Math 324: Linear Algebra

Section 4.3: Subspaces of Vector Spaces

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Last Time.

- Proving Something is not a Vector Space
- Definition of Subspaces
- Test for Subspaces

Today.

- Subspaces of $\mathbb{R}^2$
- Subspaces of $\mathbb{R}^3$
- Intersections of Subspaces

Exercise 1.

Solve the homogeneous system of equations with coefficient matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

Is the set of solutions to this system a vector space with standard operations from $\mathbb{R}^3$?

Exercise 2.

Let $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Is the set of solutions to the system $A\vec{x} = \vec{b}$ a vector space with the standard operations in $\mathbb{R}^2$?

Exercise 3.

Let A be any $n \times n$ matrix. Complete the proof of the fact that the set of solutions to the homogeneous system $A\vec{x} = \vec{0}$ is a vector space by showing that it is a subspace of $\mathbb{R}^n$.

Proof.

Let A be a $n \times n$ matrix. Let $\vec{u}$ and $\vec{v}$ be two solutions to $A\vec{x} = \vec{0}$. This means that $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$. Let c be a scalar.

(a) Show that $\vec{0}$ is in the set.

(b) Show that $\vec{u} + \vec{v}$ is a solution to $A\vec{x} = \vec{0}$. (What is $A(\vec{u} + \vec{v})$?)

(c) Show that $c\vec{u}$ is a solution to $A\vec{x} = \vec{0}$. (What is $A(c\vec{u})$?)

$\square$

Exercise 4.

Show that the set of points on the line $y = \frac{1}{3}x$ is a subspace of $\mathbb{R}^2$.

Exercise 5.

Show that the set of points on the line $y = \frac{1}{3}x + 2$ is not a subspace of $\mathbb{R}^2$.

Subspaces of $\mathbb{R}^2$

The only possible subspaces of $\mathbb{R}^2$ are:

- $V = \{\vec{0}\}$
- The set of points on a line that passes through the origin
- $V = \mathbb{R}^2$

Subspaces of $\mathbb{R}^3$

The only possible subspaces of $\mathbb{R}^3$ are:

- $V = \{\vec{0}\}$
- The set of points on a line that passes through the origin
- The set of points on a plane that passes through the origin
- $V = \mathbb{R}^3$

Example.

The set of points on the plane $x + y + z = 0$ is a subspace of $\mathbb{R}^3$.

Example.

The set of points in the intersection of the planes $x + y + z = 0$ and $x + 2y + 3z = 0$ is a subspace of $\mathbb{R}^3$.

Brain Break.

What is the best piece of advice you've ever been given?

This less-than-wise owl missed the advice of avoiding the dog.

Exercise 6.

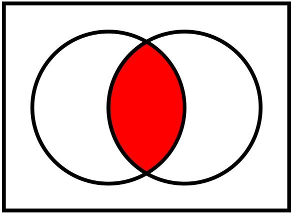
In the video for today we proved that the set
$W = \{(a+b, 3a, a-2b) : a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.
In fact, this subspace is the plane $-2x + y - z = 0$. How do we show this? Notice that every vector in W is on the plane because

$$-2x + y - z = -2(a+b) + (3a) - (a-2b) = 0.$$

Think about the other hand. What a and b could we choose with respect to a point (x, y, z) that satisfies $-2x + y - z = 0$? That is, solve for a and b given the equation
$(x, y, z) = (a+b, 3a, a-2b)$, assuming x, y and z are fixed.

Definition.

The `intersection` of two sets is the set consisting of everything that appears in both sets, the shaded region in the Venn diagram below. The intersection of V and W is denoted $V \cap W = \{\vec{x} \ : \ \vec{x} \in V \text{ and } \vec{x} \in W\}$.



Theorem 4.6.

If V and W are both subspaces of a vector space U, then the intersection of V and W is also a subspace of U.

Exercise 7.

Complete the proof of Theorem 4.6.

Proof.

Let V and W be subspaces of a vector space U. We want to show that $V \cap W$ is a subspace of U using the subspace test.

(a) Explain why $\vec{0} \in V \cap W$.

(b) Let $\vec{v}$ and $\vec{w}$ be in $V \cap W$ this means that $\vec{v}$ and $\vec{w}$ are in V and that $\vec{v}$ and $\vec{w}$ are in W. Explain why $\vec{v} + \vec{w}$ is in $V \cap W$.

(c) Let $\vec{v}$ be in $V \cap W$ and c be a scalar. Explain why $c\vec{v} \in V \cap W$ too.

□

Exercise 8.

(a) Prove that the set $V = \{(a, b, a + b) \; : \; a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

(b) Prove that the set $W = \{(a, b, 3a) \; : \; a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

(c) Find the intersection of these sets and verify that it is also a subspace of $\mathbb{R}^3$.