Previously.

- Ideals
- Principal Ideals
- Prime Ideals
- Maximal Ideals

This Section.

- Ring Homomorphisms
- First Isomorphism Theorem for Rings

Exercise 1. Consider $\theta\colon M_2(\mathbb{Z}) \to M_2(\mathbb{Z})$ defined by

$$\theta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$

(a) What is $\theta\left(\begin{bmatrix} 7 & 23 \\ 67 & -535 \end{bmatrix}\right)$?

(b) Does θ satisfy $\theta(A+B) = \theta(A) + \theta(B)$?

(c) Does θ satisfy $\theta(AB) = \theta(A)\theta(B)$?

(d) What is $\theta^{-1}\left(\begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}\right)$?

Definition. If R and S are rings with unity, we call a map $\theta : R \to S$ a `ring homomorphism` if

1. $\theta(r_1 + r_2) =$ ______________________________

2. $\theta(r_1 r_2) =$ ______________________________

3. $\theta(1_R) = 1_S$

Example. (a) $\theta : \mathbb{Z} \to \mathbb{Z}_n$, $\theta(a) = \overline{a}$

(b) $\theta : R \to R/I$, $\theta(r) = r + I$

(c) $\pi_1 : R_1 \times R_2 \to R_1$, $\pi_1(r_1, r_2) = r_1$

Theorem 3.4.1. Let $\theta : R \to R_1$ be a ring homomorphism and let $r \in R$.

1. $\theta(0) = 0$
2. $\theta(-r) = -\theta(r)$ for all $r \in R$
3. $\theta(kr) = k\theta(r)$ for all $r \in R$ and $k \in \mathbb{Z}$
4. $\theta(r^n) = \theta(r)^n$ for all $r \in R$ and $n \geq 0$ in $\mathbb{Z}$
5. If $u \in R^*$, $\theta(u^k) = \theta(u)^k$ for all $k \in \mathbb{Z}$.

Exercise 2. Consider $\theta : \mathbb{Z} \to \mathbb{Z}_9$ with $\theta(a) = \overline{a}$.
Use this map, Theorem 3.4.1, and contradiction to show there are no integers a, b, c for which

$$a^3 + b^3 + c^3 = 31.$$

Theorem 3.4.2. Let $R \neq 0$ be a commutative ring with characteristic p, and define

$$\phi : R \to R \quad \text{by} \quad \phi(r) = r^p \text{ for all } r \in R.$$

Then ϕ is a ring homomorphism.
We call this ϕ the `Frobenius Endomorphism`. If ϕ is a finite field, we call ϕ the `Frobenius Automorphism`, which is an isomorphism.

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Definition. If $\theta : R \to S$ is a ring homomorphism, then the `kernel` of θ is

$$\ker(\theta) = \{r \in R \mid \theta(r) = 0_S\}.$$

The `image` of θ is

$$\operatorname{im}(\theta) = \theta(R) = \{\theta(r) \mid r \in R\}.$$

Theorem 3.4.3. Let $\theta : R \to S$ be a ring homomorphism. Then

1. $\theta(R)$ is ________________________ of ______

2. $\ker\theta$ is ________________________ of ______

First Isomorphism Theorem for Rings (Theorem 3.4.4). Let $\theta : R \to S$ be a ring homomorphism and write $A = \ker\theta$. Then θ induces a ring isomorphism

$$\bar{\theta} : R/A \to \theta(R) \quad \text{given by} \quad \bar{\theta}(r + A) = \theta(r) \text{ for all } r \in R.$$

Note. Now's a great time to discuss some notation.
In R/A, elements look like $r + A$ where $r \in R$.
If A is clear, we can simplify our notation by writing $\overline{r}$.

Example. In $\mathbb{Z}/n\mathbb{Z}$, elements are of the form $k + n\mathbb{Z}$ for $k \in \mathbb{Z}$. These correspond to $\overline{k}$ in $\mathbb{Z}_n$.

Example. Though to make our lives easier, if $m|n$, we might want to use the longer version in the restriction map,

$$\phi : \mathbb{Z}_n \to \mathbb{Z}_m \quad \text{with} \quad \phi(k + n\mathbb{Z}) = k + m\mathbb{Z}.$$

Corollary. Let A and B be ideals of the rings R and S, respectively. Then $A \times B$ is an ideal of $R \times S$ and

$$\frac{R \times S}{A \times B} \cong \frac{R}{A} \times \frac{S}{B}.$$

Corollary. Let A be an ideal of the ring R. Then $M_n(A)$ is an ideal of $M_n(R)$ and

$$\frac{M_n(R)}{M_n(A)} \cong M_n(R/A).$$