Math 425: Abstract Algebra 1

Section 0.3: Mappings

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Last Time.

- Counterexamples
- Contrapositive
- Converse
- Sets and Operations
- Cardinality
- Power Sets
- Cartesian Products

Today.

- Mappings
- Codomain vs Range
- Image and Inverse Image
- One-to-One, Onto, Bijection
- Identity Map

Definition.

A mapping or function α from A to B is a rule that assigns to every input $a \in A$ exactly one output $\alpha(a) \in B$.
Notation:

$$\alpha\colon A \to B \text{ or } A \xrightarrow{\alpha} B.$$

Once we have verified that each input maps to exactly one output then we say the mapping is well-defined.

Definition.

Assume $\alpha : A \to B$ is a mapping.

– We call A the `domain` of α and B the `codomain` of α.
– If $C \subseteq A$, then the `image` of C is

$$\alpha(C) = \{b \in B : b = \alpha(c) \text{ for some } c \in C\}.$$

– The `range` of α is the image of the domain,

$$\mathrm{im}(\alpha) = \alpha(A) = \{\alpha(a) \in B : a \in A\}.$$

Exercise 1.

Define $\alpha : \mathbb{Z} \to \mathbb{Z}$ by $\alpha(n) = 3n + 1$.

1. Compute the image of $C = \{2, 4, 6\}$.

2. What is the range of α?

Definition.

We will call two maps $\alpha : A \to B$ and $\beta : A \to B$ equal if $\alpha(a) = \beta(a)$ for all $a \in A$.

Example.

Consider $\alpha : \mathbb{Z} \to \{0, 1\}$ and $\beta : \mathbb{Z} \to \{0, 1\}$ defined by

$$\alpha(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \text{ and } \beta(n) = \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor .$$

These are equal but have very different feeling descriptions.

Definition.

Let $\alpha \colon A \to B$ be a mapping.

(a) We call α `one-to-one` or `injective` if for all $a_1, a_2 \in A$ if $\alpha(a_1) = \alpha(a_2)$, then $a_1 = a_2$.

(b) We call α `onto` or `surjective` if for all $b \in B$ there is an $a \in A$ such that $\alpha(a) = b$.

(c) We call α a `bijection` or `bijective` if α is both one-to-one and onto.

Exercise 2.

Define $\alpha : \mathbb{Z} \to \mathbb{Z}$ by $\alpha(n) = 3n + 1$.

1. Is α one-to one?

2. Is α onto?

3. Is α a bijection?

Generic Proof of One-to-One.

Generic Proof of Onto.

Exercise 3.

Write down a mapping $\alpha\colon \mathbb{Z} \to \mathbb{Z}$ that is

(a) neither one-to-one nor onto,

(b) one-to-one and not onto,

(c) onto and not one-to-one,

(d) a bijection.

Brain Break.

What is your favorite class you've taken and why?

Definition.

The `identity map` for the set A is the map $1_A : A \to A$ defined by $1_A(a) = a$ for all $a \in A$.
If $\alpha : A \to B$ and $\beta : B \to C$ are mappings, we can write

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C,$$

and the `composition` of the maps is the mapping $\beta\alpha : A \to C$ defined by

$$\beta\alpha(a) = \beta[\alpha(a)] \text{ for all } a \in A.$$

Theorem 0.3.3.

Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D$ be mappings on sets. Then

1. (identity) $\alpha 1_A = \alpha$ and $1_B \alpha = \alpha$
2. (associativity) $\gamma(\beta\alpha) = (\gamma\beta)\alpha$
3. If α and β are both one-to-one (resp. onto), then $\beta\alpha$ is one-to-one (resp. onto) too.

Definition.

If $\alpha : A \to B$ is a mapping of sets, then we call $\beta : B \to A$ an `inverse` of α if

$$\beta\alpha = 1_A \text{ and } \alpha\beta = 1_B.$$

Example.

$\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha(x) = 3x + 1$
$\beta : \mathbb{R} \to \mathbb{R}$, $\beta(x) = \frac{1}{3}x - \frac{1}{3}$

Theorem 0.3.4.

If $\alpha : A \to B$ has an inverse, then the inverse mapping is unique.

Proof.

Let $\alpha : A \to B$ be a mapping with an inverse. Let β and β' be two inverses of α. We compute

$$\begin{array}{rcll} \beta & = & \beta 1_B & \text{Theorem 0.3.3(a)} \\ & = & \beta(\alpha\beta') & \beta' \text{ inverse of } \alpha \\ & = & (\beta\alpha)\beta' & \text{Theorem 0.3.3(b)} \\ & = & 1_A\beta' & \beta \text{ inverse of } \alpha \\ & = & \beta' & \text{Theorem 0.3.3(a).} \end{array}$$

Therefore $\beta = \beta'$, and the inverse of α is unique. □

Result.

The notation α^{-1} is valid.

Theorem 0.3.5.

Let $\alpha : A \to B$ and $\beta : B \to C$ denote mappings.

1. The identity map, $1_A : A \to A$ is invertible and $1_A^{-1} = 1_A$.
2. If α is invertible, then α^{-1} is invertible and $(\alpha^{-1})^{-1} = \alpha$.
3. If α and β are both invertible, then $\beta\alpha$ is invertible with $(\beta\alpha)^{-1} = \alpha^{-1}\beta^{-1}$.

Invertibility Theorem (Theorem 0.3.6).

A mapping $\alpha : A \to B$ is invertible if and only if α is a bijection.