Math 324: Linear Algebra

1.2: Gaussian Elimination and Gauss-Jordan Elimination

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Last Time.

- $m \times n$ matrix
- coefficient and augmented matrix
- elementary row operations
- row-echelon form and leading 1's
- reduced row-echelon form

Today.

- an algorithm for row reducing
- Gauss–Jordan elimination
- homogeneous equations

Exercise 1.

What are all of the possible 2×3 reduced row-echelon matrices?
(Use a $*$ to indicate a position that can have any value.)
For example,

$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 2.

For each of the reduced row-echelon forms determine if the corresponding linear system with that augmented matrix would have: one solution, infinitely many solutions, or no solution.

A generic row reducing algorithm - Gaussian Elimination.

Work column by column from left to right and top to bottom.

1 Swap rows and/or scale a row so that there is a 1 in the top left corner.

2 Add a multiple of row 1 to each of the rows below to zero-out the rest of the first column.

3 Repeat these steps, ignoring the first row and column.

Example.

$$\begin{bmatrix} 0 & -2 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ -2 & -1 & 2 & -3 \end{bmatrix}$$

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$$\xrightarrow{R_3 + 2R_1 \to R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

Example.

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\begin{bmatrix} 0 & -2 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ -2 & -1 & 2 & -3 \end{bmatrix}
\xrightarrow{R_1 \leftrightarrow R_2}
\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ -2 & -1 & 2 & -3 \end{bmatrix}
$$

$$
\xrightarrow{R_3 + 2R_1 \to R_3}
\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix}
$$

$$
\xrightarrow{-\frac{1}{2} R_2 \to R_2}
\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix}
$$

Example.

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$$\xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Exercise 3.

Use this algorithm to find the reduced row echelon form of the matrix:

$$\begin{bmatrix} 3 & -2 & -1 & 2 & 0 \\ -3 & 3 & -3 & 3 & 0 \\ 2 & -3 & 2 & -3 & -3 \\ -1 & 2 & 3 & -2 & 0 \end{bmatrix}.$$

Question 4.

Can a matrix be reduced to two different row-echelon forms?
Can a matrix be reduced to two different reduced row-echelon forms?

Question 5.

If a matrix is in row-echelon form, how would you go about reducing it to reduced row-echelon form?

Facts

(a) Yes, every matrix can be reduced to a *unique* reduced row echelon form.

(b) The process of translating a RE matrix to a RRE matrix is called `Gauss-Jordan elimination`.

Definition.

A `homogeneous system of linear equations` is a linear system of equations in which every constant term is zero. Every homogeneous system has the `trivial` solution, the one where every variable is zero.

Exercise 6.

Solve the homogeneous linear system with the given coefficient matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

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Question 7.

Why is it ok to just use the coefficient matrix to solve a homogeneous system but for other systems we use the augmented matrix?

Exercise 8.

Assume the given matrix is the augmented matrix of a system of linear equations, and (a) determine the number of equations and the number of variables, and (b) find the values of k such that the system is consistent.
Then assume the matrix is the coefficient matrix of a homogeneous system of linear equations and repeat (a) and (b).

$$A = \begin{bmatrix} 1 & k & 2 \\ -3 & 4 & 1 \end{bmatrix}$$

Exercise 9.

Find the values of a, b and c (if possible) such that the system of linear equations has

(a) a unique solution

(b) no solutions

(c) infinitely many solutions

$$\begin{array}{ccccccc} x & + & y & & & = & 2 \\ & & y & + & z & = & 2 \\ x & & & + & z & = & 2 \\ ax & + & by & + & cz & = & 0 \end{array}$$

Exercise 10.

Get more practice!
Open `http://sagecell.sagemath.org` and use the following code to generate a random matrix:

```
matrix([[randint(0,3) for i in [1..5]] for j in [1..4]])
```

Do this a couple times until you have one with 3-6 zeros.
Use elementary row operations to reduce this matrix to (reduced) row-echelon form.