Previously.

- Mappings between groups
- Homomorphisms
- Isomorphisms
- Image
- Kernel
- Automorphisms

This Section.

- Cosets
- Lagrange's Theorem
- Corollaries to Lagrange!

Goal. Prove that if $H \leq G$ and $|G| < \infty$, then

$$|H| \text{ divides } |G|.$$

Definition. Let $H \leq G$ and let $a \in G$. Define the two sets

1. $H * a = \{h * a | h \in H\}$ called the right coset of H by a.
2. $a * H = \{a * h | h \in H\}$ called the left coset of H by a.

Note. We often write Ha and aH if the group operation is unknown/unspecified.

Example. If $G = \mathbb{Z}$ the right cosets of $H = \langle 4 \rangle$ are

$$\begin{aligned} H + 0 &= 4\mathbb{Z} \\ H + 1 &= 4\mathbb{Z} + 1 \\ H + 2 &= 4\mathbb{Z} + 2 \\ H + 3 &= 4\mathbb{Z} + 3 \end{aligned}$$

What are the left cosets?

Example. For $G = (\mathbb{R}, +)$, $H = (\mathbb{Z}, +)$ is a subgroup.
The right coset $\mathbb{Z} + \frac{1}{2}$ is the set of all points if we shift all integers one $\frac{1}{2}$ unit to the right. That is

$$\mathbb{Z} + \frac{1}{2} = \{\dots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}.$$

Exercise 1. Let $G = \mathbb{Z}_{12}$ and $H = \langle \bar{3} \rangle = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$.
What are the right cosets of H?

Exercise 2. Let $G = S_3$ and $H = \langle(1\ 2)\rangle = \{\varepsilon, (1\ 2)\}$.
Compute the right cosets

(a) $H\varepsilon =$

(b) $H(1\ 2\ 3) =$

(c) $H(1\ 3\ 2) =$

(d) $H(1\ 2) =$

(e) $H(1\ 3) =$

(f) $H(2\ 3) =$

Any additional observations?

Exercise 3. Let $G = S_3$ and $H = \langle(1\ 2)\rangle = \{\varepsilon, (1\ 2)\}$.
Compute the left cosets

(a) $\varepsilon H =$

(b) $(1\ 2\ 3)H =$

(c) $(1\ 3\ 2)H =$

(d) $(1\ 2)H =$

(e) $(1\ 3)H =$

(f) $(2\ 3)H =$

What do you notice about the left vs right cosets in the case with $G = S_3$ and $H = \{\varepsilon, (1\ 2)\}$?

Theorem 2.6.1. Let H be a subgroup of a group G and let $a, b \in G$.

1. $H = He_G$.
2. $Ha = H$ if and only if $a \in H$.
3. $Ha = Hb$ if and only if $ab^{-1} \in H$.
4. If $a \in Hb$, then $Ha = Hb$.
5. Either $Ha = Hb$ or $Ha \cap Hb = \emptyset$.
6. The distinct right cosets of H partition G.

Note. Here's a recommendation. Go back to the previous 4 examples and verify these facts.

Corollary. Corresponding statements hold for left cosets. In particular, part (3) becomes

$$aH = bH \Leftrightarrow a^{-1}b \in H \Leftrightarrow b^{-1}a \in H.$$

Lemma. There is a 1-1 correspondence (aka bijection) between the elements in any two right cosets of H in G.

Lemma. If H is a finite subgroup of the group G and $a, b \in G$, then $|Ha| = |Hb|$.

Note. "The single most important result about finite groups!" - W. Keith Nicholson (textbook author)

Lagrange's Theorem (Theorem 2.6.2). Let G be a finite group. Then if H is a subgroup of G, we have $|H|$ divides $|G|$.

Example. Recall the previous exercises

(a) $G = \mathbb{Z}_{12}, \quad H = \langle \overline{3} \rangle = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\}$

(b) $G = S_3 = \{\varepsilon, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}, \quad H = \langle (1\ 2) \rangle = \{\varepsilon, (1\ 2)\}$

Corollary 1. If G is a finite group and $g \in G$, then $|g|$ divides $|G|$.

Warning. Corollary 1 does **not** say that G has a subgroup of order d for every divisor d of $|G|$.

Example. Consider the alternating group A_4,

$$\begin{aligned} A_4 = \{\varepsilon, &(1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), \\ &(2\ 3\ 4), (2\ 4\ 3), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \end{aligned}$$

Satisfies $|A_4| = 12$ but A_4 has no subgroup of order 6.

Corollary 2. If G is a group and $|G| = n$, then $g^n = e$ for all $g \in G$.

Proof. Let G be a group with $|G| = n$. Let $g \in G$. If $|g| = m$ then m divides n $\boxed{\text{why?}}$. So $n = mk$ for some $k \in \mathbb{Z}$. Thus

$$g^n = g^{mk} = (g^m)^k = e^k = e.$$

□

Corollary 3. If p is a prime, then every group of order p is cyclic. In fact, $G = \langle g \rangle$ for every non-identity element g in G, so the only subgroups of G are $\{e\}$ and G itself.

Exercise 4. Complete the proof.

Corollary 4. Let H and K be finite subgroups of a group G. If $|H|$ and $|K|$ are relatively prime, then $H \cap K = \{e\}$.

Exercise 5. Prove Corollary 4.

Definition. The index of H in G, denoted $|G : H|$, is defined to the number of distinct right (or left if you prefer) cosets of H in G.

Exercise 6. Compute each of the following indexes.

1. $|\mathbb{Z}_{12} : \langle \overline{4} \rangle|$

2. $|\mathbb{Z} : 2\mathbb{Z}|$

3. $|S_3 : \{\varepsilon, (1\ 2)\}|$

Corollary 5. If H is a subgroup of a finite group G, then

$$|G : H| = \frac{|G|}{|H|}.$$