Analytic Mechanics Notes

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Contents

I. Energy (Ch. 4)

Kinetic Energy and Work

- Basic facts for particle mass m speed v:

$$T = \frac{1}{2}mv^2 \tag{1}$$
$$\frac{\mathrm{d}T}{\mathrm{d}t} = m\dot{\boldsymbol{v}} \cdot \boldsymbol{v} = \mathcal{F} \cdot \boldsymbol{v} \tag{2}$$
$$\mathrm{d}T = \mathcal{F} \cdot d\mathbb{R} \tag{3}$$
$$\Delta T = \int_1^2 \mathcal{F} \cdot \mathrm{d}\mathbb{R} \equiv W(1 \to 2) \tag{4}$$

where the last equation is the **Work-KE theorem**: the change in T between two points on its path equals the work done by the **net** force as it moves between the two points.

Potential E/Conservative F

- To define $U(\mathbb{R})$ corresponding to a given conservative force, first choose reference point $\mathbb{R}_0$ at which U defined to be zero. Then

$$U(\mathbb{R}) \equiv -W(\mathbb{R}_0 \to \mathbb{R}) \equiv -\int_{\mathbb{R}_0}^{\mathbb{R}} \mathcal{F}(\mathbb{R}') \cdot \mathrm{d}\mathbb{R}' \tag{4.13}$$

- Principle of **Conservation of Energy** for One particle:

  If all the n forces $\mathcal{F}_i$ acting on a particle are conservative, each with its corresponding potential energy $U_i(\mathbb{R})$, the **total mechanical energy** defined as

$$E \equiv T + U \equiv T + U_1(\mathbb{R}) + \cdots + U_n(\mathbb{R}) \tag{5}$$

  is constant in time.

- For some general scalar multivariable function f, the change in f resulting from a small displacement $\mathrm{d}\mathbb{R}$ is just

$$df = \nabla f \cdot \mathrm{d}\mathbb{R} \tag{6}$$

- Remember that if any $\mathcal{F}$ is conservative that $\nabla \times \mathcal{F} = 0$.

2-Particle Interaction

- Interaction force only a function of $\mathbb{R}_1 - \mathbb{R}_2$, i.e.[1]

$$\mathcal{F}_{12} = \mathcal{F}_{12}(\mathbb{R}_1 - \mathbb{R}_2) \tag{7}$$

and so we can learn almost everything about $\mathcal{F}_{12}$ by *fixing* $\mathbb{R}_2$ *at any convenient point.*

- How to take gradients/relationship with Newton's third law:

$$\mathcal{F}_{12} = -\nabla_1 U(\mathbb{R}_1 - \mathbb{R}_2) \tag{8}$$
$$\nabla_1 U(\mathbb{R}_1 - \mathbb{R}_2) = -\nabla_2 U(\mathbb{R}_1 - \mathbb{R}_2) \tag{9}$$
$$\mathcal{F}_{21} = -\nabla_2 U(\mathbb{R}_1 - \mathbb{R}_2) \tag{10}$$

- Total energy $E = T_1 + T_2 + U$ is conserved[2].
- Using math and the KE theorem, we can get

$$W_{tot} = -d\mathbb{R} \cdot \nabla U(\mathbb{R}) = -dU \tag{11}$$

as the total work done: the work done by $\mathcal{F}_{12}$ during $d\mathbb{R}_1$ plus the work done by $\mathcal{F}_{21}$ during $d\mathbb{R}_2$.

[1] Notation: $\mathcal{F}_{12}$ is force on particle 1 by particle 2.

[2] U only included once since it apparently "takes both of [the kinetic energy terms] into account."

II. Oscillations (Ch. 5)

Hooke's Law

- Force exerted by a spring (1D):

$$F_x(x) = -kx \tag{12}$$

where x is displacement from equilib. $k > 0$ and so **stable equilibrium**.

- Equivalent way to state Hooke's law:

$$U(x) = \frac{1}{2}kx^2 \tag{13}$$

which can be obtained from expanding $U(x)$ in Taylor series:

$$\begin{aligned} U(x) &= U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 + \cdots && (14) \\ &= U'(0)x + \frac{1}{2}U''(0)x^2 && (15) \\ &= \frac{1}{2}U''(0)x^2 && (16) \end{aligned}$$

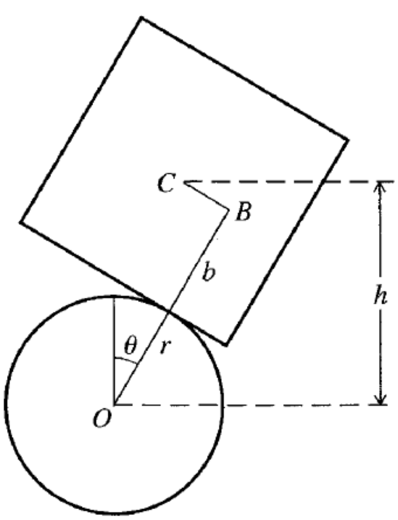


Figure 4.14 A cube, of side $2b$ and center C, is placed on a fixed horizontal cylinder of radius r and center O. It is originally put so that C is centered above O, but it can roll from side to side without slipping.

Figure 1

- **Example 4.7/5.1**: Cube balanced on a cylinder (figure 1)
  - Clarifications/confusions:
    - CYLINDER IS FIXED. In all ways possible. The *cube* can rock back and forth (because apparently it is sentient) but the cylinder itself does not rotate.
    - The angle θ is referring to the angle the cube has rotated.
    - Author claims that BC is "distance the cube has rolled around the cylinder, namely $r\theta$". Neither his explanation nor his expression are correct.
  - **(4.7) Goal:** *By examining the cube's potential energy, find out if the equilib with the cube centered above the cylinder is stable or unstable.*
  - Forces: Constraining forces: the normal and frictional forces of the cylinder on the cube; do no work so ignore. Only other force is **gravity**. *Grav potential energy is the same as for a point mass at the center of the cube.*
  - Suppose we want to stroke Taylor's cock and let $BC = r\theta$, even though it obviously is not, then

$$h = (r+b)\cos\theta + r\theta\sin\theta \tag{17}$$

  - Set $dU/d\theta = 0$ to find equilibrium position(s). it is at $\theta = 0$.
  - Check stability by seeing if $d^2U/d\theta^2 > 0$

$$\frac{d^2U}{d\theta^2} = mg(r-b) \tag{18}$$

which is stable for $b < r$.

  - **(5.1) Goal**: *Show that for small angles θ the potential energy takes the Hooke's-law form $U(\theta) = \frac{1}{2}k\theta^2$.*
  - If angle is small, expand cos/sin up to second-order and substitute in:

$$U(\theta) \approx mg(r+b) + \frac{1}{2}mg(r-b)\theta^2 \tag{19}$$

and thus blah blah who cares.

- General features of motion can be understood when recalling that $U(x)$ is a parabola. If an object has energy $E > 0$, it is common to denote the values $x = \pm A$, where A denotes the displacement from equilibrium where $U(x = \pm A) = E$ and the kinetic energy is instantaneously zero. A is called the **amplitude of the oscillations**.

Simple Harmonic Motion

- Examine Newton's second law for a mass m displaced from stable equilibrium.

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x \tag{20}$$

where ω is the angular frequency at which cart will oscillate.

Exponential Solutions

- The linear combination of the two independent solutions for equation 20 can be written as

$$x(t) = C_1 \exp^{i\omega t} + C_2 \exp^{-i\omega t} \tag{21}$$

- Since $x(t) \in \mathbb{R}$, need to choose C_1, C_2 carefully to ensure this.

Sine and Cosine Solutions

- Can use euler's formula on eq. 21 to rewrite as

$$x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t) \tag{22}$$

where $B_1 = C_1 + C_2$ and $B_2 = i(C_1 - C_2)$. This form is the definition of **S.H.M.**

- $x(t)$ is real iff B_1 and B_2 are real.
- Finding coefficients: Set t=0 to get $B_1 = x_0$. Take deriv to identify $\omega B_2 = v_0$.
- Period of $x(t)$ is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{23}$$

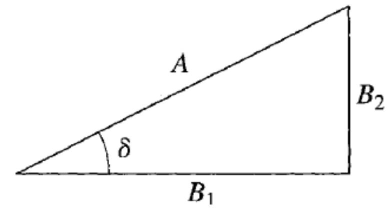


Figure 2

Phase-Shifted Cosine Solution

- Let $A = \sqrt{B_1^2 + B_2^2}$ as shown in triangle above.
- Now we have the cute form

$$\begin{aligned} x(t) &= A\left[\frac{B_1}{A}\cos(\omega t) + \frac{B_2}{A}\sin(\omega t)\right] \\ &= A\left[\cos\delta\cos(\omega t) + \sin\delta\sin(\omega t)\right] \\ &= A\cos(\omega t - \delta) \end{aligned} \tag{24}$$

Sln Real Part of Complex Exp

- Rewrite equation 21 by doing the following bullshit:

$$C_1 = \frac{1}{2}(B_1 - iB_2) \quad \textit{and} \quad C_2 = \frac{1}{2}(B_1 + iB_2) \tag{25}$$

$$C_2 = C_1^* \tag{26}$$

- Now we can write x as the following bullshit[3]

$$x(t) = C_1 \exp^{i\omega t} + C_1^* \exp^{-i\omega t} \tag{27}$$

$$= 2\Re[C_1 \exp^{i\omega t}] \tag{28}$$

- We can even do just a *little* more bullshit related to figure 2 to obtain

$$x(t) = 2\Re[C_1 \exp^{i\omega t}] = \Re[A \exp^{i(\omega t - \delta)}] \tag{29}$$

Energy Considerations

- For the simple case of eq. 24, the potential U and kinetic energy T are

$$U = \frac{1}{2}kA^2\cos^2(\omega t - \delta) \tag{30}$$

$$T = \frac{1}{2}m\dot{x} = \frac{1}{2}kA^2\sin^2(\omega t - \delta) \tag{31}$$

and the total energy is constant (true for any conservative force):

$$E = T + U = \frac{1}{2}kA^2 \tag{32}$$

2D Oscillators

- Simplest case: **isotropic harmonic oscillator.** Restoring force is direction-independent.

[3]For the retards reading this, recall $z + z^* = 2\Re z$

– Restoring force:

$$\boldsymbol{F} = -k\boldsymbol{r} \tag{33}$$

.

– Technique/steps for analyzing 2D case: First, write force equations in components.

$$\begin{aligned} \ddot{x} &= \omega^2 x \\ \ddot{y} &= \omega^2 y \end{aligned} \tag{34}$$

– We can write these in the same form as eq 24, but define the origin of time such that we can dispose of x phase shift δ_x.

$$\begin{aligned} x(t) &= A_x \cos(\omega t) \\ y(t) &= A_y \cos(\omega t - \delta) \end{aligned} \tag{5.20}$$

where $\delta = \delta_y - \delta_x$ is the **relative phase** of the y and x oscillations.

- In the case of **anisotropic oscillator**, we have

$$F_x = -k_x x \quad F_y = -k_y y \quad F_z = -k_z z \tag{5.21}$$

Damped Oscillations

- "...resistive forces that will damp the oscillations."
- Assume resistive force is proportional to v: $\boldsymbol{f} = -b\boldsymbol{v}$.
- Consider a cart attached to a spring.
  – Newton's second law:

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{5.24}$$

  – Define the following β as the **damping constant**.

$$\frac{b}{m} = 2\beta \tag{5.26}$$

which characterizes strength of damping force.

  – Denote ω as ω_0 now, to distinguish it as the **natural frequency**, freq. it oscillates w/o resistive forces present. $\omega_0 = \sqrt{k/m}$.
  – Equation of motion for damped oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{5.28}$$

- **Math** to solve the ODE of eqn 5.28:
  – Let $x(t) = \exp^{rt}$. Plugging in yields the **auxiliary equation** of 5.28.

$$r^2 + 2\beta\, r + \omega_0^2 = 0 \tag{5.30}$$

  – With the two solutions

$$\begin{aligned} r_1 &= -\beta + \sqrt{\beta^2 - \omega_0^2} \\ r_2 &= -\beta - \sqrt{\beta^2 - \omega_0^2} \end{aligned} \tag{5.31}$$

  – Thus, we have the general solution of 5.28 as

$$x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2 - \omega_0^2}\, t} + C_2 e^{-\sqrt{\beta^2 - \omega_0^2}\, t}\right) \quad (5$$

- If **undamped**, $\beta = 0$.
- **Weak Damping**: $\beta < \omega_0$, sometimes called **underdamping**.

$$\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} = i\omega_1 \tag{5.36}$$

and thus we can reduce $x(t)$ to similar form as 24.

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta) \tag{5.38}$$

- **Strong Damping**: $\beta > \omega_0$. (**overdamping**). Rate at which the motion dies out can be charactereized by $\beta - \sqrt{\beta^2 - \omega_0^2}$. Rate of decay *decreases* when β is made bigger.
- **Critical Damping**: $\beta = \omega_0$. Equation 5.33 becomes just

$$x(t) = e^{-\beta t} \tag{5.42}$$

and so we need to find a second independent solution. Turns out it is:

$$x(t) = te^{-\beta t} \tag{5.43}$$

and general solution is

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t} \tag{5.44}$$

Driven Damped Oscillations

- Denote external driving force by $F(t)$ so now we have

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{5.45}$$

and, as before, can let $b/m = 2\beta$ and, also, let $f(t) = F(t)/m$ be force per unit mass. Then

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \tag{5.48}$$

- Denote D as some linear differential operator. Key point is that, for $Dx = f$, there exist two special solutions for x denoted as x_p and x_h for *particular* and homosexual, where

$$D(x_p + x_h) = Dx_p + Dx_h = f + 0 = f$$

Complex solutions for a Sinusoidal Driving Force

- Consider driving force

$$f(t) = f_0 \cos(\omega t) \tag{5.56}$$

where ω denotes angular frequency of the *driving force*.

- Substitute this in eq 5.48, note that $\sin(\omega t)$ is also a solution, and don't be a retard to get

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t} \tag{5.60}$$

where $z(t) = x(t) + iy(t)$, x being solution using cosine, y being the sine solution.

- To solve,

$$\begin{aligned} z(t) &= Ce^{i\omega t} && (35)\\ &\text{then} && (36)\\ (-\omega^2 + 2i\beta\omega + \omega_0^2)\, Ce^{i\omega t} &= f_0 e^{i\omega t} && (37)\\ &\text{is solution if} && (38)\\ C &= \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} && (39) \end{aligned}$$

- Let $C = A^{-i\delta}$ before taking $\Re(z)$. Multiply by complex conj. to get amplitude of the oscillations caused by the driving force $f(t)$:

$$CC^* = A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \tag{5.64}$$

which is biggest when $\omega_0 \approx \omega$.

- Do math to get

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \tag{5.65}$$

- Recall that we are seeking $\Re(z) = x_p(t) = A\cos(\omega t - \delta)$, which is the *particular* solution.

- The **general solution** is

$$x(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{5.67}$$

where the two extra terms are called **transients** because they both die out exponentially with time.

- If we assume weakly damped ($\beta < \omega_0$) then we can write as

$$x(t) = A\cos(\omega t - \delta) + A_{tr} e^{-\beta t}\cos(\omega_1 t - \delta_{tr}) \tag{5.68}$$

where A_{tr} and δ_{tr} are arbitrary constants determined by the initial conditions.

- **Example 5.3**
  - Goal: plot $x(t)$ for a driven (weakly) damped oscillator. Parameters given (see mathematica).
  - Answers to questions I have along the way:
    * **Q**: What is f_0? **A:** It is the amplitude (divided by the oscillator's mass), first seen in equation 5.56 as $f(t) = f_0 \cos\omega t$. **Learned:** Distinguish between driving amplitude f_0 and oscillation amplitude A.
    * **Q**: Where did A_{tr} and δ_{tr} go? No mention of them here/ignored. They let

$$A_{tr} e^{-\beta t}\cos(\omega_1 t - \delta_{tr}) \stackrel{?}{=} e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right] \tag{40}$$

Ans: Aha, appears the deltas have been absorbed via the identity (see 2):

$$A_{tr}\cos(\omega_1 t - \delta_{tr}) = A_{tr}\left[\tfrac{B_1}{A_{tr}}\cos(\omega_1 t) + \tfrac{B_2}{A_{tr}}\sin(\omega_1 t)\right] \tag{41}$$

such that $\frac{B_1}{A_{tr}} = \cos(\delta_{tr})$ and $\frac{B_2}{A_{tr}}\sin(\delta_{tr})$. However, I'm not sure what happened to the outside factor of A_{tr}.

  - Although the general solutions of x in eq 5.67 has the transient terms, the first term, $A\cos(\omega t)$ is called an **attractor** because all solutions will eventually converge to just that term, irrespective of initial conditions. There is a *unique* attractor (for a given driving force).

- *The amplitude and phase of the attractor depend on the params of the driving force, which is the subject of the next section.*

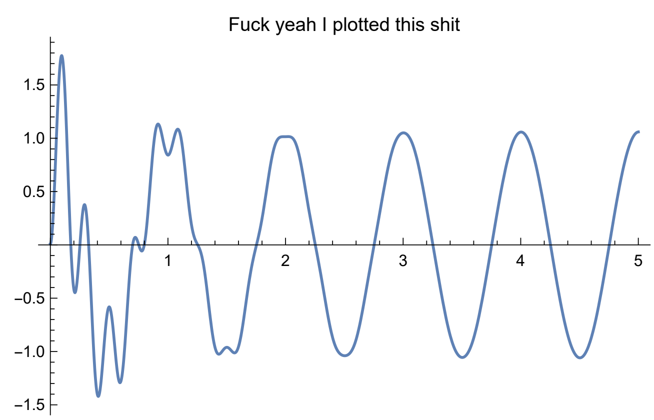


Figure 3: Plot of $x(t)$ from example 5.3

Resonance

- Apart from transient motions that we will ignore, a damped[4] system responds by oscillating at same frequency ω as driver but with different amplitude A:

$$x(t) = A\cos(\omega t - \delta)$$
$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \tag{5.71}$$

- Discuss here the case when damping constant β is very small.
- Although location of A_{max} depends on whether we vary ω_0 or ω, in both cases we want the denominator

$$(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2 \tag{5.72}$$

to be a minimum.

- Summary of frequencies seen thus far:

$$\omega_0 = \sqrt{k/m} = \text{natural/undamped} \tag{42}$$
$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \text{damped} \tag{43}$$
$$\omega = \text{driving freq.} \tag{44}$$
$$\omega_2 = \sqrt{\omega_0^2 - 2\beta^2} \Leftrightarrow \max_{\omega} A \tag{45}$$

where it's important to note that (32) is maximum over ω *with all else fixed*. General maximum is $\omega_0 \approx \omega$ which gives

$$A_{max} \approx \frac{f_0}{2\beta\omega_0} \tag{5.74}$$

The Q Factor

- Decrease $\beta \longrightarrow$ higher and more narrow resonance peak.
- Sharpness of resonance peak, characterized by the **Q factor**, is indicated by the ratio of its FWHM $\approx 2\beta$ to its position ω_0

$$Q = \frac{\omega_0}{2\beta} \tag{5.77}$$

which can be interpreted as *we want the width* 2β *to be very small compared to system's natural frequency* ω_0.

- For weakly damped, $\beta << \omega_0$, we can rewrite definition as

$$Q = \pi\frac{1/\beta}{2\pi/\omega_0} = \pi\frac{\text{decay time}}{\text{period}} \tag{46}$$

Phase at Resonance

- Phase difference by which oscillator lags behind driving force:

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \tag{5.79}$$

[4]appears that damped basically means weakly damped

Fourier Series (5.7)

- Any periodic driving force can be built up from sinusoidal forces using Fourier Series.
- If $f(t)$ is any periodic function with period τ then[5] it can be expressed as the sum

$$f(t) = \sum_{n=0}^{\infty}\Big[a_n\cos(n\omega t) + b_n\sin(n\omega t)\Big] \tag{5.82}$$
$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos(n\omega t)\mathrm{d}t \tag{5.83}$$
$$b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin(n\omega t)\mathrm{d}t \tag{5.84}$$
$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\mathrm{d}t \tag{5.85}$$
(47)

where $n \geq 1$ for equations 5.83 and 5.84

Fourier on Oscillator (5.8)

- Goal: Solve for motion of an oscillator that is driven by an *arbitrary* driving force.
- Equation of motion:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f \tag{48}$$

where β is damping constant, ω_0 natural freq, and $f = f(t)$ is any periodic driving force/mass of period τ.

- Rewrite in compact form: $Dx = f$.
- Procedure for finding $x(t)$:
  1. Find the Fourier coefficients f_n in the Fourier series for $f(t)$.
  2. Calculate the values of A_n and δ_n, defined as:

$$A_n = \frac{f_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2\omega^2}} \tag{5.92}$$
$$\delta_n = \arctan\left(\frac{2\beta_n\omega}{\omega_0^2 - n^2\omega^2}\right) \tag{5.93}$$

  3. Write down $x(t)$ as the following Fourier series:

$$x(t) = \sum_{n=0}^{\infty} A_n\cos(n\omega t - \delta_n) \tag{5.94}$$

[5]Note: $cos(2n\pi t/\tau) = cos(n\omega t)$

III. Calculus of Variations (Ch. 6)

Want an alternative equation of motion that works equally well in any coordinates, and the required alternative is provided by **Lagrange's equations**.

Two Examples (6.1)

Shortest path between two points. The total length of some arbitrary path between points 1 and 2 is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2} dx \tag{6.2}$$

since $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'(x)^2}$. Therefore, the problem is to find $y(x)$ such that eq 6.2 is a minimum.

Fermat's Principle. Required path is the path for which the time of travel of the light is minimum. Note that the refractive index may vary: $n = n(x, y)$. The problem is to find the path $y(x)$ for which the following integral (derivation included) is a minimum.

$$\begin{aligned} \int_1^2 dt = \int_1^2 \frac{ds}{v} &= \frac{1}{c}\int_1^2 n ds & (49) \\ &= \frac{1}{c}\int_1^2 n(x,y) ds = \frac{1}{c}\int_{x_1}^{x_2} n(x,y)\sqrt{1 + y'(x)^2} dx & (6.3) \end{aligned}$$

where we don't necessarily need the $1/c$ for the minimization (it's constant). Remember that the speed of light is $v = c/n$ when traversing medium with index of ref. n.

Note: When $df/dx = 0$ at a point x_0 but we don't know if max, min, or saddle, we say that x_0 is a **stationary point** of the function $f(x)$. Our problem is to find the *paths* that make integrals like eq 6.2 stationary.

Euler-Lagrange Eq (6.2)

The variational problem. Have integral of the form

$$S = \int_{x_1}^{x_2} f\left[y(x), y'(x), x\right] dx \tag{6.4}$$

and want path $y(x)$ from pre-specified point (x_1, y_1) to (x_2, y_2). The following derivation will lead us to the Euler-Lagrange equation, which lets us find the path for which eq 6.4 is stationary.

1. Denote the **correct solution** as $y = y(x)$.
2. Write any other ("wrong path") as

   $$Y(x) = y(x) + \alpha\eta(x) \tag{6.8}$$

   where $\eta(x_1) = \eta(x_2) = 0$. Also notice that this approaches the correct solution as $\alpha \to 0$.
3. Now $S = S(\alpha)$ where we've defined α such that $S(\alpha = 0)$ is a **minimum**.

   $$S(\alpha) = \int_{x_1}^{x_2} f(y + \alpha\eta,\ y' + \alpha\eta', x) dx \tag{6.9}$$

   $$\frac{\partial S}{\partial \alpha} = \int_{x_1}^{x_2} \left(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right) dx = 0 \tag{6.10}$$

   where I'm assuming we can get away with partial derivs being wrt y instead of Y since we are only trying to analyze the case where $\alpha = 0$.
4. Integrate by parts on the second term, take advantage of boundary condition[6], and substitute back in to get

   $$\int_{x_1}^{x_2} \eta(x)\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right) dx = 0 \tag{6.12}$$

5. Since eq. 6.12 must be true for all $\eta(x)$ in range $x_1 \le x \le x_2$, we know that

   $$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0 \tag{6.13}$$

   for all $x_1 \le x \le x_2$. This is the **Euler-Lagrange Equation**.

Procedure: Using EL equation.

1. Set up the problem so that the quantity whose stationary path you seek is expressed in form of eq. 6.4.
2. Write down the EL equation 6.13.
3. Solve this differential equation for the function $y(x)$ that defines the required stationary path.

[6] $\eta(x_1) = \eta(x_2) = 0$

Applications (6.3)

[Ex. 6.1] Shortest Path Between Two Points. Know that the length of the path is given by 6.2. Notice that we *began* with a quantity in the form of an integral here. We can then simply identify

$$f(y, y', x) = \left(1 + y'^2\right)^{1/2} \tag{6.15}$$

Since

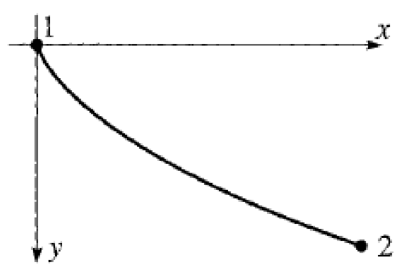
$$\frac{\partial f}{\partial y} = 0 \text{ and } \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \tag{6.16}$$

we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0 \rightarrow \frac{\partial f}{\partial y'} = \text{const} \tag{50}$$

which, after doing algebra, gives us that $y'(x)^2$ is constant as well, i.e. that $y(x) = mx + b$ after integration, yielding the equation for a straight line.[7]

[Ex 6.2] Brachistochrone. Problem: What shape should we build roller coaster track such that car released from point 1 reaches point 2 in shortest possible time?



Since $v(y) = \sqrt{2gy}$, take y as the independent variable and write path as $x = x(y)$. Therefore, the time to travel from 1 to 2 is

$$t_{12} = \int_1^2 \frac{ds}{v} \tag{6.17}$$

$$= \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{x'^2+1}}{\sqrt{y}} dy \tag{6.19}$$

Again, when we evaluate the derivatives in the EL equation, we end up with $\frac{\partial f}{\partial x'}$ as a constant. Let

$$\left(\frac{\partial f}{\partial x'}\right)^2 = \frac{x'^2}{y(1+x'^2)} = const = \frac{1}{2a} \tag{6.22}$$

which, after solving for x', we can feed into the following to obtain the path $x(y)$:

$$x(y) = \int x'(y) dy \tag{51}$$

$$= \int \sqrt{\frac{y}{2a-y}} dy \tag{6.23}$$

Let $y = a(1 - \cos\theta)$. Then

$$x = a \int (1 - \cos\theta) d\theta \tag{52}$$

$$= a(\theta - \sin\theta) + const \tag{6.25}$$

and now we have $x(\theta)$ and $y(\theta)$ (parametric equations).

[7] Note on the total derivative: If $z = z(x, y)$, then $dz \triangleq \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$. This can be used to find either $\frac{\mathrm{d}z}{\mathrm{d}x}$ or $\frac{\mathrm{d}z}{\mathrm{d}y}$, for example.

IV. Lagrange's Equations (Ch. 7)

Advantages over Newtonian formulation:

1. Take same form in any coordinate system.
2. In treating constrained systems, Lagrangian approach eliminates the forces of constraint (like normal force).

Unconstrained Motion (7.1)

For a particle moving unconstrained in 3D, subject to a conservative net $F(\mathbb{R})$:

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{7.1}$$
$$U(\mathbb{R}) = U(x,y,z) \tag{7.2}$$
$$\mathcal{L} = T - U \tag{7.3}$$
$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x \tag{7.4}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \tag{7.6}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\mathrm{d}T}{\mathrm{d}\dot{x}} = m\dot{x} = p_x \tag{7.5}$$

where eq 7.6 implies that

> **[Hamilton's Principle]:** The path followed by a particle between two points 1 and 2 in a given time interval t_1 to t_2 is such that the action integral,
>
> $$S = \int_{t_1}^{t_2} \mathcal{L} dt \tag{7.8}$$
>
> is stationary when taken along the actual path.

Generalized Coordinates Each position $\boldsymbol{r}$ specifies a unique value of (q_1, q_2, q_3) and vice versa[8] :

$$q_i = q_i(\boldsymbol{r}) \quad \text{for } i = 1,2,3 \tag{7.9}$$
$$\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, q_3) \tag{7.10}$$

which allows us to call the following expressions the **generalized force** and **generalized momentum**, respectively:

$$\frac{\partial \mathcal{L}}{\partial q_i} = (\text{ith component of g.f.}) \tag{7.15}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = (\text{ith component of g.m.}) \tag{7.16}$$
$$\text{where } \frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{53}$$

[8] **Warning:** " Although Lagrange's equations are true for any choice of generalized coordinates q_1, q_2, q_3 — and these generalized coordinates may in fact be the coordinates of a noninertial reference frame — we must be careful that when we first write down the Lagrangian $\mathcal{L} = T - U$ we do so in an inertial frame."

[Example 7.2]. One particle in 2D with Polar Coordinates.

- $\mathcal{L}(r,\phi,\dot{r},\dot{\phi}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r,\phi)$
- **The r Equation**.

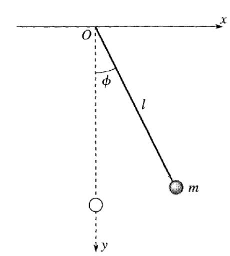
$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} \tag{54}$$
$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = m\ddot{r} \tag{7.19}$$
$$F_r = m(\ddot{r} - r\dot{\phi}^2) \tag{7.20}$$

- **The ϕ Equation**. Results in (torque) = (rate of change of angular momentum).

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{7.21}$$
$$-\frac{\partial U}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\phi}) \tag{7.22}$$
$$\nabla U = \frac{\partial U}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{7.23}$$
$$\Gamma = \frac{\mathrm{d}L}{\mathrm{d}t} \tag{7.24}$$

The Plane Pendulum (7.2)

Consider the simple pendulum shown below.



The only degree of freedom is the angle ϕ. The Lagrangian for this pendulum is

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1 - \cos\phi) \tag{7.30}$$

Whichever way we proceed, the Lagrangian is written in terms of a single generalized coordinate q and $\dot{q}$. This would be true even if we wrote in terms of x and y — i.e. $\mathcal{L}$ "knows" how many DOF there really are. Luckily, we chose the right (single) coordinate ϕ.

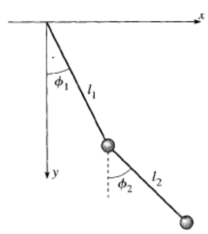
General Constrained Sys. (7.3)

Exact definitions. Consider arbitrary system of N particles, $\alpha = 1, \dots, N$ with positions $\boldsymbol{r}_\alpha$. The params $q_1, \dots, q_n$ are a set of **generalized coordinates** for the system if each

$$\boldsymbol{r}_\alpha = \boldsymbol{r}_\alpha(q_1, \dots, q_n, t) \quad [\alpha = 1, \dots, N] \tag{7.34}$$
$$q_i = q_i(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N, t) \quad [i = 1, \dots, n] \tag{7.35}$$

and require that the number of gen. coords n is the *smallest* number that allows the system to be parameterized in this way. In 3D world, $n \leq 3N$, and is usually less.

Double Pendulum. In the double pendulum below ((Fig 7.3)), system can be described in terms of four cartesian coordinates (of the bobs) or the generalized coordinates ϕ_1 and ϕ_2 shown in the figure.



$$\boldsymbol{r}_1 = (l_1 \sin\phi_1,\ l_1 \cos\phi_1) = \boldsymbol{r}_1(\phi_1) \tag{7.37}$$
$$\begin{aligned}\boldsymbol{r}_2 &= (l_1 \sin\phi_1 + l_2 \sin\phi_2,\ l_1 \cos\phi_1 + l_1 \cos\phi_2) \\ &= \boldsymbol{r}_2(\phi_1, \phi_2)\end{aligned} \tag{7.38}$$

This transformation from Cartesian (r) to generalized (ϕ) coordinates does not depend on time, which defines the generalized coordinates ϕ as **natural**.

Degrees of Freedom. Def: the number of coordinates that can be independently varied in a small displacement. For example, a gas of N particles has $3N$ DOF. When the NDOF of an N particle system in 3D is less than 3N, the system is **constrained**. If the NDOF is equal to the number of generalized coordinates needed to describe the system's configuration, the system is said to be **holonomic**[9] We assume all systems hereafter are holonomic, because without this condition, we cannot [guarantee that] Lagrange's equations give the time-evolution of the system.

[9]Nonholonomic example: a ball rolling on a horizontal table. NDOF=2 but need more than the two naive coordinates (x, y) since the *orientation* of the ball can change (this is extremely annoying because most people would assume the orientation doesn't matter because we are physicists and this is a stupid problem).

Proof: L.Eq. W/Constraints (7.4)

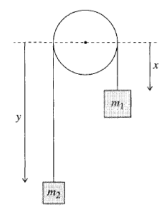
The Lagrangian approach aims to formulate equations *not* involving constraint forces[10]. Assumptions and definitions:

$$\boldsymbol{F} = \nabla U(\boldsymbol{r}, t) \tag{7.41}$$
$$\mathcal{L} = T - U \tag{7.42}$$

where the total force is $\boldsymbol{F}_{tot} = \boldsymbol{F}_{cstr} + \boldsymbol{F}$. Note that it is the *independence*[11] of the generalized coordinates that lets us use the Lagrange equations without worrying about constraints.

Examples of Lagrange's Eqs (7.5)

[7.3] Atwood's Machine. Goal: solve Lagrange E.O.M for $\ddot{x}$.



- **Let x be generalized coord.** Reasoning: Notice that heights x and y cannot vary independently. $l = x + \pi R + y$, therefore

$$y = -x + \text{const.} \tag{7.53}$$

- **Write T and U.**. Reasoning: Notice that $\dot{y} = -\dot{x}$. Therefore[12]

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

$$U = -m_1 g x - m_2 g y = -(m_1 - m_2) g x + \text{const.}$$

- **Solve for $\ddot{x}$.**

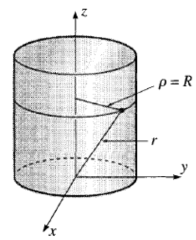
$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2) g x \tag{7.54}$$
$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \rightarrow (m_1 - m_2) g = (m_1 + m_2)\ddot{x} \tag{7.55}$$
$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g \tag{7.56}$$

[10]Constraint forces examples: (1) wire on a bead – normal force of bead; (2) particle on a surface – normal force of surface.

[11]Independent here means each generalized coordinate can be independently varied.

[12]How to think about potential energy:
- Consider each mass by itself.
- Consider the work done by gravity should the mass move from its current position to the center of the circle (i.e. a distance -x for m_1).
- This work is clearly negative, and equal to -$m_1 g x$ for m_1.

[7.4] Particle Confined to Cylinder Surface. Given that only force on the mass is $\boldsymbol{F} = -k\boldsymbol{r}$ (toward origin.)



- **Obtain generalized coordinates**. They are (z, ϕ) since $\rho = R$ fixed.
- **Write $\mathcal{L}$.** First evaluate velocity component-wise:

$$v_\rho = 0 \quad v_\phi = R\dot{\phi} \quad v_z = \dot{z}$$

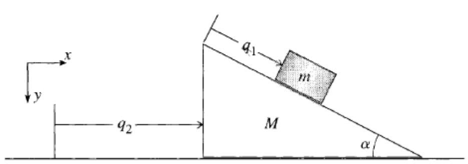
$$\mathcal{L} = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) - \frac{1}{2}k(R^2 + z^2) \tag{7.57}$$

- **Solve Lagrange Eqs for each DOF**.

$$-kz = m\ddot{z} \tag{7.58}$$
$$0 = \frac{\mathrm{d}}{\mathrm{d}t}mR^2\dot{\phi} \tag{7.59}$$

- **Analyze**. These equations tell us that (1) particle moves along $z = A\cos(\omega t - \delta)$ with S.H.O; and (2) that mass has constant angular velocity $\dot{\phi}$.

[7.5] Block Sliding on a Wedge. Goal: Find time for m to reach bottom of wedge, if released from rest atop the wedge.



- **Focus on finding** $\ddot{q}_1$ since this is what we need to obtain the answer.
- **Find** T. First in cartesian coordinates. **Analyze motion of M and m separately**[13]

$$\begin{aligned} T &= T_M + T_m \\ &= \frac{1}{2}M\dot{q}_2^2 + \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\cos\alpha) \end{aligned} \tag{7.60}$$

- **Find** U. Notice that $+\hat{y}$ is measured *downward* and $y = 0$ at the top of the wedge. Therefore, the block begins with $U = 0$ remember [14] and ends with $U = -mgy = -mgq_1\sin\alpha$, which means this is the potential energy:

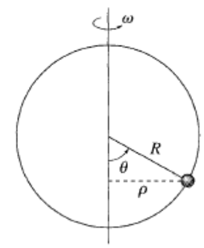
$$U = -mgq_1\sin\alpha$$

- **Find $\mathcal{L}$ and solve for $\ddot{q}_1$ and $\ddot{q}_2$**. Derivation is same as always, results:

$$\ddot{q}_2 = -\frac{m}{M+m}\ddot{q}_1\cos\alpha \tag{7.66}$$
$$\ddot{q}_1 = \frac{g\sin\alpha}{1 - \frac{m\cos^2\alpha}{M+m}} \tag{7.67}$$

which gives us our answer for the time to reach the bottom of the wedge: $t = \sqrt{2l/\ddot{q}_1}$.

[7.6] Bead on Spinning Wire Hoop.



- **The Lagrangian**. Generalized coordinate is θ. Bead has velocities

$$v_{tan} = R\dot{\theta}, \quad v_{norm} = \rho\omega = (R\sin\theta)\omega$$

which we can use to write

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) - mgR(1 - \cos\theta) \end{aligned} \tag{7.68}$$

- **Solve the Lagrange Equations**. Result is

$$\ddot{\theta} = (\omega^2\cos\theta - g/R)\sin\theta \tag{7.69}$$

- **Equilibrium points**. An equilibrium point is any value θ — call it θ_0 — such that if bead is placed at rest ($\dot{\theta} = 0$) at $\theta = \theta_0$, *then it will remain at rest at* $\theta = \theta_0$. **This is guaranteed if** $\ddot{\theta} = 0$.

  > We conclude that when the hoop is rotating slowly ($\omega^2 < g/R$), there are just two equilibrium positions, at the bottom and top of the hoop, but when it rotates fast enough ($\omega^2 > g/R$), there are two more, symmetrically placed on either side of the bottom.

[13] Note, for the small wedge, that

$$\boldsymbol{v} = (v_x,\ v_y) = (\dot{q}_2 + \dot{q}_1\cos\alpha, \dot{q}_1\sin\alpha\ )$$

[14] that single values of U mean nothing, all that matters are differences

Lagrange Multipliers (7.10)

Sometimes, we *are* interested in the constraint forces and thus want a formulation that includes them. Consider system restricted by a **constraint equation** of the form $f(x,y) = const$. The main points in the derivation of Lagrange multipliers are as follows.

- **Perturb the action**. We know by Hamilton's principle that $\delta S = 0$, where the delta is a change in the actual path followed. For a Lagrangian $\mathcal{L} = \mathcal{L}(x, \dot{x}, y, \dot{y})$, this is defined as

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\delta \dot{x} + \frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\delta \dot{y} \right) dt \tag{7.114}$$

- **Integrate by parts** on the second and fourth terms[15]

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x dt + \int \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}} \right) \delta y dt \tag{7.115}$$

- **Key Idea:** The perturbed path must also satisfy the constraint, which here was stated as f(x, y) = const. for the original path. Therefore, the displacement must leave this unchanged, i.e. that $\delta f = 0$.

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y = 0 \tag{7.116}$$

- **Final result:** Multiply eq 7.116 by some $\lambda(t)$. Add this to equation 7.115.

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \tag{7.118}$$

$$\frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}} \tag{7.119}$$

$$f(x,y) = \mathrm{const} \tag{7.120}$$

[Ex 7.8] Atwood's Machine Using Lagrange Multiplier.

[15] Let u = the partial deriv of L and let dv be $\delta \dot{x} dt = \delta(dx/dt)dt$

V. Two-Body Problems (Ch. 8)

The Problem

- Two points particles where only forces of interest are $\mathcal{F}_{12}$ and $\mathcal{F}_{21}$ their mutual conservative/central interaction.
- The forces can be derived from a potential[16] energy $U(\mathbb{R}_1, \mathbb{R}_2)$.
- If a conservative force is central, then U is independent of the direction of $(\mathbb{R}_1 - \mathbb{R}_2)$ (only depends on the magnitude).
- Denote $\mathbb{R} = \mathbb{R}_1 - \mathbb{R}_2$ as the **relative position** of body 1 to body 2.

Rel. Coords/Reduced Mass

- Want generalized coordinates to solve problem. Recall that the **center of mass position** $\boldsymbol{R}$ of the two bodies is

$$\boldsymbol{R} = \frac{m_1\mathbb{R}_1 + m_2\mathbb{R}_2}{M} \tag{8.7}$$

  which lies along the line joining the two bodies.

- Total momentum of the two bodies is $\boldsymbol{P} = M\dot{\boldsymbol{R}}$, which is constant $\to \dot{\boldsymbol{R}}$ is constant $\to$ we can choose the **CM frame** in which the CM is at rest.
- Want to find the kinetic energy in generalized coord system of $\mathbb{R}$ and $\boldsymbol{R}$. Recall that

$$\mathbb{R}_1 = \boldsymbol{R} + \frac{m_2}{M}\mathbb{R} \tag{55}$$
$$\mathbb{R}_2 = \boldsymbol{R} - \frac{m_1}{M}\mathbb{R} \tag{56}$$

  which leads to

$$T = \frac{1}{2}\left(m_1\dot{\mathbb{R}}_1^2 + m_2\dot{\mathbb{R}}_2^2\right) \tag{57}$$
$$= \frac{1}{2}\left(M\dot{\boldsymbol{R}}^2 + \mu\dot{\mathbb{R}}^2\right) \tag{58}$$

  where μ is the **reduced mass**, defined as

$$\mu = \frac{m_1 m_2}{M} \tag{8.11}$$

Equations of Motion

- Lagrange equations for $\boldsymbol{R}$ and $\mathbb{R}$ respectively:

$$\dot{\boldsymbol{R}} = \text{const} \tag{8.14}$$
$$\mu\ddot{\mathbb{R}} = -\nabla U(\mathbb{R}) \tag{8.15}$$

  and thus, to solve for the relative motion, we have only to solve Newton's second law for a single particle of mass μ and position $\mathbb{R}$, with potential energy $U(r)$.

The CM Frame

- Due to eq 8.14, we can choose the **CM-frame**, in which the CM is at rest and the total momentum is zero.

Conservation of Angular Momentum

- The angular momentum in the CM frame is

$$\begin{aligned}\boldsymbol{L} &= \frac{m_1 m_2}{M}\left(m_2\mathbb{R} \times \dot{\mathbb{R}} + m_1\mathbb{R} \times \dot{r}\right) \\ &= \mathbb{R} \times \mu\dot{\mathbb{R}}\end{aligned} \tag{8.19}$$

  which is *constant* since angular momentum conserved.

- In CM frame, whole motion remains in a fixed plane.

The Two Equations of Motion

- Do Lagrangian stuff to get equations of motion:

$$\mu r^2\dot{\phi} = \text{const} = \ell \tag{59}$$
$$\mu r\dot{\phi}^2 - \frac{\mathrm{d}U}{\mathrm{d}r} = \mu\ddot{r} \tag{60}$$

  the ϕ and r equations, respectively.

[16] A conservative force is dependent only on the position of the object. If a force is conservative, it is possible to assign a numerical value for the potential at any point. When an object moves from one location to another, the force changes the potential energy of the object by an amount that does not depend on the path taken.

Equivalent 1D Problem (8.4)

- Since $\dot{\phi} = \ell/\mu r^2$, we can rewrite

$$\mu\ddot{r} = -\frac{\mathrm{d}U}{\mathrm{d}r} + \mu r\dot{\phi}^2 = -\frac{\mathrm{d}U}{\mathrm{d}r} + F_{cf} \tag{61}$$

so the particle's radial motion is same as if particle were subject to actual force $-\frac{\mathrm{d}U}{\mathrm{d}r}$ plus a centrifugal force.

- Can use math to express F_{cf} as gradient of some U_{cf}.

$$F_{cf} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\ell^2}{2\mu r^2}\right) \tag{62}$$
$$= -\frac{\mathrm{d}U_{cf}}{\mathrm{d}r} \tag{63}$$

- Now, we can rewrite radial equation with an **effective potential**, defined as the sum of U and U_{cf}.

$$\mu\ddot{r} = -\frac{\mathrm{d}}{\mathrm{d}r}[U(r) + U_{cf}(r)] = -\frac{\mathrm{d}}{\mathrm{d}r}U_{eff}(r) \tag{8.29}$$
$$U_{eff}(r) = U(r) + U_{cf}(r) = U(r) + \frac{\ell}{2\mu r^2} \tag{8.30}$$

Conservation of Energy

- Multiply both sides of eq 8.29 and move terms to LHS to get

$$\frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = \text{const} \tag{8.34}$$
$$\frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + U(r) = E \tag{8.35}$$

which is conservation of energy.

Equation of Orbit (8.5)

- Eq 8.29 gives $r(t)$. Now we seek $r(\phi)$. Tells us about shape of the orbit.
- Main parts of derivation from $r(t) \to u(\phi)$:

$$\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3} \quad \text{Radial equation} \tag{8.37}$$
$$u = \frac{1}{r} \quad \text{Define substitution} \tag{8.38}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\ell u^2}{\mu}\frac{\mathrm{d}}{\mathrm{d}\phi} \quad \text{Chain rule on } \phi \tag{8.39}$$

(64)

which, after applying to the radial equation and shifting some terms around yields

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}F \tag{8.41}$$

- If we solve eq 8.41, we have $r(\phi) = 1/u(\phi)$.
- **Example 8.3: Radial Equation for a Free Particle:**
  - If no external forces, expect "orbit" to be straight line.
  - Transformed radial equation is

$$u''(\phi) = -u(\phi) \to u(\phi) = A\cos(\phi - \delta) \tag{8.42}$$

  - Which means we can write $r(\phi) = \frac{r_0}{\cos(\phi-\delta)}$, the equation for a line in polar coordinates.

The Kepler Orbits (8.6)

- The **Kepler Problem**: Find the possible orbits of any object subject to an inverse-square force.
- Write general inverse-square force as

$$F(r) = -\frac{\gamma}{r^2} = -\gamma u^2 \tag{8.44}$$

- Main parts of deriving $r(\phi)$ for inverse-square force:

$$u''(\phi) = -u(\phi) + \gamma\mu/\ell^2 \quad \text{Plug in to radial eq.} \tag{8.45}$$
$$u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi \quad \text{General solution} \tag{8.47}$$
$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi} \quad \text{Solve for r} \tag{8.49}$$

(65)

where we've defined constants[17] $c = \ell^2/\gamma\mu$ and $\epsilon = A\ell^2/\gamma\mu = Ac$.

- Recap: This is the (very) general solution of $r(\phi)$ in the case of some inverse-square (central) force.

The Bounded Orbits

- $\epsilon = 1$ is boundary between **bounded** ($\epsilon < 1$) and unbounded ($\epsilon \geq 1$) orbits.[18]
- With $\epsilon < 1$, r oscillates between[19]

$$r_{min} = \frac{c}{1+\epsilon} \quad \text{PERIHELION} \tag{8.50A}$$
$$r_{max} = \frac{c}{1-\epsilon} \quad \text{APHELION} \tag{8.50B}$$

- Since $r(0) = r(2\pi)$, orbit closes on itself after one revolution.
- Orbit is an ellipse. Can rewrite eq 8.49 as

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } a = c/(1-\epsilon^2); b = c/\sqrt{1-\epsilon^2}; d = a\epsilon \tag{66}$$

- We can now identify ϵ as the **eccentricity of the ellipse**.

$$\frac{b}{a} = \sqrt{1-\epsilon^2} \tag{8.53}$$

where (1) $\epsilon = 0 =$ circle, (2) $\epsilon \to 1 =$ parabola, and (3) $\epsilon > 1 =$ hyperbola.

- Now we have **Kepler's first law** (see figure 4):

> The planets follow orbits that are ellipses with the sun at one focus.

[17] c has the dimensions of length. ϵ is assumed positive
[18] Again, ϵ is ALWAYS assumed $\epsilon > 0$.
[19] Note that ratio of r max/min can be determined give ϵ

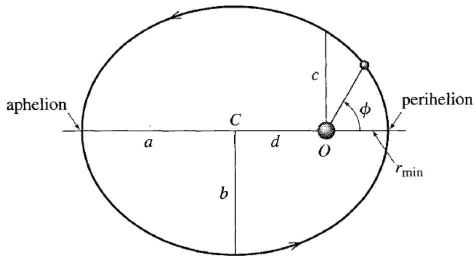


Figure 4: Center at $x = -d$. Sun at one foci at origin.

Orbital Period; 3rd Law

- See page 312 for more details. Helpful info on ellipses/kepler. Don't want to copy everything here.
- Resul: for all bodies orbiting the sun, square of period $\propto$ cube of semimajor axis.

$$\tau^2 = \frac{4\pi^2}{GM_s} a^3 \tag{8.55}$$

Relationship: E and ϵ

- Relate eccentricity of orbit to energy of the orbiting body (e.g. planet).
- Energy is equal to the *effective* potential energy when $r = r_{min}$.

$$E = U_{eff}(r_{min}) = \frac{\gamma^2 \mu}{2\ell^2} \left(\epsilon^2 - 1\right) \tag{8.58}$$

  which is true for **both** bounded and unbounded orbits.

- Note that, for a given value of ℓ, orbit of lowest possible E is circular orbit with $\epsilon = 0$.

VI. Mechanics in Noninertial Frames (Ch. 9)

Acceleration (9.1)

Modified version of Newton's second law in a reference frame $\mathcal{S}$ moving with speed V and acceleration $A = \dot{V}$ relative to some inertial frame $\mathcal{S}_0$:

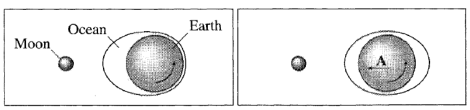
$$m\ddot{r} = \mathcal{F} - m\boldsymbol{A} = \mathcal{F} - \mathcal{F}_{inertial} \tag{9.4}$$

$$\mathcal{F}_{inertial} = -m\boldsymbol{A} \tag{9.5}$$

where $\mathcal{F} = m\ddot{r}_0$ is the net force on the ball as seen in (the inertial frame) $\mathcal{S}_0$.

The Tides (9.2)

Caused by gravitational attraction of the moon[20].



(a) Wrong (a) Right

Can obtain the **tidal force** by using eq 9.4.

- Forces on a mass m near surface of Earth:
  1. Earth gravity $m\boldsymbol{g}$.
  2. Moon gravity $-GM_{moon}\, m\hat{\boldsymbol{d}}/d^2$.[21]
  3. Net other/non-gravity $\mathcal{F}_{ng}$
- Centripetal acceleration of Earth due to moon[22] is $\boldsymbol{A} = -GM_{moon}\hat{\boldsymbol{d}}_{\mathbf{0}}/d_0^2$.
- Plugging into equation 9.4 yields

$$m\ddot{r} = m\boldsymbol{g} + \mathcal{F}_{tid} - \mathcal{F}_{ng} \tag{9.11}$$

$$\mathcal{F}_{tid} = -GM_{moon}\, m\left(\frac{\hat{\boldsymbol{d}}}{d^2} - \frac{\hat{\boldsymbol{d}}_0}{d_0^2}\right) \tag{9.12}$$

[20] and the sun a little bit.
[21] d points from object to center of moon.
[22] d_0 points from earth center to moon center.

Angular Velocity Vector (9.3)

In discussing *rotation of a rigid body*, only the following two situations concern us (for now):

1. Body rotating about a *fixed* point (in some inertial frame). e.g. a wheel spinning about a fixed axle.
2. If no fixed point (e.g. baseball spinning as it flies through air), then proceed by
   - Find motion of the CM $\boldsymbol{R}(t)$.
   - Analyze rotational motion of body relative to its CM via $\mathbb{R}(t)$.

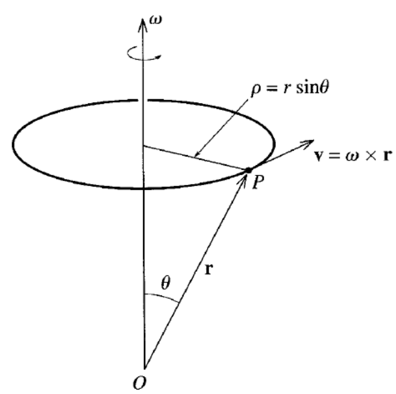
Euler's Theorem: *The most general motion of any body relative to a fixed point O is a rotation about some axis through O.*

Denote direction of axis of rotation with $\boldsymbol{u}$. Combine with the rate of rotation ω to get the **angular velocity vector**

$$\boldsymbol{\omega} = \omega\boldsymbol{u} \tag{9.21}$$

where the relationship between the angular velocity $\boldsymbol{\omega}$ and the linear velocity $\boldsymbol{v}$ is

$$\boldsymbol{v} = \boldsymbol{\omega} \times \mathbb{R} \tag{9.22}$$



and if $\boldsymbol{e}$ is *any* vector fixed in the rotating body, then its rate of change as seen from the non-rotating frame is

$$\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t} = \boldsymbol{\omega} \times \boldsymbol{e} \tag{9.23}$$

Finally, note that addition of *relative* velocities is the same for angular and linear.

$$\boldsymbol{v}_{31} = \boldsymbol{v}_{32} + \boldsymbol{v}_{21} \tag{9.24}$$

$$\boldsymbol{\omega}_{31} = \boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{21} \tag{9.25}$$

TimeDeriv Rotating Frame (9.4)

Consider an arbitrary vector $\boldsymbol{Q}$. The time rate of change of $\boldsymbol{Q}$ as measured in $\mathcal{S}_0$ in terms of its corresponding rate in $\mathcal{S}$ is

$$\left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\mathcal{S}_0} = \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\Omega} \times \boldsymbol{Q} \tag{9.30}$$

where $\boldsymbol{\Omega}$ denotes the (usually known) angular velocity of a noninertial frame $\mathcal{S}$ (relative to $\mathcal{S}_0$).

2nd Law in Rot. Frame (9.5)

Assuming now that $\boldsymbol{\Omega}$ is constant. Find that the form of Newton's second law for the rotating frame $\mathcal{S}$ is

$$m\ddot{r} = \mathcal{F} + \mathcal{F}_{cor} + \mathcal{F}_{cf} \tag{9.34/37s}$$

$$F_{cor} = 2m\dot{r} \times \boldsymbol{\Omega} \tag{9.35}$$

$$F_{cf} = m(\boldsymbol{\Omega} \times \mathbb{R}) \times \boldsymbol{\Omega} \tag{9.36}$$

and we have the **coriolis force and centrifugal force**. The take-away here is that we can use Newton's second law in rotating (and hence noninertial) reference frames, provided we remember to always add these two "fictitious" forces.

Centrifugal Force (9.6)

The Coriolis force, which is proportional to object's velocity $\boldsymbol{v} = \dot{r}$, is **zero** for any object that is at rest in the rotating frame and negligible if moving sufficiently slowly. **For the rest of the chapter, main concern is rotating frame of the earth**. In this case,

$$\frac{F_{cor}}{F_{cf}} \sim \frac{v}{R\Omega} \sim \frac{v}{V} \tag{9.38}$$

where v is the object's speed relative to the rotating frame of the earth, $R = R_{earth}$, and $V \approx 1000\ mi/h$ is the speed of a point on the equator as the earth rotates with angular velocity Ω. Therefore, for objects with $v << 1000\ mi/h$, a good approx. is to ignore $\mathcal{F}_{cor}$, which is what we will do in this section.

Note: The angle θ in the fig below eq 9.22 is called the **colatitude**. Also $\mathcal{F}_{cf}$ points radially outward from axis of rotation in the direction of $\hat{\rho}$, thus

$$F_{cf} = m\Omega^2 \rho \hat{\boldsymbol{\rho}} \tag{9.40}$$

Free-Fall Acceleration

Denote the familiar $\boldsymbol{g}$ now as just $\boldsymbol{g}_0$, the *initial* acceleration[23], relative to earth. The equation of motion is

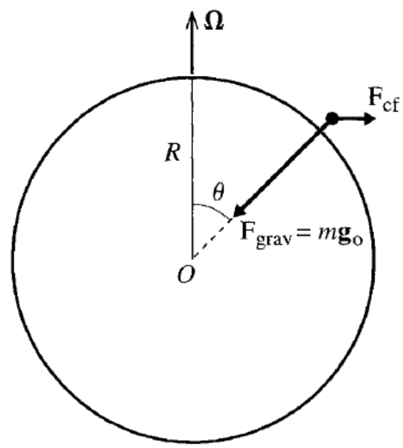
$$m\ddot{r} = \mathcal{F}_{grav} + \mathcal{F}_{cf} \equiv \mathcal{F}_{eff} \tag{9.41}$$

$$= m\boldsymbol{g}_0 + m\Omega^2 R \sin\theta \hat{\boldsymbol{\rho}} \tag{9.43}$$

$$\boldsymbol{g} = \boldsymbol{g}_0 + \Omega^2 R \sin\theta \hat{\boldsymbol{\rho}} \tag{9.44}$$

$$g_{rad} = g_0 - \Omega^2 R \sin^2\theta \tag{9.45}$$

$$g_{tang} = \Omega^2 R \sin\theta \cos\theta \tag{9.47}$$

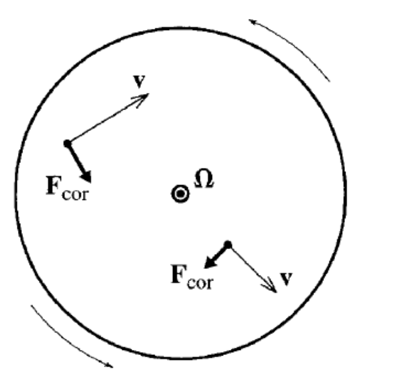


Coriolis Force (9.7)

When an object is moving, there is a second inertial force that you must include when want to use 2nd law in rotating frame:

$$\mathcal{F}_{cor} = 2m\dot{r} \times \boldsymbol{\Omega} = 2m\boldsymbol{v} \times \boldsymbol{\Omega} \tag{9.49}$$

where $\boldsymbol{v} = \dot{r}$ is object's velocity relative to the rotating frame.



[23] What this means: Immediately after being released *from rest* (and so zero velocity) the Coriolis force is zero. This will change when the object speeds up. Therefore, from here on out, $\boldsymbol{g}_0$ is the acceleration solely due to the familiar $-GMm/r^2 = m\boldsymbol{g}_0$ force, while $\boldsymbol{g}$ is defined in equation 9.44.

Notice how, in fig above, F_{cor} seems to deflect object's velocity to the right.

Coriolis Free-Fall (9.8)

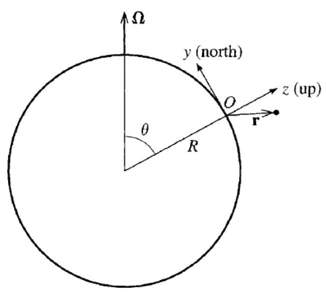
Consider freely falling object close to a point $\boldsymbol{R}$ on earth's surface. Equation of motion is

$$m\ddot{r} = m\boldsymbol{g}_0 + \mathcal{F}_{cf} + \mathcal{F}_{cor} \tag{9.50}$$

but since $\mathbb{R}$, the position of object relative to center of Earth is $\approx \boldsymbol{R}$, we can replace $\mathcal{F}_{cf} \approx m\left(\boldsymbol{\Omega} \times \boldsymbol{R}\right) \times \boldsymbol{\Omega}$. Substituting this in and recalling formula 9.44 for $\boldsymbol{g}$, we can rewrite eq 9.50 as

$$\ddot{r} = \boldsymbol{g} + 2\dot{r} \times \boldsymbol{\Omega} \tag{9.51}$$

which doesn't involve the position $\mathbb{R}$ directly at all. Therefore, choose origin at position $\boldsymbol{R}$ shown below to make life easier.



Now, can break into components and solve.

$$\dot{r} = (\dot{x}, \dot{y}, \dot{z}) \tag{67}$$

$$\boldsymbol{\Omega} = (0, \Omega\sin\theta, \Omega\cos\theta) \tag{68}$$

$$\dot{r} \times \Omega = \Big(\dot{y}\Omega\cos\theta - \dot{z}\Omega\sin\theta, \quad -\dot{x}\Omega\cos\theta, \;\; \dot{x}\Omega\sin\theta\Big) \tag{9.52}$$

and the equation of motion, eq 9.51, resolves into

$$\begin{aligned} \ddot{x} &= 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta) \\ \ddot{y} &= -2\Omega\dot{x}\cos\theta \\ \ddot{z} &= -g + 2\Omega\dot{x}\sin\theta \end{aligned} \tag{9.53}$$

Zeroth-order approximation. If we ignore Ω entirely, since it is really small, we get

$$x = 0, \quad y = 0, \quad \text{and} \quad z = h - \frac{1}{2}gt^2 \tag{9.55}$$

First-order approximation. Since there are no rules in math (amirite?) just plug in eq 9.55 into eq 9.53 and solve for x (y and z are the same as zeroth order) to get

$$x = \frac{1}{3}\Omega g t^3 \sin\theta \tag{9.57}$$

and if we are bored, can repeat this to get the second-order approximations and so on. Equation 9.57 tells us that **a freely falling object does not fall straight down**, due to the Coriolis force.

VII. Rotational Motion of Rigid Bodies (Ch. 10)

Properties of CM (10.1)

$$\mathbf{F} = M\ddot{\mathbf{R}} \tag{10.3}$$

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_{\alpha} \boldsymbol{r}'_{\alpha} \times m_{\alpha}\dot{\boldsymbol{r}}'_{\alpha} \tag{10.8}$$

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_{\alpha} m_{\alpha}\dot{\boldsymbol{r}}'^2_{\alpha} \tag{10.15}$$

$$U = U^{ext} + U^{int} \tag{10.19}$$

Rot. about Fixed Axis (10.2)

Consider rigid body rotating about z axis, and with the origin O somewhere on axis of rotation. The z-component of the angular momentum is given by

$$L_z = \sum m_{\alpha}(x_{\alpha}^2 + y_{\alpha}^2)\omega \tag{10.23}$$

$$= \sum m_{\alpha}\rho_{\alpha}^2\omega = I_z\omega \tag{10.24}$$

$$T = \frac{1}{2}\sum m_{\alpha}\rho_{\alpha}^2\omega^2 = \frac{1}{2}I_z\omega^2 \tag{10.26}$$

where I_z is the **moment of inertia** about the z-axis.

Directions of L and $\boldsymbol{\omega}$. Fig below shows how direction of L and ω may not coincide. Accordingly, we redefine notation for L and I shown below the figure.

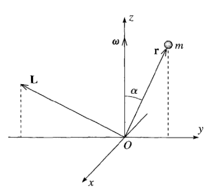


Figure 10.3 A rigid rotating body comprising a single mass m anchored to the z axis by a massless rod at a fixed angle α, shown at a moment when m happens to lie in the yz plane. As the body rotates about the z axis, m has velocity, and hence momentum, into the page (in the negative x direction) at the moment shown. Therefore the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is directed as shown and is certainly not parallel to the angular velocity $\boldsymbol{\omega}$.

where the two coefficients I_{xz} and I_{yz} are called the **product of inertia** of the body.

Inertia Tensor (10.3)

General case.

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} \quad \text{where} \tag{10.42}$$

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{69}$$

$$I_{xx} = \sum m_{\alpha}(y_{\alpha}^2 + z_{\alpha}^2) \tag{10.37}$$

$$I_{xy} = -\sum m_{\alpha}x_{\alpha}y_{\alpha} \tag{10.38}$$

and we note that $\mathbf{I}$ is **symmetric**.

Principal Axes of Inertia (10.4)

A **principal axis** is one where $\mathbf{L}$ and $\boldsymbol{\omega}$ are parallel. Mathematically: any axis with property that if $\boldsymbol{\omega}$ points along it, that

$$\mathbf{L} = \lambda\boldsymbol{\omega} \tag{10.65}$$

where λ is the *moment of inertia about the axis*. More generally, if $\mathbf{I}$ is diagonal wrt a chosen set of axes,

$$\mathbf{I} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{70}$$

then the λ_i are called the **principal moments**, the moments of inertia about the 3 principal axes. *Any* rigid body rotating about any point has 3 principal axes.

Kinetic Energy. In general,

$$T = \frac{1}{2}\omega \cdot \mathbf{L} \tag{10.67}$$

Finding Princ. Axes & Eigval. Eqs (10.5)

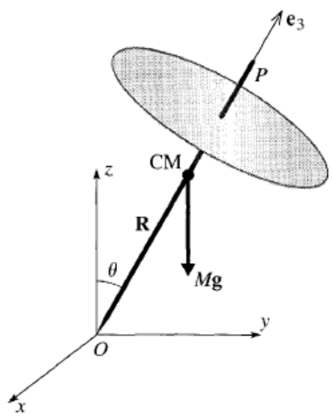
Procedure. First, find eigenvalues and then plug in each separately to find eigenvectors.

$$(\mathbf{I}-\lambda)\omega = 0 \tag{10.70}$$
$$\det(\mathbf{I}-\lambda) = 0 \tag{10.71}$$

Note that this entire process is equivalent to diagonalizing the inertia tensor **I**.

Top Precession due to Weak Torque (10.6)

Important: If a body has two perpendicular planes of reflection symmetry through O, then the 3 perpendicular axes defined by these two planes and O are principal axes. Because of the top's axial symmetric, inertia tensor is diagonal.



In the absence of gravity, $\mathbf{L} = \lambda_3 \omega e_3$, which is constant since no torque. *With* gravity, there is a torque,

$$\mathbf{\Gamma} = \mathbf{R} \times M\boldsymbol{g} \tag{71}$$
$$|\mathbf{\Gamma}| = RMg\sin\theta \tag{72}$$

and let's assume it is *small*, meaning that although $\mathbf{\Gamma} = \dot{\mathbf{L}}$, we can assume the newly nonzero ω_1, ω_2 remain negligibly small compared to ω_3. Then $\dot{\mathbf{L}} = \mathbf{\Gamma}$ becomes

$$\lambda_3 \omega \dot{e}_3 = \mathbf{R} \times M\boldsymbol{g} \tag{73}$$

Problem-Solving Tips

Potential Energy.

- The (grav) potential energy of a system is equal to the potential of the CM.

Kinetic Energy.

- For a rigid body, the only possible motion rel. to the CM is rotation, therefore

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum m_\alpha \dot{\boldsymbol{r}}_\alpha'^2 \tag{74}$$

- Don't assume kinetic energy is conserved in collisions, since may be inelastic.

Angular momentum.

- **L** is conserved *component-wise*, so don't get tripped up about whether or not some component of L is nonzero (e.g. in 10.16). Just focus on conserving the relevant components for the problem.

10.7 - 10. Rapid Summary

Euler's Equations

Frames. The **space frame** is any inertial frame (fixed in space), whose axes we label x, y, z. The **body frame** is the rotating frame, fixed w.r.t. the body, denoted with axis vectors e_1, e_2, e_3. In the body frame, angular momentum will be

$$\mathbf{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3) \quad \textit{[BODYFRAME]} \tag{10.84}$$

Euler's Equations determine the motion of a spinning body as seen in the body frame. Their derivation is sketched below, with the final result in gray.

$$\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{space} = \Gamma \tag{10.85}$$
$$= \left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} \tag{10.86}$$

$$\dot{\mathbf{L}} + \boldsymbol{\omega} \times \mathbf{L} = \Gamma \quad \textit{[EULER'S EQ]} \tag{10.87}$$

$$\begin{aligned} \lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 &= \Gamma_1 \\ \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 &= \Gamma_2 \\ \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 &= \Gamma_3 \end{aligned} \tag{10.87}$$

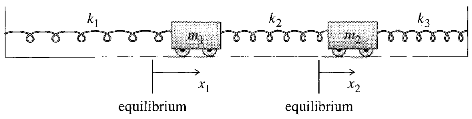
With Zero Torque

Euler Angles

$$\omega = (-\dot{\phi}\sin\theta)e_1' + \dot{\theta}e_2' + (\dot{\psi} + \dot{\phi}\cos\theta)e_3 \tag{10.99}$$
$$\mathbf{L} = (-\lambda_1\dot{\phi}\sin\theta)e_1' + \lambda_2\dot{\theta}e_2' + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)e_3 \tag{10.100}$$
$$T = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 \tag{10.105}$$
$$\mathcal{L} = T - MgR\cos\theta \tag{10.106}$$

VIII. Coupled Osc. & Normal Modes (Ch. 11)

Two Masses & Three Springs (11.1)



The equations of motion for the coupled oscillators above can be written in matrix notation as

$$\mathbf{M}\ddot{\boldsymbol{x}} = -\mathbf{K}\boldsymbol{x} \tag{11.3}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \tag{11.5}$$

where $\mathbf{M}$ is the *mass-matrix*, and $\mathbf{K}$ is the *spring-constant matrix*. Below, (11.8 11.9) gives the elements of the complex solution.

$$z_1(t) = x_1(t) + iy_1(t) = \alpha_1 e^{i(\omega t - \delta_1)} = a_1 e^{i\omega t} \tag{11.8}$$
$$z_2(t) = x_2(t) + iy_2(t) = \alpha_2 e^{i(\omega t - \delta_2)} = a_2 e^{i\omega t} \tag{11.9}$$

where it's important to notice that $\boldsymbol{x}(t) = \Re[\boldsymbol{z}(t)]$. After substituting this into 11.3 and rearranging, we obtain

$$(\mathbf{K} - \omega^2\mathbf{M})\boldsymbol{a} = 0 \tag{11.11}$$
$$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0 \tag{11.12}$$

where the two frequencies at which our system can oscillate sinusoidally, the normal frequencies, are determined by the quadratic equation 11.12 for ω^2.

Procedure seems to be

1. **Normal Frequency**: Solve 11.12 to find solutions for ω.
2. **Normal Mode:** For each such ω, solve 11.11 for $\boldsymbol{a}$.

Identical Springs & Equal Masses (11.2)

Working through the last section, now for the special case of all $k_i = k$ and all $m_i = m$ yields, after solving 11.12, the two normal frequencies

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{3k}{m}} \tag{11.15}$$

which give us the frequencies at which our carts can oscillate in sinusoidal motion.

The first normal mode. The two carts oscillate back and forth with equal amplitudes and exactly in phase, and the middle spring remains at equilibrium length all the time.

$$\begin{aligned} x_1(t) &= A\cos(\omega_1 t - \delta) \\ x_2(t) &= A\cos(\omega_1 t - \delta) \end{aligned} \quad [1^{st}\text{NORM. MODE}] \tag{11.18}$$

The second normal mode. Carts oscillate with same amplitude but exactly out of phase, so that $x_2(t) = -x_1(t)$ always.

$$\begin{aligned} x_1(t) &= A\cos(\omega_2 t - \delta) \\ x_2(t) &= -A\cos(\omega_2 t - \delta) \end{aligned} \quad [2^{nd}\text{NORM. MODE}] \tag{11.20}$$

General solution. The eq. of mot. is two 2nd-order diff. eq. for the two vars $x_1(t)$ and $x_2(t)$. Any solution can be written as a sum of the form 11.18 + 11.20, namely

$$\boldsymbol{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2) \tag{11.21}$$

Normal Coordinates. Want coordinates that vary independently of one another (uncoupled).

$$\xi_1 = \frac{1}{2}(x_1 + x_2) \tag{11.22}$$
$$\xi_2 = \frac{1}{2}(x_1 - x_2) \tag{11.23}$$

Two Weakly Coupled Oscillators (11.3)

Consider the carts. Here, "weakly coupled" means the middle spring (that connects the carts together) is weak ($k_2 << k$), and we assume that $k \leftarrow k_1 = k_2$. Working through the usual procedure yields the normal frequencies

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{k + 2k_2}{m}} \tag{11.27}$$

Comparing ω_1 and ω_2. We introduce some notation below, and then solve for z as a sum of the normal solutions.

$$\omega_0 = \frac{\omega_1+\omega_2}{2}, \qquad \omega_1 = \omega_0 - \epsilon, \qquad \omega_2 = \omega_0 + \epsilon \tag{75}$$

$$z(t) = \left[C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\epsilon t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\epsilon t} \right] e^{i\omega_0 t} \tag{11.29}$$

Analysis:

- **Small time intervals**. The first term of the factor, in large brackets, is essentially constant over any reasonably short time interval compared with the other factor $e^{i\omega_0 t}$, and solution behaves like $z(t) \approx a e^{i\omega_0 t}$.
- **Values of C_1 and C_2**. If either on of $C_i = 0$, solution is just one of the normal modes. If they're equal, $C_1 = C_2 = A/2$, (and real), solution becomes

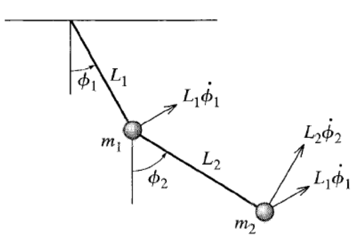
$$x(t) = \Re[z(t)] = \Re\left[A \left[\begin{smallmatrix} \cos\epsilon t \\ -i\sin\epsilon t \end{smallmatrix}\right] e^{i\omega_0 t}\right] \tag{11.30}$$

$$= A \begin{bmatrix} \cos\epsilon t \cos\omega_0 t \\ \sin\epsilon t \sin\omega_0 t \end{bmatrix} \tag{11.31}$$

$$\approx \begin{bmatrix} A\cos(\omega_0 t) \\ 0 \end{bmatrix} \qquad [0 \le t \le 1/\epsilon] \tag{11.32}$$

$$\approx \begin{bmatrix} 0 \\ A\sin(\omega_0 t) \end{bmatrix} \qquad [t \approx \pi/2\epsilon] \tag{11.33}$$

Lagrangian & Double Pendulum (11.4)



Writing $\mathcal{L}$. When angle ϕ_1 increases from 0, the mass m_1 rises by amount $L_1(1-\cos\phi)$, and similarly for ϕ_2 except we note that when m_1 move up vertically, that m_2 must also go up by the same amount.

$$\begin{aligned} U(\phi_1,\phi_2) &= (m_1+m_2)gL_1(1-\cos\phi_1) \\ &\quad + m_2 g L_2(1-\cos\phi_2) \end{aligned} \tag{11.37}$$

$$\begin{aligned} T &= \frac{1}{2}(m_1+m_2)L_1^2\dot\phi_1{}^2 \\ &\quad + m_2 L_1 L_2 \dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) \\ &\quad + \frac{1}{2}m_2 L_2^2 \dot\phi_2{}^2 \end{aligned} \tag{11.38}$$

Small-angles approximation. Now, assume that all of $\phi_1, \phi_2, \dot\phi_1, \dot\phi_2$ stay small. Then the cosine in T goes away since it is also multiplied by a doubly small factor of $\dot\phi_1\dot\phi_2$, and we taylor expand the cosine in U to $1-\phi^2/2$, reducing the previous equations to

$$U(\phi_1,\phi_2) = \frac{1}{2}(m_1+m_2)gL_1\phi_1^2 + \frac{1}{2}m_2 g L_2 \phi_2^2 \tag{11.40}$$

$$\begin{aligned} T &= \frac{1}{2}(m_1+m_2)L_1^2\dot\phi_1{}^2 \\ &\quad + m_2 L_1 L_2 \dot\phi_1\dot\phi_2 + \frac{1}{2}m_2 L_2^2\dot\phi_2{}^2 \end{aligned} \tag{11.39}$$

Solving the two lagrange equations and writing the 2 solutions in matrix form yields

$$\mathbf{M}\ddot\phi = -\mathbf{K}\phi \tag{11.43}$$

$$\mathbf{M} = \begin{bmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix} \tag{11.44}$$

and, as before, we solve the procedure outlined under eqs 11.11 and 11.12, with the only difference being

$$\phi(t) = \Re[z(t)] \quad \text{where} \quad z(t) = a e^{i\omega t} = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] e^{i\omega t}$$

Equal Lengths and Masses. Results in the solution for the two normal frequencies

$$\omega^2 = (2 \mp \sqrt{2})\omega_0^2 = (2 \mp \sqrt{2})\frac{g}{L} \tag{11.47}$$

and the corresponding solutions for each are below.

$$\phi(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} \tag{76}$$

$$= A_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \cos(\omega_1 t - \delta_1) \qquad [1^{st} \textbf{ MODE}] \tag{11.48}$$

$$= A_2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \cos(\omega_2 t - \delta_2) \qquad [2^{nd} \textbf{ MODE}] \tag{11.49}$$

→ **1st Mode**: The angles ϕ_1 and ϕ_2 oscillate in phase, with amplitude for ϕ_2 larger by a factor of $\sqrt{2}$.

→ **2nd Mode**: " " exactly out of phase, " "

The General Case. For the small-angle approximation with generalized coordinate-vector $\boldsymbol{q}$, the potential and kinetic energy take the forms

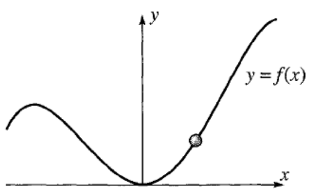
$$U = U(\boldsymbol{q}) = \frac{1}{2}\sum_{j,\,k} K_{jk} q_j q_k \tag{11.53}$$

$$T = T(\dot{\boldsymbol{q}}) = \frac{1}{2}\sum_{j,\,k} M_{jk} \dot{q}_j \dot{q}_k \tag{11.54}$$

where the matrices $\mathbf{K}$ and $\mathbf{M}$ have the exact same meaning as before. We still follow exactly the same procedure for finding ω^2 and the normal modes. The only difference is notational for the general analog of 11.43 as

$$\mathbf{M}\ddot{\boldsymbol{q}} = -\mathbf{K}\boldsymbol{q} \tag{11.60}$$

[Example 11.1] Bead on a Wire. Bead constrained to wire as shown below. Gravity exists. Concerned with small oscillations.



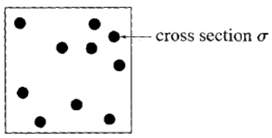
$$U(x) = mgf(x) \approx \frac{1}{2} mgf''(0)x^2$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m[1 + f'(x)^2]\dot{x}^2 \approx \frac{1}{2}m\dot{x}^2$$

IX. Collision Theory (Ch. 14)

Collision Cross Section (14.2)

- Firing projectiles at target with several hard spheres, each of cross section σ in figure



- Probability that any one projectile hits one of these targets is

$$P(\text{hit}) = \frac{\text{total area of targets}}{totalarea} \quad (77)$$
$$= \frac{n_{tar} A\sigma}{A} = n_{tar}\sigma \quad (78)$$

where n_{tar} is **number (area) density** = number of targets per unit area.

- Then number of scattered particles if we send some number N_{in} in is

$$N_{sc} = N_{inc} n_{tar} \sigma \quad (14.2)$$

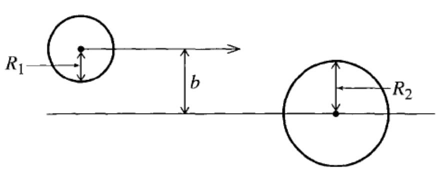
which, in practice, is used to solve for the cross section σ, since this is usually the unknown.

- Nuclear cross sections are usually given in units of **barn**, where

$$1 \text{ barn} = 10^{-28}\ m^2 \quad (79)$$

Generalizations of σ (14.3)

- *Scattering of Two Hard Spheres*.
  - They hit iff $b \leq R_1 + R_2$.



  - Center of projectile must lie inside circle with area $\sigma = \pi(R_1 + R_2)^2$. Lesson: σ is a property of the target and projectile and should be thought of as the effective area of the former for scattering the latter.
- **elastic:** internal motions of target are left unchanged, else called **inelastic.**
- *Differential Scattering Cross Section*.
  - Setup: beam along z-axis. Directions of particles given by θ and ϕ. Want to know number of particles in some cone around (θ, ϕ).
  - **Solid angle** defined as subtending area A on sphere of radius r:

$$\Delta\Omega = A/r^2 \quad (14$$
$$d\Omega = \sin\theta d\theta d\phi \quad (14$$

  - Define corresponding **differential scattering cross section** as effective cross-sectional area of target for scattering into $d\Omega$, often written as

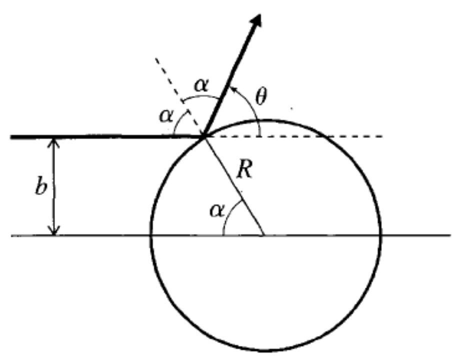
$$d\sigma = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} d\Omega$$

  - The number of particles scattered into $d\Omega$ can then be given by

$$N_{sc} = N_{inc} n_{tar} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} d\Omega \quad (14$$

where the differential cross section is typically a function of θ, ϕ.

Differential Cross Section (14.5)



- Independent of ϕ if axially symmetric. Want scattering angle θ as a function of b, or vice-versa.
- For picture above, $b = R\sin\alpha$ and $\theta = \pi - 2\alpha$
- Particles incident between b and $b + db$ that emerge between θ and $\theta + d\theta$ have

$$d\sigma = 2\pi b db \quad (14.21)$$
$$d\Omega = 2\pi \sin\theta d\theta \quad (14.22)$$

which means we can divide the two to get the differential cross section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \quad (14.23)$$

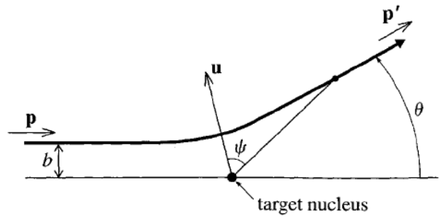
- To get diff cross section: (1) find $\theta(b)$ or $b(\theta)$, then (2) take derivatives.

Rutherford Scattering

- Scattering α particles off gold nuclei. Force on charge q from nucleus is

$$F = \frac{kqQ}{r^2} = \frac{\gamma}{r^2} \quad (14.26)$$

and so the "orbit" of the alpha is a hyperbola with nucleus at its focus, shown below.



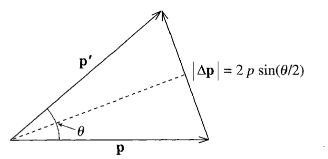


Figure 14.12 The change in momentum of the projectile is $\Delta \mathbf{p} = \mathbf{p}' - \mathbf{p}$. Since, $|\mathbf{p}| = |\mathbf{p}'|$, it is easily seen that $|\Delta \mathbf{p}| = 2p\sin(\theta/2)$.

- Orbit is symmetric about DCA intersected by $\boldsymbol{u}$. Label position of particle by ψ, the angle from $\boldsymbol{u}$, which approaches limit $\psi \to \psi_0$ as particle moves away[24]. Then

$$\theta = \pi - 2\psi_0 \tag{14.27}$$

since $2\psi_0$ is angle from horizontal (left) to eventual p' in figure.

- Math derivation with no explanations[25]:

$$\Delta \boldsymbol{p} = \boldsymbol{p}' - \boldsymbol{p} \tag{14.28}$$
$$|\Delta \boldsymbol{p}| = 2p\sin(\theta/2) \tag{14.29}$$
$$|\Delta \boldsymbol{p}| = \int_{-\infty}^{\infty} F_u dt \tag{80}$$
$$= \int_{-\psi_0}^{\psi_0} \frac{\gamma\cos\psi}{r^2}\frac{d\psi}{bp/mr^2} = \frac{2\gamma m}{bp}\cos(\theta/2) \tag{14.30}$$
$$b = \frac{\gamma}{mv^2}\cot(\theta/2) \tag{14.31}$$

- **Rutherford scattering formula**:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{kqQ}{4E\sin^2(\theta/2)}\right)^2 \tag{14.32}$$

where E is incident projectile energy $\frac{1}{2}mv^2$.

[24] Because symmetric, also have that $\psi \to -\psi_0$ as $t \to -\infty$

[25] Srsly just see where I bookmarked the pdf.

Example 14.6: Angular Dependence

– Goal: To write eq 14.32 as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta) = \frac{\sigma_0(E)}{\sin^4(\theta/2)} \tag{14.33}$$

Cross Section in Various Frames

- CM frame: projectile and target approach each other with equal/opposite momenta.
- Lab frame: Can be fixed target or both moving in opposite directions; In all cases the initial momenta are *collinear*.
- I'll save you time: All quantities individually in eq 14.2 are equal regardless of which frame (CM/lab) you're in. Equation 14.2 applies in either frame:

$$N_{sc}^{cm} = N_{inc}^{cm} n_{tar}^{cm} \sigma_{cm} \tag{81}$$
$$N_{sc}^{lab} = N_{inc}^{lab} n_{tar}^{lab} \sigma_{lab} \tag{82}$$

where, if you aren't an idiot, you should be able to reason that $N_{sc}^{cm} = N_{sc}^{lab}$ and similarly that $N_{inc}^{cm} = N_{inc}^{lab}$

- Acknowledging that θ may differ between reference frames, the following is true in either:

$$N_{sc}(\text{into } \mathrm{d}\Omega) = N_{inc} n_{tar} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} d\Omega \tag{83}$$

- The differential cross sections are *not* the same in either frame, but only because $d\Omega_{cm}$ is not necessarily equal to $d\Omega_{lab}$.

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{lab} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{cm} \frac{\mathrm{d}\Omega_{cm}}{\mathrm{d}\Omega_{lab}} \tag{84}$$

and since $d\Omega = \sin\theta d\theta d\phi = -d(\cos\theta)d\phi$ where ϕ is same in either frame,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{lab} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{cm} \frac{\mathrm{d}(\cos\theta_{cm})}{\mathrm{d}(\cos\theta_{lab})} \tag{14.45}$$

X. Center of Mass

Exhaustive Review of all CM-related material.

The **center of mass** of a system of N particles is a mass-weighted position average of the constitutent particles. The total momentum of the system can also be written in terms of the CM position.

$$\boldsymbol{R} = \frac{m_1\mathbb{R}_1 + \cdots + m_N\mathbb{R}_N}{M} \quad \textit{DISCRETE} \tag{85}$$

$$= \frac{1}{M}\int \rho\mathbb{R}dV \quad \textit{CONT.} \tag{86}$$

$$\boldsymbol{P} = \sum_\alpha m_\alpha \mathbb{R}_\alpha = M\dot{\boldsymbol{R}} \tag{87}$$

$$\mathcal{F}^{ext} = M\ddot{\boldsymbol{R}} = \dot{\boldsymbol{P}} \tag{88}$$

It is always true that, if the external torque $\boldsymbol{\Gamma}^{ext}$ about the CM is zero, then the total angular momentum $\boldsymbol{L}$ about the CM is conserved.

The **CM frame** is the inertial frame in which the CM is at rest, i.e. that $\dot{\boldsymbol{R}} = 0$.

XI. Formula Dump

Coordinate Systems

Cartesian	Polar
$x = r\cos\phi$	$r = \sqrt{x^2+y^2}$
$y = r\sin\phi$	$\phi = \arctan(y/x)$
$\mathcal{F} = m\boldsymbol{a}$	$\mathcal{F} = m(\ddot{r} - r\dot{\phi}^2)\hat{\mathbb{R}} + m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\boldsymbol{\phi}}$
Polar Vectors	**Polar Derivatives**
$\mathbb{R} = r\hat{\mathbb{R}}$	$\frac{\mathrm{d}\hat{\mathbb{R}}}{\mathrm{d}t} = \dot{\phi}\hat{\boldsymbol{\phi}}$
$\dot{\mathbb{R}} = \dot{r}\hat{\mathbb{R}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$	$\frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} = -\dot{\phi}\hat{\mathbb{R}}$
$\ddot{\mathbb{R}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\mathbb{R}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\boldsymbol{\phi}}$	
Spherical Polar Coordinates	**Cylindrical Polar**
$x = r\sin\theta\cos\phi$	$d\boldsymbol{s} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\hat{\boldsymbol{\phi}} + dz\hat{\boldsymbol{z}}$
$y = r\sin\theta\sin\phi$	$\nabla \equiv \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{z}}\frac{\mathrm{d}}{\mathrm{d}z}$
$z = r\cos\theta$	$\dot{\boldsymbol{\rho}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\boldsymbol{z}}$
$d\mathbb{R} = dr\hat{\mathbb{R}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$	
$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}$	

Ellipse

- Equation of an ellipse whose major axis is the x-axis and minor axis is the y-axis (a¿b):

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1 \tag{89}$$

  where (h, k) is the center of the ellipse.

- Parametric formula:

$$x(t) = a\cos t \tag{90}$$
$$y(t) = b\sin t \tag{91}$$

ODE Stuff

- The **Laplace transform** of a function defined for $t \geq 0$ is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \equiv F(s) \tag{92}$$

- Let $y = \sin at$ for $t > 0$ else 0.

$$\mathcal{L}\{y(t)\} = \int_0^\infty e^{-pt} \sin(at) dt = \frac{p}{p^2 + a^2} \tag{93}$$

- Let $y = g(t-a)$ for $t > a > 0$ else 0.

$$\mathcal{L}\{y(t)\} = \int_a^\infty e^{-pt} g(t-a) dt = e^{-pa} G(p) \tag{94}$$

Trig Identities

→ Double-angle formulas.

$$\begin{aligned} cos(2u) &= cos^2 u - sin^2 u & (95)\\ &= 2cos^2 u - 1 & (96)\\ &= 1 - sin^2 u & (97) \end{aligned}$$

$$tan(2u) = \frac{2tanu}{1 - tan^2 u} \tag{98}$$

→ Half-angle formulas.

$$sin^2 u = \frac{1 - cos(2u)}{2} \tag{99}$$

$$cos^2 u = \frac{1 + cos(2u)}{2} \tag{100}$$

XII. Lecture Notes

Orbits Oct 3

- Planets move in elliptical orbits:

$$\frac{\alpha}{r} = 1 + \epsilon \cos\theta \tag{101}$$
$$\alpha = a(1-\epsilon^2) \tag{102}$$
$$b^2 = a^2(1-\epsilon^2) \tag{103}$$

- Precession of planetary orbits.
- $\delta = 3GM_0/c^2 << 1$.
- Since $\delta << 1$, solution resembles ellipse:

$$u = \frac{1}{\alpha}(1+\epsilon cos\theta) - \delta u_1 + \cdots \tag{104}$$

- Expands out a ton of Taylor series.
- Moving on to scattering. Particularly hyperbolic motion when scattering in 1/r potential
- Elastic collisions:
  - No energy loss.
  - Let u_1, u_2 be initial velocities in lab frame; v_1, v_2 the final velocities in lab frame.
  - Easier to solve in CM frame. Prime quantities in this frame.

oct 10

- cross-section for scattering in CM frame.

$$\sigma(\theta) = \frac{b}{\sin\theta}\frac{\mathrm{d}b}{\mathrm{d}\theta} \tag{105}$$

- In CM frame

$$\ell = \mu\mu' b \tag{106}$$
$$= b\sqrt{2\mu E} \tag{107}$$

since $E = \frac{1}{2}\mu\mu'^2$.

- Since $\theta = \pi - 2\Theta$, need to calculate first:

$$\Theta = \int \dot{\Theta} dt \tag{108}$$
$$= \int \hat{\Theta} dr/\dot{r} \tag{109}$$
$$= \int \frac{(\ell/\mu r^2)dr}{\sqrt{2/\mu(E - U(r) - \ell^2/2\mu r^2)}} \tag{110}$$

- **Rutherford scattering formula:**
  - Suppose $U(r) = k/r$. If $k < 0$, attracting; else repelling.
  - Have

$$\Theta = \int_{r_{min}}^{\infty} \frac{(b/r^2)dr}{\sqrt{1 - 2\kappa/r - b^2/r^2}} \tag{111}$$

where $\kappa = \frac{1}{2}k/E$.

  - Let $r = b/u$ and write integral in terms of u.

$$\Theta = \int_{b/r_{min}}^{0} \frac{(u^2/b)(-b/u^2)du}{\sqrt{1 - 2\kappa u/b - u^2}} \tag{112}$$
$$\int \frac{du}{1 + \kappa^2/b^2 - (u + \kappa/b)^2} \tag{113}$$

  - Now substitute $u + \kappa/b = \sqrt{1 + \kappa/b^2}\cos\alpha$ where α is a new variable.

$$\Theta = \frac{\sqrt{1+\kappa/b^2}(-\sin\alpha)d\alpha}{\sqrt{1+\kappa/b^2}\sin\alpha} \tag{114}$$
$$= \alpha \tag{115}$$

WTF IS HAPPENING

  - Result is

$$\cos\Theta = \frac{\kappa}{\sqrt{\kappa^2 + b^2}} \tag{116}$$
$$= \sin(\Theta/2) \tag{117}$$

Oct 12

- From last lecture, know that i CM frame,

$$\sigma = \frac{\kappa^2}{4\sin^4(\theta/2)} \tag{118}$$

where θ is angle in CM frame and $\kappa = k/2E$.

- ***Dynamics in rotating frames:***
  - Want expression for acceleration

$$\frac{\mathrm{d}r}{\mathrm{d}t}|_{intertial} = \frac{\mathrm{d}r}{\mathrm{d}t}|_{rotating} + \omega \times r \tag{119}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}_{intertial} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}_{rotating} + 2\omega x \frac{\mathrm{d}r}{\mathrm{d}t}_{rot} + \dot{\omega} \times r + \omega \times (\omega \times r) \tag{120}$$

$$= F/m \tag{121}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}_{rot} = F/m - 2\omega \times v|_{rot} - \dot{\omega} \times r - \omega \times (\omega \times r) \tag{122}$$

where last three terms are Coriolis force, azimuthal force, centrifugal force.

Motion on a turntable

- Assume ω is constant. Draw a circle centered in xy plane. Radial motion in rotating frame.
- $v = ve_x$ (unit vector e) $r = xe_x$.
- Effective force

$$F_{eff} = -m\omega \times (\omega \times r) - 2m\omega \times v \tag{123}$$

$$= -m\omega^2 e_z \times (e_z \times e_x) - 2m\omega v e_z \times e_x \tag{124}$$

$$= -m\omega^2(-e_x) - 2m\omega v e_y \tag{125}$$

- if particle is moving in y direction on turntable $v = ve_y$, then

$$F_{eff} = m\omega^2 x e_x + 2m\omega v e_x \tag{126}$$

so the forces are now in the same direction.

Motion relative to Earth

- Effect of earth is actually rotating?
- $r = R + r'$ where R is origin of my rotating coordinate system and r' is measured relative to that.
- newton's second law

$$m\ddot{r}' = F_{inert} - 2\omega \times \dot{r}' - m\dot{\omega} \times (R + r') - m\omega \times (\omega \times (R + r')) \tag{127}$$

$$= F_{inert} - 2m\omega \times \dot{r} - m\dot{\omega} \times R - m\omega \times (\omega \times R) \tag{128}$$

$$= mg' - 2m\omega \times \dot{r}' \tag{129}$$

where g' is now the effective gravitational acceleration.

- $t = \sqrt{2h/g'}$ for particle in free-fall deflection from Coriolis force. $\ddot{x} = 2\omega g' t \cos\lambda$.

Lagrangian examples Oct 21

Example: spring pendulum. Natural length l and spring const k.

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{130}$$

$$U = -mgr\cos\theta + \frac{1}{2}k(r-l)^2 \tag{131}$$

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta + \frac{1}{2}k(r-l)^2 \tag{132}$$

then take derivatives to get Lagrange equations.

Example: Pendulum with moving vertical support. Pendulum origin moving up. Apparently only one degree of freedom still, since $\zeta(t)$ (vertical) is known (wtf?).

$$T = \frac{1}{2}m\left[(l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta + \dot{\zeta})^2\right] \tag{133}$$

$$= \frac{1}{2}m\left[(l^2\dot{\theta}^2 + 2(l\dot{\theta}\dot{\zeta}\sin\theta) + 2\dot{\zeta}^2\right] \tag{134}$$

$$U = -mgl\cos\theta + mg\zeta(t) \tag{135}$$

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}m\left[(l^2\dot{\theta}^2 + 2(l\dot{\theta}\dot{\zeta}\sin\theta) + 2\dot{\zeta}^2\right] + mgl\cos\theta - mg\zeta(t) \tag{136}$$

then take derivatives and stuff. Result is

$$l\ddot{\theta} + (g + \ddot{\zeta})\sin\theta = 0 \tag{137}$$

Note: effect of accelerating up is something something gravity something something equivalence principle.

Now attach pendulum to spring. Phase space variables now are $(\zeta, \theta, \dot{\zeta}, \dot{\theta})$.

$$T = \frac{1}{2}m\left[l^2\dot{\theta}^2 + 2(l\dot{\theta}\dot{\zeta}\sin\theta) + 2\dot{\zeta}^2\right] \tag{138}$$

$$U = -mgl\cos\theta + mg\zeta(t) + \frac{1}{2}k(\zeta - \zeta_0)^2 \tag{139}$$

which is basically same as before.

Mathieu equation. Suppose $\zeta(t)$ is given and $\zeta = a\cos\Omega t$ is position of support of pendulum [moves in sinusoidal way]. Then we have to solve

$$\ddot{\theta} + (\alpha + \beta\cos t)\theta = 0 \tag{140}$$

which is the **Mathieu equation**. Not a constant coefficient problem. Can't look for solutions that go like e^{-t} or anything like that. α is square of natural frequency of oscillator. Each time pendulum goes through $\theta = 0$, the tension is largest. Suppose exactly at that time, we are raising the mass. Then we are doing *work* on the mass. Suppose oscillation of support movement is twice the oscillation of $\sqrt{\alpha}$. We are putting enery in the system every cycle then. Therefore, **unstable solution** however small β is, as long as it is not zero. This is simple model for parametric

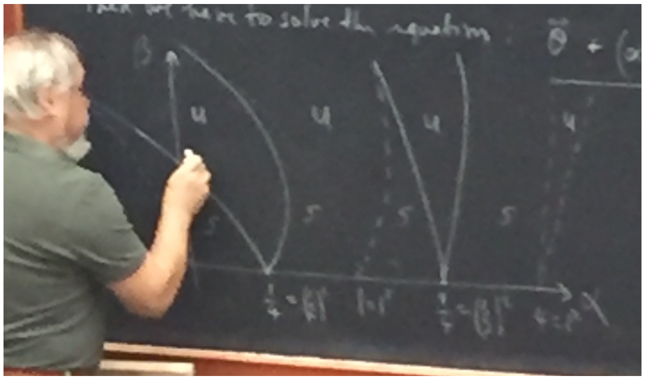


Figure 5: “And here you can see me drawing some bullshit” - Knobloch

instability.