Math 324: Linear Algebra and Matrix Theory
Constructing Mathematical Proofs

Definition. A `mathematical proof` is a convincing argument that a mathematical statement is true.

- Must contain the correct amount of detail for the intended audience.

  In this class, the intended audience is a classmate who has attended every class and done all of the work but has not thought about this particular problem.

- Must have correct, consistent logical reasoning.

- Must be written using full sentences.

Definition. An `axiom` is a mathematical statement that is accepted without proof.

- For us, we will assume all of the algebraic properties of real numbers that we learned in high school algebra are true.

- For example, if a and b are real numbers then $a + b = b + a$.

Definition. A `definition` is an agreed-upon meaning of a given term.

- For example, the definition of a matrix being a rectangular grid of numbers with a given number of rows and columns.

Definition. A `theorem` is a mathematical statement that can be proven to be true.

- A common synonym for theorem is `proposition`.

- A `lemma` is the word used for a true mathematical statement that isn't entirely useful on its own but leads up to a theorem.

- A `corollary` is a true mathematical statement that follows almost immediately from a theorem.

Definition. A `conjecture` is a mathematical statement that is believed to be true but is yet unproven.

- Think of this as the mathematical version of the "scientific hypothesis".

Note. You are not expected to see a statement and know immediately how to prove it. This would make mathematics really boring. The beauty of proofs is the creativity you get to implement by exploring the known definitions and theorems.

- Scratch work and multiple drafts are essential! Soon we will learn $\LaTeX$, a mathematical typesetting software that will make the process of drafting and organizing much easier.

Example. Consider the following example on the proof of property 3 of Theorem 2.6. I am including my scratch work to emphasize the importance of that portion of the task.

Theorem 2.6-3. *If A is a matrix and c is a scalar, then $(cA)^T = cA^T$.*

Scratch. Assumptions

- $A = [a_{ij}]$ an $m \times n$ matrix
- c a scalar

Relevant definitions

- Transpose - Swap rows and columns
- Scalar multiple - multiply every entry by c.

$$A = [a_{ij}] \Rightarrow A^T = [a_{ji}]$$

$$(cA)^T = [ca_{ij}]^T = [ca_{ji}] \quad \text{vs.} \quad cA^T = c[a_{ji}] = [ca_{ji}]$$

Proof. Let A be an $m \times n$ matrix and c a scalar. Write $A = [a_{ij}]$ where a_{ij} is the entry of A in the i-th row and j-th column. Compute

$$\begin{aligned}
(cA)^T &= (c[a_{ij}])^T && \text{substitution;} \\
&= [ca_{ij}]^T && \text{definition of scalar multiplication;} \\
&= [ca_{ji}] && \text{definition of transpose;} \\
&= c[a_{ji}] && \text{definition of scalar multiplication;} \\
&= c[a_{ij}]^T && \text{definition of transpose;} \\
&= cA^T && \text{substitution.}
\end{aligned}$$

Therefore $(cA)^T = cA^T$, as desired. ∎

There are a few key components to this proof that I want to point out.

- The first two sentences define the variables that we use throughout.
- Every step in the string of equalities has an explanation.
- There is a concluding sentence that reminds the reader of what was being shown.

Exercise 1. Prove property 2 of Theorem 2.6: If A and B are matrices of the same size then $(A + B)^T = A^T + B^T$.

Exercise 2. (Challenge) Prove property 4 of Theorem 2.6: If A is and $m \times n$ matrix and B is an $n \times r$ matrix then $(AB)^T = B^T A^T$.
Your scratch work for this may include testing the 2×2 case:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$