Math 324: Linear Algebra

2.2: Properties of Matrix Operations

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Last Time.

- Row and Column Matrices
- Linear Combinations
- Linear Systems as Matrix Products

Today.

- Properties of Matrix Addition
- Additive and Multiplicative Identities
- Properties of Matrix Multiplication
- Transposes and Properties of

Theorem 2.1.

If A, B, and C are $m \times n$ matrices, and c and d are scalars, then the following properties are true:

1. $A + B = B + A$ Commutativity (of addition)
2. $A + (B + C) = (A + B) + C$ Associativity (of addition)
3. $(cd)A = c(dA)$ Associativity (of scalar multiplication)
4. $1A = A$ Scalar multiplication identity
5. $c(A + B) = cA + cB$ Distributivity
6. $(c + d)A = cA + dA$ Distributivity

Exercise 1.

Let $A = \begin{bmatrix} 3 & -2 & 4 \\ 0 & 1 & 5 \end{bmatrix}$ $c = 2$ and $d = -1$.
Carefully work through the left-hand side and right-hand side of property (6) of Theorem 2.1.
What is the significance of this part of the Theorem?

Note.

Theorem 2.1 feels intuitive but it is absolutely necessary that we write it down and verify it.

When adding, we want to have the option to add “zero” though what we mean by zero depends on how addition is performed.

Definition.

A `zero matrix`, or `additive identity` for the set of $m \times n$ matrices is the $m \times n$ matrix that consists entirely of zeros, denoted by O_{mn}.

Theorem 2.2.

If A is an $m \times n$ matrix and c is a scalar, then the following are true.

1. $A + O_{mn} = A$
2. $A + (-A) = O_{mn}$
3. If $cA = O_{mn}$ then $c = 0$ or $A = O_{mn}$.

Exercise 2.

Use Theorems 2.1 and 2.2 to solve for the matrix X in terms of the matrices A and B if $3A - 2X = B$.
Carefully list what properties of Theorems 2.1 and 2.2 you are using when you use them.
(Hint: You might use properties (1) and (2) of Thm 2.2, and properties (3), (4), and (5) of Thm 2.1).

Theorem 2.3.

If A, B, and C are matrices of appropriate size and c is a scalar then the following are true

1. $A(BC) = (AB)C$ `Associativity (of multiplication)`
2. $A(B + C) = AB + AC$ `Distributivity`
3. $(A + B)C = AC + BC$ `Distributivity`
4. $c(AB) = (cA)B = A(cB)$

Exercise 3.

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

Have some people at the table compute $A(BC)$ with the others compute $(AB)C$.

Warnings.

- Commutativity does not apply for matrices. In general

$$AB \neq BA.$$

- We can't divide by matrices. It is possible for

$$AB = AC \text{ but } B \neq C.$$

Definition.

The `identity matrix of order` n is the $n \times n$ matrix with 1's on the diagonal and zeros everywhere else. Denote it by I_n.

Example.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 2.4.

If A is a matrix of size $m \times n$ then

1. $AI_n = A$
2. $I_m A = A$

Exercise 4.

Compute

$$I_3 \begin{bmatrix} 3 & 4 \\ 7 & 9 \\ -5 & 2 \end{bmatrix}$$

Exercise 5.

Compute $\begin{bmatrix} 6 & 2 \\ 13 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, and find a pattern.
Without going through the steps of multiplication, determine

$$\begin{bmatrix} -2 & 3 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

Brain Break

What book/movie/TV show would you change the ending to if you could?

Definition.

If k is a positive integer and A is a square matrix, define the `kth power of` A by

$$A^k = AA \cdots A$$

multiplying A by itself k times.
Define $A^0 = I$.

Question 6.

For positive integers j and k, is

$$A^{j+k} = (A^j)(A^k)?$$

Is

$$A^{jk} = (A^j)^k = (A^k)^j?$$

Definition.

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A in the same order.

Exercise 7.

Find the transpose of each matrix

$$\begin{bmatrix} -1 & 13 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 9 & 7 \\ -3 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & -5 \\ -1 & -5 & 1 \end{bmatrix}$$

Definition.

If $A^T = A$, we call A symmetric.

Theorem 2.5.

If A and B are matrices of appropriate size and c is a scalar, then the following are true

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = cA^T$
4. $(AB)^T = B^T A^T$

Exercise 8.

Verify that $(AB)^T = B^T A^T$ for

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$$

Exercise 9.

Compute AA^T, what do you notice?

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Exercise 10.

Compute $B + B^T$, what do you notice?

$$B = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$$

Exercise 11.

Use Theorem 2.5 (and commutativity of addition) to verify that $A + A^T$ is symmetric for all $n \times n$ matrices A. Also verify that cA is symmetric if A is symmetric.

Definition.

A $n \times n$ matrix B is called `skew symmetric` if $B^T = -B$.

Exercise 12.

Use Theorem 2.5 (and 2.1) to verify that $A - A^T$ is skew symmetric for all $n \times n$ matrices A.

Exercise 13.

Use the previous two exercises to write the following matrix as the sum of a symmetric matrix and a skew symmetric matrix:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 13 \\ -1 & 2 & 3 \end{bmatrix}$$

Definition.

The trace of an $n \times n$ matrix A is the sum:

$$\operatorname{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Exercise 14.

Compute $\operatorname{Tr}(I_7)$ and $\operatorname{Tr}\left(\begin{bmatrix} 3 & -2 & 4 \\ 0 & 2 & 1 \\ -3 & 15 & 3 \end{bmatrix}\right)$

Exercise 15.

Suppose A and B are $n \times n$ matrices and c a scalar.

(a) How does $\operatorname{Tr}(cA)$ compare to $\operatorname{Tr}(A)$?

(b) Is it true that $\operatorname{Tr}(A + B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)$?

(c) Is it true that $\operatorname{Tr}(AB) = \operatorname{Tr}(A)\operatorname{Tr}(B)$?

Exercise 16.

Consider square matrices of the form

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{(n-1)n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

(a) Write a 2×2 and a 3×3 matrix in the form of A.

(b) Use Sage or a calculator to square, cube, fourth power,... each of the matrices. Describe the result.

(c) Make a guess at what will happen for a 4×4 matrix of this form as you raise it to powers. Test your guess using technology.

(d) Make a conjecture about the powers of A when A is an $n \times n$ matrix.