THE ASYMPTOTIC DIMENSION OF THE GRAND ARC GRAPH IS INFINITE

MICHAEL C. KOPRESKI

Abstract. Let Σ be a compact, orientable surface of genus g, and let Γ be a relation on $\pi_0(\partial\Sigma)$ such that the prescribed arc graph $\mathcal{A}(\Sigma,\Gamma)$ is Gromov-hyperbolic and non-trivial. We show that $\operatorname{asdim}\mathcal{A}(\Sigma,\Gamma) \geq -\chi(\Sigma) - 1$, from which we prove that the asymptotic dimension of the grand arc graph is infinite. More generally, an *arc and curve model* on Σ is a graph of simple arc and curves on Σ, on which $\mathrm{PMap}(\Sigma)$ acts by permuting vertices. We prove that any connected, Gromov-hyperbolic cocompact arc and curve model $\mathcal{M}$ has $\operatorname{asdim}\mathcal{M} \geq g - \lceil \frac{1}{2}\chi(\Sigma)\rceil$, and that a broad class of arc and curve models on infinite-type surfaces has infinite asymptotic dimension.

1. Introduction

Let Σ be a compact, orientable surface with boundary, and let Γ be a relation on $\pi_0(\partial\Sigma)$. A simple, essential arc a in Σ is Γ-*allowed* if it joins boundary components in Γ. The Γ-*prescribed arc graph* $\mathcal{A}(\Sigma,\Gamma)$ is the full subgraph of $\mathcal{A}(\Sigma)$ spanned by isotopy classes of Γ-allowed arcs. We assume throughout that $\mathcal{A}(\Sigma,\Gamma)$ is *non-trivial*, *i.e.* $\chi(\Sigma) \leq -1$, $\Sigma \neq \Sigma_0^3$, and $\Gamma \neq \varnothing$.

We suppose $\mathcal{A}(\Sigma,\Gamma)$ is δ-hyperbolic. If $\Sigma = \Sigma_0^4$, then $\mathcal{A}(\Sigma,\Gamma) \subset \mathcal{A}(\Sigma_0^4)$ is a quasi-tree and $\operatorname{asdim}\mathcal{A}(\Sigma,\Gamma) = 1$. Otherwise, we prove a lower bound:

Theorem 1.1. *If $\mathcal{A}(\Sigma,\Gamma)$ is δ-hyperbolic, then $-\chi(\Sigma) - 1 \leq \operatorname{asdim}\mathcal{A}(\Sigma,\Gamma)$.*

For Ω an infinite-type surface with finite grand splitting, let $\mathcal{G}(\Omega)$ denote the *grand arc graph* on Ω [BNV22]. By applying Theorem 1.1, we obtain the following:

Theorem 1.2. *If $\mathcal{G}(\Omega)$ is non-empty and connected, then $\operatorname{asdim}\mathcal{G}(\Omega) = \infty$.*

More generally, an *arc and curve model* on a surface Ω is a connected graph whose vertices are collections of (possibly intersecting) simple arcs and curves, with an action of $\mathrm{PMap}_c(\Omega)$ induced by the permutation of its vertices. An arc and curve model $\mathcal{M}$ is *witness-cocompact* if it has a (compact) witness and in each witness has uniformly bounded geometric (self-)intersection over the vertices and edges in $\mathcal{M}$.

Theorem 1.3. *If Ω is compact and $\mathcal{M}$ is δ-hyperbolic, then $\operatorname{asdim}\mathcal{M} \geq g(\Omega) - \lceil \frac{1}{2}\chi(\Omega)\rceil$. If Ω is infinite-type, then $\operatorname{asdim}\mathcal{M} = \infty$.*

In Section 2, we use the theory of alignment-preserving maps [DT17] to show that the Gromov boundary $\partial\mathcal{A}(\Sigma, \Gamma)$ contains $\partial\mathcal{A}(\Sigma)$. From results of Gabai [Gab14] and Schleimer [Po17] we obtain a compact subspace $Z \subset \partial\mathcal{A}(\Sigma, \Gamma)$ of dimension $-\chi(\Sigma) - 2$. We then prove that $\operatorname{asdim} \mathcal{A}(\Sigma, \Gamma) \geq \dim Z + 1$, extending a result for proper δ-hyperbolic spaces, whence Theorem 1.1 follows.

In Section 3, we show that witness subsurfaces $W \subset \Omega$ for $\mathcal{G}(\Omega)$ of arbitrarily large complexity admit prescribing relations Γ such that $\mathcal{A}(W, \Gamma)$ quasi-isometrically embeds into $\mathcal{G}(\Omega)$, where Ω is an infinite-type surface with finite grand splitting. In fact, W may be chosen so that either $\mathcal{A}(W, \Gamma)$ has large coarse rank or it is δ-hyperbolic: Theorem 1.2 thus follows from Theorem 1.1 and the monotonicity of asymptotic dimension.

Section 4 generalizes the techniques in Sections 2 and 3 to witness-cocompact arc and curve models, which include prescribed arc graphs, the grand arc graph, the the marking complex, and many other multiarc and curve graphs. In addition to tools developed in Section 2, we utilize properties of the hierarchically hyperbolic structure of such graphs in the finite-type setting [Kop23a]. Theorem 1.3 follows analogously in the witness-cocompact case.

Remark. For the reader interested in only Theorem 1.3 (which does imply Theorem 1.2 and a weaker version of Theorem 1.1, albeit with more technology than necessary), it suffices to read Sections 2.1 and 4.

1.1. **Background.** An orientable surface Ω has *infinite topological type* if its fundamental group is not finitely generated, or equivalently if $\operatorname{int}(\Omega)$ has infinite genus or infinitely many punctures (we typically assume $\partial\Omega = \varnothing$). Beginning with a 2009 blog post of Calegari [Cal09], mapping class groups of infinite-type surfaces have been objects of considerable contemporary study: see [AV20, CPV21] for surveys of recent results and open problems.

An infinite-type surface Ω is classified by its genus and *end space*, which is obtained as the inverse limit of the complementary components of a compact exhaustion [Ric63]; its mapping class group $\operatorname{Map}(\Omega)$ is a non-compactly generated Polish group. Given mild assumptions, Mann and Rafi [MR20] classify when $\operatorname{Map}(\Omega)$ admits a generating set that is *coarsely bounded (CB)*, or bounded in any left-invariant metric, and hence a well defined quasi-isometry type in the sense of [Ros14]. Mann–Rafi also define a preorder on the ends of Ω corresponding to topological complexity. We denote by $\mathscr{M}(\Omega)$ the non-empty subspace of maximal ends with respect to this preorder.

When $\operatorname{Map}(\Omega)$ is locally CB (and in particular when it is CB-generated), Bar-Natan and Verberne define the *grand splitting* $\mathcal{S}(\Omega)$, a canonical and $\operatorname{Map}(\Omega)$-invariant partition of $\mathscr{M}(\Omega)$ into finitely many disjoint sets $E_i \in \mathcal{S}(\Omega)$, each of which is either a singleton or Cantor set. A *grand arc* in Ω is a bi-infinite simple arc converging to ends in distinct sets in the grand splitting [BNV22].

Definition 1.4 (Bar-Natan–Verberne)**.** Let Ω be an infinite-type surface. The *grand arc graph* $\mathcal{G}(\Omega)$ is the simplicial graph with vertices corresponding to isotopy classes of grand arcs and edges determined by disjointness.

The grand arc graph $\mathcal{G}(\Omega)$ is an arc and curve model for Ω which generalizes the ray graph defined by Calegari [Cal09] on $S^2 \setminus$ Cantor set and for surfaces with stable endspace extends the omnipresent arc graph defined by Fanoni–Ghaswala–McLeay [FGM21]. $\mathrm{Map}(\Omega)$ acts naturally on $\mathcal{G}(\Omega)$ by isometries. Bar-Natan–Verberne classify the δ-hyperbolicity of $\mathcal{G}(\Omega)$ and show that when $\mathcal{G}(\Omega)$ is δ-hyperbolic, the action of $\mathrm{Map}(\Sigma)$ is quasi-continuous, extends continuously to $\partial\mathcal{G}(\Omega)$, and has loxodromic elements.

Notation. We typically denote by Σ a compact, orientable surface, and by Ω an arbitrary orientable surface that may have either finite or infinite topological type.

1.1.1. *Prescribed arc graphs, witnesses.* Prescribed arc graphs were defined by the author in [Kop23b] as arc and curve models of finite-type surfaces that quasi-isometrically embed into $\mathcal{G}(\Omega)$. Excepting trivial cases they are connected and infinite-diameter and their δ-hyperbolicity is fully determined by the prescribing relation Γ:

Theorem 1.5 ([Kop23b, Thm. 1.3])**.** *Assume that $\mathcal{A}(\Sigma, \Gamma)$ is non-trivial. Then if $\chi(\Sigma) \geq -2$ or $\Sigma = \Sigma_0^{n+1}$ and Γ is a n-pointed star then $\mathcal{A}(\Sigma, \Gamma)$ is δ-hyperbolic. Otherwise, $\mathcal{A}(\Sigma, \Gamma)$ is (uniformly) δ-hyperbolic if and only if Γ is not bipartite.*

We note that if $\Gamma \subset \Gamma'$ then every Γ-allowed arc is Γ'-allowed, which induces a simplicial map $\iota : \mathcal{A}(\Sigma, \Gamma) \to \mathcal{A}(\Sigma, \Gamma')$. This map is 3-coarsely surjective [Kop23b, Lem. 2.10]. In particular, since the prescribed arc graph with the complete relation is exactly $\mathcal{A}(\Sigma)$, $\mathcal{A}(\Sigma, \Gamma)$ always coarsely surjects onto $\mathcal{A}(\Sigma)$.

A compact, essential (π_1-injective, non-peripheral) subsurface without pants components is a *witness* for a given arc and curve model if each component intersects every vertex. We call a witness $W \subset \Sigma$ for $\mathcal{A}(\Sigma, \Gamma)$ a Γ-*witness.*

1.1.2. *Boundaries of non-proper δ-hyperbolic spaces.* In general, if $\mathcal{A}(\Sigma, \Gamma)$ is non-trivial then it is non-proper, and likewise for any admissible arc and curve model with sufficient complexity. For a geodesic δ-hyperbolic space X, by ∂X we always mean the sequential boundary of X; when X is non-proper, ∂X may be non-compact. In this setting, ∂X does not coincide with the geodesic boundary, but is instead homeomorphic to the quasi-geodesic boundary [Has22]. We will make use of the following statement by Hasegawa, from a construction of Kapovich–Benakli [KB02, Rmk. 2.16]:

Remark 1.6 ([Has22, Prop. 4]). Fixing $x_0 \in X$, for any $z \in \partial X$ there exists a $(1+4\delta, 12\delta)$-quasi-geodesic ray $\rho : [0, \infty) \to X$ based at x_0 with $[\rho(n)] = z$.

Any quasi-isometry between geodesic δ-hyperbolic spaces $X \to Y$ extends to a map $X \cup \partial X \to Y \cup \partial Y$ that restricts to a homeomorphism on the boundaries (*e.g.* applying the proof of [DK18, Thm. 11.108]). Given $x, y \in X \cup \partial X$, let $(x|y)_{x_0}$ denote their Gromov product at x_0. We occassionally omit the basepoint, which is changeable up to bounded error.

1.1.3. *Ending laminations.* Let $\chi(\Sigma) \leq -1$, hence fix a (finite-area) hyperbolic metric for Σ with geodesic boundary. We recall that a *geodesic lamination* on Σ is a closed subset $L \subset \Sigma$ which decomposes (in fact, uniquely) into pair-wise disjoint simple geodesic leaves. L is *minimal* if it has no proper sublaminations, or equivalently, if every leaf is dense in L.

Definition 1.7. Given a connected subspace $X \subset \Sigma$ with non-trivial π_1-image, if $Y \subset \Sigma$ is the smallest essential subsurface containing X up to isotopy, then Y is *filled* by X. If $Y = \Sigma$, then X is *filling*.

Definition 1.8. The space of *ending laminations* $\mathcal{EL}(\Sigma)$ is the set of filling minimal laminations on Σ, equipped with the coarse Hausdorff topology. Similarly, let $\mathcal{EL}_0(\Sigma)$ denote the space of minimal laminations that fill a subsurface containing $\partial\Sigma$, again with the coarse Hausdorff topology.

$\mathcal{EL}(\Sigma)$ and $\mathcal{EL}_0(\Sigma)$ give explicit descriptions for the hyperbolic boundaries of $\mathcal{C}(\Sigma)$ and $\mathcal{A}(\Sigma)$, respectively (see [Kla18] and [Po17]):

Theorem 1.9 (Klarreich, Schleimer)**.** $\mathcal{EL}(\Sigma) \cong \partial\mathcal{C}(\Sigma)$ *and* $\mathcal{EL}_0(\Sigma) \cong \partial\mathcal{A}(\Sigma)$.

1.1.4. *Markings.* In Section 4, we will make use of *markings* on surfaces in the sense of [MM00]. For an essential simple closed curve $a \subset \Omega$, let $\mathcal{C}(a)$ denote the curve graph of the annulus with core a and π_a the corresponding (set-valued) subsurface projection.

Definition 1.10. A *marking* $\mu = \{(a_i, t_i)\}$ on a surface Ω is an essential simple multicurve $\{a_i\}$, denoted $\operatorname{base}\mu$, along with a collection of (possibly empty) diameter 1 subsets $t_i \subset \mathcal{C}(a_i)$; for $a_i \in \operatorname{base}\mu$, let $\operatorname{trans}_\mu(a_i) = t_i$ denote the associated *transversal.*

A marking μ is *complete* if $\operatorname{base}\mu$ is a pants decomposition and every transversal is non-empty. If μ is complete and for each component $(a, t) \in \mu$ $t = \pi_a b$ for some simple closed curve $b \neq a$ disjoint from $\operatorname{base}\mu \setminus \{a\}$ that intersects a minimally, then μ is *clean.*

Definition 1.11 ([Kop23a, Def. 2.2])**.** A marking μ on a surface Ω is *locally clean* if the maximal submarking μ' with only non-empty transversals is complete and clean in each component of $\Omega \setminus (\mu \setminus \mu')$ that it intersects.

Let $\Delta \subset \Omega$ be an essential, non-pants subsurface. Like multicurves, markings have a subsurface projection $\pi_\Delta(\mu) \subset \mathcal{C}(\Delta)$. If Δ is an annulus parallel to some $a \in \operatorname{base}\mu$, then $\pi_\Delta(\mu) := \operatorname{trans}_\mu(a) \subset \mathcal{C}(\Delta)$. Otherwise, $\pi_\Delta(\mu) := \pi_\Delta(\operatorname{base}\mu)$. We say Δ intersects μ if and only if $\pi_\Delta(\mu) \neq \varnothing$. For an essential simple closed curve $c \subset \Omega$, again let $\pi_c(\mu)$ denote the projection to the annulus with core c.

Definition 1.12. Let μ, ν be two markings on Ω. Then their *geometric intersection number* $i(\mu,\nu)$ is defined as follows:

$$i(\mu,\nu) := i(\operatorname{base}\mu, \operatorname{base}\nu) + \sum_{a \in \operatorname{base}\mu \cup \operatorname{base}\nu} \operatorname{diam}_{\mathcal{C}(a)}(\pi_a\mu \cup \pi_a\nu)$$

1.1.5. *Alignment-preserving maps.* We briefly recall the theory of alignment-preserving maps from [DT17]. Let X be a geodesic metric space. Then a triple $(x,y,z) \in X^3$ is *K-aligned* if $d(x,y) + d(y,z) \leq d(x,z) + K$. A Lipschitz map between geodesic metric spaces $f : X \to Y$ is *coarsely alignment preserving* if there exists $K \geq 0$ for which f maps any 0-aligned triple in X to a K-aligned triple in Y.

Suppose that $f : X \to Y$ is a coarsely alignment preserving map between geodesic δ-hyperbolic spaces. Then we define $\partial_Y X \subset \partial X$ to be

$$\partial_Y X := \{[\gamma] \in \partial X \mid \gamma : \mathbb{R}^+ \to X \text{ quasi-geodesic, } \operatorname{diam}_Y(f\gamma(\mathbb{R}^+)) = \infty\}.$$

Theorem 1.13 (Dowdall–Taylor, [DT17, Thm. 3.2])**.** *Let $f : X \to Y$ be a coarsely surjective, coarsely alignment preserving map between geodesic δ-hyperbolic spaces. Then f admits an extension to a homeomorphism $\partial f : \partial_Y X \to \partial Y$ such that if $x_n \to \omega \in \partial_Y X$, then $f(x_n) \to \partial f(\omega)$.*

2. Asymptotic dimension of $\mathcal{A}(\Sigma, \Gamma)$

When $\Sigma = \Sigma_0^4$, then $\mathcal{A}(\Sigma,\Gamma) \subset \mathcal{A}(\Sigma_0^4)$ is an infinite-diameter connected subgraph of a quasi-tree, hence likewise a quasi-tree: $\operatorname{asdim}\mathcal{A}(\Sigma,\Gamma) = 1$. For $\Sigma \neq \Sigma_0^4$, we first prove $\mathcal{EL}_0(\Sigma) \cong \partial\mathcal{A}(\Sigma) \subset \partial\mathcal{A}(\Sigma,\Gamma)$ for Γ not bipartite.

Lemma 2.1. *If Γ is not bipartite then for any $\Gamma' \supset \Gamma$ the induced coarse surjection $\iota : \mathcal{A}(\Sigma,\Gamma) \to \mathcal{A}(\Sigma,\Gamma')$ is uniformly coarsely alignment-preserving.*

Proof. We first claim that if Γ is not bipartite, then geodesics in $\mathcal{A}(\Sigma,\Gamma)$ are uniformly (independent of Γ) Hausdorff close to unicorn paths with coarsely the same endpoints, and *vice versa.* If Γ is not bipartite and if $\Sigma = \Sigma_1^2$ then Γ is not two loops, then the claim holds by [Kop23b, §3]. If instead $\Sigma = \Sigma_1^2$ and $\Gamma = \ell_1 \cup \ell_2$ is two loops, then $\iota : \mathcal{A}(\Sigma,\ell_1) \to \mathcal{A}(\Sigma,\Gamma)$ is a quasi-isometry by [Kop23b, Lem. 5.2] and we apply the Morse lemma. We observe that if Γ is not bipartite then neither is Γ'.

We claim that for any geodesic γ between $a, b \in \mathcal{A}(\Sigma,\Gamma)$, $\iota\gamma$ is uniformly Hausdorff close to a geodesic between $\iota(a), \iota(b)$, whence the proof follows. Let γ' be a unicorn path close to γ, in the sense above. ι is Lipschitz, hence

$\iota\gamma, \iota\gamma'$ are close; since $\iota\gamma'$ is a unicorn path in $\mathcal{A}(\Sigma, \Gamma')$, choose a geodesic γ'' close to $\iota\gamma'$. By the Morse lemma, there exists a geodesic γ''' between $\iota(a), \iota(b)$ that is close to γ'', hence close to $\iota\gamma$. $\square$

Applying Theorem 1.13, we obtain the desired embedding.

Corollary 2.2. *If Γ is not bipartite and $\Gamma' \supset \Gamma$, then there exists an embedding $(\partial\iota)^{-1} : \partial\mathcal{A}(\Sigma, \Gamma') \to \partial\mathcal{A}(\Sigma, \Gamma)$.* $\square$

By [Kop23b, §5], if $\Sigma \neq \Sigma_0^4$ and $\mathcal{A}(\Gamma, \Sigma)$ is δ-hyperbolic, then (i) $\Sigma = \Sigma_0^{n+1}$ and Γ is an n-pointed star, (ii) $\Sigma = \Sigma_1^2$ and Γ is a non-loop edge, or (iii) Γ is not bipartite. In case (i), by [Kop23b, Lem. 5.4] $\mathcal{A}(\Sigma, \Gamma)$ is quasi-isometric to $\mathcal{A}(\Sigma, \ell_0)$, where ℓ_0 is a single loop and hence not bipartite. Thus for cases (i) and (iii), Corollary 2.2 implies $\partial\mathcal{A}(\Sigma) \subset \partial\mathcal{A}(\Sigma, \Gamma)$. In case (ii), every Γ-witness is in fact a witness for the usual arc graph: by [Kop23a] $\mathcal{A}(\Sigma, \Gamma)$ and $\mathcal{A}(\Sigma)$ have the same quasi-isometry type, hence $\partial\mathcal{A}(\Sigma, \Gamma) \cong \partial\mathcal{A}(\Sigma)$.

Proposition 2.3. *Let $\Sigma \neq \Sigma_0^4$ and $\mathcal{A}(\Sigma, \Gamma)$ be δ-hyperbolic. $\partial\mathcal{A}(\Sigma) \cong \mathcal{EL}_0(\Sigma)$ embeds canonically into $\partial\mathcal{A}(\Sigma, \Gamma)$.* $\square$

2.1. **A lower bound.** From [Gab14], we have the following:

Theorem 2.4 (Gabai)**.** *Let S be the $(n + 4)$-times punctured sphere for $n \geq 0$. Then $\mathcal{EL}(S)$ is homeomorphic to the n-dimensional Nöbeling space.*

For any Σ with $\chi(\Sigma) \leq -2$, let $n = n(\Sigma) = -\chi(\Sigma) - 2$ and let Γ be a prescribing relation such that $\mathcal{A}(\Sigma, \Gamma)$ is δ-hyperbolic. We may choose an essential $(n + 4)$-punctured sphere S that contains all of the punctures of Σ, thus $\mathcal{EL}(S) \subset \mathcal{EL}_0(\Sigma) \cong \partial\mathcal{A}(\Sigma)$. Then applying Proposition 2.3 and Theorem 2.4, $\partial\mathcal{A}(\Sigma, \Gamma)$ contains the n-dimension Nöbeling space, and in particular, a compact subspace $Z \subset \mathcal{EL}(S)$ of topological dimension n by the universal embedding property of Nöbeling spaces [Nöb30].

For the remainder of the section, we will prove the following generalization of a result for proper δ-hyperbolic spaces (*e.g.* [BL08, Prop. 6.2]):

Proposition 2.5. *Let X be a geodesic δ-hyperbolic space with $Z \subset \partial X$ compact. Then* $\operatorname{asdim} X \geq \dim Z + 1$.

Since $\partial\mathcal{A}(\Sigma, \Gamma)$ contains a $n(\Sigma)$-dimensional compact subspace for $\chi(\Sigma) \leq -2$, Theorem 1.1 follows (vacuously for $\chi(\Sigma) > -2$).

By δ-hyperbolic, we mean that geodesic triangles are δ-slim. Let X be a geodesic δ-hyperbolic space and let $Z \subset \partial X$ be compact. A metric $d : \partial X \times \partial X \to [0, \infty)$ is *visual* if there exist k_1, k_2 and $a > 0$ such that

$$k_1 a^{-(\xi|\xi')} \leq d(\xi, \xi') \leq k_2 a^{-(\xi|\xi')}.$$

Such metrics always exist [BH99, Prop. III.H.3.21] and are compatible with the usual topology on the (sequential) boundary: $d(\xi_i, \xi) \to 0$ if and only if $(\xi_i|\xi) \to \infty$, which is equivalent to $\xi_i \to \xi$.

Notation. Where unambiguous, we denote by $|xx'|$ the distance between $x, x' \in X$ a metric space. Given a specified basepoint $o \in X$, let $|x| := |ox|$.

Definition 2.6. For (Z, d) a bounded metric space, the *hyperbolic cone* $\operatorname{Co} Z$ is the topological cone $Z \times [0, \infty)/Z \times \{0\}$ endowed with the following metric. Let $\mu = \pi / \operatorname{diam}(Z)$. For any $x = (z, t), x' = (z', t') \in \operatorname{Co} Z$, consider a geodesic triangle $\bar{o}\bar{x}\bar{x}' \subset H^2$ with $|\bar{o}\bar{x}| = t, |\bar{o}\bar{x}'| = t'$, and $\angle_{\bar{o}}(\bar{x}, \bar{x}') = \mu|zz'|$. Then let $|xx'| := |\bar{x}\bar{x}'|$.

This metric is compatible with the usual topology on $\operatorname{Co} Z$. In addition, $\operatorname{Co} Z$ is δ-hyperbolic, $Z \hookrightarrow \partial \operatorname{Co} Z$ via the geodesic rays $\gamma_z : t \mapsto (z, t)$, and d is visual for $Z \subset \partial \operatorname{Co} Z$ with respect to $\operatorname{Co} Z$ [Buy06, Prop. 6.1]. We fix $o = Z \times \{0\}$ as a basepoint for $\operatorname{Co} Z$. Analogously to [Buy06, Prop. 6.2], we have the following:

Lemma 2.7. *Let X be a geodesic δ-hyperbolic space and let $Z \subset \partial X$ be compact. Then $\operatorname{Co} Z$ quasi-isometrically embeds into X.*

Proof. Fix a basepoint $x_0 \in X$ and let $\delta' = \delta(\operatorname{Co} Z)$. Since d is visual for both X and $\operatorname{Co} Z$, up to rescaling X we may assume that $(z|z')_{x_0}$ and $(z|z')_o$ are uniformly close for all $z, z' \in Z$. For each $z \in Z$, fix a representative $(\kappa_0, 12\delta)$-quasi-geodesic ray $\rho_z \in z$ eminating from x_0 by Remark 1.6, where $\kappa_0 = 1 + 4\delta$. Let $\iota : \operatorname{Co} Z \to X$ be the map $(z, t) \mapsto \rho_z(t)$.

Since $\gamma_z \in z$ is geodesic, $(z|\gamma_z(t))_o > |\gamma_z(t)| - \delta'$ and $|\gamma_z(t)| = t$. Likewise, since $\rho_z \in z$ is $(\kappa_0, 12\delta)$-quasi-geodesic, $(z|\rho_z(t))_{x_0} > |\rho_z(t)| - M - \delta$, where $M = M(\kappa_0, 12\delta)$ is the Morse constant, and $|\rho_z(t)| = \kappa_z(t)t + O_\delta(1)$ with $\frac{1}{\kappa_0} \le \kappa_z(t) \le \kappa_0$. Let $y = (z, t), y' = (z', t') \in \operatorname{Co} Z$. By [BS00, Lem. 5.1], we have

$$
\begin{aligned}
|yy'| &= |\gamma_z(t)\gamma_{z'}(t')| \\
&= |\gamma_z(t)| + |\gamma_{z'}(t')| - 2\min\{|\gamma_z(t)|, |\gamma_{z'}(t')|, (z|z')_o\} + O_{\delta'}(1) \\
&= t + t' - 2\min\{t, t', (z|z')_o\} + O_{\delta'}(1)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
|\iota(y)\iota(y')| &= |\rho_z(t)\rho_{z'}(t')| \\
&= \kappa_z(t)t + \kappa_{z'}(t')t' - 2\min\{\kappa_z(t)t, \kappa_{z'}(t')t', (z|z')_{x_0}\} + O_\delta(1).
\end{aligned}
$$

ι is a quasi-isometric embedding. □

Applying the argument in [BL08, Prop. 6.5], we obtain that $\operatorname{asdim} \operatorname{Co} Z \ge \dim Z + 1$. Proposition 2.5 then follows from Lemma 2.7. //

3. Asymptotic dimension of $\mathcal{G}(\Omega)$

We prove Theorem 1.2. Let Ω be a surface of infinite topological type with finite grand splitting $\mathcal{S}(\Omega)$.

Definition 3.1. An essential, connected, compact subsurface $\Sigma \subset \Omega$ is *fully separating* if every component of $\partial\Sigma$ is separating.

Any compact subsurface can be enlarged to one that is fully separating: *e.g.* we may glue 1-handles between boundary components adjacent to the same complementary component and take the compact surface filled by the result.

Lemma 3.2. *Suppose that $\Sigma \subset \Omega$ is a fully separating non-annular witness for $\mathcal{G}(\Omega)$ and $|\mathcal{S}(\Omega)| = m$. There exists a minimally m-partite relation Γ on $\pi_0(\partial\Sigma)$ such that $\mathcal{A}(\Sigma, \Gamma)$ quasi-isometrically embeds into $\mathcal{G}(\Omega)$. In particular, Γ is not bipartite if $|\mathcal{S}(\Omega)| > 2$.*

Proof. Since Σ is a witness for $\mathcal{G}(\Omega)$, it must separate distinct sets in $\mathcal{S}(\Omega)$. In particular, each boundary component is adjacent to a complementary component containing ends in at most one set in $\mathcal{S}(\Omega)$. Color each component $c \in \pi_0(\partial\Sigma)$ with the corresponding set $e(c) \in \mathcal{S}(\Omega)$, if one exists; let Γ be the complete m-partite relation on these colors (components without a corresponding class are left isolated).

Fix a hyperbolic metric on Σ. For each colored boundary component c, choose a parameterization $c : [0, 1) \to \Sigma$ and a simple ray ρ_c disjoint from $\mathrm{int}(\Sigma)$ with origin $c(0)$ and converging to an end in $e(c)$. Let $a \in \mathcal{A}(\Sigma, \Gamma)$ be an arc that terminates on $c_1, c_2 \in \pi_0(\partial\Sigma)$. Let α be the geodesic representative for a with endpoints $c_i(t_i)$ and define $\delta_i = c_i|_{[0,t_i]}$ to be the subpath of c_i between $c_i(0)$ and $c_i(t_i)$. Let $\alpha^\dagger$ denote the extension of α from both endpoints by $\bar{\delta}_i * \rho_{c_i}$, for $i = 1, 2$ as appropriate. $\alpha^\dagger$ is a simple arc converging to ends in $e(c_1), e(c_2)$ respectively, which are distinct in $\mathcal{S}(\Omega)$ by our choice of Γ. $\alpha^\dagger$ is a grand arc. The map $a \mapsto [\alpha^\dagger]$ preserves disjointness hence extends to a simplicial (1-Lipschitz) map $\psi : \mathcal{A}(\Sigma, \Gamma) \to \mathcal{G}(\Omega)$.

We show that ψ is a quasi-isometric embedding by constructing a coarse Lipschitz retraction $\pi : \mathcal{G}(\Omega) \to \mathcal{A}(\Sigma, \Gamma)$. For a grand arc $w \in \mathcal{G}(\Omega)$, fix a representative ω that is geodesic in Σ. Let $\omega^\pm$ denote the first and last intersections of ω with Σ and let $\hat{\omega}$ denote the shortest path between ω^- and ω^+ in $(\omega \cap \Sigma) \cup \partial\Sigma$. Since ω converges to maximal ends distinguished by $\mathcal{S}(\Omega)$, $\omega^\pm$ lie on boundary components with distinct colors: isotoping $\hat{\omega}$ into the interior of Σ rel $\omega^\pm$, $\hat{\omega}$ is Γ-allowed and we define $\pi : w \mapsto [\hat{\omega}]$. From the constructions of ψ, π, it is immediate that $\pi\psi$ is identity on $\mathcal{A}(\Sigma, \Gamma)$. We verify that π is Lipschitz. Let $w, w' \in \mathcal{G}(\Omega)$ be disjoint grand arcs and let $\pi(w) = [\hat{\omega}]$ and $\pi(w') = [\hat{\omega}']$ as above. Since $\hat{\omega}$ is constructed as a shortest path, it contains at most $|\pi_0(\partial\Sigma)| - 1$ segments that are components of $\omega \cap \Sigma$. Each of these segments intersects $\hat{\omega}'$ at most twice and in subsegments of $\hat{\omega}'$ parallel to $\partial\Sigma$, and the same statement holds exchanging $\hat{\omega}$ and $\hat{\omega}'$. Thus $i(\hat{\omega}, \hat{\omega}') \leq 4|\pi_0(\partial\Sigma)| - 4$. Finally, since $d([\hat{\omega}], [\hat{\omega}']) \leq i(\hat{\omega}, \hat{\omega}') + 1$ by [Kop23b, Prop. 2.6], we obtain that π is $(4|\pi_0(\partial\Sigma)| - 3)$-Lipschitz. ☐

Witnesses for $\mathcal{G}(\Sigma)$ exist [BNV22, Lem. 2.7] and their enlargements are likewise witnesses, hence there exist fully separating witnesses $\Sigma \subset \Omega$ of arbitrarily large complexity. If $|\mathcal{S}(\Omega)| > 2$ and Γ is chosen as in Lemma 3.2, then $\mathcal{A}(\Sigma, \Gamma)$ is δ-hyperbolic by Theorem 1.5 and by Lemma 3.2 and Theorem 1.1 $\operatorname{asdim} \mathcal{G}(\Omega) > n$ for all n.

Suppose instead that $|\mathcal{S}(\Omega)| = 2$. If Ω has infinite genus or infinitely many non-maximal ends, then there exists an infinite collection of pairwise-disjoint annular witnesses separating the sets $\{e, f\} = \mathcal{S}(\Omega)$. Choosing finite subcollections defines quasi-flats of arbitrarily large dimension [Sch, Exercise 3.13], hence again $\operatorname{asdim} \mathcal{G}(\Omega) = \infty$. Alternatively, $\mathcal{G}(\Omega)$ contains an asymphoric hierarhically hyperbolic space of arbitrarily high rank, hence has infinite asymptotic dimension [Kop23a, Prop. 1.11].

Finally, suppose that $|\mathcal{S}(\Omega)| = 2$ and Ω has finite genus and finitely many non-maximal ends. Ω must have at least one infinite set $e \in \mathcal{S}(\Omega)$; let $f \in \mathcal{S}(\Omega)$ be the other set. For any n, choose a $(n+1)$-holed sphere $\Sigma \subset \Omega$ with n boundary components partitioning e and the remaining component separating e from f and any genus or non-maximal ends. Then Σ is a fully separating witness for $\mathcal{G}(\Omega)$ and Γ, defined as in Lemma 3.2, is a n-pointed star. $\mathcal{A}(\Sigma, \Gamma)$ is δ-hyperbolic by Theorem 1.5: we conclude by Lemma 3.2 and Theorem 1.1. //

4. Asymptotic dimension of arc and curve models

We generalize the preceding arguments to a broad class of arc and curve models for finite and infinite-type surfaces. We first consider *cocompact* arc and curve models, which we demonstrate to be equivariantly quasi-isometric to cocompact marking graphs in the finite-type case; it follows that such arc and curve models are hierarchically hyperbolic by [Kop23a]. We then compute asymptotic dimension lower bounds for such models, which we use to construct lower bounds in the infinite-type setting.

4.1. Cocompact arc and curve models.

Let Ω be a connected and non-pants hyperbolic surface of finite or infinite type. We first provide an extension of arc and curve systems and markings on Ω that subsumes both. Let $\mathcal{K}(\Omega) := K(V(\mathcal{AC}(\Omega)))$ denote the set of finite collections of (not necessarily disjoint) simple arcs and curves on Ω and let $\mathrm{PMap}_c(\Omega) \leq \mathrm{Map}(\Omega)$ denote the subgroup of compactly supported pure mapping classes.

Definition 4.1. An *arc and curve model* $\mathcal{M}$ on Ω is a connected simplicial graph with $V(\mathcal{M}) \subset \mathcal{K}(\Omega)$ that admits an action of $\mathrm{PMap}_c(\Omega)$ induced by the permutation of its vertices. $\mathcal{M}$ is *cocompact* if this action is cocompact.

Remark 4.2. If Ω is finite-type, then *(i)* $\mathrm{PMap}_c(\Omega) = \mathrm{PMap}(\Omega)$ and *(ii)* $\mathcal{M}$ is cocompact if and only if $i(u, u)$ and $i(u, v)$ are uniformly bounded for $u \in V(\mathcal{M})$ and $(u, v) \in E(\mathcal{M})$.

Cocompact arc and curve models include most familiar graphs on finite-type surfaces, such as the arc and curve graph $\mathcal{AC}(\Sigma)$, connected subgraphs preserved by $\mathrm{PMap}(\Sigma)$ (*e.g.* the curve graph and arc graph), and the pants graph. Masur and Minsky's marking complex $\mathcal{MC}(\Sigma)$ is likewise included, as we will show below.

4.1.1. *Cocompact marking graphs.* A *marking graph* $\mathcal{L}$ on a compact surface Σ is a connected simplicial graph whose vertices are locally clean markings on Σ [Kop23a]. As above, $\mathcal{L}$ is *cocompact* if $\mathrm{PMap}(\Sigma)$ acts on $\mathcal{L}$ cocompactly by permuting its vertices. We prove that cocompact arc and curve models and cocompact marking graphs are identical, up to $\mathrm{PMap}(\Sigma)$-equivariant quasi-isometry.

For an essential subsurface $W \subset \Sigma$ and u a collection of simple arcs and curves, let $\pi_W(u) \subset \mathcal{C}W$ denote union of the (set-valued) projections of the elements in u, and let $d_W(u,v) := \mathrm{diam}(\pi_W(u) \cup \pi_W(v))$. We show:

Proposition 4.3. *Let $\mathcal{M}$ be a cocompact arc and curve model on a compact surface Σ. Then there exists a cocompact marking graph $\mathcal{L}_\mathcal{M}$ on Σ with an identical witness set and an equivariant coarse quasi-isometry $\zeta : \mathcal{M} \to \mathcal{L}_\mathcal{M}$ that coarsely preserves projections to witness curve graphs.*

Lemma 4.4. *There is a uniform proper increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ depending only on Σ such that for any $a, b \in \mathcal{AC}(\Sigma)$ and any essential subsurface $W \subset \Sigma$, $i(a,b) \geq \phi(d_W(a,b))$.*

Proof. When a, b are both curves, the claim follows from the Choi-Rafi formula for curves [Wat16, Thm. 2.10]. In general, there exists some increasing (affine) function $\phi' : \mathbb{R}_+ \to \mathbb{R}_+$ so that $i(a,b) > \phi'(i(a',b'))$ for every $a' \in \pi_\Sigma(a), b' \in \pi_\Sigma(b)$. We conclude by observing that $\pi_W \circ \pi_\Sigma$ and π_W are uniformly boundedly close. $\square$

Remark 4.5. It follows from Remark 4.2 and Lemma 4.4 that for any cocompact arc and curve model $\mathcal{M}$, witness subsurface projections $\pi_W : V(\mathcal{M}) \to 2^{\mathcal{C}W}$ are uniformly Lipschitz with uniformly bounded vertex images; in general, any subsurface projection has uniformly bounded diameter vertex images.

Definition 4.6. A *generalized marking* $\mu = \{(a_i, t_i)\}$ on a surface Σ is a multicurve $\mathrm{base}\,\mu = \{a_i\}$ along with bounded subsets $t_i \subset \mathcal{C}(a_i)$ for each a_i. μ is *locally complete* if $t_i \neq \varnothing$ only if the complementary component of $\mathrm{base}\,\mu \setminus \{a_i\}$ has complexity $\xi = 1$.

Definition 4.7. A locally clean marking $\mu' = \{(a_i', t_i')\}$ is *compatible* with a generalized marking $\mu = \{(a_i, t_i)\}$ if $a_i' = a_i$ (up to reindexing), $t_i = \varnothing$ only if $t_i' = \varnothing$, and $\mathrm{diam}(t_i', t_i)$ is minimal among all choices of clean t_i'.

Only a locally complete generalized marking admits compatible locally clean markings. Exactly analogously to [MM00, Lem. 2.8], we have the following for generalized markings:

Lemma 4.8. *For any locally complete generalized marking $\mu = \{(a_i, t_i)\}$, there is a constant n_0 depending only on $\max_i(\mathrm{diam}(t_i))$ and at least one and at most $n_0^{|\mu|}$ compatible locally clean markings. For any compatible marking $\mu' = \{(a_i, t_i')\}$, $d_{a_i}(t_i, t_i') < n_1$ for a universal constant n_1.*

For each $u \in V(\mathcal{M})$, we first equivariantly construct a corresponding locally clean marking μ_u. Choose representatives for $u = \{a_i\}$ in pair-wise minimal position; let $\Gamma_u = \bigcup_i a_i$, viewed as a 1-complex (add basepoints to isolated simple curves as necessary). For a subcomplex $\Gamma' \subset \Gamma_u$, let $\partial_\Sigma(\Gamma')$ denote the essential, non-peripheral boundary components of a regular neighborhood of $\Gamma' \cup \partial\Sigma$, or equivalently the set of non-peripheral boundary components of the essential subsurface filled by Γ'. Fix an enumeration of the edges in Γ_u and let $\Gamma_{u,j} \subset \Gamma_u$ be an exhaustion of Γ_u by adding the jth successive edge; let F_u denote the essential subsurface filled by u and $F_{u,j}$ the essential subsurface filled by $\Gamma_{u,j}$. Let m_u be the multicurve $\bigcup_j \partial_\Sigma(\Gamma_{u,j})$. Obtain the generalized marking $\tilde{\mu}_u$ by adding to each component $c_i \in m_u$ the transversal $\pi_{c_i}(u)$.

Claim 4.9. *$\tilde{\mu}_u$ satisfies the following:*

(i) The submarking $\tilde{\mu}_u \setminus \partial F_u$ is contained in F_u and complete on F_u.
(ii) $\tilde{\mu}_u \cap \partial F_u$ has empty transversals, and for all other $(c_i, t_i) \in \tilde{\mu}_u$, $d_{c_i}(t_i, \pi_{c_i}(u))$ is uniformly bounded independently of u.
(iii) For $c_i \in \mathrm{base}\, \tilde{\mu}_u$, $i(c_i, u)$ is uniformly bounded independently of u.

In particular, $\tilde{\mu}_u$ is locally complete and any compatible locally clean marking satisfies the same.

Proof of claim. Let $e_j = \overline{\Gamma_{u,j} \setminus \Gamma_{u,j-1}}$ be the jth edge in Γ_u and let $\Gamma_{u,0} = \varnothing$. Let $m_{u,k} = \bigcup_{j=1}^k \partial_\Sigma(\Gamma_{u,j})$; note that $m_{u,0} = \varnothing = F_{u,0}$. Inducting on $k \leq |E(\Gamma_u)|$, we first claim that $m_{u,k} \setminus \partial F_{u,k}$ is a pants decomposition for $F_{u,k}$. In particular, $m_u \setminus \partial F_u$ is a pants decomposition of F_u and (i) follows: since u fills F_u, it intersects every component c_i of $m_u \setminus \partial F_u$, which then has transversal $\pi_{c_i}(u) \neq \varnothing$ in $\tilde{\mu}_u$.

By construction $m_{u,k} \setminus \partial F_{u,k}$ is contained in $F_{u,k}$. The claim holds vacuously for $k = 0$; suppose it holds for $k - 1 \geq 0$ and consider the edge e_k. Recall that $\partial_\Sigma(\Gamma_{u,j})$ is the set of non-peripheral boundary curves in $F_{u,j}$, hence if $F_{u,k}$ and $F_{u,k-1}$ are isotopic then $m_{u,k} = m_{u,k-1}$ and the claim holds by induction. Assume that $F_{u,k} \supsetneq F_{u,k-1}$. If e_k is disjoint from $F_{u,k-1}$, then $F_{u,k}$ is obtained from $F_{u,k-1}$ by adding either a disjoint annulus or a disjoint pair of pants; in either case, the claim holds. Else, $F_{u,k-1}$ is obtained by adding a 1-handle with core e_k, hence by adjoining a pair of pants along a (non-peripheral) boundary curve $c \in \partial F_{u,k-1} \cap m_{u,k-1}$: $m_{u,k} \setminus \partial F_{u,k} = (m_{u,k-1} \setminus \partial F_{u,k-1}) \cup \{c\}$ is a pants decomposition for $F_{u,k}$.

To prove (ii), observe that by construction ∂F_u does not intersect u, hence $\mu_u \cap \partial F_u$ has empty transversals; the remaining transversals are uniformly bounded by Remark 4.5. Finally, every component $c \in m_u$ is a boundary curve for some $F_{u,k}$, which is filled by $\Gamma_{u,k} \subset \Gamma_u$: any essential intersection between c and u implies an essential self-intersection, whence (iii) follows. □

Let μ_u be a choice of locally clean marking compatible with $\tilde{\mu}_u$. While μ_u and $\tilde{\mu}_u$ are non-canonical, we enforce that the choices are $\mathrm{PMap}(\Sigma)$-equivariant: construct μ_u for representatives $u \in V(\mathcal{M})/\mathrm{PMap}(\Sigma)$ and declare $\mu_{\varphi u} := \varphi\mu_u$ for $\varphi \in \mathrm{PMap}(\Sigma)$.

Proof of Prop. 4.3. Let $V(\mathcal{L}_\mathcal{M}) = \{\mu_u : u \in V(\mathcal{M})\}$ and obtain $E(\mathcal{L}_\mathcal{M})$ by pushing forward the edge relation on $\mathcal{M}$ via $u \mapsto \mu_u$; let $\zeta : \mathcal{M} \to \mathcal{L}_\mathcal{M}$ be the $\mathrm{PMap}(\Sigma)$-equivariant surjective Lipschitz map induced by $u \mapsto \mu_u$. Since $\mathcal{M}$ is cocompact, ζ implies that $\mathcal{L}_\mathcal{M}$ is cocompact.

Let $W \subset \Sigma$ be an essential, non-pants subsurface. Recall that u is filling on F_u, hence Claim 4.9(i)-(ii) implies W intersects μ_u if and only if it intersects F_u if and only if it intersects u: the witness set for $\mathcal{L}_\mathcal{M}$ is identical to that of $\mathcal{M}$. Claim 4.9(ii) implies that the projection of u and μ_u to annular witnesses parallel to μ_u is uniformly close; similarly, by Claim 4.9(iii) $i(u, \operatorname{base}\mu_u)$ is uniformly bounded, hence Lemma 4.4 implies that the projection of u, μ_u to all other witnesses is likewise uniformly close.

It remains to show that ζ is a coarse Lipschitz retraction, hence a quasi-isometry. Since $\mathcal{L}_\mathcal{M}$ has finitely many $\mathrm{PMap}(\Sigma)$-orbits of edges, it suffices to show that the fibers of ζ are uniformly bounded. The following lemma completes the proof. □

Lemma 4.10. *The fibers $E_\mu = \zeta^{-1}(\mu)$ are uniformly bounded over $\mu \in V(\mathcal{L}_\mathcal{M})$.*

Proof. Let $u, v \in E_\mu$, hence $F_u = F_v$ and μ (excluding boundary components) is complete on F_u. It suffices to show that $i(u, v)$ is uniformly bounded. Let u', v' denote the subsets of non-boundary elements in u, v respectively. By Claim 4.9(iii) u', v' intersect each component of $\operatorname{base}\mu$ uniformly many times, independently of u, v and μ, hence u', v' have at most uniformly many components in each pair of pants $P \subset F_u \setminus \mu$. Up to Dehn twists along components in $\operatorname{base}\mu$, these components are among finitely many arcs and curves in P; by Claim 4.9(ii) projections to curves in $\operatorname{base}\mu$ are uniformly close to the corresponding transversal in μ (independently of u, v, μ), hence the order of these twists (and hence the intersection number) for any two components in P is uniformly bounded. It follows that $i(u, v)$ is uniformly bounded independently of u, v, μ. □

Proposition 4.11. *The map $\mathcal{M} \mapsto \mathcal{L}_{\mathcal{M}}$ induces a bijection between the* $\mathrm{PMap}(\Sigma)$*-equivariant quasi-isometry types of cocompact arc and curve models on Σ and of cocompact marking graphs on Σ.*

Proof. Since $\mathcal{M}$ and $\mathcal{L}_{\mathcal{M}}$ are equivariantly quasi-isometric, the induced map is well-defined and injective. We show surjectivity. Given a locally clean marking $\mu = \{(a_i, t_i)\}$, where $t_i = \varnothing$ or $\pi_{a_i}(b_i)$ for some (unique) clean transverse curve b_i, let $u_\mu = \{a_i\} \cup \{b_i\}$. Let $\mathcal{L}$ be a cocompact marking graph on Σ, and let $\mathcal{M}_{\mathcal{L}}$ be the graph obtained as the (connected) push-forward of $\mathcal{L}$ by the map $\psi : \mu \mapsto u_\mu$. ψ is $\mathrm{PMap}(\Sigma)$-equivariant, hence $\mathcal{M}_{\mathcal{L}}$ is cocompact. As above, $\psi : \mathcal{L} \to \mathcal{L}_{\mathcal{M}}$ is a surjective Lipschitz retraction, hence a quasi-isometry: any two markings in the fiber $E_u = \{\mu : u = u_\mu\}$ differ by at most uniformly many flip moves, each of which increases the intersection number by at most 2. Hence $\mathcal{L}_{\mathcal{M}_{\mathcal{L}}}$ is equivariantly quasi-isometric to $\mathcal{L}$, which suffices. □

4.1.2. *Geometry in the finite-type case.* By [Kop23a, Thm. 2.10], cocompact marking graphs on finite-type surfaces are hierarchically hyperbolic with respect to witness subsurface projection, and this geometry is determined up to equivariant quasi-isometry by the set of connected witnesses [Kop23a, Thm. 2.12]. Let $\mathscr{X}^{\mathcal{M}}$ and $\hat{\mathscr{X}}^{\mathcal{M}}$ denote the sets of witnesses and connected witnesses of an arc and curve model, respectively. Applying Proposition 4.3, we obtain:

Theorem 4.12. *Let $\mathcal{M}$ be a cocompact arc and curve model on a finite-type hyperbolic, non-pants surface Σ. Then $(\mathcal{M}, \mathscr{X}^{\mathcal{M}})$ is an asymphoric hierarchically hyperbolic space with respect to subsurface projection to witness curve graphs $\pi_W : \mathcal{M} \to 2^{\mathcal{C}W}, W \in \mathscr{X}^{\mathcal{M}}$.*

A *(connected) witness set* on Σ is any collection of essential compact (connected) subsurfaces without pants components and closed under enlargement and the action of $\mathrm{PMap}(\Sigma)$. From [Kop23a, Thm. 2.12] and Propositions 4.3 and 4.11 we have:

Theorem 4.13. *The map $\mathcal{M} \mapsto \hat{\mathscr{X}}^{\mathcal{M}}$ induces a bijection between coarsely* $\mathrm{PMap}(\Sigma)$*-equivariant quasi-isometry types of cocompact arc and curve models on Σ and connected witness sets on Σ.*

4.2. **Witness-cocompactness.** In the infinite-type setting, we require a related "local" condition, determined by the existence of projections to cocompact arc and curve models on each witness subsurface. Crucially, these projections admit Lipschitz sections, whence we will obtain asymptotic dimension lower bounds in Section 4.3.

4.2.1. *Witness projections.* Let $\Sigma \subset \Omega$ be a compact, essential, and connected subsurface. Let $\mathcal{K}(\Omega, \Sigma) \subset \mathcal{K}(\Omega)$ denote the subset of collections of arcs and curves intersecting Σ. We construct a projection $\rho_\Sigma : \mathcal{K}(\Omega, \Sigma) \to \mathcal{K}(\Sigma)$ as follows (see *e.g.* [Sch, §5.2]). Let $\iota : \Sigma \hookrightarrow \Omega$ be the inclusion map and let $p : \Omega_\Sigma \to \Omega$ be the covering space associated to $\pi_1(\Sigma) \cong \operatorname{im} \iota_* < \pi_1(\Omega)$ with Gromov closure $\overline{\Omega}_\Sigma$. Let $\tilde{\iota} : \Sigma \hookrightarrow \Omega_\Sigma$ be the (unique) lift of ι, and $\bar{\iota}$ its inclusion into $\overline{\Omega}_\Sigma$. Fix any homeomorphism $\sigma : \overline{\Omega}_\Sigma \to \Sigma$ that is a homotopy inverse for $\bar{\iota}$; note that σ is unique up to homotopy, hence isotopy.

$$\begin{tikzcd}
& \Omega_\Sigma \arrow[r, hook] \arrow[d, "p"] & \overline{\Omega}_\Sigma \arrow[dll, bend right, "\sigma"'] \\
\Sigma \arrow[ur, hook, "\tilde{\iota}"] \arrow[r, hook, "\iota"'] & \Omega &
\end{tikzcd}$$

Given $\omega \in \mathcal{K}(\Omega, \Sigma)$, let $\rho_\Sigma(\omega)$ be the isotopy class defined by the closures of non-peripheral components of $\sigma p^{-1}(\omega)$.

One verifies that $\rho_\Sigma(\omega)$ is independent of the choice of representative for ω and σ. Likewise, ρ_Σ is independent of the choice of embedding of Σ: if $\iota' : \Sigma \hookrightarrow \Omega$ is isotopic to ι, then the lift $\bar{\iota}'$ is isotopic to $\bar{\iota}$ and thus a homotopy inverse for σ.

The natural action of $\mathrm{PMap}(\Sigma)$ on $\mathcal{K}(\Sigma)$ defines an action of $\mathrm{Map}(\Sigma, \partial\Sigma) \twoheadrightarrow \mathrm{PMap}(\Sigma)$. Similarly, $\mathrm{Map}(\Sigma, \partial\Sigma) \curvearrowright \mathcal{K}(\Omega, \Sigma)$ via the homomorphism $\mathrm{Map}(\Sigma, \partial\Sigma) \to \mathrm{PMap}_c(\Omega)$ obtained by extending by identity.

Lemma 4.14. *$\rho_\Sigma : \mathcal{K}(\Omega, \Sigma) \to \mathcal{K}(\Sigma)$ is $\mathrm{Map}(\Sigma, \partial\Sigma)$-equivariant.*

Proof. Let $\varphi_0 \in \mathrm{Map}(\Sigma, \partial\Sigma)$, fixing a representative. Let $\varphi \in \mathrm{PMap}_c(\Omega)$ be its extension by identity; since φ is (compactly) supported in Σ, it lifts to a quasi-isometry on Ω_Σ that extends to a homeomorphism $\overline{\varphi}$ on $\overline{\Omega}_\Sigma$. Since $\bar{\iota}\varphi_0 = \overline{\varphi}\,\bar{\iota}$ and $\sigma, \bar{\iota}$ are homotopy inverses, $\varphi_0\sigma$ and $\sigma\overline{\varphi}$ are homotopic and thus isotopic. For $\omega \in \mathcal{K}(\Omega, \Sigma)$, $\overline{\sigma p^{-1}(\varphi\omega)} = \overline{\sigma\overline{\varphi}p^{-1}(\omega)}$ is isotopic to $\overline{\varphi_0\sigma p^{-1}(\omega)}$, whence the claim follows. $\square$

Corollary 4.15. *Let $\phi \in \mathrm{PMap}(\Sigma)$. Then there exists $\psi \in \mathrm{PMap}(\Omega)$ preserving $\mathcal{K}(\Omega, \Sigma)$ such that for any $\omega \in \mathcal{K}(\Omega, \Sigma)$, $\phi\rho_\Sigma(\omega) = \rho_\Sigma(\psi\omega)$.* $\square$

Given an arc and curve model $\mathcal{M}$ on Ω, let $V(\mathcal{M}), E(\mathcal{M})$ denote its vertex and edge sets, respectively. If Σ is a witness for $\mathcal{M}$ then $V(\mathcal{M}) \subset \mathcal{K}(\Omega, \Sigma)$ and ρ_Σ defines a projection $V(\mathcal{M}) \to \mathcal{K}(\Sigma)$.

Definition 4.16. A connected arc and curve model $\mathcal{M}$ on Ω is *witness-cocompact* if

(i) $\mathcal{M}$ admits a (compact) witness,
(ii) $\mathrm{PMap}_c(\Omega)$ preserves $V(\mathcal{M})$ and extends to an action on $\mathcal{M}$, and
(iii) for any witness $\Delta \subset \Omega$, there exists L_Δ such that if $(a, b) \in E(\mathcal{M})$, then $i(\rho_\Delta(a), \rho_\Delta(b)) \leq L_\Delta$.

Remark 4.17. When Ω is finite-type, it deformation retracts to a compact witness $\overline{\Omega}$. Since in addition $i(\rho_\Delta(a), \rho_\Delta(b)) \leq i(a, b) = i(\rho_{\overline{\Omega}}(a), \rho_{\overline{\Omega}}(b))$, (i) is tautological and in (iii) we may choose $L_\Delta = L_{\overline{\Omega}}$ to be uniform.

Witness-cocompact arc and curve models include many graphs of contemporary interest on infinte-type surfaces, including the ray graph, the omnipresent arc graph, and the grand arc graph.

4.2.2. *Arc and curve models on witnesses.* Let $\Sigma \subset \Omega$ be a witness for a witness-cocompact arc and curve model $\mathcal{M}$ on Ω. We construct a cocompact arc and curve model $\mathcal{M}_\Sigma$ on Σ for which the projection ρ_Σ restricts to a Lipschitz map $\mathcal{M} \to \mathcal{M}_\Sigma$, along with a Lipschitz coarse section $\iota : \mathcal{M}_\Sigma \to \mathcal{M}$. It follows that $\mathcal{M}_\Sigma$ quasi-isometrically embeds into $\mathcal{M}$.

Let $V(\mathcal{M}_\Sigma) = \rho_\Sigma(V(\mathcal{M}))$ and let $(a, b) \in E(\mathcal{M}_\Sigma)$ if and only if $a \neq b$ and there exist $\tilde{a} \in \rho_\Sigma^{-1}(a), \tilde{b} \in \rho_\Sigma^{-1}(b)$ such that $(\tilde{a}, \tilde{b}) \in E(\mathcal{M})$. It is immediate that $\rho_\Sigma : V(\mathcal{M}) \to V(\mathcal{M}_\Sigma)$ extends to a surjective 1-Lipschitz map $\rho_\Sigma : \mathcal{M} \to \mathcal{M}_\Sigma$, hence in particular since $\mathcal{M}$ is connected so is $\mathcal{M}_\Sigma$. Likewise, since $\mathcal{M}$ satisfies Definition 4.16(iii), so does $\mathcal{M}_\Sigma$ for uniform $L = L_\Sigma$. By Corollary 4.15 $\mathrm{PMap}(\Sigma)$ acts naturally on $\mathcal{M}_\Sigma$, hence $\mathcal{M}_\Sigma$ is a cocompact arc and curve model on Σ.

Fix any $\mathrm{Map}(\Sigma, \partial\Sigma)$-equivariant section $\iota : V(\mathcal{M}_\Sigma) \to V(\mathcal{M})$, and let $\tilde{a} = \iota(a) \in \rho_\Sigma^{-1}(a)$. We show that ι is Lipschitz, hence extends to a Lipschitz coarse section $\iota : \mathcal{M}_\Sigma \to \mathcal{M}$ for ρ_Σ. Since for any $(a, b) \in E(\mathcal{M}_\Sigma)$, $i(a, b) \leq L$, there are finitely many $\mathrm{Map}(\Sigma, \partial\Sigma)$-orbits of edges in $\mathcal{M}_\Sigma$. Let

$$M = \max_{(a,b) \in E(\mathcal{M}_\Sigma)/G} d_{\mathcal{M}}(\tilde{a}, \tilde{b})$$

where $G = \mathrm{Map}(\Sigma, \partial\Sigma)$. Then ι is M-Lipschitz. We have shown:

Proposition 4.18. *Let $\Sigma \subset \Omega$ be a witness for a witness-cocompact arc and curve model $\mathcal{M}$ on Ω. There exists a cocompact arc and curve model $\mathcal{M}_\Sigma$ on Σ which quasi-isometrically embeds into $\mathcal{M}$.* □

Remark 4.19. A connected arc and curve model $\mathcal{M}$ on Ω with a natural action of $\mathrm{PMap}_c(\Omega)$ is witness-cocompact if and only if it has a witness and the projection $\mathcal{M}_\Delta$ is cocompact for every witness $\Delta \subset \Omega$.

4.3. **Asymptotic dimension lower bounds.** We first consider the asymptotic dimension of cocompact arc and curve models on Σ, a finite-type non-pants hyperbolic surface. Up to deformation retraction, we assume Σ is compact.

Remark 4.20. If Σ is a (closed) torus, then any cocompact arc and curve model is quasi-isometric to the curve graph, hence a quasi-tree with asdim = 1. Otherwise, if Σ admits a non-empty cocompact arc and curve model (and in particular, a witness subsurface), then $\chi(\Sigma) \leq -1$ and $\Sigma \not\cong \Sigma_0^3$.

Let $\mathcal{M}$ be a cocompact arc and curve model on Σ; recall that $(\mathcal{M}, \mathscr{X}^{\mathcal{M}})$ is an asymphoric hierarchically hyperbolic space by Theorem 4.12. Then in particular the *rank* ν of $(\mathcal{M}, \mathscr{X}^{\mathcal{M}})$ corresponds to the largest cardinality of a set of pairwise disjoint, connected witnesses in $\mathscr{X}^{\mathcal{M}}$. Since $(\mathcal{M}, \mathscr{X}^{\mathcal{M}})$ is asymphoric, $\operatorname{asdim} \mathcal{M} \geq \nu$ [BHS21, Thm. 1.15] and $\mathcal{M}$ is δ-hyperbolic if and only if $\nu = 1$ [BHS21, Cor. 2.15]. The lower bound here will prove sufficient except when $\nu = 1$; we note that an identical bound can be achieved by explicitly constructing quasi-flats.

4.3.1. *The δ-hyperbolic case.* Adapting the arguments in Section 2, we prove the following:

Theorem 4.21. *Let Σ be a genus g compact surface, possibly with boundary. If $\mathcal{M}$ is a (non-empty) δ-hyperbolic cocompact arc and curve model on Σ, then* $\operatorname{asdim} \mathcal{M} \geq g - \lceil \frac{1}{2}\chi(\Sigma) \rceil$.

If $\Sigma \cong \Sigma_1$, then the claim is immediate by Remark 4.20. Otherwise, we may assume $\mathcal{M}$ is a cocompact marking graph by Proposition 4.3. For any $\mathcal{M}'$ a cocompact marking graph on Σ with $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}^{\mathcal{M}'}$, there exists a functorial canonical coarse surjection $\iota : \mathcal{M} \to \mathcal{M}'$ such that $\pi_W \circ \iota$ is uniformly coarsely π_W for any $W \in \mathscr{X}^{\mathcal{M}'}$ [Kop23a, §2.1]. In particular, $\mathscr{X}^{\mathcal{MC}(\Sigma)}$ is every essential, non-peripheral subsurface in Σ and $\mathscr{X}^{\mathcal{C}\Sigma} = \{\Sigma\}$, hence we have canonical maps $\mathcal{MC}(\Sigma) \to \mathcal{M} \to \mathcal{C}\Sigma$.

Lemma 4.22. *Let $\mathcal{M}, \mathcal{M}'$ be cocompact marking graphs on Σ, a compact surface, such that $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}^{\mathcal{M}'}$, and let $\iota : \mathcal{M} \to \mathcal{M}'$ be the canonical coarse surjection. If $\mathcal{M}$ is δ-hyperbolic, then ι is coarsely alignment-preserving.*

We note that if $\mathcal{M}$ is δ-hyperbolic, then $\nu(\mathcal{M}') \leq \nu(\mathcal{M}) \leq 1$, hence $\mathcal{M}'$ is δ-hyperbolic. Recall that a path $\rho \subset X$ is a *D-hierarchy path* for a hierarchically hyperbolic space $(X, \mathscr{G})$ if it is a (D, D)-quasi-geodesic and $\pi_\alpha \rho$ is a unparameterized (D, D)-quasi-geodesic for all $\alpha \in \mathscr{G}$.

Proof. Since $(\mathcal{M}, \mathscr{X}^{\mathcal{M}})$ is hierarchically hyperbolic, there exists $D > 0$ such that for any $x, y \in \mathcal{M}$, there exists a D-hierarchy path joining x, y [BHS19, Thm. 4.4]. Let $(x, z, y) \in \mathcal{M}^3$ be aligned and let γ be the geodesic from x to y passing through z and ρ the hierarchy path between x, y. By the Morse lemma, there exists a constant $M(D, \delta)$ such that γ, ρ are $M(D, \delta)$-Hausdorff close, hence $d(z, \rho) \leq M(D, \delta)$. For any $W \in \mathscr{X}^{\mathcal{M}}$, $\pi_W \rho$ is an unparameterized (D, D)-quasi-geodesic. Applying the Morse lemma and that π_W is L-Lipschitz for uniform L, it follows that $(\pi_W(x), \pi_W(z), \pi_W(y))$ are K-aligned where $K = 2(M(D, \delta_0) + LM(D, \delta))$ is uniform over $\mathcal{M}^3, \mathscr{X}^{\mathcal{M}}$ and δ_0 is a uniform hyperbolicity constant for curve graphs [HPW15].

It follows that π_W for $W \in \mathscr{X}^{\mathcal{M}}$ and likewise $\pi_{W'}$ for $W' \in \mathscr{X}^{\mathcal{M}'}$ are K'-alignment preserving for uniform K'. Since $\mathscr{X}^{\mathcal{M}'} \subset \mathscr{X}^{\mathcal{M}}$, the distance formulas for $\mathcal{M}, \mathcal{M}'$ imply the claim. $\square$

Suppose that $\mathcal{M}$ is a δ-hyperbolic cocompact marking graph on a compact surface Σ with genus g. By Lemma 4.22, the canonical map $\iota : \mathcal{M} \to \mathcal{C}\Sigma$ is coarsely alignment preserving, hence by Theorem 1.13 $\partial\mathcal{C}\Sigma$ embeds into $\partial\mathcal{M}$. To prove Theorem 4.21 it suffices to find a compact subspace $Z \subset \partial\mathcal{C}\Sigma$ such that $\dim Z \geq n := g - 1 - \lceil\frac{1}{2}\chi(\Sigma)\rceil$, since by Proposition 2.5 $\dim Z + 1 \leq \operatorname{asdim}\mathcal{M}$. Recall that $\partial\mathcal{C}\Sigma \cong \mathcal{EL}(\Sigma)$.

Proposition 4.23. *Let Σ be a genus g compact hyperbolic surface and S the $(n+4)$-times punctured sphere, where $n = g-1-\lceil\frac{1}{2}\chi(\Sigma)\rceil$. Then $\mathcal{EL}(S)$ embeds into $\partial\mathcal{C}\Sigma \cong \mathcal{EL}(\Sigma)$.*

Proof. For simplicity, we replace the boundary components of Σ with punctures, noting that $\mathcal{C}\Sigma \cong \mathcal{C}(\Sigma \setminus \partial\Sigma)$. Choose a hyperelliptic involution η on Σ that fixes at most one puncture and let $h : \Sigma \to S'$ be the orbifold covering map obtained by quotienting by η. Obtain S by removing the cone points of S': one verifies that S has $n+4$ punctures. By [RS09], h induces a quasi-isometric embedding $h_* : \mathcal{C}S \to \mathcal{C}\Sigma$, which has quasi-convex image by the Morse lemma. Hence $\mathcal{EL}(S) \cong \partial\mathcal{C}S \subset \partial\mathcal{C}\Sigma$. □

When Σ is a sphere with four boundary components, Theorem 4.21 is vacuously true. Otherwise, from Theorem 2.4 and the universal embedding property of Nöbeling spaces, we obtain the desired subspace $Z \subset \mathcal{EL}(S) \subset \partial\mathcal{C}\Sigma$ and Theorem 4.21 follows. //

4.3.2. *Lower bounds for infinite-type surfaces.* Given a witness-cocompact arc and curve model $\mathcal{M}$ on an infinite-type surface Ω, let $w_{\mathcal{M}} \in \mathbb{N} \cup \{\infty\}$ denote the least upper bound on cardinalities for a set of pairwise-disjoint connected witnesses for $\mathcal{M}$. We consider two cases:

(i) *$w_{\mathcal{M}}$ is infinite.* For arbitrarily large $m \in \mathbb{N}$, we may choose a compact, essential subsurface $\Sigma \subset \Omega$ containing at least m disjoint witnesses. Σ is a witness for $\mathcal{M}$, and any witness for $\mathcal{M}$ contained in Σ is a witness for $\mathcal{M}_\Sigma$ by construction. It follows that $\mathcal{M}_\Sigma$ is an asymphoric hierarchically hyperbolic space of rank $\nu \geq m$, hence by Proposition 4.18 $\operatorname{asdim}\mathcal{M} \geq \operatorname{asdim}\mathcal{M}_\Sigma \geq m$. $\operatorname{asdim}\mathcal{M} = \infty$.

(ii) *$w_{\mathcal{M}} = m$ is finite.* Fix a collection of pairwise disjoint witnesses $\{W_i\}$ with cardinality m. Fix W_0 among these such that W_0 lies in a complementary component Ω_0 of $\bigcup_{i>0} W_i$ of infinite type. Let $\Sigma \subset \Omega_0$ be an enlargement of W_0 of arbitrarily negative $\chi(\Sigma)$: Σ is a witness for $\mathcal{M}$. Moreover, since any witness for $\mathcal{M}_\Sigma$ is a witness for $\mathcal{M}$ disjoint from the $W_{i>0}$, any two connected witnesses for $\mathcal{M}_\Sigma$ must intersect: $\mathcal{M}_\Sigma$ is an asymphoric hierarchically hyperbolic space of rank $\nu = 1$, hence δ-hyperbolic. By Proposition 4.18 and Theorem 4.21, $\operatorname{asdim}\mathcal{M} \geq \operatorname{asdim}\mathcal{M}_\Sigma \geq -\frac{1}{2}\chi(\Sigma)$, hence $\operatorname{asdim}\mathcal{M} = \infty$.

Theorem 4.24. *Let $\mathcal{M}$ be a witness-cocompact arc and curve model on an infinite-type surface Ω. Then* $\operatorname{asdim}\mathcal{M} = \infty$. □

Theorem 1.3 follows from Theorems 4.21 and 4.24. //

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