Math 324: Linear Algebra

Section 6.1: Linear Transformations

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Last Time.

- Exam 2: Vector Spaces, Matrix Spaces, Inner Products

Today.

- Linear Transformations
- Properties of Linear Transformations
- Matrix Transformations

Definition.

A transformation from V to W, a.k.a. function or mapping, denoted by $T\colon V \to W$ is a pairing that assigns to each element in the domain V exactly one element in the codomain W.

Example.

In Calculus I and II, we use functions whose domains are subsets of $\mathbb{R}$ and codomains are $\mathbb{R}$, such as

- $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$
- or $g\colon [-1,1] \to \mathbb{R}$ defined by $g(x) = \sqrt{1-x^2}$.

Note.

Notice that the codomain is different from the range which is the elements in the codomain that are *hit* by the map.

Exercise 1.

Consider the mapping $T : \mathbb{R}^2 \to P_1$ defined by

$$T((a, b)) = (a^2 + b^2 x).$$

(a) Compute $T((3, -2))$.

(b) What is the domain of T?

(c) What is the codomain of T?

(d) What is the range of T?

Definition.

If $T : V \to W$ is a transformation, the `image` of $\vec{v}$ under T is the value $T(\vec{v})$ in W that has been assigned to $\vec{v}$.
For $\vec{w} \in W$, the `preimage` of $\vec{w}$, denoted $T^{-1}(\vec{w})$, is the collection of *all* vectors $\vec{v}$ in V for which $T(\vec{v}) = \vec{w}$. In set notation,

$$T^{-1}(\vec{w}) = \{\vec{v} \in V : T(\vec{v}) = \vec{w}\}.$$

Exercise 2.

Again using the mapping $T : \mathbb{R}^2 \to P_1$ defined by

$$T((a, b)) = (a^2 + b^2 x).$$

(a) What is the image of $(-2, 2)$?
(b) What is the preimage of $1 + x$?
(c) What is the preimage of $-2x$?

Definition.

If V and W are vector spaces, a `linear transformation` from V to W is a mapping $T: V \to W$ that satisfies

1. $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in V$,
2. and $T(c\vec{v}) = cT(\vec{v})$ for all scalars c and $\vec{v} \in V$.

Example.

The mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{v}) = 2\vec{v}$ is a linear transformation because

1. $T(\vec{v}_1 + \vec{v}_2) = 2(\vec{v}_1 + \vec{v}_2) = 2\vec{v}_1 + 2\vec{v}_2 = T(\vec{v}_1) + T(\vec{v}_2)$
2. $T(c\vec{v}) = 2(c\vec{v}) = c(2\vec{v}) = cT(\vec{v})$

Exercise 3.

Which of the following functions are linear transformations?

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3,\ T(x, y) = (2y - x, x, y)$

(b) $T : \mathbb{R}^2 \to P_1,\ T(a, b) = a^2 + b^2x$

(c) $T : M_{2,2} \to \mathbb{R},\ T(A) = \det(A)$

(d) $T : \mathbb{R}^3 \to M_{2,2},\ T(x, y, z) = \begin{bmatrix} x & y \\ z & x + y + z \end{bmatrix}$

Example.

Here are some important linear transformations, you may want to make sure you believe they are linear transformations.

(a) $T : M_{m,n} \to M_{n,m},\ T(A) = A^T$

(b) $D : P_n \to P_{n-1},\ D(p) = p'$
(works when the domain and codomain are the collection of all differentiable functions too)

(c) $I : P_n \to \mathbb{R},\ I(p) = \int_a^b p(x)\ dx$

Theorem 6.1.

Let $T\colon V \to W$ be a linear transformation and $\vec{u}, \vec{v} \in V$. Then the following are true

1. $T(\vec{0}_V) = \vec{0}_W$
2. $T(-\vec{v}) = -T(\vec{v})$
3. $T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$
4. $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_nT(\vec{v}_n)$

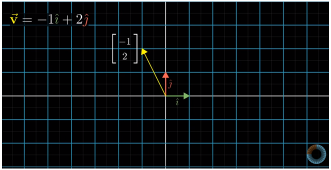
Exercise 4.

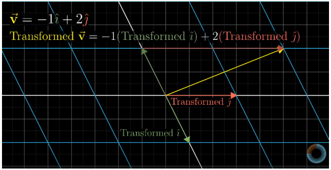
Prove properties 1 and 2, being sure to reference Theorem 4.4, which states $0\vec{v} = \vec{0}$ and $-\vec{v} = (-1)\vec{v}$.

Note.

The last property of Theorem 6.1 indicates the strength of linear transformations.

In particular if $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for the vector space V, then any linear transformation $T\colon V \to W$ can be described solely by $T(\vec{v}_1)$, $T(\vec{v}_2)$, $\ldots$, $T(\vec{v}_n)$, as in the video





Exercise 5.

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfy $T(1,0,0) = (1,1)$, $T(0,1,0) = (2,1)$, and $T(0,0,1) = (0,-3)$.

(a) What is $T(1,2,3)$?

(b) What is $T(x,y,z)$?

Brain Break.

What is your favorite fruit?

My favorite fruit is probably a raspberry.

Exercise 6.

Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ satisfy $T(1,2) = (1,2,0,0)$ and $T(2,1) = (0,0,2,1)$.

(a) What is $T(1,0)$?

   i. First, verify that $S = \{(1,2),(2,1)\}$ is a basis for $\mathbb{R}^2$.
   ii. Write $(1,0)$ as a linear combination of $(1,2)$ and $(2,1)$.
   iii. Use Theorem 6.1 and part (ii) to determine $T(1,0)$.

(b) What is $T(0,1)$?

(c) Use parts (a) and (b) to determine $T(-5,2)$.

(d) Use parts (a) and (b) to determine $T(x,y)$.

Note.

Another very important linear transformation is a `matrix transformation`, $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\vec{x}) = A\vec{x}$ where A is some fixed $m \times n$ matrix.

Theorem 6.2.

Let A be an $m \times n$ matrix. Then the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{v}) = A\vec{v}$ is a linear transformation.

Exercise 7.

Verify that $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(c\vec{v}) = cT(\vec{v})$ if T is a matrix transformation.

Exercise 8.

Define the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T(\vec{v}) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vec{v}.$$

(a) What are n and m?

(b) What is $T(1, 1, 1, 1)$?

(c) What is the preimage of $(1, 1, 1)$?
(Hint: You're trying to find vectors $\vec{v} \in \mathbb{R}^n$ such that $T(\vec{v}) = (1, 1, 1)$.)

Exercise 9 (Optional).

Let T be a linear transformation from P_2 to P_2 such that $T(1) = 2 + x$, $T(x) = 2 - x$, and $T(x^2) = x^2$.
What is $T(6 - 3x + 5x^2)$?

Exercise 10 (Optional).

A `fixed point` of a linear transformation $T : V \to V$ is any vector $\vec{v}$ such that $T(\vec{v}) = \vec{v}$.

(a) What are the fixed points of $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, 2y)$?

(b) What are the fixed points of the matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$?