Math 324: Linear Algebra

2.3: The Inverse of a Matrix

Mckenzie West

Last Updated: December 29, 2023

Last Time.

- Inverses of Matrices
- Gauss-Jordan Elimination
- Inverses of 2×2 matrices

Today.

- Properties of Inverses
- Using Inverses to Solve System of Equations

Recall.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, A is invertible if and only if $ad - bc \neq 0$. If A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Note.

For the exercises today, compute the inverses of all 2×2 matrices by hand.
You may use a calculator or Sage to compute the inverse of the larger matrices. Recall the process of $[A|I] \to [I|A^{-1}]$.

Exercise 1 (Warm-up).

Compute the inverse of the matrix if it exists:

(a) $\begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

Exercise 2.

Find all x and y that make the matrix singular $\begin{bmatrix} -x & 2 \\ y & 5 \end{bmatrix}$

Remember to parametrize your solutions.

Exercise 3.

Find x so that $A^{-1} = A^2$ for $A = \begin{bmatrix} -2 & -3 \\ x & 1 \end{bmatrix}$

(Hint: This means $A(A^2) = I$.)

Theorem 2.8.

If A is an invertible matrix, k is a positive integer, and c is a nonzero scalar, then A^{-1}, A^k, cA, and A^T are invertible and the following are true:

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$

Exercise 4.

Verify property (1) of Theorem 2.8 by computing A^{-1} and $(A^{-1})^{-1}$ for $A = \begin{bmatrix} 1 & -9 \\ -3 & 5 \end{bmatrix}$.

Exercise 5.

Verify property (3) of Theorem 2.8 by computing $(2A)^{-1}$ and $\frac{1}{2}(A^{-1})$ for $A = \begin{bmatrix} 1 & -5 \\ -3 & 10 \end{bmatrix}$.

Exercise 6.

Follow the steps to verify a simplified version of property (2) of Theorem 2.8.

Claim.

If A is an invertible matrix then A^3 is invertible and $(A^3)^{-1} = (A^{-1})^3$.

(a) Remind yourself of the definition of an inverse.

(b) To verify the result, begin by computing A^3B and BA^3 for $B = (A^{-1})^3$. Carefully note the location of the exponents here.

(c) If the two product equal I, then you can conclude that the inverse of A^3 is $(A^{-1})^3$.

Brain Break.

What is your favorite zoo animal?

Theorem 2.9.

If A and B are invertible matrices of order n then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Note.

ORDER IS INCREDIBLY IMPORTANT.

Exercise 7.

Let

$$A = \begin{bmatrix} -2 & 1 & -2 \\ -1 & 2 & -3 \\ 1 & -1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 2 \\ -3 & -1 & 3 \end{bmatrix}.$$

Use the *Gauss-Jordan method* to compute A^{-1} and B^{-1}.
Multiply these in the right order to compute $(AB)^{-1}$.
*Use Sage to row reduce and verify that you actually did get $(AB)^{-1}$.
**Recall the *Gauss-Jordan method* from last time: if A is invertible, $[A|I]$ row reduces to $[I|A^{-1}]$.

Theorem 2.10.

If C is an invertible matrix, then the following properties hold:

1. If $AC = BC$, then $A = B$ `Right cancellation`
2. If $CA = CB$, then $A = B$ `Left cancellation`

Exercise 8.

Suppose A is invertible and $A^2 = A$, what must be true about A?
(Hint: Write the equation as $AA = AI$.)
Is the same true for invertible matrices with $A^3 = A$?

Note.

ORDER IS INCREDIBLY IMPORTANT.

Exercise 9.

Consider $A = \begin{bmatrix} -6 & 3 \\ -4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -3 \\ -2 & -3 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$.
Verify that although $A \neq B$, $AC = CB$.

Recall.

We can write a system of linear equations as the matrix product $A\vec{x} = \vec{b}$, where

- A is the coefficient matrix;
- $\vec{x}$ is the column vector consisting of the variables;
- and $\vec{b}$ is the column vector consisting of the constant terms.

Theorem 2.11.

If A is an invertible matrix, then the system of linear equations $A\vec{x} = \vec{b}$ has a unique solution given by $\vec{x} = A^{-1}\vec{b}$.

Exercise 10.

Use an inverse matrix to solve each system of linear equations:

(a)
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ x_1 & + & 2x_2 & - & x_3 & = & 4 \\ x_1 & - & 2x_2 & + & x_3 & = & -2 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & - & x_3 & = & 3 \\ x_1 & - & 2x_2 & + & x_3 & = & -3 \end{array}$$

Exercise 11.

Use a matrix inverse to solve the following.
A group took a trip to the zoo on a bus, at \$3 per child and \$3.20 per adult for a total of \$118.40.
They took the train back from the zoo at \$3.50 per child and \$3.60 per adult for a total of \$135.20.
How many children, and how many adults?

Exercise 12.

Compute the inverse of each of the following matrices, if it exists. (Use a calculator or Sage.)

(a) $\begin{bmatrix} 4 & 0 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} -3 & 4 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

We call a matrix of this form `upper triangular` because the only non-zero entries are in the upper right-hand triangle.
Make a conjecture about the shape of the inverses of upper triangular matrices – and when they exist.