# Laplacian of a Symmetric Rank-2 Tensor in Spherical Coordinates

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## 1 Components

$$\left(\nabla^2 \mathbf{Q}\right)_{rr} = \nabla^2_{sph} Q_{rr} - \frac{4}{r^2}\left[\frac{\partial Q_{r\theta}}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial Q_{r\phi}}{\partial \phi}\right] + \frac{2}{r^2}\left[-2Q_{rr} - \cot\theta Q_{r\theta} + Q_{\theta\theta} + Q_{\phi\phi}\right] \tag{1}$$

$$\begin{aligned}\left(\nabla^2 \mathbf{Q}\right)_{r\theta} = \nabla^2_{sph} Q_{r\theta} &+ \frac{2}{r^2}\left[\frac{\partial Q_{rr}}{\partial \theta} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial Q_{r\phi}}{\partial \phi} - \frac{\partial Q_{\theta\theta}}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial Q_{\theta\phi}}{\partial \phi}\right] \\ &+ \frac{1}{r^2}\left[-\cot\theta Q_{rr} - (5+\cot^2\theta)Q_{r\theta} - \cot\theta Q_{\theta\theta} + 2\cot\theta Q_{\phi\phi}\right]\end{aligned} \tag{2}$$

$$\begin{aligned}\left(\nabla^2 \mathbf{Q}\right)_{r\phi} = \nabla^2_{sph} Q_{r\phi} &+ \frac{2}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial Q_{rr}}{\partial \phi} + \frac{\cos\theta}{\sin^2\theta}\frac{\partial Q_{r\theta}}{\partial \phi} - \frac{\partial Q_{\theta\phi}}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial Q_{\phi\phi}}{\partial \phi}\right] \\ &- \frac{1}{r^2}\left[(5+\cot^2\theta)Q_{r\phi} + 3\cot\theta Q_{\theta\phi}\right]\end{aligned} \tag{3}$$

$$\begin{aligned}\left(\nabla^2 \mathbf{Q}\right)_{\theta\theta} = \nabla^2_{sph} Q_{\theta\theta} &+ \frac{4}{r^2}\left[\frac{\partial Q_{r\theta}}{\partial \theta} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial Q_{\theta\phi}}{\partial \phi}\right] \\ &+ \frac{2}{r^2}\left[Q_{rr} - \cot\theta Q_{r\theta} - \frac{1}{\sin^2\theta}Q_{\theta\theta} + \cot^2\theta Q_{\phi\phi}\right]\end{aligned} \tag{4}$$

$$\begin{aligned}\left(\nabla^2 \mathbf{Q}\right)_{\theta\phi} = \nabla^2_{sph} Q_{\theta\phi} &+ \frac{2}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial Q_{r\theta}}{\partial \phi} + \frac{\partial Q_{r\phi}}{\partial \theta} + \frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial \phi}\Big(Q_{\theta\theta} - Q_{\phi\phi}\Big)\right] \\ &- \frac{1}{r^2}\left[3\cot\theta Q_{r\phi} + \Big(2 + 4\cot^2\theta\Big)Q_{\theta\phi}\right]\end{aligned} \tag{5}$$

$$\begin{aligned}\left(\nabla^2 \mathbf{Q}\right)_{\phi\phi} = \nabla^2_{sph} Q_{\phi\phi} &+ \frac{4}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial Q_{r\phi}}{\partial \phi} + \frac{\cos\theta}{\sin^2\theta}\frac{\partial Q_{\theta\phi}}{\partial \phi}\right] \\ &+ \frac{2}{r^2}\left[Q_{rr} + 2\cot\theta Q_{r\theta} + \cot^2\theta Q_{\theta\theta} - \frac{1}{\sin^2\theta}Q_{\phi\phi}\right]\end{aligned} \tag{6}$$

## 2 Derivation

$$\begin{aligned}
\nabla \mathbf{Q} &= \nabla_k \big[Q_{ij}\hat{e}_i \otimes \hat{e}_j\big] \otimes \hat{e}_k \\
&= \big[(\nabla_k Q_{ij})\hat{e}_i \otimes \hat{e}_j + Q_{ij}(\nabla_k \hat{e}_i) \otimes \hat{e}_j + Q_{ij}\hat{e}_i \otimes (\nabla_k \hat{e}_j)\big] \otimes \hat{e}_k \\
&= \big[Q_{ij,k}\hat{e}_i \otimes \hat{e}_j + Q_{ij}\Gamma_{ilk}\hat{e}_l \otimes \hat{e}_j + Q_{ij}\Gamma_{ljk}\hat{e}_i \otimes \hat{e}_l\big] \otimes \hat{e}_k \\
&= \big[Q_{ij,k} + Q_{lj}\Gamma_{ilk} + Q_{il}\Gamma_{jlk}\big]\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k \\
&= Q_{ij;k} \quad \hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k
\end{aligned} \tag{7}$$

$$\begin{aligned}
\nabla^2 \mathbf{Q} &= \nabla \cdot \nabla \mathbf{Q} \\
&= \nabla_m [Q_{ij;k}\hat{e}_i \otimes \hat{e}_j]\hat{e}_m \cdot \hat{e}_k \\
& \qquad \vdots \quad \textit{Apply product rule} \\
&= \big[Q_{ij,kk} + 2Q_{lj,k}\Gamma_{ilk} + 2Q_{il,k}\Gamma_{jlk} + Q_{lj}\Gamma_{ipk}\Gamma_{plk} + Q_{pl}\Gamma_{ipk}\Gamma_{jlk} \\
&\quad + Q_{il}\Gamma_{jpk}\Gamma_{plk} + Q_{lp}\Gamma_{jpk}\Gamma_{ilk} + Q_{lj}(\nabla_k \Gamma_{ilk}) + Q_{il}(\nabla_k \Gamma_{jlk})\big]\hat{e}_i \otimes \hat{e}_j
\end{aligned} \tag{8}$$

## 3 Useful Information

$$\nabla_k \hat{e}_j = \Gamma_{ijk}\hat{e}_i \tag{9}$$

$$\Gamma_{\theta r\theta} = \Gamma_{\phi r\phi} = -\Gamma_{r\theta\theta} = -\Gamma_{r\phi\phi} = \frac{1}{r}, \qquad \Gamma_{\phi\theta\phi} = -\Gamma_{\theta\phi\phi} = \frac{\cot\theta}{r} \tag{10}$$

$$\nabla_r = \frac{\partial}{\partial r}, \quad \nabla_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}, \quad \nabla_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{11}$$

$$\nabla^2_{sph} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{12}$$

$$Q_{ij} = Q_{ji} \quad \Rightarrow \quad \big(\nabla^2\mathbf{Q}\big)_{ij} = \big(\nabla^2\mathbf{Q}\big)_{ji} \tag{13}$$

## 4 References


Kip Thorne and Roger Blandford, 2017, *Modern Classical Physics*, Chapters 1 and 11

See also http://www.pmaweb.caltech.edu/Courses/ph136/yr2012/


To calculate these expressions for yourself, the following *Mathematica* notebook may be helpful: “Laplacian of a Tensor in Spherical Coordinates Mathematica Notebook” which is available on https://mckimb.github.io/code/