Math 324: Linear Algebra

Section 4.3:Subspaces of Vector Spaces

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Last Time.

- Properties of $\vec{0}$
- Abstract Vector Spaces
- Important Vector Spaces

Today.

- Proving Something is not a Vector Space
- Definition of Subspaces
- Test for Subspaces

Theorem 4.4.

Let $\vec{v}$ be any element of a vector space V, and let c be a scalar. Then the following properties are true.

1. $0\vec{v} = \vec{0}$
2. $c\vec{0} = \vec{0}$
3. If $c\vec{v} = \vec{0}$, then $c = 0$ or $\vec{v} = \vec{0}$.
4. $(-1)\vec{v} = -\vec{v}$

Exercise 1.

Prove Theorem 4.4 property 1 following the steps on Canvas.

Exercise 2.

Verify all four of these results for the goofy vector space from last class: (first remind yourself of what $\vec{0}$ and $-\vec{v}$ were)

$$V = \{(x, y) \ : \ x, y \in \mathbf{R} \textit{ and } y > 0\} \textit{ with the operations:}$$

$$\begin{array}{lrcl} \textit{Addition:} & (x_1, y_1) \oplus (x_2, y_2) & = & (x_1 + x_2, y_1 y_2) \\ \textit{Scalar Multiplication:} & c * (x, y) & = & (cx, y^c) \end{array}$$

Brain Break.

Would you rather fight 1 horse-sized duck or 100 duck-sized horses?

Definition.

A nonempty subset W of a vector space V is called a `subspace` of V when W is a vector space under the operations of addition and scalar multiplication defined in V.

Example.

(a) $W = \{(x, x) \ : \ x \in \mathbf{R}\}$ is a subspace of $V = \mathbf{R}^2$

(b) $W = P_1$ is a subspace of $V = P_2$ (polynomials)

Warning.

$W = \mathbf{R}^2$ is NOT a subspace of $V = \mathbf{R}^3$ because elements of $\mathbf{R}^2$ are ordered pairs while elements of $\mathbf{R}^3$ are ordered triples.

Note.

One nice thing about subspaces is that they use the same operations as something that is a vector space. Therefore, axioms 2, 3, 7, 8, 9, and 10 are already known.

Theorem 4.5 - Subspace Test.

A subset W of a vector space V is a subspace if and only if the following hold.

1. $\vec{0} \in W$ (or W is non-empty).

2. If $\vec{u}, \vec{v} \in W$, then $\vec{u} + \vec{v} \in W$.

3. If $\vec{u} \in W$ and c is a scalar then $c\vec{u} \in W$.

Example.

Proofs using Theorem 4.5 should use the following format. Note: this is an essential type of proof to know.
Claim: The set $\{(x, 2x) \ : \ x \in \mathbf{R}\}$ with the standard operations on $\mathbf{R}^2$ is a vector space.

Proof.

Let $W = \{(x, 2x) \ : \ x \in \mathbf{R}\}$ with the standard operations on $\mathbf{R}^2$.
We use The Subspace Test (Theorem 4.5).

1. Notice that $\vec{0} = (0, 0)$ is in W as it is of the form $(x, 2x)$ with $x = 0$.
2. Let $\vec{u}, \vec{v} \in W$. Write $\vec{u} = (x, 2x)$ and $\vec{v} = (y, 2y)$ for $x, y \in \mathbf{R}$. Then
$\vec{u} + \vec{v} = (x, 2x) + (y, 2y) = (x + y, 2x + 2y) = (x + y, 2(x + y))$, which is in the form of W.

(Continued on the next slide.)

Example (Cont.).

3. Let c be a scalar. Then $c\vec{u} = c(x, 2x) = (cx, c(2x)) = (cx, 2(cx))$, which is in the form of W.

Therefore W satisfies the subspace test and is a subspace of $\mathbf{R}^2$, so it is a vector space. $\square$

Exercise 3.

Use the subspace test to determine which of the following are vector spaces. For those that are, write a formal proof as in the example. For those that are not, give an example to explain why they are not.

(a) The set $\{(x, 0) \ : \ x \in \mathbf{R}\}$ with the standard operations on $\mathbf{R}^2$.

(b) The set $\{(a, -a, a + 2) \ : \ a \in \mathbf{R}\}$ with the standard operations on $\mathbf{R}^3$.

(c) The set D_n of $n \times n$ diagonal matrices with the standard operations on M_n.

(d) The set of 2×2 matrices satisfying $A^2 = A$ with the standard operations on M_n.

(e) The set of 2×2 matrices of the form $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ with the standard operations on M_2.

Exercise ?? (Cont.)

Use the subspace test to determine which of the following are vector spaces. For those that are, write a formal proof as in the example. For those that are not, give an example to explain why they are not.

(f) The set $W = \{ax^2 \ : \ a \in \mathbf{R}\}$ with the standard polynomial operations.

(g) The set of all even continuous functions. Recall that a function is even if $f(-x) = f(x)$.