Math 324: Linear Algebra

Section 4.1: Vectors in $\mathbb{R}^n$

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Today.

- Vectors in 2-d
- Vectors in n-dimensional space

Exercise 1.

Geometrically (tail-to-tip) and algebraically compute each of the following for $\vec{u} = (-2, 1.5)$ and $\vec{v} = (1, 6)$.

(a) $\vec{u} - \vec{v}$

(b) $3\vec{v}$

(c) $\frac{1}{2}\vec{u} + \vec{v}$

Theorem 4.1.

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in the plane and let c and d be scalars. Then the following are true

1. $\vec{u} + \vec{v}$ is a vector in the the plane (additive closure)
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutativity of addition)
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associativity of addition)
4. $\vec{u} + (0, 0) = \vec{u}$ (additive identity)
5. $\vec{u} + (-\vec{u}) = (0, 0)$ (additive inverse)
6. $c\vec{u}$ is a vector in the plane (scalar closure)
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (distributivity)
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$ (distributivity)
9. $c(d\vec{u}) = (cd)\vec{u}$ (associativity of scalars)
10. $1(\vec{u}) = \vec{u}$ (multiplicative identity)

Exercise 2.

Use the algebraic definitions of vector addition and scalar multiplication to prove properties **2.** and **9.** of Theorem 4.1:

2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

9. $c(d\vec{u}) = (cd)\vec{u}$

Definition.

An n-dimensional vector is an ordered list of n numbers–an n-tuple–$\vec{x} = (x_1, x_2, \ldots, x_n)$, that represents a terminal point of an arrow from the origin in n-dimensional space. Denote n-dimensional space by $\mathbb{R}^n$ (the book uses R^n).

Note.

It sure is hard to imagine what 4-dimensional vectors look like but fortunately we have a very concrete algebraic representation.

Definition.

Two vectors in $\mathbb{R}^n$ are equal if and only if their corresponding components are equal.

Definition.

The standard operations in $\mathbb{R}^n$ are the following definitions of vector addition and scalar multiplication:
Let $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbb{R}^n$ and let c be a scalar. We define the sum of $\vec{u}$ and $\vec{v}$ as:

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

Define the scalar multiple of $\vec{u}$ by c as:

$$c\vec{u} = (cu_1, cu_2, \ldots, cu_n).$$

Definition.

The `negative` of $\vec{u}$ is defined as:

$$-\vec{u} = -1(\vec{u}) = (-u_1, -u_2, \ldots, -u_n),$$

and the `difference` of $\mathbf{u}$ and $\mathbf{v}$ is:

$$\mathbf{u} - \mathbf{v} = \vec{u} + (-\mathbf{v}) = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n).$$

Lastly, the `zero vector` in $\mathbb{R}^n$ is the point at the origin,

$$\vec{0} = (0, 0, \ldots, 0).$$

Exercise 3.

Let $\vec{u} = (-2, 4, 0, -4)$, $\vec{v} = (0, 5, 5, 4)$, $\vec{w} = (1, -4, 3, 3)$ and $c = -2$, compute:

(a) $\vec{u} + \vec{v}$

(b) $c\vec{w}$

(c) $2\vec{w} - \vec{v}$

Theorem 4.2.

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then the following are true

1. $\vec{u}+\vec{v}$ is a vector in the the plane (additive closure)
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ (commutativity of addition)
3. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ (associativity of addition)
4. $\vec{u}+\vec{0}=\vec{u}$ (additive identity)
5. $\vec{u}+(-\vec{u})=\vec{0}$ (additive inverse)
6. $c\vec{u}$ is a vector in the plane (scalar closure)
7. $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ (distributivity)
8. $(c+d)\vec{u}=c\vec{u}+d\vec{u}$ (distributivity)
9. $c(d\vec{u})=(cd)\vec{u}$ (associativity of scalars)
10. $1(\vec{u})=\vec{u}$ (multiplicative identity)

Exercise 4.

Let $\vec{u} = (-1, 3, 1)$, $\vec{v} = (5, -5, 6)$, and $\vec{w} = (3, 7, 4)$. Find $\vec{x}$ given the equation:

(a) $\vec{x} = 3\vec{u} + 4\vec{v} - 7\vec{w}$

(b) $\frac{1}{2}(\vec{w} + \vec{x}) = 2\vec{u} - \vec{v}$

Definition.

We call the zero vector, $\vec{0}$, in $\mathbb{R}^n$ the `additive identity` and the negative, $-\vec{u}$, the `additive inverse` of $\vec{u}$.
These vectors have very special properties as outlined in the following Theorem.

Theorem 4.3.

Let $\vec{u} \in \mathbb{R}^n$ and c a scalar. Then the following are true:

1. The additive identity is unique. (If $\vec{u} + \vec{v} = \vec{u}$, then $\vec{v} = \vec{0}$.)
2. The additive inverse is unique. (If $\vec{u} + \vec{v} = \vec{0}$, then $\vec{v} = -\vec{u}$.)
3. $0\vec{u} = \vec{0}$
4. $c\vec{0} = \vec{0}$
5. If $c\vec{u} = \vec{0}$ then $c = 0$ or $\vec{v} = \vec{0}$.
6. $-(-\mathbf{u}) = \vec{u}$

Exercise 5.

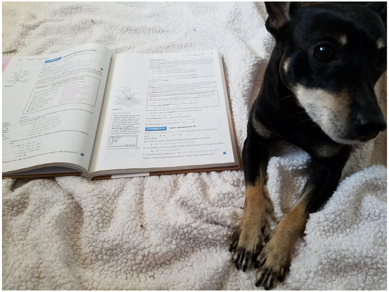
Prove property **1.** of Theorem 4.3:

1. If $\vec{u} + \vec{v} = \vec{u}$, then $\vec{v} = \vec{0}$.

Challenge: Prove this without using the fact that $\vec{u}, \vec{v} \in \mathbb{R}^n$. Instead just use properties from Theorem 4.2.

Brain Break.

Do you have any pets? If so what kind, how many, what are their names? If not, do you want any?



This is my dog, Pepper, studying up on Section 4.1.

Exercise 6.

Complete the proof of property **3.** of Theorem 4.3 by justifying each step using Theorem 4.2 or properties of real numbers.

Proof.

Let $\vec{u} \in \mathbb{R}^n$. Notice that,

$$\begin{aligned} 0\vec{u} &= (0+0)\vec{u} \qquad (1) \\ 0\vec{u} &= 0\vec{u} + 0\vec{u}. \qquad (2) \end{aligned}$$

Then we get, $0\vec{u} + (-0\vec{u}) = (0\vec{u} + 0\vec{u}) + (-0\vec{u})$. (3)
Therefore,

$$\begin{aligned} \vec{0} &= 0\vec{u} + (0\vec{u} + (-0\vec{u})) \qquad (4) \\ \vec{0} &= 0\vec{u} + \vec{0} \qquad (5) \\ \vec{0} &= 0\vec{u}. \qquad (6) \end{aligned}$$

□

Definition.

A `linear combination` of the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in \mathbb{R}^n$ is a vector of the form

$$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n,$$

where $c_1, c_2, \ldots, c_n$ are scalars.

Example.

The vector $\vec{x} = (-13, 23, -3)$ is a linear combination of $\vec{u}_1 = (-3, 1, 1)$, $\vec{u}_2 = (0, -4, -2)$ and $\vec{u}_3 = (-3, 12, -6)$ because

$$\vec{x} = 3\vec{u}_1 - \vec{u}_2 + \frac{4}{3}\vec{u}_3.$$

Exercise 7.

Write $\vec{x} = (3, 2)$ as a linear combination of $\vec{u} = (-1, 1)$ and $\vec{v} = (3, 4)$, if possible.

Exercise 8.

Prove that the zero vector, $\vec{0}$, can be written as a linear combinination of the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m \in \mathbb{R}^n$.
Hint: Don't over-think this one, but do make sure to reference Theorems 4.2 and 4.3 as needed.

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Hint: Don't over-think this one, but do make sure to reference Theorems 4.2 and 4.3 as needed.

Definition.

We call $\vec{0} = 0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_m$ the `trivial solution`. Any other solution is called a `nontrivial solution`.

Exercise 9.

Is there a nontrivial way of writing $\vec{0}$ as a linear combination of:

(a) $\vec{u} = (-1, 7, 0)$, $\vec{v} = (-1, 5, 0)$, and $\vec{w} = (-4, 6, 0)$?

(b) $\vec{u}_1 = (7, 6, 2)$, $\vec{u}_2 = (2, 2, 0)$, and $\vec{u}_3 = (0, 7, -2)$?

(c) $\vec{u}_1 = (7, 6, 2)$, $\vec{u}_2 = (2, 2, 0)$, $\vec{u}_3 = (0, 7, -2)$, and $\vec{u}_4 = (3, 2, 1)$?

Exercise 10.

Prove that every vector $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ can be written as a linear combination of $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ as in Exercise **??** part (b). Is the same true for the vectors in part (a)? Why or why not?