Math 324: Linear Algebra

Section 3.1: The Determinant of a Matrix
Section 3.2: Determinants and Elementary Operations

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Last Time.

- The determinant of a 2×2 matrix
- Minors
- Cofactors
- The determinant of an $n \times n$ matrix.

Today.

- Determinants of Triangular Matrices
- How Elementary Row Operations Affect Determinants

Definition.

An `upper triangular matrix` is a square matrix for which every entry below the main diagonal is zero. A `lower triangular matrix` is a square matrix for which every entry above the main diagonal is zero.
A `diagonal matrix` is a matrix that is both upper triangular and lower triangular.

Exercise 1.

Compute the determinant of each of the following triangular matrices, using the appropriate cofactor expansion.

$$\begin{bmatrix} -3 & 0 \\ 12 & 8 \end{bmatrix} \qquad \begin{bmatrix} -6 & 15 & -4 \\ 0 & -1 & 6 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Theorem 3.2.

If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal. That is,

$$\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

Exercise 2.

Complete the proof of this Theorem on your worksheet.
Note that due to the inductive definition of determinants, we often use induction to prove properties of determinants.

Exercise 3.

Use Theorem 3.2 to compute:

(a) $\det(I_n)$

(b) $\det(O_n)$

(c) $\det(4I_3)$

Brain Break.

What fictional character would you like to meet?

Exercise 4.

Each person at your table should claim one of these four matrices.

$$\begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ -3 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}.$$

(a) Compute the determinant of your matrix.

(b) Compute the determinant of your matrix after performing the elementary row operation $R_1 \leftrightarrow R_2$.

(c) Reset your matrix, now compute the determinant after performing the elementary row operation $R_2 - 4R_1 \to R_2$.

(d) Reset your matrix again, now compute the determinant after performing the elementary row operation $3R_1 \to R_1$.

Exercise 5.

Using your answers from Exercise 4, fill in the blanks of Theorem 3.3 that describe how elementary row operations affect determinants.

Theorem 3.3.

Let A and B be square matrices.

1. When B is obtained from A by interchanging two rows of A, $\det(B) =$ __________.
2. When B is obtained from A by adding a multiple of a row of A to another row of A, $\det(B) =$ __________.
3. When B is obtained from A by multiplying a row of A by a nonzero constant c, $\det(B) =$ __________.

Example.

We now have another way to compute determinants, row reducing to upper triangular and keeping track of our path:

$$\begin{bmatrix} 0 & 4 & -2 \\ 0 & 1 & 3 \\ -3 & 2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -3 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 4 & -2 \end{bmatrix} \xrightarrow{R_3 - 4R_2 \to R_3} \begin{bmatrix} -3 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -14 \end{bmatrix}.$$

Therefore, the determinant of the starting matrix is

$$\begin{vmatrix} 0 & 4 & -2 \\ 0 & 1 & 3 \\ -3 & 2 & 0 \end{vmatrix} = - \begin{vmatrix} -3 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 4 & -2 \end{vmatrix} = - \begin{vmatrix} -3 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -14 \end{vmatrix}$$

$$= -(-3)(1)(-14) = -72.$$

Exercise 6.

Use elementary row operations to compute:

$$\begin{vmatrix} 4 & 2 & 6 \\ 0 & 2 & 3 \\ 2 & 5 & 5 \end{vmatrix}$$

Definition.

The following are `elementary column operations`:

1. swap two columns, denoted $C_i \leftrightarrow C_j$,
2. add a multiple of one column to another, denoted $C_i + cC_j \to C_i$,
3. multiply a column of B by a nonzero scalar c, denoted $cC_i \to C_i$.

We call two matrices `column equivalent` if one of them can be obtained from the other by elementary column operations.

Note.

Elementary **column** operations affect the determinant in the exact same way that their related elementary **row** operations do.

Exercise 7.

Make use of elementary column operations to compute the determinant:

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 3 & -6 \\ 4 & 0 & -8 \end{vmatrix}$$

Exercise 8.

Make use of elementary row and/or column operations to compute each determinant:

(a) $\begin{vmatrix} 6 & 5 & 4 \\ 0 & 3 & 2 \\ 3 & 5 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 0 & 0 & 5 \\ -2 & 3 & 3 & 4 \\ 1 & 3 & -3 & 4 \\ -2 & 3 & -2 & 2 \end{vmatrix}$