Math 324: Linear Algebra

Section 5.2: Inner Product Spaces

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Last Time.

- Dot Product
- Length
- Cauchy-Schwarz Inequality
- Triangle Inequality

Today.

- Orthogonal Complement
- Inner Products

You may have seen the following definition last time.

Definition.

Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are said to be `orthogonal` if $\vec{u} \cdot \vec{v} = 0$.

Recall.

The angle between $\vec{u}$ and $\vec{v}$ satisfies

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}.$$

Thus orthogonal vectors have angle θ, with $\cos(\theta) = 0$. Or $\theta = \frac{\pi}{2}$, and are thus perpendicular.

Exercise 1.

Find all vectors (x, y, z) that are orthogonal to $(-2, 1, 0)$.

(a) Compute $(x, y, z) \cdot (-2, 1, 0)$.

(b) Set the dot product equal to zero and parameterize.

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The equation $-2x + y = 0$ is equivalent to $y = 2x$, so if x and z are given parameters, the orthogonal complement of $(-2, 1, 0)$ is the set

$$\{(t, 2t, s) \ : \ s, t \in \mathbb{R}\}. \tag{1}$$

(c) Verify that no matter the values of s and t, $(t, 2t, s) \cdot (-2, 1, 0) = 0$.

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(c) Verify that no matter the values of s and t, $(t, 2t, s) \cdot (-2, 1, 0) = 0$.

(d) The set in (**??**) is called the `orthogonal complement` of $(-2, 1, 0)$. Plot $(-2, 1, 0)$ and the orthogonal complement in Sage (`https://sagecell.sagemath.org/`) using:

```
s,t = var("s,t")
P = parametric_plot3d((t,2*t,s), (t,-2,2),(s,-2,2))
P += arrow((0,0,0), (-2,1,0), width=3, color="red")
P.show(aspect_ratio=1)
```

Recall.

The four main properties of the dot product are:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v}$.
4. $\vec{v} \cdot \vec{v} \geq 0$ and $\vec{v} \cdot \vec{v} = 0$ if and only if $\vec{v} = \vec{0}$.

Definition.

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in a vector space V, and let c be any scalar. An `inner product` on V is a function that associates a real number $\langle \vec{u}, \vec{v} \rangle$ for each pair of vectors $\vec{u}$ and $\vec{v}$ and satisfies the following axioms.

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
2. $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
3. $c \langle \vec{u}, \vec{v} \rangle = \langle c\vec{u}, \vec{v} \rangle$
4. $\langle \vec{v}, \vec{v} \rangle \geq 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.

A vector space with an inner product is called an `inner product space`.

Example.

The dot product is an inner product in $\mathbb{R}^n$, so $\mathbb{R}^n$ is an inner product space.

Exercise 2.

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$. Verify that

$$\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 2u_2v_2$$

is an inner product on $\mathbb{R}^2$.

(a) Is $\langle \vec{u}, \vec{v} \rangle$ a real number?

(b) Does $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$?

(c) Does $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$?

(d) Does $c \langle \vec{u}, \vec{v} \rangle = \langle c\vec{u}, \vec{v} \rangle$?

(e) Is it true that $\langle \vec{v}, \vec{v} \rangle \geq 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$?

Exercise 3.

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$. Use a specific example to show that

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 - u_2 v_2$$

is NOT an inner product on $\mathbb{R}^2$.
That is, pick out an explicit (numerical values) example of $\vec{u}$ and $\vec{v}$ for which one of the four inner product axioms fails.

Brain Break.

How many people are in your family?
(Define family how you wish. To me, this means parents and siblings. Maybe you want to count aunts/uncles/cousins/grandparents/etc. Let me know what you decide.)

This is my family in town for my graduate school graduation.

Exercise 4.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be matrices in the vector space $M_{2,2}$. Show that the function

$$\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

is an inner product on $M_{2,2}$.

Note.

For a finite dimensional vectors space, inner products essentially always look like dot products with positive scalars. Infinite dimensional vector spaces are not always so straight forward.

Exercise 5.

Let f and g be real valued continuous functions in the vector space $C[0,1]$. I claim that the following is an inner product,

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\ dx.$$

Compute $\langle 3x+2, e^x\rangle$.
(Yes, you can—and should—use your calculator or an online integral calculator).

Definition.

Let $\vec{u}$ and $\vec{v}$ be vectors in an inner product space V. Then similarly to in the case of the dot product, we can make similar definitions for length, distance, angle, orthogonal, and orthogonal projection.

1. The `length` (or `norm`) of $\vec{u}$ is $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$.
2. The `distance` between $\vec{u}$ and $\vec{v}$ is $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$.
3. The `angle` between $\vec{u}$ and $\vec{v}$ is given by

$$\cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}, \quad 0 \leq \theta \leq \pi.$$

4. We say $\vec{u}$ and $\vec{v}$ are `orthogonal` if $\langle \vec{u}, \vec{v} \rangle = 0$.

Exercise 6.

Consider the inner product on $M_{2,2}$ from exercise **??**:

$$\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}.$$

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ 2 & -2 \end{bmatrix}$. Compute

(a) $\|A\|^2$

(b) $d(A, B)^2$

(c) Cosine of the angle between A and B.

(d) Are A and B orthogonal?

Note.

The inequalities also persist for inner products. (Note that in proving the Triangle Inequality for example, we didn't actually use the definition of the dot product.)

Theorem 5.8.

Let $\vec{u}$ and $\vec{v}$ be vectors in an inner product space V.

1. Cauchy–Schwarz Inequality: $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\|\|\vec{v}\|$
2. Triangle Inequality: $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
3. Pythagorean Theorem: $\vec{u}$ and $\vec{v}$ are orthogonal if and only if

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Note.

One of the great things that we can use this for is to prove these relationships for integrals without actually talking about integrals!

Exercise 7.

Let f and g be real valued continuous functions in the vector space $C[0,1]$. Use the inner product,

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\ dx.$$

Verify the Cauchy-Schwarz Inequality for $f = 3x + 2$ and $g = e^x$. (Yes, you can—and should—use your calculator).