Math 324: Linear Algebra

Section 5.1: Length and Dot Product in $\mathbb{R}^n$

Mckenzie West

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Last Time.

- Nullspace of a matrix
- Dimension of solution spaces
- Solutions of systems of equations

Today.

- Dot Product
- Length
- Cauchy-Schwarz Inequality
- Orthogonality
- Triangle Inequality
- Pythagorean Theorem

Definition.

Let $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ be two vectors in $\mathbb{R}^n$. Define the `dot product` of $\vec{u}$ and $\vec{v}$ to be the **scalar**

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Exercise 1 (Warmup).

Let $\vec{u} = (1, 2)$ and $\vec{v} = (3, -1)$, compute $\vec{u} \cdot \vec{v}$.
What is $(2\vec{u}) \cdot \vec{v}$?

Theorem 5.3.

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$ and c is a scalar, then the following are true:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$

Exercise 2.

Prove property **1.** of Theorem 5.3 using $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$. Make sure to be precise about what $\vec{u} \cdot \vec{v}$ and $\vec{v} \cdot \vec{u}$ are.

Definition.

The `length` of a vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is the scalar

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Theorem 5.3.

If $\vec{v}$ is a vector in $\mathbb{R}^n$, then the following are true:

4. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

5. $\vec{v} \cdot \vec{v} \geq 0$ and $\vec{v} \cdot \vec{v} = 0$ if and only if $\vec{v} = \vec{0}$

Theorem 5.1.

Let $\vec{v}$ be a vector in $\mathbb{R}^n$ and let c be a scalar. Then

$$\|c\vec{v}\| = |c|\|\vec{v}\|,$$

where $|c|$ is the absolute value of the scalar c.

Alternate Proof Using Dot Products.

Let $\vec{v}$ be a vector in $\mathbb{R}^n$ and c a scalar. Then

$$\begin{aligned}
\|c\vec{v}\| &= \sqrt{(c\vec{v})\cdot(c\vec{v})} \\
&= \sqrt{c^2(\vec{v}\cdot\vec{v})} \\
&= |c|\sqrt{\vec{v}\cdot\vec{v}} \\
&= |c|\|\vec{v}\|.
\end{aligned}$$

Therefore, $\|c\vec{v}\| = |c|\|\vec{v}\|$, as desired. $\square$

Exercise 3.

Let $\vec{u} = (1, 0, 3)$ and $\vec{v} = (-2, -1, 2)$.
Compute each of the following:

(a) $\vec{u} \cdot \vec{v}$

(b) $\|\vec{u}\|$

(c) $\|\vec{u} - \vec{v}\|$

(d) $\vec{w} = \frac{\vec{u}}{\|\vec{u}\|}$ and $\|\vec{w}\|$

(e) $\vec{w} \cdot \vec{u}$ (same $\vec{w}$ as part **??**)

Definition.

A vector $\vec{v}$ in $\mathbb{R}^n$ is called a unit vector if $\|\vec{v}\| = 1$.

Theorem 5.2.

If $\vec{v}$ is a nonzero vector in $\mathbb{R}^n$, then the vector

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

has length 1 and has the same direction as $\vec{v}$. The vector $\vec{u}$ is called the unit vector in the direction of $\vec{v}$.

Exercise 4.

Let $\vec{u} = (1, 2)$ and $\vec{v} = (3, 1)$. What is the distance between the endpoints of these vectors?
Repeat the process for generic $\vec{u} = (x_1, y_1)$ and $\vec{v} = (x_2, y_2)$.

Definition.

The `distance` between two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is defined to be

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

Exercise 5.

Let $\vec{u} = (1, 3, -2)$ and $\vec{v} = (-1, 0, -1)$. Compute $d(\vec{u}, \vec{v})$.

Exercise 6.

Verify that for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$,

1. $d(\vec{u}, \vec{v}) \geq 0$
2. $d(\vec{u}, \vec{v}) = 0$ if and only if $\vec{u} = \vec{v}$
3. $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$.

Explain they this makes sense.

Exercise 7.

Use the dot product version of length to verify that

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u} \cdot \vec{v}).$$

Definition.

The `angle between the vectors` $\vec{u}$ `and` $\vec{v}$ is the angle $0 \leq \theta \leq \pi$ satisfying

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}.$$

Exercise 8.

Find the angle between $\vec{u} = (3, -1, 0, 2)$ and $\vec{v} = (-6, 2, 0, -4)$.

Warning, for our definition of the angle between the curves to work, we need to know that

$$\left|\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right| = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|\|\vec{v}\|} \leq 1.$$

Theorem 5.4: The Cauchy–Schwarz Inequality.

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|.$$

Note.

The proof of this is a neat application of the quadratic formula and is in the book.

Exercise 9.

Verify the Cauchy–Schwarz inequality using $\vec{u} = (1, 1, 3)$ and $\vec{v} = (3, -2, 1)$.

Brain Break.

What is your favorite word?

Definition.

Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are `orthogonal` if $\vec{u} \cdot \vec{v} = 0$.

Exercise 10.

Verify that $\vec{u} = (1, 0, 0)$ and $\vec{v} = (0, 2, 3)$ are orthogonal.

Exercise 11.

Verify that $\vec{u} = (1, 2, 3, 4)$ and $\vec{v} = (1, 0, 1, -1)$ are orthogonal.

Theorem 5.5: The Triangle Inequality.

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

Exercise 12.

Let $\vec{u} = (-2, 3)$ and $\vec{v} = (1, 1)$. Draw a picture of these vectors as well as $\vec{u}+\vec{v}$.

Explain why the Triangle Inequality is true in this case.

Geometrically speaking, what does the Triangle Inequality say?

Theorem 5.6: The Pythagorean Theorem.

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $\vec{u}$ and $\vec{v}$ are orthogonal if and only if

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

If we consider each vector $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ as $n \times 1$ column vectors, then

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \end{bmatrix}.$$

Exercise 13.

Use this method to compute the dot product of

$$\vec{u} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}.$$