Math 324: Linear Algebra

Section 6.1: Linear Transformations
Section 6.2: The Kernel and Range of a Linear Transformation

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Last Updated: December 29, 2023

Last Time.

- Linear Transformations
- Properties of Linear Transformations

Today.

- Matrix Transformations
- The Kernel of a Transformation
- The Range of a Transformation
- Rank and Nullity

Recall.

If V and W are vector spaces, a `linear transformation` from V to W is a mapping $T\colon V \to W$ that satisfies

(a) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in V$,

(b) and $T(c\vec{v}) = cT(\vec{v})$ for all scalars c and $\vec{v} \in V$.

Example.

A very important linear transformation is a `matrix transformation`, $T\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\vec{x}) = A\vec{x}$ where A is some fixed $m \times n$ matrix.

Definition.

The `kernel` of the linear transformation $T\colon V \to W$ is the collection of all vectors $\vec{v}$ in V such that $T(\vec{v}) = \vec{0}$. The notation for this is,

$$\ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$

Note.

The kernel of a linear transformation is the preimage of $\vec{0}$.

Exercise 1.

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (0, y, z)$. What is $\ker(T)$?

Definition.

The `identity transformation` $T\colon V \to V$ is the map $T(\vec{v}) = \vec{v}$. Sometimes we denote this map by Id_V.
The `zero transformation` $T\colon V \to W$ is the map $T(\vec{v}) = \vec{0}$.

Exercise 2.

What is $\ker(Id_V)$?
What is the kernel of the zero transformation $T\colon V \to W$?

Exercise 3.

Consider the transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by the matrix

$$A = \begin{bmatrix} -1 & -2 & 4 \\ 2 & 3 & 4 \end{bmatrix}$$

What is $\ker(T)$? Carefully write out what it means for $T(\vec{v}) = \vec{0}$. Answer this question by finding a basis for the kernel.

Exercise 3.

Consider the transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by the matrix

$$A = \begin{bmatrix} -1 & -2 & 4 \\ 2 & 3 & 4 \end{bmatrix}$$

What is $\ker(T)$? Carefully write out what it means for $T(\vec{v}) = \vec{0}$. Answer this question by finding a basis for the kernel.

The Kernel of a Matrix Transformation.

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation defined by the $m \times n$ matrix A, then $\ker(T) = \text{null}(A)$.

Theorem 6.3.

The kernel of a linear transformation $T\colon V \to W$ is a subspace of the domain, V.

Exercise 4.

Prove Theorem 6.3 via the subspace test.

(a) State why $\ker(T)$ contains $\vec{0}$.

(b) Show that if $\vec{u}$ and $\vec{v}$ are in $\ker(T)$—meaning $T(\vec{u}) = \vec{0}$ and $T(\vec{v}) = \vec{0}$—then $\vec{u} + \vec{v} \in \ker(T)$ too.

(c) Show that if c is a scalar, then $c\vec{u}$ is also in $\ker(T)$.

Brain Break.

When you were a kid, what did you want to be when you grew up?

Image source https://www.flexjobs.com/blog/post/childhood-dream-job-pay-well/

Recall.

The range of a transformation $T\colon V \to W$ is the collection of all possible images under the transformation. We write

$$\operatorname{range}(T) = \{T(\vec{v}) : \vec{v} \in V\}.$$

Theorem 6.4.

The range of a linear transformation $T\colon V \to W$ is a subspace of the codomain W.

Proof Idea.

Take two generic elements of the range of T, $T(\vec{u})$ and $T(\vec{v})$. Then by the definition of linear transformation,

$$T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v}).$$

Therefore the sum of the two generic elements is also in the form of range(T). Similarly, this can be one with scalar multiples.

The Range of a Matrix Transformation.

If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation defined by the $m \times n$ matrix A, then $\operatorname{range}(T) = \operatorname{col}(A)$.

Exercise 5.

Consider the transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by the matrix

$$A = \begin{bmatrix} -1 & -2 & 4 \\ 2 & 3 & 4 \end{bmatrix}$$

What is $\operatorname{range}(T)$? Answer this question by finding a basis for the range.

Definition.

Let $T\colon V \to W$ be a linear transformation. The `nullity` of T, denoted nullity(T), is the dimension of ker(T).
The `rank` of T, denoted rank(T), is the dimension of range(T).

Exercise 6.

Consider the derivative transformation, $D\colon P_2 \to P_1$. What is the nullity and rank of D?

Theorem 6.5.

Let $T\colon V \to W$ be a linear transformation from an n-dimensional vector space V into a vector space W. Then

$$\dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T).$$

Exercise 7.

Does this match up with the fact that if A is an $m \times n$ matrix then

$$n = \operatorname{rank}(A) + \operatorname{nullity}(A)?$$

Exercise 8.

Find the rank and nullity of the matrix transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$A = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$