# Math 324: Linear Algebra

## Section 6.2: The Kernel and Range of a Linear Transformation

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**Last Time.**

– Matrix Transformations

– The Kernel of a Transformation

– The Range of a Transformation

– Rank and Nullity

**Today.**

– One-to-One Transformations

– Onto Transformations

– Isomorphisms

## Recall.

The kernel of a linear transformation $T: V \to W$ is the collection of all vectors in $V$ that map to the zero vector in $W$. We denote this by

$$\ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}_W\}$$

The range of $T$ is all of the vectors in $W$ that are 'hit' by $T$. This is can be denoted using either of the following

$$\begin{aligned} \text{range}(T) &= \{T(\vec{v}) : \vec{v} \in V\} \\ &= \{\vec{w} \in W : T(\vec{v}) = \vec{w} \ \exists \ \vec{v} \in V\}. \end{aligned}$$

(Note: Here we read the $\exists$ symbol as 'for some'.)

## Recall.

We also saw last time that if $T$ is a matrix transformation with matrix $A$, then $\ker(T) = N(A)$ and $\text{range}(T) = \text{col}(A)$.
Some natural questions that may arise are whether or not

- the kernel/nullspace have only the trivial solution;
- the range/column space is all of the codomain. (That is, the columns span the space.)

The first of these ideas we will call `one-to-one` and the second is called `onto`. They will be precisely defined in this worksheet.

### Definition.

A linear transformation, $T\colon V \to W$, is one-to-one if the preimage of every vector in range$(T)$ has exactly one vector. That is, for every $\vec{u}, \vec{v} \in V$, if $T(\vec{u}) = T(\vec{v})$ then $\vec{u} = \vec{v}$.

### Exercise 1.

Show that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, x - y, 0)$ is one-to-one.

(a) Let $\vec{v}_1 = (x_1, y_1)$ and $\vec{v}_2 = (x_2, y_2)$ be two vectors in $\mathbb{R}^2$ such that $T(\vec{v}_1) = T(\vec{v}_2)$.

(b) Write out what it means for $T(\vec{v}_1) = T(\vec{v}_2)$ using the definition of $T$.

(c) Show that $x_1 = x_2$ and $y_1 = y_2$. Thus proving $\vec{v}_1 = \vec{v}_2$.

(d) Conclude that $T$ is one-to-one.

**Theorem 6.6.**

Let $T : V \to W$ be a linear transformation. Then $T$ is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

**Note.**

Notice how powerful this is. It says that knowing how many things map to $\vec{0}$ immediately tells you how many things map to everything else in the range.

## Exercise 2.

Use Theorem 6.6 to show that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, x - y, 0)$ is one-to-one.

(a) Let $\vec{v} = (x, y)$ be a vector in $\mathbb{R}^2$ such that $T(\vec{v}) = \vec{0}$.

(b) Carefully write out what $T(\vec{v})$ is and show that in fact $\vec{v} = \vec{0}$ must be true.

## Exercise 3.

Determine whether the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is one-to-one.

**Question 4.**

If $T : V \to W$ is a one-to-one linear transformation, how does $\dim(V)$ compare to $\dim(W)$?

- Can a transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ be one-to-one?
- Can a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ be one-to-one?
- Can a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be one-to-one?

**Definition.**

A linear transformation $T : V \to W$ is `onto` if $\text{range}(T) = W$.

**Theorem 6.7.**

Let $T : V \to W$ be a linear transformation where $W$ is finite dimensional. Then $T$ is onto if and only if $\text{rank}(T) = \dim(W)$.

**Exercise 5.**

Discuss this theorem, make sure you agree and understand the difference between $T$ being onto and $\text{rank}(T) = \dim(W)$.

**Brain Break.**

What would you go viral for on the Internet?

### Theorem 6.8.

Let $T : V \to W$ be a lienar transformation with vector space $V$ and $W$, both dimension $n$. Then $T$ is one-to-one if and only if $T$ is onto.

### Exercise 6.

Prove Theorem 6.8,

(a) Assume first that $T$ is one-to-one.

   i. What does theorem 6.6 say about $\ker(T)$?

   ii. You know $\dim(V) = \dim(W) = n$ and $n = \text{rank}(T) + \text{nullity}(T)$. Use this and part (i) to conclude that $\text{rank}(T) = n$.

   iii. Use Theorem 6.7 to conclude $T$ is onto.

(b) Now, reset and assume $T$ is onto. Reverse the arguments from (a) with the starting point of $\text{rank}(T) = n$ to get to $\ker(T) = \{\vec{0}\}$.

### Definition.

A linear transformation $T : V \to W$ that is one-to-one and onto is called an `isomorphism`. Moreover, if there exists at least one isomorphism between two vector spaces $V$ and $W$, then we call $V$ and $W$ `isomorphic`.

### Note.

Note that isomorphic means "same". It often does not mean "equal".

### Example.

The vector spaces $P_2$ and $\mathbb{R}^3$ are isomorphic because $T : P_2 \to \mathbb{R}^3$ defined by $T(ax^2 + bx + c) = (a, b, c)$ is an isomorphism. (Check this.)

## Exercise 7.

For (a)-(d), consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$.

i. Determine $n$ and $m$.

ii. Is $T$ one-to-one, onto, or neither?

iii. Is $T$ an isomorphism?

(a) $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

**Theorem 6.9.**

Two finite-dimensional vector spaces $V$ and $W$ are isomorphic if and only if they are of the same dimension.

**Note.**

Holy schmoley is this awesome! Theorem 6.9 says so many things we know, e.g. $R^2$ is essentially the same as $M_{2,1}$, $M_{1,2}$, $P_1$, a plane in $\mathbb{R}^3$, a plane in $\mathbb{R}^4$,.... the list goes on.

**Exercise 8.**

List at least 4 vector spaces that are isomorphic to $\mathbb{R}^6$.

## Question 9.

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation defined by the matrix $A$, what do each of the following tell you about the dimensions and the invertibility of $A$?

(a) $T$ is neither one-to-one nor onto

(b) $T$ is one-to-one

(c) $T$ is onto

(d) $T$ is an isomorphism