Math 324: Linear Algebra

Section 4.6: Null Spaces

Mckenzie West

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Last Time.

- Row and Column Spaces
- Rank of a Matrix

Today.

- Null space of a matrix
- Dimension of solution spaces
- Solutions of systems of equations

Exercise 1 (Warmup).

Find a basis for row(A) and col(A) for the matrix A, which has the given reduced row echelon form:

$$A = \begin{bmatrix} 1 & 2 & -3 & -10 & -2 \\ 0 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & -10 & 3 \\ 1 & 2 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 2.

Solve the system

$$\begin{array}{rcrcrcrcrcl}
x_1 & + & 2x_2 & - & 3x_3 & - & 10x_4 & - & 2x_5 & = & 0 \\
 & & & & x_3 & + & 2x_4 & + & x_5 & = & 0 \\
3x_1 & + & 6x_2 & + & x_3 & - & 10x_4 & + & 3x_5 & = & 0 \\
x_1 & + & 2x_2 & + & 2x_3 & & & + & x_5 & = & 0.
\end{array}$$

(Hint: Use the matrix on the previous slide.)

Theorem 4.16.

If A is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$ called the `nullspace` of A and is denoted $N(A)$, or sometimes $\text{null}(A)$. Setwise, we can write

$$N(A) = \{\vec{x} \in \mathbb{R}^n \ : \ A\vec{x} = \vec{0}\}.$$

The `nullity` of A is defined to be the dimension of $N(A)$.

The proof for this theorem is recorded to YouTube, please watch.
`https://youtu.be/7Z4Bqle4pLY`

Exercise 3.

Consider the matrix,

$$B = \begin{bmatrix} -2 & -1 & -1 & -1 \\ 3 & 1 & 2 & 0 \\ 3 & 0 & 3 & -3 \end{bmatrix}.$$

(a) Find the rank and nullity of B.

(b) Find a subset of the column vectors of B that forms a basis for the column space of B. (Recall the video from yesterday.)

Exercise 4.

Recall the matrix A from Exercises **??** and **??**, for which $\operatorname{rank}(A) = 3$ and $\operatorname{nullity}(A) = 2$:

$$A = \begin{bmatrix} 1 & 2 & -3 & -10 & -2 \\ 0 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & -10 & 3 \\ 1 & 2 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, recall the matrix B from Exercise **??**, for which $\operatorname{rank}(B) = \operatorname{nullity}(B) = 2$:

$$B = \begin{bmatrix} -2 & -1 & -1 & -1 \\ 3 & 1 & 2 & 0 \\ 3 & 0 & 3 & -3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

With these examples in mind, what can you deduce about the relationship between rank and nullity?

Theorem 4.17.

If A is an $m \times n$ matrix of rank r, then the dimension of the solution space of $A\vec{x} = \vec{0}$ is $n - r$. That is,

$$n = \operatorname{rank}(A) + \operatorname{nullity}(A).$$

Question 5.

What is the rank of an invertible $n \times n$ matrix?
Conversely, if the $n \times n$ matrix A has rank n, is it invertible?

Equivalent Conditions for Square Matrices.

Let A be an $n \times n$ matrix, then the following are equivalent:

1. A is invertible.

2. $A\vec{x} = \vec{b}$ has a unique solution for an $n \times 1$ matrix $\vec{b}$

3. $A\vec{x} = \vec{0}$ has only the trivial solution

4. A is row equivalent to I_n

5. $\det(A) \neq 0$

6. $\text{rank}(A) = n$

7. The n row vectors of A are linearly independent.

7b. The n row vectors of A span $\mathbb{R}^n$.

8. The n column vectors of A are linearly independent.

8b. The n column vectors of A span $\mathbb{R}^n$.

Brain Break.

What's the most spontaneous thing you've ever done?

Note.

Morally speaking, there *should* be a relationship between solutions to the homogeneous system $A\vec{x} = \vec{0}$ and to the inhomogeneous system $A\vec{x} = \vec{b}$.

Theorem 4.18.

Let $\vec{x}_p$ be a fixed solution to the system $A\vec{x} = \vec{b}$. Then EVERY solution to $A\vec{x} = \vec{b}$ is of the form $\vec{u} = \vec{x}_p + \vec{x}_h$ where $\vec{x}_h$ is a solution to the corresponding homogeneous system, $A\vec{x} = \vec{0}$.

Exercise 6.

Let $A = \begin{bmatrix} -2 & -1 & 2 \\ 3 & \frac{3}{2} & -3 \\ 3 & \frac{3}{2} & -3 \end{bmatrix}$.

(a) Solve the homogeneous system $A\vec{x} = \vec{0}$.
That is, find a basis for $N(A)$.

(b) Let $\vec{b} = (3, -9/2, -9/2)$. Show that $\vec{x}_p = (-1, 1, 1)$ is a particular solution to $A\vec{x} = \vec{b}$. That is, check that

$$A \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -9/2 \\ -9/2 \end{bmatrix}.$$

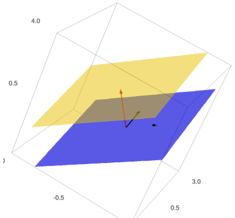
Exercise ?? (Cont).

(c) Then Theorem 4.18 says the set of solutions to $A\vec{x} = \vec{b}$ can be written as

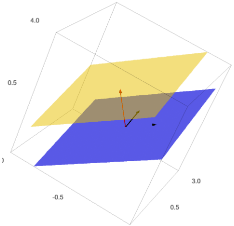
$$\{(-1, 1, 1) + s\vec{u} + t\vec{v}\}$$

where $\vec{u}$ and $\vec{v}$ are the basis vectors you found for $N(A)$. (Fill those in.)

(d) The image below depicts Theorem 4.18 geometrically. The blue plane is $N(A)$. The gold, slightly transparent plane is the set of all solutions to $A\vec{x} = \vec{b}$. Lastly, the red vector between the planes is the particular solution $\vec{x}_p = (-1, 1, 1)$.



Exercise ?? (Cont).



(e) Open `https://people.uwec.edu/westmr/teaching/1920-1-324/docs/sage4-6.txt` and copy everything to `https://sagecell.sagemath.org/` so you can move/rotate the picture yourself.

Theorem 4.19.

The system $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in the column space of A.

Proof.

Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ where $\vec{a}_j$ represents the j-th column of A. Notice that

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n.$$

Therefore $A\vec{x} = \vec{b}$ if and only if $\vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$ is a linear combination of the columns of A. $\square$

Exercise 7.

The system, $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 1 & 3 & 10 \\ -2 & 7 & 32 \\ -1 & 3 & 14 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 18 \\ 29 \\ 12 \\ 8 \end{bmatrix}$$

is consistent. Thus Theorem 4.19 says that $\vec{b}$ can be written as a linear combination of the columns of A. Find such a combination. Hint: Row reduce the augmented matrix $[A|\vec{b}]$. The entries in the last column will indicate which values to use.

Exercise 8.

The dimension of the row space of a 5×3 matrix A is 2.

(a) What is the dimension of the column space of A?

(b) What is the rank of A?

(c) What is the nullity of A?

(d) What is the dimension of the solution space of the homogeneous system $A\vec{x} = \vec{0}$?