Math 324: Linear Algebra

Section 4.5: Basis

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Last Time.

– Linear Independence

Today.

– Basis

Definition.

A set of vectors $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in a vector space V is called a `basis` for V if both $(\mathbf{1})$ S spans V and $(\mathbf{2})$ S is linearly independent.

Note.

Bases (base-eaze) are "Goldilocks sets" in that they aren't too small $(\mathbf{1})$ and they aren't too big $(\mathbf{2})$.
Recall that for out usual vector spaces, using the vectors as columns of a matrix,

$$\begin{array}{rcl} \text{spanning} & \longleftrightarrow & \text{no zero rows in RREF} \\ \text{linearly independent} & \longleftrightarrow & \text{no free variables} \end{array}$$

Exercise 1.

Explain why $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathbb{R}^3$.

Definition.

The basis $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is called the `standard basis for` $\mathbb{R}^3$. The `standard basis for` $\mathbb{R}^n$ is the set $S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of vectors where,

$$\begin{array}{rcl} \vec{e}_1 & = & (1,0,\ldots,0) \\ \vec{e}_2 & = & (0,1,\ldots,0) \\ & \vdots & \\ \vec{e}_n & = & (0,0,\ldots,1). \end{array}$$

Exercise 2.

Show that $S = \{(1,2),(5,-2)\}$ is a basis for $\mathbb{R}^2$. That is, show that S is linearly independent and spans $\mathbb{R}^2$.

Exercise 3.

Which of the following are bases for $\mathbb{R}^3$? Explain why or why not.

(a) $\{(1,0,1),(1,1,0),(0,1,1),(1,1,1)\}$

(b) $\{(4,0,2),(1,1,3)\}$

(c) $\{(1,1,1),(1,1,0),(1,0,0)\}$

(d) $\{(1,2,3),(4,5,6),(7,8,9)\}$

Exercise 4.

Explain why $S = \{1, x, x^2, x^3\}$ is a basis for P_3, called the `standard basis for` P_3.

Example.

The `standard basis for` $M_{2,2}$ is the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Exercise 5.

What do you think the standard basis for $M_{m,n}$ is?

Definition.

If a basis exists for a vector space, we call it `finite dimensional` otherwise a vector space is said to be `infinite dimensional`.

Example.

The vector space P of all polynomials is infinite dimensional, as is $C(-\infty, \infty)$ the collection of all real valued continuous functions.

Brain Break.

What's your current favorite show to binge watch?

Turns out Pepper doesn't have any tiger toys, so we're settling for Lion King.

Theorem 4.9.

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of the vectors in S.

Note.

The proof of this theorem is in the book. It uses the fact that since S is a basis, its elements are linearly independent and so there is only 1 way to represent the zero vector.

Example.

Every vector $\vec{u} = (x, y)$ in $\mathbb{R}^2$ can be written uniquely in terms of the vectors in the basis $S = \{(1,2), (5,-2)\}$ from exercise **??**. How might you show there is a unique solution to

$$c_1(1,2) + c_2(5,-2) = (x,y),$$

no matter the value of x and y?

I suggest writing down the matrix A so that $A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Theorem 4.10.

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

Note.

The proof of Theorem 4.10 is available in the book. It relies on the fact that you can write the vectors in the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ $(m > n)$ in terms of the vectors in the basis S and insert them into the equation $k_1\vec{u}_1 + k_2\vec{u}_2 + \cdots + k_m\vec{u}_m = \vec{0}$. Ultimately you will get a homogeneous system with more variable then equations, so there is a nontrivial solution.

Exercise 6.

Explain by looking, no computations only references to theorems, why the following are linearly dependent:

(a) $S = \{(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$ in $\mathbb{R}^3$

(b) $S = \{x-1, x+1, x^2-1, x^2+1\}$ in P_2

(c) $S = \{(6,2),(18,6)\}$ in $\mathbb{R}^2$

(d) $S = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \right\}$.