Previously.

- Groups
- Properties of Groups

This Section.

- Subgroups
- The Subgroup Test
- The center of a group, $Z(G)$

Definition. A subset H of a group G is call a subgroup of G if H is also a group using the same operation as G. We denote subgroups using the notation $H \le G$.
If $H \le G$ and $H \neq G$, we call H a proper subgroup of G.

Example. Every group G with two or more elements automatically has two subgroups:

(a) $G \le G$

(b) $\{e\} \le G$ (the trivial subgroup)

Example. (a) $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{R}, +)$

(b) $\mathbb{Z} \le \mathbb{Q} \le \mathbb{R} \le \mathbb{C}$

(c) $(A_3, \circ)$ is a subgroup of $(S_3, \circ)$
Recall: $S_3 = \{\varepsilon, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ and $A_3 = \{\varepsilon, (1\ 2\ 3), (1\ 3\ 2)\}$

Exercise 1. Is it a Subgroup?!

$$\text{Is } (\mathbb{Q} \setminus \{0\}, \cdot) \le (\mathbb{R}, +)?$$

Exercise 2. Is it a Subgroup?!

$$\text{Is } (\{\bar{0}, \bar{2}\}, +) \le (\mathbb{Z}_4, +)?$$

Exercise 3. Is it a Subgroup?!

$$\text{Is } (\mathbb{Z}_2, +) \leq (\mathbb{Z}_3, +)?$$

Exercise 4. Is it a Subgroup?!

$$\text{Is } (\{i, -i\}, \cdot) \leq (\mathcal{U}_4, \cdot)?$$

Recall $\mathcal{U}_4 = \{1, i, -1, -i\}$.

Exercise 5. Is it a Subgroup?!

$$\text{Is } (\{\varepsilon, (1\ 2)\}, \circ) \leq (S_3, \circ)?$$

Subgoup Test (Theorem 2.3.1). A subset H of a group G is a subgroup of G if and only if the following conditions are satisfied.

1. $1_G \in H$, where 1_G is the identity element of G.

2. If $h \in H$ and $h_1 \in H$, then $hh_1 \in H$.

3. If $h \in H$, then $h^{-1} \in H$, where $h^{-1} \in G$ denotes the inverse of h in G.

Note that implicit in these statements, if $H \leq G$ then H and G have the same unity and inverses persist.

Exercise 6. Show $(2\mathbb{Z},+) \leq (\mathbb{Z},+)$ using the subgroup test.

Exercise 7. Is $(2\mathbb{Z}+1,+) \leq (\mathbb{Z},+)$?

Exercise 8. We have the `special linear group` and `general linear group`, respectively below:

$$\mathrm{SL}_2(\mathbb{R}) = \{M \in \mathrm{M}_n(\mathbb{R}) | \det M = 1\},$$

$$\mathrm{GL}_2(\mathbb{R}) = \{M \in \mathrm{M}_n(\mathbb{R}) | \det M \neq 0\}.$$

Show $(\mathrm{SL}_2(\mathbb{R}),\cdot) \leq (\mathrm{GL}_2(\mathbb{R}),\cdot)$

Definition. The center of the group G is the set

$$Z(G) = \{z \in G | zg = gz \ \forall g \in G\}.$$

Question 9. How would you describe $Z(G)$ with words?

Theorem 2.3.3. If G is any group, then $Z(G)$ is a subgroup of G.
Moreover, $Z(G)$ is always abelian.

Proof.

- (Identity)

- (Closure)

- (Inverses)

- ∘ (Abelian)

Definition. Given $H \leq G$ and $g \in G$, we define the set

$$gHg^{-1} = \{ghg^{-1} \mid h \in G\}$$

Proposition. If $H \leq G$ and $g \in G$, then $gHg^{-1} \leq G$.
Thus, we can call gHg^{-1} the conjugate subgroup of H by g.

Exercise 10. Consider $G = S_3$ and $H = \langle (1\ 2) \rangle = \{\varepsilon, (1\ 2)\}$. Let $g = (1\ 2\ 3)$. Compute the conjugate subgroup of H by g.

Proposition. If $H, K \leq G$ then $H \cap K \leq G$.

Exercise 11. Let $G = \mathbb{Z}$, $H = 2\mathbb{Z}$ and $K = 3\mathbb{Z}$. What is $H \cap K$?