Math 425: Abstract Algebra 1

Section 0.2: Sets

Mckenzie West

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Last Time.

- Course Introduction
- Direct Proofs
- Cases

Today.

- Contradiction
- Counterexamples
- Contrapositive/Converse
- Sets and Operations
- Cardinality
- Power Sets
- Cartesian Products

Method of Proof by Contradiction.

To prove "if P then Q" using contradiction, assume P is true and Q is false then show that something is terribly wrong.

Example.

Prove that there are no integers x and y satisfying $24x + 12y = 1$.

Proof.

Let x and y be integers. Assume, toward contradiction, that $24x + 12y = 1$.

Exercise 1.

Using the last proof as inspiration, what do you think must be true about m and n if

$$mx + ny = 1$$

is satisfied for integers x and y?
Thank about what property(ies) $m = 24$ and $n = 12$ had that caused a problem. Why does $m = 4$ and $n = 9$ have an integer solution with $x = -2$ and $y = 1$?

Counterexamples.

Sometimes we can prove a universal statement is false by finding a simple counterexample.

Example.

Prove the result or give a counterexample.

If $n^2 = 1$ *then* $n = 1$.

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If $n^2 = 1$ *then* $n = 1$.

Solution.

This statement is false because if $n = -1$ then $n^2 = 1$ while $n \neq 1$.

Definition.

The `contrapositive` of the statement $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$.
The `converse` of the statement $P \Rightarrow Q$ is $Q \Rightarrow P$.

Note.

The contrapositive of a statement is logically equivalent to it - and is sometimes easier to prove!
The converse of a statement is NOT logically equivalent to it.

Exercise 2.

1. If n is an even integer then n^2 is a multiple of 4.

(a) State the converse and the contrapositive.
(b) Which of the converse and contrapositive are true?
(c) If either is false, give a counterexample.

2. If $m + n = 25$, where n and m are integers, then exactly one of n and m is greater than 12.

(a) State the converse and the contrapositive.
(b) Which of the converse and contrapositive are true?
(c) If either is false, give a counterexample.

3. For all integers n, if $n \geq 2$, then $n^3 \geq 2^n$.

(a) State the converse and the contrapositive.
(b) Which of the converse and contrapositive are true?
(c) If either is false, give a counterexample.

Definition.

A `set` is a collection of objects called `elements`.

Notation.

If S is a set with an element a, then we write $a \in S$.

Example.

- $\mathbb{Z}$
- $\mathbb{R}$
- $\mathbb{C}$
- $\mathbb{Q}$
- $\mathbb{N}$
- $\mathbb{Z}^+$

Example.

Sets can be represented in many ways:

- Listing all elements: $\{1, 4, 7, 322\}$
- Giving a formula: $\{3a - 2 \in \mathbb{Z} \mid a \in \mathbb{Z}\}$
- Giving a criteria: $\{a \in \mathbb{R} \ : \ 2^a > 5\}$

Exercise 3.

Determine the elements of the following sets:

(a) $\{n \in \mathbb{Z} \ : \ |n - 2| \leq 1\}$

(b) $\{10k \ : \ k \in \mathbb{Z}\}$

(c) $\{m + 1 \mid m \in \mathbb{N} \text{ and } m \text{ is even}\}$

Definition.

If A has n-elements then we say the `cardinality` of A is n and we write $|A| = n$. Such sets are called `finite` sets. Sets with an infinite number of elements are `infinite` sets.

Example.

$|\emptyset| = 0$

Exercise 4.

Compute $|A|$ and $|B|$ given

$$A = \{1, \{2\}\} \text{ and } B = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

Definition.

For sets A and B we say that A `is a subset of` B if every element of A is also an element of B. In mathematical notation we would write

$$\forall a \in A,\ a \in B.$$

Notation.

If A is a subset of B we write $A \subseteq B$.

Exercise 5.

Let $A = \{x \in \mathbb{R} \mid x^2 - 1 < 1\}$ and $B = \{x \in \mathbb{R} \mid x < 3\}$. Show that $A \subseteq B$.

Definition.

The `power set` of a set A is the set $P(A)$ consisting of all subsets of A.

Example.

$B = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} = P(\{1, 2\})$

Exercise 6.

If $A = \{4, 5, 6\}$, what is $P(A)$ and $|P(A)|$?

Principle of Set Equality.

If A and B are sets then

$$A = B \quad \text{if and only if} \quad A \subseteq B \text{ and } B \subseteq A$$

Definition.

The intersection of the sets A and B, denotes $A \cap B$ is the set of all elements that appear in both A and in B.

Definition.

The union of the sets A and B, denotes $A \cup B$ is the set of all elements that appear in at least one of A or B.

Exercise 7.

Use the Principle of Set Equality to show that for sets A, B, and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Definition.

The `Cartesian Product` of the sets A and B is the set

$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}.$$

Note that the elements, (a, b), are ordered pairs.

Exercise 8.

Compute $A \times B$ if $A = \{3, 4\}$ and $B = \{a, b, c\}$.

Exercise 9.

(a) Compute $A \times A$ if $A = \{2, 4, 6\}$.

(b) Consider $A = [-2, 6)$ and $B = (5, 9)$ as intervals in $\mathbb{R}$. What is $A \times B$?