Math 324: Linear Algebra

Section 5.3: Orthonormal Bases

Mckenzie West

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Last Time.

- Orthogonal Projection
- Inner Products

Today.

- Orthogonal Sets

Definition.

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in a vector space V, and let c be any scalar. An `inner product` on V is a function that associates a real number $\langle \vec{u}, \vec{v} \rangle$ for each pair of vectors $\vec{u}$ and $\vec{v}$ and satisfies the following axioms.

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
2. $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
3. $c \langle \vec{u}, \vec{v} \rangle = \langle c\vec{u}, \vec{v} \rangle$
4. $\langle \vec{v}, \vec{v} \rangle \geq 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.

A vector space with an inner product is called an `inner product space`.

Example.

Every inner product on $\mathbb{R}^n$ is of the form

$$\langle (u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n) \rangle = \lambda_1 a_1 u_1 v_1 + \lambda_2 u_2 v_2 + \cdots + \lambda_n u_n v_n,$$

where the coefficients $a_1, a_2, \ldots, a_n$ are positive real numbers. We think of this as a `weighted dot product` because it is calculated in almost the same way as a dot product but gives specific weights to each of the coordinates.

Exercise 1.

Use the weighted inner product on $\mathbb{R}^3$ given by

$$\langle (u_1, u_2, u_3), (v_1, v_2, v_3) \rangle = \frac{1}{2} u_1 v_1 + 4 u_2 v_2 + 1.7 u_3 v_3$$

to compute

(a) $\langle (4, 6, 1), (-2, 0, 3) \rangle$

(b) $\|(4, 6, 1)\|^2 = \langle (4, 6, 1), (4, 6, 1) \rangle$

(c) $d((4, 6, 1), (-2, 0, 3))^2 = \|(4, 6, 1) - (-2, 0, 3)\|^2$

Definition.

A set S of vectors in an inner product space V is said to be orthogonal if each pair of vectors in S is orthogonal.
If, in addition, each vectors in the set is a unit vector then S is called orthonormal.

Example.

The standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ for $\mathbb{R}^3$ is an orthonormal set under the dot product because:

$$\begin{array}{ccc} \|\vec{e}_1\| = 1 & \vec{e}_1 \cdot \vec{e}_2 = 0 & \vec{e}_1 \cdot \vec{e}_3 = 0 \\ & \|\vec{e}_2\| = 1 & \vec{e}_2 \cdot \vec{e}_3 = 0 \\ & & \|\vec{e}_3\| = 1 \end{array}$$

(Notice that I tested every length and every pair for the dot products.)

Exercise 2.

(a) Show that the set $S = \{(\frac{3}{5}, \frac{4}{5}), (-\frac{4}{5}, \frac{3}{5})\}$ is an orthonormal set of vectors in $\mathbb{R}^2$. (If an inner product is not specified, assume you're using the standard dot product.)

(b) Show that S is a basis for $\mathbb{R}^2$.

Exercise 3.

Using the inner product for P_2 given by the analogue of the dot product,

$$\left\langle a_0 + a_1x + a_2x^2, b_0 + b_1x + b_2x^2 \right\rangle = a_0b_0 + a_1b_1 + a_2b_2,$$

determine whether $\{1 + 2x^2, -x, 3 + x^2\}$ is an orthogonal set. If it is orthogonal is it orthonormal?

Brain Break

Would you rather have mozzarella sticks for fingers or sponges for feet?

Some answers to your questions:

1. If you eat the mozzarella sticks they do not grow back.
2. There are supports in the sponges that allow you to walk.
3. Neither is not an acceptable answer.

(This was a bit of a controversial question among my graduate school cohort.)


Definition.

If a basis for a vector space V of dimension n is an orthonormal set, then we call it an `orthonormal basis`.

Theorem 5.11.

If $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthonormal basis for an inner product space V, then the representation of $\vec{w} \in V$ in terms of B is given by

$$\vec{w} = \langle \vec{w}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{w}, \vec{v}_2 \rangle \vec{v}_2 + \cdots + \langle \vec{w}, \vec{v}_n \rangle \vec{v}_n.$$

Exercise 4.

Write $\vec{w} = (3, -2)$ in terms of $S = \{(\frac{3}{5}, \frac{4}{5}), (-\frac{4}{5}, \frac{3}{5})\}$, which you proved to be an orthonormal basis for $\mathbb{R}^2$.

Theorem 5.10.

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then S is linearly independent.

Note.

In the beginning of the proof of this, we let $c_1, c_2, \ldots, c_n$ be scalars such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}.$$

You then should use the orthogonality by taking the inner product of $\vec{v}_i$ with

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

for each i.
You also make use of the fact that $\langle \vec{0}, \vec{v} \rangle = 0$ for all $\vec{v}$, no matter the inner product space V.

Exercise 5.

Let $S = \{(\frac{3}{5}, \frac{4}{5}), (-\frac{4}{5}, \frac{3}{5})\}$. Use $\vec{v}_1 = (\frac{3}{5}, \frac{4}{5})$ and $\vec{v}_2(-\frac{4}{5}, \frac{3}{5})$. What is

(a) $(3\vec{v}_1 - 2\vec{v}_2) \cdot \vec{v}_1$?

(b) $(3\vec{v}_1 - 2\vec{v}_2) \cdot \vec{v}_2$?

(c) $(3\vec{v}_1 - 2\vec{v}_2) \cdot (-\vec{v}_1 + 5\vec{v}_2)$?

Note.

You can create a vector in Sage
`http://sagecell.sagemath.org` using

```
v1 = vector((3/5,4/5))
v2 = vector((-4/5,3/5))
v3 = 3*v1-2*v2
print(f"v3 = {v3}")
ansa = v3.dot_product(v1)
print(f"answer to (a) = {ansa}")
```