Definition. A `mapping` or `function` α from A to B is a rule that assigns to every input $a \in A$ exactly one output $\alpha(a) \in B$.
Notation:

$$\alpha\colon A \to B \text{ or } A \xrightarrow{\alpha} B.$$

Once we have verified that each input maps to exactly one output then we say the mapping is `well-defined`.

Example:

1. (Calculus) The map $\alpha : \mathbb{R} \to [-1, 1]$ defined by $\alpha(x) = \sin(x)$.

2. (Linear Algebra) The map $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\beta(\vec{v}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \vec{v}$

Exercise 1. You define a mapping $\gamma : \mathbb{Z} \to \{0, 1\}$.

Definition. Assume $\alpha : A \to B$ is a mapping.

– We call A the `domain` of α and B the `codomain` of α.

– If $C \subseteq A$, then the `image` of C is

$$\alpha(C) = \{b \in B : b = \alpha(c) \text{ for some } c \in C\}.$$

– The `range` of α is the image of the domain,

$$\text{im}(\alpha) = \alpha(A) = \{\alpha(a) \in B : a \in A\}.$$

Exercise 2. Define $\alpha : \mathbb{Z} \to \mathbb{Z}$ by $\alpha(n) = 3n + 1$.

1. Compute the image of $C = \{2, 4, 6\}$.

2. What is the range of α?

Definition. Let $\alpha\colon A \to B$ be a mapping.

(a) We call α `one-to-one` or `injective` if:

$$\text{for all } a_1, a_2 \in A \text{ if } \alpha(a_1) = \alpha(a_2), \text{ then } a_1 = a_2.$$

(b) We call α `onto` or `surjective` if

$$\text{for all } b \in B \text{ there is an } a \in A \text{ such that } \alpha(a) = b.$$

(c) We call α a `bijection` or `bijective` if α is both one-to-one and onto.

Exercise 3. Define $\alpha : \mathbb{Z} \to \mathbb{Z}$ by $\alpha(n) = 3n + 1$.

1. Is α one-to one?

2. Is α onto?

3. Is α a bijection?

Generic Proof of One-to-One.
Statement: The map $\alpha : A \to B$ defined by $\alpha(a) = ...$ is one-to-one.

Proof. Let $a_1, a_2 \in A$ and assume $\alpha(a_1) = \alpha(a_2)$.

$$\vdots$$
use the definition of α and whatever theorems
$$\vdots$$

Therefore $a_1 = a_2$. And so we can conclude that α is one-to-one. □

Generic Proof of Onto.
Statement: The map $\alpha : A \to B$ defined by $\alpha(a) = ...$ is onto.

Proof. Let $b \in B$.

$$\vdots$$
do some reverse engineering to pick just the right a
$$\vdots$$

Therefore with this a as we have defined it, $\alpha(a) = b$. And so we can conclude that α is onto. □

Exercise 4. Prove that the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x - 2$ is one-to-one.

Exercise 5. Prove that the map $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3 - 1$ is onto.

Exercise 6. Prove that the map $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2$ is neither one-to-one or onto.
You really should use counterexamples here, not generic "Let $a \in \mathbb{R}$" statements. That is, you should write something like "Consider $x_1 = 1$ and $x_2 = -1$."

Exercise 7. Write down a mapping $\alpha\colon \mathbb{Z} \to \mathbb{Z}$ that is

(a) neither one-to-one nor onto,

(b) one-to-one and not onto,

(c) onto and not one-to-one,

(d) a bijection.

Definition. The identity map for the set A is the map $1_A : A \to A$ defined by $1_A(a) = a$ for all $a \in A$.
If $\alpha : A \to B$ and $\beta : B \to C$ are mappings, we can write

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C,$$

and the composition of the maps is the mapping $\beta\alpha : A \to C$ defined by

$$\beta \circ \alpha(a) = \beta\alpha(a) = \beta[\alpha(a)] \text{ for all } a \in A.$$

Exercise 8. Let $\alpha : \mathbb{N} \to \mathbb{R}$ be defined by $\alpha(n) = \sqrt{n}$ and let $\beta : \mathbb{R} \to \mathbb{Z}$ be defined by $\beta(x) = \lfloor x \rfloor$ (the largest integer less than or equal to x.)

(a) Which of the following are allowable compositions and which are not?

(i) $1_{\mathbb{N}} \circ \alpha$

(ii) $\alpha \circ 1_{\mathbb{N}}$

(iii) $1_{\mathbb{R}} \circ \alpha$

(iv) $\alpha \circ 1_{\mathbb{R}}$

(v) $\alpha\beta$

(vi) $\beta\alpha$

(b) Describe $\beta\alpha$. (Find a formula for $\beta\alpha(n)$.)

Theorem 0.3.3 Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D$ be mappings on sets. Then

1. (identity) $\alpha 1_A = \alpha$ and $1_B\alpha = \alpha$

2. (associativity) $\gamma(\beta\alpha) = (\gamma\beta)\alpha$

3. If α and β are both one-to-one (resp. onto), then $\beta\alpha$ is one-to-one (resp. onto) too.

Definition. If $\alpha : A \to B$ is a mapping of sets, then we call $\beta : B \to A$ an `inverse` of α if

$$\beta\alpha = 1_A \text{ and } \alpha\beta = 1_B.$$

Exercise 9. Define $\alpha : \mathbb{R} \to \mathbb{R}$ by $\alpha(x) = 3x + 1$ and $\beta : \mathbb{R} \to \mathbb{R}$ by $\beta(x) = \frac{1}{3}x - \frac{1}{3}$. Show that $\alpha = \beta^{-1}$ by computing both $\alpha\beta(x)$ and $\beta\alpha(x)$.

Theorem 0.3.4 If $\alpha : A \to B$ has an inverse, then the inverse mapping is unique.

Proof. Let $\alpha : A \to B$ be a mapping with an inverse. Let β and β' be two inverses of α. We compute

$$\begin{array}{rcll} \beta & = & \beta 1_B & \text{Theorem 0.3.3(a)} \\ & = & \beta(\alpha\beta') & \beta' \text{ inverse of } \alpha \\ & = & (\beta\alpha)\beta' & \text{Theorem 0.3.3(b)} \\ & = & 1_A\beta' & \beta \text{ inverse of } \alpha \\ & = & \beta' & \text{Theorem 0.3.3(a).} \end{array}$$

Therefore $\beta = \beta'$, and the inverse of α is unique. □

Result. The notation α^{-1} is valid.
Theorem 0.3.5 Let $\alpha : A \to B$ and $\beta : B \to C$ denote mappings.

1. The identity map, $1_A : A \to A$ is invertible and $1_A^{-1} = 1_A$.

2. If α is invertible, then α^{-1} is invertible and $(\alpha^{-1})^{-1} = \alpha$.

3. If α and β are both invertible, then $\beta\alpha$ is invertible with $(\beta\alpha)^{-1} = \alpha^{-1}\beta^{-1}$.

Invertibility Theorem (Theorem 0.3.6) A mapping $\alpha : A \to B$ is invertible if and only if α is a bijection.

Proof. ($\Rightarrow$) Assume that $\alpha : A \to B$ is invertible. Denote its inverse by $\beta : B \to A$. We now show α is one-to-one and onto.
Let $a_1, a_2 \in A$ such that $\alpha(a_1) = \alpha(a_2)$. By definition of inverse, we have

$$a_1 = \beta\alpha(a_1) \quad \text{and} \quad a_2 = \beta\alpha(a_2).$$

So now we can use substitution to find that

$$a_1 = \beta\alpha(a_1) = \beta\alpha(a_2) = a_2.$$

Thus $a_1 = a_2$, so α is one-to-one.
Now let $b \in B$. Then $a = \beta(b) \in A$. Furthermore,

$$\alpha(a) = \alpha\beta(b) = b,$$

by definition of inverse. Therefore b is in the image of α, so α is onto. $\square$

Definition. We will call two maps $\alpha : A \to B$ and $\beta : A \to B$ equal if $\alpha(a) = \beta(a)$ for all $a \in A$.

Example. Consider $\alpha : \mathbb{Z} \to \{0, 1\}$ and $\beta : \mathbb{Z} \to \{0, 1\}$ defined by

$$\alpha(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \text{ and } \beta(n) = \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor.$$

These are equal but have very different feeling descriptions.