Math 324: Linear Algebra

Section 3.4: Applications of Determinants

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Last Time.

- More Properties of Determinants
- The Adjoin of a Matrix
- Cramer's Rule

Today.

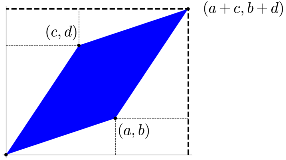
- Areas and Volumes
- Lines and Planes
- Cross Products

Area of a Parallelogram

The area of the parallelogram defined by (a, b) and (c, d) is

$$\text{Area} = \left\| \begin{matrix} a & b \\ c & d \end{matrix} \right\|,$$

where $\|A\|$ denotes the absolute value of the determinant.



Exercise 1.

Geometrically verify this formula using areas of rectangles and triangles.
Then find the area of the parallelogram defined by $(3, 1)$ and $(2, 3)$.

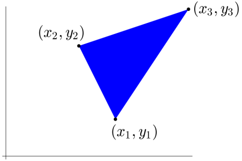
Area of a Triangle in the *xy*-Plane

The area of the triangle with vertices

$$(x_1, y_1), (x_2, y_2), \text{ and } (x_3, y_3)$$

can be calculated using

$$\text{Area} = \frac{1}{2}\left\|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right\|.$$



Exercise 2.

Find the area of the triangle with vertices $(0, 1), (3, 2)$, and $(5, 7)$.

Note.

If three points are `co-linear`–meaning they all lie on the same line–then they will form a triangle with zero area.

Exercise 3.

Which of the following sets of points are co-linear? Use the area formula from the last slide.

(a) $(0, 1), (2, 5), (4, -2)$

(b) $(-2, -7), (0, -1), (3, 8)$

(c) $(2, 1), (2, 2), (2, 3)$

Note.

Given two points, any other point (x, y) that is co-linear to these two will produce a triangle of zero area.

Two-Point Form of the Equation of a Line

An equation for the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$0 = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}.$$

Exercise 4.

Use the determinant formula to find an equation for the line passing through the points

(a) $(-3, 2)$ and $(1, 5)$

(b) $(1, 1)$ and $(-10, 1)$

Exercise 5.

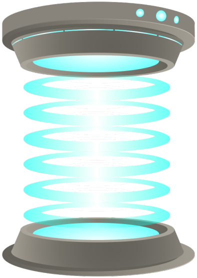
Use the determinant formula to verify the point slope formula of the line passing through the points (x_1, y_1) and (x_2, y_2):

$$y - y_1 = m(x - x_1)$$

where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Brain Break.

If teleportation were possible, what would be the first place you'd teleport to?



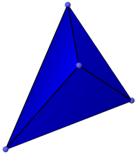
Volume of a Tetrahedron.

The volume of the tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is given by

$$\text{Volume} = \frac{1}{6} \left\| \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \right\| .$$

Exercise 6.

Find the volume of the Tetrahedron with vertices $(1, 0, 1)$, $(1, 3, 2)$, $(4, 7, 1)$, and $(-1, 0, 0)$.



Note.

If four points are `co-planar`–meaning they all lie on the same plane–then they will form a tetrahedron with zero area.

Exercise 7.

Which of the following sets of points are co-planar? Use the volume formula from the last slide.

(a) $(-2, -2, 1)$, $(1, -3, -1)$, $(-1, -2, -3)$, $(-2, -3, 3)$

(b) $(-2, 3, 2)$, $(0, -2, 1)$, $(3, 3, -3)$, $(3, -2, -2)$

(c) $(-3, -1, 1)$, $(1, 3, 1)$, $(2, 2, 2)$, $(-1, -1, 2)$

Note.

Given three points in 3-d space, any other point (x, y, z) that is co-planar to these three will produce a tetrahedron of zero volume.

Three-Point Form of the Equation of a Plane

An equation for the line passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is given by

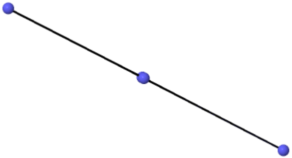
$$0 = \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}.$$

Exercise 8.

Use the determinant formula to find an equation for the plane passing through the points $(3, -3, 3)$, $(-1, 0, -2)$, and $(-2, -1, -2)$

Exercise 9.

The following three points lie on a single line in 3-d: $(-1, 0, -2)$, $(-2, -1, -2)$, $(-3, -2, -2)$. What happens when you try to find the plane that contains the three points?



A line in 3-d requires two linear equations: $a_1 x + b_1 y + c_1 z = d_1$ and $a_2 x + b_2 y + c_2 z = d_1$. Find an equation for the line containing these three points.