Math 324: Linear Algebra

Section 4.4: Linear Combinations and Spans

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Last Time.

- Subspaces of $\mathbb{R}^2$
- Subspaces of $\mathbb{R}^3$
- Intersections of Subspaces

Today.

- Linear Combination
- Span

Definition.

A vector $\vec{v}$ in a vector space V is a `linear combination` of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in V$ if there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n.$$

Example.

The vector $\vec{v} = (1, 4, 5)$ is a linear combination of $\vec{u}_1 = (1, 2, 3)$ and $\vec{u}_2 = (3, 4, 7)$ because

$$\vec{v} = 4\vec{u}_1 - \vec{u}_2.$$

Example.

The polynomial $p(x) = 3 + x - 2x^2$ is a linear combination of $q_1(x) = 1 + x$, $q_2(x) = 1 + x^2$, $q_3(x) = 1 + x + x^2$, and $q_4(x) = x^2$ because

$$\begin{aligned} p(x) &= 3 + x - 2x^2 \\ &= 5(1 + x) + 2(1 + x^2) - 4(1 + x + x^2) + 0x^2 \\ &= 5q_1(x) + 2q_2(x) + (-4)q_3(x) + 0q_4(x) \end{aligned}$$

Exercise 1.

Can the vector $\vec{v} = (1, 0, 5)$ be written as a linear combination of $\vec{u}_1 = (-1, 1, 1)$ and $\vec{u}_2 = (2, 1, 3)$?
I suggest you turn this into a question about solutions to a linear system of equations.

Exercise 2.

Can the polynomial $p(x) = 2 + x + x^2$ be written as a linear combination of $q_1(x) = 1 + x$ and $q_2(x) = 1 + x^2$?
You should also turn this into a question of solutions to a linear system.

Exercise 3.

Can $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ be written as a linear combination of
$M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$?

Definition.

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a set of vectors in a vector space V, then the `span of` S, denoted $\mathrm{span}(S)$, is the set of all linear combinations of the vectors in S,

$$\mathrm{span}(S) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \; : \; c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

Exercise 4.

Compute span(S) for $S = \{(1,2,3),(4,5,6)\}$ as a subset of $\mathbb{R}^3$. That is, describe all vectors, $\vec{v} = (x, y, z)$, that can be written in the form

$$(x, y, z) = a(1, 2, 3) + b(4, 5, 6).$$

Use the following code to row reduce a matrix with variables in Sage: (`https://sagecell.sagemath.org/`)

```
R.<x,y,z> = QQ[]
A = matrix([[1,4,x],[2,5,y],[3,6,z]])
print(A.echelon_form())
```

Go to the next slide to see the solution.

Exercise 4.

Compute $\operatorname{span}(S)$ for $S = \{(1,2,3),(4,5,6)\}$ as a subset of $\mathbb{R}^3$. That is, describe all vectors, $\vec{v} = (x,y,z)$, that can be written in the form

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Use the following code to row reduce a matrix with variables in Sage: (`https://sagecell.sagemath.org/`)

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print(A.echelon_form())
```

Notice that the last row of the augmented matrix A reduces to $\begin{bmatrix} 0 & 0 & x-2y+z \end{bmatrix}$. We want the system to have solutions. This means that

$$\operatorname{span}(S) = \{(x,y,z) : 0 = x - 2y + z\}.$$

Exercise 5.

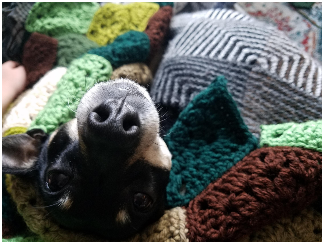
Continuing, open `https://sagecell.sagemath.org/` and use the code below to plot the plane $x - 2y + z = 0$ along with the vectors in $S = \{(1, 2, 3), (4, 5, 6)\}$.

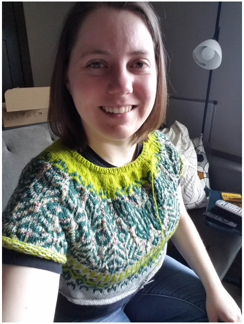
```
x,y = var("x,y")
P = plot3d(-x+2*y,(x,-2,6),(y,-2,6),opacity=.5)
P += arrow((0,0,0),(1,2,3),color="red",thickness=5)
P += arrow((0,0,0),(4,5,6),color="green",thickness=5)
P.show()
```

Brain Break.

Do you have any art-related hobbies?

I'm a bit knitting obsessed. Here's Pepper enjoying two homemade blankets and Me trying on a half(quarter)-finished sweater.





Exercise 6.

Compute $\operatorname{span}(S)$ for $S = \{1 + x, 1 - x\}$ as a subset of P_1. That is, provide a meaningful description of all polynomials that can be written as $a(1 + x) + b(1 - x)$.

Definition.

Let V be a vector space. We call the set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ a `spanning set` for V if $\text{span}(S) = V$.

Example.

$S = \{1 + x, 1 - x\}$ is a spanning set for P_1

Non-Example.

The set $S = \{(1, 2, 3), (4, 5, 6)\}$ is NOT a spanning set for $\mathbb{R}^3$ because $\text{span}(S)$ is the plane defined by $x - 2y + z = 0$.
How might you prove it is not a spanning set? Pick a vector not on the plane, say $\vec{v} = (1, 0, 0)$ and prove that it is not a linear combination of $(1, 2, 3)$ and $(4, 5, 6)$.

Exercise 7.

Use proof by contradiction to show that $\vec{v} = (1, 0, 0)$ is not a linear combination of $\vec{u}_1 = (1, 2, 3)$ and $\vec{u}_2 = (4, 5, 6)$.

Theorem 4.7.

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a set of vectors in a vector space V, then span(S) is a subspace of V.
Moreover, span(S) is the `smallest` subspace of V that contains S. This means that if W is any other subspace of V that contains S, then W also contains span(S).

Exercise 8.

Prove that span(S) is a subspace of V.

Proof.

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a subset of a vector space V. To show that span(S) is a subspace of V, we use the Subspace Test:

(a) Write $\vec{0}$ as a linear combination of the vectors in S. *Be sure to reference any previous Theorems.*

(b) Let $\vec{u}$ and $\vec{w}$ be elements of span(S). Then we can write $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$ and $\vec{w} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + c_k\vec{v}_k$. Show that $\vec{u} + \vec{w}$ can be written as a linear combination of the vectors in S.

(c) Let c be a scalar. Show that $c\vec{u}$ can be written as a linear combination of the scalars in S.

□