Math 324: Linear Algebra

1.1: Introduction to Systems of Linear Equations

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Today.

- Linear Equations
- Solving Linear Equations and Parametrization
- Linear Systems of Equations
- Solving Linear Systems of Equations
- Consistent vs Inconsistent Systems

Definition.

A linear equation in n variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ are real numbers called coefficients and b is a real number called the constant term.
We call a_1 the leading coefficient and x_1 the leading variable.

Exercise 1.

Which of the following equations are linear equations? For the linear equations, identify the constant term, leading coefficient, and leading variable.

(a) $x - 2y = 1$

(b) $\sin x + \cos y = 1$

(c) $x - xy + y = 3$

(d) $5x_1 = 6 + 3x_2$

(e) $3(x_1+5) = 2(-6-x_2)-2x_3$

(f) $\dfrac{2z}{z+3} = \dfrac{3}{z-10} + 2$

Definition.

A solution to a linear equation in n variables is a sequence of n real numbers $s_1, s_2, \ldots, s_n$ arranged so that when you substitute the values $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ the equation is satisfied.

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Exercise 2.

Find a solution to $x - 2y = 1$. (Be creative here.)
How many solutions does this equation have? Can you describe them all?

Definition.

The collection of all solutions of a linear equation is called a solution set. If you're asked to solve a linear equation, you should find all of the solutions and describe them.

Example.

The solutions to the linear equation $x + y = 4$ are all of the form $x = 4 - y$. We write these parametrically as $y = t$, $x = 4 - t$.
We call y a free variable because it is free to be whatever it wants to be.
We call t a parameter and specific solutions can be found by assigning values to this parameter.

Note

Typically, leading variables are not used as free variables.

Exercise 3.

Write all solutions to $5x = 6 + 3y$ in parametric form.

Exercise 4.

Write all solutions to $6x - 2y + 3z = 1$ in parametric form. (Note: you may need more than one parameter.)

Definition.

A `system of` m `linear equations in` n `variables` is a set of m equations each of which is linear in up to n variables $x_1, \ldots, x_n$.

Example.

A system of 2 equations in 3 variables:

$$\begin{array}{rcrcrcr} 3x & - & 2y & + & 3z & = & 4 \\ x & & & + & z & = & 5 \end{array}$$

Definition.

A `solution` to a system of m linear equations in n variables is any sequence of n real numbers that is a solution to every linear equation in the system. To `solve` a linear system is to find all of the solutions to the system.

A system is called `consistent` if it has at least one solution and `inconsistent` if it has none.

Exercise 5.

Find a solution to the system

$$\begin{array}{rcrcrcr} 3x & - & 2y & + & 3z & = & 4 \\ x & & & + & z & = & 5 \end{array}$$

Exercise 6.

Come up with a linear system of 2 equations and 2 variables that is inconsistent.

Exercise 7.

For each of the following systems, graph the two lines in the xy-plane. Where do they intersect? How many solutions does this system of linear equations have?

(a)
$$\begin{aligned} 3x &- y = 1 \\ 2x &- y = 0 \end{aligned}$$

(b)
$$\begin{aligned} 3x &- y = 1 \\ 3x &- y = 0 \end{aligned}$$

(c)
$$\begin{aligned} 3x &- y = 1 \\ 6x &- 2y = 2 \end{aligned}$$

Question 8.

Is it possible for a system of linear equations in two variables to have two solutions?

Exercise 9.

Consider the system of equations:

$$\begin{array}{rcrcrcrcl} -2x & - & 2y & + & 2z & - & w & = & 2 \\ & & 2y & - & 2z & + & w & = & 0 \\ 2x & & & & & & & = & 2 \end{array}$$

Find all solutions to the system.

Exercise 10.

In a chemical reaction, atoms reorganize in one or more substances. For instance, when methane gas (CH_4) combines with oxygen (O_2) and burns, carbon dioxide (CO_2) and water (H_2O) form. Chemists represent this process by a chemical equation of the form

$$(x_1)CH_4 + (x_2)O_2 \to (x_3)CO_2 + (x_4)H_2O.$$

Write a system of linear equations in the four variables x_1, x_2, x_3, x_4 corresponding to the fact that the amount of each element–carbon, hydrogen, and oxygen–must be equal before and after the reaction.