

Previously.

- Subgroups generated by one or more elements of a group
- Cyclic groups
- The order of an element
- Subgroup lattices

This Section.

- Mappings between groups
- Homomorphisms
- Isomorphisms
- Image
- Kernel
- Automorphisms

Goal. Study “sensible” functions from one group to another.

Definition. Let $(G, *)$ and (H, \diamond) be groups. Then a mapping $\phi: G \rightarrow H$ is a [\[group homomorphism\]](#) if $\phi(g_1 * g_2) = \phi(g_1) \diamond \phi(g_2)$ for all $g_1, g_2 \in G$.

Exercise 1. Consider the groups \mathbb{Z} and $2\mathbb{Z}$ and the map $\phi: \mathbb{Z} \rightarrow 2\mathbb{Z}$ defined by $\phi(n) = -2n$ for all $n \in \mathbb{Z}$.

Show that ϕ is a homomorphism.

Example. The [trivial homomorphism](#),

$$\phi: G \rightarrow H, \quad \phi(g) = e_H \quad \forall g \in G$$

Remark. We might leave operations out and write

$$\phi(ab) = \phi(a)\phi(b).$$

Exercise 2. Let $H = \{\varepsilon, (1\ 2)\}$ and $G = S_3$. Define $\pi: H \rightarrow G$ by $\pi(\sigma) = \sigma$ for all $\sigma \in H$. Show that π is a homomorphism.

Exercise 3. $G = (\mathbb{Z}_2, +)$, $H = (\mathbb{Z}_5^*, \cdot)$

$$\phi: \mathbb{Z}_2 \rightarrow \mathbb{Z}_5^*, \quad \phi(\bar{0}_2) = \bar{0}_5 \quad \text{and} \quad \phi(\bar{1}_2) = \bar{4}_5.$$

Answer each of the following to verify that ϕ is a homomorphism:

(a) Is $\phi(\bar{0}_2 + \bar{0}_2)$ equal to $\phi(\bar{0}_2) \cdot \phi(\bar{0}_2)$?

(b) Is $\phi(\bar{0}_2 + \bar{1}_2)$ equal to $\phi(\bar{0}_2) \cdot \phi(\bar{1}_2)$?

(c) Is $\phi(\bar{1}_2 + \bar{0}_2)$ equal to $\phi(\bar{1}_2) \cdot \phi(\bar{0}_2)$?

(d) Is $\phi(\bar{1}_2 + \bar{1}_2)$ equal to $\phi(\bar{1}_2) \cdot \phi(\bar{1}_2)$?

Exercise 4. Let $\sigma \in S_4$. Define $\phi: S_4 \rightarrow S_4$ by $\phi(\tau) = \sigma\tau\sigma^{-1}$ for all $\tau \in S_4$. Show that ϕ is a homomorphism by showing that $\phi(\tau_1\tau_2) = \phi(\tau_1)\phi(\tau_2)$ for all $\tau_1, \tau_2 \in S_4$.

Definition. A homomorphism that is both injective and surjective is called an **isomorphism**. If an isomorphism exists from G to H , we call G and H **isomorphic** and we write $G \cong H$ ($\$G \backslash \text{cong } H\$$).

Example. Considering the homomorphism from before, $\phi: \mathbb{Z} \rightarrow 2\mathbb{Z}$ defined by $\phi(n) = -2n$ for all $n \in \mathbb{Z}$. Show that ϕ is:

- (One-to-One)

- (Onto)

Therefore $\mathbb{Z} \cong 2\mathbb{Z}$.

Exercise 5. Let $G = (\mathbb{R}, +)$ and $H = (\mathbb{R}_{>0}, \cdot)$. Define $\phi: G \rightarrow H$ by

$$\phi(x) = e^x \quad \forall x \in \mathbb{R}.$$

(a) Is ϕ a homomorphism?

(b) Is ϕ one-to-one?

(c) Is ϕ onto?

(d) Is ϕ an isomorphism?

Exercise 6. Show that $\mathbb{Z}_3 \cong C_3$ (here $C_3 = \{1, a, a^2\}$).

Properties of Homomorphisms

Theorem 2.5.1. Let $\phi: G \rightarrow H$ be a group homomorphism. Then

(a) $\phi(e_G) = e_H$ (ϕ preserves identities)

(b) $\phi(g^{-1}) = \phi(g)^{-1} \quad \forall g \in G$ (ϕ preserves inverses)

(c) $\phi(g^k) = \phi(g)^k \quad \forall g \in G, k \in \mathbb{Z}$ (ϕ preserves powers)

Corollary. Let $\phi: G \rightarrow H$ be a homomorphism. If $g \in G$ has $|g| = n < \infty$, then $|\phi(g)| < \infty$. Moreover $|\phi(g)|$ divides $|g|$.

Example. $\phi: \mathbb{Z}_8 \rightarrow \mathbb{Z}_4$, $\phi(\bar{a}_8) = \bar{a}_4$

Warning: A map $\phi: \mathbb{Z}_n \rightarrow \mathbb{Z}_m$ with $\phi(\bar{a}_n) = \bar{a}_m$ only exists if $m \mid n$.

Exercise 7. Show $(\mathbb{Z}_4, +) \cong (\{1, -1, i, -i\}, \cdot)$ by writing down a group isomorphism.

Exercise 8. Show \mathbb{Z}_4 has a subgroup isomorphic to \mathbb{Z}_2 . (Meaning there is an isomorphism between that subgroup and \mathbb{Z}_2 .)

Exercise 9. Give some reason why...

(a) $K_4 \not\cong (\mathbb{Z}_4, +)$

(b) $S_3 \not\cong C_6$

(c) $(\mathbb{Z}_{12}, +) \not\cong (\mathbb{Q}^+, \cdot)$

Exercise 10. Is $(2\mathbb{Z}, +) \cong (3\mathbb{Z}, +)$?

Image of a Homomorphism

Definition. Let $\phi: G \rightarrow H$ be a group homomorphism.

The **image of ϕ** is denoted $\phi(G)$ or $\text{im}(\phi)$ and is defined to be the set

$$\{\phi(g) \in H \mid g \in G\} = \{h \in H \mid \exists g \in G \text{ s.t. } \phi(g) = h\}.$$

Corollary of Thm 2.5.1. Let $\phi: G \rightarrow H$ be a homomorphism. Then $\text{im}(\phi) \leq H$.

Exercise 11. Consider $\phi: \mathbb{Z} \rightarrow GL_2(\mathbb{R})$ defined by

$$\phi(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

(a) Show that ϕ is a homomorphism.

(b) Verify $\text{im}(\phi) = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}$. (Call this set H .)

(c) Conclude H is a group.

Isomorphisms are Equivalence Relations

Theorem 2.5.3. Let G , H , and K denote groups.

1. The identity map $id_G: G \rightarrow G$ is an isomorphism for every group G .
2. If $\sigma: G \rightarrow H$ is an isomorphism then the inverse mapping $\sigma^{-1}: H \rightarrow G$ is an isomorphism.
3. If $\sigma: G \rightarrow H$ and $\tau: H \rightarrow K$ are isomorphisms then $\tau\sigma: G \rightarrow K$ is an isomorphism.

Corollary 1. This isomorphism relation, \cong is an equivalence relation on groups. That is, for all groups G , H , and K ,

1. $G \cong G$,
2. if $G \cong H$, then $H \cong G$, and
3. if $G \cong H$ and $H \cong K$, then $G \cong K$.

Group of Homomorphisms

Corollary. If G is a group, then the set of all isomorphisms $G \rightarrow G$ forms a group under composition.

Proof. Notice that the set of all isomorphisms $G \rightarrow G$ is a subset of S_G . Therefore we can use the subgroup test. Theorem 2.5.3 completes the proof. \square

Question 12. How does Theorem 2.5.3 show that we have (1) Non-empty, (2) Closure, (3) Inverses?

Definition. Let G be a group.

1. An **automorphism of G** is an isomorphism from G to itself.
2. The set $\text{Aut}(G)$ is the set of all automorphisms of G .

Note. The previous Corollary says $\text{Aut}(G) \leq S_G$, so $\text{Aut}(G)$ is a group.

Exercise 13. (a) If G is abelian, then $\phi: G \rightarrow G$ defined by $\phi(g) = g^{-1}$ is an automorphism of G .

- i. Verify ϕ is a homomorphism.
- ii. Check ϕ is injective.
- iii. Check ϕ is surjective.

(b) Let $G = S_3$ (which is not abelian). Show that ϕ from (a) is not a homomorphism.

Exercise 14. Compute the automorphism group of the cyclic group of order 6.

$$\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$$

- (a) Show that $\lambda : \mathbb{Z}_6 \rightarrow \mathbb{Z}_6$ defined by $\lambda(\bar{n}) = -\bar{n}$ is an automorphism of \mathbb{Z}_6 .
- (b) Verify that if $\phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_6$ is an automorphism, then $\phi(\bar{1}) = \bar{1}$ or $\phi(\bar{1}) = \bar{5}$.
- (c) Conclude that $\text{Aut}(\mathbb{Z}_6) = \{id_{\mathbb{Z}_6}, \lambda\}$.
- (d) Discuss why $\text{Aut}(\mathbb{Z}_6) \cong \mathbb{Z}_2$.

Exercise 15. Let G be a group and $a \in G$.

Define $\sigma_a : G \rightarrow G$ by $\sigma_a(g) = aga^{-1}$.

We call σ_a an **inner automorphism of G** .

- (a) Verify σ_a is a homomorphism.
- (b) Check σ_a is injective.
- (c) Check σ_a is surjective.

Definition. The set $\text{Inn}(G) = \{\sigma_a | a \in G\}$ is the set of all inner automorphisms of G .

Exercise 16. Prove that $\text{Inn}(G) \leq \text{Aut}(G)$.

- (a) Find a fixed $a \in G$ for which $\sigma_a = id_G$.
- (b) If $\sigma_a, \sigma_b \in \text{Inn}(G)$ what c satisfies $\sigma_a \sigma_b = \sigma_c$?
- (c) For each $\sigma_a \in \text{Inn}(G)$ what might be σ_a^{-1} ?

Kernel of a Homomorphism

Definition. Let $\phi: G \rightarrow H$ be a group homomorphism.

The **kernel of ϕ** is denoted $\ker(\phi)$ and is defined to be the set

$$\{g \in G \mid \phi(g) = e_H\}.$$

Exercise 17. Let $\phi: G \rightarrow H$ be a homomorphism. Prove that $\ker(\phi) \leq G$.

Exercise 18. Let $\phi: \mathbb{Z} \rightarrow \mathbb{Z}_5$ be defined by $\phi(n) = \bar{n}$. Compute $\ker(\phi)$.