Potential Exam-able Topics

Chapter 4 - Vector Spaces

- Writing a vector as a linear combination of other vectors (in $\mathbb{R}^n$ or other vector spaces)
- Finding the additive identity, additive inverse of an arbitrary vector, and scalar identity of a vector space
- Determining whether a set is a subspace
- Determining whether a set spans a vector space
- Determining whether a set is linearly independent
- Determining whether a set is a basis
- Finding the dimension of a vector space
- Finding rank and nullity
- Finding a basis for the row space column space or nullspace of a matrix
- Relating determinants, rank, invertibility, and uniqueness of solutions

Chapter 5 - Inner Products and Orthogonality

- Finding norms, distances, and angles of vectors in $\mathbb{R}^n$ using the dot product
- Use a given inner product in a vector space V to find norms, distances, and angles of vectors.
- Determine whether two vectors are orthogonal, or find a basis for the space of all vectors orthogonal to a given vector
- Compute the orthogonal projection of $\vec{u}$ onto $\vec{v}$
- Interpret orthogonal projection in $\mathbb{R}^n$ geometrically
- Determine whether a set is orthogonal or orthonormal
- Computing orthogonal complements
- Interpret the process of solving the least squares problem in terms of column spaces and orthogonal projections

Linear Algebra Exam 1 Computational Review Questions

- Please note that this collection of sample exercises gives you an idea of what type of problems you may see on the exam. There could be other types of problems that were done in class or in homework that may appear on the exam. Also, your exam will have fewer questions than are listed here.

- If you want to do a practice exam, then pick several of these/pre-class/homework problems and give yourself a time limit in a quiet space to do them.

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1. Determine whether the statement is True or False. If True, explain why. If False, give a counter example or explain why it is incorrect.

   (a) $\{(x, y) \ : \ x, y \geq 0\}$ is a subspace of $\mathbb{R}^2$

   (b) $\{(1, 0), (0, 2), (1, 3)\}$ spans $\mathbb{R}^2$

   (c) If an $n \times n$ matrix A is invertible, then nullity$(A) = 1$.

   (d) If V and W are subspace of a vector space U, then the intersection $V \cap W$ is also a subspace of U.

   (e) $\langle \vec{u}, \vec{v} \rangle = u_1 v_1 - u_2 v_2$ is an inner product on $\mathbb{R}^2$

2. Let $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Find a basis for the row space of A.

   (b) For what value of n is the row space of A a subspace of $\mathbb{R}^n$?

   (c) Find a basis for the nullspace of A.

   (d) Find the rank and nullity of A.

3. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}$, and define

$$\begin{aligned} \vec{u} \oplus \vec{v} &= (\sqrt[3]{u} + \sqrt[3]{v} - 2)^3 \\ c \odot \vec{v} &= (c\sqrt[3]{v} - 2c + 2)^3 \end{aligned}$$

   This is a vector space with addition $\oplus$ and scalar multiplication $\odot$.

   (a) Find the additive identity $\vec{0}$.

   (b) Find the additive inverse of $\vec{u} = 1$.

   (c) Find the scalar identity. (As in the scalar c such that $c \odot \vec{v} = \vec{v}$.)

4. In the vector space $C[0,1]$, let the inner product be

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\ dx$$

and let

$$f(x) = 2x - 1 \text{ and } g(x) = x^2.$$

   (a) Find $\langle f, g\rangle$.
   (b) Find $\|f\|$.
   (c) Find the angle between f and g.
   (d) Find the orthogonal projection of g onto f.

5. Using the dot product in $\mathbb{R}^4$.

   (a) Find the unit vector in the direction of $(-2,5,1,4)$.
   (b) Find the projection of $(1,1,1,1)$ onto $(-2,5,1,4)$.
   (c) In $\mathbb{R}^3$, find a vector orthogonal to both $(1,1,1)$ and $(-2,5,1)$.

6. A and B are matrices such that $\det(2A) = 8\det(A)$ and $\det(AB) = 6$. What is the rank of A?

7. Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -3 & -1 \\ 3 & 5 & 7 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \text{ and } \vec{c} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

   (a) How many solutions does the system $A\vec{x} = \vec{b}$ have? Interpret the solution set geometrically. If there is one solution, write it in the form (x,y,z). If there are infinitely many solutions express them parametrically.
   (b) How many solutions does the system $A\vec{x} = \vec{c}$ have? Interpret the solution set geometrically. If there is one solution, write it in the form (x,y,z). If there are infinitely many solutions express them parametrically.

8. Find the orthogonal complement, $S^\perp$, of the subspace S of $\mathbb{R}^3$ spanned by the two column vectors in the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}.$$

9. Find a basis for each of the four fundamental subspaces, $\text{row}(A)$, $\text{col}(A)$, $\text{null}(A)$ and $\text{null}(A^T)$, of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

10. If you really feel confident with these and want more exercises, you may look at Chapter 4 and 5 Cumulative Test (p289) Exercises 1–11, 14, 15, 17–24

Linear Algebra Exam 1 Proof Review Questions

- You will be expected to prove 1-2 of the following on the exam. (They will appear in their exact form.)

1. Let A and B be fixed 2×2 matrices. Prove that $W = \{X \in M_{2,2} : XAB = ABX\}$ is a subspace of $M_{2,2}$.

2. Prove that the set of $n \times n$ lower triangular matrices is a subspace of some vector space.

3. Prove that if $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for a vector space V and c is a nonzero scalar, then the set $T = \{c\vec{v}_1, c\vec{v}_2, \ldots, c\vec{v}_n\}$ is also a basis for V.

4. Prove that if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a set of linearly independent vectors and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{u}\}$ is linearly dependent, then $\vec{u}$ is a linear combination of the $\vec{v}$'s.

   - Hint: You cannot use Theorem 4.8 directly on $\vec{u}$ because you only know that ONE of the vectors is a linear combination of the others, you need to justify why that one is $\vec{u}$.

   - Hint: $\vec{u}$ is not necessarily the zero vector. If you find yourself concluding that $\vec{u} = \vec{0}$, you are doing something wrong.

5. If $S = \{\vec{u}, \vec{v}\}$ is a linearly independent set in a vector space V, then $T = \{\vec{u} - \vec{v}, \vec{u} + \vec{v}\}$ is linearly independent.

6. If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in an inner product space V, then $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

7. If $\vec{u}$ and $\vec{v}$ are nonzero vectors, then $\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}$ is orthogonal to $\vec{v}$.

8. If $S = \{\vec{u}, \vec{v}\}$ is an orthogonal set of nonzero vectors, then S is linearly independent.

9. Let P be an $n \times n$ matrix. If the row vectors of P form an orthonormal basis for $\mathbb{R}^n$, then $P^T P = I$.