Math 324: Linear Algebra

Section 3.3: Properties of Determinants

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Last Updated: December 29, 2023

Last Time.

– Determinants of Triangular Matrices

– How Elementary Row Operations Affect Determinants

Today.

– Determinants of Products

Theorem 3.4.

If A is a square matrix and any one of the following conditions is true, then $\det(A) = 0$:

1. An entire row (or an entire column) consists entirely of zeros.
2. Two rows (or columns) are equal.
3. One row (or column) is a multiple of another row (or column).

Exercise 1.

Prove property 1 using an appropriate cofactor expansion. I'm not going to help you with this one, but I do want your group to write it down on the worksheet.

Question 2.

How could you use property 1 and elementary row (or column) operations to prove properties 2 and 3?

Question 3.

Think of the properties from Theorem 3.4 in terms of coefficient matrices and linear systems of equations. What would it mean for one row to be a multiple of another row?

Note.

The ability to use elementary row operations to compute a determinant significantly reduce the effort needed to do the computation.
We can even count out the number of steps one would need for a generic matrix in each type of computation:

	Cofactor Expansion Method		Row Reduction Method	
Order n	Additions	Multiplications	Additions	Multiplications
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

Question 4.

If these are the only options, what method do you think your calculator uses to compute determinants?

Exercise 5.

Compute the determinants of the following elementary matrices,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 6.

If E is an elementary matrix and B is any matrix, how does $|EB|$ compare to $|E|$ and $|B|$?
You may want to reference Theorem 3.3 from last time as well as the previous exercise.

Exercise 5.

Compute the determinants of the following elementary matrices,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 6.

If E is an elementary matrix and B is any matrix, how does $|EB|$ compare to $|E|$ and $|B|$?
You may want to reference Theorem 3.3 from last time as well as the previous exercise.

Lemma 3.5.

If E is an elementary matrix and B is a square matrix both of the same size, then $|EB| = |E||B|$.

Theorem 3.5.

If A and B are square matrices of order n, then $\det(AB) = \det(A)\det(B)$.

Exercise 7.

Use the following Sage code to produce two random 2×2 matrices: (`https://sagecell.sagemath.org/`)

```
A = matrix([[randint(1,5) for j in [1,2]] for i in [1,2]])
print(f"A=\n{A}")
B = matrix([[randint(1,5) for j in [1,2]] for i in [1,2]])
print(f"B=\n{B}")
```

Then use your matrices to verify the Theorem.

Brain Break.

Why did you choose to attend University of Wisconsin-Eau Claire?

Exercise 8.

Guided Proof of Theorem 3.5, use the steps below to write the proof on your worksheet.

(a) Let A and B be square matrices of order n.

(b) Consider two cases (i) A is invertible or (ii) A is singular.

(c) (i) Assume A is invertible and invoke Theorem 2.15 to write A as a product of elementary matrices. Then use Lemma 3.5 on slide **??** to prove that $|AB| = |A||B|$.
NOTE: In this case, you may assume for this proof that Lemma 3.5 says $|E_r \cdots E_2 E_1 B| = |E_r| \cdots |E_2||E_1||B|$. Though in general, we would need to prove this by induction.

Exercise 8 (Cont.).

(d) (ii) Assume A is singular. Use the inverse of Theorem 2.9 to note that AB is also singular. Use Theorem 3.4 and Lemma 3.5 to explain why the determinant of a singular matrix is 0. Conclude that $|AB| = |A||B| = 0$.
NOTE: If you did not watch the video for today, do that. I proved that the converse of Theorem 2.9 is true. And thus the inverse is also true. There is also an important corollary that will undoubtedly come in handy.

(e) Conclude that no matter what A is, $|AB| = |A||B|$.

Exercise 9.

In light of Theorem 3.5, how does $\det(A^{-1})$ compare to $\det(A)$? Can you prove it?

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In light of Theorem 3.5, how does $\det(A^{-1})$ compare to $\det(A)$? Can you prove it?

Theorem 3.8.

If A is an $n \times n$ invertible matrix, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Equivalent Conditions for a Nonsingular Matrix.

If A is an $n \times n$ matrix, then the following are equivalent:

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.
3. $A\mathbf{x} = O$ has only the trivial solution.
4. A is row equivalent to I_n.
5. A can be written as the product of elementary matrices.
6. $\det(A) \neq 0$. (**Theorem 3.7**)

Exercise 10.

Which of the following are invertible?

(a) $\begin{bmatrix} 4 & 5 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 0 \\ 4 & 3 & 0 \\ 5 & 4 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 2 & 1 & -1 & 0 \end{bmatrix}$