Houston, We Have an Optimization Problem

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Overview

In this project, we attempt to model the path of a spaceship from an initial point to a space station. We also consider the possibility of asteroids in the path of the spaceship. This makes the problem much harder. Starting with a single stationary asteroid, we extend our model to more complicated situations, while optimizing the use of gas in our rocket.

The problem

We consider a spaceship positioned at (0,0) in a two-dimensional world. This spaceship has the aim of getting back to a space station that will be positioned at (1,1) in exactly one hour. Because fuel is expensive, the goal is to use as little gas as possible while making it back to the space station. On its way, the spaceship encounters asteroids and must figure out how to avoid them while still conserving gas and making it back in time.

We model this scenario as an optimization problem, with gas as the control variable. The state space equations represent position and velocity in the x and y directions. We construct a cost functional in a way that prevents the spaceship from hitting the asteroids while achieving the goals above.

The approach

$$J[u] = \int_0^{t_f} \left[\frac{1}{2}u_x(t)^2 + \frac{1}{2}u_y(t)^2 + \sum_{i=1}^{n} \frac{W_i}{\left(\frac{(x-c_{xi})^2}{r_i} + \frac{(y-c_{yi})^2}{r_i}\right)^{20} + 1} \right] dt$$

$$S = \begin{bmatrix} x & y & \dot{x} & \dot{y} \end{bmatrix}^T \qquad \dot{S} = \begin{bmatrix} \dot{x} & \dot{y} & u_x & u_y \end{bmatrix}^T$$

We use the cost functional defined above with W_i being the weight of the ith asteroid and c_{x_i}, c_{y_i}, and r_i defining the center and radius of the ith asteroid. This functional assumes that the asteroids remain stationary, we worked to extend this to the case where the asteroids are moving which required some adjustments. But, to start we will assume this simpler approach. Using Pontryagin's Maximum Principle we see

$$P = \begin{bmatrix} P_0 & P_1 & P_2 & P_3 \end{bmatrix}^T \qquad \dot{P} = \begin{bmatrix} C_x & C_y & -P_0 & -P_1 \end{bmatrix}^T$$

We also are able to solve for $u_x = P_2$ and $u_y = P_3$. Using this we can solve for S using solve_bvp. In our system we define the following boundary conditions

$$\begin{aligned} x(0) &= 0 & x(t_f) &= 1 \\ y(0) &= 0 & y(t_f) &= 1 \\ \dot{x}(0) &= v_x & \dot{x}(t_f) &= 0 \\ \dot{y}(0) &= v_y & \dot{y}(t_f) &= 0 \end{aligned}$$

This ensures that our spaceship is starting at the origin and travels to the space station. We are also assuming that our spaceship could have been moving with initial velocities of v_x and v_y in the x and y direction before we begin our optimization problem. We force our spaceship to stop moving once it arrives at the space station.

Model 1: Single Stationary Meteor

After setting up all of our equations, we wanted to start the project as simple as possible. This would mean flying our spaceship from its initial position back to the space station with the absence of asteroids. Optimizing over the amount of fuel spent should lead us to fly directly back to the space station in a straight line. After coding up our solution, we did get this result.

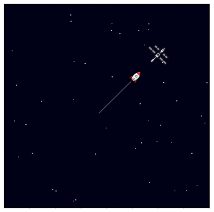


Figure 1: Spaceship going to space station with no asteroids

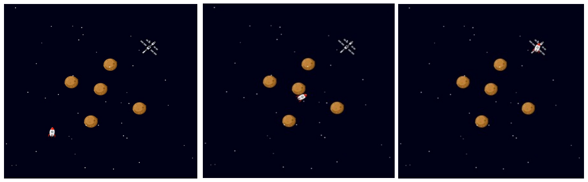
We then wanted to add complexity to our model by putting a singular stationary asteroid in our spaceship's path. We had some difficulty when we first attempted this. To avoid the asteroid, we needed to add a constraint to our system. Traditionally this could be modeled with an inequality constraint; however, due to the complexities of inequality constraints in optimal control, we brought the constraint into our cost functional as a soft constraint. To make this work, we had to adjust the coefficient of this part of the Lagrangian to penalize it enough to avoid the asteroid. Eventually this worked, and we were able to find the optimal path back to the space station to avoid the asteroid.

Once we had the optimal path, our goal was to find a better way to visualize it. So, we animated the result to see how the spaceship avoids the asteroid and what path it follows.

Figures 2-4: Spaceship avoiding a single asteroid

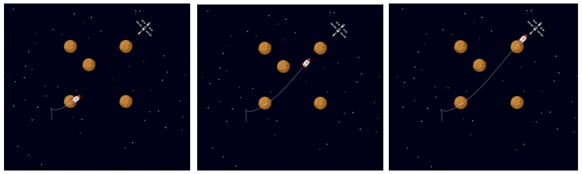
Model 2: Multiple Stationary Asteroids

After finding the optimal path with one asteroid we added several more to represent a more realistic scenario. We had to make some adjustments in our problem setup and adjust our initial conditions. We experimented with putting the asteroids in several places and changing the size. To ensure our spaceships did not hit the asteroids, we also had to adjust the weights associated with the cost of hitting each asteroid.



Figures 5-7: Spaceship avoiding multiple asteroids

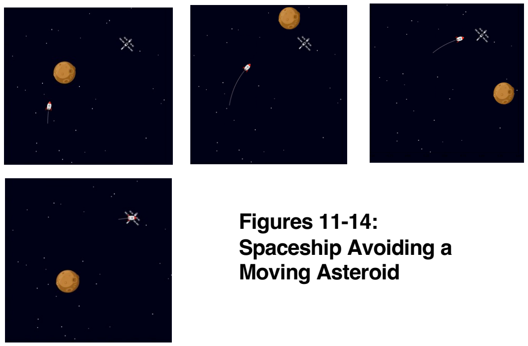
Though many of the scenarios we ran resulted in a realistic and successful path, there was one simulation in which the spaceship did not avoid the asteroids. Rather than prioritizing avoidance of the asteroids, it seems that the spaceship was more concerned about conserving gas. This suggests the need for a higher penalty for getting close to the obstacles or a lower penalty for using gas.



Figures 8-10: Spaceship failing to avoid multiple asteroids

Model 3: Moving Asteroids

Finally, we wanted to model the case where the asteroids were moving. We first wanted to do this with one moving asteroid, for which we know the path. Thus, the center of the asteroid is changing at each step, changing the setup of our problem. We then animated our results to ensure the solution to the model was reasonable.



Figures 11-14: Spaceship Avoiding a Moving Asteroid

Conclusions and next steps

Overall, we were able to get several models set up and working. Our results show optimal paths minimizing gas for our spaceship to return to the space station. These results begin from simple cases and extend to situations with a moving asteroid. This has been a complicated problem to set up and to implement. Our results are also improved through the animations to better understand how the system is moving overtime instead of a static picture.

There is a lot we can still do and add to this project. Several future explorations could include having multiple moving asteroids, having asteroids move in different paths, changing the speed of the moving asteroids, adding a third dimension to the problem and more.

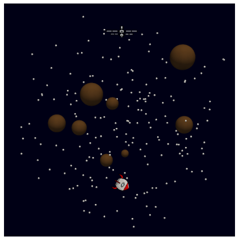


Figure 15: Setup of a spaceship avoiding asteroids in 3-dimensional space

References

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[2] Johnston, Hunter. 'The Theory of Functional Connections: A Journey from Theory to Application'. *arXiv [Math.OC]*, 2021, http://arxiv.org/abs/2105.08034. ArXiv.

[3] Li, S.; Yan, Y.; Zhang, K.; Li, X. Fuel-optimal ascent trajectory problem for launch vehicle via theory of functional connections. Int. J. Aerosp. Eng. 2021, 2021, 2734230.