Math 324: Linear Algebra

Section 3.1: The Determinant of a Matrix

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Last Time.

- Proof by Induction

Today.

- The determinant of a 2×2 matrix
- Minors
- Cofactors
- The determinant of an $n \times n$ matrix.

Definition.

Generally speaking, the `determinant` of an $n \times n$ matrix is a real number that we can associate to it that indicates whether $A\vec{x} = \vec{0}$ has nontrivial solutions.

Note.

Recall that $A\vec{x} = \vec{0}$ has a unique solution if and only if A is invertible. Therefore, the determinant can also indicate whether A is invertible.

Question 1.

In light of this discussion, what do you think the determinant of the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is?

Notation.

If A is an 2×2 matrix, denote the determinant of A by $|A|$, $\det(A)$,

$$\text{or } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Exercise 2.

Compute each of the following:

(a) $\begin{vmatrix} -5 & 2 \\ 4 & -3 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -\frac{1}{3} \\ -2 & -5 \end{vmatrix}$ (c) $\begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix}$

Question 3.

What are the possible values for $|A|$?

Definition.

If A is a square matrix, then the (i, j)-`minor`, M_{ij}, is the determinant of the matrix obtained by deleting the ith row and jth column of A.

Exercise 4.

Determine the given minor:

(a) $(1, 2)$-minor of $\begin{bmatrix} 0 & 3 & 3 \\ 4 & 3 & 5 \\ -1 & 0 & 3 \end{bmatrix}$

(b) $(3, 3)$-minor of $\begin{bmatrix} -1 & 2 & 4 \\ 2 & -1 & -3 \\ 1 & 4 & 5 \end{bmatrix}$

(c) $(2, 1)$-minor of $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Definition.

If A is a square matrix, then the (i,j)-`cofactor`, C_{ij}, is a signed version of the (i,j)-minor given by $C_{ij} = (-1)^{i+j} M_{ij}$.

Exercise 5.

Determine the given cofactor:

(a) $(1,2)$-cofactor of $\begin{bmatrix} 0 & 3 & 3 \\ 4 & 3 & 5 \\ -1 & 0 & 3 \end{bmatrix}$

(b) $(3,3)$-cofactor of $\begin{bmatrix} -1 & 2 & 4 \\ 2 & -1 & -3 \\ 1 & 4 & 5 \end{bmatrix}$

(c) $(2,1)$-cofactor of $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Note.

The sign of the cofactor alternates across rows and columns:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$3 \times 3 \qquad\qquad 4 \times 4 \qquad\qquad n \times n$$

Definition.

If A is an $n \times n$ matrix ($n \geq 2$), the `determinant` of A is the value

$$\det(A) = |A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

Note.

We call this an inductive definition because we use the determinant of one size smaller matrix to define the determinant of the next.

Exercise 6.

Use the definition given here to compute

$$\begin{vmatrix} 0 & 3 & 3 \\ 4 & 3 & 5 \\ -1 & 0 & 3 \end{vmatrix}$$

Brain Break.

What band were you obsessed with in middle school?

Note.

Notice how convenient it was that one of the entries in the first row of the last matrix was 0. We didn't actually have to compute the $(1, 1)$-cofactor to compute the determinant.

We can actually take advantage of rows (and columns!) that have a lot of zeros when we take determinants.

Theorem 3.1 (Expansion by Cofactors).

Let A be a square matrix of order n. Then the determinant of A is given by:

(a) the i-th row expansion

$$\det(A) = |A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}.$$

(b) or the j-th column expansion

$$\det(A) = |A| = c_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}.$$

Exercise 7.

Compute the determinant of the following matrix in two different ways: (a) the 3rd row expansion and (b) the 4th column expansion:

$$\begin{bmatrix} -4 & -2 & 5 & 0 \\ -3 & -1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ -5 & 0 & 0 & 3 \end{bmatrix}.$$

Hopefully you got the same answer in both cases. Why do you think that happened?

Exercise 8.

Use your choice of row/column cofactor expansion(s) to compute each of the following:

(a) $\begin{vmatrix} -2 & 1 & 0 \\ 4 & 0 & -2 \\ -4 & -1 & 5 \end{vmatrix}$

(b) $\begin{vmatrix} 2 & 2 & -2 & 1 \\ 0 & -1 & 5 & -2 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 4 \end{vmatrix}$

(c) $\begin{vmatrix} -5 & 5 & 0 & -4 \\ 5 & 4 & 0 & 0 \\ -1 & -2 & 2 & -3 \\ -4 & 0 & 0 & 2 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & -3 & -2 & 0 \\ -3 & 0 & 1 & 1 \\ 5 & 0 & 0 & 0 \\ -5 & 0 & -4 & -4 \end{vmatrix}$

Exercise 9.

Determine the values of λ for which the determinant of the given matrix is zero:

(a) $\begin{bmatrix} \lambda+2 & 0 \\ -5 & \lambda-3 \end{bmatrix}$

(b) $\begin{bmatrix} \lambda & 1 \\ 7 & \lambda+6 \end{bmatrix}$

(c) $\begin{bmatrix} \lambda+1 & 0 & 0 \\ 1 & \lambda-2 & 1 \\ 1 & 0 & \lambda \end{bmatrix}$

Exercise 10.

Evaluate the determinant to verify the equation:

(a) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$

(b) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$

Exercise 11.

Show that the system of linear equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

has a unique solution if and only if the determinant of the coefficient matrix is nonzero.