Math 324: Linear Algebra

Mathematical Statements
A Definition of True

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Last Time.

- Consumer Preference Models
- Cryptography-Encoding and Decoding using Matrices
- Leontief Input-Output Models
- Least Squares Approximation

Today.

- Statements
- Exploration and Counterexamples
- Quantifiers
- Conditional Statements
- Compound Statements
- Negation

Definition.

A `mathematical statement` is a declarative sentence that is either true or false but not both.

Example.

- There is a matrix A such that $2A = 3A - 2I_2$.
- For all $m \times n$ matrices $A + B = B + A$.
- All systems of linear equations have a solution.
  While this statement is not correct—there are examples of systems without solutions—it is a statement.

Non-Examples.

- $AB = 7I$
  We do not know what A and B are. Really we also don't know what I is. are we even working with matrices here?
- Find a least squares approximation for the data
  $(2, 1),\ (3, 2),\ (1, 1),\ (5, 7)$.
  This is a directive.
- Linear algebra is awesome.
  While in general I believe this is true, I recognize it is my opinion.

Exercise 1.

Can you add any context or change the three sentences above to each be statements?

Exercise 2.

Share the statements you wrote down for your pre-class homework. Decide as a group if they are indeed statements.

Determining if a Statement is True or False - Exploration.

- Make a preliminary guess, ask some questions.
- Examples! Write down some examples test the statement. If you find one that shows the statement is false, this is called a `counterexample`.
- Use prior knowledge. Are there any theorems or things you have learned that relate to the statement?
- Brainstorm together. Collaborate.

Exercise 3.

Use methods of exploration to test the statement:
For all $n \times n$ matrices A and B,

$$(A+B)^2 = A^2 + 2AB + B^2.$$

(a) Before testing or anything, think about this statement. Would you guess it is true or false?

(b) Does is look like something you have seen before?

(c) Test several examples of A and B. Remember that matrices are weird and don't always function like normal, especially when multiplying.

Exercise 4.

Consider a variation on the statement from exercise 3.
There exist $n \times n$ matrices A and B such that

$$(A+B)^2 = A^2 + 2AB + B^2.$$

Is this statement true?

Note.

Notice that the statement in exercise 3 refers to all $n \times n$ matrices. Since it is so broad, the statement is false. On the other hand exercise 4 is incredibly more specific and has much more likely of a chance to be true.

Definition.

We call the phrases *for all $n \times n$ matrices* or *for some $n \times n$ matrices* quantifiers. Formally, a `quantifier` is an expression that indicates the scope of a statement.
Informally, we may denote "for all" using $\forall$ and "for some" or "there exists" by $\exists$.

Exercise 5.

Add a quantifier that makes each of the following into statements.

(a) $3A - 2B = O$

(b) $(A+B)^T = A^T + B^T$

(c) $[A|I]$ row reduces to $[I|A^{-1}]$

Note.

To disprove a "for all" statement, you just have to find one example where the statement is false.
To prove a "there exists" statement, you just have to find one example where the statement holds.
However, proving a "for all" or disproving a "there exists" almost always requires a more detailed write-up as examples will not suffice.

Brain Break.

Which season of the year is your favorite?

Definition.

A `conditional statement` (or `implication`) is a statement that can be written in the form "If P then Q," where P and Q are sentences.
We call P the `hypothesis` and Q the `conclusion`.
We can denote such a statement by $P \Rightarrow Q$, which we often read as "P implies Q."

Note.

Such a statement means that Q is true whenever P is true. Notice though if P is false, the truth value of Q does not matter.

Exercise 6.

Consider the statement.

If Pepper sits, then they get a treat.

(a) What are the hypothesis and conclusion?

(b) There are four possible cases for this statement relating to whether each of the hypothesis and conclusion are true or false. What are they, both generally speaking and in the case of this statement?

Exercise 7.

Consider the conditional statement:
If A is a 2×2 matrix satisfying $A^2 = A$, then $A = I_2$.

(a) What are the hypothesis and conclusion?

(b) Give an example satisfying each of the possible true/false conditions (if possible):

P	Q
T	T
T	F
F	T
F	F

Note.

The statement in Exercise **??** is false because we can come up with an example where the hypothesis is true and the conclusion is false.

Definition.

Suppose P and Q are statements.
`conjunction`: We call the statement *P and Q* true exactly when both P and Q are true. Denote this by $P \wedge Q$.
`disjunction`: We call the statement *P or Q* true exactly when at least one of P or Q is true. Denote this by $P \vee Q$.

Exercise 8.

Let P be the statement *the matrix A is singular* and Q the statement *the matrix A is* 2×2.

(a) Write down a matrix A that satisfies both P and Q.

(b) Write down a matrix A that satisfies P but not Q.

(c) Write down a matrix A that satisfies Q but not P.

(d) Write down a matrix A that satisfies neither P nor Q.

(e) Which or your matrices satisfy $P \wedge Q$? $P \vee Q$?

Definition.

The negation of the statement P is the statement "not P" and is denoted by $\neg P$.
We say $\neg P$ is true whenever P is false and $\neg P$ is false whenever P is true.

Example.

The negation of the statement *Pepper is sitting* is the statement *Pepper is not sitting*.

Exercise 9.

Write the negation of each of the following.

(a) $4 \leq 5$

(b) 7 is an even number

(c) $3 + 4 = 2 + 5$

(d) I_5 is singular

Exercise 10.

For the statements

$$P\colon \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \text{ is square} \qquad Q\colon \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \text{ is invertible,}$$

write each of the following statements as English sentences and determine whether they are true or false. Notice that P is true and Q is false.

(a) $P \wedge Q$.

(b) $P \vee Q$.

(c) $P \wedge \neg Q$.

(d) $\neg P \vee \neg Q$.

Negating Compound Statements.

- $\neg(P \wedge Q) = \neg P \vee \neg Q$
- $\neg(P \vee Q) = \neg P \wedge \neg Q$

Exercise 11.

Explore the compound statements and their negations in the context of the statements

P: Dr. Pepper is delicious. $\qquad$ Q: The clock is wrong.

(a) $P \wedge Q$

(b) $\neg P \vee Q$

(c) $\neg(P \wedge Q)$

(d) $\neg(P \vee \neg Q)$