Math 324: Linear Algebra

2.1: Operations with Matrices

Mckenzie West

Last Updated: December 29, 2023

Last Time.

- Matrix Equality
- Matrix Addition, Subtraction and Scalar Multiplication
- Matrix Multiplication

Today.

- Row and Column Matrices
- Linear Combinations
- Linear Systems as Matrix Products

Definition.

A matrix that has only one column is called a `column matrix` or `column vector`.
A matrix that has only one row is called a `row matrix` or `row vector`.

Note.

- Column matrices are often denoted by either bold lowercase letters $\mathbf{a}$ or by lowercase letters with an arrow over the top $\vec{a}$.
- Every $m \times n$ matrix A consists of m row vectors and n column vectors which may be denoted by $\mathbf{a}_i$. It is essential to specify *row* or *column* when doing this.
- When using $\mathbf{a}_j$, $1 \leq j \leq n$, to denote column vectors, we can write

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

- What would it look like if we used $\mathbf{a}_i$ to denote the rows of A?

Exercise 1.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $\mathbf{a}_1$ and $\mathbf{a}_2$ be the columns of A.
What are $\mathbf{a}_1$ and $\mathbf{a}_2$ explicitly?

Exercise 2.

Compute the product

$$\begin{bmatrix} 1 & 2 & -3 \\ 7 & -5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Note.

From the exercise, we see that the linear system of equations:

$$\begin{aligned} x + 2y - 3z &= 3, \\ 7x - 5y + 2z &= 0, \end{aligned}$$

can be written as a matrix equation:

$$\begin{bmatrix} 1 & 2 & -3 \\ 7 & -5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Let A be the coefficient matrix of a system, $\vec{x}$ the column vector of variables, and $\vec{b}$ the column matrix containing the constant terms. Then a linear system of equations can be written as the equality

$$A\vec{x} = \vec{b}.$$

Exercise 3.

Write the system of linear equations in the form $A\vec{x} = \vec{b}$ and solve the matrix equation for $\vec{x}$.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 9 \\ -x_1 & + & 3x_2 & - & x_3 & = & -6 \\ 2x_1 & - & 5x_2 & + & 5x_3 & = & 17 \end{array}$$

Brain Break.

What sport is your favorite to watch?

Definition.

A `linear combination` of the column matrices $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ with coefficients $x_1, x_2, \ldots, x_n$ is the vector $\vec{b}$ given by

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

Example.

The column matrix $\vec{b} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$ is a linear combination of $\vec{a}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ because

$$4\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

Exercise 4.

Can $\vec{b} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ be written as a linear combination of

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}, \text{ and } \vec{a}_3 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}?$$

Hint: Work entry by entry - think in terms of matrix equality and linear equations.