Previously.

- Dihedral Groups
- Groups of order 2p

This Section.

- Normal Subgroups
- Products of Groups
- Simple Groups

Recall from Section 2.6:

Definition. If $H \leq G$ and $a \in G$, then we call $Ha = \{ha : h \in H\}$ a `right coset` of H. Similarly we call $aH = \{ah : h \in H\}$ a `left coset` of H.

Recall. Let $G = S_3$ and $H = \langle(1\ 2)\rangle = \{\varepsilon, (1\ 2)\}$.

The right cosets of H

- $H\varepsilon = \{\varepsilon, (1\ 2)\}$
- $H(1\ 2\ 3) = \{(1\ 2\ 3), (2\ 3)\}$
- $H(1\ 3\ 2) = \{(1\ 3\ 2), (1\ 3)\}$

The left cosets of H are:

- $\varepsilon H = \{\varepsilon, (1\ 2)\}$
- $(1\ 2\ 3)H = \{(1\ 2\ 3), (1\ 3)\}$
- $(1\ 3\ 2)H = \{(1\ 3\ 2), (2\ 3)\}$

Exercise 1.
What are the left and right cosets of $K = \{\varepsilon, (1\ 2\ 3), (1\ 3\ 2)\}$?

Definition. A subgroup H of G is called a `normal subgroup` of G if $gH = Hg$ for all $g \in G$. If H is a normal subgroup of G, we might say `H is normal in G` and write $H \trianglelefteq G$.

Theorem 2.8.2. If G is abelian and $H \leq G$, then $H \trianglelefteq G$.

Theorem 2.8.1. If G is a group, every subgroup of the center, $Z(G)$, is normal in G. In particular, $Z(G) \trianglelefteq G$.

Normality Test (Theorem 2.8.3). The following conditions are equivalent for a subgroup H of a group G.

1. $H \trianglelefteq G$.

2. $gHg^{-1} \subseteq H$ for all $g \in G$.

3. $gHg^{-1} = H$ for all $g \in G$.

Exercise 2. Let $G = GL_2(\mathbb{R})$ and $H = SL_2(\mathbb{R})$. Show $H \trianglelefteq G$.

Corollary 1. If $G = \langle g_1, g_2, \ldots, g_n \rangle$, then a subgroup H of G is normal if and only if $g_i H g_i^{-1} \subseteq H$ for all $1 \leq i \leq n$.

Exercise 3. Let $G = D_{12} = \langle r, f \rangle$ and $H = \{e, r^4, r^8\}$. Show that $H \trianglelefteq G$.

Theorem. If $a \in G$, then $\langle a \rangle \trianglelefteq G$ if and only if $gag^{-1} \in \langle a \rangle$ for all $g \in G$.

Theorem. If $a_1, a_2, \ldots, a_n \in G$ and $H = \langle a_1, a_2, \ldots, a_n \rangle$, then $H \trianglelefteq G$ if and only if $ga_ig^{-1} \in H$ for all $g \in G$ and all $1 \leq i \leq n$.

Exercise 4. Let $G = D_{15} = \langle r, f$ and $H = \langle r^3 \rangle = \{1, r^3, r^6, r^9, r^{12}\}$. Show that $H \trianglelefteq G$.

Theorem 2.8.4. If $H \leq G$ with $|G : H| = 2$, then $H \trianglelefteq G$.

Exercise 5. Show $A_n \trianglelefteq S_n$. Using Theorem 2.8.4.

Definition. If H is a subgroup of G and $g \in G$, we call gHg^{-1} a `conjugate` of H in G.

Corollary 2. If H is a subgroup of G, then gHg^{-1} is also a subgroup of G, isomorphic to H, for all $g \in G$. Moreover if G has no other subgroups isomorphic to H then $H \trianglelefteq G$.

Exercise 6. What are the conjugate subgroups of $H = \{\varepsilon, (1\ 2)\}$ in S_3?

Definition. The `product` of the subgroups $H, K \leq G$ is the set

$$HK = \{hk \ :\ h \in H,\ k \in K\}.$$

Exercise 7. $S_3 = \{\varepsilon, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$, $H = \{\varepsilon, \tau\}$, $K = \{\varepsilon, \tau\sigma\}$
Compute HK.

Theorem 2.8.5. Let H and K be subgroups of a group G.

1. If H or K is normal in G, then $HK = KH$ is a subgroup of G.

2. If both H and K are normal in G, then $HK \trianglelefteq G$ too.

Example. $G = D_6$, $H = \{e, r^2, r^4\}$, $K = \{e, r^3\}$

Theorem 2.8.6. If $H \trianglelefteq G$ and $K \trianglelefteq G$ satisfy $H \cap K = \{e_G\}$, then $HK \cong H \times K$.

Proof Idea.. Define $\phi : H \times K \to HK$ by $\phi(h, k) = hk$.

- (Onto) True no matter H and K.
- (One-to-One) True because $H \cap K = \{e_G\}$
- (Homomorphism) True because H and K are normal and $H \cap K = \{e_G\}$.

Exercise 8. $G = \mathbb{Z}_6$, $H = \{\overline{0}, \overline{2}, \overline{4}\}$, $K = \{\overline{0}, \overline{3}\}$.
Compute $H + K$ and make a correspondence with the elements of HK.

Exercise 9. $G = D_6$, $H = \{e, r^2, r^4\}$, $K = \{e, r^3\}$
Verify $H \times K \cong HK$.

Exercise 10. We saw $SL_2(\mathbb{R})$, the set of 2×2 matrices with determinant 1, is a normal subgroup of $GL_2(\mathbb{R})$, the multiplicative group of 2×2 invertible matrices.

(a) Verify that $D_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in \mathbb{R}^\times \right\}$ is also an normal subgroup of $GL_2(\mathbb{R})$.

(b) Compute $D_2(\mathbb{R}) \cap SL_2(\mathbb{R})$.

(c) Make conclusions about $D_2(\mathbb{R}) \cdot SL_2(\mathbb{R})$ and $D_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

(d) Show that $D_2(\mathbb{R})$ is the center of $GL_2(\mathbb{R})$.

Corollary 1. If G is a finite group and $H, K \leq G$ with $H \cap K = \{e_G\}$, then $|HK| = |H||K|$.

Corollary 2. If G is a finite group and $H, K \trianglelefteq G$ with $H \cap K = \{e_G\}$ and $|HK| = |G|$, then $G \cong H \times K$.

Theorem. If m and n are relatively prime integers and G is a cyclic group of order mn, then $G \cong C_m \times C_n$.

Theorem. Let G be an abelian group of order p^2 for some prime p. Then either $G \cong C_{p^2}$ or $G \cong C_p \times C_p$.

Example. A non-abelian group where every subgroup is normal.
$Q = \{\pm 1, \pm i, \pm j \pm k\}$ with relations

$$i^2 = j^2 = k^2 = -1 = ijk$$

$$ij = k = -ji$$

$$jk = i = -kj$$

$$ki = j = -ik$$

Definition. A group G is `simple` if its only normal subgroups are $\{e_G\}$ and G.

Example. $\mathbb{Z}_p$ is simple for all primes p

Theorem 2.8.7. An abelian group $G \neq \{e_G\}$ is simple if and only if $|G|$ is prime.

Theorem 2.8.8. If $n \geq 5$, then A_n is simple.

Proof Idea.. To summarize the argument:

- Every non-identity element of A_n can be written as a product of 3-cycles.

$$(ij)(ij) = \varepsilon \quad (i\ j)(i\ k) = (i\ k\ j) \quad (i\ j)(k\ l) = (i\ l\ k)(i\ j\ k).$$

- If $H \lhd A_n$ and H contains a 3-cycle, then H contains all 3 cycles. So by the previous note, $H = A_n$.

- Use a contradiction to the minimality of the number of elements of $\{1, 2, \ldots, n\}$ that are changed by $\tau \in H$.