Math 324: Linear Algebra

Section 3.3: Properties of Determinants
Section 3.4: Cramer's Rule

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Last Time.

– Determinants of Products

Today.

– More Properties of Determinants

– Applications of Determinants

3 - Last Time

Theorem 3.5.

If A and B are square matrices of order n, then

$$\det(AB) = \det(A)\det(B).$$

Theorem 3.8.

If A is an $n \times n$ invertible matrix, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

4 - Last Time with a New Exercise

Equivalent Conditions for a Nonsingular Matrix.

If A is an $n \times n$ matrix, then the following are equivalent:

1. A is invertible.
2. $A\vec{x} = \vec{b}$ has a unique solution for every $n \times 1$ column matrix $\vec{b}$.
3. $A\vec{x} = O$ has only the trivial solution.
4. A is row equivalent to I_n.
5. A can be written as the product of elementary matrices.
6. $\det(A) \neq 0$. (**Theorem 3.7**)

Exercise 1.

Compute the determinants of each of the following to determine which are invertible.

(a) $\begin{bmatrix} 4 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 0 \\ 4 & 3 & 0 \\ 5 & 4 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ -1 & 1 & 0 & -1 \end{bmatrix}$

Exercise 2.

Each person claim one of the following three matrices,

$$\begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ -2 & -1 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 5 \\ 1 & -1 \end{bmatrix}$$

For your matrix A, using the definition or properties of the determinant find:

(a) $|A^T|$, (b) $|A^2|$, (c) $|AA^T|$, (d) $|2A|$, and (e) $|A^{-1}|$.

Exercise 3.

Fill in the blanks of the Theorem:

Theorems 3.6 and 3.9.

Let A be an $n \times n$ matrix.

3.6: If c is a scalar, then $\det(cA) =$ __________.

3.9: Moreover $\det(A^T) =$ __________.

3.?: If k is a positive integer, then $\det(A^k) =$ __________.

Recall.

The (i, j)-cofactor of the $n \times n$ matrix $A = [a_{ij}]$ is

$$C_{ij} = (-1)^{i+j} \det(M_{ij}),$$

where M_{ij} is the matrix obtained by removing the i-th row and j-th column of A.

Definition.

The `matrix of cofactors` of the $n \times n$ matrix A is the $n \times n$ matrix given by:

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

Definition.

The `adjoin` of the $n \times n$ matrix A is the transpose of the matrix of cofactors of A and is denoted $\mathrm{adj}(A)$. In particular:

$$\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Theorem 3.10.

If A is an $n \times n$ invertible matrix then $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$.

Exercise 4.

Find $\operatorname{adj}(A)$ for

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Compute $A \operatorname{adj}(A)$.
What is A^{-1}?

Exercise 5.

Proof IDEA for Theorem 3.10.

Take $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$.

Then the (1,1)-entry of $A \operatorname{adj}(A)$ is

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \underline{\hspace{8cm}}.$$

Then the (2,2)-entry of $A \operatorname{adj}(A)$ is

$$a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = \underline{\hspace{8cm}}.$$

Then the (3,3)-entry of $A \operatorname{adj}(A)$ is

$$a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = \underline{\hspace{8cm}}.$$

Exercise 5 (Cont.).

On the other hand the (1,2)-entry of $A\,\mathrm{adj}(A)$ is

$$\begin{aligned}
&a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} \\
&\qquad = -a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\
&\qquad = \text{ middle row cofactor expansion for } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&\qquad = 0 \quad \text{(why:)}\underline{\hspace{7cm}}.
\end{aligned}$$

What are the $(1,3)$, $(2,1)$, $(2,3)$, $(3,1)$, and $(3,2)$-entries of the product $A\,\mathrm{adj}(A)$?

Brain Break.

Today the Jewish celebration of Purim will end. Purim is an annual holiday commemorating the salvation of the Jewish people from the ancient Persian Empire, as written in the Megillah (book of Esther). On Purim, observers celebrate friendship and community by sending gifts of food and drink to friends. Who would you send such gifts to and why?

* Today is also the Hindu holiday of Holi and the Sikh holiday of Hola Mohalla.

Information found at https://www.chabad.org/holidays/purim/article_cdo/aid/645309/jewish/What-Is-Purim.htm

and https://www.interfaith-calendar.org/2020.htm

Cramer's Rule

Consider the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

which has coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. `Cramer's rule` says the following:

1. Check that A is invertible.
2. Let $A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$, obtained by replacing the given columns of A by the constant terms.
3. Compute the determinants $|A|$, $|A_1|$, and $|A_2|$.
4. Then $x_1 = \dfrac{|A_1|}{|A|}$ and $x_2 = \dfrac{|A_2|}{|A|}$ is the unique solution to the system.

Exercise 6.

Use Cramer's rule to solve the system:

$$\begin{aligned} x + y &= 2 \\ 3x - y &= 0. \end{aligned}$$

Theorem 3.11: Cramer's Rule.

If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the unique solution to the system is

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots,\ x_n = \frac{\det(A_n)}{\det(A)}$$

where the i-th column of A_i is the column of constants in the system of equations, all other columns of A_i equal those of A.

Exercise 7.

Use Cramer's Rule to solve the system of linear equations for $\vec{x}$.

$$\begin{array}{rcrcrcr} 2x & + & y & - & 3z & = & 4 \\ 4x & & & + & 2z & = & 0 \\ -2x & + & 3y & & & = & 8 \end{array}$$

Proposition.

Let A be an invertible matrix with integer entries. Then $\det(A) = \pm 1$ if and only if all of the entries of A^{-1} are integers.

Exercise 8.

Prove the proposition using the following steps:

(a) Forward direction.
   (i) Assume $|A| = \pm 1$.
   (ii) Use the formula $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$ and the definition of cofactors to explain why A^{-1} has integer entries.

(b) Reverse Direction
   (i) Assume that A and A^{-1} have integer entries.
   (ii) Use Theorem 3.5 to explain why $|A||A^{-1}| = 1$.
   (iii) Use the definition of the determinant to explain why $|A|$ and $|A^{-1}|$ are both integers.
   (iv) Conclude that $|A| = |A^{-1}| = \pm 1$.

Exercise 9.

Use Cramer's Rule to solve the system of linear equations for x and y in terms of k. Then determine which values of k will make the system inconsistent.

$$\begin{array}{rcrcl} kx & + & (1-k)y & = & 1 \\ (1-k)x & + & ky & = & 3 \end{array}$$

Exercise 10.

The table shows annual personal health care expenditure (in billions of dollars) in the United States from 2007 through 2009.

Year, t	2007	2008	2009
Amount, y	1465	1547	1623

(a) Create a system of linear equations for the data to fit the curve $y = at^2 + bt + c$ where $t = 7$ corresponds to 2007, and y is the amount of expenditure.

(b) Use Cramer's Rule to solve the system.

(c) Graph the curve and the data.

(d) Briefly describe how well the polynomial fits the data.