Math 425: Abstract Algebra 1

Section 1.4: Permutations

Mckenzie West

Last Updated: September 19, 2022

Last Time.

- Arithmetic Modulo n
- Some further results

Today.

- Permutations
- Notation for Permutations
- Composition of Permutations
- Cycles
- Disjoint Cycles
- Transpositions
- Even vs Odd Permutations

Definition.

A permutation of $T_n = \{1, 2, \ldots, n\}$ is a mapping $\sigma\colon T_n \to T_n$ that is both one-to-one and onto (a bijection).

We call the collection of all permutations of T_n the symmetric group of order n, and we write

$$S_n := \{\sigma\colon T_n \to T_n \mid \sigma \text{ is a permutation}\}.$$

Note.

We can define a permutation on any set X to be a bijection $\sigma\colon X \to X$. And the set of all permutation on X is the set of symmetries of X:

$$S_X := \{\sigma\colon X \to X \mid \sigma \text{ is a bijection}\}.$$

Note.

The set S_n has an operation, composition. With this operation on S_n, we have

(a) an identity, ε, the identity map,

(b) associativity, $\sigma \circ (\tau \circ \gamma) = (\sigma \circ \tau) \circ \gamma$, and

(c) inverses, if $\sigma \in S_n$, then $\sigma^{-1} \in S_n$.

A look ahead to the future: This is why we can call S_n a *group*.

5- Notation 1 - Two-Line Notation

Example.

In the case of $\sigma : T_4 \to T_4$, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ means

(a) $\sigma(1) = 3$ (b) $\sigma(2) =$ (c) $\sigma(3) =$ (d) $\sigma(4) =$

Definition (`Two-Line Notation`).

For $\sigma : T_n \to T_n$, we can write

6- Notation 2 - One-Line Notation

Example.

For the previous example, the permutation in two-line notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

can instead be written as

$$\sigma = 3\ 2\ 4\ 1.$$

Definition (`One-Line Notation`).

For $\sigma : T_n \to T_n$, we can write

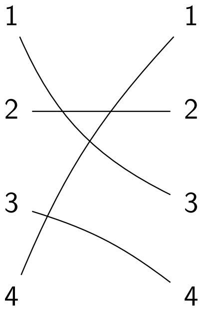
Example.

There are 6 permutations in S_3, written in one-line notation as:

8 - Notation 3 - Braid or Arrow Notation

Example.

For $\sigma = 3\ 2\ 4\ 1$:



9- Notation 4 - Permutation Matrices

Recall: Standard basis for $\mathbb{R}^n$, $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$
We can view permutations as "permuting" the indices of the $\vec{e}_i$.

Example.

The permutation $\sigma = 3\ 2\ 4\ 1$ corresponds to the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T(\vec{e}_1) = \vec{e}_3, \quad T(\vec{e}_2) = \vec{e}_2, \quad T(\vec{e}_3) = \vec{e}_4, \quad T(\vec{e}_4) = \vec{e}_1$$

10- Notation 4 - Permutation Matrices

Definition.

A `permutation matrix` is an $n \times n$ matrix that has exactly one 1 in each row and column and every other entry is 0.

Note.

Every permutation matrix has determinant ± 1, and can be constructed by swapping columns (or rows) of the identity matrix.

Brain Break.

Black Licorice?

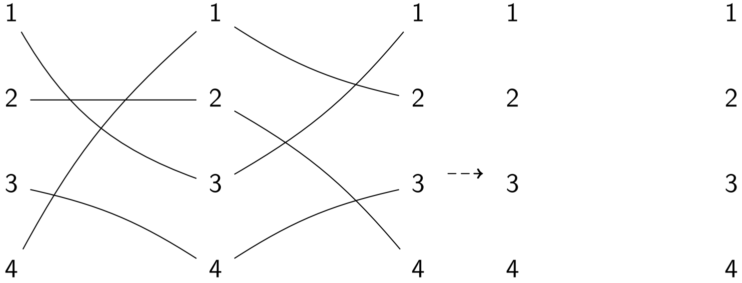
Composition (aka Multiplicaiton in S_n).

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$.

Compute $\tau\sigma$

Example.

Same operation in terms of braids, computing $\tau\sigma$:



σ τ $\tau\sigma$

14- Notation 5 - Cycle Notation

Definition.

The r-cycle $(x_1\ x_2\ \ldots\ x_r)$ in S_n is the permutation that sends

$$\begin{array}{ccc} x_1 & \mapsto & x_2 \\ x_2 & \mapsto & x_3 \\ x_3 & \mapsto & x_4 \\ & \vdots & \\ x_{r-1} & \mapsto & x_r \\ x_r & \mapsto & x_1. \end{array}$$

15- Notation 5 - Cycle Notation

Example.

$(2\ 4\ 1) \in S_5$ does the following

$$\begin{array}{rcl} 2 & \mapsto & 4 \\ 4 & \mapsto & 1 \\ 1 & \mapsto & 2 \end{array}$$

Question 1.

What does this permutation do to 3 and 5?

Note.

There are several equivalent ways to write $(2\ 4\ 1)$:

Exercise 2.

Write our original σ and τ in cycle notation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

18- Multiplying Cycles - Right to Left

Example.

Let $\sigma = (1\ 3\ 2)$, and $\tau = (1\ 5\ 3)$

Exponents.

We write exponents to mean the repeated composition of :

$$\sigma^k := \underbrace{\sigma\sigma\cdots\sigma}_{k}.$$

Exercise 3.

Let $\sigma = (1\ 4\ 2\ 6) \in S_6$. Compute σ^k for $k = 2, 3, 4, 5, \ldots$.

Exercise 4.

Recall that permutations are bijective maps, so they ALL have inverses! (Woo!)

(a) Find the inverse of the cycle $(1\ 2)$.

(b) Find the inverse of the cycle $(1\ 4\ 2\ 5\ 3\ 6)$.

Note.

The notation for the `identity permutation` is ε.

Inverses.

The inverse of the cycle $(x_1\ x_2\ \ldots\ x_r)$ is the cycle $(x_r\ x_{r-1}\ \ldots\ x_2\ x_1)$.

Exercise 5.

If $\sigma = (1\ 2\ 3)$ and $\tau = (3\ 2\ 1)$, verify that $\sigma\tau = \varepsilon$ and $\tau\sigma = \varepsilon$.

Theorem 0.3.5(3) - Specialized.

Let $\sigma, \tau \in S_n$, then

$$(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}.$$

Exercise 6.

Let $\sigma = (1\ 5\ 3\ 4)$ and $\tau = (2\ 6)$.
Use Theorem 0.3.5 to compute $(\sigma\tau)^{-1}$.

Definition.

Two cycles $(x_1\ x_2\ \dots\ x_r)$ and $(y_1\ y_2\ \dots\ y_s)$ are `disjoint` if

$$\{x_1, x_2, \dots, x_r\} \cap \{y_1, y_2, \dots, y_s\} = \emptyset.$$

Theorem.

Disjoint cycles commute. That is if σ and τ are disjoint cycles then $\sigma\tau = \tau\sigma$.

Theorem 1.4.5 (Cycle Decomposition Theorem).

Every $\sigma \in S_n$ with $\sigma \neq \varepsilon$ can be written as a product of disjoint cycles.

Exercise 7.

Write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 4 & 2 & 8 & 1 & 6 \end{pmatrix} \in S_8$$

as a product of disjoint cycles.

Definition.

A `transposition` is a cycle of length 2.

Theorem 1.4.6.

If $n \geq 2$, then every cycle in S_n can be written as a product of transpositions.

"Proof."

$$(x_1\ x_2\ \ldots\ x_r) = (x_1\ x_r)(x_1\ x_{r-1}) \cdots (x_1\ x_3)(x_1\ x_2) \tag{1}$$

$\square$

Exercise 8.

Verify that $(1\ 2\ 5\ 3) = (1\ 3)(1\ 5)(1\ 2)$.

Definition.

A permutation $\sigma \in S_n$ is called `even` if it can be written as a product of an even number of transpositions.
Similarly, permutations can be called `odd`.

The Parity Theorem (Theorem 1.4.7).

If a permutation has two factorizations

$$\sigma = \gamma_n \cdots \gamma_2 \gamma_1 = \mu_m \cdots \mu_s \mu_1,$$

where each of γ_i and μ_j are transpositions, then $m \equiv n \pmod 2$ (m and n have the same parity).

Definition.

The `alternating group of degree` n is the set of even permutations in S_n. We call it A_n.

Exercise 9.

Determine A_3.

Question 10.

How do you think $|A_n|$ compares with $|S_n|$?

Exercise 11.

Let $f = (7\ 3)(2\ 6\ 1\ 9\ 4\ 8\ 7)$ and $g = (6\ 7\ 8\ 3\ 1\ 4\ 9)$.

(a) Begin by re-writing f and g so that the smallest number comes first in each cycle.

(b) Compute

i. fg

ii. g^{-1}

iii. f^{-1}

iv. fgf^{-1}

Definition.

The `order` of a permutation, $\sigma \in S_n$ is the smallest positive integer k such that $\sigma^k = \varepsilon$.

Exercise 12.

(a) What is the order of (1 2)(3 4)?

(b) What is the order of (1 2)(3 4 5)?

Question 13.

Assuming $\sigma \in S_n$ can be written as k disjoint cycles $\rho_1\rho_2\cdots\rho_k$ where cycle ρ_i has order n_i. Conjecture a value for the order of $\sigma \in S_n$.

Note.

You might want to first try some more examples.