Math 324: Linear Algebra

2.3: The Inverse of a Matrix

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Last Time.

- Properties of Matrix Addition
- Additive and Multiplicative Identities
- Properties of Matrix Multiplication
- Transposes and Properties of

Today.

- Inverses of Matrices
- Gauss-Jordan Elimination
- Inverses of 2×2 matrices

Exercise 1 (Warm-up).

Compute the product of the matrices:

(a) $\begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Note.

We are NEVER going to be able to divide by matrices.
But maybe we can multiply to get the identity matrix (also known as 1 in matrix land).

Definition.

An $n \times n$ matrix A is `invertible` (or `nonsingular`) when there is an $n \times n$ matrix B satisfying

$$AB = BA = I_n.$$

The matrix B is called an `inverse` of A.
If A does not have an inverse, we call it `noninvertible` (or `singular`).

Example.

The matrix $A = \begin{bmatrix} 13 & 1 \\ -1 & 0 \end{bmatrix}$ is invertible because

$$A \begin{bmatrix} 0 & -1 \\ 1 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 \\ 1 & 13 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 2.

The following matrices are singular. Is there a pattern?

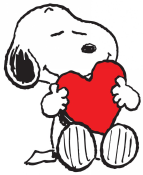
$$\begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ \frac{1}{2} & -\frac{5}{6} \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 17 & 0 \end{bmatrix} \begin{bmatrix} 5 & -5 \\ -1 & 1 \end{bmatrix}$$

Note.

An invertible matrix only has one inverse.

Theorem 2.7.

If A is an invertible matrix, then its inverse is unique and can be denoted by A^{-1}.



The Gauss-Jordan Method for Inverses.

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $\begin{bmatrix} A & \vdots & I \end{bmatrix}$. This process is called `adjoining` I to A.
2. If possible, row reduce A to I using elementary row operations on the entire matrix $\begin{bmatrix} A & \vdots & I \end{bmatrix}$. The result will be the matrix $\begin{bmatrix} I & \vdots & A^{-1} \end{bmatrix}$. If this is not possible, then A is noninvertible (or singular).
3. Check your work by multiplying $AA^{-1} = I = A^{-1}A$.

Exercise 3.

By hand, without a calculator, determine whether each of the following are invertible. If so, what is the inverse?

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Exercise 4.

Use the Gauss-Jordan method to find the inverse of the matrix if it exists. You should use a calculator or Sage to row reduce.

(a) $A = \begin{bmatrix} 2 & -4 & 8 \\ -7 & -3 & 10 \\ 10 & -3 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} -3 & 5 & -2 \\ 3 & -5 & 1 \\ -4 & 7 & -6 \end{bmatrix}$

(c) $C = \begin{bmatrix} 5 & 8 & 5 & 6 \\ 1 & -10 & 9 & -6 \\ -3 & -2 & 1 & -5 \\ 6 & -4 & -7 & -2 \end{bmatrix}$

Brain Break.

Always start these by reminding your group members of your name. Names are hard.

What acquired skill have you always wanted to learn?

Exercise 5.

Let's discover the nice formula for the inverse of a 2×2 matrix.

(a) Take a generic 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and adjoin it by I_2.

(b) Do the following elementary row operations:

   i. $aR_2 \to R_2$

   ii. $R_2 - cR_1 \to R_2$

(c) At this stage, what must be true in order for A to have an inverse?

Exercise 4. (Continued)

(d) Do the following elementary row operations:

iii. $\frac{1}{ad-bc}R_2 \to R_2$

iv. $R_1 - bR_2 \to R_1$

v. $\frac{1}{a}R_1 \to R_1$

(e) Simplify the (1,1)-entry of the inverse. Did you get:

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}?$$

Warning/Challenge.

We cheated on step i of our elementary row operations by scaling by a — a could have been 0.

Find the inverse of the matrix $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$.

Hint: The same formula works. Why?

Proposition.

If A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover if $ad - bc \neq 0$, then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Exercise 6.

Which is invertible? Find the inverse of those that are:

(a) $\begin{bmatrix} 8 & -2 \\ -1 & -9 \end{bmatrix}$

(b) $\begin{bmatrix} 8 & -2 \\ 4 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 7 \\ 0 & 7 \end{bmatrix}$

(d) $\begin{bmatrix} -9 & -5 \\ 8 & -10 \end{bmatrix}$

Exercise 7.

Find all x and y that make the matrix singular $\begin{bmatrix} 2 & x \\ -3 & y \end{bmatrix}$

Exercise 8.

Find x so that $A = A^{-1}$ if $A = \begin{bmatrix} -2 & x \\ 3 & 2 \end{bmatrix}$

Exercise 9.

Under what conditions will the diagonal matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

be invertible?

In the case that A is invertible, what is its inverse?

Exercise 10.

Let $A = \begin{bmatrix} -3 & -2 \\ 3 & -1 \end{bmatrix}$.

(a) Verify that $A^2 + 4A + 9I = O$.

(b) Verify that $A^{-1} = \frac{-1}{9}(A + 4I)$.

(c) Show that for any square matrix satisfying $A^2 + 4A + 9I = O$, the inverse of A is given by $A^{-1} = \frac{-1}{9}(A + 4I)$.