Previously.

- Factoring degree 2 and 3 polynomials
- Unique Factorization
- Factoring in $\mathbb{C}[x]$, $\mathbb{R}[x]$, $\mathbb{Q}[x]$, $\mathbb{Z}[x]$

This Section.

- Algebraic structure of quotients of polynomial rings

Exercise 1. Describe the quotient ring $\mathbb{Q}[x]/A$ where $A = (x^2 - 2)$ is the ideal containing all multiples of $h = x^2 - 2$.

Theorem 4.3.2. Let h be a monic polynomial of degree $m \geq 1$ in $F[x]$, there F is a field. Then

$$F[x]/(h) \cong \{a_0 + a_1t + a_2t^2 + \cdots + a_{m-1}t^{m-1} \; | a_i \in F, h(t) = 0\}.$$

Moreover, this representation is unique. That is,

$$a_0 + a_1t + a_2t^2 + \cdots + a_{m-1}t^{m-1} = b_0 + b_1t + b_2t^2 + \cdots + b_{m-1}t^{m-1}$$

if and only if $a_i = b_i$ for all i.

Exercise 2. Consider $R = \mathbb{Z}_2[x]/(x^2 + x + 1)$. Complete the addition and multiplication tables for R: (use $t = \bar{x} \in R$)

$+$	0	1	t	$1+t$
0				
1				
t				
$1+t$				

$\cdot$	0	1	t	$1+t$
0				
1				
t				
$1+t$				

Exercise 3. Describe the quotient $\mathbb{Z}_2[x]/(x^2)$. That is write an addition and multiplication table as in exercise 2.

Exercise 4. Describe the quotient $\mathbb{Z}_2[x]/(x^2-1)$. That is write an addition and multiplication table as in exercise 2.

Exercise 5. Describe the ring $R = \mathbb{Z}_2[x]/(x^3+1)$. That is write an addition and multiplication table as in exercise 2.

Exercise 6. Describe the ring $R = \mathbb{R}[x]/(x^2+1)$. That is write an addition and multiplication table as in exercise 2.

Exercise 7. Let $R = \mathbb{Z}_3[x]$ and $A = \{f \in R \mid f(1) = 0\}$.

1. Use the definition of ideal to show A is an ideal.

2. Find the monic polynomial $h \in R$ such that $A = (h)$.

3. Describe R/A.

Theorem 4.3.3. Let h be a monic polynomial of degree $m \geq 1$ in $F[x]$, where F is a field. Then $F[x]/(h)$ is a field if and only if h is irreducible.

Exercise 8. Find a field of order 9.

Exercise 9. Consider $F = \mathbb{Q}$ and $h = x^2 + 7x + 3$. Describe $F[x]/(h)$.

Theorem 4.3.4 (Kronecker's Theorem). Let F be a field and $h \in F[x]$ an irreducible polynomial. Then there is some field K containing F that has a root of h.

Exercise 10. Consider $F = \mathbb{Q}$ and $h = x^2 + bx + c$ with $b^2 - 4c$ not a square. Describe $F[x]/(h)$.

Exercise 11. Consider $F = \mathbb{Q}$ and $h = x^3 + 2$. Describe $F[x]/(h)$.