Math 324: Linear Algebra

Section 4.4: Linear Independence

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Last Time.

- Linear Combination
- Span

Today.

- Linear Independence

Exercise 1.

One thing we want to avoid in a spanning set is redundancies, for example consider $S = \{(1, -1, 0), (-1, 1, 0), (0, 1, 1)\}$.
A very easy redundancy check is for non-trivial solutions to:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}.$$

Solve the homogeneous system given by

$$c_1(1, -1, 0) + c_2(-1, 1, 0) + c_3(0, 1, 1) = (0, 0, 0).$$

Definition.

A subset $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ of a vector space V is called `linearly independent` if the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

has only the trivial solution,

$$c_1 = 0,\ c_2 = 0,\ \ldots,\ c_k = 0.$$

If there are non-trivial solutions, then we call S `linearly dependent`.

Example.

Is the set $S = \{1+x, 1-x, 2+3x, 1+2x+3x^2\}$ linearly independent?
Consider the homogeneous system:

$$a(1+x) + b(1-x) + c(2+3x) + d(1+2x+3x^2) = 0.$$

Then solve for a, b, c, d by considering each power of x:

$$\left.\begin{array}{ccccccccc} a & + & b & + & 2c & + & d & = & 0 \\ a & - & b & + & 3c & + & 2d & = & 0 \\ & & & & & & 3d & = & 0 \end{array}\right\} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ 1 & -1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{rref} \begin{bmatrix} 1 & 0 & \frac{5}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Since there is a free variable in the third column, the set is linearly **dependent**.

Note (Spanning Sets and Linear Independence Tests).

Encode each of the vectors as columns of a matrix.
Row reduce the matrix.
The following conclusions can be made.

- Zero Row $\Rightarrow$ Does NOT Span the whole space
- Free Variable (column w/o a leading 1) $\Rightarrow$ Linearly Dependent

Exercise 2.

Use the guide on the previous slide to answer the following.

(a) Is the set $S = \{(1,2),(3,4),(5,6)\}$ linearly independent? Does S span $\mathbb{R}^2$?

(b) Is the set $S = \{(1,2,0),(3,0,4)\}$ linearly independent? Does S span $\mathbb{R}^3$?

(c) Is the set $S = \{1, 1-x-x^2, x+x^2\}$ linearly independent? Does S span P_2?

(d) Is the set $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ linearly independent? Does S span M_2?

Theorem 4.8.

A set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\vec{v}_j$ can be written as a linear combination of the other vectors in S.

Example.

Recall the linearly dependent set from the last example, $S = \{1 + x, 1 - x, 2 + 3x, 1 + 2x + 3x^2\}$. The row reduced matrix was

$$\begin{bmatrix} 1 & 0 & \frac{5}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Notice that elementary row operations do not impact the relationships between the columns. Therefore reading from the third column, we see $\vec{v}_3 = \frac{5}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2$, or

$$2 + 3x = \frac{5}{2}(1 + x) - \frac{1}{2}(1 - x).$$

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Recall the linearly dependent set from the last example, $S = \{1+x, 1-x, 2+3x, 1+2x+3x^2\}$. The row reduced matrix was

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$$2+3x = \frac{5}{2}(1+x) - \frac{1}{2}(1-x) \boxed{+0(1+2x+3x^2)}.$$

Exercise 3.

Determine whether each of the following sets is linearly independent. If it is linearly dependent, write one of the vectors as a linear combination of the others as in the previous example.

(a) $S = \{2 - x, 2x - x^2, 6 - 5x + x^2\}$

(b) $S = \{(1, -1), (2, 1)\}$

(c) $S = \{(1, -1), (2, 1), (3, -2)\}$

(d) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

Brain Break.

Describe yourself in 3 words. (No, they don't have to be interview words.)

As I write this, Pepper is sticking with:
Loud, Anti-Aviary, and Persistently-Adorable.

Corollary 4.8.

Two vectors $\vec{u}$ and $\vec{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

Exercise 4.

Explore this statement using the set $S = \{(1,2),(3,6)\}$. How would you use the fact that $(3,6)$ is a scalar multiple of $(1,2)$ to show that there is a nontrivial way to write $\vec{0}$ in terms of the vectors in S?

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Two vectors $\vec{u}$ and $\vec{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

Exercise 5.

Prove the reverse direction of Corollary 4.8.

Proof.

Let $\vec{u}$ and $\vec{v}$ be vectors in a vector space V. Assume that $\vec{u}$ is a scalar multiple of $\vec{v}$.
Prove that $S = \{\vec{u}, \vec{v}\}$ is linearly dependent by finding c, d not both zero, such that $c\vec{u} + d\vec{v} = \vec{0}$. □

General Method of Proving Linear Independence

(a) Set up an equation of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}.$$

(b) Show that $c_1 = c_2 = \cdots = c_k = 0$ is the only possibility.

(c) Conclude that the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

Exercise 6.

Prove that if $S = \{\vec{u}, \vec{v}\}$ is linearly independent, then $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is linearly independent.

Proof.

Let $S = \{\vec{u}, \vec{v}\}$ be a linearly independent set in a vector space V. We want to show that $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is also linearly independent. To that end, let c, d be scalars such that

$$c(\vec{u} + \vec{v}) + d(\vec{u} - \vec{v}) = \vec{0}.$$

Now prove that c and d actually have to be 0 by rearranging and using the fact that S is linearly independent. $\square$