Math 324: Linear Algebra

Direct Proofs and Cases

Mckenzie West

Last Updated: December 29, 2023

Last Time.

- Proper Proof Techniques
- Direct Proofs

Today.

- Proofs using Cases

Exercise 1.

Take time to read the proof from your classmates.
Indicate (a) 2 things their proof does well and (b) 1-2 things that could be improved. Then return the commented version to Dr. West.

Example.

Here is one possible proof the the statement from last time.

Claim.

If the product AB is a square matrix, then the product BA is defined.

Proof.

Let A and B be matrices such that AB is defined and square of size $n \times n$. By definition of matrix multiplication, this means that A has n rows and B has n columns. Then the middle dimensions for the product BA match and so the product is defined. $\square$

Do you think this is sufficient or would you add more?

Definition.

You may have seen previously that if A is a square matrix satisfying $A^2 = A$, then we call A `idempotent`.

Exercise 2 (Exercise 5 from last time).

Follow the steps to prove that if A and B are idempotent matrices satisfying $AB = BA$, then AB is also idempotent.

(a) Start with some scratch work, you want to show $(AB)^2 = AB$. What is $(AB)^2$? Where can you use the fact that $AB = BA$?

(b) Write down your assumptions.

(c) State what it means for A and B to be idempotent.

(d) Write down a sentence stating what you want to show.

(e) Go through the algebraic steps. Note: You may want to display them as in guideline (9).

(f) Write a concluding sentence as suggested in guideline (13).

Brain Break.

When you wake up, what is the very first thing you do?

Theorem 2.2 (3).

Let A be an $m \times n$ matrix and c a scalar. If $cA = O_{mn}$, then $c = 0$ or $A = O_{m,n}$.

Exercise 3.

Before getting to the proof of this theorem, let's think about how it's going to go.
Because we see the word 'or', we're going to want to use cases.
What might be some reasonable cases to consider? Note: There is more than one option.

Theorem 2.2 (3).

Let A be an $m \times n$ matrix and c a scalar. If $cA = O_{mn}$, then $c = 0$ or $A = O_{m,n}$.

Exercise 4.

Do some scratch work for each collection of cases:

(a) $c = 0$ or $c \neq 0$

(b) $A = O_{m,n}$ or $A \neq O_{m,n}$

Note.

Keep in mind the fact that $A = O_{m,n}$ if every entry of A is 0. On the other hand $A \neq O_{m,n}$ if there exists at least one entry of A that is not 0.

Theorem 2.2 (3).

Let A be an $m \times n$ matrix and c a scalar. If $cA = O_{m,n}$, then $c = 0$ or $A = O_{m,n}$.

Exercise 5.

Add some explanation to the proof using the given cases.

Let A be an $m \times n$ matrix and c a scalar such that $cA = O_{m,n}$. We consider two cases, $\boxed{c = 0 \text{ or } c \neq 0}$.

Case 1: Assume $c = 0$. Then the statement is true. (why)

Case 2: Assume $c \neq 0$. We want to show that $A = O_{m,n}$. (explain why every entry of A must be 0 – this may include saying something like $ca_{ij} = 0$, so $a_{ij} = 0$.)

Theorem 2.2 (3).

Let A be an $m \times n$ matrix and c a scalar. If $cA = O_{m,n}$, then $c = 0$ or $A = O_{m,n}$.

Exercise 6.

Let's start over and try the other set of cases.

Let A be an $m \times n$ matrix and c a scalar such that $cA = O_{m,n}$. We consider two cases, $\boxed{A = O_{m,n} \text{ or } A \neq O_{m,n}}$.

Case 1: Assume $A = O_{m,n}$. Then the statement is true. (why)

Case 2: Assume $A \neq O_{m,n}$. We want to show that $c = 0$. (explain why c must be 0 – this may include saying something like there is an entry $a_{ij} \neq 0$ of A.....$ca_{ij} = 0$, so $c = 0$.)

Exercise 7.

Which of these two cases do you prefer? Keep in mind there is no right answer but it will be useful to discuss this with your group.

Exercise 8.

Consider the following statement — which we may have seen a few times previously.

Claim.

If $A^2 = A$, then A is singular or $A = I$.

(a) To prove this we use cases. What might you consider as the cases?

(b) Using the cases: A is singular or A is invertible, work out some scratch work. Follow the foundation of the proofs in exercises 5 and 6 to prove the claim.