# Math 324: Linear Algebra

## 1.1: Introduction to Systems of Linear Equations
## 1.2: Gaussian Elimination and Gauss-Jordan Elimination

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## Last Time.

- Linear Equations
- Solving Linear Equations and Parametrization
- Linear Systems of Equations
- Solving Linear Systems of Equations
- Consistent vs Inconsistent Systems

## Today.

- $m \times n$ matrix
- coefficient and augmented matrix
- elementary row operations
- row-echelon form and leading 1's
- reduced row-echelon form

Today we will use matrices and elementary row operations to simplify the process of solving a system of linear equations.

## Question 1.

Last time you found solutions to the system of equations:

$$\begin{array}{rcrcrcl} 3x & - & 2y & + & 3z & = & 4 \\ x & & & + & z & = & 5 \end{array}$$

what were some steps you took to do this?

**Example.**

I don't know about you, but I think that writing variables is a waste of time, when all we're looking at is canceling coefficients. Let's save ourselves some time by turning everything into a nicely arranged table:

$$\begin{array}{rcrcrcr} 3x & - & 2y & + & 3z & = & 4 \\ x & & & + & z & = & 5 \end{array} \dashrightarrow \begin{bmatrix} 3 & -2 & 3 & 4 \\ 1 & 0 & 1 & 5 \end{bmatrix}$$

## Definition.

The tables from the previous page are called `matrices`.
If $m$ and $n$ are positive integers, an $m \times n$ `matrix` is a rectangular array with $m$ rows and $n$ columns:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

where the entries $a_{ij}$ are real numbers.
We say the `size` of the matrix is $m \times n$.
A `square` matrix is one where $m = n$, in this case we call the entries $a_{11}, a_{22}, \dots, a_{nn}$ the `main diagonal`.

## Exercise 2.

What is the size of each of the following matrices:

(a) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 0 \\ -54 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$

**Definition.**

There are two basic matrices that can be constructed using a linear system of equations:

the `coefficient matrix` which contains only the coefficients of the linear system, not the constant terms;

the `augmented matrix` which is the matrix that contains both the coefficients and the constant terms.

**Question 3.**

If we're solving a system of linear equations which of these two matrices are we going to use?

## Exercise 4.

Write down the coefficient matrix and augmented matrix for the following system of linear equations:

$$\begin{array}{rcrcrcr} x & & & - & 2z & = & 5 \\ 2x & + & 3y & & & = & -3 \\ -2x & + & y & - & 2z & = & 3 \end{array}$$

### Definition (Elementary Row Operations).

There are three `elementary row operations` that we use on a matrix:

1. Interchange two rows.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of on equation to another equation.

### Note.

Each of these operations corresponds to an equivalent action on the linear equations.

**Exercise 5.**

Use elementary row operations on the augmented matrix and back substitution to solve the system of linear equations.

$$\begin{array}{rcrcrcl} x & + & 2y & - & 4z & = & -5 \\ & & y & + & z & = & 1 \\ -2x & - & y & - & z & = & 5 \end{array}$$

### Definition.

A matrix is in `row-echelon form` if it satisfies the following properties:

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a `leading 1`).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 of the lower row.

### Example.

The following matrix is in row echelon form, verify this

$$\begin{bmatrix} 1 & 4 & 0 & 11 \\ 0 & 1 & 14 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Exercise 6.**

Which of the following matrices is in row-echelon form?

(a) $\begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 0 & -6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & -1 & 5 & -4 & -2 & -4 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

## Question 7.

Can we use elementary row operations to reduce a matrix to row echelon form?

## Question 8.

How will row-echelon form help us solve linear systems of equations?

### Definition.

A matrix is in `reduced row-echelon form` if it is in row-echelon form and in every column with a leading 1 all other entries are zero.

### Exercise 9.

Which of the following matrices are in reduced row-echelon form?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 13 & 0 \\ 0 & 30 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

## Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

**Example.**

In a chemical reaction, atoms reorganize in one or more substances. For instance, when methane gas ($CH_4$) combines with oxygen ($O_2$) and burns, carbon dioxide ($CO_2$) and water ($H_2O$) form. Chemists represent this process by a chemical equation of the form

$$(x_1)CH_4 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O.$$

With hopes of solving, we can create a system of linear equations, one for each element:

$$\begin{array}{rrcl} C: & 1x_1 + 0x_2 & = & 1x_3 + 0x_4 \\ H: & 4x_1 + 0x_2 & = & 0x_3 + 2x_4 \\ O: & 0x_1 + 2x_2 & = & 2x_3 + 1x_4 \end{array}$$

We then rearrange this system into the form we have been working with:

$$\begin{array}{rrcl} C: & 1x_1 + 0x_2 & = & 1x_3 + 0x_4, \\ H: & 4x_1 + 0x_2 & = & 0x_3 + 2x_4, \\ O: & 0x_1 + 2x_2 & = & 2x_3 + 1x_4, \end{array}$$

becomes

$$\begin{array}{rcrcrcrcl} x_1 & & & - & x_3 & & & = & 0, \\ 4x_1 & & & & & - & 2x_4 & = & 0, \\ & & 2x_2 & - & 2x_3 & - & x_4 & = & 0. \end{array}$$

As an augmented matrix, we get the following matrix and reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

## Exercise 10.

Write the solutions to the system in parametric form.

## Exercise 11.

Balance the chemical equation:

$$(x_1)Cu + (x_2)HNO_3 \rightarrow (x_3)Cu(NO_3)_2 + (x_4)NO + (x_5)H_2O$$

That is, find a parametric representation of the solutions to the linear system as in the previous example.
If you want to end up with 7,200 units of water, $H_2O$, how much of the initial compounds must you start with?

### Exercise 12.

Discuss a general method you would use to reduce a matrix to row echelon form.

### Exercise 13.

Open `http://sagecell.sagemath.org` and use the following code to generate a random matrix:

```
matrix([[randint(-2,2) for i in [1..5]] for j in [1..4]])
```

Do this a couple times until you have one with 3-6 zeros.
Use elementary row operations to reduce this matrix to (reduced) row-echelon form.