Math 324: Linear Algebra

2.4: Elementary Matrices

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Last Updated: December 29, 2023

Last Time.

- Properties of Inverses
- Using Inverses to Solve System of Equations

Today.

- Theorem 2.15
- Elementary Matrices
- Row Equivalence
- Writing a Matrix as a Product of Elementary Matrices
- Elementary Matrices and Inverses

Exercise 1 (Warm-up).

Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$. Use A^{-1} to solve the system of equations

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}.$$

Theorem 2.15 (1-4).

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. A is invertible.
2. $A\vec{x} = \vec{b}$ has a unique solution for every $n \times 1$ column matrix $\vec{b}$.
3. $A\vec{x} = \vec{0}$ has only the trivial solution.
4. A is `row-equivalent` to I_n, meaning there are elementary row operations that transform A to I_n.

Exercise 2.

What Theorem says (1) implies (2)?
Why does (2) imply (3)?
Why does (3) imply (4)?
Why does (4) imply (1)? (Cite the method used here.)
Given these, there is a loop of implications implying equivalence:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$$

Definition.

An $n \times n$ `elementary matrix` is any matrix that can be obtained by performing a single elementary row operation on the identity matrix.

Exercise 3.

Which of the following matrices are elementary matrices?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Exercise 4.

Compute each of the products

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 3 & -5 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -5 & -5 & 0 \\ 1 & 5 & 4 & -3 & 5 \end{bmatrix}$

How can you read the products from the elementary matrix without actually having to go through the process of matrix multiplication?

Theorem 2.12.

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

Definition.

We say A is `row equivalent` to B if there exist elementary matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = B$.

Exercise 5.

Let $A = \begin{bmatrix} 0 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

(a) Manually, without technology, use elementary row operations to reduce A to row-echelon form.

(b) Encode each of those elementary row operations as an elementary matrix.

(c) Multiply the elementary matrices you just found by A in the correct order to verify that you got the right sequence.

Brain Break.

∼NAMES∼

What song always gets stuck in your head?

Note.

Elementary row operations can always be undone.

Exercise 6.

What would you do to undo the following ERO's?

(a) $R_2 \leftrightarrow R_3$ (b) $7R_3$ (c) $R_1 + 5R_3 \to R_1$

Exercise 7.

What is the inverse of each of the following elementary matrices?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Theorem 2.13.

If E is an elementary matrix, then E^{-1} exists and is also an elementary matrix.

Theorem 2.14.

An $n \times n$ matrix A is invertible if and only if it can be written as a product of elementary matrices.

Exercise 8.

Let $A = \begin{bmatrix} 0 & 1 \\ 5 & 4 \end{bmatrix}$

(a) Manually, without technology, use elementary row operations to transform A to reduced row-echelon form. (Note: A is invertible so it can be reduced to I_2)

(b) Encode each of those elementary row operations as an elementary matrix.

(c) Multiply the elementary matrices you just found by A in the correct order to verify that you got the right sequence.

(d) Taking inverses of the ERO's in the correct order, solve for A. Thus writing A as a product of elementary matrices.

Question 9.

Using Exercise **??** as motivation, why is Theorem 2.14 true?

Theorem 2.15.

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. A is invertible.
2. $A\vec{x} = \vec{b}$ has a unique solution for every $n \times 1$ column matrix $\vec{b}$.
3. $A\vec{x} = \vec{0}$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. A can be written as a product of elementary matrices.

Exercise 10.

We can also find inverses by multiplying elementary matrices using the same process as we did to write A as a product of elementary matrices.

Again, by hand, writing down each step, determine the reduced row-echelon form of A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$$

Then write the product $(E_k \cdots E_2 E_1)A = I_3$.

What do you think A^{-1} is?

Multiply $A(E_k \cdots E_2 E_1)$. What did you get?

Exercise 11.

(Yes-this was part of the pre-class homework. Make sure you all agree on the answers.)
Determine whether each statement is True or False.
A statement is `True` if it is always true no matter what.
A statement is `False` if you can write down some example where it fails.
For every true statement, explain why it is true. For every false statement, give an example of when it fails.

(a) The identity matrix is an elementary matrix.

(b) If E is an elementary matrix, then $2E$ is an elementary matrix.

(c) The inverse of an elementary matrix is an elementary matrix.

Question 12.

If A is row equivalent to B and B is row equivalent to C, does this mean that A is row equivalent to C? Why or why not?

Question 13.

If A is row equivalent to B and A is nonsingular, does this mean that B is also nonsingular? Why or why not?

Recall.

Last time we saw that if $A^2 = A$ and A is invertible then $A = I$.
In general if $A^2 = A$, then we call A `idempotent`.

Exercise 14.

Which of the following matrices are idempotent?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise 15.

Determine a and b such that A is idempotent.

$$A = \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$$

Question 16.

If A is idempotent and invertible, is A^{-1} also idempotent? Why or why not?

Question 17.

If A and B are idempotent, is AB also idempotent? Why or why not?