Math 324: Linear Algebra

Section 6.3: Matrices for Linear Transformations

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Last Time.

- One-to-One Transformations
- Onto Transformations
- Isomorphisms

Today.

- The matrix of a transformation.
- Compositions

Recall.

Linear transformations can be defined by their action on a basis.

Exercise 1.

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies $T(1,0) = (3,2,1)$ and
$T(0,1) = (1,1,1)$.
What is $T(3,-8)$?

Theorem 6.10.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that, for the standard basis vectors $\vec{e_i}$ of $\mathbb{R}^n$,

$$T(\vec{e_1}) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \; T(\vec{e_2}) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \; \ldots, T(\vec{e_n}) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Then the $m \times n$ matrix whose n columns correspond to $T(\vec{e_i})$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

satisfies $T(\vec{v}) = A\vec{v}$ for every $\vec{v} \in \mathbb{R}^n$. We call A the `standard matrix` for T.

Exercise 2.

Consider the transformation from exercise 1, $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,0) = (3,2,1)$ and $T(0,1) = (1,1,1)$.

(a) Find the standard matrix for T.

(b) Use the standard matrix to compute $T(-2,6)$.

Exercise 3.

Find the standard matrix for $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (2x+y, 2x-z)$.

Exercise 4 (Optional).

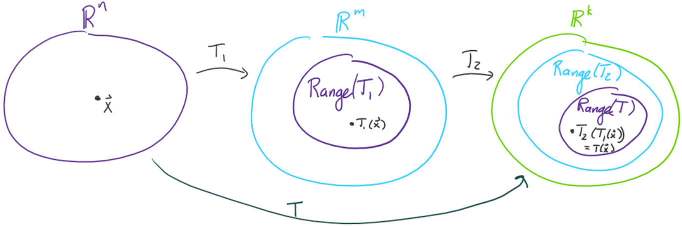
Consider the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is given by rotation 60° counterclockwise. Find the standard matrix for T using a little trigonometry and the action of this transformation on the basis vectors $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$.

Definition.

The composition of the transformations $T_1\colon \mathbb{R}^n \to \mathbb{R}^m$ with $T_2\colon \mathbb{R}^m \to \mathbb{R}^k$ is the transformation $T\colon \mathbb{R}^n \to \mathbb{R}^k$ defined by

$$T(\vec{v}) = T_2(T_1(\vec{v})).$$

Denote the composition by $T = T_2 \circ T_1$.



Exercise 5.

Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^5$ be defined by $T_1(x, y, z) = (x, y, z, 2x, y + z)$.
Let $T_2 : \mathbb{R}^5 \to \mathbb{R}^2$ be defined by
$T_2(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_3 + x_4 - x_5)$.
Let $T = T_2 \circ T_1$.

(a) What is the domain of T?

(b) What is the codomain of T?

(c) Compute $T(1, 0, 0)$, $T(0, 1, 0)$, and $T(0, 0, 1)$.

Theorem 6.11.

Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations with standard matrices A_1 and A_2, respectively. Then the composition $T : \mathbb{R}^n \to \mathbb{R}^k$ is also a linear transformation whose standard matrix is $A = A_2 A_1$.

Exercise 6.

Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^5$ be defined by $T_1(x, y, z) = (x, y, z, 2x, y + z)$.
Let $T_2 : \mathbb{R}^5 \to \mathbb{R}^2$ be defined by
$T_2(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_3 + x_4 - x_5)$.
Let $T = T_2 \circ T_1$.

(a) Compute, via Theorem 6.10, the standard matrix A_1 of T_1, A_2 of T_2, and A of T.

(b) Verify that $A = A_2 A_1$.

Brain Break.

Would you rather have a slice of Chicago style pizza, New York style pizza, or Italian style pizza?

I am a fan of an Italian margarita pizza fired in a brick oven.

Exercise 7.

Prove Theorem 6.11. Particularly, prove the result stating that $T = T_2 \circ T_1$ is a linear transformation.

(a) Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations.

(b) Recall that in order to prove $T = T_2 \circ T_1$ is a linear transformation, we need to show $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(c\vec{u}) = cT(\vec{u})$.

(c) Using the definition of T, e.g. $T(\vec{u} + \vec{v}) = T_2(T_1(\vec{u} + \vec{v}))$, prove the distributivity of T.

(d) Similarly prove the scalar factorization.

Exercise 8.

Consider the transformations $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T_1(x, y, z) = (x - y, x - z) \text{ and } T_2(x, y) = (3x, 4y, 0).$$

(a) Find the standard matrices for T_1 and T_2.

(b) Use your answers to (a) to find the standard matrices of $T = T_2 \circ T_1$ and of $T' = T_1 \circ T_2$.

Definition.

If $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $T_2 : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations such that for every $\vec{v} \in \mathbb{R}^n$,

$$T_2 \circ T_1(\vec{v}) = \vec{v} \text{ and } T_1 \circ T_2(\vec{v}) = \vec{v}$$

then T_2 is called the `inverse` of T_1, and T_1 is said to be invertible.

Exercise 9.

Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T_1(x, y) = (x, x + y) \text{ and } T_2(x, y) = (x, y - x)$$

Show that T_2 is the inverse of T_1.

Theorem 6.12.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A. Then the following are equivalent.

1. T is an invertible linear transformation.
2. T is an isomorphism.
3. A is an invertible matrix.

Moreover, if T is invertible, then the standard matrix for T^{-1} is A^{-1}.

Exercise 10.

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (3x - 2y, x + y + 2z, 2x + y + z)$. Determine whether T is invertible. If so, find its inverse:

(a) Find the standard matrix, A, for T.

(b) Find the inverse of A.

(c) Then $T^{-1}(\vec{v}) = A^{-1}\vec{v}$.

(d) Write T^{-1} in the form $T^{-1}(x, y, z) = (a, b, c)$ where a, b and c are linear functions of x, y and z.