Math 324: Linear Algebra

Section 4.6: Row and Column Spaces

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Last Time.

- Dimension

Today.

- Row and Column Spaces
- Rank of a Matrix

Exercise 1 (review of section 4.5).

Consider the subspace of $\mathbb{R}^4$, given by

$$W = \{(a, b, a, b) \ : \ a, b \in \mathbb{R}\}.$$

(a) Find a basis for W.

(b) Determine the dimension of W.

Definition.

Let A be an $m \times n$ matrix.

1. The `column space` of A, denoted $\text{col}(A)$, is the subspace of $\mathbb{R}^m$ that is spanned by the columns of A.
2. the `row space` of A, denoted $\text{row}(A)$, is the subspace of $\mathbb{R}^n$ that is spanned by the rows of A.

Example.

For $A = \begin{bmatrix} 8 & -1 & 7 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & -3 & -3 & 9 \end{bmatrix}$,

- $\text{col}(A)$ is a subspace of $\mathbb{R}^3$ because each column has 3 entries
- $\text{row}(A)$ is a subspace of $\mathbb{R}^4$ because each row has 4 entries

Theorem 4.13.

If an $m \times n$ matrix A is row-equivalent to an $m \times n$ matrix B, then $\text{row}(A) = \text{row}(B)$.

Theorem 4.14.

If a matrix A is row-equivalent to a B that is in row-echelon form, then the non-zero rows of B form a basis for $\text{row}(A)$.

Exercise 2.

Let $A = \begin{bmatrix} 8 & -1 & 7 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & -3 & -3 & 9 \end{bmatrix}$.

Find a basis for $\text{row}(A)$. What is the dimension of $\text{row}(A)$?

Exercise 3.

Let $A = \begin{bmatrix} 8 & -1 & 7 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & -3 & -3 & 9 \end{bmatrix}$.

Consider the matrix A^T.

(a) Convince yourself that $\mathrm{col}(A) = \mathrm{row}(A^T)$.

(b) Find a basis for $\mathrm{row}(A^T)$ using Theorem 4.14.

(c) What is a basis for and the dimension of $\mathrm{col}(A)$?

Exercise 4.

Find a basis for col(A) and row(A) for

$$A = \begin{bmatrix} 5 & 1 & -9 & -5 & 2 \\ 2 & 8 & 4 & -2 & 9 \\ 3 & -5 & -11 & -3 & -3 \\ -3 & 2 & 8 & 3 & 7 \end{bmatrix}$$

Exercise 5.

Find a basis for col(A) and row(A) for

$$A = \begin{bmatrix} 7 & -5 \\ 2 & 7 \\ -2 & 3 \\ 9 & 9 \end{bmatrix}$$

Brain Break.

What one non-essential item do you wish you had more of?

Pepper wants more Barkbox.

Theorem 4.15.

If A is an $m \times n$ matrix, then the row space and the column space of A have the same dimension.

Exercise 6.

Convince yourself of this fact by considering the relationship between dimension of row/column spaces and leading 1's in the reduced row echelon form of A.

Definition.

The `rank of a matrix` A, denoted $\operatorname{rank}(A)$, is the dimension of the row (or column) space of A.

Example.

$\operatorname{rank}(I_n) = n$
$\operatorname{rank}(O_{m,n}) = 0$

Exercise 7.

Determine

$$\operatorname{rank}\left(\begin{bmatrix} 7 & 9 & -1 \\ 1 & -2 & -2 \\ 4 & 0 & -1 \\ 7 & -3 & -2 \end{bmatrix}\right)$$

Exercise 8.

Give examples of matrices such that

(a) $\operatorname{rank}(A+B) < \operatorname{rank}(A)$ and $\operatorname{rank}(A+B) < \operatorname{rank}(B)$

(b) $\operatorname{rank}(A+B) = \operatorname{rank}(A)$ and $\operatorname{rank}(A+B) = \operatorname{rank}(B)$

(c) $\operatorname{rank}(A+B) > \operatorname{rank}(A)$ and $\operatorname{rank}(A+B) > \operatorname{rank}(B)$

Is it possible for $\operatorname{rank}(A+B) < \operatorname{rank}(A)$ and $\operatorname{rank}(A+B) > \operatorname{rank}(B)$?