

Math 324: Linear Algebra

1.1: Introduction to Systems of Linear Equations

1.2: Gaussian Elimination and Gauss-Jordan Elimination

Mckenzie West

Last Updated: December 29, 2023

Last Time.

- Linear Equations
- Solving Linear Equations and Parametrization
- Linear Systems of Equations
- Solving Linear Systems of Equations
- Consistent vs Inconsistent Systems

Today.

- $m \times n$ matrix
- coefficient and augmented matrix
- elementary row operations
- row-echelon form and leading 1's
- reduced row-echelon form

Today we will use matrices and elementary row operations to simplify the process of solving a system of linear equations.

Question 1.

Last time you found solutions to the system of equations:

$$\begin{array}{rccccccc} 3x & - & 2y & + & 3z & = & 4 \\ & & x & & + & z & = & 5 \end{array}$$

what were some steps you took to do this?

Example.

I don't know about you, but I think that writing variables is a waste of time, when all we're looking at is canceling coefficients. Let's save ourselves some time by turning everything into a nicely arranged table:

$$\begin{array}{rcccccl} 3x & - & 2y & + & 3z & = & 4 \\ x & & & + & z & = & 5 \end{array} \dashrightarrow \begin{bmatrix} 3 & -2 & 3 & 4 \\ 1 & 0 & 1 & 5 \end{bmatrix}$$



Definition.

The tables from the previous page are called **matrices**.

If m and n are positive integers, an $m \times n$ **matrix** is a rectangular array with m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

where the entries a_{ij} are real numbers.

We say the **size** of the matrix is $m \times n$.

A **square** matrix is one where $m = n$, in this case we call the entries $a_{11}, a_{22}, \dots, a_{nn}$ the **main diagonal**.

Exercise 2.

What is the size of each of the following matrices:

(a) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 0 \\ -54 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$

Definition.

There are two basic matrices that can be constructed using a linear system of equations:

- the **coefficient matrix** which contains only the coefficients of the linear system, not the constant terms;

- the **augmented matrix** which is the matrix that contains both the coefficients and the constant terms.

Question 3.

If we're solving a system of linear equations which of these two matrices are we going to use?

Exercise 4.

Write down the coefficient matrix and augmented matrix for the following system of linear equations:

$$\begin{array}{rclclcl} x & & & - & 2z & = & 5 \\ 2x & + & 3y & & & = & -3 \\ -2x & + & y & - & 2z & = & 3 \end{array}$$

Definition (Elementary Row Operations).

There are three **elementary row operations** that we use on a matrix:

1. Interchange two rows.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of one equation to another equation.

Note.

Each of these operations corresponds to an equivalent action on the linear equations.

Exercise 5.

Use elementary row operations on the augmented matrix and back substitution to solve the system of linear equations.

$$\begin{array}{rrcrcl} x & + & 2y & - & 4z & = & -5 \\ & & y & + & z & = & 1 \\ -2x & - & y & - & z & = & 5 \end{array}$$

Definition.

A matrix is in **row-echelon form** if it satisfies the following properties:

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 of the lower row.

Example.

The following matrix is in row echelon form, verify this

$$\begin{bmatrix} 1 & 4 & 0 & 11 \\ 0 & 1 & 14 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 6.

Which of the following matrices is in row-echelon form?

(a) $\begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 0 & -6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & -1 & 5 & -4 & -2 & -4 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Question 7.

Can we use elementary row operations to reduce a matrix to row echelon form?

Question 8.

How will row-echelon form help us solve linear systems of equations?

Definition.

A matrix is in **reduced row-echelon form** if it is in row-echelon form and in every column with a leading 1 all other entries are zero.

Exercise 9.

Which of the following matrices are in reduced row-echelon form?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 13 & 0 \\ 0 & 30 & 1 \end{bmatrix}$

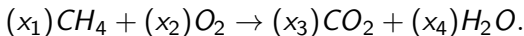
(c) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Example.

In a chemical reaction, atoms reorganize in one or more substances. For instance, when methane gas (CH_4) combines with oxygen (O_2) and burns, carbon dioxide (CO_2) and water (H_2O) form. Chemists represent this process by a chemical equation of the form



With hopes of solving, we can create a system of linear equations, one for each element:

$$\begin{array}{lcl} \text{C} : & 1x_1 + 0x_2 & = 1x_3 + 0x_4 \\ \text{H} : & 4x_1 + 0x_2 & = 0x_3 + 2x_4 \\ \text{O} : & 0x_1 + 2x_2 & = 2x_3 + 1x_4 \end{array}$$

We then rearrange this system into the form we have been working with:

$$C : 1x_1 + 0x_2 = 1x_3 + 0x_4,$$

$$H : 4x_1 + 0x_2 = 0x_3 + 2x_4,$$

$$O : 0x_1 + 2x_2 = 2x_3 + 1x_4,$$

becomes

$$\begin{array}{rclclcl} x_1 & & & - & x_3 & & = & 0, \\ 4x_1 & & & & & - & 2x_4 & = & 0, \\ & 2x_2 & - & 2x_3 & - & x_4 & = & 0. \end{array}$$

As an augmented matrix, we get the following matrix and reduced row echelon form:

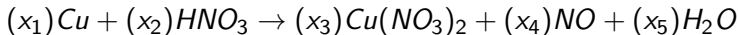
$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

Exercise 10.

Write the solutions to the system in parametric form.

Exercise 11.

Balance the chemical equation:



That is, find a parametric representation of the solutions to the linear system as in the previous example.

If you want to end up with 7,200 units of water, H_2O , how much of the initial compounds must you start with?

Exercise 12.

Discuss a general method you would use to reduce a matrix to row echelon form.

Exercise 13.

Open <http://sagecell.sagemath.org> and use the following code to generate a random matrix:

```
matrix([[randint(-2,2) for i in [1..5]] for j in [1..4]])
```

Do this a couple times until you have one with 3-6 zeros.

Use elementary row operations to reduce this matrix to (reduced) row-echelon form.