Previously.

- Groups
- Properties of Groups

This Section.

- Subgroups
- The Subgroup Test
- The center of a group, $Z(G)$

Definition. A subset H of a group G is call a `subgroup` of G if H is also a group using the same operation as G. We denote subgroups using the notation $H \leq G$.
If $H \leq G$ and $H \neq G$, we call H a `proper subgroup of G`.

Example. Every group G with two or more elements automatically has two subgroups:

(a) $G \leq G$

(b) $\{e\} \leq G$ (the `trivial subgroup`)

Example. (a) $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{R}, +)$

(b) $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$

(c) $(A_3, \circ)$ is a subgroup of $(S_3, \circ)$
Recall: $S_3 = \{\varepsilon, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ and $A_3 = \{\varepsilon, (1\ 2\ 3), (1\ 3\ 2)\}$

Exercise 1. Is it a Subgroup?!

$$\text{Is } (\mathbb{Q} \setminus \{0\}, \cdot) \leq (\mathbb{R}, +)?$$

Exercise 2. Is it a Subgroup?!

$$\text{Is } (\{\bar{0}, \bar{2}\}, +) \leq (\mathbb{Z}_4, +)?$$

Exercise 3. Is it a Subgroup?!

$$\text{Is } (\mathbb{Z}_2, +) \leq (\mathbb{Z}_3, +)?$$

Exercise 4. Is it a Subgroup?!

$$\text{Is } (\{i, -i\}, \cdot) \leq (\mathcal{U}_4, \cdot)?$$

Recall $\mathcal{U}_4 = \{1, i, -1, -i\}$.

Exercise 5. Is it a Subgroup?!

$$\text{Is } (\mathbb{Z}_2, +) \leq (\mathbb{Z}_2 \times \mathbb{Z}_2, \text{component-wise } +)?$$

Exercise 6. Is it a Subgroup?!

$$\text{Is } (\{\varepsilon, (1\ 2)\}, \circ) \leq (S_3, \circ)?$$

Subgoup Test (Theorem 2.3.1). A subset H of a group G is a subgroup of G if and only if the following conditions are satisfied.

1. $1_G \in H$, where 1_G is the identity element of G.

2. If $h \in H$ and $h_1 \in H$, then $hh_1 \in H$.

3. If $h \in H$, then $h^{-1} \in H$, where $h^{-1} \in G$ denotes the inverse of h in G.

Note that implicit in these statements, if $H \leq G$ then H and G have the same unity and inverses persist.

Exercise 7. Show $(2\mathbb{Z}, +) \leq (\mathbb{Z}, +)$ using the subgroup test.

Exercise 8. Is $(2\mathbb{Z}+1, +) \leq (\mathbb{Z}, +)$?

Exercise 9. We have the `special linear group` and `general linear group`, respectively below:

$$SL_2(\mathbb{R}) = \{M \in Mat_n(\mathbb{R}) |\, \det M = 1\},$$
$$GL_2(\mathbb{R}) = \{M \in Mat_n(\mathbb{R}) |\, \det M \neq 0\}.$$

Show $(SL_2(\mathbb{R}), \cdot) \leq (GL_2(\mathbb{R}), \cdot)$

Definition. The `center` of the group G is the set

$$Z(G) = \{z \in G | zg = gz \ \forall g \in G\}.$$

Question 10. How would you describe $Z(G)$ with words?

Theorem 2.3.3. If G is any group, then $Z(G)$ is a subgroup of G.
Moreover, $Z(G)$ is always abelian.

Proof.

- (Identity)
- (Closure)
- (Inverses)
  - (Abelian)

Definition. Given $H \leq G$ and $g \in G$, we define the set

$$gHg^{-1} = \{ghg^{-1} \mid h \in G\}$$

Proposition. If $H \leq G$ and $g \in G$, then $gHg^{-1} \leq G$.
Thus, we can call gHg^{-1} the `conjugate subgroup of` H `by` g.

Exercise 11. Consider $G = S_3$ and $H = \langle (1\ 2) \rangle = \{\varepsilon, (1\ 2)\}$. Let $g = (1\ 2\ 3)$. Compute the conjugate subgroup of H by g.

Proposition. If $H, K \leq G$ then $H \cap K \leq G$.

Exercise 12. Let $G = \mathbb{Z}$, $H = 2\mathbb{Z}$ and $K = 3\mathbb{Z}$. What is $H \cap K$?