Math 109 Course Packet

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Contents

Part I

R Review Material

R.1 Rational Exponents & Radicals

Fact 1

For any real number **a** *the following holds:*

$$a^{\frac{m}{n}} = \sqrt[\underline{\quad}]{a^{\underline{\quad}}}$$

Exercise R.1. Rewrite $3^{\frac{2}{3}}$ as a radical.

Exercise R.2. Rewrite $\sqrt[5]{16}$ as a rational exponent.

R.1.1 Simplifying Radicals

We have two techniques on how to simplify radicals. One is called the *factor tree method*, and the other involves looking at powers of primes.

Exercise R.3 (Factor Tree Method)**.** Simplify $\sqrt{72}$.

Exercise R.4 (Powers of Primes Method)**.** Simplify $\sqrt{72}$.

Exercise R.5. Simplify $\sqrt[3]{500}$.

R.2 Polynomials

R.2.1 Basics

Definition R.6. Let n be a positive integer (not a fraction), and $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ be real numbers with $a_n \neq 0$. The following is called a *polynomial*:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

- We call $a_0, a_1, \ldots, a_n$ the __________,
- a_n is specifically called the __________,
- $a_n x^n$ is called the __________,
- a_0 is called the __________,
- and n is the __________ of the polynomial.

Names based on the number of terms

# of Terms	Name
1	
2	
3	
4 or more	

Names based on the degree

Degree	Name
0th	
1st	
2nd	
3rd	
4th	
5th	

Exercise R.7. What special name(s) would the following polynomials have?

- $3x^2 + x - 1$ Name: __________
- $7 + -4x^3$ Name: __________

R.2.2 Adding and Subtracting

When adding or subtracting polynomials it boils down to combining the coefficients of *like-terms* that is terms that have the same power of x.

Exercise R.8. Add the following two polynomials together:

$$(3x^2 + x + 1) + (3 - 6x - 2x^2)$$

Exercise R.9. Compute the following subtraction of polynomials:

$$(1 - x - 5x^2) - (-3x^2 + 4x - 3)$$

R.2.3 Multiplying

The basic premise of multiplication of polynomials is multiplying coefficients and adding powers.

$$(Ax^n)(Bx^m) = ABx^{n+m}$$

Exercise R.10. Compute the following multiplication:

$$(3x)(2x)$$

Definition R.11. When multiplying two binomials we use an acronym to help remember which terms to multiply together, which is **F.O.I.L.**.

F____________ **O**____________ **I**____________ **L**____________

Exercise R.12. Multiply the following binomials:

$$(2x+7)(3x-4)$$

Exercise R.13. Simplify the following:

$$(5x+2)^2$$

Exercise R.14. Multiply the following binomials:

$$(3x+1)(3x-1)$$

Exercise R.15. Multiply the following:

$$(5x-7)(6x^2-5x+3)$$

Exercise R.16. Multiply the following:

$$(3x^2-4)(6x^2-7x+2)$$

R.3 Factoring

R.3.1 Greatest Common Factor

Definition R.17. Given a collection of terms, the *Greatest Common Factor* (or *GCF*) is the largest factor that divides everything in the collection.

Our process will involve breaking the problem into finding the GCF of the numbers and the GCF of the variables, and then mulitplying them together.

Fact 2

Given two real numbers a and b with $a \leq b$, the GCF of x^a and x^b is ____________

Exercise R.18. Find the Greatest Common Factor (GCF) of $8z^5$ and $36z^3$

We can use this to do the most basic type of factoring for polynomials which is to factor out the GCF of all the terms in the polynomial (if there is one).

Exercise R.19. Factor out the GCF from

$$45x^5y^7 + 33x^3y^3 + 78x^2y^4$$

Exercise R.20. Factor out the GCF from

$$87x^8y^3 + 18x^5y + 21x^4y^2$$

R.3.2 Factoring by grouping

When there is not a GCF to factor out, we need different methods to factor polynomials. We will mainly focus on factoring trinomials and specifically quadratics which are of the form $ax^2 + bx + c$.

Leading coefficient $a = 1$

In this situation, our quadratic will look like $x^2 + bx + c$. Our method for factoring these will be to find two numbers m and n such that $m + n = b$ and $m \cdot n = c$. We'll see what we do with this m and n in the following exercise.

Exercise R.21. Factor the following polynomial

$$y^2 + 6y + 5$$

Exercise R.22. Factor the following polynomial

$$x^2 + 8x - 20$$

Leading coefficient $a \neq 1$

The process for $a \neq 1$ turns out to be almost exactly the same. We still want an m and n such that $m + n = b$, but now we have $m \cdot n = a \cdot c$.

Exercise R.23. Factor the following polynomial:

$$4x^2 + 19x + 21$$

Exercise R.24. Factor the following polynomial:

$$-6x^2 + 17x - 7$$

R.3.3 Special polynomials

Definition R.25. The following is called a *perfect square trinomial* and factors as follows:

$$A^2 + 2AB + B^2 = \underline{\hspace{2cm}}$$

Similarly,

$$A^2 - 2AB + B^2 = \underline{\hspace{2cm}}$$

Exercise R.26. Factor the following polynomial:

$$16x^2 + 24x + 9$$

Exercise R.27. Factor the following polynomial:

$$25x^2 - 90x + 81$$

Definition R.28. The following is called a *difference of squares* and factors as follows:

$$A^2 - B^2 = (A + B)(A - B)$$

Exercise R.29. Factor the following:

$$x^2 - 9$$

Exercise R.30. Factor the following:

$$16x^2 - 25y^2$$

Exercise R.31. Factor the following:

$$16x^4 - 81$$

1 Linear & Rational Equations

1.1 Linear equations

Definition 1.1. An equation in one variable is called a *linear equation* if the highest power of the variable is __________

1.1.1 Determining type

When dealing with any type of equation, they come in 3 forms.

1) Identity: An equation that is always true no matter the value of the variable.

   Examples:________________________________

2) Inconsistent: An equation that is always false no matter the value of the variable.

   Examples:________________________________

3) Conditional: An equation that is only true for specific values of the variable.

   Examples:________________________________

Exercise 1.2. Determine what type of equation, and solve if it is consistent.

$$9 - 10x + 7x = 2 - 3x + 6$$

Exercise 1.3. Determine what type of equation, and solve if it is consistent.

$$9x - 2 - 2x = 5 - 7 + 7x$$

Exercise 1.4. Determine what type of equation, and solve if it is consistent.

$$3x - 1 = 11$$

1.1.2 Solving linear equations

Exercise 1.5. Solve the following linear equation:

$$6(x + 2) - 2 = -6 - 3(x - 5)$$

Definition 1.6. Given a collection of denominators, the *Least Common Denominator* (or *LCD*) is the smallest term that is divisible by all of the denominators.

Note. The LCD needs to be at least as big as the largest denominator. This may seem strange since it is the "Least" which we usually associate with smallest.

Exercise 1.7. Solve the following linear equation:

$$\frac{9x}{4} + \frac{5}{2} = \frac{-1}{2}$$

1.2 Rational Equations

Definition 1.8. A *rational equation* is an equation that involves variables in denominators.

When solving rational equations, we will follow a similar process to the previous problem by using the Least Common Denominator. It will involve denominators with variables.

Fact 3

Given two real numbers a and b with $a \leq b$, the LCD of x^a and x^b is _________

Exercise 1.9. Solve the following equation:

$$\frac{7}{2x} + \frac{5}{6x} = \frac{7}{4}$$

Exercise 1.10. Solve the following equation:

$$-\frac{2}{3x} - \frac{5}{6x} = \frac{3}{2}$$

Exercise 1.11. Solve the following equation:

$$\frac{-7}{x} = \frac{-5}{x+4}$$

Exercise 1.12. Solve the following equation:

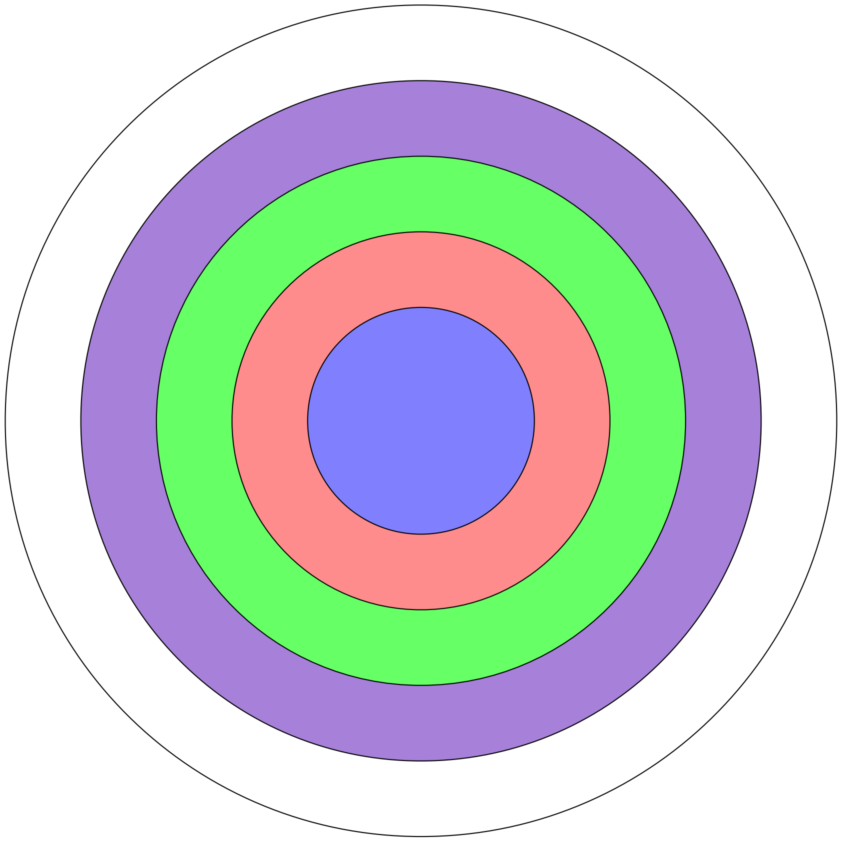
$$\frac{-1}{x+3} - \frac{6}{x+1} = \frac{-2}{x^2+4x+3}$$

Exercise 1.13. Solve the following equation:

$$\frac{-3}{x-2} + \frac{5}{x-4} = \frac{-1}{x^2-6x+8}$$

2 Complex Numbers

2.1 The Basics



Question. How do we deal with $\sqrt{-1}$?

Definition 2.1. The *imaginary constant* is defined to be $i = \sqrt{-1}$

Note. This will now allow us to deal with square roots with negative values. If $n > 0$, then

$$\sqrt{-n} = \underline{\hspace{6cm}}$$

Exercise 2.2. Simplify $\sqrt{-75}$.

Note. We can simplify powers of i as follows:

$$\begin{aligned} i &= i \\ i^2 &= \underline{\hspace{4cm}} \\ i^3 &= \underline{\hspace{4cm}} \\ i^4 &= \underline{\hspace{4cm}} \\ i^5 &= \underline{\hspace{4cm}} \\ i^6 &= \underline{\hspace{4cm}} \\ i^7 &= \underline{\hspace{4cm}} \\ &\vdots \end{aligned}$$

Exercise 2.3. Simplify i^{62}

Exercise 2.4. Simplify i^{19}

Note. We can also continue the pattern in reverse to get negative powers of i:

$$\begin{aligned} i^0 &= \underline{\hspace{4cm}} \\ i^{-1} &= \underline{\hspace{4cm}} \\ i^{-2} &= \underline{\hspace{4cm}} \\ i^{-3} &= \underline{\hspace{4cm}} \\ &\vdots \end{aligned}$$

Definition 2.5. The *complex numbers* (denoted by $\mathbb{C}$) is the set of all numbers of the form $a + bi$ where a and b are real numbers and i is the imaginary constant.
We call $a + bi$ the standard form of a complex number, calling a the real part and b the imaginary part.

2.2 Addition & Subtraction

Exercise 2.6. Compute the following:

$$(3 + 2i) + (4 - 7i)$$

Exercise 2.7. Compute the following:

$$(5 + 7i) - (-3 + 2i)$$

2.3 Multiplication

Exercise 2.8. Compute the following:

$$3(7 - 5i)$$

Exercise 2.9. Compute the following:

$$7i(2+4i)$$

Exercise 2.10. Compute the following:

$$(-3+7i)(2+5i)$$

Exercise 2.11. Compute the following:

$$(3+2i)(3-2i)$$

Definition 2.12. The two complex numbers $a+bi$ and $a-bi$ are called *complex conjugates.* Their product is always a^2+b^2 which is real.

2.4 Division

In order to divide complex numbers we will need to use the complex conjugate of the complex number we are dividing by (the denominator).

Example 2.13. Give the complex conjugates for the following complex numbers:

- $5-2i \longleftrightarrow$ ___________
- $7+11i \longleftrightarrow$ ___________
- $-4-2i \longleftrightarrow$ ___________

Exercise 2.14. Divide the following and write your answer in standard form.

$$\frac{16+22i}{3+i}$$

Exercise 2.15. Divide the following and write your answer in standard form.

$$(17 + 5i) \div (-2 + i)$$

3 Solving Quadratic Equations

3.1 Solving by Factoring

Fact 4 (*The Zero Product Rule*)

If $AB = 0$, then ________________

Exercise 3.1. Solve the following for x:

$$x^2 + 4x - 21 = 0$$

Exercise 3.2. Solve the following for y:

$$y^2 + 8y + 12 = 0$$

Exercise 3.3. Solve the following for x:

$$9x^2 - 26x - 3 = 0$$

3.1.1 The Square Root Property

Exercise 3.4. Solve $x^2 = 36$ for x.

Exercise 3.5. Solve $x^2 = 81$ for x.

3.2 Completing the Square

This will be using the square root property from above and perfect square trinomials (see Definition R.25). We will change our quadratics as perfect square trinomials, so that we can take a square root to simplify the problem.

Exercise 3.6. If $x^2 + 4x - 45 = 0$, solve for x.

Exercise 3.7. If $x^2 - 12x + 11 = 0$, solve for x.

3.3 The Quadratic Formula

Now completing the square works to solve any quadratic equation, but what if we tried to complete the square on the general quadratic?

Example 3.8. If $ax^2 + bx + c = 0$, solve for x.

Proposition 3.9 (*The Quadratic Formula*)

If $ax^2 + bx + c = 0$, *then*

$$x =$$

Definition 3.10. For a quadratic equation $ax^2 + bx + c = 0$, the *discriminant* is $b^2 - 4ac$, and it tells us about the types of solutions to said equation.
There are 3 (really 4) cases:

- $b^2 - 4ac < 0 \Rightarrow$ ____________________
- $b^2 - 4ac = 0 \Rightarrow$ ____________________
- $b^2 - 4ac > 0$ and
  - it IS a perfect square $\Rightarrow$ ____________________________
  - it IS NOT a perfect square $\Rightarrow$ _____________________

Exercise 3.11. Find the discriminant and solve $-4x^2 + 5x + 3 = 0$.

Exercise 3.12. Find the discriminant and solve $4n^2 - 20n + 25 = 0$.

4 More Equations and Some Applications

4.1 Using the Greatest Common Factor

Exercise 4.1. Solve $4x^4 = 16x^2$

Exercise 4.2. Solve $9x^4 = 81x^2$

4.2 Factoring by Grouping

Exercise 4.3. Solve $x^3 - 5x^2 - 4x + 20 = 0$.

Exercise 4.4. Solve $x^3 - 8x^2 - 9x + 72 = 0$.

Exercise 4.5. Solve $x^3 + 7x^2 - x - 7 = 0$.

4.3 Polynomials in Quadratic Form

Exercise 4.6. Solve $(x-5)^2 + (x-5) - 2 = 0$.

Exercise 4.7. Solve $(x-7)^2 - 7(x-7) + 10 = 0$.

4.4 Absolute Value Equations

Definition 4.8. The *absolute value* of a number n, written $|n|$, is the distance from n to 0 on the number line (i.e. it is the positive version of n).

Example 4.9. If $|x| = 5$, then either $x = 5$ or $x = -5$

This means that absolute value equations are just two problems in one. A positive equation and a negative equation.

Exercise 4.10. If $|-12x - 7| - 8 = 2$, solve for x.

Exercise 4.11. Solve $2|x + 1| + 7 = 10$ for x.

Exercise 4.12. Solve the following for x:

$$|3x - 5| - 11 = -9$$

4.5 Radical Equations

Our goal will be to get rid of the radical in the problem.

Exercise 4.13. Solve the following for x:

$$\sqrt{8x - 23} + 2 = x$$

Exercise 4.14. Solve:

$$\sqrt{-2x+8} = x$$

Exercise 4.15. Solve:

$$\sqrt{-x+4} + \sqrt{5x+1} = 3$$

Exercise 4.16. Solve:

$$\sqrt{-2x+2}+\sqrt{x+2}=-3$$

4.6 Rational Exponent Equations

Recall that the square root is the same as the ______ power, so in general we take

$$\left(\left(\quad\right)^{m/n}\right)^{\underline{\quad}/\underline{\quad}}$$

Exercise 4.17. Solve $x^{7/6}=128$ for x.

Exercise 4.18. Find x when $x^{-3/2}=\dfrac{1}{125}$

Exercise 4.19. Solve for x when $(x-1)^{-4/3} = \dfrac{1}{81}$

Exercise 4.20. Solve the following for x:

$$7x^4 - 189x^{13/4} = 0$$

Exercise 4.21. Solve the following for x:

$$4x^3 - 36x^{7/3} = 0$$

4.7 More Rational Equations

Note. We still need to check what makes our denominators equal to 0

Exercise 4.22. If $\dfrac{6x+5}{x+2} - \dfrac{1}{x} = -\dfrac{2}{x^2+2x}$, find x.

5 Linear, Compound, and Absolute Value Inequalities

We need a shared language to discuss these problems because the answers are not just single numbers they are sets (or collections) of numbers like the following:

Example 5.1. The set of all real numbers greater than or equal to 1.

$$-\infty \longleftrightarrow \infty$$

5.1 Interval Notation

Set Builder Notation	Interval Notation
$\{x \mid a < x < b\}$	________
$\{x \mid a \leq x \leq b\}$	________
$\{x \mid a < x \leq b\}$	________
$\{x \mid a \leq x < b\}$	________
$\{x \mid a \leq x\}$	________
$\{x \mid x < b\}$	________

Exercise 5.2. Write the number line from Example 5.1 in interval notation.

Definition 5.3.

- ◦ _____ – denotes the *union* which means “or”.
- ◦ _____ – denotes the *intersection* which means “and”.

Exercise 5.4. Use interval notation to express the following:
The set of all real numbers greater than -1 and less than 0.

Exercise 5.5. Use interval notation to express the following:
The set of all real numbers greater than -7 and less than or equal to 5.

Exercise 5.6. Use interval notation to express the following:
The set of all real numbers greater than or equal to 4 or less than 2.

5.2 Properties of Inequalities

Addition Property

If $a < b$, then for a real number c ______________.

Multiplication Property

- If $a < b$ and $c > 0$, then _________.
- If $a < b$ and $c < 0$, then _________.

Exercise 5.7. If $x + 10 \leq 4$, solve for x.

Exercise 5.8. Find x when $-5x - 2 \leq 8$.

Exercise 5.9. Solve $-16x + 8 \geq -9x - 13$ for x.

Exercise 5.10. Find x when $-10x + 12 > -8x - 10$.

5.3 Compound Inequalities

Exercise 5.11. Solve $2x - 3 \leq 4x + 3 < 2x + 5$ for x.

Exercise 5.12. Find x if $-4x - 10 < -2x - 8 \leq -4x + 10$.

5.4 Absolute Value Inequalities

Recall that absolute values turn one problem into two!

Exercise 5.13. Solve $|x-6|+2=4$ for x.

With inequalities we either have an "and" or an "or" problem.

Note. Here is a trick to remember:

- And – ____________
- Or – ____________

Exercise 5.14. Solve $|2x+6|<18$ for x.

Exercise 5.15. Solve $-7|x+12| \leq -14$ for x.

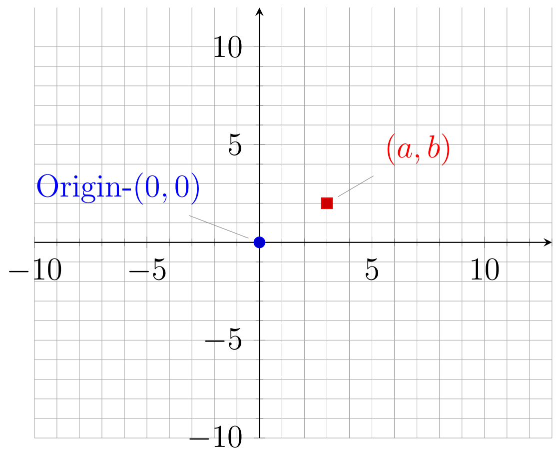
Exercise 5.16. Solve $2|x-12|+1 \leq 23$ for x.

Exercise 5.17. Solve $|x-5| \leq -2$ for x.

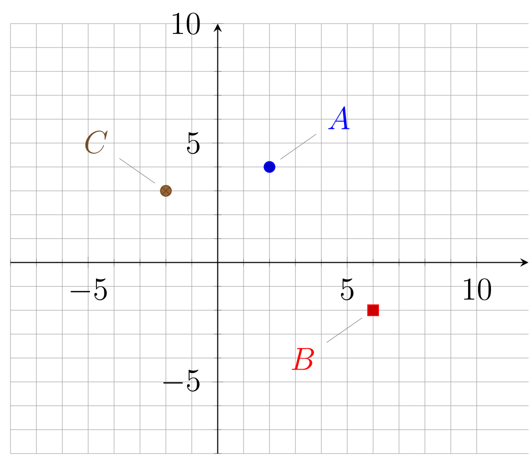
Part II

6 The Rectangular Coordinate System

6.1 The x-y plane

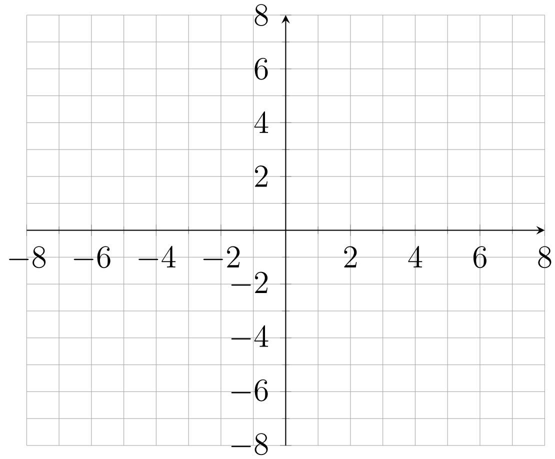


Exercise 6.1. Find the coordinates of the following points:

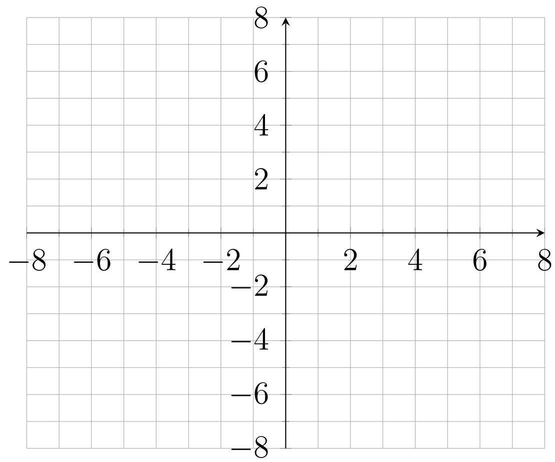


6.2 Graphing Lines

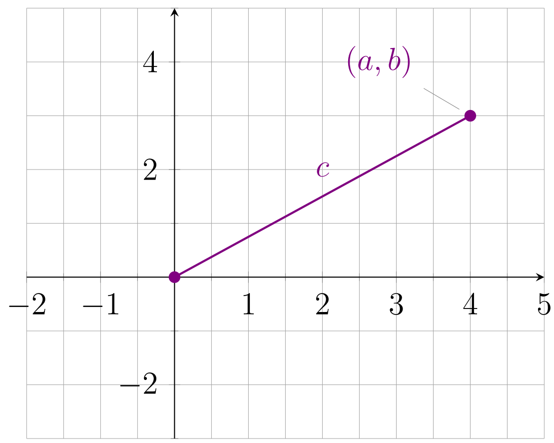
Exercise 6.2. Graph $y = 4x - 5$ on the x-y plane.



Exercise 6.3. Graph $y = \frac{4}{3}x - 2$ on the x-y plane.



6.3 Distances



Question. How would we compute the value of c the length of the purple line?

The Pythagorean Theorem

Theorem 6.4 (*The Pythagorean Theorem*)

Given a right triangle with a hypotenuse of length c and side lengths a and b, then the following holds

$$\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_$$

This means that $c =$ ___________. Which is the distance between the points $(0,0)$ & (a,b). Looking at it a slightly different way we can say that

$$c = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

We can generalize this to the distance between any two points on the plane.

Proposition 6.5 (*The Distance Formula*)

Given two points (x_1, y_1) & (x_2, y_2) in the Cartesian plane, the distance D between these points is as follows:

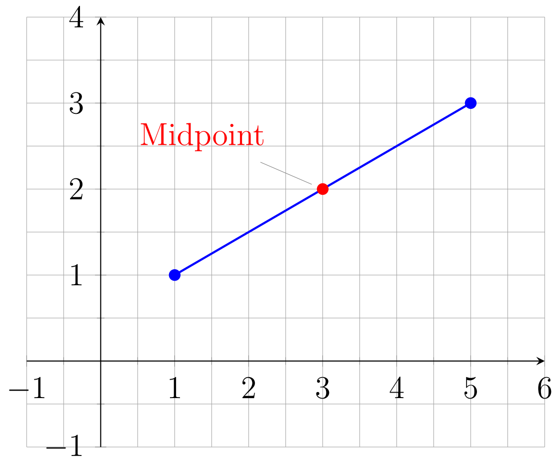
$$D = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Exercise 6.6. Find the distance between $(-6,1)$ & $(5,-1)$.

Exercise 6.7. Find the distance between $(2, 5)$ & $(-4, -5)$.

Exercise 6.8. Find the distance between $(2, -3)$ & $(-4, -7)$.

6.4 Midpoints



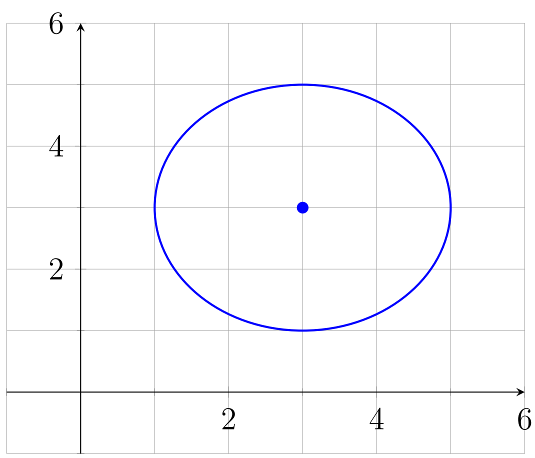
Definition 6.9. The *midpoint* between two points (x_1, y_1) & (x_2, y_2) in the Cartesian plane is computed as follows:

$$\text{Midpoint} = \left(\underline{\hspace{3cm}}, \underline{\hspace{3cm}}\right)$$

Exercise 6.10. Find the midpoint between $(12, 13)$ & $(-11, 7)$.

Exercise 6.11. Find the midpoint between $(8, 7)$ & $(12 - 5)$.

7 Circles



Definition 7.1. A *circle* is a the set of all points (x, y) a distance r from the center (h, k). We call r the _________.

Using the distance formula (see Proposition 6.5), we can come up with an equation for the equation of the a circle as follows

$$\underline{\hspace{4cm}} = \underline{\hspace{0.8cm}}$$
$$\underline{\hspace{3.6cm}} = \underline{\hspace{0.9cm}}$$

We call this the *standard form of a circle*

Exercise 7.2. Write the following in standard form:

$$x^2 - 4x + y^2 - 2y - 31 = 0$$

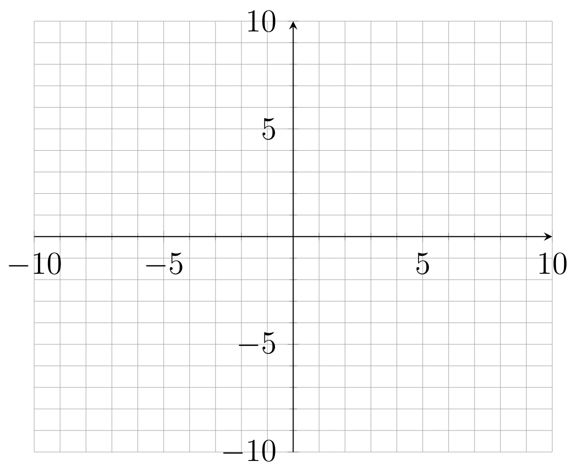
Exercise 7.3. Write $2x^2 + 12x + 2y^2 + 8y - 24 = 0$ in standard form.

Exercise 7.4. Write the following in standard form:

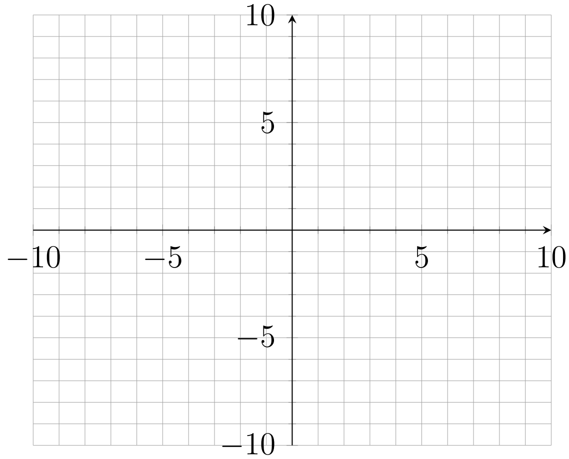
$$3x^2 - 12x + 3y^2 + 6y + 3 = 0$$

7.1 Graphing circles

Exercise 7.5. Graph $(x+4)^2 + (y+2)^2 = 16$.



Exercise 7.6. Graph $(x+1)^2 + (y-1)^2 = 4$.



8 Functions

Definition 8.1. A *relation* is any set of order pairs where the *domain* is the set of _______ values in the ordered pair, and the *range* is the set of _________ values.

Exercise 8.2. Find the domain and range of

$$\{(0, 9.1),\ (10, 6.7),\ (20, 10.7)\}.$$

Definition 8.3. A *function* is a relation where each element in the domain, corresponds to exactly one element of the range (i.e. no domain element can go to two different range elements).

Example 8.4. $\{(1,2),\ (3,4),\ (6,5),\ (8,5)\}$ is a function

Non-Example 8.5. $\{(1,2),\ (3,4),\ (5,6),\ (5,8)\}$ is a NOT function

8.1 Equations as functions

Definition 8.6. We will call ____ the *independent variable* which corresponds to the __________, and we will call ____ the *dependent variable* which corresponds to the ________.

Note. This means that an equation is a functions if every ____ goes to exactly one ____.

Exercise 8.7. Is $2x + y = 6$ a function?

Exercise 8.8. Is $x^2 + y^2 = 1$ a function?

8.2 Function Notation

Example 8.9. We can rewrite the function $y = x^2 - 2x + 7$, in *function notation*, as follows:

$$f(x) = x^2 - 2x + 7$$

We read $f(x)$ as "f of x", so $f(-)$ is "f of $-$"

Function notation is useful if in evaluation.

Exercise 8.10. For $f(x) = x^2 - 2x + 7$, find $f(-5)$.

Exercise 8.11. For $f(x) = x^2 - 2x + 7$, find $f(2)$.

Exercise 8.12. For $f(x) = x^2 - 2x + 7$, find $f(x + 4)$.

Solving using function notation

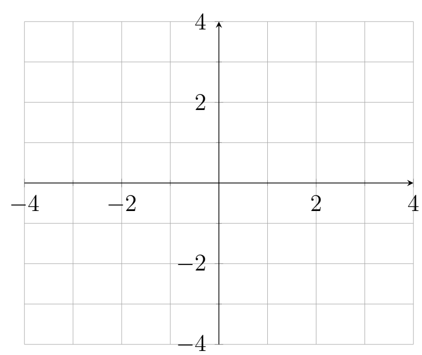
Exercise 8.13. For $f(a) = 4a^2 - 29a$, solve $f(a) = -30$.

Exercise 8.14. Find when $h(x) = -12$ if $h(x) = x^2 - 7x$.

8.3 Graphing Functions

Recall that "$f(x) =$" $\leftrightarrow$ "$y =$", so the output of a function is the y-value.

Exercise 8.15. Graph $f(x) = 2x + 1$.

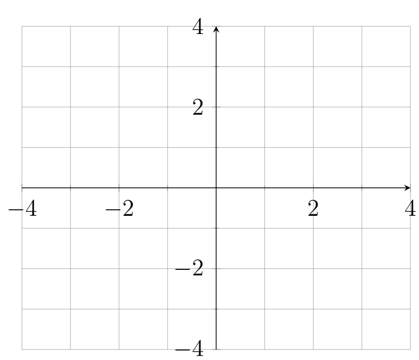


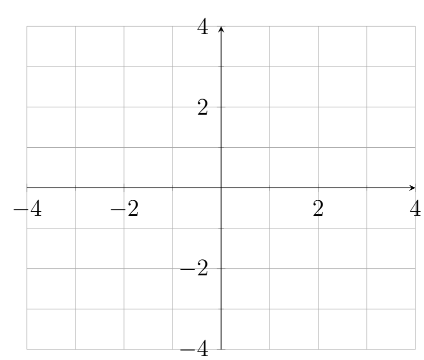
Note. Every function has a graph, but not every graph has a function.

8.3.1 Vertical Line Test

Proposition 8.16 (*Vertical line test*)

If any vertical line crosses a graph more than once, it does not represent a function.



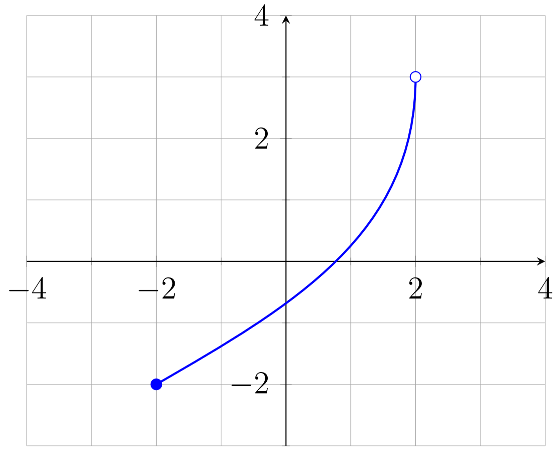


Here are some other symbols to be familiar with:

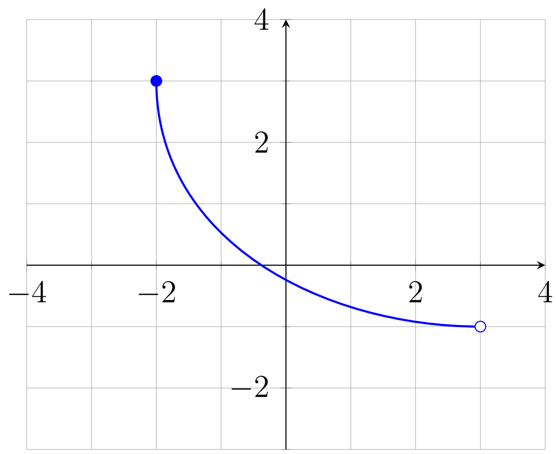
- ___________ - ends at that point, but does NOT include it.
- ____________ - ends at that point AND includes it.
- ________ - the graph continues in that direction forever.

8.4 Finding Domain and Range from a Graph

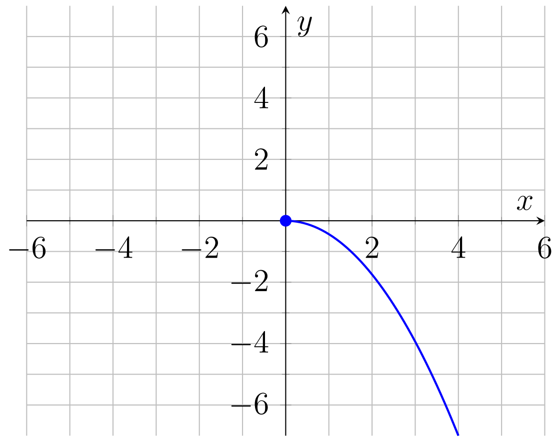
Exercise 8.17. Find the domain and range of the following:



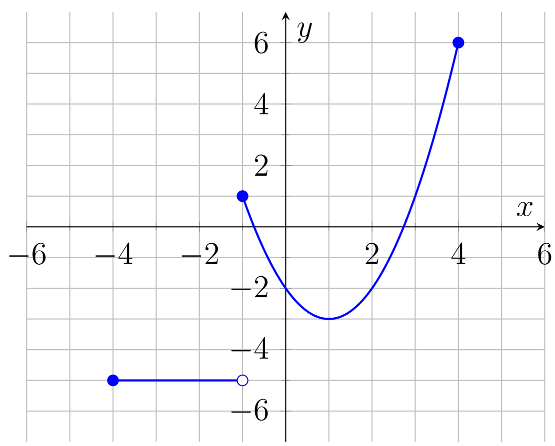
Exercise 8.18. Find the domain and range of the following:



Exercise 8.19. Find the domain and range of the following:



Exercise 8.20. Find the domain and range of the function shown below:



8.5 Finding domains from equations

For some of these questions instead of asking what numbers work, the easier question is what numbers "don't work".

Exercise 8.21. Find the domain of

$$f(x) = \frac{1}{(x+3)(x-2)}$$

Exercise 8.22. Find the domain of

$$g(x) = \sqrt{x+2}$$

Exercise 8.23. Find the domain of

$$q(x) = \frac{1}{\sqrt{8-9x}}$$

9 Linear Functions

9.1 Slope

Definition 9.1. The *slope* of a line is the following ratio

$$m = \underline{\hspace{2cm}} = \underline{\hspace{4cm}}$$

If (x_1, y_1) & (x_2, y_2) are two points on a line, then the slope is

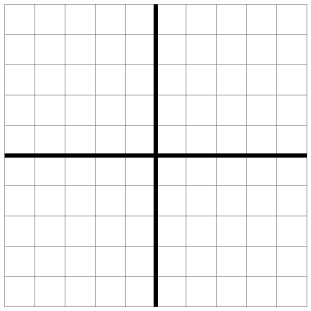
$$m = \underline{\hspace{3cm}}$$

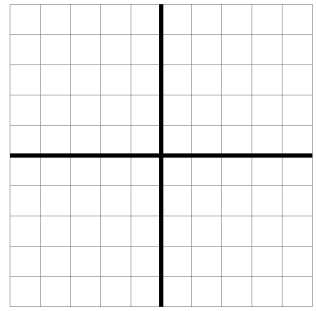
Exercise 9.2. Find the slope of the line between $(-3, 4)$ & $(-4, -2)$.

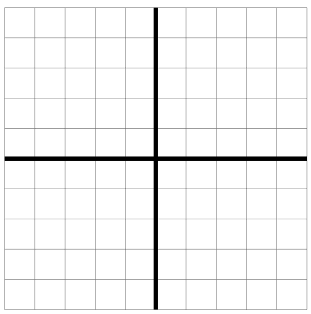
Exercise 9.3. Find the slope of the line between $(4, -2)$ & $(-1, 5)$.

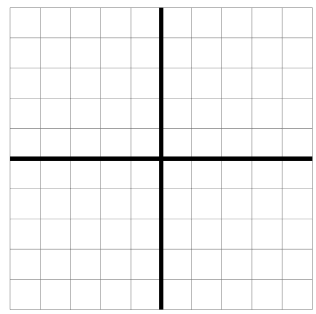
Exercise 9.4. Find the slope of the line between $(7, -3)$ & $(1, 11)$.

9.2 Graphs

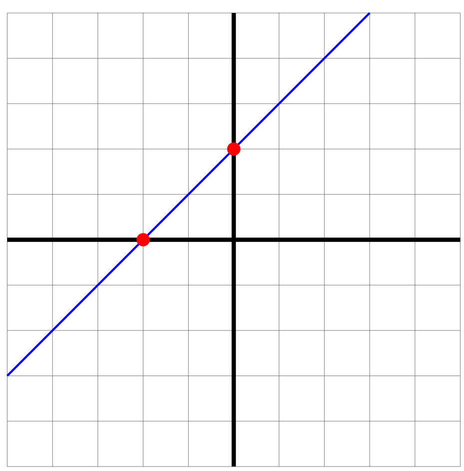








9.2.1 Slope-Intercept form



Definition 9.5 (Slope-intercept form)**.** A line with slope m and y-intercept b has the equation

$$y = \_\_\_\_\_\_\_\_\_\_$$

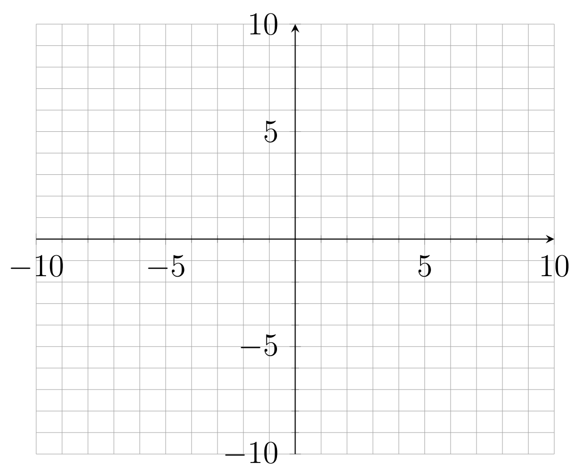
Exercise 9.6. What are the slope and y-intercept of $y = 3x - 4$?

Exercise 9.7. What is the x intercept of $f(x) = 2x - 8$?

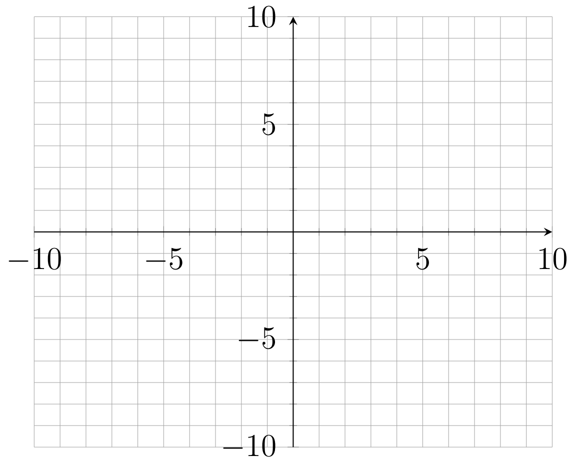
Exercise 9.8. Find the slope-intercept form of the line with slope 7 and y-intercept 2.

9.3 Graphing Lines

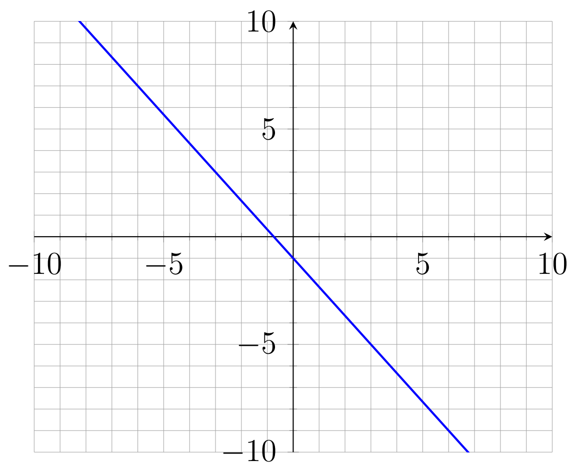
Exercise 9.9. Graph $f(x) = 2x + 5$



Exercise 9.10. Graph $h(x) = -3x - 6$



Exercise 9.11. Find $y = mx + b$ for the following:



10 More on Lines

10.1 Finding the equation of a line

In general, to find the equation of a line we need the slope and any point on the line.

Definition 10.1 (Point-slope form)**.** If we have a point (x_1, y_1) on a line with slope m, then the *point-slope form* is

$$\underline{\hspace{3cm}} = \underline{\hspace{4cm}}$$

Exercise 10.2. Find the equation of a line passing through $(2, -5)$ with slope $m = 6$ in slope-intercept form.

Exercise 10.3. Find the equation of a line passing through $(3, 4)$ with slope $m = -5$ in slope-intercept form.

Exercise 10.4. Find the equation of a line passing through $(3, 4)$ & $(-1, -6)$ in slope-intercept form.

10.2 Standard Form

Definition 10.5 (Standard form)**.** If A, B, and C are integers (with $A > 0$), then the standard form of a line is

$$\underline{\hspace{3cm}} = \underline{\hspace{1.2cm}} \quad \text{or} \quad \underline{\hspace{4cm}} = \underline{\hspace{1.1cm}}$$

Exercise 10.6. Find the standard form of the line passing through $(-8, -1)$ & $(-1, -2)$.

Exercise 10.7. Find the standard form of the line passing through $(-7, -4)$ & $(-2, 3)$.

10.3 Horizontal & Vertical Lines

Note. A horizontal line has a slope of ____. This means that their equation is

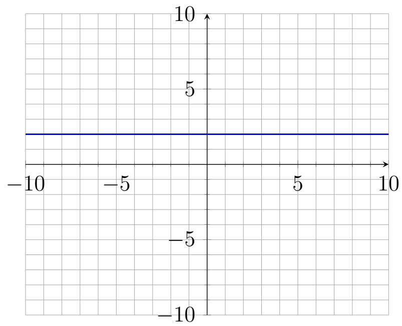
$$y = \underline{\hspace{2.2cm}} = \underline{\hspace{2cm}} = \underline{\hspace{1.1cm}}$$

Every horizontal line can be written this way, which makes sense because the __________ never changes.

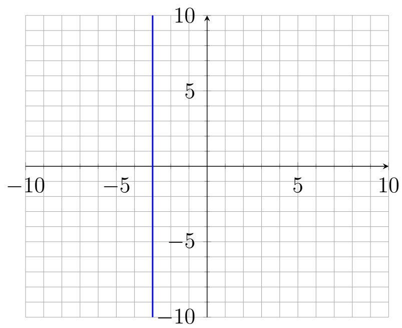
Note. In a similar way, the __________ never changes for vertical lines, their equation looks like

$$\underline{\hspace{1.1cm}} = \underline{\hspace{1.1cm}}$$

Exercise 10.8. What is the equation of the following line?



Exercise 10.9. What is the equation of the following line?



10.4 Parallel and Perpendicular Lines

Definition 10.10. Two lines are called *parallel* (denoted by $\|$) if they ________________

Proposition 10.11

- *Two distinct non-vertical lines are parallel if and only if* __________________________.
- *If two vertical lines are* __________, *then they are parallel.*

Exercise 10.12. Find the slope-intercept form of a line passing through $(-2, 5)$ and parallel to $y = 3x + 1$.

Exercise 10.13. Find the line parallel to $y = 5x - 9$ passing through $(4, -2)$ in slope-intercept form.

Definition 10.14. Two lines are called *perpendicular* (denoted by $\perp$) if they ___________________.

Proposition 10.15

- *Two distinct non-vertical lines are perpendicular if and only if the product of their slopes is ______.*
- *Horizontal and vertical lines are _________ perpendicular.*

Note. Another way to understand the first point is if the one slope is $m = \frac{a}{b}$, then the perpendicular slope is $m_{\perp} =$ ______

Exercise 10.16. Find a line through $(-2, -6)$ and perpendicular to $y = -\frac{1}{3}x + 4$ in slope-intercept form.

Exercise 10.17. Find the equation of the line perpendicular to $y = \frac{1}{4}x - 4$ and passing through $(-8, 5)$ in slope-intercept form.

Exercise 10.18. Are the following two lines parallel, perpendicular, or neither?

$$y = 3x - 5 \quad y + \frac{1}{3}x = 2$$

10.5 Real-World Applications

Exercise 10.19. You are considering using a moving company to help you move. They charge \$60 per hour plus a base fee of \$290. If C is the total cost of hiring the moving company and H is the number of hours worked, what is the relationship between H and C?

11 Transformations of Graphs/Functions

11.1 The Basic Transformations

Let $f(x)$ be a function. Then there are 4 (or 6) basic transformations:

1) Vertical Shifts

2) Horizontal Shifts

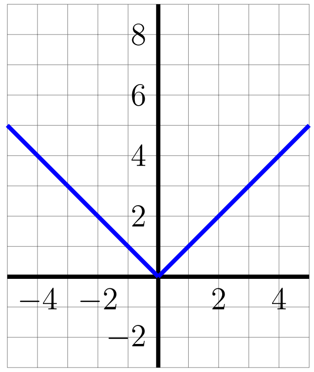
3) Vertical stretches & shrinks/compressions

3.5) Vertical (x-axis) Reflection

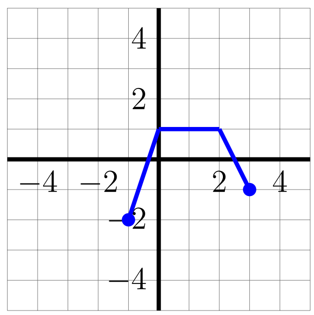
4) Horizontal stretches & shrinks/compressions

4.5) Horizontal (y-axis) Reflection

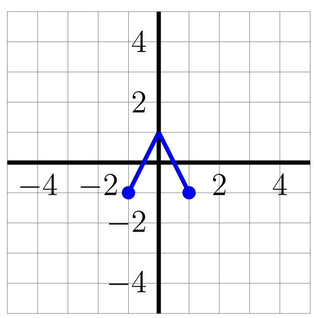
Example 11.1. Vertical shift: Transform $f(x)$ to $f(x)+2$



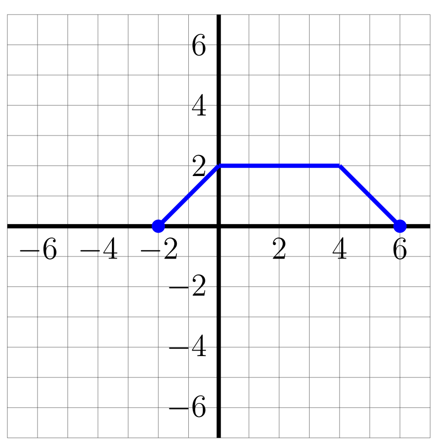
Example 11.2. Horizontal shift: Find $f(x+3)$ using $f(x)$ below



Example 11.3. Vertical Stretch & Shrink/Compression: Find $3f(x)$ using $f(x)$ below



Example 11.4. Horizontal Stretch & Shrink/Compression: Find $f(-2x)$ using $f(x)$ below



11.2 Describing transformations

Exercise 11.5. Describe the graph of $y = 3f(x-2)+4$ in terms of the graph of $y = f(x)$.

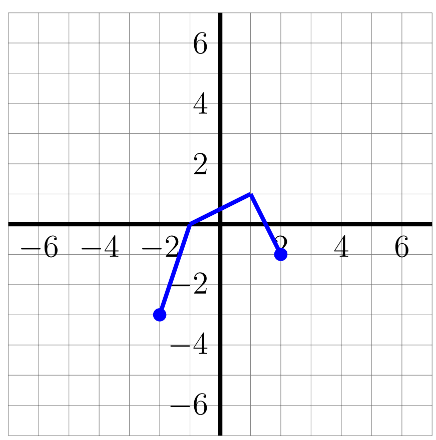
Exercise 11.6. Describe the graph of $y = \frac{1}{4}f(-x+2)$ in terms of the graph of $y = f(x)$.

11.3 Graphing multiple transformations

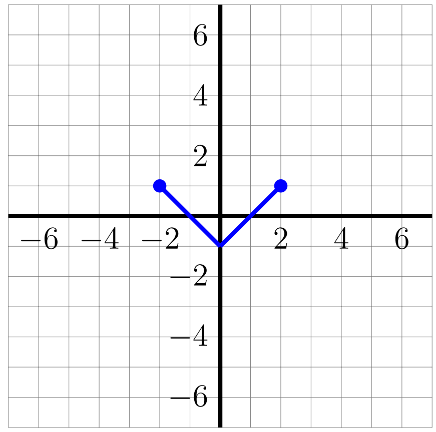
Note. Here is a rule of thumb for transformations:

- Work inside to outside.
- Multiplication/division before addition/subtraction

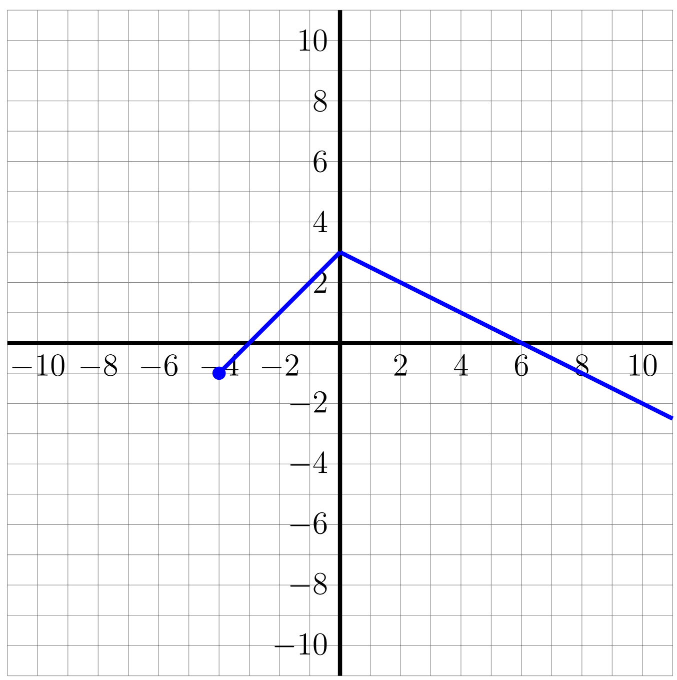
Exercise 11.7. Graph $y = f(-x+3)-4$ using the graph of $y = f(x)$ below:



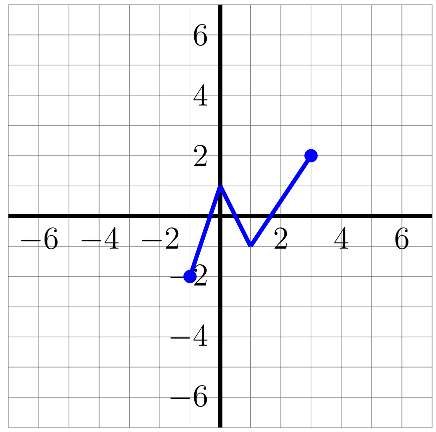
Exercise 11.8. Graph $y = 3f(x+2) - 1$ using the graph of $y = f(x)$ below:



Exercise 11.9. Graph $y = 3f(2x-2) + 2$ using the graph of $y = f(x)$ below:



Exercise 11.10. Using $y = f(x)$ below, graph $y = f(-x-3)+1$



12 More on Functions & Piece-wise Functions

12.1 Even and Odd Functions

There are two ways to see whether a functions is even, odd or neither.

Algebraically

Definition 12.1.

- If $f(x)$ is odd, then $f(-x) =$ ________.
- If $f(x)$ is even, then $f(-x) =$ _______.

Exercise 12.2. Is $f(x) = x^2 + 6$ even, odd, or neither?

Exercise 12.3. Is $g(x) = x^3 - x$ even, odd, or neither?

Exercise 12.4. Is $h(x) = x^5 + 1$ even, odd, or neither?

Notice the following

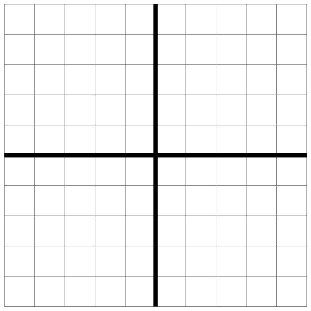
- $f(x)$ had all even powers.
- $g(x)$ had all odd powers.
- $h(x)$ had both even and odd powers.

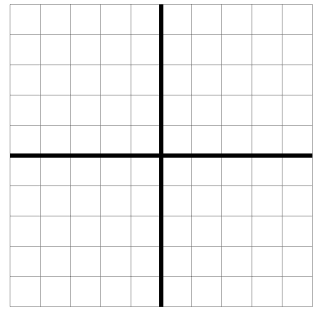
Fact 5

- *If all the powers in a polynomial are ________, then it is as an even function.*
- *If all the powers in a polynomial are _______, then it is as an odd function.*

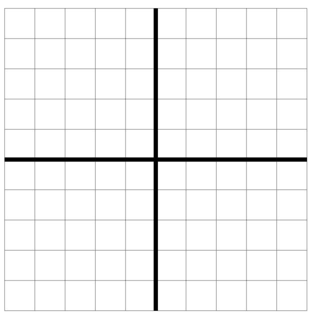
Graphically

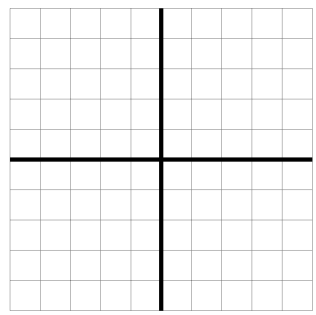
Recall that if $f(x)$ is even, then $f(-x) = f(x)$. This means that x and $-x$ have the same value. So $f(x)$ is even if it has ________________





If $f(x)$ is instead odd, then $f(-x) = -f(x)$. This means that x and $-x$ have opposite values, so we say that $f(x)$ is odd if it has ____________.





12.2 Piece-wise functions

Definition 12.5. A *piece-wise defined function* or *piece-wise* function is a function that is defined by different functions on different parts of its domain. Said another way, piece-wise functions are made of "pieces" of other functions.

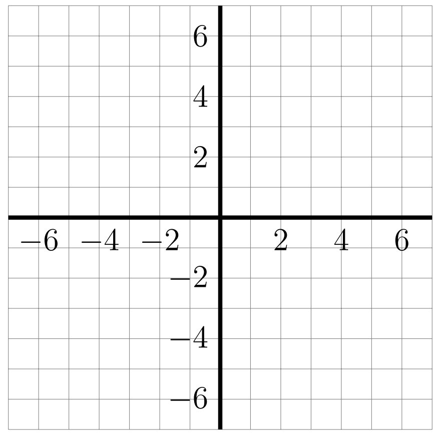
Example 12.6.

$$f(x) = \begin{cases} 2x - 1 & \text{if } x < -1 \\ x + 1 & \text{if } x \geq -1 \end{cases}$$

Graphing

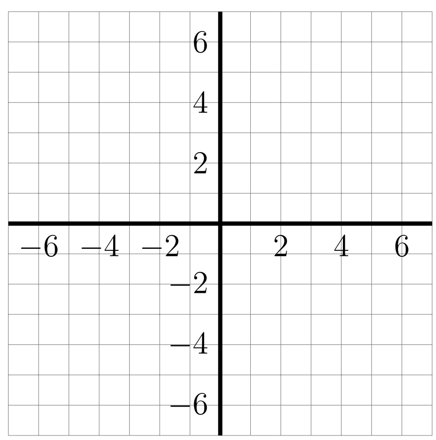
Exercise 12.7. Graph

$$f(x) = \begin{cases} 2x - 1 & \text{if } x < -1 \\ x + 1 & \text{if } x \geq -1 \end{cases}$$



Exercise 12.8. Graph

$$g(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$



Evaluating

Exercise 12.9. Find $f(2)$ for

$$f(x) = \begin{cases} -2x & \text{if } x < 1 \\ x + 2 & \text{if } 1 \leq x < 3 \\ x^2 + 2 & \text{if } 3 \leq x \end{cases}$$

Exercise 12.10. For

$$g(x) = \begin{cases} -6x + 4 & \text{if } x < 3 \\ 3x - 5 & \text{if } x \geq 3 \end{cases}$$

find $g(3)$.

13 Algebra of Functions

13.1 Basics

Let $f(x)$ & $g(x)$ be two functions and D_f & D_g be their domains

Functions	Domains
Sum: $(f+g)(x) =$ ____________	________
Difference: $(f-g)(x) =$ ____________	________
Product: $(fg)(x) =$ ____________	________
Quotient: $\left(\dfrac{f}{g}\right)(x) =$ ________	____________________

Exercise 13.1. For $f(x) = x^2 - 1$ & $g(x) = 3x - 4$, find:

- $(f+g)(x) =$
- $(f-g)(x) =$
- $(fg)(x) =$
- $\left(\dfrac{f}{g}\right)(x) =$

13.2 Composite Functions

Definition 13.2. Given two functions $f(x)$ and $g(x)$, the *composition of* f *and* g (f compose g) is the following:

$$(f \circ g)(x) = \underline{\qquad\qquad}$$

Exercise 13.3. For $f(x) = 4x + 3$ and $g(x) = 2x^2 + 5x$ find $(f \circ g)(x)$

Exercise 13.4. What if we found $(g \circ f)(x)$ instead?

Exercise 13.5. For $f(x) = 6x - 1$ and $g(x) = 4x^2 + x$, find both $(f \circ g)(x)$ and $(g \circ f)(x)$.

13.3 Evaluating compositions

There are two ways evaluate compositions: algebraically & graphically.

Algebraically

Exercise 13.6. Given f and g below, find $f(g(3))$.

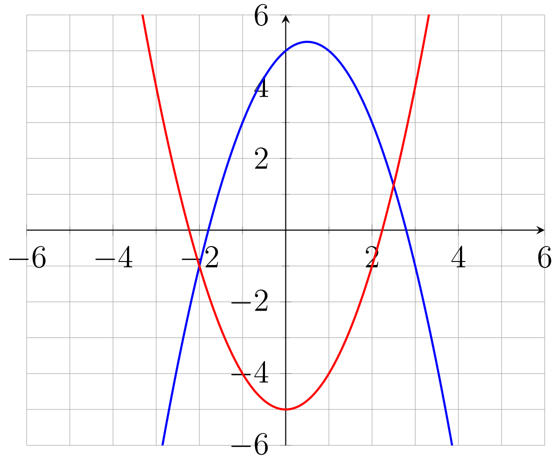
$$f(x) = -4x - 6$$
$$g(x) = x - 3$$

Exercise 13.7. Find $(f \circ g)(2)$ if $f(x) = -2x^2 + 4x + 6$ & $g(x) = 2x - 3$.

Graphically

This will be very similar to the second algebraic technique.

Exercise 13.8. Find $f(g(2))$ using the graphs of f and g below:



13.4 Domains of Compositions

Here is a nice visualization of compositions to understand their domains.

Fact 6

If we have two functions $f(x)$ and $g(x)$ with D_f and D_g their respective domains, then the domain of $(f \circ g)(x)$ is

$$D_{f \circ g} = \underline{\hspace{6cm}}$$

Exercise 13.9. Find the domain of $(f \circ g)(x)$ when

$$f(x) = \frac{1}{8x - 16} \quad \& \quad g(x) = \sqrt{2x + 10}$$

Exercise 13.10. Find the domain of $f(g(x))$ when

$$f(x) = \frac{4}{x+2} \quad \& \quad g(x) = \frac{-2}{3x-2}$$

13.5 Decomposing functions

Exercise 13.11. Find $f(x)$ and $g(x)$ such that

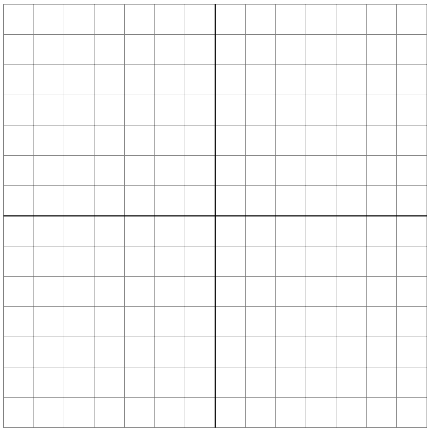
$$(f \circ g)(x) = \sqrt{x^2+1}$$

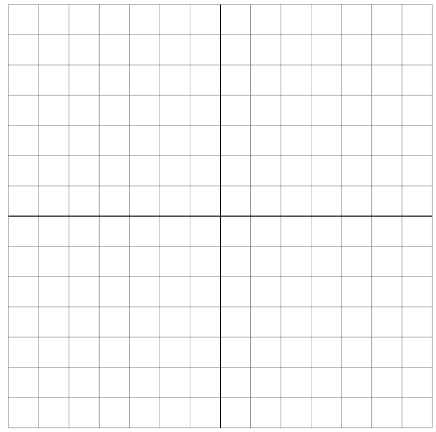
Exercise 13.12. For $h(x) = \dfrac{-5}{(-x-3)^9}$, find $f(x)$ and $g(x)$ such that $f(g(x)) = h(x)$.

Part III

14 Quadratics & Their Applications

Question. What is a quadratic's graph?





We call these shapes ___________.

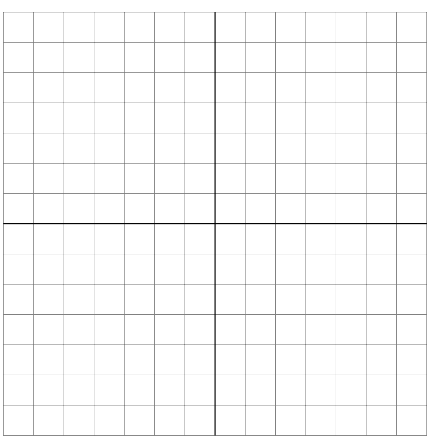
14.1 Standard form

Definition 14.1. The following is the *standard form* of a parabola with *vertex* (h, k):

$$f(x) = \underline{\hspace{3cm}}$$

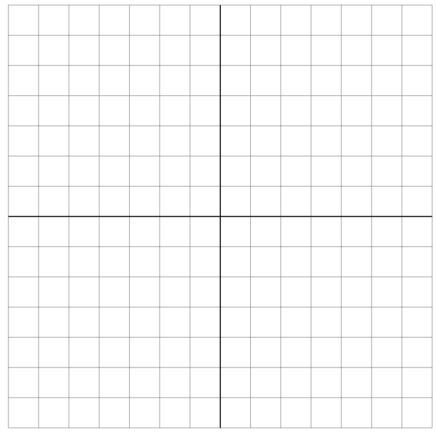
If ________ then the quadratic opens up, and if ________ then the quadratic opens down. We call ________ the *axis of symmetry*.

Basic quadratic

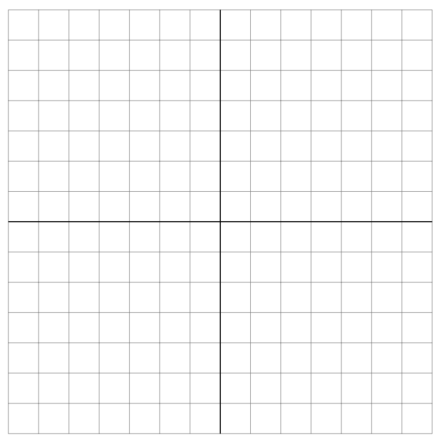


Vertex: ________
Axis of Symmetry: ________
Scale: ________

Exercise 14.2. Graph $f(x) = (x-2)^2 + 1$ below.



Exercise 14.3. Graph $f(x) = -2(x-1)^2 + 2$ below.



14.2 General form

Definition 14.4. The *general form* of a quadratic is as follows:

$$f(x) = \underline{\hspace{4cm}}$$

- Vertex: ______________
- Axis of Symmetry: _________.

So standard form is good for graphing, but we tend to see most quadratics in general form. As such, we will need to be able to change between the two forms.

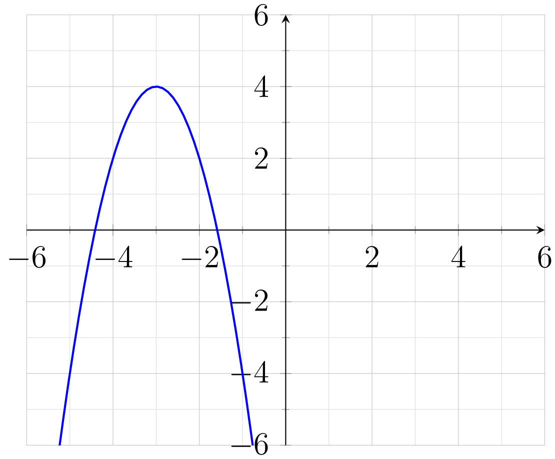
Exercise 14.5. What is the standard form of

$$f(x) = x^2 - 4x + 2$$

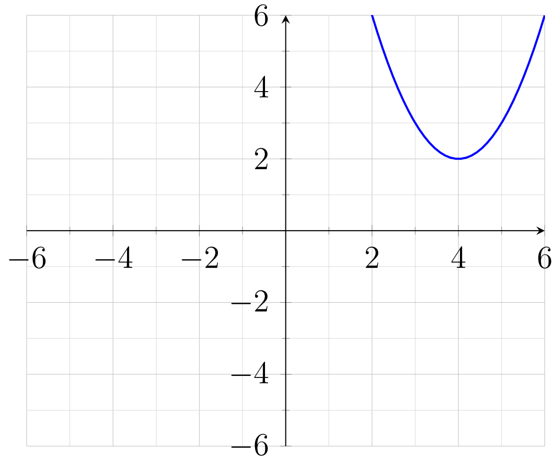
Exercise 14.6. Write $f(x) = 2x^2 + 12x + 3$ in standard form.

14.3 Determining the equation from the graph

Exercise 14.7. Find the equation of the following parabola in both standard and general form:



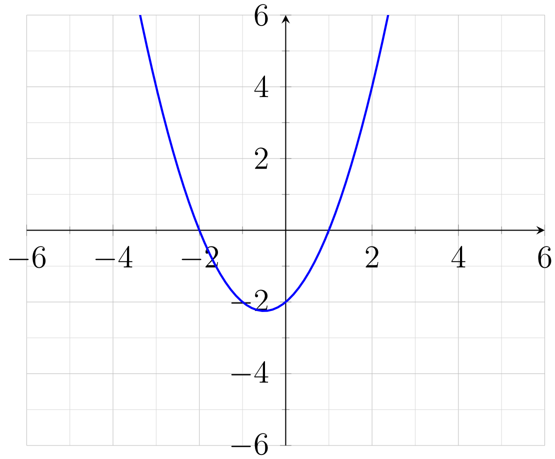
Exercise 14.8. Find both the standard and general form of $f(x)$ below:



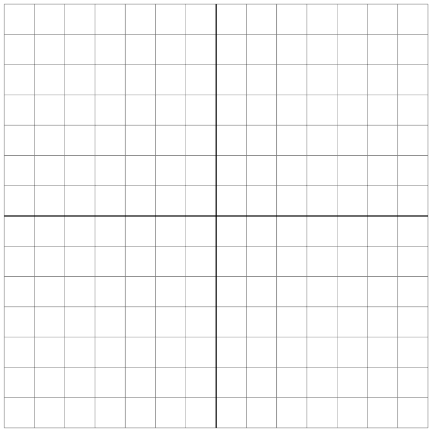
14.4 Using intercepts to find the equation

Sometimes the vertex is not a nice point that we can see, but the intercepts are nice such as the following problem:

Exercise 14.9. Find the general form of $f(x)$ below:



14.5 Finding domain & range from equations



Domain: __________

(if $a > 0$) Range: __________

(if $a < 0$) Range: __________

Exercise 14.10. Find the domain and range of $f(x) = (x+3)^2 - 4$

Exercise 14.11. Find the domain and range of $f(x) = -3x^2 + 12x + 17$

14.6 Finding intercepts from the equation

Exercise 14.12. Find the x- and y-intercepts of $f(x) = 2x^2 - 4x - 6$.

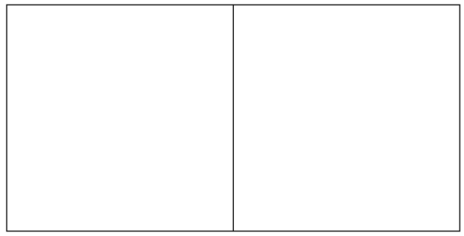
14.7 Maximum and Minimum values of quadratics

Note. The max/min value always occur at the ____________________.

- If $a > 0$ it has a ________________,
- and if $a < 0$ it has a ________________.

Exercise 14.13. A farmer is building a fence to enclose a rectangular area next to a wall. They have 164 ft of fence to use. What is the largest area they can enclose in fence?

Exercise 14.14. Find the dimensions with the maximum area that is fenced in using the shape below and using 234 ft of fence.



15 Polynomials

Definition 15.1. A *power function* is a function that can be written as

$$f(x) = \underline{\hspace{2cm}}$$

Example 15.2.

- ____________
- ____________________

Non-Example 15.3.

- _____________
- ________________

Having defined polynomials before (see Definition R.6), we can define them sightly differently using power functions as follows:

Definition 15.4. A *polynomial* is a sum of power functions such that all of the powers are natural numbers (non-negative integers) with the following form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Recall that a_n is called the leading coefficient, $a_n x^n$ is the leading term and n is the degree of $f(x)$.

Example 15.5.

- ____________
- ________________
- ____________________

Non-Example 15.6.

- __________________

15.1 Graphs of Polynomials

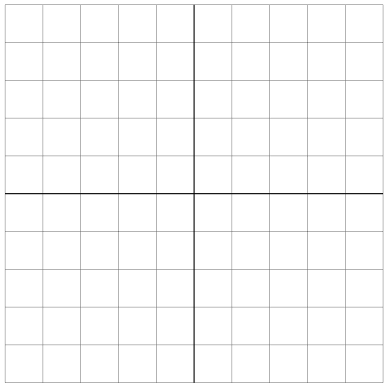
The graphs of polynomials have 3 key properties:

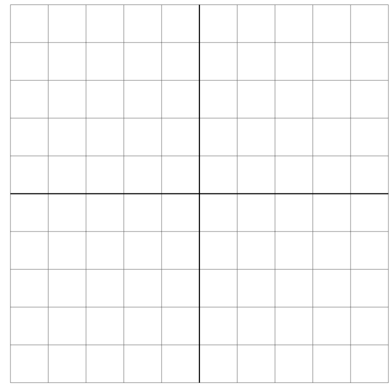
1) _________ - the graph has no sharp corners.
2) ____________ - no jumps or breaks in the graph.
3) ________________

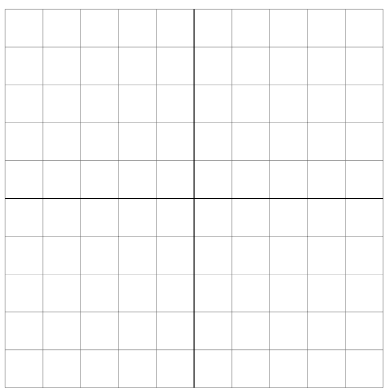
Question. What is an ___________?

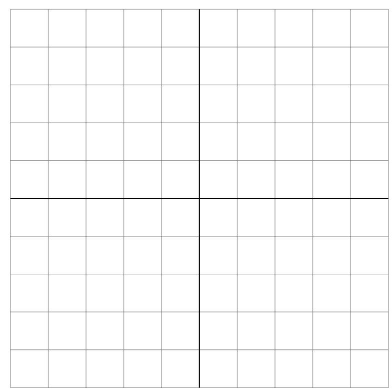
- ________________________ - vertical lines that a function approaches by cannot cross.
- ___________________________ - horizontal lines that a function approaches as $x \to \infty$ or $x \to -\infty$.

Here are some visualizations of these different properties:



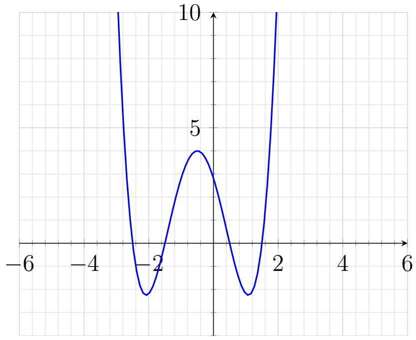




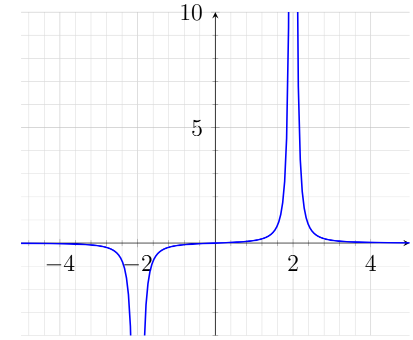


Exercise 15.7. Which of the following are polynomials?

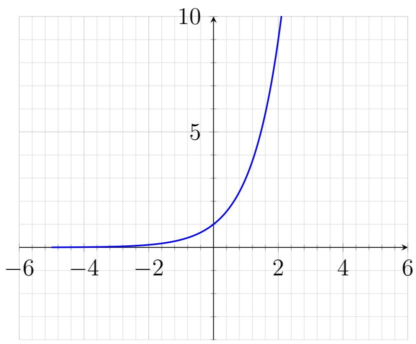
(A)



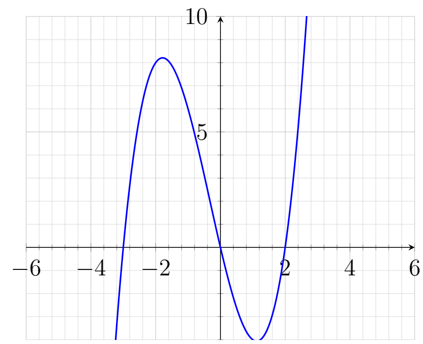
(B)



(C)



(D)



15.2 End behavior of a polynomial

The end behavior of a polynomial describes what $f(x)$ is doing as $x \to$ ______ or $x \to$ ________.

Question. What do we need from a polynomial's equation to determine the end behavior?

Proposition 15.8

If $f(x) = a_n x^n + a_{n-1}^{n-1} + \cdots$, then the end behavior of $f(x)$ can be categorized as follows:

	n *even*	n *odd*
$a_n > 0$		
$a_n < 0$		

or

	n *even*	n *odd*
$a_n > 0$		
$a_n < 0$		

Exercise 15.9. What is the end behavior of $f(x) = 18x^3 + 2$

Exercise 15.10. Describe the end behavior of

$$f(x) = -5x(x-4)(x+5)(2x+3)$$

Exercise 15.11. Which of the following factors would cause the graph of $f(x)$ to decrease as x goes to negative infinity?

$$f(x) = (2x^2+5)(x-7)$$

- ○ -3
- ○ $5x^2$
- ○ $4(2x-7)$

15.3 Identifying intercepts

Recall that x-intercepts are where __________, and the y-intercept is at ______.

Exercise 15.12. What are the intercepts of

$$f(x) = (x-5)(x+3)(x-1)$$

15.4 Finding zeros

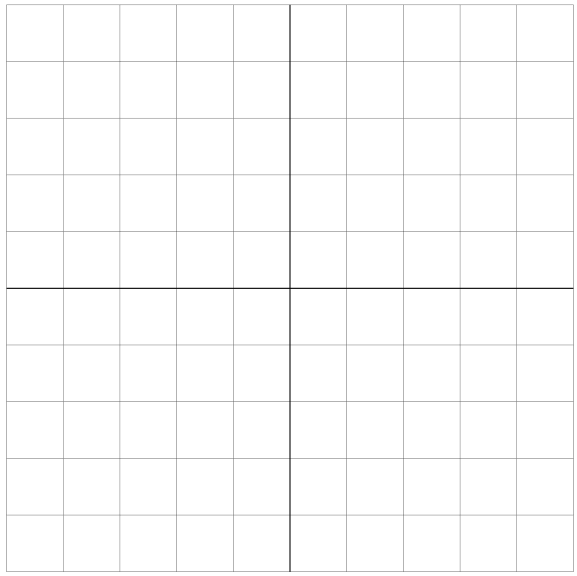
There are two steps to finding the zeros of a polynomial:

1) ________________________________
2) _____________________________

Exercise 15.13. Find the zeros of $f(x) = x^5 - 14x^4 + 49x^3$

15.5 More on graphs and degrees

Definition 15.14. A *turning point* is a point where $f(x)$ changes from ____.



Here are two facts relating to the degree of a polynomial and its graph.

Fact 7

A polynomial of degree n can have at most ________ turning points.

Fact 8

A polynomial of degree n can have at most ____ x-intercepts.

Exercise 15.15. If a polynomial has 7 x-intercepts, what can you say about the degree?

Exercise 15.16. If the degree of a polynomial is 8, what can you say about the number of turning points?

15.6 More about zeros

What follows is one of the most important theorems in algebra, but we need one definition first.

Definition 15.17. If b is a zero of a polynomial $f(x)$, then the *multiplicity* of b is the number of times that b appears as a zero to $f(x)$.

Theorem 15.18 (*Fundamental Theorem of Algebra*)

If we allow complex numbers, then any polynomial can be completely factored into linear factors of its zeros. That is if $f(x)$ is a polynomial and $b_1, b_2, \ldots, b_k$ are the zeros of $f(x)$ with multiplicities $m_1, m_2, \ldots, m_k$, we can write $f(x)$ as follows:

$$f(x) = a(x - b_1)^{m_1}(x - b_2)^{m_2} \cdots (x - b_k)^{m_k}$$

Example 15.19. The polynomial $f(x) = (x-3)^2(x+4)^6$ has zeros at ________ and __________ with multiplicities ____ and ____, respectively.

Note. A consequence of the Fundamental Theorem of Algebra is

$$m_1 + m_2 + \cdots + m_k = \underline{\hspace{2cm}}$$

The multiplicities have an effect on how the graph acts around the corresponding zero (x-intercept).

- If the multiplicity of the zero b is odd, then

  ______________ or ______________

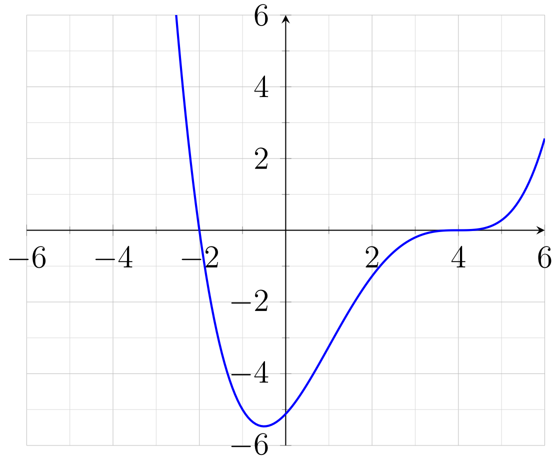
- If the multiplicity of the zero b is even, then

  ______________ or ______________

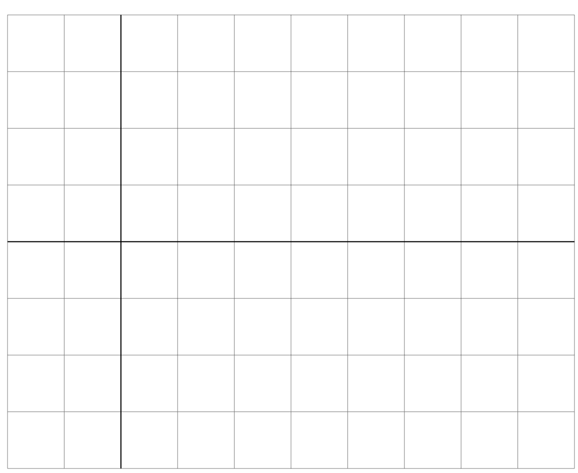
Additionally, the flatter the graph is around the zero the larger the multiplicity.

Using these ideas we can make conclusions about the multiplicity of zeros based on the graph of a polynomial and some information about the degree.

Exercise 15.20. Given that $f(x)$ below is a degree 4 polynomial, find the zeros and their multiplicities:



15.7 The Intermediate Value Theorem



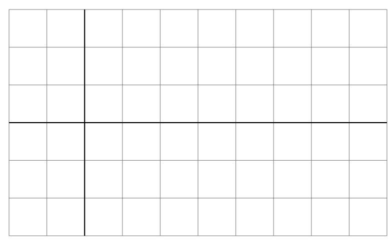
Theorem 15.21 (*Intermediate Value Theorem*)

For a continuous function (think polynomial) $f(x)$ on $[a, b]$ and a number z in between $f(a)$ and $f(b)$, there exists a c in $[a, b]$ such that $f(c) = z$.

Question. Do we care?

This theorem is useful for ________________ because if ____ is between $f(a)$ and $f(b)$ and f is continuous on $[a, b]$, then the IVT says that there is a c in $[a, b]$ such that $f(c) =$ ____.

Note. ____ is between $f(a)$ and $f(b)$ $\Longleftrightarrow$ $f(a)$ and $f(b)$ ____________________

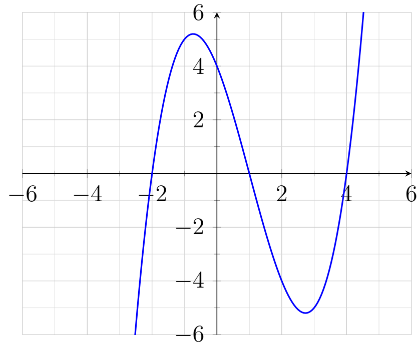


Exercise 15.22. Show that $f(x) = 3x^2 - 1$ has a zero in $[-1, 0]$ using IVT.

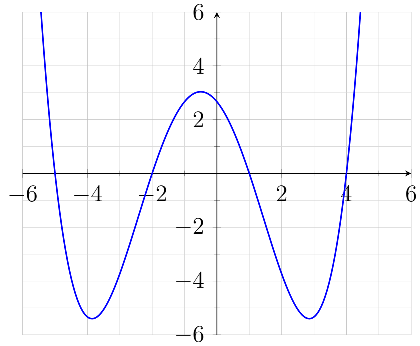
Exercise 15.23. Does IVT guarantee that $f(x) = x^2 - 4x + 4$ have a zero between $[1, 4]$?

15.8 Drawing conclusions from graphs of polynomials

Exercise 15.24. Describe the leading coefficient and degree of the following polynomial:

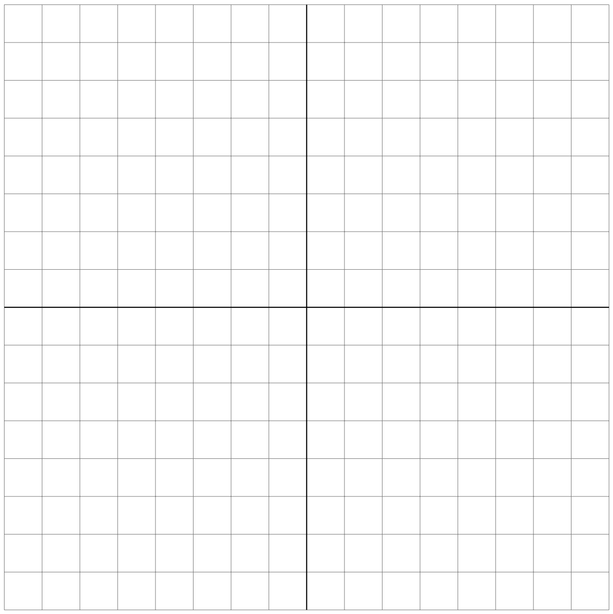


Exercise 15.25. Describe the leading coefficient and degree of the following polynomial:

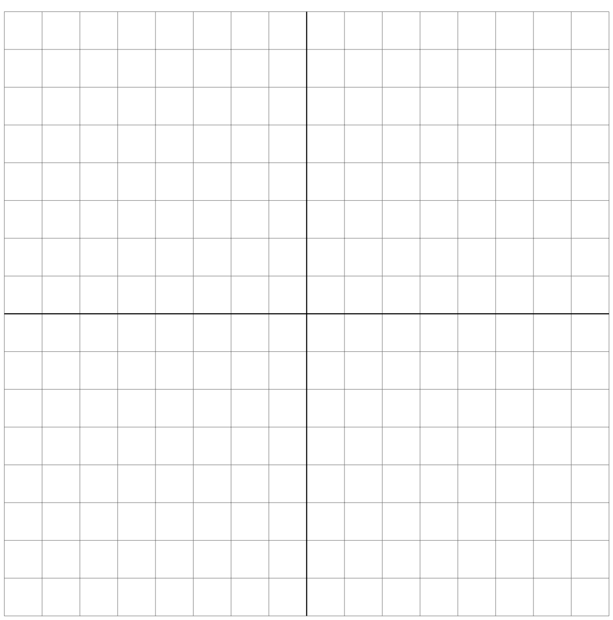


15.9 Graphing Polynomials

Exercise 15.26. Graph $f(x) = 4(x-2)(x+1)(x+6)$.

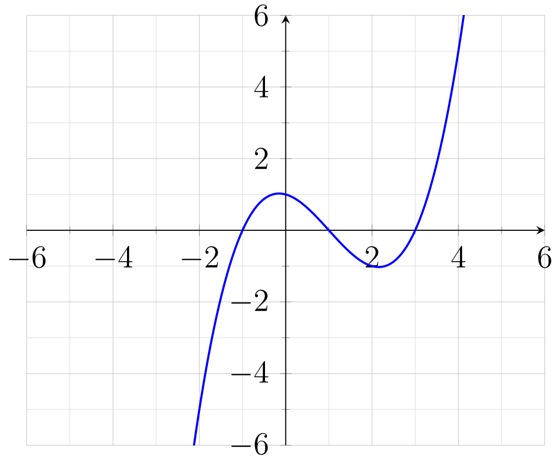


Exercise 15.27. Graph $f(x) = -2(x-6)(x-2)(x+3)$.

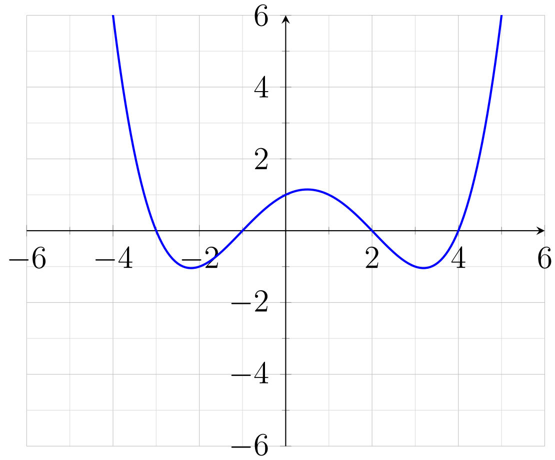


15.10 Finding the equation from the graph

Exercise 15.28. The graph of the polynomial $f(x)$ is given below. If $f(x)$ is degree 3, find the equation of $f(x)$



Exercise 15.29. The graph of the polynomial $f(x)$ is given below. If $f(x)$ is degree 4, find the equation of $f(x)$



15.11 Finding the equation from key information

Exercise 15.30. Write a polynomial, $p(x)$, in factored form given the following requirements:

- degree 4;
- leading coefficient of 1;
- has zeros at $(1, 0)$, $(-4, 0)$, and $(2, 0)$;
- y-intercept is $(0, -24)$.

Exercise 15.31. Write a polynomial, $p(x)$, in factored form given the following requirements:

- degree 3;
- leading coefficient of 1;
- has zeros at $(8, 0)$, $(-4, 0)$, and $(1, 0)$;
- y-intercept is $(0, 32)$.

Exercise 15.32. Which of the following is the equation of a polynomial of degree 4 with zeros at $(4, 0)$ & $(3, 0)$ and y-intercept of $(0, 96)$.

- ○ $y = (x + 4)(x + 3)(x + 8)(x + 1)$
- ○ $y = (x + 4)(x - 3)(x - 2)(x - 1)$
- ○ $y = (x - 4)(x - 3)(x + 8)(x - 1)$
- ○ $y = (x - 4)(x - 3)(x + 2)(x + 4)$

16 Division of Polynomials & the Remainder Theorem

16.1 Long Division

First I want to recall how we long divide numbers

Exercise 16.1. Use long division to find $24 \div 2$.

We are now going to do the exact same thing with polynomials

Exercise 16.2. Divide $x^2 + 14x + 45$ by $x + 9$.

Exercise 16.3. Divide $24x^3 + 2x^2 - 20x - 6$ by $4x + 3$.

Exercise 16.4. Divide 25 by 2.

Exercise 16.5. Divide $40x^2 + 31x - 33$ by $5x + 7$.

Exercise 16.6. Divide $-60x^3 - 40x^2 + 54x + 38$ by $6x + 4$.

Exercise 16.7. Divide $6x^3 - 8x + 4$ by $2x + 6$.

Exercise 16.8. The volume of a hexagonal prism is $12x^5 + 8x^4 + 9x^3 - 42x^2 - 33x - 44$ and the area of the base is $2x^2 + 3x + 4$. Find the height of the prism.

16.2 Synthetic division

This is a way to divide polynomial when the divisor is of the form ________

Exercise 16.9. Synthetically divide $x^2 + 14x + 45$ by $x + 9$

Exercise 16.10. Use synthetic division to find $(x^4 - 10x^2 - 13x - 50) \div (x + 5)$

Exercise 16.11. Synthetically divide $x^4 - 2x^2 - 4x - 48$ by $x - 3$

16.3 The Remainder Theorem

Theorem 16.12 (*Remainder Theorem*)

If a polynomial $f(x)$ is divided by $x - c$, then remainder is ______.

Exercise 16.13. What is the remainder of $g(x) = 3x^3 - 20x^2 + 29x + 30$ divided by $x + 2$

16.4 The Factor Theorem

Theorem 16.14 (*Factor Theorem*)

$x - c$ is a factor of $f(x)$ if and only if the remainder of $f(x)$ when divided by $x - c$ is ____.

Another way we can say this is $x - c$ is a factor of $f(x)$ if and only if __________.

Exercise 16.15. Consider $f(x) = x^3 - 4x^2 - 47x + 210$. When $f(x)$ is divided by $x + 7$, the remainder is 0. What other binomials have a reminder of zero?

Exercise 16.16. For what values of x is $x^3 - 12x^2 + 39x - 28 = 0$?

17 Zeros of Polynomials

First we need a rule for guessing zeros.

17.1 Rational zero theorem

Theorem 17.1 (*Rational Zero Theorem*)

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every rational zero (fraction zero) of $f(x)$ is of the form $\frac{p}{q}$, where p is a factor ________________ *and q is a factor of* ________________________

Exercise 17.2. What are the possible rational zeros of $4x^5 + 12x^4 - x - 3$

Exercise 17.3. What are the possible rational zeros of $x^3 + 2x^2 - 5x - 6$

Exercise 17.4. Find the rational zeros of $f(x) = x^3 - 5x^2 - 33x - 27$.

17.2 Polynomials with complex zeros

Exercise 17.5. Find all of the zeros of $f(x) = x^3 + 13x^2 + 57x + 85$.

17.3 Linear Factorization Theorem

Theorem 17.6 (*Linear Factorization Theorem*)

Any polynomial $f(x)$ can be written as

$$f(x) = \underline{\hspace{6cm}}$$

where a_n is the leading coefficient, the degree of $f(x)$ is n, and $c_1, c_2, \ldots, c_n$ are the complex zeros of $f(x)$ (not necessarily distinct).

Exercise 17.7. Find a third degree polynomial that has an output of 16 when $x = 2$ and zeros of 1 and $-2i$.

Exercise 17.8. Find a third degree polynomial that has an output of 20 when $x = 0$ and zeros of 4 and $-1 + 3i$.

17.4 Descartes' Rule of Signs

The following is occasionally useful when guessing zeros for the rational zero theorem.

Proposition 17.9

1) *The number of positive real zeros of a polynomial $f(x)$ is either the number of sign changes of ________ or less a positive even integer.*

2) *The number of negative real zeros of a polynomial $f(x)$ is either the number of sign changes of __________ or less a positive even integer.*

Exercise 17.10. For $f(x) = 7x^4 + 2x^3 + 4x^2 - 6x + 5$, what are the possible combinations of positive, negative, and imaginary zeros of $f(x)$.

Exercise 17.11. For $f(x) = x^4 - 2x^3 + 4x^2 - 3x + 17$, what are the possible combinations of positive, negative, and imaginary zeros of $f(x)$.

Exercise 17.12. For $f(x) = x^6 - 2x^5 + 7x^4 + 5x^3 - 7x^2 + 3x - 10$, what are the possible combinations of positive, negative, and imaginary zeros of $f(x)$.

18 Rational Functions

$$\text{rational numbers} \longleftrightarrow \text{fractions of integers}$$
$$\text{rational functions} \longleftrightarrow \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Definition 18.1. A function $f(x)$ is a *rational function* if

$$f(x) =$$

where $p(x)$ and $q(x)$ are ____________

Example 18.2.

- ____________________

- ___________________

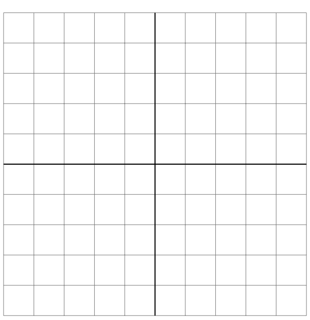
Non-Example 18.3.

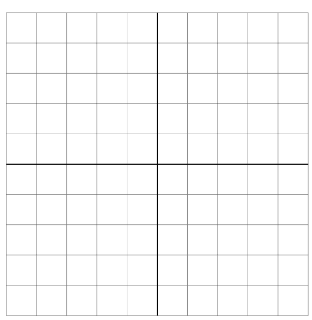
- _____________

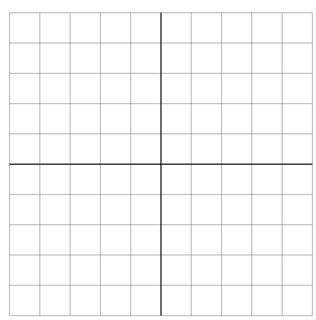
Note. Here is some useful notation that we will see when working with rational functions:

- _________ - x approaches a from the right/above.
- _________ - x approaches a from the left/below.
- _________ - x approaches infinity.
- __________ - x approaches negative infinity.

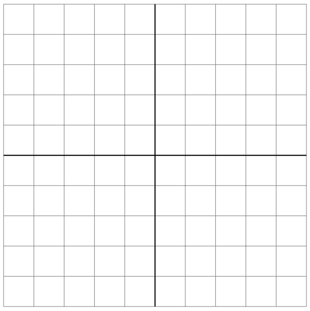
Definition 18.4. A *vertical asymptote* (VA) is a vertical line $x = a$ such that $f(x) \to$ ______ as $x \to$ _____

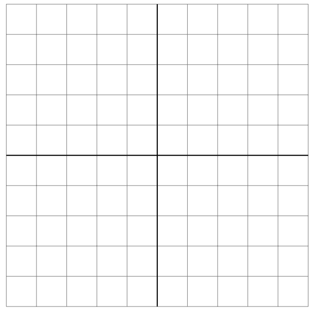


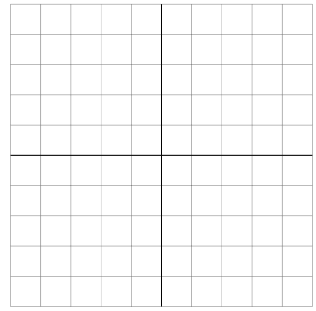




Definition 18.5. A *horizontal asymptote* (HA) is a horizontal line $y = b$ such that $f(x) \to$ ______ as $x \to$ _______

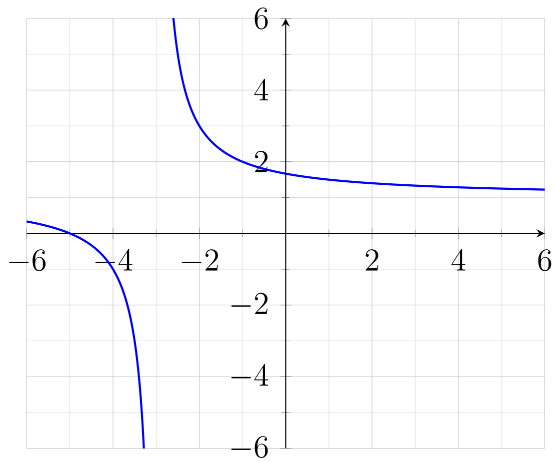




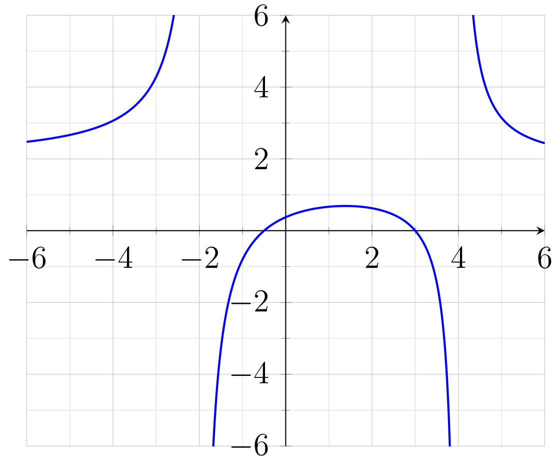


18.1 Describe the behavior of rational functions using their graph

Exercise 18.6. Describe the behavior of $f(x)$ below:



Exercise 18.7. Describe the behavior of $f(x)$ below:



18.2 Finding vertical asymptotes

Fact 9

The function $f(x) = \dfrac{p(x)}{q(x)}$ *has a VA at* $x = a$ *if* ________________________.

If both ________________________, *then either* a *is a* ____________________________________ *if we can simplify* $f(x)$ *such that* a *no longer makes the denominator* ____.

If the denominator is still ____, *but the numerator is not, then it is a VA and is called an* ______________________.

Exercise 18.8. Consider

$$f(x) = \frac{x^3 - x^2 - 2x}{x^2 + x}.$$

Does $f(x)$ have any removable discontinuities?

Exercise 18.9. Consider

$$f(x) = \frac{x^2 - 36}{x^3 - 2x^2 - 24x}.$$

What are the vertical asymptotes of $f(x)$ (if any)?

18.3 Finding horizontal asymptotes

Fact 10

Let $f(x) = \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$. *To find the horizontal asymptote we need to compare the degrees m and n:*

- _________ $\Rightarrow$ *there is no HA.*
- _________ $\Rightarrow$ *the x-axis ($y = 0$) is the HA.*
- _________ $\Rightarrow$ $y = \dfrac{a_n}{b_m}$ *is the HA.*

Exercise 18.10. What is the HA (if one) of

$$f(x) = \frac{9x}{3x+1}$$

Exercise 18.11. What is the HA (if one) of

$$g(x) = \frac{7x^5 + 4}{3x^6 + x + 3}$$

Exercise 18.12. What is the HA (if one) of

$$h(x) = \frac{x^2 + 1}{x + 1}$$

As we saw, some rational functions do not have a HA, but some have the following instead

Definition 18.13. If $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function and $\deg(p(x)) = \deg(q(x)) + 1$, then $f(x)$ has a *slant asymptote* (SA) which is a line $y = mx + b$ that $f(x)$ approaches as $x \to$ ______

Question. How do you find a SA?

We use division! Since the denominator's degree is one less than the numerator's degree the quotient will have to have degree 1.

Exercise 18.14. If

$$h(x) = \frac{x^2 + 1}{x + 1}$$

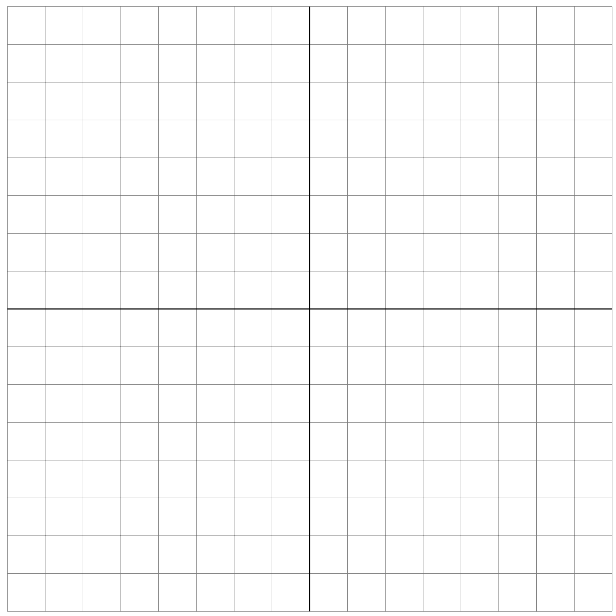
what is its SA?

18.4 Graphing rational functions

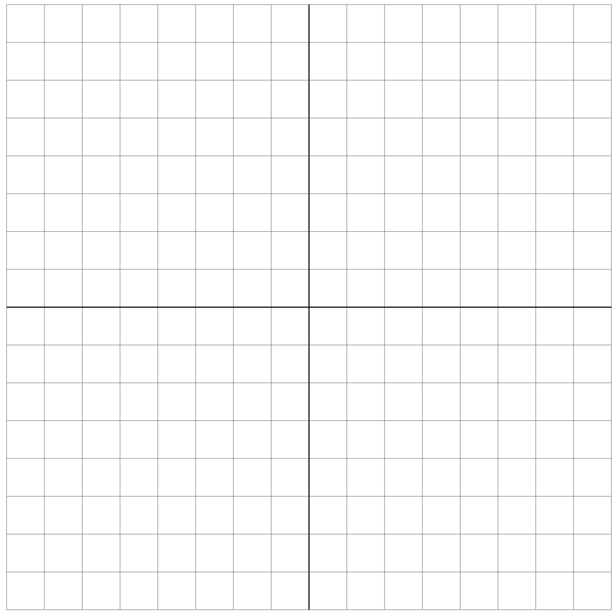
Here is the general strategy if $f(x) = \dfrac{p(x)}{q(x)}$ is fully simplified.

1) Find the y-intercept (if there is one) by finding ______.
2) Find x-intercept (if any) by solving __________.
3) Find VA by solving __________.
4) Find HA (if there is one) by comparing degrees.
   4.5) Find the SA (if there is one) using division.
5) Plot extra points between the intercepts and VA as needed.

Exercise 18.15. Graph $f(x) = \dfrac{x-1}{x-4}$

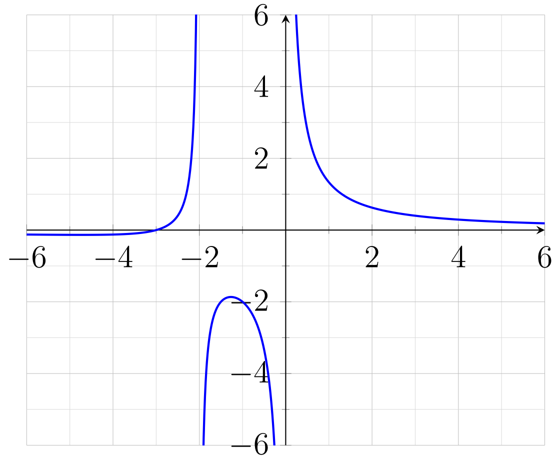


Exercise 18.16. Graph $f(x) = \dfrac{(x-2)(x-5)}{(x-3)^2}$

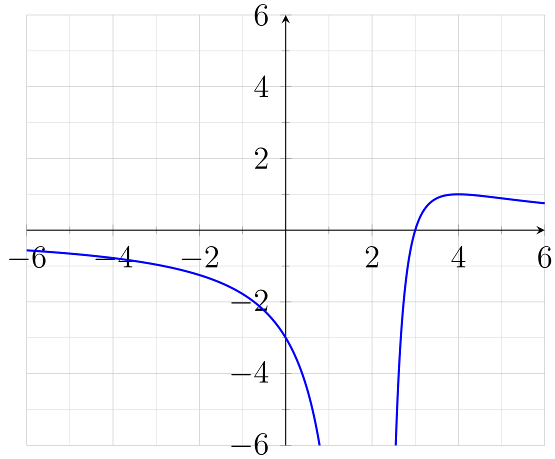


18.5 Identifying rational functions from their graphs

Exercise 18.17. Identify the rational function graphed below. Note the x-intercept of $x = -3$ and $(-1, -2)$ is a point on the graph.



Exercise 18.18. Identify the rational function graphed below. Note the x-intercept of $x = 3$ and y-intercept at -3.



19 Polynomial and Rational Inequalities

Just as linear inequalities start with solving the "equation", the same is true of polynomial and rational inequalities.

19.1 Polynomial inequalities

Here is the procedure for solving inequalties such as $f(x) > 0,\ f(x) < 0,\ f(x) \geq 0,\ f(x) \leq 0$ where $f(x)$ is a polynomial:

1. Set one side to ____
2. Solve __________
3. Plot those solutions on a _____________
4. Find test values in between these numbers and test to find the ______
5. Choose correct interval(s) based on original inequality ________________ and ________________

Exercise 19.1. Solve $x^2 - x > 20$ for x. (Give your answer in interval notation)

Exercise 19.2. Solve $2x^2 \leq -6x - 4$ for x. (Give your answer in interval notation)

Exercise 19.3. Solve $x^3 + 3x^2 \leq x + 3$ for x. (Give your answer in interval notation)

Position of a free-falling object near earth's surface

Fact 11

The position of a free falling object (in feet) at time t can be described by the following quadratic:

$$s(t) = \underline{\hspace{3cm}}$$

where _____ is the initial velocity and _____ is the initial position.

Exercise 19.4. Given the following position function, find the interval of time that the object is more than 64 feet above the ground.

$$s(t) = -16t^2 + 80t$$

19.2 Rational inequalities

What follows is the procedure for solving rational inequalities.

1. Write the inequality with 0 on one side
2. Combine the fractions on the other side as ________________
3. Set the numerator and denominator equal to ____
4. Plot on number line
5. Test values
6. Choose correct interval(s)

Exercise 19.5. Solve the following for x:

$$\frac{2x}{x+1} \geq 1$$

Exercise 19.6. Solve the following for x:

$$\frac{x-2}{x+2} \leq 2$$

Exercise 19.7. Find the solution to the following inequality in interval notation:

$$\frac{2}{x+3} \leq \frac{1}{x-3}$$

Exercise 19.8. Find the solution to the following inequality in interval notation:

$$\frac{3}{x+7} \geq \frac{2}{x-5}$$

Part IV

20 Inverse Functions

Consider the composition of the following two functions.

$$f(x) = x + 300 \qquad g(x) = x - 300$$

Definition 20.1. If for two functions $f(x)$ and $g(x)$, $(g \circ f)(x) =$ ____ & $(f \circ g)(x) =$ ____ for all x in their domains, then we call $g(x)$ the *inverse of* $f(x)$ denoted $f^{-1}(x)$.

Exercise 20.2. Verify that $f(x) = 4x - 7$ and $g(x) = \dfrac{x+7}{4}$ are inverses.

Exercise 20.3. Verify that $f(x) = -2x - 1$ and $g(x) = \dfrac{-x}{2} - \dfrac{1}{2}$ are inverses.

20.1 Finding Inverses

Example 20.4. Let's deduce the inverse for $f(x) = 3x + 2$

20.2 Algebraically finding the inverse

Here are the steps for calculating the inverse

1) Replace $f(x)$ by y
2) Interchange ("switch") the x's and y's
3) Solve for y
4) Replace y by $f^{-1}(x)$

    4.5 Verify by composing the two functions

Exercise 20.5. Find the inverse of $f(x) = 2x + 7$

Exercise 20.6. Find the inverse of $f(x) = \dfrac{9x-7}{4x+3}$

Exercise 20.7. Find the inverse of $f(x) = \dfrac{2x+3}{4x-7}$

Exercise 20.8. Find the inverse of

$$f(x) = -9\sqrt{x-8} + 5$$

for $x \geq 8$

Note. Notice that since we switch the x's and y's when finding the inverse, it makes sense that

$$\_\_\_\_\_\_\_\_\_\_ \leftrightarrow \_\_\_\_\_\_\_\_\_\_\_\_$$

Exercise 20.9. Based on the following table of values for $f(x)$. Find the requested values of $f^{-1}(x)$.

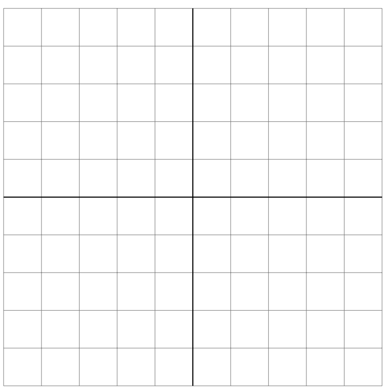
x	1	3	7	9	11	12
$f(x)$	-5	1	2	4	8	10

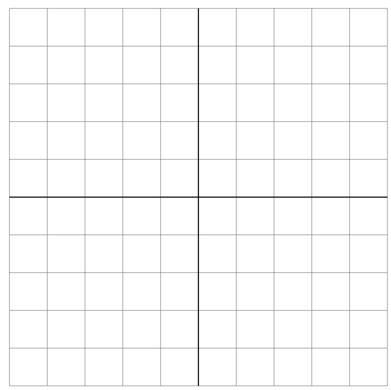
- $f^{-1}(1) =$
- $f^{-1}(-5) =$
- $f^{-1}(8) =$

20.3 Finding inverses graphically

Proposition 20.10 (*Horizontal Line Test*)

A function $f(x)$ has an inverse, $f^{-1}(x)$, if no horizontal line crosses its graph more than once.

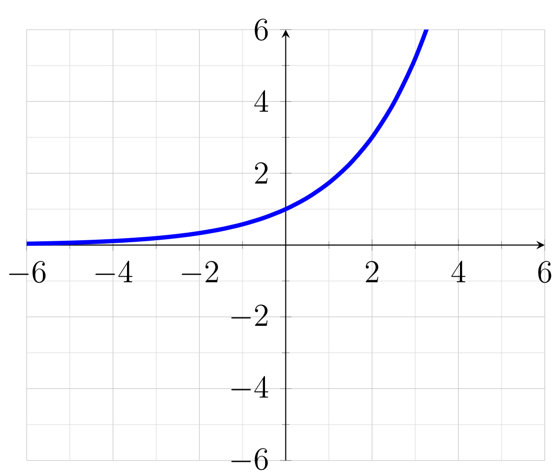




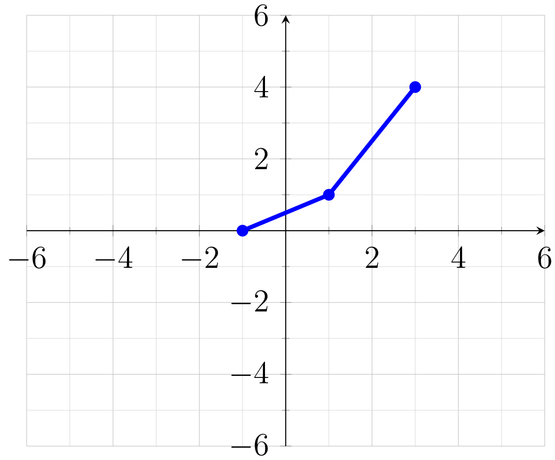
Definition 20.11. We call functions that pass the Horizontal Line Test *one-to-one*

20.4 Finding the graph

Exercise 20.12. Find the graph of $f^{-1}(x)$ if $f(x)$ is given below

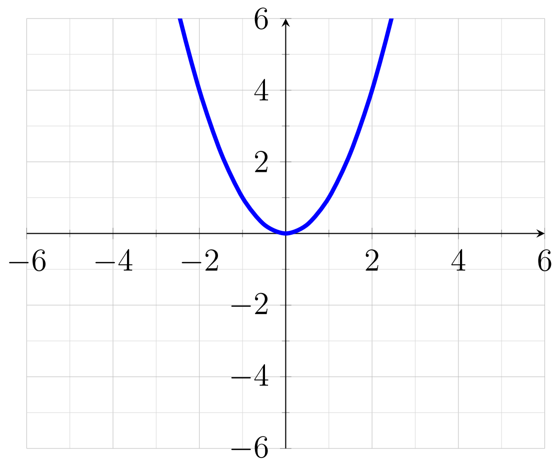


Exercise 20.13. Find the graph of $f^{-1}(x)$ if $f(x)$ is given below

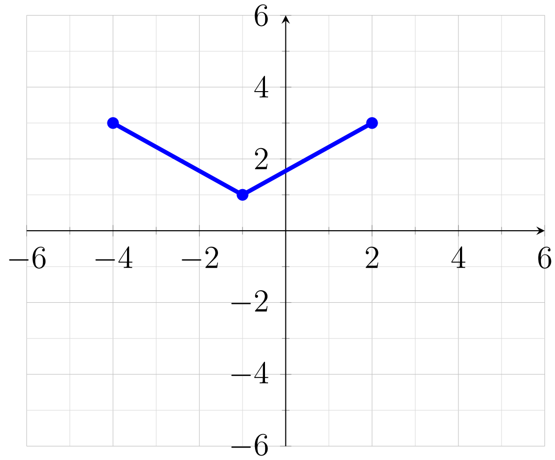


20.5 Restricting the domain

Example 20.14. Consider the function $f(x) = x^2$



Exercise 20.15. How could we restrict the domain of the following function to make it one-to-one?



21 Exponential Functions

Definition 21.1. A (basic) *exponential function* can be written as follows

$$f(x) = \_\_\_\_\_\_\_\_$$

where ____ is called the base and ________________

Example 21.2.

- $f(x) =$ ____________
- $f(x) =$ ________
- $f(x) =$ ________

Non-Example 21.3.

- $f(x) =$ _________
- $f(x) =$ _____

Exercise 21.4. Let $f(x) = \frac{1}{2}(4)^{x-1}$. Evaluate $f(3)$ without a calculator.

Exercise 21.5. Let $f(x) = 3\left(\frac{1}{2}\right)^{x+1}$. Find $f(1)$.

Exercise 21.6. Let $f(x) = -8\,(2)^{3x} + 3$. Evaluate $f(0)$.

21.1 Finding the equation

Initial point/value is known

Exercise 21.7. Construct an exponential function that contains $(0, -2)$ & $(3, -128)$.

Exercise 21.8. The graph of an exponential function has a y-intercept of 7 and contains the point $(4, 112)$. Find the equation.

Initial point/value is unknown

Exercise 21.9. Find the exponential function that contains $(2, 48)$ & $(3, 192)$.

Exercise 21.10. Find the exponential function that contains $(1, 300)$ & $(2, 100)$.

21.2 Word problems

Exercise 21.11. The number of users online at a website has grown exponentially since its launch. After 2 months, there are 200 users. After 4 months, there are 800 users. Find the function that models the number of users x months after launch.

Exercise 21.12. The selling price of a car is \$13,500. Each year it loses 11% of its value. Find the exponential that models its value after x years.

21.3 Compound Interest

Proposition 21.13 (*Compound Interest Formula*)

If an initial investment of P (typically referred to as principle*) at an interest rate of r compounded n times a year, then the amount after t years would be*

$$A = \underline{\hspace{3cm}}$$

Exercise 21.14. If \$5000 is borrowed at an interest rate of 12.3% compound semi-annually, what is the total amount of money needed to pay it back in 3 years? Round to nearest dollar.

Exercise 21.15. A loan is paid off in 15 years with a total of \$192,000. If it had a 4% interest rate compounded monthly, what was the initial amount borrowed?

21.4 The Natural Number

Definition 21.16. *Euler's Constant* (or the *Natural Number*) is a special constant denoted by the letter e. It can be found as follows

$$\underline{\hspace{3cm}}$$

Exercise 21.17. For $f(x) = -7e^x$, find $f(2)$ rounded to one decimal place.

21.5 Continuous growth or decay

Proposition 21.18

Something that is changing continuously either growing or decaying can be modeled as follows:

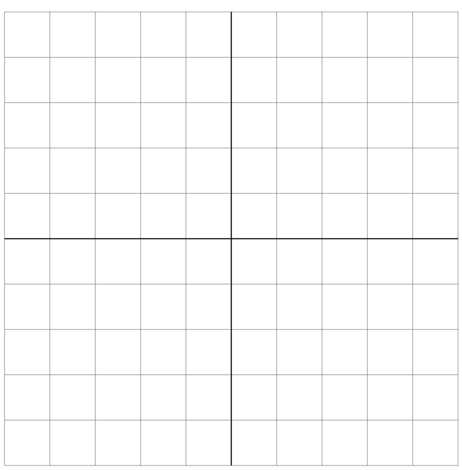
$$A = \underline{\hspace{1.5cm}}$$

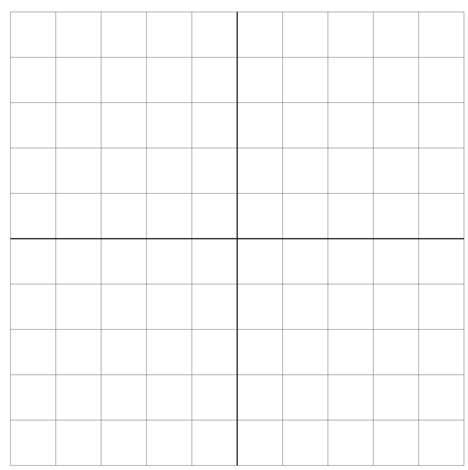
Where P is the initial amount, r is the rate of change ($r < 0$ means it is decaying and $r > 0$ if it is growing), and t is the amount of time passed.

Exercise 21.19. A motorcycle bought at \$10,000 depreciates continuously at 9% per anum. What is its value after 7 years?

21.6 Graphs of Exponentials

The base b of an exponential $b > 1$ or $0 < b < 1$ which have two respective styles of graphs:





Characteristics of basic exponentials

If the we have a basic exponential $f(x) = b^x$, it has the following properties:

1) D_f : ____________ & R_f : _________

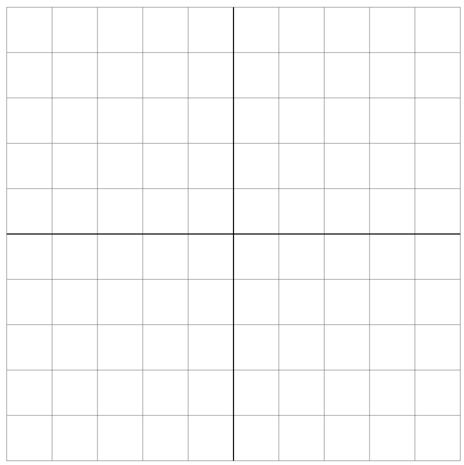
2) y-intercept: ________ & x-intercept: ________

3) If $b > 1$, ________________. If $0 < b < 1$, ________________.

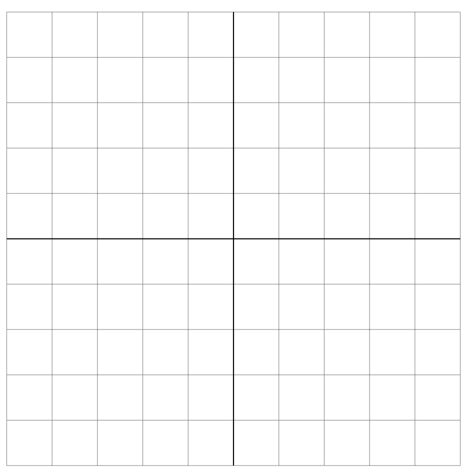
4) $f(x)$ is _____________, so __________________.

5) f has a Horizontal Asymptote at ________.

Exercise 21.20. Graph $f(x) = 3^{x-1}$, by moving key points.



Exercise 21.21. Graph $f(x) = \left(\frac{1}{2}\right)^{x+2}$, by moving key points.



21.7 Finding domain and range

- For exponentials their domain is always ___________
- The range depends, if $f(x) = a(b)^{x-c} + d$
  - If $a > 0$, then the range is ________
  - If $a < 0$, then the range is __________

Exercise 21.22. What is the domain and range of

$$f(x) = -\frac{5}{2}(3)^{x+2} - 1$$

Exercise 21.23. What is the domain and range of

$$f(x) = 7(e)^{x-5} + 4$$

21.8 Writing the equation from a description

Exercise 21.24. If the function $y = e^{3x}$ is vertically stretched by a factor of 4, reflected over the x-axis, and then shifted down 1 unit, what is the resulting function?
Write your answer as $y = Ce^{ax} + b$

22 Logarithmic Functions

Logarithms are the inverses (see Definition 20.1) of exponentials, that is they are the "mathematical opposite".

Definition 22.1. The *logarithm with base b* is

$$y = \underline{\hspace{2cm}}$$

with the restrictions that ________, ________ & ________.

Note. A logarithm output an exponent, that is

$$\underline{\hspace{2cm}}$$

This shows that $g(x) = \underline{\hspace{2cm}}$ is the inverse of $f(x) = \underline{\hspace{1cm}}$

Exercise 22.2. Write the following in exponential form:

$$3 = \log_7 x$$

Exercise 22.3. Write the following in exponential form:

$$2 = \log_b 25$$

Exercise 22.4. Write the following in exponential form:

$$\log_4 26 = y$$

Exercise 22.5. Write the following in logarithmic form:

$$2^5 = x$$

Exercise 22.6. Write the following in logarithmic form:

$$b^3 = 27$$

Exercise 22.7. Write the following in logarithmic form:

$$e^y = 33$$

Two special logarithms

- $\log_e \longleftrightarrow$ ______________________
- $\log_{10} \longleftrightarrow$ ________________________

22.1 Evaluating logarithms

Exercise 22.8. What is the $\log_2 16$?

Exercise 22.9. What is the $\log 100$?

Exercise 22.10. What is the $\log_3 \left(\frac{1}{27}\right)$?

Exercise 22.11. What is the $\log_{\frac{1}{6}} 6$?

Logarithm identities involving 1

- ____________ because _________
- ____________ because _________

Characteristics of basic logarithms

If the we have a basic exponential $f(x) = \log_b x$, it has the following properties:

1) D_f : _________ & R_f : ____________
2) x-intercept: ________ & y-intercept: ________
3) f has a Vertical Asymptote at ________.
4) If $b > 1$, ________________. If $0 < b < 1$, ________________.

22.2 Finding the domain

Need what is inside the logarithm to be __________ (i.e., ______).

Exercise 22.12. What is the domain of

$$f(x) = \log(x - 5)$$

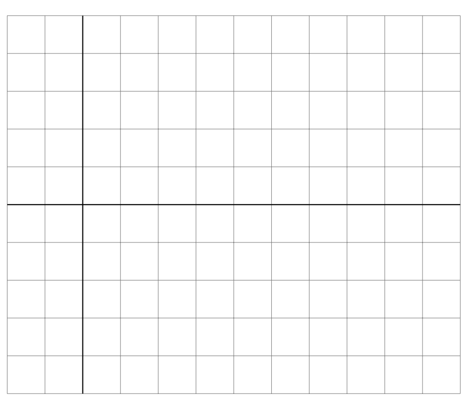
Exercise 22.13. What is the domain of

$$f(x) = \log_3(3 - 4x)$$

22.3 Graphing basic functions

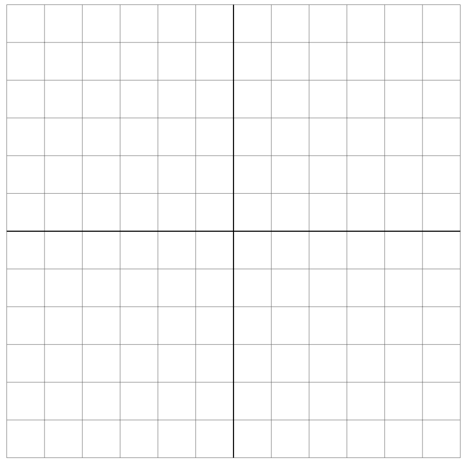
Exercise 22.14. Graph $f(x) = \log_2 x$

Exercise 22.15. Graph $f(x) = \log_{\frac{1}{3}} x$

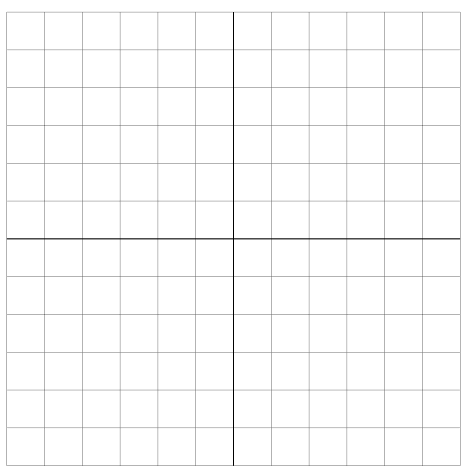


22.4 Graph transformations

Exercise 22.16. Graph $f(x) = -\log_5(x+1)$



Exercise 22.17. Graph $f(x) = \log_4(-x) + 2$



22.5 Writing a logarithm from a description

Exercise 22.18. The graph of $f(x) = \log_6(x)$ is stretched by a factor of 3, reflected over the x-axis, reflected over the y-axis, and shifted up 4 units.

Find the equation of $g(x)$ described above

Exercise 22.19. The graph of $f(x) = \log_5(x)$ is vertically stretched by a factor of 2, shifted to the right by 5, and shifted down 3.

Find the equation of $g(x)$ described above

23 Properties of Logarithms

There are many properties of logarithms that will be important to us, and several of them are very similar to rules for exponents.

23.1 Inverse property

Recall that $f(f^{-1}(x)) = x$ & $f^{-1}(f(x)) = x$. As such we have the following:

Fact 12

____________ & ____________

Exercise 23.1. Evaluate $\ln(e^7)$.

Exercise 23.2. Evaluate $3^{\log_3(5)}$

23.2 Product Rule

Recall that

$$b^M \cdot b^N = \underline{\hspace{2cm}}$$

As such we get the following:

Fact 13

$$\log_b(mn) = \underline{\hspace{4cm}}$$

Exercise 23.3. Expand: $\ln(x(3x-2))$

Exercise 23.4. Condense: $\log_2(3x) + \log_2(7y)$

23.3 Quotient Rule

Recall that

$$\frac{b^M}{b^N} = \underline{\hspace{3cm}}$$

As such we get the following:

Fact 14

$$\log_b\left(\frac{m}{n}\right) = \underline{\hspace{6cm}}$$

Exercise 23.5. Expand: $\log_8\left(\dfrac{23}{y}\right)$

Exercise 23.6. Condense: $\log_5(8x) - \log_5(4)$

23.4 Power Rule

Recall that

$$(b^M)^P = \underline{\hspace{2.5cm}}$$

As such we get the following:

Fact 15

$$\log_b(m^p) = \underline{\hspace{4cm}}$$

Exercise 23.7. Expand: $\log\left(3^9\right)$

Exercise 23.8. Condense: $10\log_6(y)$

23.5 Expanding logarithmic expressions

Exercise 23.9. Expand: $\log\left(\frac{3}{5}x^2\right)$

Exercise 23.10. Expand: $\log_2\left(\dfrac{3x^4}{t}\right)$

Exercise 23.11. Expand: $\ln\left(\sqrt{8t}\right)$

23.6 Condensing logarithmic expressions

Exercise 23.12. Condense: $\ln(30) + 2\ln(5) - \ln(15)$

Exercise 23.13. Condense: $2\ln(t) + 5\ln(z) - \ln(t)$

Exercise 23.14. Condense: $\log(a) + 3\log(2b)$

23.7 Change-of-base Rule

Not every calculator has function for a logarithm of any base. Thankfully there is a way to "change" the base of your logarithm:

Fact 16

$$\log_b m = \underline{\hspace{3cm}}$$

Note. This is generally used to change to the natural log or the common log, but can be useful in other situations

Exercise 23.15. Write $\log_6 8$ using natural logarithms.

Exercise 23.16. Find $\log_3 7$ to the nearest tenth.

24 Exponential & Logarithmic Equations

24.1 Exponentials

Notice that if $b^M = b^N$, then __________ because $y = b^x$ is ____________
There are two main methods for solving.

Method 1: equal base or one-to-one method

1) Rewrite the equation in the form __________

2) Set __________

3) Solve for the variable.

Exercise 24.1. If $3^{8x} = 3^{(x-5)}$, solve for x.

Exercise 24.2. Solve $5^{(3x-6)} = 125$ for x.

Exercise 24.3. What is the value of x that makes $4^{(x-5)} = 64^{(x+2)}$ true?

Exercise 24.4. Determine the value of x satisfying

$$64^{(3x-3)} = 16^{(3x)}$$

Method 2: logarithms

1) Isolate ________________ or get _____________ exponential on each side.
2) Take the ____________ (or _________) of both sides.
3) ______________________________
4) Solve for the variable.

Exercise 24.5. Solve $5^x = 135$.

Exercise 24.6. Solve $11e^{-4x} = 21$.

Exercise 24.7. Solve $7e^{2x} - 5 = 58$ to the nearest tenth.

Exercise 24.8. Solve $7^{(x+4)} = 8^x$

Exercise 24.9. Find the solution to $15^{(x-11)} = 11^x$ to the nearest tenth.

24.2 Logarithms

There is only one method for solving logarithmic equations:

1) Express the equation as either

______________ or __________________

2) Rewrite as

__________ or __________

3) Solve for the variable.

4) Check your solutions!

Exercise 24.10. Solve $\log_2(x-4)=3$

Exercise 24.11. Solve $\log_4(-12x+108)-\log_4(3)=1$

Exercise 24.12. Solve $\log_2(12x + 48) + \log_2(3) = 2$

Exercise 24.13. Solve $\log_2(x + 2) + \log_2(x - 5) = 3$

Exercise 24.14. Solve $\log_3(x+5) + \log_3(x+1) = \log_3(21)$

Exercise 24.15. Solve $\log_3(x+5) + \log_3(x-5) = \log_3(24)$

Exercise 24.16. Solve $\log(x+3) - \log(x-2) = \log(6)$

24.3 Application Problems

Exercise 24.17. Thomas invests \$50,000 into an account that compounds interest continuously at a rate of 15%. How long will it take for the investment to triple?

Exercise 24.18. Mai invests \$20,000 at age 20. They hope the investment will be worth \$500,000 when they turn 40. If the interest compounds continuously, what would the rate of growth (interest rate) need to be for this to occur?

Exponential growth

Exercise 24.19. Researchers record that a certain bacteria population grew from 250 to 750,000 in 48 hours. At this rate of growth, how many bacteria were there at 8 hours?

Exercise 24.20. A sample of bacteria is growing at a rate of 8% per hour, compound continuously. How long will it take for the population to double in size?

Exercise 24.21. A pride of lions is growing at a rate of 0.5% per year, compounded continuously. If the rate remains the same, how long will it take for the population to be 275% of its current size (round up to the next whole number).

Exponential decay

Exercise 24.22. A patient takes a medication with a particular half-life. Initially, there are 11 mg of the medication in their system, but after 70 minutes there are only 7 mg left. After how many minutes will there be only 3 mg left?

Exercise 24.23. A small object has an initial temperature of $135°F$ and it is dropped into a tub with a temperature of $60°F$. The function $f(t) = Ce^{-kt} + 60$ represents the temperature of the object after t minutes

After 6 minutes the object is $85°F$. What is the approximate value of k to 3 decimal places?

Exercise 24.24. A small object has an initial temperature of $75°F$ and it is dropped into a lake with a temperature of $33°F$. The function $f(t) = Ce^{-kt} + 33$ represents the temperature of the object after t minutes

After 2 minutes the object is $55°F$. What will the temperature be after 4 minutes?

THE END