Mathematical Physics II

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Chapter 1

Introduction

1.1 Linear versus non-linear worlds

Phenomena as diverse as the motion of planets and stars in the universe, the optical excitation of atoms and molecules, turbulence in plasmas, fluctuations in stock markets and the dynamics of weather systems can all be described mathematically in terms of differential equations. Finding exact or approximate solutions to these equations is therefore of enormous practical importance. It is fortunate that many systems of practical interest can be described adequately by linear differential equations for which analytical solutions are usually available. We are often interested in both homogeneous and inhomogeneous linear differential equations, which physically represent some kind of force applied to a system. For example, inhomogeneous differential equations can be used to model suspension systems in cars, earthquake resistant buildings, and for modelling the interaction of electromagnetic fields with atoms and molecules. The general approach to solving inhomogeneous linear differential equations is the method of Green's functions. In the first part of this course we will explore this approach for some important classes of differential equation, including the driven harmonic oscillator and Laplace's equation.

While linear systems do include many systems of interest, there are other systems for which nonlinearity can be extremely important. Examples include light propagation in certain types of material, turbulence, electronics and the propagation of shallow water waves. In the latter part of this course we will investigate the solutions of some important nonlinear differential equations in order to explore some of the general features of nonlinear systems. In particular, we will explore how nonlinearity in the wave equation leads to the emergence of unusually coherent solutions called solitons, and nonlinearity in a mechanical oscillator leads to the emergence of chaos. The fact that many nonlinear systems exhibit chaotic behaviour has wide reaching consequences for our ability to make long term predictions about physical systems.

In this first lecture, we will review some important concepts and ideas concerning differential equations and their solution using elementary methods. This will form the foundation for introducing Green's functions as a more general approach to such problems.

1.2 Review of linear differential equations

We will first review some important linear second order differential equations and the methods for their solution. Recall that for a differential equation to be **linear**, the dependent variable or its derivatives should not be raised to any power greater than unity. In Physics, we are often interested in solving differential equations for functions of many variables, however in the following we will focus mainly on one-dimensional examples. An important point is that the solution (or solutions) of a differential equation are **only defined** if one specifies the boundary conditions.

Example 1 - The one-dimensional simple harmonic oscillator

The one-dimensional simple harmonic oscillator (SHO) is probably one of the first differential equations you encountered, and it is often employed to describe many physical systems, the classic example being the oscillation of a pendulum. The relevant differential equation for some generalized displacement, $\mathcal{X}$, can be written as

$$\frac{d^2}{dt^2}\mathcal{X} + \omega_0^2\mathcal{X} = 0, \tag{1.1}$$

where ω_0 is the natural frequency of oscillation. For a pendulum $\mathcal{X}$ would correspond to the angle of deflection, in which case it can be shown that for a pendulum of length l, $\omega_0 = \sqrt{g/l}$, where g is the acceleration due to gravity. The general solution to Eq. 1.1 is

$$\mathcal{X}(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t), \tag{1.2}$$

where A and B are constants which are determined by the boundary conditions. Equivalently, one may write

$$\mathcal{X}(t) = C\sin(\omega_0 t + \delta), \tag{1.3}$$

where C and δ are again constants which are determined by the boundary conditions. In both cases we have two undetermined constants and so must have two boundary conditions. For example, if the pendulum passes the origin at $t = 0$, then $\mathcal{X}(0) = 0$, and therefore $\delta = 0$ in Eq. 1.3. If the angular velocity of the pendulum $\mathcal{X}'(0) = m$, then $C = m/\omega_0$ and the complete solution is defined.

Example 2 - Exponential growth and decay

A closely related differential equation to the SHO is,

$$\frac{d^2}{dt^2}\mathcal{X} - \gamma_0^2\mathcal{X} = 0, \tag{1.4}$$

which differs only in the sign of the second term. The general solution to this equation is,

$$\mathcal{X}(t) = Ae^{\gamma_0 t} + Be^{-\gamma_0 t}. \tag{1.5}$$

Therefore, solutions to this equation correspond to exponential growth or decay.

Example 3 - Driven one-dimensional simple harmonic oscillator

The examples given so far have been homogeneous differential equations. If we consider applying a time dependent force, $f(t)$, to a SHO the relevant differential equation becomes inhomogeneous, i.e.

$$\frac{d^2}{dt^2}\mathcal{X} + \omega_0^2\mathcal{X} = f(t). \tag{1.6}$$

An elementary method for solving equations of this type is the **method of undetermined coefficients**. We write the general solution to an inhomogeneous differential equation as the sum of a complementary function (CF) and a particular integral (PI),

$$\mathcal{X}(t) = \mathcal{X}_{\mathrm{CF}} + \mathcal{X}_{\mathrm{PI}}. \tag{1.7}$$

The complementary function is simply the general solution for the homogeneous version of the differential equation (i.e. with $f(t) = 0$ in Eq. 1.6). The form of the particular integral depends on the functional form of the driving force and there are prescribed rules for obtaining this function.

Taking a particular example, consider $f(t) = \cos(\omega' t)$. The CF is given by the solution to the homogeneous equation (Eq. 1.1) that we obtained previously

$$\mathcal{X}_{\mathrm{CF}} = A\sin(\omega_0 t) + B\cos(\omega_0 t). \tag{1.8}$$

Providing $\omega' \neq \omega_0$ then the correct form for the PI is,

$$\mathcal{X}_{\mathrm{PI}} = C\sin\left(\omega' t\right) + D\cos\left(\omega' t\right). \tag{1.9}$$

If $\omega' = \omega_0$ then this PI is not independent of the CF, therefore one should use the following function for the PI instead,

$$\mathcal{X}_{\mathrm{PI}} = Ct\sin\left(\omega' t\right) + Dt\cos\left(\omega' t\right). \tag{1.10}$$

Considering first the case $\omega' \neq \omega_0$, we substitute Eq. 1.7 into Eq. 1.6 to give,

$$\mathcal{X}(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) + \frac{1}{\omega_0^2 - \omega'^2}\cos\left(\omega' t\right). \tag{1.11}$$

The term on the right hand side increases in amplitude as one drives closer to the natural frequency. The total response of the system is a sum of terms with different frequencies but constant amplitudes. Therefore, the system will exhibit a beating effect: i.e. the higher frequency oscillation will be modulated by a slower envelope. When $\omega' = \omega_0$ the solution is given by,

$$\mathcal{X}(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) + \frac{1}{2\omega_0}t\sin\left(\omega' t\right). \tag{1.12}$$

The term on the right hand side grows linearly with time if one drives at natural frequency (i.e. resonance). As for the preceding examples, the undetermined constants are fixed by the boundary conditions.

Example 4 - One-dimensional wave equation

The 1D wave equation (i.e. one spatial dimension and one time) can represent such diverse systems as strings on a guitar, sound waves or lattice vibrations in a crystal. The differential equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{1.13}$$

where u is a generalized displacement and c is the wave velocity. One can solve this equation directly by separation of variables. In this case one obtains a general solutions of the following form,

$$u(x,t) = Ae^{i(kx-\omega t)} + Be^{i(kx+\omega t)}, \tag{1.14}$$

where the wavenumber is given in terms of the wavelength of the wave $k = 2\pi/\lambda$ and $\omega = 2\pi/T$ (where T is the period of oscillation). These solutions represent left and right traveling waves moving with speed $c = \omega/k$. For certain boundary conditions one can also obtain solutions which represent standing waves which are formed from a linear combination of right and left moving waves. The solutions in the case can be written as

$$u(x,t) = A\sin(kx)\cos(\omega t). \tag{1.15}$$

Another way to solve Eq. 1.13 was first pointed out by D'Alembert in 1746. If we consider writing the wave equation in terms of the following variables

$$\epsilon = x - ct \tag{1.16}$$

$$\eta = x + ct \tag{1.17}$$

then it is straightforward to show via the chain rule that Eq. 1.13 is equivalent to:

$$\frac{\partial^2 u}{\partial \epsilon \partial \eta} = 0. \tag{1.18}$$

Therefore, the solutions must consist of a pure functions of ϵ and η, i.e.

$$u(x,t) = g(\epsilon) + f(\eta), \tag{1.19}$$

where f and g are arbitrary functions. An example could be a gaussian wavepacket or a top-hat function. However, an important point is that this general solution is only valid if the wave speed c is a constant for all ω and k. This is known as a non-dispersive system. We will explore the consequences of dispersion in wave equations later in the course.

1.3 The Dirac delta function

The Dirac delta function, $\delta(x)$, can be thought of as a function which is zero everywhere except at $x = 0$ where it is infinite. Mathematicians would point out that the delta function is not strictly a function but a distribution, but this definition will serve our purposes. The delta function is defined such that,

$$\int_{-\eta}^{+\eta} \delta(x)\, dx = 1, \tag{1.20}$$

where η is infinitesimally small. In other words, the area under a delta function is unity. In this course, we will make extensive use of delta functions, but normally they will be written as $\delta(x - x')$, which simply moves the spike in the function from $x = 0$ to $x = x'$. In this case,

$$\int_{x'-\eta}^{x'+\eta} \delta\left(x - x'\right) dx = 1. \tag{1.21}$$

A useful property of the delta function is that it is symmetric, i.e.

$$\delta\left(x - x'\right) = \delta\left(x' - x\right). \tag{1.22}$$

Another useful property is that any well behaved function $f(x)$ can be expanded in terms of delta functions in the following way,

$$f(x) = \int_{-\infty}^{\infty} f\left(x'\right) \delta\left(x - x'\right) dx'. \tag{1.23}$$

This can be understood since one can choose a small enough integration range such that the function is approximately independent of x' (i.e. constant). Since the relevant integration range of the delta function is infinitesimal the function can be factored out of the integral as a constant. This is known as the **integral representation** of $f(x)$.

Finally, consider again the driven SHO but with a delta function as the driving term,

$$\frac{d^2}{dt^2}\mathcal{X} + \omega_0^2\mathcal{X} = F\left(t\right) = \delta\left(t - t'\right). \tag{1.24}$$

What does this correspond to physically? The term on the right can be regarded as a force applied to the system. An infinite force applied for an infinitesimal amount of time at time t' corresponding to a single unit of impulse (i.e. $\int F(t)dt = 1$). In other words, we give the pendulum a prod! Therefore, $\mathcal{X}$ in this equation becomes a function of both t and t', i.e. $\mathcal{X}(t, t')$. Next week we will see why being able to solve this equation allows us to solve the general inhomogeneous SHO equation for arbitrary driving forces.

1.4 Final comments

A few final points to highlight:

- Linear differential equations describe many physical systems (pendulums, waves, populations, economies)
- It is possible to solve differential equations (homogeneous and inhomogeneous) using standard techniques
- The Dirac delta function, although not really a function, has many useful properties which we can use to solve differential equations

Chapter 2

Green's functions

2.1 Recap

Last week we revised some important linear homogeneous and inhomogeneous differential equations and their solutions. We also introduced the idea of the Dirac delta function. This week we will begin to explore how the use of delta functions enables us to solve general linear inhomogeneous differential equations with arbitrary driving forces.

2.2 Solution of general linear differential equation

Consider a linear differential equation of the following form,

$$[\hat{L} - \lambda] y\left(x\right) = f\left(x\right), \tag{2.1}$$

where $\hat{L}$ is a linear differential operator and λ is a constant. Our aim is to solve this equation for particular functions $f\left(x\right)$ and particular boundary conditions. Last week we saw examples of how this equation can be solved for either homogeneous, $f\left(x\right) = 0$, or inhomogeneous problems using elementary approaches you should be familiar with. At the end of the last lecture, we also introduced the following type of equation:

$$[\hat{L} - \lambda] y\left(x\right) = \delta\left(x - x'\right), \tag{2.2}$$

where the forcing function on the right hand side has been replaced by a delta function. The effect of the delta function is to apply an impulsive force at x'. Therefore, the solution to this equation now depends on both x and x'. It is traditionally given the symbol G and is called a Green's function after it's inventor George Green. From now on, when we refer to the Green's function equation for a given differential equation we will mean an equation of the following form,

$$[\hat{L} - \lambda] G\left(x, x'\right) = -\delta\left(x - x'\right). \tag{2.3}$$

Note we have introduced a minus sign on the right hand side of this defining equation. Whether to include the minus sign in the definition of the Green's function is largely a matter of taste and text books often do not include it. However, this was George Green's original definition for reasons which will become apparent later in this lecture, and it is the definition we will always use in this course.

One may ask why would solving Eq. 2.3 to find the Green's function be useful? Well, if you recall from last week an arbitrary function $f(x)$ can be written as an integral over delta functions known as the integral representation (Eq. 1.23). One intuitive way to picture this is that an arbitrary function can be built as a sum of individual spikes at certain positions with appropriate weighting factors. Therefore, Eq. 2.1 can be written as:

$$[\hat{L} - \lambda] y(x) = \int f(x') \, \delta(x - x') \, dx'. \tag{2.4}$$

Now substituting Eq. 2.3 into Eq. 2.4 we find

$$[\hat{L} - \lambda] y(x) = -\int f(x') \, [\hat{L} - \lambda] G(x, x') \, dx'. \tag{2.5}$$

Since the operator $[\hat{L} - \lambda]$ does not depend on x' it can be taken outside of the integral on the right, yielding

$$[\hat{L} - \lambda] y(x) = -[\hat{L} - \lambda] \int f(x') \, G(x, x') \, dx'. \tag{2.6}$$

Therefore, one finds that,

$$y(x) = -\int f(x') \, G(x, x') \, dx'. \tag{2.7}$$

This equation shows that the solution for an arbitrary forcing function can be determined as an integral over the Green's function. This has the simple interpretation that the response due to an arbitrary force can be expressed in terms of a sum over the responses due to a series of point-like sources. This is only possible if the differential equation is **linear** so that the solutions due to different delta function like driving forces can be added linearly.

You are already familiar with a Green's function but you probably didn't know it. Poisson's equation is a differential equation relating the electrostatic potential, Φ, to the charge density, ρ. If we consider that we have a single point charge then Poisson's equation becomes (in natural units),

$$\nabla^2 \Phi = -\rho = -\delta(\mathbf{r} - \mathbf{r}'). \tag{2.8}$$

Of course we already know the solution to this equation, it is simply Coulomb's law,

$$\Phi(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.9}$$

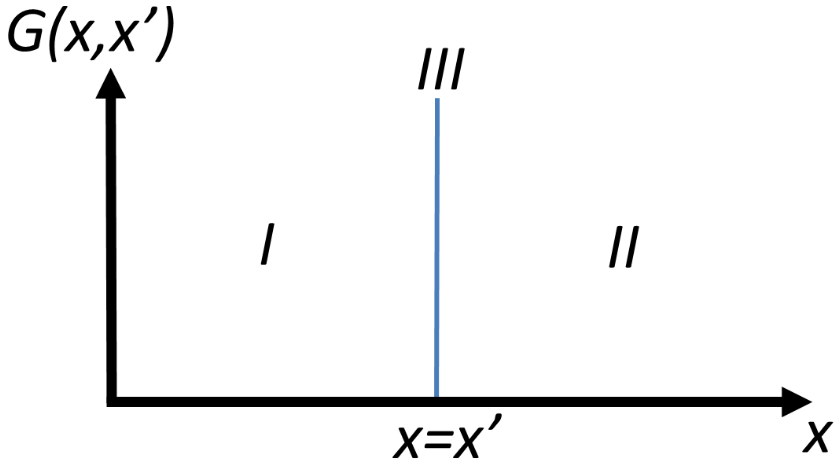

Figure 2.1: The three regions for the variable x relative to the delta function at x'.

Eq. 2.8 is analogous to the definition of the Green's function (Eq. 2.3) and Eq. 2.9 is the solution. Given Eq. 2.9 we could determine the electrostatic potential for an arbitrary charge distribution simply be integrating over point charge contributions. This is exactly what Eq. 2.7 represents. By now it should be clear that the minus sign was introduced in the definition of the Green's function (Eq. 2.3) to make the link with Poisson's equation more obvious.

2.3 Determining the Green's function

We are interested in solving an inhomogeneous differential equation of the following form,

$$[\hat{L} - \lambda]y\left(x\right) = f\left(x\right), \tag{2.10}$$

subject to prescribed boundary conditions. The first step in obtaining the Green's function is to write out the defining equation,

$$[\hat{L} - \lambda]G\left(x, x'\right) = -\delta\left(x - x'\right). \tag{2.11}$$

In order to solve this equation for $G\left(x, x'\right)$ we consider three different regions for the variable x: $x < x'$ (region I), $x > x'$ (region II) and $x \simeq x'$ (region III). In regions I and II the right hand side of Eq. 2.11 is zero and we simply solve the homogeneous differential equation. In region III we have the delta function on the right hand side. This can be pictured schematically as shown in Fig. 2.1.

We proceed by solving first for $G_I(x, x')$ (where the subscript on G labels the region), corresponding to the left hand side of Fig. 2.1. Similarly, we solve for $G_{II}(x, x')$, corresponding to the right hand side of Fig. 2.1. Some (but not all) of the undetermined constants in the general solutions in these two regions are fixed by applying the appropriate boundary conditions. Finally we consider region III, which is the bridge between the other two regions of solution. There are restrictions on how the solutions in regions I and II must connect together in region III which allows us to fully determine the remaining constants. The first condition imposed in region III is that the Green's function should be continuous at $x = x'$,

$$G_I\left(x', x'\right) = G_{II}\left(x', x'\right). \tag{2.12}$$

Secondly, we require that integral of the delta function over an infinitesimal region either side of x' is unity, i.e.

$$\int_{x'-\eta}^{x'+\eta} [\hat{L} - \lambda] G\left(x, x'\right) dx = -\int_{x'-\eta}^{x'+\eta} \delta\left(x - x'\right) dx = -1. \tag{2.13}$$

The integrand on the left hand side of Eq. 2.13 involves derivatives of $G(x, x')$. Only the highest order derivative of the Green's function is proportional to the delta function on the right hand side of the equation, whereas all lower derivatives are finite near $x = x'$. Therefore, as the integral is performed over an infinitesimally small region, only the highest order derivative contributes to the integral, i.e.

$$\int_{x'-\eta}^{x'+\eta} [\hat{L} - \lambda] G\left(x, x'\right) dx = \int_{x'-\eta}^{x'+\eta} a_n \frac{d^n}{dx^n} G\left(x, x'\right) dx, \tag{2.14}$$

where n is the order of the differential equation and a_n are the coefficients. This simplifies to

$$\left.\frac{d^{n-1}}{dx^{n-1}} G\left(x, x'\right)\right|_{x'-\eta}^{x'+\eta} = \frac{d^{n-1}}{dx^{n-1}} G_{II}\left(x', x'\right) - \frac{d^{n-1}}{dx^{n-1}} G_I\left(x', x'\right) = -\frac{1}{a_n}. \tag{2.15}$$

Eq. 2.15 states that the $(n-1)^{\text{th}}$ derivative of the Green's function has an integer discontinuity of $1/a_n$ at $x = x'$. All lower order derivatives are continuous.

Once the Green's function is fully determined we can solve the differential equation for arbitrary forcing function in the following way,

$$y(x) = -\int f\left(x'\right) G\left(x, x'\right) dx'. \tag{2.16}$$

The best way to see how all of this works is to work through some examples.

Worked example

Determine the Green's function for the following differential equation

$$y''(x) = f(x), \tag{2.17}$$

in the region between 0 and 1 subject to the boundary conditions $y(0) = 0$ and $y'(1) = 0$. For brevity, from now on we will use primes to denote differentiation (both for functions and Green's functions). The Green's function equation can be written as

$$G''\left(x, x'\right) = -\delta\left(x - x'\right) \tag{2.18}$$

Now, as described above, we divide the solution into three regions (Fig. 2.1) and consider each region in turn.

Region I

For $0 < x < x'$, $\delta(x - x') = 0$, therefore,

$$G_{\mathrm{I}}''\left(x, x'\right) = 0. \tag{2.19}$$

Integrating once we get,

$$G_{\mathrm{I}}'\left(x, x'\right) = m, \tag{2.20}$$

where m is a constant. Integrating again we find,

$$G_{\mathrm{I}}\left(x, x'\right) = mx + c, \tag{2.21}$$

where c is a constant. Applying the boundary condition on the left at $x = 0$,

$$G_{\mathrm{I}}\left(0, x'\right) = c = 0, \tag{2.22}$$

hence $c = 0$. Therefore,

$$G_{\mathrm{I}}\left(x, x'\right) = mx. \tag{2.23}$$

Region II

For $x' < x < 1$, $\delta(x - x') = 0$, therefore:

$$G_{\mathrm{II}}''\left(x, x'\right) = 0 \tag{2.24}$$

Integrating once we get,

$$G_{\mathrm{II}}'\left(x, x'\right) = m', \tag{2.25}$$

where m' is a constant. Integrating again we find,

$$G_{\mathrm{II}}\left(x, x'\right) = m'x + c', \tag{2.26}$$

where c' is a constant. Applying the boundary condition on the right at $x = 1$,

$$G_{\mathrm{II}}'\left(1, x'\right) = m' = 0, \tag{2.27}$$

hence $m' = 0$. Therefore,

$$G_{\mathrm{II}}\left(x, x'\right) = c'. \tag{2.28}$$

Region III

The solutions in regions I and II above are obtained in terms of two constants, which can be determined by enforcing the constraints in region III. First, using the fact that the first derivative of the Green's function has an integer discontinuity (Eq. 2.15), we have,

$$G'_{\mathrm{II}}\left(x', x'\right) - G'_{\mathrm{I}}\left(x', x'\right) = 0 - m = -1. \tag{2.29}$$

Therefore, $m = 1$. Secondly, we also require continuity of the Green's function,

$$G_{\mathrm{II}}\left(x', x'\right) = c' = G_{\mathrm{I}}\left(x', x'\right) = x', \tag{2.30}$$

hence $c' = x'$.

Therefore, with all constants determined the Green's function is,

$$G\left(x, x'\right) = \begin{cases} x & : x < x' \\ x' & : x > x' \end{cases} \tag{2.31}$$

2.4 Final comments

A few final points to highlight:

- Green's functions provide a way to solve inhomogeneous differential equations for arbitrary forcing functions
- The procedure for determining the Green's function is to divide the problem into three regions and solve subject to appropriate boundary conditions

Chapter 3

Simple harmonic oscillator

3.1 Recap

Last week we saw our first worked example showing how one can obtain the Green's function for a differential equation. This week we will consider several more examples based around the simple harmonic oscillator.

3.2 Green's function for the simple harmonic oscillator

We will solve the following differential equation,

$$\mathcal{X}'' + \omega_0^2 \mathcal{X} = f(t), \tag{3.1}$$

where the prime on $\mathcal{X}$ in this case represents differentiation with respect to t. We first construct the defining equation for the Green's function,

$$G''\left(t,t'\right) + \omega_0^2 G\left(t,t'\right) = -\delta\left(t-t'\right). \tag{3.2}$$

We will solve for the Green's function subject to the boundary condition that the oscillator is at rest before the impulse is applied at t', i.e.

$$G\left(t,t'\right) = G'\left(t,t'\right) = \ 0 \quad : t < t' \ . \tag{3.3}$$

First we split the problem into regions as before.

Region I

The boundary conditions in this problem fully specify the solution in region I,

$$G_{\mathrm{I}}'\left(t,t'\right) = G_{\mathrm{I}}\left(t,t'\right) = 0. \tag{3.4}$$

Region II

For $t > t'$, $\delta(t - t') = 0$, therefore

$$G''_{\mathrm{II}}\left(t, t'\right) + \omega_0^2 G_{\mathrm{II}}\left(t, t'\right) = 0. \tag{3.5}$$

The general solution to this homogeneous differential equation is (Eq. 1.3)

$$G_{\mathrm{II}}\left(t, t'\right) = A \sin\left(\omega_0\left(t - \delta\right)\right). \tag{3.6}$$

Since we know that $G_{\mathrm{II}}(t', t') = 0$, it follows that $\delta = t'$. Therefore,

$$G_{\mathrm{II}}\left(t, t'\right) = A \sin\left(\omega_0\left(t - t'\right)\right), \tag{3.7}$$

and

$$G'_{\mathrm{II}}\left(t, t'\right) = A\omega_0 \cos\left(\omega_0\left(t - t'\right)\right). \tag{3.8}$$

Region III

In this region the differential equation is

$$G''\left(t, t'\right) + \omega_0^2 G\left(t, t'\right) = -\delta\left(t - t'\right). \tag{3.9}$$

Applying the condition that the first derivative of the Green's function has an integer discontinuity at t' (Eq. 2.15) we find,

$$G'_{\mathrm{II}}\left(t', t'\right) - G'_{\mathrm{I}}\left(t', t'\right) = -1. \tag{3.10}$$

Substituting Eq. 3.4 and Eq. 3.8 into Eq. 3.11 we get

$$G'_{\mathrm{II}}\left(t', t'\right) - G'_{\mathrm{I}}\left(t', t'\right) = A\omega_0 \cos(0) - 0 = -1, \tag{3.11}$$

therefore, $A = -1/\omega_0$. Finally, we can write the full Green's function for the simple harmonic oscillator,

$$G\left(t, t'\right) = \begin{cases} 0 & : t < t' \\ -\frac{1}{\omega_0} \sin\omega_0\left(t - t'\right) & : t > t' \end{cases} \tag{3.12}$$

3.3 Using the Green's function to solve for particular driving forces

Example 1 - Application of impulsive forces

Consider that the system is a rest for $t < 0$ and is then subject to instantaneous impulsive forces at $t = 0$ and $T = \pi/\omega_0$ of equal magnitudes A. The two impulsive forces can be written as

$$f(t) = A\Big[\delta(t) + \delta(t - T)\Big]. \tag{3.13}$$

In general, to calculate the response of the system we integrate the driving force with the Green's function,

$$\mathcal{X}(t) = -\int_{-\infty}^{t} f\left(t'\right) G\left(t, t'\right) dt'. \tag{3.14}$$

Substituting in the particular driving force and Green's function defined above we find

$$\mathcal{X}(t) = \int_{0}^{t} A\left[\delta\left(t'\right) + \delta\left(t' - T\right)\right] \frac{1}{\omega_0} \sin\left(\omega_0\left(t - t'\right)\right) dt'. \tag{3.15}$$

The delta functions in the above equations are zero except at certain values of t' and so pick out particular values of the integrand. The solution for $t > T$ is given by,

$$\mathcal{X}(t) = \frac{A}{\omega_0}\left[\sin\left(\omega_0 t\right) + \sin\left(\omega_0\left(t - T\right)\right)\right]. \tag{3.16}$$

As $T = \pi/\omega_0$ the two terms are exactly π out of phase and cancel. Therefore for $t > T$, $\mathcal{X}(t) = 0$. Note that for $t > 0$ and $t < T$ the response is given by the first term only.

Example 2 - Application of constant force

Consider that the system is a rest for $t < 0$ and then we apply a constant driving force

$$f\left(t\right) = A. \tag{3.17}$$

In general, to calculate the response of the system we integrate the driving force with the Green's function,

$$\mathcal{X}(t) = -\int_{-\infty}^{t} f\left(t'\right) G\left(t, t'\right) dt'. \tag{3.18}$$

Substituting in the particular driving force and Green's function defined above we find

$$\mathcal{X}(t) = \int_{0}^{t} A\frac{1}{\omega_0} \sin\left(\omega_0\left(t - t'\right)\right) dt' \tag{3.19}$$

and finally,

$$\chi\left(t\right) = \frac{A}{\omega_0^2}\cos\left(\omega_0\left(t - t'\right)\right)\Big|_0^t = \frac{A}{\omega_0^2}\left[1 - \cos\left(\omega_0 t\right)\right]. \tag{3.20}$$

Note that this represents oscillation about an equilibrium which is displaced from the origin.

3.4 Inhomogeneous damped simple harmonic oscillator

The damped simple harmonic oscillator is described by the following differential equation,

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{X} + b\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{X} + k\mathcal{X} = F\cos\left(\omega t\right). \tag{3.21}$$

It is also possible to obtain the Green's function for this equation, allowing the general solution of the inhomogeneous damped simple harmonic oscillator to be expressed as an integral equation. The derivation requires complex integration and is beyond the scope of this course, however the solution is given below,

$$G\left(t,t'\right) = \left\{ \begin{array}{cl} 0 & : t < t' \\ -\frac{1}{m\omega'}e^{-\gamma(t-t')}\sin\omega'\left(t-t'\right) & : t > t' \end{array} \right. \tag{3.22}$$

where $\gamma = b/2m$ and $\omega' = \sqrt{(-b^2+4km)/2m}$.

3.5 Final comments

A few final points to highlight:

- The Green's function for the simple harmonic oscillator allows us to determine the solution for arbitrary driving forces.

Chapter 4

String under tension

4.1 Recap

Last week derived the Green's function for the 1D simple harmonic oscillator and explored its solutions for different driving forces. This week we will consider another important differential equation which describes a 1D string under tension.

4.2 Green's function for a 1D string under tension

The following differential equation describes the lateral displacement, u, of a string of mass density ρ under tension T and subject to a distributed force $f(x)$,

$$\frac{\partial^2 u}{\partial t^2} - T\frac{\partial^2 u}{\partial x^2} = f(x). \tag{4.1}$$

We will assume that the string is static so that the first term on the left disappears leaving,

$$u'' = -\frac{1}{T} f(x), \tag{4.2}$$

where as before the prime indicates differentiation with respect to the independent variable (in this case x). This equation is the one-dimensional form of **Laplace's equation**.

We would like to find the Green's function for Eq. 4.2 so that we can write the solution for an arbitrary force in terms of an integral. We will consider boundary conditions corresponding to the string being held fixed with $u = 0$ at $x = 0$ and $x = L$. We first write down the defining Green's function equation,

$$G''\left(x, x'\right) = -\delta\left(x - x'\right), \tag{4.3}$$

and then split the problem into three regions as before.

Region I

In region I $G''_I(x,x') = 0$, therefore,

$$G_I\left(x,x'\right) = mx + c. \tag{4.4}$$

However, since $G_I(0,x') = 0$, $c = 0$ and,

$$G_I\left(x,x'\right) = mx. \tag{4.5}$$

Region II

In region II $G''_{II}(x,x') = 0$, therefore,

$$G_{II}\left(x,x'\right) = m'x + c'. \tag{4.6}$$

However, since $G_{II}(L,x') = 0$, $m' = -c'/L$ and,

$$G_{II}\left(x,x'\right) = c'\left(1 - \frac{x}{L}\right). \tag{4.7}$$

Region III

We first apply the condition on the first derivative of the Green's function,

$$G'_{II}\left(x',x'\right) - G'_I\left(x',x'\right) = -\frac{c'}{L} - m = -1. \tag{4.8}$$

Rearranging we find,

$$m = 1 - \frac{c'}{L}. \tag{4.9}$$

Then we apply the continuity condition,

$$G_I\left(x',x'\right) = G_I\left(x',x'\right) = mx' = c'\left(1 - \frac{x'}{L}\right). \tag{4.10}$$

By combining Eq. 4.9 and Eq. 4.10 we find $c' = x'$ and

$$m = \left(1 - \frac{x'}{L}\right). \tag{4.11}$$

Therefore, the Green's function is fully determined as,

$$G\left(x,x'\right) = \begin{cases} \left(1 - \frac{x'}{L}\right)x & : x < x' \\ \left(1 - \frac{x}{L}\right)x' & : x > x' \end{cases} \tag{4.12}$$

Using Eq. 2.16, the displacement of the string in response to an arbitrary force $f(x')$ is then given by,

$$u(x) = -\int_0^x -\frac{1}{T} f\left(x'\right) G_{II}\left(x,x'\right) dx' + \int_x^L -\frac{1}{T} f\left(x'\right) G_I\left(x,x'\right) dx', \tag{4.13}$$

where the integral has been split into two regions, one where x' is smaller than x, and one where x' is larger than x. We then use the appropriate Green's function in each integral. Substituting the Green's function into this equation we obtain,

$$u(x) = \frac{1}{T}\int_0^x f(x')\left(1 - \frac{x}{L}\right)x'dx' + \frac{1}{T}\int_x^L f(x')\left(1 - \frac{x'}{L}\right)xdx'. \quad (4.14)$$

4.3 Solving for particular applied forces

Example 1 - Application of a force at a point

If a force is applied at a single point, i.e. $f(x) = A\delta(x - x')$, then the Green's function gives the response across the full length of the string. Looking at the form of the solution (Eq. 4.12) we see that the displacement of the string depends linearly on x between the boundary points at $x = 0$ and $x = L$ and the point of application of the force $x = x'$. Therefore, the shape of the loaded string is a triangle, with the maximum displacement occurring at $x = x'$. Depending on where the force is applied the maximum displacement is different. In general for a force applied at x' the displacement is $A/T\left(\frac{x'}{L} - 1\right)x'$. This is a maximum at $x = L/2$ and decreases quadratically to zero as one approaches the fixed points at $x = 0$ and $x = L$.

Example 2 - Application of a constant force

Now we consider application of a constant force, i.e. $f(x) = A$. Substituting into Eq. 4.14 and integrating we find,

$$u(x) = \frac{A}{T}\frac{x}{2}(L - x). \quad (4.15)$$

The displacement of the string is a quadratic function with maximum displacement at $x = L/2$. This function can be regarded as resulting from a sum over many triangular displacement patterns like that described above corresponding to numerous point-like forces applied over the full length of the string.

4.4 Final comments

A few final points to highlight:

- The advantage of using the Green's function to solve for the displacement of a loaded string is that it has the boundary conditions built into it automatically

- The Green's function solution also has the natural interpretation that the displacement due to a continuous distributed load can be regarded as being due to a sum of point like loads

Chapter 5

Eigenfunction representation

5.1 Recap

Over the last few weeks we have explored various examples demonstrating how Green's functions can be used to solve inhomogeneous differential equations. In this last lecture on Green's functions we will explore the eigenfunction representation of Green's functions.

5.2 Solution of inhomogenerous differential equation in terms of eigenfunctions

We are interested in finding solutions to the general inhomogeneous differential equation,

$$[\hat{L} - \lambda]u\left(x\right) = f\left(x\right), \tag{5.1}$$

where $\hat{L}$ is a differential operator. We know that, for given boundary conditions, there is a complete set of eigenvalues and eigenfunctions associated with the operator $\hat{L}$,

$$\hat{L}\psi_n\left(x\right) = \lambda_n\psi_n\left(x\right). \tag{5.2}$$

As these eigenfunctions form a complete basis we can expand the functions $u\left(x\right)$ and $f\left(x\right)$ in Eq. 5.1 in the following way,

$$u\left(x\right) = \sum_n a_n\psi_n \tag{5.3}$$

$$f\left(x\right) = \sum_n b_n\psi_n, \tag{5.4}$$

where a_n and b_n are in general complex coefficients. Substituting Eq. 5.3 and Eq. 5.4 into Eq. 5.1 we find,

$$\sum_n a_n\left(\lambda_n - \lambda\right)\psi_n = \sum_n b_n\psi_n. \tag{5.5}$$

Therefore,

$$a_n = \frac{b_n}{\lambda_n - \lambda}. \tag{5.6}$$

To find an expression for b_n we multiply Eq. 5.4 by $\psi^*_{n'}$ and integrate over all space,

$$\int \psi^*_{n'}(x) f(x)\, dx = \sum_n b_n \int \psi^*_{n'} \psi_n dx = \sum_n b_n \delta_{nn'} = b_{n'}, \tag{5.7}$$

giving,

$$b_{n'} = \int \psi^*_{n'}(x) f(x)\, dx \tag{5.8}$$

and

$$a_n = \frac{\int \psi^*_n(x) f(x)\, dx}{\lambda_n - \lambda}. \tag{5.9}$$

Therefore, the solution for any driving force can be obtained if the eigenfunctions and eigenvalues of the operator $\hat{L}$ are known. For example, if we consider $f(x) = \delta(x - x')$ then,

$$a_n = \frac{\psi^*_n(x')}{\lambda_n - \lambda}, \tag{5.10}$$

and

$$u(x, x') = \sum_n \frac{\psi^*_n(x') \psi_n(x)}{\lambda_n - \lambda}. \tag{5.11}$$

If we consider a negative delta function as the driving force then we can define the Green's function in the following way,

$$G(x, x'; \lambda) = \sum_n \frac{\psi^*_n(x') \psi_n(x)}{\lambda - \lambda_n}. \tag{5.12}$$

5.3 Example - Green's function for a bowed stretched string

The differential equation describing the motion of a stretched string subject to a time varying force, e.g. as when playing a violin, is

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = f(x,t), \tag{5.13}$$

where c is the wave velocity. The relevant boundary conditions are that the string is fixed between two points at $x = 0$ and $x = L$. If the driving force due to bowing the string takes the form,

$$f(x,t) = f(x)\, e^{(-i\omega t)}, \tag{5.14}$$

then the response of the string will also have the same time dependence, i.e.

$$u(x,t) = u(x)\, e^{(-i\omega t)}. \tag{5.15}$$

Therefore, one can reduce the problem to a time independent differential equation,

$$\frac{d^2 u}{dx^2} + k^2 u = f(x), \tag{5.16}$$

where $k = \omega/c$. This has the associated the Green's function equation,

$$\frac{d^2 u}{dx^2} + k^2 u = -\delta\left(x = x'\right). \tag{5.17}$$

We could proceed to solve this equation using the approach described in previous lectures. However, instead here we will use Eq. 5.12 to expand the Green's function in terms of the eigenfunctions of the associated operator.

We first consider the eigenfunction equation,

$$\frac{\partial^2}{\partial x^2} u_n = -\lambda_n u_n, \tag{5.18}$$

with the boundary conditions $u(0) = u(L) = 0$. The normalised eigenfunctions take the form,

$$u_n = \sqrt{\frac{2}{L}} \sin\left(k_n x\right), \tag{5.19}$$

where,

$$\lambda_n = k_n^2 = \left(\frac{n\pi}{L}\right)^2. \tag{5.20}$$

Direct substitution of Eq. 5.19 and Eq. 5.20 into Eq. 5.12 gives,

$$G\left(x, x'; k\right) = \frac{2}{L} \sum_n \frac{\sin\left(k_n x'\right) \sin\left(k_n x\right)}{k^2 - k_n^2} \tag{5.21}$$

We can use the Green's function to solve a for a particular driving force. As an example, we will consider $f(x) = x^2$ in Eq. 5.16. In this case we can write the solution as,

$$u(x) = -\int_0^L f\left(x'\right) G\left(x, x'\right) dx', \tag{5.22}$$

$$u(x) = -\int_0^L \frac{2}{L} \sum_n \frac{\sin\left(k_n x'\right) \sin\left(k_n x\right)}{k^2 - k_n^2} x'^2 dx', \tag{5.23}$$

$$u(x) = -\frac{2}{L} \sum_n \frac{\sin\left(k_n x\right)}{k^2 - k_n^2} \int_0^L \sin\left(k_n x'\right) x'^2 dx', \tag{5.24}$$

$$u\left(x\right) = -\frac{2}{L}\sum_{n}\frac{\sin\left(k_n x\right)}{k^2 - k_n^2}\int_0^1 \sin\left(n\pi\eta\right)\eta^2 d\eta, \tag{5.25}$$

where the substitution $\eta = x/L$ was employed to get to the last line. This integral can be evaluated to give the final solution as,

$$u\left(x\right) = -\frac{2}{L}\sum_{n}\frac{\sin\left(\left(\frac{n\pi}{L}\right)x\right)}{k^2 - \left(\frac{n\pi}{L}\right)^2}\left[\frac{\left(2 - n^2\pi^2\right)\left(-1\right)^n - 2}{n^3\pi^3}\right]. \tag{5.26}$$

5.4 Final comments

A few final points to highlight:

- The Green's function for an inhomogeneous differential equation can be expressed in terms of the eigenfunctions and eigenvalues of the associated differential operator (the eigenfunction representation)
- The solution for an arbitrary driving force is then expressed as an infinite series in terms of the eigenfunctions of homogeneous equation (like a Fourier series)

Chapter 6

Dispersion in wave equations

6.1 Recap

In the last five lectures we have explored a number of linear inhomogeneous differential equations and solved them using Green's functions. The Green's function method is only applicable to **linear** differential equations since is relies on that fact that the response of a system due to different driving forces can be added together linearly to give the total response. In the rest of this course we will explore some of the interesting effects one finds in the behaviour of nonlinear systems. The first example we will discuss over this and the next lecture is emergence of solitons in nonlinear wave equations. As a prelude, in this lecture we will discuss the effects of dispersion in linear wave equations.

6.2 The wave equation

First let's recall the 1D wave equation (i.e. one spatial dimension and one time) that we revised in the first lecture

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{6.1}$$

where u is a generalized displacement and c is the wave velocity. The general solutions to this equation are of the form

$$e^{i(kx \pm \omega t)} = e^{ik\left(x \pm \frac{\omega}{k}t\right)}, \tag{6.2}$$

where the wavenumber is given in terms of the wavelength of the wave, i.e $k = 2\pi/\lambda$, and $\omega = 2\pi/T$ (where T is the period of oscillation). These solutions correspond to waves which move to the left or right with a velocity given by,

$$c = \frac{\omega(k)}{k}. \tag{6.3}$$

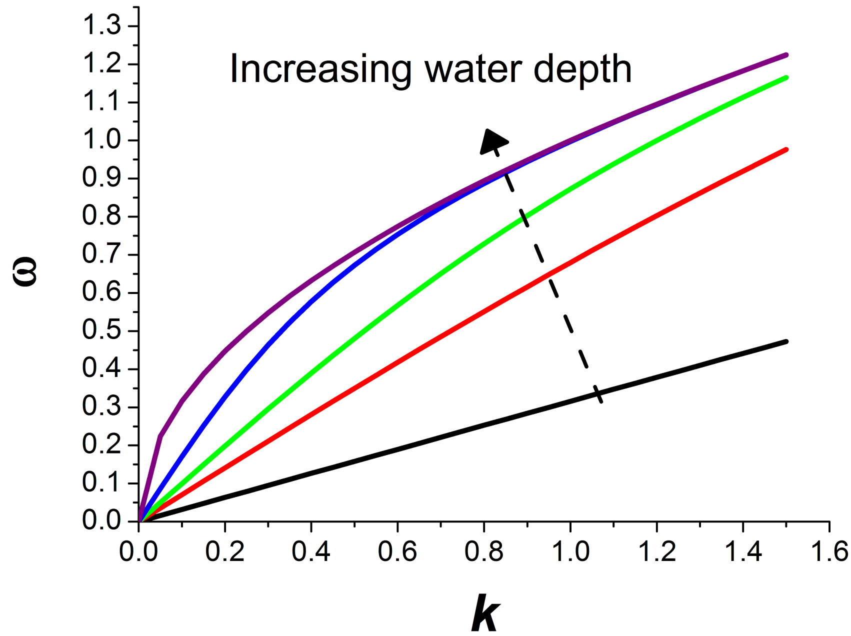

Figure 6.1: Dispersion relations for surface waves on water with various depths. As the depth increases the deviation from a linear dispersion relation increases.

This velocity is known as the phase velocity since it is the velocity with which any point on the waveform (e.g. a trough or a peak) moves through space. The function $\omega(k)$ is known as the dispersion relation. If $\omega(k)$ is directly proportional to k, then the phase velocity is a constant. This is known as a **non-dispersive** system. However, in general ω is not simply proportional to k so the phase velocity also depends on k. This is known as a **dispersive** system. Fig. 6.1 shows examples of dispersion relations for surface waves on water with various depths. Very shallow water waves are to a good approximation non dispersive while deep water waves, e.g. on the sea, are highly dispersive.

An important consequence of a system being non-dispersive is that waves of the form given in Eq. 6.2 travel with the same velocity independent of the frequency. Wavepackets of arbitrary shape can be considered as a linear combination of these waves with different frequencies and amplitudes (i.e. a Fourier series). Therefore, a disturbance of any shape will propagate with constant velocity in a non-dispersive system and its shape will remain unchanged. Another way to show this is via the D'Alembert solution to the wave equation (see Chapter 1) which is valid only for constant c. The

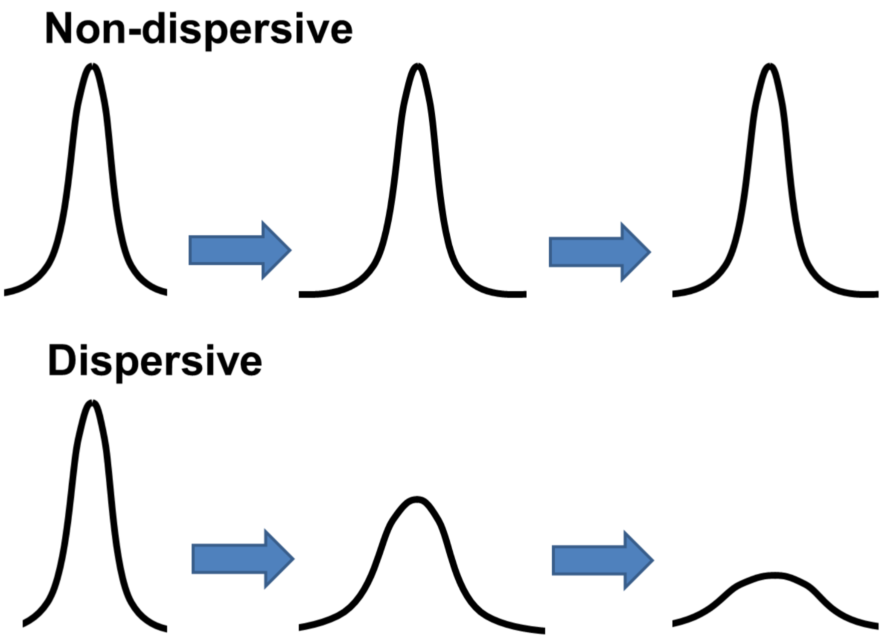

Figure 6.2: The propagation of a wavepacket in non-dispersive and dispersive systems. In a non-dispersive system the wavepacket retains its shape while in a dispersive system it gradually spreads out.

D'Alembert solution is,

$$u(x,t) = g(x - ct) + f(x + ct)\,, \tag{6.4}$$

where g and f are arbitrary functions. This equation also shows that if two wavepackets collide they will simply pass through each other, with the amplitude adding together linearly in the region where the two wavepackets overlap.

The behaviour of wavepackets in dispersive systems is quite different to that in non-dispersive systems. In this case, the various Fourier components which make up the wavepacket will propegate with different velocities. Therefore, over time the wavepacket will spread out and lose its shape (Fig. 6.2).

6.3 Dispersion of surface water waves

We will now consider how these ideas apply to a real physical system, namely the propagation of surface water waves. This covers such diverse situations as ripples on puddles or waves the lake outside or in the sea. A thorough treatment of these problems requires the solution of the differential equations of fluid dynamics which is outside of the scope of this course. However, we can learn a lot about wave propagation in water simply by employing dimensional analysis.

Example 1 - Deep water waves

The relevant physical quantities for this deep water waves are ω, k, and the acceleration due to gravity g. These quantities involve 2 independent dimensions [L] and [T]. Therefore Buckingham's Π theorem tells us that there is only one dimensionless parameter, which is

$$\Pi = \frac{\omega^2}{gk}. \tag{6.5}$$

Therefore,

$$\frac{\omega^2}{gk} = \text{constant}, \tag{6.6}$$

and,

$$\omega = C\sqrt{gk}, \tag{6.7}$$

where C is a constant which cannot be determined by dimensional analysis. In fact, if one solves the fluid dynamics equations one finds that $C = 1$. Therefore, waves on deep water are dispersive with a phase velocity $c = \sqrt{gk}/k = \sqrt{g/k}$. The phase velocity increases as the wavenumber decreases, i.e. longer wavelength waves travel faster.

Example 2 - General water waves

In this case, the relevant physical quantities are ω, k, g, and the depth of the water h_0. These involve only 2 independent dimensions [L] and [T], therefore there are two dimensionless parameters,

$$\Pi_1 = \frac{\omega^2}{gk}, \tag{6.8}$$

and

$$\Pi_2 = kh_0. \tag{6.9}$$

Therefore,

$$\frac{\omega^2}{gk} = \Phi\left(kh_0\right), \tag{6.10}$$

where Φ is a function and,

$$\omega^2 = gk\Phi\left(kh_0\right). \tag{6.11}$$

We know that in the limit $h_0 \to \infty$, $\Phi(kh_0) \to 1$ in order that we converge to the deep water result given above. In fact, solution of the fluid dynamics equations gives $\Phi(kh_0) = \tanh(kh_0)$, i.e.,

$$\omega^2 = gk \tanh(kh_0). \tag{6.12}$$

If we have shallow water waves, such that $\lambda \gg h_0$ or equivalently $k \ll h_0$, then we can perform a Taylor expansion of the tanh term on the right side of Eq. 6.12, yielding,

$$\omega^2 = gk^2 h_0 \left(1 - \frac{1}{3}(kh_0)^2 + \frac{2}{15}(kh_0)^4 + \dots\right). \tag{6.13}$$

If one considers only the first term we wave a non-dispersive system as $\omega \propto k$. For small k the higher order terms contribute a small amount of dispersion. Therefore, shallow waves are said to be **weakly dispersive** – i.e. they have a phase velocity which is almost independent of frequency. The curves shown in Fig. 6.1 were plotted using Eq. 6.12 which shows graphically the transition from linear to nonlinear dispersion as the water depth increases. Therefore, a wavepacket propagating in weakly dispersive shallow water will gradually spread out and lose its shape, but less quickly than it would in deeper water (compare ripples on the surface of a deep sea to those on a puddle).

Example 3 - Surface waves including surface tension

One may ask how surface tension will affect the conclusions above. Again we can address this with dimensional analysis. The relevant physical quantities are now ω, k, g, h_0, the water density ρ and the surface tension σ (which has the dimensions of energy per unit area). These involve 3 independent dimensions [L], [T] and [M], and so there are 3 dimensionless parameters,

$$\Pi_1 = \frac{\omega^2}{gk}, \tag{6.14}$$

$$\Pi_2 = kh_0, \tag{6.15}$$

and

$$\Pi_3 = \frac{\sigma k^2}{g\rho}. \tag{6.16}$$

Therefore,

$$\frac{\omega^2}{gk} = \Phi(kh_0)\,\mathcal{X}\left(\frac{\sigma k^2}{g\rho}\right). \tag{6.17}$$

where Φ and $\mathcal{X}$ are functions. Solution of the fluid dynamics equations yields,

$$\omega^2 = gk\left(1 + \frac{\sigma k^2}{g\rho}\right)\tanh(kh_0). \tag{6.18}$$

After Taylor expanding the tanh term as before for shallow water waves we find,

$$\omega^2 = gk^2h_0 \left(1 - \frac{1}{3}(kh_0)^2 + \frac{\sigma k^2}{g\rho} + \dots\right). \tag{6.19}$$

Therefore, the system remains weakly dispersive and the surface tension actually works to weaken the degree of dispersion. In the next lecture we will see how adding nonlinearity into a weakly dispersive wave equation gives rise to solutions involving wavepackets which do not spread out at all - known as solitons.

6.4 Final comments

A few final points to highlight:

- In a non-dispersive system $\omega \propto k$ and waves propagate with the same velocity independent of wavelength
- In a non-dispersive system wavespackets retain their shape as they propagate while in a dispersive system they gradually spread out
- Dimensional analysis can be used to determine dispersion relations for surface water waves
- Shallow water waves can be shown to be weakly dispersive

Chapter 7

Solitons

7.1 Recap

In the last lecture we employed dimensional analysis to investigate dispersion in surface water waves. We showed that shallow water waves, i.e. when $k < h_0$, are only weakly dispersive. Therefore, wavespackets should gradually spread out as they propagate. In this lecture, we will consider how the incorporation of nonlinear effects into the governing differential equation modifies the behaviour of shallow surface water waves.

7.2 The modified wave equation

If we wish to describe dispersion of shallow water waves with the standard wave equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{7.1}$$

then we must allow c to be a function of k. Alternatively, one can construct a modified wave equation which has constant coefficients but has dispersion built in to it. An example of such a modified wave equation is,

$$\eta_{tt} - c_0^2\eta_{xx} - \frac{1}{3}c_0^2 h_0^2 \eta_{xxxx} = 0, \tag{7.2}$$

where we have used abbreviated notation for the differentials (e.g. $\eta_{tt} = \partial^2\eta/\partial t^2$) and $c_0^2 = gh_0$. It it straightforward to show by substitution that the solutions of this equation are the same as that of the original wave equation, i.e.

$$e^{i(kx \pm \omega t)}, \tag{7.3}$$

and that

$$\omega^2 = c_0^2 k^2 \left(1 - \frac{1}{3}h_0^2 k^2\right). \tag{7.4}$$

This is the same dispersion relation as that given by Eq. 6.13 if one truncates after the second term in the Taylor expansion. Therefore the modified wave equation (Eq. 7.2) allows us to describe weak dispersion of shallow water waves using an equation with constant coefficients. However, this comes at the expense of making the equation higher order than the original wave equation.

Another example of a modified wave equation for describing shallow water waves is,

$$\eta_t + c_0\eta_x + \frac{1}{6}c_0 h_0^2 \eta_{xxx} = 0. \tag{7.5}$$

which has right-moving solutions of the form

$$e^{i(kx-\omega t)}, \tag{7.6}$$

with the following dispersion relation,

$$\omega = c_0 k\left(1 - \frac{1}{6}h_0^2 k^2\right). \tag{7.7}$$

This is the same as that given by Eqs. 6.13 noting that $(1-x)^{\frac{1}{2}} \simeq 1 - \frac{1}{2}x$. Unlike Eq. 7.2 the dispersion for waves traveling in the opposite direction does not have the correct form, therefore it can only be used to model shallow water waves moving in the positive x direction. However, it has lower order derivatives which makes it more convenient to solve, and it can even be solved when nonlinearity is included.

7.3 Addition of nonlinearity

In order to describe the propagation of shallow waves including and weak dispersion and nonlinearity we can modify Eq. 7.5 in the following way,

$$\eta_t + c_0\left(1 + \frac{\delta\eta}{h_0}\right)\eta_x + \frac{1}{6}c_0 h_0^2 \eta_{xxx} = 0. \tag{7.8}$$

This is known as the **Korteweg-de Vries equation** named after the two people who first studied it in 1895, although a very similar equation was proposed first by Boussinesq in 1877. The constant δ determines the degree of nonlinearity and we will consider the case for $\delta = 3/2$. One can show that the solution to this equation take the form,

$$\eta = \eta_0 \mathrm{sech}^2\left[\gamma\left(x - vt\right)\right], \tag{7.9}$$

where η_0 is the maximum amplitude of the wave,

$$\gamma = \left(\frac{3\eta_0}{4h_0^3}\right)^{\frac{1}{2}}, \tag{7.10}$$

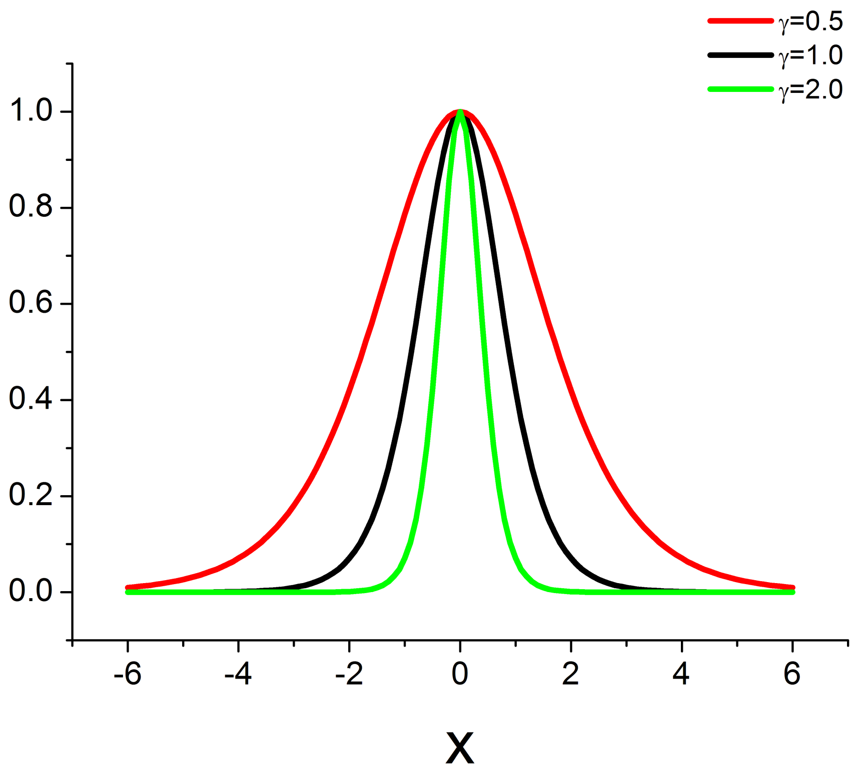

Figure 7.1: The function $\operatorname{sech}^2(\gamma x)$ plotted for several difference values of γ.

and

$$v = c_0 \left(1 + \frac{\eta_0}{2h_0}\right). \tag{7.11}$$

Note that the solution has a uniquely defined shape. This is in contrast to what is found for the linear wave equation, where any shape of disturbance is allowed (as shown by D'Alembert).

So that we can understand what this solution corresponds to, let us first examine the characteristics of the function,

$$f = \operatorname{sech}^2(\gamma x) = \left(\frac{2}{e^{\gamma x} + e^{-\gamma x}}\right)^2. \tag{7.12}$$

Fig. 7.1 shows a sketch of this function for different values of γ. We can see immediately that $f(0) = 1$ and $f(\infty) = 0$. We can also see that the function is even (i.e. symmetric about $x = 0$). In other words, the function is peaked at $x = 0$ and decays towards zero on either side. The full width at half the maximum (FWHM) is given by,

$$\text{FWHM} = \frac{2}{\gamma} \cosh^{-1}\left(\sqrt{2}\right) \simeq \frac{1.76}{\gamma} \simeq 1.76 \left(\frac{4h_0^3}{3\eta_0}\right)^{\frac{1}{2}}. \tag{7.13}$$

With this information, we can easily interpret the solution described by Eqs. 7.9–7.11. The solution is a bell-shaped disturbance which propagates in the positive x-direction with velocity v. **The velocity of propagation is larger for disturbances of larger amplitude and decreases as the depth of the water is increased**. The width of the disturbance is inversely proportional to the square root of the amplitude. In other words disturbances of bigger amplitude become narrower.

Without the inclusion of nonlinearity the weak dispersion in the wave equation would cause a propagating disturbance of type described above to spread out. However, the nonlinearity cancels the effect of dispersion exactly allowing for disturbances which preserve their shape indefinitely. In other words, dispersion tends to make wavepackets spread out, but nonlinear effects tend to keep them together. The balance of these two effects give rise to propagating disturbances which are extremely stable. These solutions are known as **solitary waves** or **solitons**.

7.4 Examples of solitons in water waves

7.4.1 Example 1 - Propagation of disturbance in canals

Actually the solitary wave phenomenon was first observed in a canal in 1834 by John Scott Russell, a Scottish naval engineer who built the Great Eastern in collaboration with Isambard Kingdom Brunel. Here is his account,

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

7.4.2 Example 2 - Tsunamis

Tsunamis produced by earthquakes are another example of a solitary wave. Although the sea may not appear to be very shallow, the equations outlined above still apply approximately since the wavelength of a wave formed by an earthquake is very long (and k is therefore small).

Consider a tsunami produced by an earthquake in middle of the ocean which has a depth of 4 km. We can first estimate the velocity of wave propagation from the dispersion relation for shallow water waves (without nonlinearity), i.e. Eq. 6.13. As the wavelength is large we can approximate by just the first term: $\omega = k\sqrt{gh_0}$ and $c = \omega/k = \sqrt{gh_0}$, therefore $c \simeq 2 \times 10^2$ ms^{-1}.

Now if we consider nonlinearity, we can use Eqs. 7.9–7.11 to describe the behaviour of the propagating solitary wave. If we assume that the earthquake produces a disturbance of amplitude of the order 1 m, we can see from examining Eq. 7.11 that the velocity will be approximately the same as for the linear case as $\eta_0 \ll h_0$. We can also use Eq. 7.10 and Eq. 7.13 to estimate the breadth of the solitary wave FWHM $\simeq$ 500 km.

7.5 Collisions of solitons

For the linear wave equation the sum of any two solutions is also a solution. Therefore, when wavepackets collide their amplitudes are simply superimposed in the region where they overlap. In other words, the wavepackets simply pass through each other as if the other were not there. After the 'collision', the wavepackets continue to propagate in exactly the same way they would have done if no 'collision' had taken place.

For the soliton solutions of the nonlinear wave equation the situation is very different. The linear sum of two separate soliton solutions is not a solution to the nonlinear wave equation. Therefore, one must explicitly solve the nonlinear wave equation in order to describe the collision between two solitons. This solution shows that **colliding solitons behave like colliding particles** rather than like linear wavepackets. For example, when a larger amplitude (and therefore faster moving) soliton collides with a smaller amplitude (slower moving) soliton it transfers momentum to the slower moving soliton. After the collision the previously slow moving soliton is now moving faster and has larger amplitude, while the previously faster moving soliton is now moving more slowly. This is most clearly appreciated by viewing an animation of a numerical solution of colliding solitons (e.g. see `http://www.acs.psu.edu/drussell/Demos/Solitons/solitons.html`).

7.6 Final comments

A few final points to highlight:

- Weak dispersion causes wavepackets to spread out but nonlinearity cancels this effect leading extremely coherent propagating disturbances known as solitons

- Solitons can not take arbitrary shapes and have defined relationships between their width, height and speed of propagation
- In collisions solitons behave like particles, transferring momentum from one soliton to another, rather than like linear wavepackets which propagate through each other unperturbed

Chapter 8

The Logistic equation

8.1 Recap

Many physical systems can be described by linear differential equations, and as we saw in the first part of this course their response to arbitrary driving forces can be completely determined by the method of Green's functions. In the last lecture, we considered adding nonlinearity to a differential equation (the wave equation) which led to the emergence of unusually coherent solutions known as solitons. In the last two lectures we will explore another important consequence of nonlinearity, namely the emergence of chaos, which has far reaching implications in fields as diverse as astronomy, plasma physics, medicine, philosophy, weather prediction and economics.

8.2 The mechanistic picture of the universe

The mechanistic picture of the universe was formulated by Laplace over 200 years ago, inspired by Newton's laws of motion and the theory of gravitation. Laplace proposed that if the mass, position and motion of every object in the universe was known it would be possible to predict all future events until the end of time, simply by applying the laws of mechanics. The uncomfortable side effect of this fully deterministic picture is that there would be no free will, i.e. every choice we make would have been determined at the beginning of the universe. However, over the last century discoveries in the field of nonlinear dynamics led to the idea that even fairly simple nonlinear systems can exhibit such complex behaviour that long term predictions become impossible. The study of such unpredictable, or chaotic, systems became known as **chaos theory**. These ideas have important consequences for understanding the behaviour of complex systems such as stock markets and weather systems, and destroy the mechanistic picture proposed by Laplace leaving open the question of free will.

8.3 The logistic equation

To illustrate the emergence of chaotic behaviour we will consider some relatively simple mathematical models which have their origins in modelling population growth. Let us first consider a simple model for the growth of a population,

$$x_{n+1} = kx_n, \tag{8.1}$$

where x_n is the population in year n and k is a constant greater than 0. Eq. 8.1 is **linear map** of the population in one year to the population in the next year. Consider the long term behaviour of the population for different values of k. For $k < 1$, whatever the initial value of population, the population will tend towards zero as $n \to \infty$. The asymptotic value of the population is known as an attractor and in this case we have a fixed point attractor at $x = 0$. For $k > 1$, whatever the initial value of population, the population will tend towards infinity as $n \to \infty$. Therefore for $k > 1$ we have an attractor at $x = \infty$.

Clearly, a model for population growth that ultimately ends in extinction or an infinite population is not physically very reasonable. In reality, limitations on food supply and space would limit the population to some maximum. Therefore let us consider the following modification to the mathematical model,

$$f_{n+1} = kf_n\left(1 - f_n\right), \tag{8.2}$$

where f_n is the population in year n given as a fraction of the maximum possible population. This is known as the logistic equation. Unlike Eq. 8.1, this is a **nonlinear map** of the population in one year to the population in the next year. The addition of nonlinearity has some interesting and unexpected consequences which we shall now explore.

Exercise - Dynamical properties of the logistic equation

- Using a calculator iterate Eq. 8.2 for different values of the initial population f_1 =0.2, 0.4, 0.6, 0.8 and for different values of the constant k =1.00, 2.00, 3.10, 3.50, 3.55, 3.60. In each case find the value of f (or values) to which the population converges in the limit of large n (i.e. the attractor).

- For $k = 3.6$ consider starting from initial populations which differ only slightly, e.g. f_1 =0.5000 and 0.5001. How does the population differ after 10 years, 20 years or 40 years?

What you should find:

- For $k = 1$ and $k = 2$ the value of the population in the limit of large n converges to a single value which does not depend on the initial population. Therefore these systems have fixed point attractors. For $k = 1$ the attractor is at 0 and for $k = 2$ the attractor is at 0.5. In general for $k < 3$ there is an attractor at $1 - 1/k$.

- For $k = 3.1$ you should find that the population does not settle on a single value but oscillates between two stable values, i.e. the attractor consists of two points. This phenomena is known as **period doubling** and is a sign that a system is close to exhibiting chaotic behaviour. The number of points the attractor has doubles again at $k = 3.5$ (4 points) and again at $k = 3.55$ (8 points). The values of k at which the period doubles get closer and closer together until close to $k = 3.6$ the period has doubled an infinite number of times. In this case the attractor is known as a **strange attractor** due to the uncountable number of points it contains. In this case the population does not exhibit periodic oscillations and does converge not to a single value. The behaviour of this system is said to be chaotic.

- For $k = 3.6$, i.e. in the chaotic regime, the population at some time in the future is very sensitive to the initial conditions. This is the main characteristic of chaotic systems which makes their long term behaviour unpredictable. This is known as the **butterfly effect**, due to the idea popularised by Edward Lorenz that a the flapping of a distant butterfly's wings could determine when and where a hurricane forms several weeks later.

8.4 The bifurcation diagram

A useful way to visualise the attractors of a particular system is a bifurcation diagram. The diagram plots the points on the attractor for each value of k. Figure 8.1 shows a bifurcation diagram for the logistic equation (Eq. 8.2). In the chaotic region ($k > 3.6$) the diagram becomes very complex but also exhibits remarkable fine structure. In particular, there is a narrow region near $k = 3.8$ where the system becomes non-chaotic and instead oscillates between three values. The diagram also has fractal-like properties - i.e. any portion of the diagram when magnified looks similar in structure to the whole diagram and contains an infinite amount of detail. There is a very useful Java application available at `http://highlyrefined.net/benwei/logistic/` which allows you to explore this bifurcation diagram and to zoom in to examine parts of it in more detail.

8.5 Summary

We first considered a simple model for population growth based on a linear mapping (Eq. 8.1). It was found to asymptotically tend towards a single

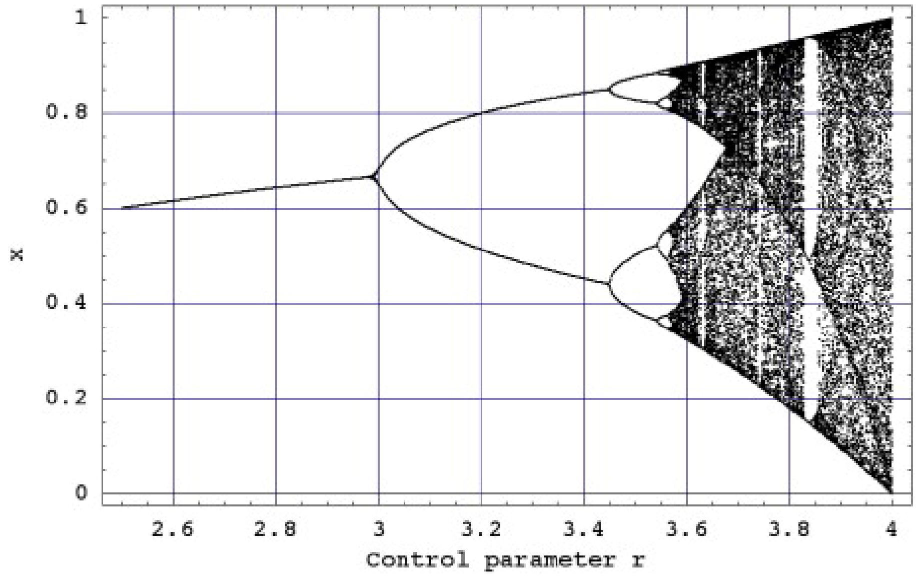

Figure 8.1: A bifurcation diagram for the logistic equation.

point depending on the value of some model parameter, in other words it has fixed point attractors. The long time evolution of the equation was insensitive to the initial conditions, therefore long term prediction is possible.

We then considered a simple modification of this model that involves a nonlinear mapping (Eq. 8.2). Depending on the parameter k we observed regions characterised by an attractor with a single stable fixed point, or regions with multiple points. For particular ranges of k the attractor was found to consist of an infinite number of points (strange attractor). In this case the population exhibits non-periodic behaviour which is extremely sensitive to the initial conditions, therefore long term predictions are not possible.

It is important understand the distinction between chaotic behaviour and stochastic (or random) behaviour. A system which is chaotic may be entirely deterministic (e.g. Eq. 8.2), however it is unpredictable due to the sensitive dependence on initial conditions. In others words, you will always get the same result if you start with the same initial conditions. On the other hand, a stochastic system would yield different results even if you start from the same initial conditions. In reality, most physical systems contain elements of both stochastic and chaotic behaviour and disentangling them can be difficult. However, there are many examples which have been documented including hydrodynamic systems, optics, biology and electrical circuits [e.g. see Phys. Rev. Lett. 48, 714 (1982)].

8.6 Final comments

A few final points to highlight:

- Simple nonlinear systems may respond chaotically
- Chaos is not random, but has an intricate fractal order
- The response of a chaotic system is incredibly sensitive to initial conditions (butterfly effect)
- Chaos limits our ability to predict the future

Chapter 9

The Driven Mechanical Oscillator

9.1 Recap

In the last lecture we introduced the idea that chaos can emerge in relatively simple systems where nonlinearity is present. This leads to unpredictability. We considered some simple models for population growth to illustrate these ideas. We will now return to a system we considered earlier in the course, the mechanical oscillator, to investigate whether chaotic behaviour emerges from the introduction of nonlinearity in this type of system.

9.2 The driven mechanical oscillator

The equation of motion for a one-dimensional driven mechanical oscillator is,

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}x}V(x) = F\cos(\omega t), \tag{9.1}$$

where m is the mass of the oscillator, b is a damping factor, V is the potential energy, F is the amplitude of the applied force and ω is the driving frequency. For sufficiently small amplitude oscillations it is possible to approximate $V(x)$ by its Taylor expansion. In the case when only the quadratic term is retained, the differential equation reduces to a linear differential equation which can be solved analytically. For example, earlier in this course we used the method of Green's functions to solve Eq. 9.1 for the special case where $V(x) = x^2$ and $b = 0$ (i.e. the simple harmonic oscillator). More generally, it may not be possible to solve a nonlinear differential equation analytically and therefore one must rely on numerical solutions. In the following we will first discuss the character of the solutions of Eq. 9.1 in the linear case before exploring consequences of the addition of nonlinearity.

9.3 The simple harmonic oscillator

By making a suitable choice of dimensionless parameters, the differential equation for the simple harmonic oscillator can be written as,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + x = F\cos(\omega t) . \tag{9.2}$$

We solved this equation previously using the method of Green's functions which for particular initial conditions gives an integral equation for $x(t)$ for all times in the future. A useful way to visualise the solutions is with a phase space plot, i.e. a plot the trajectory of the oscillator in terms of x and $\dot{x}$. To illustrate this idea it is instructive to consider the phase space trajectory for the case where $F = 0$ and $b = 0.4$ (Fig. 9.1a). One can see that regardless of the initial conditions the trajectories tend towards $x = \dot{x} = 0$. In analogy with our investigation of the logistic equation, the point $x = \dot{x} = 0$ in phase space can be regarded as a stable fixed point attractor.

If we now consider a driven and damped harmonic oscillator, the trajectory shows that the oscillator initially experiences some transient dynamics which are dependent on the initial conditions (Fig. 9.1b). However, after these transients have died out the oscillator settles in a periodic orbit known as a **limit cycle**. The frequency of the periodic orbit is the same as that of the driving force ω. The points in phase space to which the oscillator is attracted towards is known as a **periodic attractor**. A useful way to visualise the periodic nature of the orbit is with a **Poincaré diagram**, which plots points in phase space at periodic intervals of $2\pi/\omega$. In other words, it is a stroboscopic picture of the trajectory in phase space. Figure 9.1c shows a Poincaré diagram for the driven damped harmonic oscillator, which consists of only a single point demonstrating that the system settles into a stable periodic orbit with the same frequency as the driving force.

9.4 The nonlinear oscillator

As an example of a nonlinear oscillator we will consider the pendulum which is governed by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + \sin(x) = F\cos(\omega t) . \tag{9.3}$$

For small amplitude oscillations (i.e. small F) we expect that the solutions for the simple harmonic oscillator described above will be valid. However, as the amplitude of the driving force is increased the effects of nonlinearity should become important. Eq. 9.3 can be integrated numerically in order to explore the effects of nonlinearity on the dynamics.

Figure 9.2 shows phase trajectories for the pendulum for several different values of F. The trajectories are only plotted some time after $t = 0$ so that

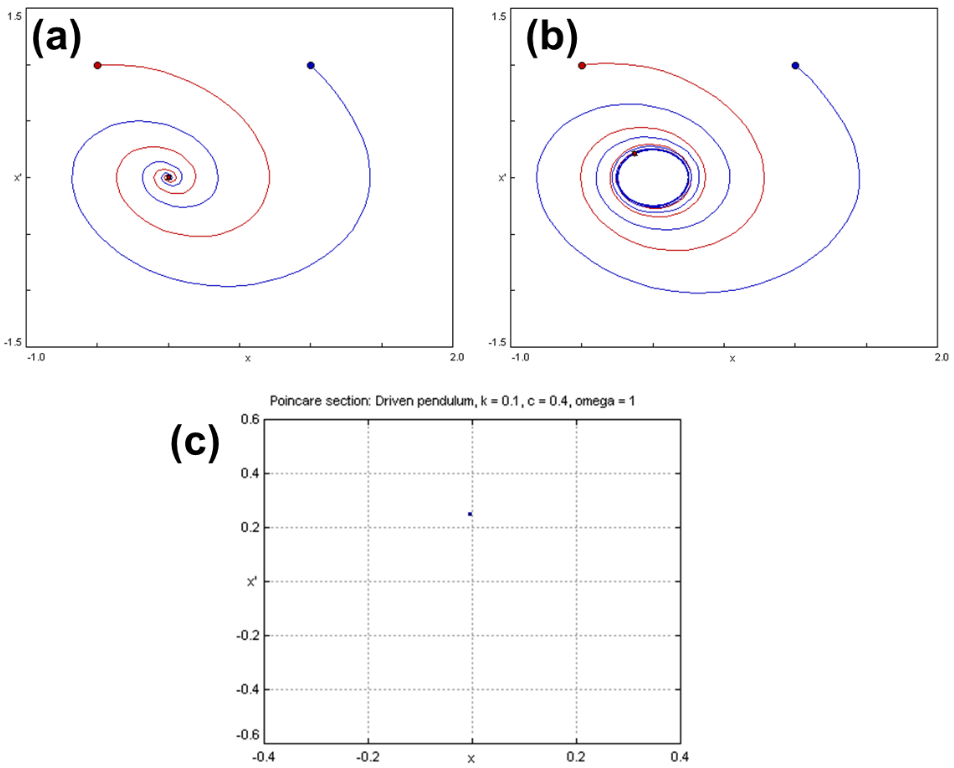

Figure 9.1: (a) Phase space trajectories for a undriven damped simple harmonic oscillator. Circles indicate the start of the trajectory and triangles indicate the end. Both trajectories shown tend towards stable equilibrium at $x = \dot{x} = 0$. (b) Phase space trajectories for a driven damped simple harmonic oscillator. Both trajectories settle into a periodic orbit after transients have died away. (c) A Poincaré diagram for the driven damped harmonic oscillator.

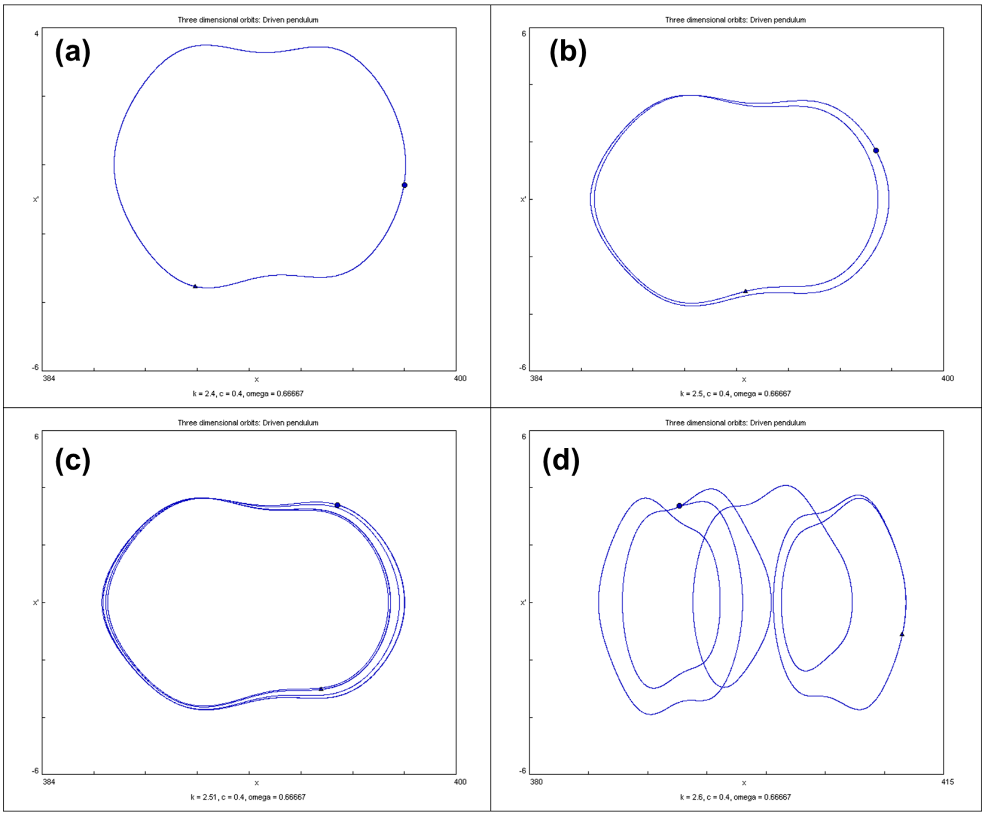

Figure 9.2: Phase trajectories for a damped pendulum for different values of the driving amplitude (a) $F = 2.40$, (b) $F = 2.50$, (c) $F = 2.51$ and (d) $F = 2.60$.

transients have a chance to die out. For values of F up to about 2.40 one sees a the oscillator settles into a stable orbit in a similar way to that observed for the simple harmonic oscillator. However, for $F = 2.50$ the trajectory is qualitatively different, tracing out an orbit in phase space which circles the origin twice and is periodic with a time period which is twice that of the driving force. A very small further increase in F to 2.51 causes the period of the orbit to double again to four times that of the driving force. The final panel of the figure shows that for $F = 2.60$ there is no discernable periodicity of the oscillators motion in phase space.

The period doubling observed above is remarkably reminiscent of what we saw in our exploration of the logistic equation. As the amplitude of the force is increased we see repeated period doubling until at some critical value of the amplitude the period becomes infinite, i.e. there is no longer any periodicity. In this regime the trajectory in phase space is observed to be very sensitive to the initial conditions. The consequence of this is that although the equation of motion is entirely deterministic, its long term behaviour is

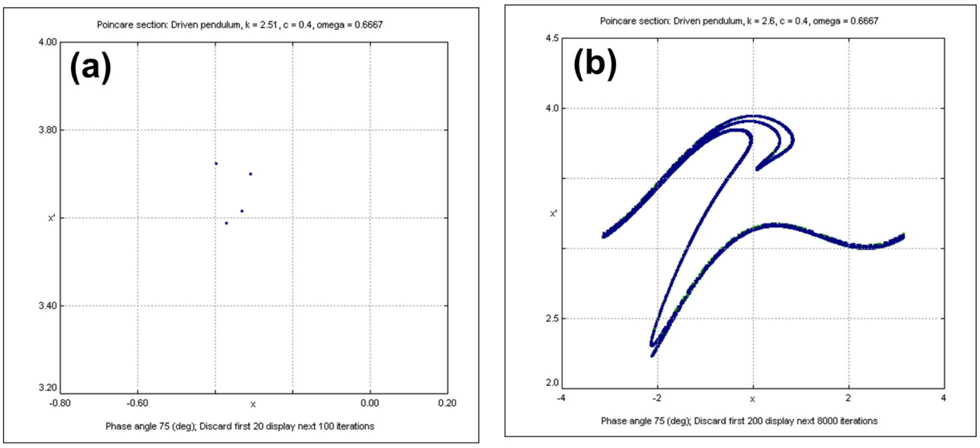

Figure 9.3: Poincaré diagrams for damped pendulum for (a) $F = 2.51$ and for (b) $F = 2.60$.

unpredictable (i.e. the butterfly effect). These observations suggest that the system is chaotic. This becomes more clear if one examines the Poincaré diagrams which provide graphical representations of the periodic attractors. Figure 9.3 shows Poincaré diagrams for $F = 2.51$ where there are four points comprising the attractor and for $F = 2.60$ where there are an infinite number of points (only 8000 are actually shown here). The strange attractor shown in Fig. 9.3b has an incredible level of order and fine structure, similar to what is seen in the bifurcation diagrams for the logistic equation.

The figures in these notes were produced using a Java applet which can be accessed and downloaded online from `http://www.clickrepair.net/chaos/software.html`. This applet is designed to accompany the book *Exploring Chaos: Theory and Experiment, B. Davies* and contains many useful tools for exploring chaos in the logistic equation and in various mechanical oscillator models.

9.5 Summary

The effect of nonlinearity on the dynamics of a mechanical oscillator and the effect of nonlinearity in the logistic equation exhibit some remarkable parallels. In both cases one can characterise the steady state behaviour of the system in terms of attractors (fixed or periodic). As a key parameter (F for the mechanical oscillator and k for the logistic equation) is increased one observes a sequence of period doublings until at some critical value the system no longer exhibits any periodicity. At this point the system is said to be chaotic and is characterised by a strange attractor that has an astounding level of complexity and detail. An important feature of the chaotic regime is that the behaviour of the system is incredibly sensitive to initial conditions, making long term predictions impossible. As the parameter is increased further still one can observe windows of stability within the chaos. Even though chaotic systems are unpredictable they do posses a great deal of order and structure. The fact that such similar features can be observed in very different systems points to the universality of chaos.

9.6 Final comments

A few final points to highlight:

- The simple harmonic oscillator can be solved using Green's functions giving the trajectory of the oscillator at all times in the future
- Nonlinearity even in simple mechanical systems can lead to the emergence of chaotic behaviour
- The path to chaos shows universal features regardless of the detailed nature of the system: period doubling, the butterfly effect, fine structure of the attractors, ...