0)^2}{2}}$$

APPROXIMATELY A S.H.O.

CHOOSE APPROPRIATE ORIGIN ($x' = x - x_0$)

+ $V(x_0)$ POTENTIAL ENERGY REFERENCE.

$$U(x') = V(x) - \underbrace{V(x_0)}_{\text{CONSTANT}}$$

$$V(x') \sim \frac{K x'^2}{2}$$

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2\right)\psi(x) = E\,\psi(x)$$

"×2"

$$\left(-\frac{\hbar^2}{m}\frac{\partial^2}{\partial x^2} + m\omega^2 x^2 - 2E\right)\psi(x) = 0$$

MAKE IT DIMENSIONLESS

PHOTO ELECTRIC EFFECT

$$E = hF = \hbar\omega$$

$\div \hbar\omega$

$$\left(-\frac{\hbar}{m\omega}\frac{\partial^2}{\partial x^2} + \underbrace{\frac{m\omega}{\hbar}}_{?} x^2 - \underbrace{\frac{2E}{\hbar\omega}}_{\frac{\text{ENERGY}}{\text{ENERGY}}}\right)\psi(x) = 0$$

$$x^2 \sim \text{LENGTH}^2 \Rightarrow \frac{m\omega}{\hbar} \propto \frac{1}{\text{LENGTH}^2}$$

$$\frac{m\omega}{\hbar} \triangleq \frac{1}{a^2} \Rightarrow \frac{\hbar}{m\omega} \triangleq a^2$$

$$\left(-a^2\frac{\partial^2}{\partial x^2} + \frac{x^2}{a^2} - \varepsilon\right)\psi(x) = 0$$

DIMENSIONAL RELATIONS

$$\boxed{\varepsilon \triangleq \frac{2E}{\hbar\omega} \qquad a \triangleq \sqrt{\frac{\hbar}{m\omega}}}$$

REALLY DON'T NEED THE PARTIAL DERIVATIVE

REWRITE AS

$$\left(-a^2\frac{d^2}{dx^2} + \frac{x^2}{a^2} - \varepsilon\right)\psi(x) = 0 \qquad (4)$$

VARIABLE SUBSTITUTION

$x = a\,u$

LENGTHS

$dx = a\,du \qquad \frac{d}{dx} = \frac{1}{a}\frac{d}{du}$

$$\left(-a^2\frac{d^2}{dx^2} + \frac{x^2}{a^2} - \varepsilon\right)\psi(x) = 0$$

$$\left(-a^2\frac{1}{a}\frac{d}{du}\left(\frac{1}{a}\frac{d}{du}\right) + \frac{(a\,u)^2}{a^2} - \varepsilon\right)\psi(u) = 0$$

(5) $$\boxed{\left(-\frac{d^2}{du^2} + u^2 - \varepsilon\right)\psi(u) = 0}$$

WAVE THE HANDS ...

A LOT

CONSIDER $u \to \pm\infty$

$\Rightarrow u^2 \gg \varepsilon$

$$\left(-\frac{d^2}{du^2} + u^2 - \varepsilon\right)\psi(u) = 0$$

$$\sim \quad \frac{d^2\psi(u)}{du^2} = u^2\psi(u) \qquad u \to \pm\infty$$

THIS QUITE A BIT DIFFERENT THAN THE CLASSICAL S.H.O

$$\frac{\partial^2 \psi}{\partial x^2} = -\psi(x)$$ WHICH HAD TRIGONOMETRIC SOLUTIONS

WHAT TYPE OF FUNCTION (FOR LARGE u)

$$\frac{d^2\psi(u)}{du^2} = u^2\psi(u) \quad ?$$

WELL WE MIGHT REMEMBER

$$\frac{de^x}{dx} = e^x$$

SO TRY

$$\psi(u) = e^{\alpha u^2/2}$$

$$\frac{d\psi(u)}{du} = \alpha u\, e^{u^2/2} = \alpha u\, \psi(u)$$

$$\frac{d^2\psi(u)}{du^2} = \alpha\psi(u) + \alpha u(\alpha u)\psi(u)$$
$$= \alpha\psi(u) + \alpha^2 u^2\psi(u) \overset{?}{=} u^2\psi(u)$$

AND FOR LARGE u

$$\alpha^2 u^2 \psi(u) >> \alpha\psi(u)$$

SO IT'S ON THE RIGHT TRACK

FOR LARGE u

$$\alpha^2 u^2\psi(u) \sim u^2\psi(u) \Rightarrow \alpha = \pm 1$$

$$\Rightarrow \psi(u) \sim e^{\pm u^2/2}$$

WANT BOUNDED SOLUTIONS

$$\psi(u) \to 0 \quad u \to \pm\infty$$

$\Rightarrow \quad \alpha = -1$

$$\psi(u) \sim e^{-u^2/2} \quad \text{FOR LARGE } u$$

LET THE GUESSING GAME CONTINUE

ANSATZ

WITH NO LOSS OF GENERALITY

$$\psi(u) = h(u)\, e^{-u^2/2}$$

($h(u)$ COULD ALWAYS BE $\dfrac{f(u)}{e^{-u^2/2}}$)

SUBSITITUTE BACK INTO (5)

$$\Big(\underbrace{-\frac{d^2}{du^2}}_{} + \underbrace{(u^2-\varepsilon)\Big)\,\psi(u)}_{\longrightarrow \text{ NOT TOO BAD}} = 0$$

$$\frac{d}{du}\psi(u) = \frac{d}{du}\left(h(u)\, e^{-u^2/2}\right)$$

$$= \frac{dh}{du} e^{-u^2/2} + h(u) - u e^{-u^2/2}$$

3 TERMS

$$\frac{d}{du}\left(\frac{d\psi}{du}\right) = \frac{d^2h}{du^2} e^{-u^2/2} - u e^{-u^2/2} \frac{dh}{du}$$

$$- u e^{-u^2/2} \frac{dh}{du} + h(u)\left(-1 + u^2\right) e^{-u^2/2}$$

$$\frac{d^2\psi}{du} = \left(\frac{d^2h}{du^2} - 2u\frac{dh}{du} + (u^2-1)\,h(u)\right) e^{-u^2/2}$$

$$(u^2-\varepsilon)\,\psi(u) = (u^2-\varepsilon)\,h(u)\,e^{-u^2/2}$$

$$-\frac{d^2\psi}{du^2} + (u^2-\varepsilon)\psi = 0$$

$$\left[-\frac{d^2h}{du^2} + 2u\frac{dh}{du} + \left[(1-u^2) + (u^2-\varepsilon)\right]h(u)\right]e^{-u^2/2} = 0$$

$e^{-u^2/2} \neq 0$ IN GENERAL $\Rightarrow$

$$(6)\quad \boxed{\frac{d^2h}{du^2} - 2u\frac{dh}{du} + (\varepsilon-1)\,h(u) = 0}$$

SOLVE FOR $h(u)$, GUESS A POWER SERIES

$$h(u) = \sum_{n=0}^{\infty} c_n u^n$$

TERMS IN $h(u)$, $\frac{dh}{du}$ AND $\frac{d^2h}{du^2}$

$$\frac{dh}{du} = \sum_{n=0}^{\infty} n\,c_n\,u^{n-1}$$

$$\frac{d^2h}{du} = \sum_{n=0}^{\infty} (n-1)(n)\,c_n\,u^{n-2}$$

(7) $$\boxed{-2u\frac{dh}{du} = -2u\sum_{n=0}^{\infty} n c_n u^{n-1} = -2\sum_{n=0}^{\infty} n c_n u^n}$$

NOW THE TRICKERY

HAVE

$$\frac{d^2h}{du^2} = \sum_{n=0}^{\infty} (n-1)(n)\, c_n u^{n-2}$$

WANT ALL COEFFIENTS TO MATCH I.E. WANT A u^n

NOTICE FOR $n=0$ AND $n=1$ TERMS VANISH

SO CAN WRITE

$$\frac{d^2h}{du^2} = \sum_{n=2}^{\infty} (n-1)(n)\, c_n u^{n-2}$$

FOR A MOMENT LET'S JUST CALL $n = n'$

IT'S JUST AN INDEX AFTER ALL

$$\frac{d^2h}{du^2} = \sum_{n'=2}^{\infty} (n'-1)(n')\, c_{n'} u^{n'-2}$$

NOW LET $n = n'-2 \Rightarrow \boxed{n' = n+2}$

$n' = 2 \Rightarrow n = 0$

$\boxed{n'-1} = (n+2) - 1 = \boxed{n+1}$

(8) $$\boxed{\frac{d^2h}{du^2} = \sum_{n=0}^{\infty} (n+1)(n+2)\, c_{n+2} u^n}$$

ALL SUMS COEFFIENTS OF u^n

SUBSTITUTE INTO (6)

9

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)\,C_{n+2} - 2n\,C_n + (\mathcal{E}-1)\,C_n\right]U^n = 0$$

TRUE IF

$$(n+1)(n+2)\,C_{n+2} + (\mathcal{E} - 2n - 1)\,C_n = 0$$

$$(9) \quad \boxed{C_{n+2} = \frac{(2n+1-\mathcal{E})}{(n+1)(n+2)}\,C_n}$$

A RECURSION RELATIONSHIP BETWEEN COEFFICIENTS

IF ONE KNOWS $C_0 \Rightarrow C_{2,4,6,8}$ EVENS

$C_1 \Rightarrow C_{3,5,7,9}$ ODDS

2^{nd} ORDER ODE $\Rightarrow$ 2 BOUNDARY CONDITIONS

$C_0 \Rightarrow \psi(0)$

$C_1 \Rightarrow \psi'(0)$

$$(10) \quad \boxed{\psi(U) = \sum_{n=0}^{\infty} C_n U^n e^{-U^2/2}}$$

BUT $\psi(U) \rightarrow 0$ AS $U \rightarrow \pm\infty$

IF $\sum C_n U^n$ IS AN INFITE SERIES

CAN WE GUARANTEE THIS?

(RATIO) TEST FOR CONVERGENCE