Laplacian of a Symmetric Rank-2 Tensor in Spherical Coordinates

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October 4, 2018

1 Components

$$\left(\nabla^2\mathbf{Q}\right)_{rr} = \nabla^2_{sph}Q_{rr} - \frac{4}{r^2}\left[\frac{\partial Q_{r\theta}}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial Q_{r\phi}}{\partial\phi}\right] + \frac{2}{r^2}\left[-2Q_{rr} + Q_{\theta\theta} + Q_{\phi\phi} - \cot\theta Q_{r\theta}\right] \tag{1}$$

$$\begin{aligned}\left(\nabla^2\mathbf{Q}\right)_{r\theta} = \nabla^2_{sph}Q_{r\theta} &- \frac{2}{r^2}\left[\frac{\partial Q_{rr}}{\partial\theta} + \frac{\partial Q_{\theta\theta}}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial Q_{\theta\phi}}{\partial\phi} + \frac{\cos\theta}{\sin^2\theta}\frac{\partial Q_{r\phi}}{\partial\phi}\right] \\ &+ \frac{1}{r^2}\left[2\cot\theta Q_{\phi\phi} - \cot\theta Q_{rr} - \cot\theta Q_{\theta\theta} - (5+\cot^2\theta)Q_{r\theta}\right]\end{aligned} \tag{2}$$

$$\begin{aligned}\left(\nabla^2\mathbf{Q}\right)_{r\phi} = \nabla^2_{sph}Q_{r\phi} &+ \frac{2}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial Q_{rr}}{\partial\phi} + \frac{\cos\theta}{\sin^2\theta}\frac{\partial Q_{r\theta}}{\partial\phi} - \frac{\partial Q_{\theta\phi}}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial Q_{\phi\phi}}{\partial\phi}\right] \\ &- \frac{1}{r^2}\left[(5+\cot^2\theta)Q_{r\phi} + 3\cot\theta Q_{\theta\phi}\right]\end{aligned} \tag{3}$$

$$\begin{aligned}\left(\nabla^2\mathbf{Q}\right)_{\theta\theta} = \nabla^2_{sph}Q_{\theta\theta} &+ \frac{4}{r^2}\left[\frac{\partial Q_{r\theta}}{\partial\theta} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial Q_{\theta\phi}}{\partial\phi}\right] \\ &+ \frac{2}{r^2}\left[Q_{rr} - \cot\theta Q_{r\theta} - \frac{1}{\sin^2\theta}Q_{\theta\theta} + 2\cot^2 Q_{\phi\phi}\right]\end{aligned} \tag{4}$$

$$\begin{aligned}\left(\nabla^2\mathbf{Q}\right)_{\theta\phi} = \nabla^2_{sph}Q_{\theta\phi} &+ \frac{2}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial Q_{r\theta}}{\partial\phi} + \frac{\partial Q_{r\phi}}{\partial\theta} + \frac{\cos\theta}{\sin^2\theta}\frac{\partial Q_{\theta\phi}}{\partial\phi} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial Q_{\phi\phi}}{\partial\phi}\right] \\ &- \frac{1}{r^2}\left[3\cot\theta Q_{r\phi} + 2\cot^2\theta Q_{\theta\phi} + \frac{1}{\sin^2\theta}Q_{\theta\phi}\right]\end{aligned} \tag{5}$$

$$\begin{aligned}\left(\nabla^2\mathbf{Q}\right)_{\phi\phi} = \nabla^2_{sph}Q_{\phi\phi} &+ \frac{4}{r^2}\left[\frac{\partial Q_{r\phi}}{\partial\phi} + \cot\theta\frac{\partial Q_{\theta\phi}}{\partial\phi}\right] \\ &+ \frac{2}{r^2}\left[Q_{rr} - \frac{1}{\sin^2\theta}Q_{\phi\phi} + \cot^2\theta Q_{\theta\theta} + 2\cot\theta Q_{r\theta}\right]\end{aligned} \tag{6}$$

2 Derivation

$$
\begin{aligned}
\nabla \mathbf{Q} &= \nabla_k \big[Q_{ij} \hat{e}_i \otimes \hat{e}_j \big] \otimes \hat{e}_k \\
&= \big[(\nabla_k Q_{ij}) \hat{e}_i \otimes \hat{e}_j + Q_{ij} (\nabla_k \hat{e}_i) \otimes \hat{e}_j + Q_{ij} \hat{e}_i \otimes (\nabla_k \hat{e}_j) \big] \otimes \hat{e}_k \\
&= \big[Q_{ij,k} \hat{e}_i \otimes \hat{e}_j + Q_{ij} \Gamma_{ilk} \hat{e}_l \otimes \hat{e}_j + Q_{ij} \Gamma_{ljk} \hat{e}_i \otimes \hat{e}_l \big] \otimes \hat{e}_k \\
&= \big[Q_{ij,k} + Q_{lj} \Gamma_{ilk} + Q_{il} \Gamma_{jlk} \big] \hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k \\
&= Q_{ij;k} \quad \hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\nabla^2 \mathbf{Q} &= \nabla \cdot \nabla \mathbf{Q} \\
&= \nabla_m [Q_{ij;k} \hat{e}_i \otimes \hat{e}_j] \hat{e}_m \cdot \hat{e}_k \\
&\qquad \vdots \quad \textit{Apply product rule} \\
&= \big[Q_{ij,kk} + 2Q_{lj,k} \Gamma_{ilk} + 2Q_{il,k} \Gamma_{jlk} + Q_{lj} \Gamma_{ipk} \Gamma_{plk} + Q_{pl} \Gamma_{ipk} \Gamma_{jlk} \\
&\quad + Q_{il} \Gamma_{jpk} \Gamma_{plk} + Q_{lp} \Gamma_{jpk} \Gamma_{ilk} + Q_{lj} (\nabla_k \Gamma_{ilk}) + Q_{il} (\nabla_k \Gamma_{jlk}) \big] \hat{e}_i \otimes \hat{e}_j
\end{aligned} \tag{8}
$$

3 Useful Information

$$\nabla_k \hat{e}_j = \Gamma_{ijk} \hat{e}_i \tag{9}$$

$$\Gamma_{\theta r \theta} = \Gamma_{\phi r \phi} = -\Gamma_{r \theta \theta} = -\Gamma_{r \phi \phi} = \frac{1}{r}, \qquad \Gamma_{\phi \theta \phi} = -\Gamma_{\theta \phi \phi} = \frac{\cot \theta}{r} \tag{10}$$

$$\nabla_r = \frac{\partial}{\partial r}, \quad \nabla_\theta = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \nabla_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tag{11}$$

$$\nabla^2_{sph} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \tag{12}$$

$$Q_{ij} = Q_{ji} \quad \Rightarrow \quad \left(\nabla^2 \mathbf{Q} \right)_{ij} = \left(\nabla^2 \mathbf{Q} \right)_{ji} \tag{13}$$

4 References

Kip Thorne and Roger Blandford, 2017, *Modern Classical Physics*, Chapters 1 and 11

See also http://www.pmaweb.caltech.edu/Courses/ph136/yr2012/

To calculate this expression for yourself, the following *Mathematica* notebook may be helpful: