Math 324: Linear Algebra

Section 7.1: Eigenvalues and Eigenvectors
Section 7.2: Diagonalizability

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Last Time.

- Eigenvalues and Eigenvectors
- Characteristic Polynomials and Equations

Today.

- Eigenvalues of Triangular Matrices
- Eigenvalues of Linear Transformations

Recall.

Theorem 7.2.

Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.
2. The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)\vec{x} = \vec{0}$.
3. The eigenspace of A corresponding to λ is the nullspace of the matrix $\lambda I - A$.

Question 1.

What is the maximum number of eigenvalues that an $n \times n$ matrix can have?

Theorem 3.2.

If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal. That is,

$$\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

Note.

We saw this back in Chapter 3. Triangular matrices have easy determinants.
Additionally, if A is triangular, so is $\lambda I - A$.

Question 2.

If A is triangular with diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$, what are the diagonal entries of $\lambda I - A$?

Theorem 7.3.

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

Exercise 3.

Find the eigenvalues and eigenspaces of

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 4.

Find the eigenvalues and eigenspaces of

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Brain Break.

If you won the lottery but only had one day to spend the money, what would you buy?

Pepper says “blankets!”

Definition.

A square matrix A is `diagonalizable` if there is a diagonal matrix D and and invertible matrix P such that $D = P^{-1}AP$.

Exercise 5.

Let $A = \begin{bmatrix} -4 & -15 \\ 2 & 7 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.
Verify that $P^{-1}AP$ is diagonal, and confirm that A is diagonalizable.

Exercise 6.

Is $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ diagonalizable?

Theorem 7.5.

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.
Moreover if $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$ is a matrix whose columns are these n linearly independent eigenvectors, then $P^{-1}AP$ is diagonal.

Exercise 7.

The matrix $A = \begin{bmatrix} 0 & 2 \\ 5 & -3 \end{bmatrix}$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -5$.

There are eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

(a) Let $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -5 \end{bmatrix}$ and compute $P^{-1}AP$.

(b) Let $Q = \begin{bmatrix} \vec{v}_2 & \vec{v}_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -5 & 1 \end{bmatrix}$ and compute $Q^{-1}AQ$.

Exercise 8.

Determine whether the matrix is diagonalizable. (Find all eigenspaces and add up their dimensions.)

(a) $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Note.

It may be helpful to use Sage for this:

```
A = matrix([[1,0],[3,4]])
print(A.eigenvalues())
print(A.right_eigenspaces())
```

Notice that the eigenspaces command indicates a 'right' eigenspace, as in we're multiplying on the right.
And the output includes each of the eigenvalues followed by their corresponding eigenspaces with basis.