Math 324: Linear Algebra

2.1: Operations with Matrices

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Last Updated: December 29, 2023

Last Time.

– Polynomial Curve Fitting

– Network Analysis

Today.

– Matrix Equality

– Matrix Addition, Subtraction and Scalar Multiplication

– Matrix Multiplication

We begin by recalling some matrix notation:

- An $m \times n$ matrix is a grid with m rows and n columns.
- Matrices are usually denoted using capital letters.
- The i, j-entry of a matrix A is the entry in the i-th row and j-th column. It may also be denoted by a_{ij}.
- Other ways to denote a matrix include a compact version, $A = [a_{ij}]$, and a rectangular array,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}.$$

  Which one we use may depend on the circumstance.

Definition.

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are `equal` if they have the same size $(m \times n)$ and $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Exercise 1.

Find x and y:

$$\begin{bmatrix} -5 & x \\ 3y-2 & 8 \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ 13 & 8 \end{bmatrix}$$

Exercise 2.

Are the following matrices equal?

$$\begin{bmatrix} 54 \\ -654 \end{bmatrix} = \begin{bmatrix} 54 & -654 \end{bmatrix}$$

Definition.

The `sum` of two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the $m \times n$ matrix $A + B = [a_{ij} + b_{ij}]$.

Note.

We're using the fact that we know how to add real numbers in order to add matrices.

Definition.

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a real number, then the `scalar multiple` of A by c is the $m \times n$ matrix given by $cA = [ca_{ij}]$.

Exercise 3.

Solve for x, y and z in the matrix equation:

$$4\begin{bmatrix} x & y \\ z & -1 \end{bmatrix} = 2\begin{bmatrix} y & z \\ -x & 1 \end{bmatrix} + 2\begin{bmatrix} 4 & x \\ 5 & -x \end{bmatrix}$$

Hint: Simplify the right-hand and left-hand sides using scalar multiplication and matrix addition, then remember back to what it means for two matrices to be equal.

Question 4.

What does it mean, in terms of addition and scalar multiplication, to subtract one matrix from another?

Brain Break.

Remind you group of your name.
What would you name your boat if you owned one?

Definition (Matrix Multiplication).

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the product of A and B is an $m \times p$ matrix

$$AB = [c_{ij}],$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Exercise 5.

We begin by asking which products we can take. Go back to the definition and make careful note about the size of each matrix.

Exercise 6.

Here are six matrices. List several products that we can take along with the size of the resulting matrix. Then list several products that are not allowed.

$$A = \begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 7 \\ 0 & 3 \\ -1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -1 \\ -3 & 5 \end{bmatrix} \quad E = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad F = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

Exercise 7.

Compute CA, where $A = \begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 7 \\ 0 & 3 \\ -1 & 2 \end{bmatrix}$.

Exercise 8.

Compute DE, where $D = \begin{bmatrix} 1 & -1 \\ -3 & 5 \end{bmatrix}$ and $E = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

Exercise 9.

Compute AD, where $A = \begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & -1 \\ -3 & 5 \end{bmatrix}$.

Exercise 10.

Let A be a matrix such that $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) What is the size of A?

(b) Using variables as placeholders for the entries of A, multiply the matrices on the left-hand side of the equation.

(c) Set the entries on the LHS equal to the corresponding entries on the RHS to create a system of linear equations.

(d) Solve the system of linear equations to find A.

Exercise 11.

Let B be a matrix such that $B\begin{bmatrix}-3 & 0\\ 1 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\ -1 & 0\\ 0 & 1\end{bmatrix}$

(a) What is the size of B?

(b) Using variables as placeholders for the entries of B, multiply the matrices on the left-hand side of the equation.

(c) Set the entries on the LHS equal to the corresponding entries on the RHS to create a system of linear equations.

(d) Solve the system of linear equations to find B.