# Math 425: Abstract Algebra 1

## Section 0.4: Relations

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## Last Time.

- Mappings
- Codomain vs Range
- Image and Inverse Image
- One-to-One, Onto, Bijection
- Identity Map
- Inverse Mappings
- Invertibility Theorem

## Today.

- Relations
- Equivalence Relations
- Equivalence Classes

**Definition.**

If $A$ is a set, any subset of $A \times A$ is called a `relation` on $A$.

**Note.**

We often call this set $R$ and we denote the elements as $a \equiv b$.

**Example.**

(a) $A = \mathbb{Z}$, $R = \mathbb{Z} \times \mathbb{Z}$

(b) $A = \mathbb{R}$, $a \sim b$ exactly when $a > b$

(c) $A = \{0, 1, 2\}$, $a \equiv b$ exactly when $a = b$

## Definition.

A relation $\equiv$ on a set $A$ is called an `equivalence relation` if it satisfies all of the following conditions for all $a, b, c \in A$,

1. $a \equiv a$ (`reflexivity`),
2. If $a \equiv b$ then $b \equiv a$ (`symmetric`),
3. If $a \equiv b$ and $b \equiv c$, then $a \equiv c$ (`transitive`).

## Exercise 1.

For integers $m$ and $n$, say $m \equiv n$ if and only if $m - n$ is even. Show that $\equiv$ defines and equivalence relation.

**Definition.**

Given an equivalence relation $\equiv$ on a set $A$, we define the `equivalence class` of $a$ to be the set

$$[a] = \{x \in A \mid x \equiv a\}.$$

**Exercise 2.**

What are the equivalence classes from the previous exercise?
($m \equiv n$ if and only if $m - n \in 2\mathbb{Z}$)

## Exercise 3.

What are the equivalence classes for each of the following relations?

(a) Equivalence given by equality.

(b) $A = \mathbb{R}$, $a \equiv b$ if and only if $a^2 = b^2$.

## Notation.

The book uses $A_{\equiv}$ to be the set of unique equivalence classes of $A$ under the equivalence relation $\equiv$. We may also call this set the `quotient set` of $A$ by $\equiv$.

## Exercise 4.

Let $U = \{1, 2, 3\}$ and $A = U \times U$. Define the relation $\equiv$ on $A$ by $(a, b) \equiv (c, d)$ if $b = d$.

(a) What is $A$?

(b) Show that $\equiv$ defines an equivalence relation on $A$.

(c) Determine $A_{\equiv}$.

## Definition.

Given an equivalence $\equiv$ on a set $A$, the `natural map` is the mapping $\phi\colon A \to A_{\equiv}$ given by $\phi(a) = [a]$ for all $a \in A$.

## Exercise 5.

Fill in the arrows for the natural map from the previous slide

| $A$ | $\xrightarrow{\phi}$ | $A_{\equiv}$ |
| --- | --- | --- |
| (1,1) | | |
| (1,2) | | [(1,1)] |
| (1,3) | | |
| (2,1) | | |
| (2,2) | | [(2,2)] |
| (2,3) | | |
| (3,1) | | |
| (2,3) | | [(3,3)] |
| (3,3) | | |

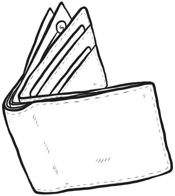
**Theorem 0.4.1.**

Let $\equiv$ be an equivalence on a set $A$ and let $a$ and $b$ denote elements of $A$. Then

1. $a \in [a]$ for all $a \in A$.
2. $[a] = [b]$ if and only if $a \equiv b$.
3. If $a \in [b]$ then $[a] = [b]$.
4. If $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$.

**Brain Break.**

What is your preferred wallet style/type?



### Note.

We say that an equivalence relation `partitions` a set because every element of that set will appear in *exactly one* equivalence class.
We call the collection of distinct equivalence classes the `quotient set` of $A$ by $\equiv$.

### Exercise 6.

Explore this property for the equivalence relation given by $a \equiv b$ if and only if $a^2 = b^2$.

**Definition.**

Two sets $X$ and $Y$ are disjoint if $X \cap Y = \emptyset$, and a collection of sets is pairwise disjoint if any two distinct sets in the family are disjoint.

**Definition.**

If $A$ is a nonempty set and $\mathcal{P}$ is a family of subsets of $A$, we say that $\mathcal{P}$ partitions $A$ if

(a) $\emptyset \notin \mathcal{P}$

(b) $\mathcal{P}$ is pairwise disjoint

(c) The union of the sets in $\mathcal{P}$ is $A$

## Exercise 7.

What are all of the partitions of $\{a, b, c\}$?

**Theorem 0.4.2.**

If $\equiv$ is any equivalence on a nonempty set $A$, then the collection of all equivalence classes of $A$ under $\equiv$ partitions $A$.

**Exercise 8.**

What is the partition of $\mathbb{Z}$ given by $m \equiv n$ if and only if $m - n$ is even?

### Exercise 9.

How does this partitioning question change if we use $m \equiv n$ if and only if $m - n$ is divisible by 3?