Math 324: Linear Algebra

Section 4.5: Dimension

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Last Time.

– Basis

Today.

– Dimension

Theorem 4.11.

If a vector space V has a basis with n vectors then every basis for V has n vectors.

Exercise 1.

Fill details into this terse proof of Theorem 4.11.

Proof.

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for the vector space V. Let $T = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ be another basis for V.

(a) Since S is a basis and T is linearly independent, by Theorem 4.10 we must have $m \leq n$.

(b) Similarly $n \leq m$.

Therefore, $m = n$, as desired. $\square$

Exercise 2.

How many vectors must a basis for each of the following have?

(a) $\mathbb{R}^2$

(b) $\mathbb{R}^3$

(c) $\mathbb{R}^n$

(d) P_2

(e) P_n

(f) $M_{2,2}$

(g) $M_{m,n}$

Definition.

If the vector space V has a basis with n elements, then we say that the `dimension of V` is n, denoted $\dim(V) = n$.
The vector space consisting of only the zero vector is said to have dimension 0.

Exercise 3.

What is the dimension of each of the following vector spaces?

(a) $\mathbb{R}^2$

(b) $\mathbb{R}^3$

(c) $\mathbb{R}^n$

(d) P_2

(e) P_n

(f) $M_{2,2}$

(g) $M_{m,n}$

Example.

Consider the subspace $W = \{(a+b, a-b, 3*a) : a, b \in \mathbb{R}\}$ of $\mathbb{R}^3$. A basis for this space is $S = \{(1,1,3), (1,-1,0)\}$ because every vector in W can be written as a combination of these two

$$(a+b, a-b, 3a) = a(1,1,3) + b(1,-1,0),$$

and the set S is linearly independent. (How would you check independence?)
Thus W is 2-dimensional.

Exercise 4.

(a) Find a basis for the subspace $W = \{(t, 2t) : t \in \mathbb{R}\}$ of $\mathbb{R}^2$. What is $\dim(W)$?

(b) Find a basis for the subspace $W = \{(a, b, b, b) : a, b \in \mathbb{R}\}$ of $\mathbb{R}^4$. What is $\dim(W)$?

Brain Break.

If you have a tattoo, what does it mean to you? If you don't, what tattoo would you get?

This lion is on my foot. It is the mascot for St. Olaf College, where I got my undergraduate degree.

Theorem 4.12.

Let V be a vector space of dimension n.

1. If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a linearly independent set of n vectors in V, then S is a basis for V.
2. If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ has n vectors and spans V, then S is a basis for V.

Exercise 5.

Discuss this Theorem in the context of $V = \mathbb{R}^3$.
If $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set, why does S also span $\mathbb{R}^3$?
If $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $\mathbb{R}^3$, why is it also linearly independent?

Theorem: Pruning.

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a spanning set of a vector space V. Then there is a subset T of S that is a basis for V.

Exercise 6.

Consider the set $S = \{(1,2,4),(1,3,5),(2,5,9),(0,1,3),(0,2,6)\}$. Find a subset of S that is a basis for $\mathbb{R}^3$.