Homework 5

Due date: February 20, 2023, 12pm

Learning Objectives

Use the inference rules, laws of logic, and proof techniques to formally prove an argument.

Exercises

1. (18 points)
   Given the argument below, determine what went wrong in each of the following proof attempts. If suitable, refer to the step that is incorrect.
   Premises:
   (1) $s \rightarrow (t \vee p)$
   (2) $\neg s \rightarrow (p \vee \neg t)$
   (3) $\neg s \vee \neg t \vee \neg u$
   (4) $(\neg s \rightarrow w) \wedge (\neg t \rightarrow \neg w)$
   (5) $p \rightarrow (\neg u \vee z)$

   Claim: $\neg z \rightarrow \neg u$

   a) Proof by contradiction

Step 1	Assume $\neg z \rightarrow \neg u$ does not hold.
Step 2	The assumption $\neg(\neg z \rightarrow \neg u)$ can be transformed as follows: $\neg(\neg z \rightarrow \neg u)$ $\Leftrightarrow \neg(z \vee \neg u)$ due to implication law and double negation $\Leftrightarrow \neg z \wedge u$ (6) due to De Morgan's law and double negation
Step 3	Applying simplification to (6), we obtain (7) $\neg z$ (8) u
Step 4	We apply disjunctive syllogism to (3) and (8) to deduce (9) $\neg s \vee \neg t$
Step 5	Applying simplification to (9), we can deduce (10) $\neg s$ (11) $\neg t$
Step 6	Applying simplification to (4), we can deduce (12) $\neg s \rightarrow w$ (13) $\neg t \rightarrow \neg w$
Step 7	From (10), (12), and modus ponens, it follows (14) w
Step 8	From (11), (13), and modus ponens, it follows (15) $\neg w$
Step 9	We have derived the contradiction (14) and (15). Therefore, your assumption that $\neg z \rightarrow \neg u$ does not hold is false. That means, $\neg z \rightarrow \neg u$ holds.

Step 3 is incorrect – from simplification, we only obtain $\neg z$, but step 3 also denotes that we obtain u, which is incorrect.

b) Proof by cases

Step 1 Case 1: s and t are both true

Step 2 In this case, the following holds:
(6) s
(7) t

Step 3 We can apply the disjunctive syllogism to (6) and (3) to deduce
(8) $\neg t \vee \neg u$

Step 4 We apply disjunctive syllogism to (7) and (8) to deduce
(9) $\neg u$

Step 5 Using addition and (9), we deduce
(10) $\neg u \vee z$

Step 6 Due to double negation and commutative law, (10) is logically equivalent to
(11) $\neg\neg z \vee \neg u$

Step 7 Due to the implication law, the following holds
(11) $\Leftrightarrow$ $\neg z \rightarrow \neg u$
Therefore, the claim $\neg z \rightarrow \neg u$ holds in Case 1.

Step 8 Case 2: s and t are both false

Step 9 In this case, the following holds:
(12) $\neg s$
(13) $\neg t$

Step 10 Applying simplification to (4), we can deduce
(14) $\neg s \rightarrow w$

Step 11 Applying modus ponens to (12) and (14), we deduce
(15) w

Step 12 Applying simplification to (4), we can deduce
(16) $\neg t \rightarrow \neg w$

Step 13 Applying modus ponens to (13) and (16), we deduce
(17) $\neg w$

(15) and (17) is a contradiction. Therefore, Case 2 cannot occur and nothing needs to be shown in this case.
For case 1, there appears to be an issue in step 5 where addition yields $\neg u \vee z$, however it is unknown where the z is derived from, since it is not one of the premises or the case statements.

Case 2 appears to have an issue in step 10 where simplification is applied to step 4, which would be valid if it were stated that the commutative law was applied beforehand.

Does not check all cases -> where s = T, t = F OR s = F, t = T

c) Proof by contradiction

Step 1 Assume $\neg z \rightarrow \neg u$ does not hold.

Step 2 The assumption $\neg(\neg z \rightarrow \neg u)$ can be transformed as follows:
$\neg(\neg z \rightarrow \neg u)$
$\Leftrightarrow \neg(z \vee \neg u)$ due to implication law and double negation
$\Leftrightarrow \neg z \wedge u$ (6) due to De Morgan's law and double negation

Step 3 Applying simplification to (6), we obtain
(7) $\neg z$
(8) u

Step 4 We apply disjunctive syllogism to (3) and (8) to deduce
(9) $\neg s \vee \neg t$ since we are doing proof by cases, we must do each statement and prove it separately since there is an OR operator

Step 5 Since $\neg s \vee \neg t$ has to hold, we distinguish the case $\neg s$ and the case $\neg t$.

Step 6 Case $\neg s$ (this is true)

Step 7 Applying simplification to (4), we can deduce
(10) $\neg s \rightarrow w$

Step 8 From (10) and the case $\neg s$ and modus ponens, it follows
(11) w

Step 9 Case $\neg t$ (this is true)

Step 10 Applying simplification to (4), we can deduce
(12) $\neg t \rightarrow \neg w$

Step 11 From (12) and the case $\neg t$ and modus ponens, it follows
(13) $\neg w$

We have derived the contradiction (11) and (13). Therefore, the assumption that $\neg z \rightarrow \neg u$ does not hold is false. That means, $\neg z \rightarrow \neg u$ holds.

Step 12 Step 2 is incorrect because double negation is not applied to the first part of the step. There is also the possibility that step 5 is correct, since a statement $\neg s \vee \neg t$ could mean that either $\neg s$ is true, $\neg t$ is true, or both are true (though, that's what proof by cases shows anyways).

2. (10 points)
Use a proof by contrapositive to prove the following argument
Premises:
(1) $t \vee \neg w$
(2) $u \vee w \vee \neg p$
(3) $t \vee u \rightarrow \neg s$
Claim: $p \rightarrow \neg s$

Assuming $\neg q$, prove $\neg p$ -> for this problem, assume $\neg(\neg s)$ and prove $\neg p$

1. double negation ⇔ s (4)
2. modus tollens to (3), (4) ⇔ $\neg (t \vee u)$ (5)
3. de morgan's to (5) ⇔ $\neg t \vee \neg u$ (6)
4. disjunctive syllogism to (1), (2) ⇔ $u \vee \neg p$ (7)
5. commutative law & simplification to (6) ⇔ $\neg u$ (8)
6. disjunctive syllogism to (8), (7) $\neg p$

Since assuming $\neg(\neg s)$ we have proven $\neg p$, then $p \rightarrow \neg s$ holds.

3. (12 points)
   An island contains two types of people: knight who always tell the truth and knaves who always lie. Two natives A and B approach you but only A speaks.
   A says: Both of us are knaves.
   Write a formal proof to show that A is a knave, and B is a knight. Specify the proof technique that you apply.

   (1) Knights always tells the truth
   (2) Knaves always lie
   (3) A and B are knaves.
   Claim: A is a knave and B is a knight

   Proof by contrapositive: Assume that B is NOT a knight (prove that A is NOT a knave).
   1. If B is not a knight, then B is a knave.
   2. If B is a knave, then that means that A is telling the truth.
   3. If A is telling the truth, then A is a knight, since knights always tell the truth.
   4. Since A is a knight, then A is NOT a knave.
   5. Since A is not a knave given that B is not a knight, then A is a knave and B is a knight holds.

Submission

Upload your solution as a PDF file to the course website.

Questions:
Can we get p is true AND q is true from simplification, even though only p is denoted?

Simplification

$$\frac{p \wedge q}{p}$$

The simplification in step 3 and step 7 are confusing because step 3 gets p is true and q is true, but step 7 only gets that p is true
For simplification, we can infer that p is true AND q is true (by themselves), since the AND operator means that they are both true.

Step 3 Applying simplification to (6), we obtain
(7) $\neg z$
(8) u

Step 4 We apply disjunctive syllogism to (3) and (8) to deduce
(9) $\neg s \vee \neg t$

Step 5 Since $\neg s \vee \neg t$ has to hold, we distinguish the case $\neg s$ and the case $\neg t$.

Step 6 Case $\neg s$

Step 7 Applying simplification to (4), we can deduce
(10) $\neg s \rightarrow w$

For addition, does q need to be one of the premises in order to infer p v q? The example uses addition, but z is not one of the premises.

For addition, as long as p is true, we can pull any premise/statement and pair it w/ p and long as the OR operator is used.

(9) $\neg u$

Step 5 Using addition and (9), we deduce
(10) $\neg u \vee z$

Addition

$$\frac{p}{p \vee q}$$