Previously.

- Normal Subgroups
- Products of Groups
- Simple Groups

This Section.

- Factor Groups
- Commutator Subgroups

Goal. Extend this idea of "modular arithmetic" to other groups.

Exercise 1. Consider $G = \mathbb{Z}$ and $H = 4\mathbb{Z}$.

1. First, find all right cosets of H.

2. We define the sum of the cosets as follows:

$$(a + H) + (b + H) = \{m + n \mid m \in a + H \text{ and } n \in b + H\}.$$

Verify that $(a + H) + (b + H) = (a + b) + H$ for all $a, b \in \mathbb{Z}$.

Notice, if $G = \mathbb{Z}$ and $H = 4\mathbb{Z}$, then

- $(a + H) = \overline{a} \in \mathbb{Z}_4$,
- $(a + H) + (b + H) = \overline{a} + \overline{b} = \overline{a + b} = (a + b) + H$

Lemma. The following conditions are equivalent for a subgroup K of G.

1. K is normal in G

2. $aK * bK = (a * b)K$ is a well-defined operation of left cosets.

Theorem 2.9.1. Let $K \trianglelefteq G$ and write $G/K = \{aK \mid a \in G\}$, the set of left cosets of K. Then

1. G/K is a group under the operation $(aK)(bK) = (ab)K$.
2. The mapping $\varphi : G \to G/K$ defined by $\varphi(a) = aK$ is an onto homorphism.
3. If G is abelian, then G/K is abelian.
4. If $G = \langle a \rangle$, then G/K is also cyclic with $G/K = \langle Ka \rangle$.
5. If $|G : K|$ is finite then $|G/K| = |G : K|$. If $|G|$ is finite, then $|G/K| = \frac{|G|}{|K|}$.

Definition. If K is a normal subgroup of the group G, then the group G/K is called the `factor group`, or `quotient group`, of G by K.
We call the homomorphism $\varphi : G \to G/K$ with $\varphi(a) = Ka$ the `coset map`.

Example. The "trivial" examples:

(a) $G/G = \{G\}$

(b) $G/\{e_G\} = \{\{a\} \mid a \in G\} \cong G$

Exercise 2. Consider the group $G = S_3$ and $K = \{\varepsilon, (1\ 2\ 3), (1\ 3\ 2)\}$.

Exercise 3. Consider the group $G = S_4$ and $K = \{\varepsilon, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.

Exercise 4. Consider the group $G = \{..., \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \dots\}$ under the operation multiplication and $K = \langle 8 \rangle$.

Exercise 5. Consider $G = D_4 = \{e, r, r^2, r^3, f, fr, fr^2, fr^3\}$.

(a) Show that $Z = Z(G) = \{e, r^2\}$.

(b) Certify $Z \trianglelefteq D_4$.

(c) Determine the cosets of Z in D_4.

(d) Complete the Cayley table for D_4/Z.

(e) What is the group D_4/Z?

Theorem 2.9.2. If G is a group and $G/Z(G)$ is cyclic, then G is abelian.

Exercise 6. In general, how do we know if G/H is abelian? That is, explore what it means for $HaHb = HbHa$.

Definition. For $a, b \in G$ we define the commutator of a and b to be

$$[a, b] = aba^{-1}b^{-1}.$$

Note. If G is abelian, then for all $a, b \in G$, $[a, b] = e_G$.

Fact. If $H \trianglelefteq G$, then G/H is abelian if and only if H contains every commutator.

Definition. The commutator subgroup of G is the group

$$\begin{aligned} G' &= \{\text{all finite products of commutators from } G\} \\ &= \langle [a, b] \mid a, b \in G \rangle. \end{aligned}$$

Note. $[a, b]^{-1} = [b, a]$

Theorem 2.9.3. Let G be a group and let H be a subgroup of G.

1. G' is a normal subgroup of G and G/G' is abelian.

2. $G' \subseteq H$ if and only if H is normal in G and G/H is abelian.

Exercise 7. Compute D_4'.