Math 324: Linear Algebra

Section 4.2: Abstract Vector Spaces

Mckenzie West

Last Updated: December 29, 2023

Last Time.

- Vectors in $\mathbf{R}^2$
- Vectors in $\mathbf{R}^n$

Today.

- Abstract Vector Spaces
- Important Vector Spaces
- Proving Something is not a Vector Space

Definition.

Let V be a set with two operations, `vector addition` and `scalar multiplication` (by real numbers, $\mathbf{R}$).
We call V a `vector space` if the following hold for all $\vec{u}, \vec{v}, \vec{w} \in V$ and all $c, d \in \mathbf{R}$:

1. $\vec{u} + \vec{v} \in V$ `(additive closure)`
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ `(commutativity of addition)`
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ `(associativity of addition)`
4. $\exists\ \vec{0} \in V$ s.t. $\vec{u} + \vec{0} = \vec{u}$ `(additive identity)`
5. $\exists\ (-\vec{u}) \in V$ s.t. $\vec{u} + (-\vec{u}) = \vec{0}$ `(additive inverse)`
6. $c\vec{u} \in V$ `(scalar closure)`
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ `(distributivity)`
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$ `(distributivity)`
9. $c(d\vec{u}) = (cd)\vec{u}$ `(associativity of scalars)`
10. $1(\vec{u}) = \vec{u}$ `(multiplicative identity)`

Example.

For all integers n, the set of n-dimensional vectors, $\mathbf{R}^n$, is a vector space with the standard operations by Theorem 4.2.

Example.

For all integers m and n, the collection of $m \times n$ matrices, denoted $M_{m,n}$, with the standard operations is a vector space by Theorems 2.1 and 2.2.

Exercise 1.

Show that $\{(x, x) \ : \ x \in \mathbf{R}\}$ (read this as, *the collection of ordered pairs* (x, x) *such that* x *is in* $\mathbf{R}$) is a vector space with the standard operations in $\mathbf{R}^2$ by verifying all 10 axioms of a vector space.

Exercise 2.

Show that P_2, the set of polynomial of degree at most 2, along with the zero polynomial, $p(x) = 0$, is a vector space.

Note.

Formally the degree of the polynomial $p(x)$ tells us the maximum possible values of x satisfy $p(x) = 0$. And so the zero polynomial either does not have a degree or is considered to have degree ∞.

Exercise 3.

Let V be the collection of all continuous functions $f(x)$ such that $f(0) = 0$. The sum of two functions f and g is the function $f + g$ where $(f + g)(x) := f(x) + g(x)$. The scalar multiple of the function f by c is the function cf where $(cf)(x) := c(f(x))$.
Is V a vector space?
(Don't get too far into the calculus, but do make sure to convince yourself of the additive and scalar closure properties.)

Exercise 4.

Can you think of any other vector spaces?

Important Vector Spaces with Notation

$$
\begin{aligned}
\mathbf{R} &= \text{set of all real numbers}\\
\mathbf{R}^n &= \text{set of all } n\text{-tuples, as described in the previous class}\\
C(-\infty,\infty) &= \text{set of all continuous functions with domain } \mathbf{R}\\
C[a,b] &= \text{set of all continuous functions defined on } [a,b]\\
P &= \text{set of all polynomials}\\
P_n &= \text{set of all polynomials of degree} \leq n\\
&\ \text{along with the zero polynomial}\\
M_{m,n} &= \text{set of all } m \times n \text{ matrices}\\
M_n &= \text{set of all } n \times n \text{ matrices}
\end{aligned}
$$

Exercise 5.

Watch: Several of the videos linked in the Canvas question. These videos complete the proof that the following set and operations form a vector space.
Show that $V = \{(x, y)\ :\ x, y \in \mathbf{R} \text{ and } y > 0\}$ is a vector space with the operations:

$$\begin{array}{lrcl} \text{Addition:} & (x_1, y_1) \oplus (x_2, y_2) & = & (x_1 + x_2, y_1 y_2) \\ \text{Scalar Multiplication:} & c * (x, y) & = & (cx, y^c) \end{array}$$

Note.

Sometimes we use different notation for addition, eg. $\oplus$, and scalar multiplication, eg. $*$, to distinguish it from the standard operations.

Brain Break.

What is your favorite candy?

I am pretty sure Pepper's favorite snack food is peanut butter.

Example.

To show that something is not a vector space, we simply have to give an example where one of the axioms fails. For example the collection S of all continuous functions $f(x)$ such that $f(0) = 1$ is not a vector space because:

> If $f(x) = 1$ then $(2f)(0) = 2 \neq 1$, so $2f \notin S$. Thus S is not closed under scaling. $\square$

Exercise 6.

What other properties does the set from the previous example fail?

Exercise 7.

Consider the set with $V = \mathbf{R}^2$ where we have the standard definition of addition but scalar multiplication is defined via:

$$c(x, y) = (0, cy).$$

Which axioms does this set/operations fail?

Exercise 8.

Which of the following are vector spaces? (Hint: None are, why not?)

(a) The set $\{(x, y) \;:\; x, y \in \mathbf{R} \text{ and } x, y \geq 0\}$ with the standard operations on $\mathbf{R}^2$.

(b) The set of 2×2 matrices of the form $\begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$ with the standard operations on M_2.

(c) The set 2×2 nonsingular matrices with the standard operations on M_2.

(d) The set of degree exactly 2 polynomials.

Exercise 9 (Challenge - Optional).

Rather than the standard definitions of addition and scalar multiplication on $\mathbf{R}^2$ use the given definition. Which operations produce a vector space on $\mathbf{R}^2$? Justify your answers.

(a)
$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1, y_2) \\ c(x_1, y_1) &= (cx_1, cy_1) \end{aligned}$$

(b)
$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (y_1 + y_2, x_1 + x_2) \\ c(x_1, y_1) &= (cx_1, cy_1) \end{aligned}$$