# Math 425: Abstract Algebra 1

## Section 1.1: Induction

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## Last Time.

- Relations
- Equivalence Relations
- Equivalence Classes

## Today.

- Induction

**Principle of Mathematical Induction.**

Let $P(n)$ be a statement for each integer $n \geq m$. Suppose the following conditions are satisfied,

1. $P(m)$ is true, and
2. $P(k) \Rightarrow P(k+1)$ for every $k \geq m$.

Then $P(n)$ is true for every $n \geq m$.

## Example.

**Claim:** For all $n \geq 4$,

$$2^n < n!.$$

**Example.**

Show that $3^{3n} + 1$ is a multiple of 7 for all odd $n \geq 1$.

**Brain Break.**

What candies (if any) do you think all of the colors taste the same?

## Induction in Algebra.

We assume the distributive property $a(b+c) = ab + ac$.
Prove that this property can be extended, that is show that for all integers $n \geq 2$, for all $a, b_1, \ldots, b_n \in \mathbb{Z}$, we have

$$a(b_1 + b_2 + \cdots + b_n) = ab_1 + ab_2 + \cdots + ab_n.$$

## Exercise 1.

Consider the statement, for all $n \geq 1$,

$$n^3 + (n+1)^3 + (n+2)^3$$

is a multiple of 9.

Use Induction to prove this statement.

(a) What is the base case?

(b) What is the inductive hypothesis?

(c) Use the inductive hypothesis to complete the proof.
Hint: Do NOT do any expanding of the terms $(k+1)^3$ and $(k+2)^3$.

## Exercise 2.

Let $a_n = 2^{3n} - 1$ for $n \geq 0$. Guess a common divisor for each $a_n$ and prove your assertion. (The common divisor should be greater than 1.)

(a) Use a calculator or math software co compute $a_1, a_2, a_3, a_4$. What common divisor do you notice?

(b) Use induction to prove your claim.

## Another Induction Principle.

Let $P(n)$ be a statement for each integer $n \geq m$. Suppose the following conditions are satisfied,

1. $P(m)$ and $P(m+1)$ are true, and
2. If $k \geq m$ and both $P(k)$ and $P(k+1)$ are true then $P(k+2)$ is true.

Then $P(n)$ is true for every $n \geq m$.

## Exercise 3.

Let $a_n$ denote a number for each integer $n \geq 0$ and assume that $a_{n+2} = a_{n+1} + 2a_n$ holds for every $n \geq 0$.
Show that if $a_0 = 1$ and $a_2 = 2$, then $a_n = 2^n$ for each $n \geq 0$.