**Previously.**

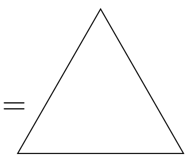
- Cosets
- Lagrange's Theorem
- Corollaries to Lagrange!

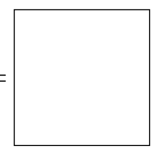
**This Section.**

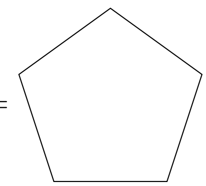
- Dihedral Groups
- Groups of order 2p

**Definition.** A regular $n$-gon is an $n$-sided polygon whose sides are all congruent. Denote this figure by $P_n$.

**Example.**

$P_3 =$ 

$P_4 =$ 

$P_5 =$ 

**Definition.** A symmetry of $P_n$ is any action on $P_n$ by a sequence of flips and/or rotation which return $P_n$ to its original position in the plane.

**Definition.** The dihedral group $D_n$ is the group of symmetries of the figure $P_n$. (Operation is composition.)

**Definition.** Let $n \geq 2$. The dihedral group $D_n$ is the group of order $2n$ presented as follows:

$$D_n = \{e, r, r^2, \ldots, r^{n-1}, f, fr, fr^2, \ldots, fr^{n-1}\},$$

where $|r| = n$, $|f| = 2$, and $rf = fr^{-1}$.

**Note.** The last relation $rf = fr^{n-1}$ can be written in many other ways, such as

- $frf = r^{n-1}$
- $rfr = f$ (this is the one the book uses)
- $r^i f = fr^{-i}$ for all $i \in \mathbb{Z}$
- $fr^i f = r^{-i}$ for all $i \in \mathbb{Z}$
- $r^i fr^i = f$ for all $i \in \mathbb{Z}$

**Note.** Here's the generator notation for $D_n$:

$$D_n = \left\langle f, r \mid f^2 = e, r^n = e, rf = fr^{-1} \right\rangle.$$

**Theorem 2.6.3.** Let $G$ be a group and $p$ a prime. If $|G| = 2p$, then $G$ is cyclic or $G \cong D_p$.

**Exercise 1.** Let $D_4 = \{e, r, r^2, r^3, f, fr, fr^2, fr^3\}$. The subgroup of order 4 are

- $\{\varepsilon, r^2, f, fr^2\}$
- $\{\varepsilon, r, r^2, r^3\}$
- $\{\varepsilon, r^2, fr, fr^3\}$

Which of these are cyclic, and which are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$?

**Example.** The subgroups of $D_4 = \{e, r, r^2, r^3, f, fr, fr^2, fr^3\}$.

- Subgroups of order 2: $\langle f\rangle$, $\langle fr^2\rangle$, $\langle r^2\rangle$, $\langle fr^3\rangle$, $\langle fr\rangle$,
- Subgroup of order 1: $\{e\}$

The full subgroup lattice of $D_4$.