Previously.

- Mappings
- Codomain vs Range
- Image and Inverse Image
- One-to-One, Onto, Bijection
- Identity Map
- Inverse Mappings
- Invertibility Theorem

This Section.

- Relations
- Equivalence Relations
- Equivalence Classes

Definition. If A is a set, any subset of $A \times A$ is called a `relation` on A.

Note. We often call this set R and we denote the elements as $a \equiv b$ or $a \sim b$.

Exercise 1. Describe the relation. That is, determine the subset R of $A \times A$ that is being described, and write it in set notation.

(a) $A = \mathbb{Z}$, every integer is related to every other integer

(b) $A = \mathbb{R}$, $a \sim b$ exactly when $a > b$

(c) $A = \{0, 1, 2\}$, $a \equiv b$ exactly when $a = b$

Definition. A relation $\equiv$ on a set A is called an `equivalence relation` if it satisfies all of the following conditions for all $a, b, c \in A$,

1. $a \equiv a$ (`reflexivity`),

2. If $a \equiv b$ then $b \equiv a$ (`symmetric`),

3. If $a \equiv b$ and $b \equiv c$, then $a \equiv c$ (`transitive`).

Exercise 2. For integers m and n, say $m \equiv n$ if and only if $m - n$ is even. Show that $\equiv$ defines an equivalence relation.

Definition. Given an equivalence relation $\equiv$ on a set A, we define the `equivalence class` of a to be the set

$$[a] = \{x \in A \mid x \equiv a\}.$$

Exercise 3. What are the equivalence classes from the previous exercise? ($m \equiv n$ if and only if $m - n \in 2\mathbb{Z}$)

Exercise 4. What are the equivalence classes for each of the following relations?

(a) Equivalence given by equality.

(b) $A = \mathbb{R}$, $a \equiv b$ if and only if $a^2 = b^2$.

Notation. The book uses $A_{\equiv}$ to be the set of unique equivalence classes of A under the equivalence relation $\equiv$. We may also call this set the `quotient set` of A by $\equiv$.

Exercise 5. Let $U = \{1, 2, 3\}$ and $A = U \times U$. Define the relation $\equiv$ on A by $(a, b) \equiv (c, d)$ if $b = d$.

(a) First, let's write down all of A. You should find that $|A| = 9$.

(b) Assuming that $\equiv$ is an equivalence relation on A, determine

(i) $[(1, 1)]$

(ii) $[(1, 2)]$

(iii) $[(1,3)]$

(c) What do you notice about $[(1,1)]$, $[(1,2)]$, and $[(1,3)]$?

Definition. Given an equivalence $\equiv$ on a set A, the natural map is the mapping $\phi\colon A \to A_{\equiv}$ given by $\phi(a) = [a]$ for all $a \in A$.

Exercise 6. Fill in the arrows for the natural map for the relation described in exercise 5.

A	$\xrightarrow{\phi}$	$A_{\equiv}$
(1,1)		
(1,2)		[(1,1)]
(1,3)		
(2,1)		
(2,2)		[(1,2)]
(2,3)		
(3,1)		
(2,3)		[(1,3)]
(3,3)		

Theorem 0.4.1 Let $\equiv$ be an equivalence relation on a set A and let a and b denote elements of A. Then

1. $a \in [a]$ for all $a \in A$.

2. $[a] = [b]$ if and only if $a \equiv b$.

3. If $a \in [b]$ then $[a] = [b]$.

4. If $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$.

Exercise 7. Let's prove part (c) of this theorem. Note that $[a]$ and $[b]$ are sets, so we will probably have the best time if we use the Principle of Set Equality.

Proof. Let $\equiv$ be an equivalence relation on a set A and let a and b denote elements of A. Assume $a \in [b]$. We will show that $[a] = [b]$.

$\square$

Note. We say that an equivalence relation `partitions` a set because every element of that set will appear in *exactly one* equivalence class.
We call the collection of distinct equivalence classes the `quotient set` of A by $\equiv$.

Exercise 8. Explore this property for the equivalence relation on $\mathbb{Z}$ given by $a \equiv b$ if and only if $a^2 = b^2$. What is the partition of $\mathbb{Z}$ given by this relation?

Definition. Two sets X and Y are `disjoint` if $X \cap Y = \emptyset$, and a collection of sets is `pairwise disjoint` if any two distinct sets in the family are disjoint.

Definition. If A is a nonempty set and $\mathcal{P}$ is a family of subsets of A, we say that $\mathcal{P}$ `partitions` A if

(a) $\emptyset \notin \mathcal{P}$

(b) $\mathcal{P}$ is pairwise disjoint

(c) The union of the sets in $\mathcal{P}$ is A

Exercise 9. What are all of the partitions of $\{a, b, c\}$?

Theorem 0.4.2 If $\equiv$ is any equivalence on a nonempty set A, then the collection of all equivalence classes of A under $\equiv$ partitions A.

Exercise 10. What is the partition of $\mathbb{Z}$ given by $m \equiv n$ if and only if $m - n$ is even?

Exercise 11. How does this partitioning question change if we use $m \equiv n$ if and only if $m - n$ is divisible by 3?