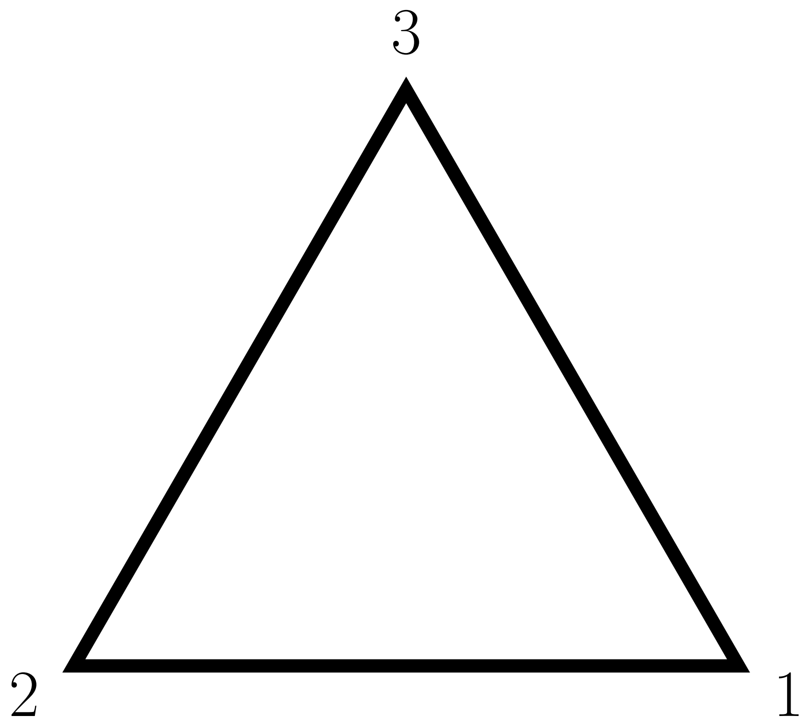
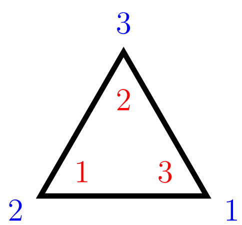
Math 425: Abstract Algebra I
Symmetries of Regular $n$-gons

**Part 1: Symmetries of a Regular Triangle**
Instructions: Cut yourself a triangle that is this same shape and size. (One is provided on the last page of this document.) Label its vertices (corners) with the numbers 1, 2, 3 on both the front and the back.




In class, we're going to explore the symmetries of this triangle (a regular 3-gon). The group of which is called $D_3$.
Let $r$ = clockwise rotation by 120° and $f$ = flip along the vertical axis.
Note that we can think of these $r$ and $f$ as permutations of $\{1, 2, 3\}$.
For example, consider $r$, then using two-line notation we have.




$\leftrightarrow \quad r = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Let's complete the following table

| $Elt$ | $Two-line$ | $cycle$ | $Symmetry$ |
|---|---|---|---|
| $e$ | $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ | $\varepsilon$ | do nothing |
| $r$ | $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ | $(1\ 2\ 3)$ | 120° clockwise |
| $r^2$ | $\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$ | | |

| $Elt$ | $Two-line$ | $cycle$ | $Symmetry$ |
|---|---|---|---|
| $f$ | $\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$ | | |
| $fr$ | $\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$ | | |
| $fr^2$ | $\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$ | | |

1. Did we miss any other symmetries? Why or why not? Is this set of symmetries equal to $S_3$?

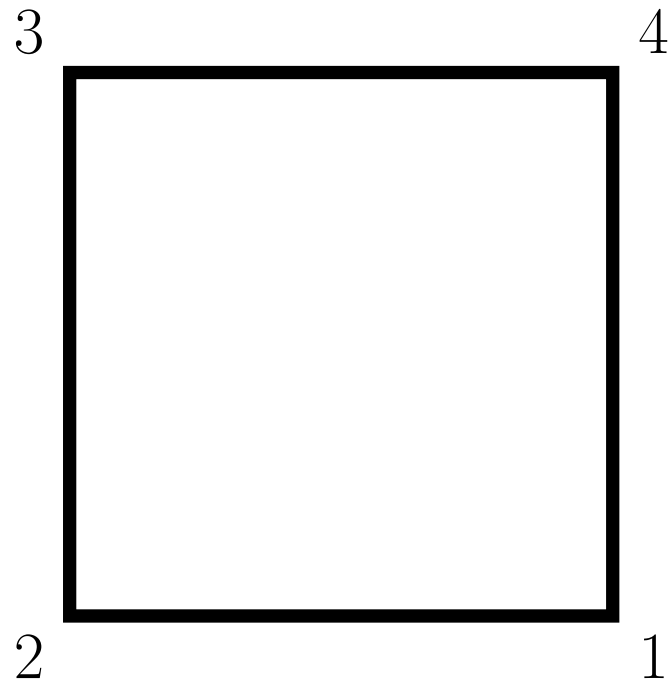
2. Use your cut-out triangle to find the symmetry $r^2frf^3r^4f$, in 2-line notation $\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$.

3. Use your cut-out triangle to prove that $f^2 = e$, $r^3 = e$, and $frf = r^2$.

4. Use the identities in question 3 to reduce $r^2frf^3r^4f$ to $r^2f$. Can you use an identity to make $r^2f$ look like one of the 6 symmetries in the table?

## Part 2: Symmetries of a Regular Rectangle (aka Square)

Instructions: Cut yourself a square that is this same shape and size. (One is provided on the last page of this document.) Label its vertices (corners) with the numbers 1, 2, 3, 4 on both the front and the back.

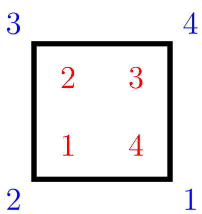



In class, we're going to explore the symmetries of this square (a regular 4-gon). The group of which is called $D_4$.

Let $r$ = clockwise rotation by 90° and $f$ = flip along the vertical axis.

Note that we can think of these $r$ and $f$ as permutations of $\{1, 2, 3, 4\}$.

For example, consider $r$, then using two-line notation we have.




$$\leftrightarrow \quad r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Let's complete the following table

| *Elt* | *Two − line* | *cycle* | *Symmetry* |
|---|---|---|---|
| $e$ | $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ | $\varepsilon$ | do nothing |
| $r$ | $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ | (1 2 3 4) | 90° clockwise |
| $r^2$ | $\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$ | | |
| $r^3$ | $\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$ | | |

| *Elt* | *Two − line* | *cycle* | *Symmetry* |
|---|---|---|---|
| $f$ | $\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$ | | |
| $fr$ | $\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$ | | |
| $fr^2$ | $\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$ | | |
| $fr^3$ | $\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$ | | |

1. Did we miss any other symmetries? Why or why not? Is this set of symmetries equal to $S_4$?

2. Use your cut-out square to find the symmetry $rfrfr^{-3}$, in 2-line notation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$.

3. Use your cut-out square to prove that $f^2 = e$, $r^4 = e$, and $frf = r^3$.

4. Use the identities in question 3 to reduce $rfrfr^{-1}$ to $r$.

5. Is $fr^3f = r = r^{-3}$? Is $fr^2f = r^2 = r^{-1}$? Make a conjecture about $fr^if$ for $i \in \mathbb{Z}$.

**PRINT AND CUT** Then duplicate the numbers on the reverse side of the shape too.