Math 324: Linear Algebra
Direct Proofs and Cases

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Last Updated: December 29, 2023

Last Time.
- Proofs using Cases

Today.
- If and only If Statements
- Contrapositive, Converse, and Inverse

Exercise 1 (Warmup).

Everyone at your table should read the pre-class exercise of another group member.
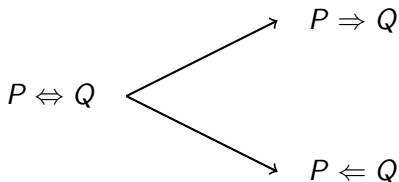What do you like about their proof? What do you think could be improved?

Exercise 2.

Discuss the advantages of using LaTeX to typeset proofs.
What are some challenges to using LaTeX/Overleaf?

The general format of `if and only if proofs` is to turn it into two smaller proofs:

$$P \Leftrightarrow Q \qquad \begin{array}{c} \nearrow \quad P \Rightarrow Q \\ \\ \searrow \quad P \Leftarrow Q \end{array}$$

We then prove both of these statements inside the larger proof. Assume P then show Q is true. Then do a total reset, forget everything that just happened. Assume Q is true then show P is true.

Notation

We read $P \Rightarrow Q$ as "If P then Q" or as "P implies Q".

Exercise 3.

**Claim:** Let A be a square matrix. The matrix A is invertible if and only if the homogeneous system $A\vec{x} = \vec{0}$ has only the trivial solution.

(a) This is an if and only if proposition, i.e., it is of the form $P \Leftrightarrow Q$. Identify the two statements P and Q as well as anything that will be referenced for the entire proof.

(b) Write the two propositions of the form $P \Rightarrow Q$ and $P \Leftarrow Q$ in full sentences. In each case, what will you need to assume about A and what will you need to prove?

(c) Do some scratch work for each of the directions of the proof so you know how the proof will go.

(d) Complete the proof of the claim on the worksheet.

Brain Break.

What's the best class you've ever taken?
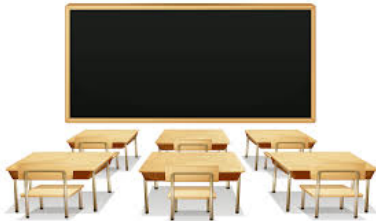
Definition.

The contrapositive of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$. Recall that $\neg Q$ means "not Q" or "the negation of Q".

Example.

The contrapositive of the statement:

> If it is raining then the grass is wet.

is the statement:

> If the grass is not wet then it is not raining.

Note.

The contrapositive of a statement is logically equivalent to the original statement. Therefore, we can prove it instead.

Exercise 4.

What is the contrapositive of this statement about $n \times n$ matrices A and B?

If AB is an invertible matrix then B is also invertible.

Note.

We can prove the contrapositive of this statement using the statement from Exercise **??**.

Definition.

The converse of the statement $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

Example.

The converse of the statement:

If it is raining then the grass is wet.

is the statement:

If the grass is wet then it is raining.

Note.

The converse is NOT logically equivalent to the original statement. As you see in the example. There are other non-rain reasons that the grass could be wet.

Exercise 5.
What is the converse of the same statement about $n \times n$ matrices A and B?

       If B is a singular matrix then AB is also singular.

Exercise 6.
Find an example of square matrices A and B such that AB is singular but B is invertible. Thus showing that the converse is false.

Definition.

The inverse of the statement $P \Rightarrow Q$ is the statement $\neg P \Rightarrow \neg Q$.

Example.

The inverse of the statement:

> If it is raining then the grass is wet.

is the statement:

> If it is not raining then the grass is not wet.

Note.

The inverse is NOT logically equivalent to the original statement. As you see in the example. There are other non-rain reasons that the grass could be wet. But look, the inverse is the contrapositive of the converse, so they are logically equivalent.

Exercise 7.
What is the inverse of the same statement about $n \times n$ matrices A and B?

If B is a singular matrix then AB is also singular.

Exercise 8.
Does the same counterexample of the converse also function as a counterexample to the inverse?

Summary

With the original statement $P \Rightarrow Q$, we have:

contrapositive	$\neg Q \Rightarrow \neg P$	equivalent to $P \Rightarrow Q$
converse	$Q \Rightarrow P$	NOT equivalent to $P \Rightarrow Q$
inverse	$\neg P \Rightarrow \neg Q$	NOT equivalent to $P \Rightarrow Q$

Exercise 9.

Consider the proposition:

Proposition..

Let A be a square matrix. If A is a singular then $A\vec{x} = \vec{0}$ has infinitely many solutions.

(a) What is the contrapositive?

(b) What is the converse? Is it True?

(c) What is the inverse? Is it True?

Question 10.
What is the relationship between if and only if statements and
contrapositive/converse/inverse?

Exercise 11.

Consider the proposition:

Proposition..

Let A and B be a $m \times n$ matrices and $O_{m,n}$ the $m \times n$ zero matrix. If $A + B = A$ then $B = O_{m,n}$.

(a) What is the contrapositive?

(b) Do some scratch work toward either the original statement or the converse to convince yourself that they are true.

(c) What is the converse? Is it True?

(d) What is the inverse? Is it True?