Previously.

- Factor Groups
- Commutator Subgroups

This Section.

- Kernels
- The first isomorphism theorem

Definition. Let $\alpha : G \to H$ be a group homomorphism. The `image of` α is the set

$$\operatorname{im}\alpha = \alpha(G) = \{\alpha(g) \mid g \in G\}.$$

The `kernel of` α is the set

$$\ker\alpha = \{k \in G \mid \alpha(k) = e_H\}.$$

Exercise 1. Let $\alpha : \mathbb{Z} \to \mathbb{C}^*$ be defined by $\alpha(n) = i^n$ where $i^2 = -1$.
Compute $\operatorname{im}(\alpha)$.

Compute $\ker(\alpha)$.

Theorem 2.10.1. Let $\alpha : G \to H$ be a group homomorphism. Then

1. $\alpha(G)$ is a subgroup of H.
2. $\ker(\alpha)$ is a **normal** subgroup of G

Exercise 2. Use Theorem 2.10.1 to prove the following.

(a) Prove $SL_2(\mathbb{R}) \lhd GL_2(\mathbb{R})$.

(b) Prove $A_n \lhd S_n$.

Theorem 2.10.3. Let $\alpha : G \to H$ be a group homomorphism. Then α is injective if and only if $\ker(\alpha) = \{e_G\}$.

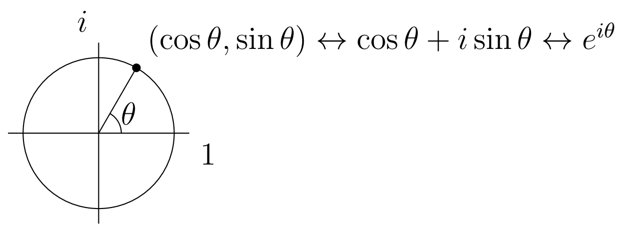
The First Isomorphism Theorem (2.10.4). Let $\alpha : G \to H$ be a group homomorphism. Then

$$G/\ker\alpha \cong im(G) = \alpha(G).$$

Exercise 3. Prove that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ for all integers $n \geq 1$.

Exercise 4. Prove that $\mathbb{R}/\mathbb{Z}$ (under addition) is isomorphic to $\mathbb{C}^0$ (under multiplication).

Recall. If $a + bi \in \mathbb{C}$ for $a, b \in \mathbb{R}$, we can plot this as the point (a, b).
Moreover, any point on unit circle corresponds to the complex number $e^{i\theta}$ where θ is the angle between the vector defined by the point and the positive x-axis:



Claim: The unit circle,

$$\mathbb{C}^0 = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : 0 \leq \theta < 2\pi\} = \{e^{i\theta} : \theta \in \mathbb{R}\}$$

is a subgroup of $\mathbb{C}^\times$ (operation is multiplication).

Claim. If $K_1 \lhd G_1$ and $K_2 \lhd G_2$, then $K_1 \times K_2 \lhd G_1 \times G_2$ and

$$(G_1 \times G_2)/(K_1 \times K_2) \cong G_1/K_1 \times G_2/K_2.$$

Exercise 5. Let $D_3 = \{e, r, r^2, f, fr, fr^2\}$ with $|r| = 3$, $|f| = 2$, and $rf = fr^3$.
How many homomorphisms are there from D_3 to C_6?
Hint: The only normal subgroups of D_3 are $\{e\}$, $\langle r \rangle = \{e, r, r^2\}$, and D_3.

Recall. The `inner automorphism` of the group G corresponding to the element $a \in G$ is the isomorphism

$$\sigma_a : G \to G \quad \text{defined by} \quad \sigma_a(g) = aga^{-1}\ \forall g \in G.$$

Moreover the set of inner automorphisms, $\mathrm{Inn}(G) = \{\sigma_a \mid a \in G\}$, is a subgroup of $\mathrm{Aut}(G)$ because $\sigma_a \circ \sigma_b = \sigma_{ab}$.

Exercise 6. There's a natural map $\alpha : G \to \mathrm{Inn}(G)$ defined by $\alpha(a) = \sigma_a$ for all $a \in G$. Prove that α is a homomorphism. What does the First Isomorphism Theorem tell us?

Theorem 2.10.5. If G is any group then $G/Z(G) \cong \mathrm{Inn}(G)$.

Exercise 7. Use Theorem 2.10.5 to show that $\mathrm{Inn}(S_3) \cong S_3$.

Then show that $|\mathrm{Aut}(S_3)| \leq 6$. By considering the possible images of $\sigma = (1\ 2\ 3)$ and $\tau = (1\ 2)$.

Conclude $S_3 \cong \mathrm{Aut}(S_3)$.