ZEROS FOR A FAMILY OF COMPLEX HARMONIC POLYNOMIALS

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A complex-valued harmonic polynomial has the form,

$$p(z) = f(z) + \overline{g(z)}$$

where f and g are analytic functions of a single complex variable z. Functions of this type behave in interesting ways as they are no longer functions of one variable, but do not quite have the same properties as functions of two variables.

The Fundamental Theorem of Algebra is a well-known result stating for any single-variable polynomial of degree $n \geq 1$, there are exactly n zeros within the complex plane. For example, consider now the function $f(z) = z^3 - z + \frac{3}{5}z^5 - \frac{1}{3}z^3$. By the Fundamental Theorem of Algebra, is has 5 zeros.

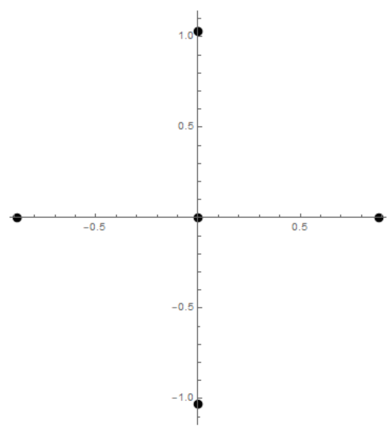

FIGURE 1. Location of the 5 zeros for $f(z) = z^3 - z + \frac{3}{5}z^5 - \frac{1}{3}z^3$

We now consider the *harmonic polynomial* where the last two terms are now taken as their complex conjugate.

$$p_c(z) = z^3 - z + c\overline{\left(\frac{3}{5}z^5 - \frac{1}{3}z^3\right)}$$

A simple plot shows when $c = 1$, $p_1(z)$ has 11 zeros, despite 5 being the degree of the polynomial. Furthermore, if we plot $p_3(z) = z^3 - z + 3\overline{\left(\frac{3}{5}z^5 - \frac{1}{3}z^3\right)}$, the number of zeros decreases to 7.

This is a surprising result. Not only is the number of zeros greater than the polynomial's highest power, but the outcome also changes depending on its coefficients. These

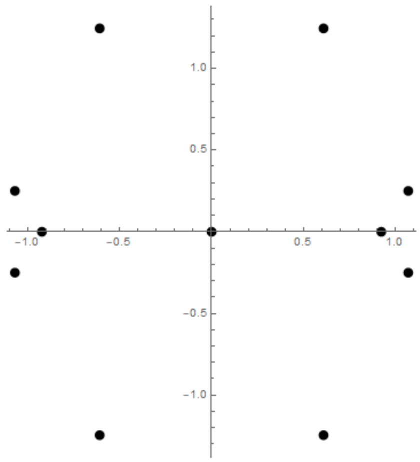

FIGURE 2. The 11 zeros of $p_1(z)$

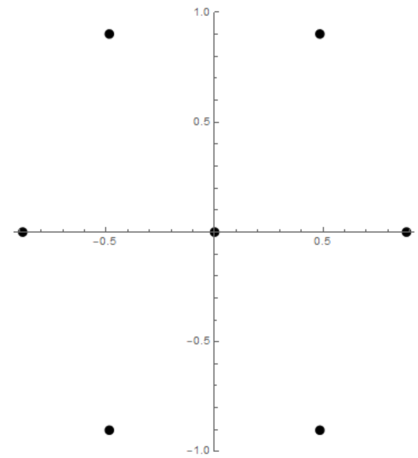

FIGURE 3. The 7 zeros of $p_3(z)$

types of polynomials clearly do not fit within the rules of the Fundamental Theorem of Algebra. Since polynomials of this nature do not follow the usual patterns, we must look at other tools to determine its number of zeros.

To analyze how the number of zeros changes, we are looking at a family of functions that have the following form:

$$p_c(z) = z^k - z + c\,\overline{\left(\frac{k}{j+k} z^{j+k} - \frac{1}{j+1} z^{j+1}\right)}$$

where $c \in \mathbb{R}_{>0}$, $j, k \in \mathbb{N}$, and $k \neq 1$.

This polynomial was specially chosen to have four terms as much of the previous research in this area had been on analyzing trinomials. This family also has a critical curve (the border separating the sense-preserving and sense reversing regions, see Definition 2) that is a circle centered at the origin. This will be an essential part to prove the following.

Theorem 1. *For our family*

$$p_c(z) = z^k - z + c\,\overline{\left(\frac{k}{j+k} z^{j+k} - \frac{1}{j+1} z^{j+1}\right)}$$

the number of zeros is:

(a) $3k + j$ when $c < \left(\frac{j}{j+2} \cdot \frac{j+1}{j+k}\right)^{\frac{j}{k-1}}$.

(b) $j + k + 2$ when $c > \left(\frac{2k+j}{j} \cdot \frac{j+1}{j+k}\right)^{\frac{j}{k-1}}$.

Example 1. As an example, let $k = 3$ and $j = 2$ such that $p_c(z) = z^3 - z + c\overline{\left(\frac{3}{5}z^5 - \frac{1}{3}z^3\right)}$. By Theorem 1 (a), there are $3k + j = 11$ zeros when $c < 0.3$ and $j + k + 2 = 11$ zeros for $c > 2.4$.

Before we can prove these results, we will need to introduce a few definitions that will be used later on in the proof.

Definition 1. If $p(z) = h(z) + \overline{g(z)}$, we define the *complex dilatation* by

$$\omega(z) = \frac{g'(z)}{h'(z)}$$

The region where $|\omega(z)| < 1$ is called the *sense-preserving region* and zeros within the region we assign with a positive order. Zeros in the *sense-reversing region* (where $|\omega(z)| > 1$) are given a negative order. We can think of the sense-preserving region as the area where the function $h(z)$ is dominant and the sense-reversing region is where $\overline{g(z)}$ dominates.

For $p_c(z)$, $h(z) = z^k - z$ and $g(z) = c\left(\frac{k}{j+k}z^{j+k} - \frac{1}{j+1}z^{j+1}\right)$.

Definition 2. The set where the complex dilatation is exactly 1 is called the *critical curve.*

Proposition 1. *For the family $p_c(z)$ the radius of the critical curve is $\left(\frac{1}{c}\right)^{\frac{1}{j}}$.*

Proof. Let $g(z) = c\left(\frac{k}{j+k}z^{j+k} - \frac{1}{j+1}z^{j+1}\right)$ and $h(z) = z^k - z$ with $g'(z) = ckz^{j+k-1} - z^j$ and $h'(z) = kz^{k-1} - 1$. The complex dilatation of $p_c(z)$ is as follows.

$$\begin{aligned} \omega(z) &= \frac{ckz^{j+k-1} - z^j}{kz^{k-1} - 1} \\ &= \frac{cz^j(kz^{k-1} - 1)}{kz^{k-1} - 1} \\ &= cz^j \end{aligned}$$

The modulus of the critical curve is equal to 1 if and only if,

$$|z|^j = \frac{1}{c}$$

Or equivalently,

$$|z| = \left(\frac{1}{c}\right)^{\frac{1}{j}}$$

Thus, the circle where $|z| = \left(\frac{1}{c}\right)^{\frac{1}{j}}$ is our critical curve. $\square$

The *order* of a zero is loosely interpreted to be the number of zeros located at the same value z_0. In a real-valued function, we can think of the function $f(x) = x^2 - 2x + 1$ to have a zero of order 2 at $x = 1$ because in its factored form $f(x) = (x-1)(x-1)$ we see that plugging in 1 for x makes the function zero in two distinct factors. A more precise definition, especially for the harmonic case, is given as follows.

Definition 3. The order of a zero z_0 of a complex-valued harmonic function is defined via its power series expansion about the zero. That is, let

$$f(z) = h(z) + \overline{g(z)} = a_0 + \sum_{j=r}^{\infty} a_j (z - z_0)^j + \overline{\left(b_0 + \sum_{j=s}^{\infty} b_j (z - z_0)^j \right)}$$

where $a_r, b_s \neq 0$. If z_0 is in the sense-preserving region of the plane, then $r \leq s$ and the order of the zero at z_0 is r. If z_0 is in the sense-reversing region, then $s \leq r$ and the order of the zero is defined to be $-s$. [1]

For the type of polynomials we are working with, zeros that lie in the sense-preserving region are given *positive order*, while zeros within the sense-reversing region have *negative order*. A zero that lies on the critical curve has an *undefined order*. In order to avoid this case, we exclude parameter values for polynomials that have zeros on the critical curve.

Now that we have a clear picture of what the order of a zero is, the following theorem is a useful tool in determining the number of zeros contained in a certain region.

In some cases, polynomials exhibit critical curves that are shaped like hypocycloids, a highly symmetrical geometric figure. The predictable nature of these curves makes

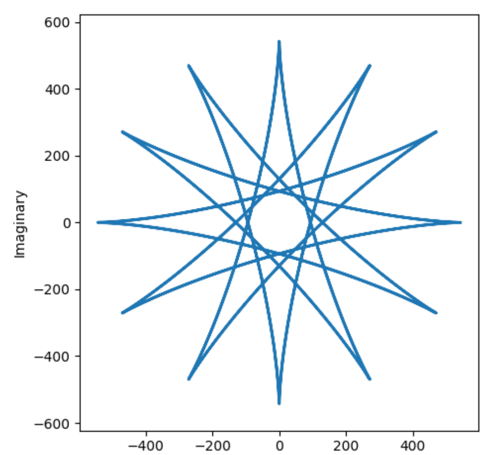

FIGURE 4. Hypocycloid-shaped critical curve

determining the number of zeros relatively straightforward, as the behavior of the zeros on both sides of the critical curve follows a well-understood pattern. However, this is not true for every polynomial. In particular, for the family of polynomials under consideration in this paper, the critical curves are not regular hypocycloids and exhibit more complex behavior.

Figures 5 and 6 show examples of critical curves for two polynomials in this family, where the irregularities in the shape of the curves present challenges in determining the number of zeros in a direct manner. Since there is no formal calculation of the shape

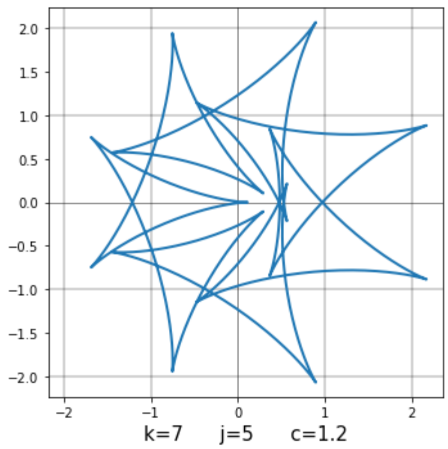

FIGURE 5. Irregular critical curve

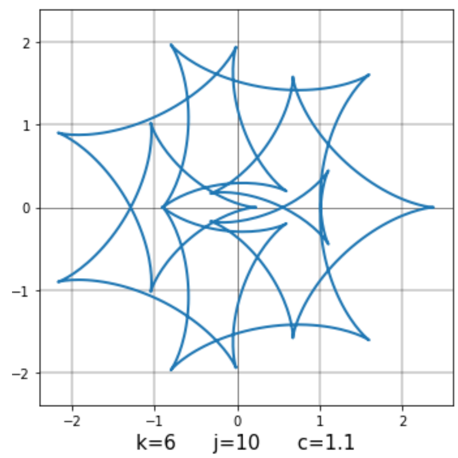

FIGURE 6. Irregular critical curve

of the critical (besides numerically plotting on a computer), we look to the following theorem as a tool to find the zeros of our family.

Theorem 2 (Rouche's Theorem)**.** *Let* $p(z) = f(z) + h(z)$ *with no zeros on the critical curve. If* $|f(z)| > |h(z)|$ *on some simple closed curve* C*,* $f(z)$ *and* $p(z)$ *have the same sum of the order of zeros within* C*.*

We will show the sum of the order of zeros for $p_c(z)$ in the total complex plane. To do so, we will use Rouche's Theorem on a curve C where the radius of C is sufficiently large to contain all of the zeros.

Proposition 2. *The sum of the order of zeros for* $p_c(z)$ *is* $-j-k$*.*

Proof. Let $p_c(z) = z^k - z + c\left(\overline{\frac{k}{j+k} z^{j+k} - \frac{1}{j+1} z^{j+1}}\right)$. Since we are working on a curve with an arbitrarily large, we see that the term with z^{j+k} is the dominating term since it has the highest power. We define $f(z) = c\frac{k}{j+k}\overline{z}^{j+k}$ and the let the remaining terms be $h(z) = z^k - z - c\frac{j}{j+1}\overline{z}^{j+1}$. We now show that $|f(z)| > |h(z)|$ on the curve C.

$$
\begin{aligned}
& & |h(z)| &= \left| z^k - z - c\frac{j}{j+1}\overline{z}^{j+1} \right| \\
&\text{(triangle inequality)} & &\leq |z|^k - |z| - c\frac{j}{j+1}\overline{|z|}^{j+1} \\
&\text{(since } |z| \text{ is arbitrarily large)} & &\leq |z|^{j+k-1} - |z|^{j+k-1} - c\frac{j}{j+1}\overline{|z|}^{j+k-1} \\
&\text{(for any } c\text{)} & &\leq c\frac{k}{j+k}\overline{|z|}^{j+k}
\end{aligned}
$$

Thus, since $|f(z)| > |h(z)|$ on the curve C, we know that $p_c(z)$ and $f(z)$ have the same order of zeros. It is clear to see that $\overline{z}^{j+k}$ has an order of $j+k$ zeros at the origin. Since f is a function of $\overline{z}$, we know its order is negative. Thus, f and p both have an order $-j-k$ zeros. $\square$

To find the true number of zeros for $p_c(z)$, we will perform Rouche's theorem on the critical curve. Through this, we find the number of positive zeros within the sense-preserving region and then determine the total number of zeros for the polynomial. Because c can be any positive real number, our primary term $f(z)$ changes depending on that value.

Proposition 3. *For small c, there are k zeros within the critical curve.*

Proof. Let $f(z) = z^k$ and let $h(z) = -z + \frac{ck}{j+k} z^{j+k} - \frac{c}{j+1} z^{j+1}$ such that $p(z) = f(z) + h(z)$. We will show using Rouche's Theorem that the this is exactly k positive zeros of $p_c(z)$ within the critical curve (where $C = \frac{1}{c}^{\frac{1}{j}}$). Observe,

$$\begin{aligned}
& & |h(z)| &= \left| -z + \frac{ck}{j+k} z^{j+k} - \frac{c}{j+1} z^{j+1} \right| \\
&\text{(triangle inequality)} & &< |z| + \frac{ck}{j+k} |z|^{j+k} + \frac{c}{j+1} |z|^{j+1} \\
&(|z| = c^{-\frac{1}{j}}) & &= c^{-\frac{1}{j}} + \frac{ck}{j+k} c^{-\frac{j+k}{j}} + \frac{c}{j+1} c^{-\frac{j+1}{j}} \\
& & &= c^{-\frac{1}{j}} + \frac{k}{j+k} c^{\frac{j}{j}} c^{-\frac{j+k}{j}} + \frac{1}{j+1} c^{\frac{j}{j}} c^{-\frac{j+1}{j}} \\
& & &= c^{-\frac{1}{j}} + \frac{k}{j+k} c^{-\frac{k}{j}} + \frac{1}{j+1} c^{-\frac{1}{j}} \\
& & &= \frac{j+2}{j+1} c^{-\frac{1}{j}} + \frac{k}{j+k} c^{-\frac{k}{j}}
\end{aligned}$$

To find the values of c where this is true, we now set this less than $|f(z)| = |z|^k$.

$$\begin{aligned}
& & |h(z)| &< |f(z)| \\
& & \frac{j+2}{j+1} c^{-\frac{1}{j}} + \frac{k}{j+k} c^{-\frac{k}{j}} &< |z|^k \\
&(|z| = c^{-\frac{1}{j}}) & \frac{j+2}{j+1} c^{-\frac{1}{j}} + \frac{k}{j+k} c^{-\frac{k}{j}} &< c^{-\frac{k}{j}} \\
& & \frac{j+2}{j+1} c^{-\frac{1}{j}} &< \frac{j}{j+k} c^{-\frac{k}{j}} \\
& & \frac{j+2}{j+1} c^{\frac{k-1}{j}} &< \frac{j}{j+k}
\end{aligned}$$

$$c^{\frac{k-1}{j}} < \frac{j}{j+2} \cdot \frac{j+1}{j+k}$$
$$c < \left(\frac{j}{j+2} \cdot \frac{j+1}{j+k}\right)^{\frac{j}{k-1}}$$

Thus, for $c < \left(\frac{j}{j+2} \cdot \frac{j+1}{j+k}\right)^{\frac{j}{k-1}}$, $f(z)$ and $p_c(z)$ have k positive zeros within the critical curve. □

Proposition 4. *If* $c < \left(\frac{j(j+1)}{(j+k)(j+2)}\right)^{\frac{j}{k-1}}$, *then* $p_c(z)$ *has a total of* $3k+j$ *zeros.*

Proof. Let n be the number of negative zeros outside the critical curve. As shown above there are k positive zeros within the critical curve, and $-j-k$ zeros total. A simple algebra statement can help us determine how many zeros lie outside the critical curve. Let n be this unknown value. Observe,

$$k - n = -j - k$$
$$-n = -j - 2k$$
$$n = 2k + j$$

We can conclude that there are $2k+j$ zeros outside the critical curve. With k zeros within the curve and $2k+j$ outside, this brings us to a total of $k+j+2k = 3k+j$ zeros of $p_c(z)$ for small c.

Let us now return to our example, when $k = 3$ and $j = 2$.

$$p_{0.2}(z) = z^3 - z + 0.2\overline{\left(\tfrac{3}{5}z^5 - \tfrac{1}{3}z^3\right)}$$

From our earlier proof the total order of zeros is $-j-k = -2-3 = -5$. We know that since $c = .2$ is sufficiently small, there are $k = 3$ zeros within the critical curve. We wish to find the number of zeros that lie *outside* the critical curve. Let n be this unknown value. Doing some simple algebra, we can conclude that there are $8 = 2k+j$ zeros outside of the critical curve. With 3 zeros inside the curve, and 8 zeros outside, this brings us to a total of 11 zeros of $p_c(z)$ for small c. This value is indeed $3k+j$ as found earlier. □

Proposition 5. *For large* c, *there is one zero inside of the critical curve.*

Proof. Let $f(z) = -z$ and let $h(z) = z^k + \frac{ck}{j+k}z^{j+k} - \frac{c}{j+1}z^{j+1}$ such that $p_c(z) = f(z)+h(z)$. We will show using Rouche's Theorem that there is exactly one zero of $p_c(z)$ within the critical curve (where $C = \frac{1}{c}^{\frac{1}{j}}$). Observe,

$$|h(z)| = \left|z^k + \frac{ck}{j+k}z^{j+k} - \frac{c}{j+1}z^{j+1}\right|$$

(triangle inequality) $$\leq |z|^k + \frac{ck}{j+k}|z|^{j+k} + \frac{c}{j+1}|z|^{j+1}s$$

$$(|z| = c^{-\frac{1}{j}}) \qquad = \left(\frac{1}{c}^{\frac{1}{j}}\right)^{k} + \frac{ck}{j+k}\left(\frac{1}{c}^{\frac{1}{j}}\right)^{j+k} + \frac{c}{j+1}\left(\frac{1}{c}^{\frac{1}{j}}\right)^{j+1}$$
$$= c^{-\frac{k}{j}} + \frac{k}{j+k}c \cdot c^{-\frac{j+k}{j}} + \frac{1}{j+1}c \cdot c^{-\frac{j+1}{j}}$$
$$= c^{-\frac{k}{j}} + \frac{k}{j+k}c^{-\frac{k}{j}} + \frac{1}{j+1}c^{-\frac{1}{j}}$$
$$= \frac{2k+j}{j+k}c^{-\frac{k}{j}} + \frac{1}{j+1}c^{-\frac{1}{j}}$$

To find the values of c where this is true, we now set this less than $|f(z)| = |z|$.

$$\frac{2k+j}{j+k}c^{-\frac{k}{j}} + \frac{1}{j+1}c^{-\frac{1}{j}} < |z|$$
$$(|z| = c^{-\frac{1}{j}}) \qquad \frac{2k+j}{j+k}c^{-\frac{k}{j}} + \frac{1}{j+1}c^{-\frac{1}{j}} < c^{-\frac{1}{j}}$$
$$\frac{2k+j}{j+k}c^{-\frac{k}{j}} < \frac{j}{j+1}c^{-\frac{1}{j}}$$
$$\frac{2k+j}{j+k} < \frac{j}{j+1}c^{\frac{k-1}{j}}$$
$$\frac{2k+j}{j} \cdot \frac{j+1}{j+k} < c^{\frac{k-1}{j}}$$
$$\left(\frac{2k+j}{j} \cdot \frac{j+1}{j+k}\right)^{\frac{j}{k-1}} < c$$

Thus, by Rouche's Theorem, we know that $p_c(z)$ has one positive order zero when c is larger than $\left(\frac{j+1}{j} \cdot \frac{2k+j}{j+k}\right)^{\frac{j}{k-1}}$. □

Proposition 6. *For $c > \left(\frac{j+1}{j} \cdot \frac{2k+j}{j+k}\right)^{\frac{j}{k-1}}$, there are a total of $j + k + 2$ zeros.*

Proof. Let m be the number of negative zeros outside the critical curve. As seen above, there is 1 positive zero within the critical curve, and a total of $-j - k$ zeros in the complex plane. Doing some simple algebra, we can determine the value m. Observe,

$$1 - m = -j - k$$
$$-m = -j - k - 1$$
$$m = j + k + 1$$

Therefore, there are $j + k + 1$ zeros outside the critical curve and 1 zero outside the critical curve, giving us a total of $j + k + 2$ zeros for $p_c(z)$.

We return again to $p_3(z) = z^3 - z + 3\overline{\left(\frac{3}{5}z^5 - \frac{1}{3}z^3\right)}$. We know the total order of zeros is -5. Since $c = 3$ is sufficiently large, there is 1 zero within the critical curve. We wish to find the number of zeros that lie *outside* the critical curve. Doing some simple algebra, we can conclude that there are 6 zeros outside of the critical curve. With 1 zero inside the curve, and 6 zeros outside, this brings us to a total of 7 zeros of $p_c(z)$ for large enough c. This value is indeed $j + k + 2$ as we previously found. □

Further Investigation

In this paper we have shown how the number of zeros behaves for sufficiently large and sufficiently small values of c. For small c, the zeros of the polynomial are closely aligned with the behavior of simpler polynomials, showing a predictable distribution. As c becomes large, the zeros tend to gather within a small disk around the origin, showing a distinct pattern in their distribution. However, for intermediate values of c, we have yet to establish a concrete method for determining the number of zeros, and the transition between these two extreme behaviors remains unclear.

Additionally, it would be valuable to explore how the methods developed in this paper can be applied to other polynomials with similar structures. Investigating different polynomials could help determine whether the behavior we observed is specific to this class or if similar patterns emerge. By addressing these directions, we can further our understanding of the behavior of zeros in complex polynomials and extend the applicability of our findings to broader classes of polynomials.

References

[1] Jen Brooks, Michael Dorff, et al. (2020) *Zeros of a One-Parameter Family of Harmonic Polynomials.* Proceedings of the American Mathematical Society, Series B.