Math 324: Linear Algebra

Section 7.1: Eigenvalues and Eigenvectors

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Last Time.

- The matrix of a transformation.
- Compositions

Today.

- Eigenvalues and Eigenvectors

Remark.

We can think of a 2×2 matrix A as a transformation of $2d$ space. We might be curious about what vectors are fixed by A:

$$A\vec{x} = \vec{x}.$$

More generally, we ask what lines are fixed by A:

$$A\vec{x} = \lambda\vec{x},$$

where λ, the Greek letter "lambda", is a fixed scalar.

Definition.

Let A be an $n \times n$ matrix. The scalar λ is called an `eigenvalue` of A if there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$.
In this case, we call $\vec{x}$ an `eigenvector` of A corresponding to λ.

Exercise 1.

Verify that $\vec{x}_1 = (1, 0, 1)$, $\vec{x}_2 = (3, 0, -2)$, and $\vec{x}_3 = (0, 3, 1)$ are all eigenvectors of

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 0 & -2 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

What are the corresponding eigenvalues?

Exercise 2.

(a) Is $-2\vec{x}_1 = (-2, 0, -2)$ an eigenvector of A?

(b) Is $\vec{x}_1 + \vec{x}_2$ an eigenvector of A?

(c) Is $\vec{x}_2 + \vec{x}_3$ an eigenvector of A?

Exercise 3.

Let $\vec{x}_1$ and $\vec{x}_2$ be eigenvectors of A corresponding to some fixed eigenvalue λ.
Verify that $\vec{x}_1 + \vec{x}_2$ is also an eigenvector of A corresponding to λ.

Exercise 4.

Let $\vec{x}$ be an eigenvector of A corresponding to some fixed eigenvalue λ.
Verify that $c\vec{x}$ is also an eigenvector of A corresponding to λ for all scalars $c \neq 0$.

Theorem 7.1.

If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector:

$$V = \{\vec{x} : A\vec{x} = \lambda\vec{x}\}$$

is a subspace of $\mathbb{R}^n$ called the `eigenspace` of A corresponding to λ.

Note.

Certainly $A\vec{0} = \vec{0} = \lambda\vec{0}$ is always true, so $\vec{0}$ is included in the set V as in Theorem 7.1. But it would be silly to call $\vec{0}$ an eigenvector because it is always fixed.

Exercise 5.

Consider the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Find the eigenspace of A corresponding to $\lambda = 3$.

(a) Let $\vec{x}$ be a generic vector in $\mathbb{R}^3$ such that $A\vec{x} = 3\vec{x}$.

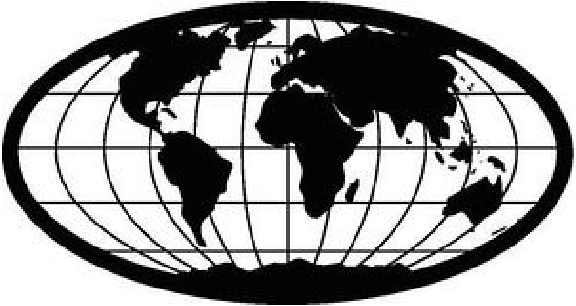
(b) Subtract $A\vec{x}$ so that all variables are on the same side, $3\vec{x} - A\vec{x} = \vec{0}$.

(c) Factor the $\vec{x}$ out to the right. Be careful when you do this, what is the constant 3 in the land of matrices?

(d) Did you get $(3I - A)\vec{x} = \vec{0}$? Look, a homogeneous system! Compute $3I - A$ and solve the system.

Brain Break.

Where are you from?



Question 6 (Finding Eigenvalues).

Let A be an $n \times n$ matrix with eigenvalue λ.
This means there is some nonzero $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$.

(a) Proceed as in the last exercise, subtract that $A\vec{x}$ over and factor out the $\vec{x}$.

(b) What does it mean for the matrix $\lambda I - A$ to have a non-zero solution to the homogeneous system $(\lambda I - A)\vec{x} = \vec{0}$?

(c) It means that $\lambda I - A$ is singular (non-invertible). What is a quick way to determine whether a matrix is singular?

(d) A matrix is singular exactly when its determinant is zero! Compute $\det(\lambda I - A)$, set it equal to zero and solve for λ.

Theorem 7.2.

Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.
2. The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)\vec{x} = \vec{0}$.
3. The eigenspace of A corresponding to λ is the nullspace of the matrix $\lambda I - A$.

Definition.

The equation $\det(\lambda I - A) = 0$ is called the `characteristic equation` of A.
The polynomial $\det(\lambda I - A)$ is called the `characteristic polynomial` of A.

Exercise 7.

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -3 & 4 \\ -3 & 5 \end{bmatrix}.$$

(a) Compute $\lambda I - A$.

(b) Compute the characteristic polynomial, $\det(\lambda I - A)$.

(c) Solve the characteristic equation, $\det(\lambda I - A) = 0$.

(d) For each eigenvalue, λ, explicitly compute the nullspace of the matrix $\lambda I - A$.