Math 324: Linear Algebra

Section 5.4: Orthogonal Complements

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Last Time.

- Properties of Inner Products
- Orthogonal Sets

Today.

- Orthogonal Complements
- Least Squares Analysis

Key Idea 1.

$A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b} \in \operatorname{col}(A)$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Exercise 1.

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Show that $\vec{b} = \begin{bmatrix} -7 \\ -5 \\ 3 \end{bmatrix}$ is in $\operatorname{col}(A)$ by solving the system $A\vec{x} = \vec{b}$ and writing $\vec{b}$ as a combination or the columns using the entries of $\vec{x}$ as the coefficients.

Key Idea 2.

If $A\vec{x} = \vec{0}$, then $\vec{x}$ is orthogonal to every row of A:

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

That is, $\vec{a}_i \cdot \vec{x} = 0$ for all rows $\vec{a}_i$ of A.

Exercise 2.

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Show that $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to every row of A by computing $A\vec{x}$.

Key Idea 3.

If $\vec{u}$ is orthogonal to each of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then $\vec{u}$ is orthogonal to every linear combination, $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$:

$$\begin{aligned}\vec{u}\cdot(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n)&\\ &= c_1(\vec{u}\cdot\vec{v}_1) + c_2(\vec{u}\cdot\vec{v}_2) + \cdots + c_n(\vec{u}\cdot\vec{v}_n)\\ &= c_1(0) + c_2(0) + \cdots + c_n(0) = 0.\end{aligned}$$

Morally, this means that to check $\vec{u}$ is orthogonal to every vector in V, we just need to check $\vec{u}$ is orthogonal to every vector in a basis for V.

Exercise 3.

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Explain why $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to every linear combination of the rows of A using Exercise 2.

Key Idea 4.

If V is a subspace of $\mathbb{R}^n$, we define the `orthogonal complement` of V to be

$$\begin{array}{rcl} V^\perp & = & \{\text{all vectors orthogonal to all vectors in } V\} \\ & = & \{\text{all vectors orthogonal to the basis vectors of } V\} \end{array}.$$

A very important note is that $\dim(V) + \dim(V^\perp) = n$.

Exercise 4.

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Explain why $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is in $\text{row}(A)^\perp$.

Key Idea 5.

If V is a subspace of $\mathbb{R}^n$ with basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$, we can find $V^\perp$ by computing

$$\text{null}\left(\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix}\right),$$

when we consider each $\vec{v}_1$ as a row vector.
Note $\dim(V) = m = \text{rank}(A)$ and $\dim(V^\perp) = \text{nullity}(A) = n - m$.

Exercise 5.

Let $V = \text{span}\{(3, 2, -1), (0, 2, 2)\}$. Find a basis for $V^\perp$.

Key Idea 6.

The Four Fundamental Subspaces of an $m \times n$ matrix A are

$V = \text{row}(A)$	$V^{\perp} = \text{null}(A)$	$\leftarrow$ subspaces of $\mathbb{R}^n$
$W = \text{col}(A) = \text{row}(A^T)$	$W^{\perp} = \text{null}(A^T)$	$\leftarrow$ subspaces of $\mathbb{R}^m$

Exercise 6.

Find the four fundamental subspaces of

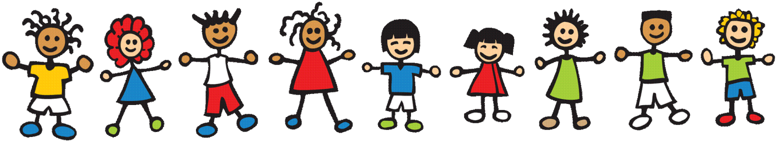
$$A = \begin{bmatrix} -2 & -3 & 3 & -2 \\ 1 & 2 & 2 & 1 \\ 0 & -3 & -2 & 0 \\ -1 & 5 & 0 & -1 \\ 2 & -1 & 4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note: a reduction of the transpose is on the next slide.

$$A^T \xrightarrow{\text{RREF}} \begin{bmatrix} 19 & 0 & 0 & -8 & 10 \\ 0 & 19 & 0 & -35 & 58 \\ 0 & 0 & 19 & -47 & 35 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Brain Break.

What's the first memory you have?



Key Idea 7.

If V is a subspace of $\mathbb{R}^n$ and $\vec{u} \in R^n$ we define the `projection of` $\vec{u}$ `onto` V, denoted $\operatorname{proj}_V \vec{u}$ to be the closest point in V to $\vec{u}$.
The $\operatorname{proj}_V \vec{u} - \vec{u}$ is orthogonal to V.
Therefore if $V = \operatorname{col}(A)$ then $\operatorname{proj}_V \vec{u} - \vec{u}$ is in $V^{\perp} = \operatorname{null}(A^T)$.
By definition, this means that

$$A^T(\operatorname{proj}_V \vec{u} - \vec{u}) = \vec{0} \tag{1}$$

But wait, $\operatorname{proj}_V \vec{u}$ is in $V = \operatorname{col}(A)$, so there is some $\vec{x}$ such that

$$A\vec{x} = \operatorname{proj}_V \vec{u}. \text{ (key idea 1)}$$

Continuing from (**??**), this means that we want to find $\vec{x}$ such that

$$\left.\begin{array}{rcl} A^T(A\vec{x} - \vec{u}) & = & \vec{0} \\ A^T A\vec{x} - A^T\vec{u} & = & \vec{0} \end{array}\right\} \Rightarrow A^T A\vec{x} = A^T\vec{u}$$

Key Idea 7 (Cont.).

Therefore if $V = \mathrm{col}(A)$, to find the projection of $\vec{u}$ onto V, first solve

$$A^T A\vec{x} = A^T\vec{u},$$

for $\vec{x}$.

Then

$$\mathrm{proj}_V\,\vec{u} = A\vec{x}.$$

Exercise 7.

Let $V = \mathrm{col}\left(\begin{bmatrix}1 & 0\\ 2 & 0\\ 0 & 3\\ 0 & 4\end{bmatrix}\right)$. Compute $\mathrm{proj}_V\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)$.

Definition.

For a set of points $(x_1, y_1),\ (x_2, y_2), \ldots,\ (x_n, y_n)$, the `least squares regression line` is given by the linear function $f(x) = a_0 + a_1 x$ that minimizes the sum of squared error:

$$\begin{aligned}[f(x_1) - y_1]^2 + [f(x_2) - y_2]^2 + \cdots + [f(x_n) - y_n]^2 \\ = \|(f(x_1), f(x_2), \ldots, f(x_n)) - (y_1, y_2, \ldots, y_n)\|^2.\end{aligned}$$

Definition.

Given an $m \times n$ matrix A and a vector $\vec{b} \in \mathbb{R}^m$, the `least squares problem` is to find $\vec{x} \in \mathbb{R}^n$ such that $\|A\vec{x} - \vec{b}\|^2$ is minimized.

That is, we solve for the $\vec{x}$ in the equation $A^T A\vec{x} = A^T\vec{b}$.

Exercise 8 (Optional).

Solve the least squares problem for

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

That is, solve for the $\vec{x}$ in the equation $A^T A\vec{x} = A^T\vec{b}$.

Exercise 9 (Optional).

The table shows the number of doctoral degrees y (in thousands) awarded in the United States from 2005 through 2008. Find the least squares regression line for the data. Then use the model to predict the number of degrees awarded in 2015. The t represent the year with $t = 5$ corresponding to 2005.

Year	2005	2006	2007	2008
Doctoral Degrees y	52.6	56.1	60.6	63.7

The least squares regression line is a line $y = a_0 + a_1 x$, and we're trying to solve the least squares problem with

$$A = \begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}, \vec{x} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} 52.6 \\ 56.1 \\ 60.6 \\ 63.7 \end{bmatrix}.$$