

GEOMETRIC MODELS AND ASYMPTOTIC DIMENSION FOR INFINITE-TYPE SURFACE MAPPING CLASS GROUPS

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ABSTRACT. Let S be an infinite-type surface and let $G \leq \text{Map}(S)$ be a locally bounded Polish subgroup. We construct a metric graph \mathcal{M} of simple arcs and curves on S preserved by the action of G and for which the vertex orbit map $G \rightarrow V(\mathcal{M})$ is a coarse equivalence; if G is boundedly generated, then \mathcal{M} is a Cayley–Abels–Rosendal graph for G and the orbit map is a quasi-isometry. In particular, if S contains a non-displaceable subsurface and $G \geq \text{PMap}_c(S)$ is boundedly generated or $G \in \{\overline{\text{PMap}_c(S)}, \text{PMap}(S), \text{Map}(S)\}$ and is locally bounded, then $\text{asdim } \mathcal{M} = \text{asdim } G = \infty$. This result completes the classification of the asymptotic dimension of stable boundedly generated infinite-type surface mapping class groups begun by Grant–Rafi–Verberne.

1. INTRODUCTION AND MAIN RESULTS

Let S be a surface of infinite topological type. A *(metric) arc and curve model* for $G \leq \text{Map}(S)$ is a connected (metric) graph whose vertices are collections of (possibly intersecting) simple arcs and curves on S , with an isometric action of G induced by the permutation of its vertices.

Theorem 1.1. *Let S be an infinite-type surface and let $G \leq \text{Map}(S)$ be a locally bounded Polish subgroup.*

- (1) *There exists a metric arc and curve model \mathcal{M} for G for which the orbit map restricted to $V(\mathcal{M})$ is a continuous coarse equivalence.*
- (2) *If additionally G is boundedly generated, then \mathcal{M} is a Cayley–Abels–Rosendal graph for G and the orbit map is a continuous quasi-isometry.*

In particular, the coarse equivalence and quasi-isometry types of $G \leq \text{Map}(S)$ are described by a (metric) arc and curve model, whenever they are well-defined. A compact subsurface $\Delta \subset S$ is *non-displaceable* by G if there exists no $f \in G \leq \text{Map}(S)$ such that $\Delta \cap f\Delta = \emptyset$. From [Theorem 1.1](#) we obtain:

Theorem 1.2. *Let S be an infinite-type surface and $G \leq \text{Map}(S)$ a Polish subgroup with a non-displaceable subsurface and containing $\text{PMap}_c(S)$. If G is boundedly generated or $G \in \{\overline{\text{PMap}_c(S)}, \text{PMap}(S), \text{Map}(S)\}$ and locally bounded, then $\text{asdim } G = \infty$.*

[Theorem 1.2](#) answers [\[GRV21, Qn. 1.8\]](#) of Grant–Rafi–Verberne and completes their characterization of the asymptotic dimension of stable boundedly generated infinite-type surface mapping class groups.

Corollary 1.3. *For stable S with boundedly generated $\text{Map}(S)$, $\text{asdim Map}(S) = \infty$ if and only if S has a non-displaceable subsurface or an essential shift; otherwise, $\text{Map}(S)$ is coarsely bounded and $\text{asdim Map}(S) = 0$.*

To our knowledge, our construction obtains the first examples of an arc and curve model admitting a geometric (Švarc–Milnor-type) action of the mapping class group of an arbitrary infinite-type surface; see [SC24] for a construction of curve graphs for translatable surfaces.

1.1. Outline. In Section 2, we state some known results about the asymptotic dimension of mapping class groups of infinite-type surfaces, specifically from [GRV21], following which we introduce some background and relevant tools for the coarse geometry of Polish groups from [Ros21] and [BDHL25]. In Section 3, we describe witness-cocompactness, a key tool for computing asymptotic dimension, and sketch the main theorem in [Kop24]:

Theorem 1.4. *Let S be an infinite-type surface and let \mathcal{M} be a witness-cocompact arc and curve model for $\text{PMap}_c(S)$. Then $\text{asdim } \mathcal{M} = \infty$.*

In Section 4, we introduce *coarse Cayley–Abels–Rosendal graphs*, which extend the Cayley–Abels–Rosendal graphs of [BDHL25] for locally bounded Polish groups; in particular, we prove a Švarc–Milnor-type result (Proposition 4.3). Section 5 constructs the model \mathcal{M} satisfying Theorem 1.1. Finally, Section 6 proves Theorem 1.2.

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2. PRELIMINARIES

We first review Rosendal’s work on Polish topological groups and introduce the necessary background on coarse structures from [Ros21]. We then recall Cayley–Abels–Rosendal graphs for topological groups [BDHL25] and several facts on the topology of boundedly generated mapping class groups. Lastly, we summarize the relevant results from [GRV21].

2.1. Coarse structure. The Polish groups considered herein are typically not finitely or compactly generated. Nonetheless, following Rosendal we may associate to every topological group G a canonical left-invariant *coarse structure*, which generalizes the (quasi)geometric structure classically associated to a group. This coarse structure will permit a well-defined coarse equivalence and quasi-isometry type for *locally bounded* and *boundedly generated* Polish groups, respectively (Section 2.2).

Definition 2.1 ([Ros21, Defn. 2.2]). A *coarse structure* on a set X is a collection \mathcal{E} of subsets $E \subseteq X \times X$ satisfying the following:

- The diagonal $\{(x, x) \mid x \in X\}$ is in \mathcal{E} .
- If $F \in \mathcal{E}$ and $E \subseteq F$, then $E \in \mathcal{E}$.

- if $E, F \in \mathcal{E}$, then $E \cup F, E^{-1}, E \circ F \in \mathcal{E}$, where $E^{-1} = \{(y, x) \mid (x, y) \in E\}$ and $E \circ F := \{(x, z) \mid \exists y \in X, (x, y) \in F, (y, z) \in E\}$.

Example 2.2 ([Ros21, Example 2.3]). The simplest examples of coarse structures arise from pseudometrics on a set X : Given a pseudometric d on X , we may define the coarse structure induced by d as follows:

$$\mathcal{E}_d := \{E \subset X \times X \mid E \subseteq E_\alpha \text{ for some } \alpha < \infty\}$$

where $E_\alpha := \{(x, y) \mid d(x, y) < \alpha\}$.

Definition 2.3 ([Ros21, Defn. 2.12]). A subset $A \subseteq X$ of a coarse space (X, \mathcal{E}) is said to be *coarsely bounded* if $A \times A \in \mathcal{E}$.

A topological group has a canonical left-invariant coarse structure:

Definition 2.4 ([Ros21, Defn. 2.10]). For a topological group G , the *left-coarse structure* \mathcal{E}_L is defined by

$$\mathcal{E}_L := \bigcap \{\mathcal{E}_d \mid d \text{ is a continuous, left-invariant pseudometric on } G\}.$$

For a topological group G with its left-coarse structure \mathcal{E}_L , we denote by \mathcal{CB} the collection of all coarsely bounded subsets of G . This collection is actually an ideal of sets additionally closed under the operations of topological closure, inversion and products.

Definition 2.5 ([Ros21, Defn. 2.10]). Given an ideal \mathcal{A} , we can define the coarse structure $\mathcal{E}_{\mathcal{A}}$ on the group G as follows:

$$\mathcal{E}_{\mathcal{A}} := \{E \mid E \subseteq E_A \text{ for some } A \in \mathcal{A}\}$$

where $E_A := \{(x, y) \in G \times G \mid x^{-1}y \in A\}$.

In particular, we can consider the coarse structure $\mathcal{E}_{\mathcal{CB}}$ on G , associated to the ideal \mathcal{CB} of coarsely bounded subsets of G .

Lemma 2.6 ([Ros21, Cor. 2.23]). *For any topological group G , $\mathcal{E}_L = \mathcal{E}_{\mathcal{CB}}$.*

Henceforth, we will always endow a topological group with the left coarse structure $\mathcal{E}_L = \mathcal{E}_{\mathcal{CB}}$ and a pseudometric space (X, d) with the coarse structure \mathcal{E}_d .

Definition 2.7 ([Ros21, Defn. 2.43]). Let (X, \mathcal{E}) and (Y, \mathcal{F}) be coarse spaces.

- A map $\varphi : X \rightarrow Y$ is said to be *bornologous* if $(\varphi \times \varphi)(\mathcal{E}) \subseteq \mathcal{F}$
- A map $\varphi : X \rightarrow Y$ is said to be *expanding* if $(\varphi \times \varphi)^{-1}(\mathcal{F}) \subseteq \mathcal{E}$
- A map $\varphi : X \rightarrow Y$ is said to be a *coarse embedding* if it is both bornologous and expanding.
- Let Z be a set. Two maps $\alpha, \beta : Z \rightarrow X$ are said to be *close* if there exists $E \in \mathcal{E}$ such that $(\alpha(z), \beta(z)) \in E$ for all $z \in Z$.
- A bornologous map $\varphi : X \rightarrow Y$ is said to be a *coarse equivalence* if there exists a bornologous map $\psi : Y \rightarrow X$ such that $\psi \circ \varphi$ is close to Id_X and $\varphi \circ \psi$ is close to Id_Y .
- A subset $A \subseteq X$ is said to be *cobounded* if there exists $E \in \mathcal{E}$ such that

$$X = E[A] := \{x \in X \mid (x, y) \in E \text{ for some } y \in A\}$$

- A map $\varphi : X \rightarrow Y$ is *cobounded* if $\varphi(X)$ is cobounded in Y .

Remark 2.8. The terms in [Definition 2.7](#) agree with their usual (metric) definitions when \mathcal{E} and \mathcal{F} are *metrizable* i.e. $\mathcal{E} = \mathcal{E}_d$ and $\mathcal{F} = \mathcal{E}_{d'}$ for metrics d, d' on X, Y respectively.

Lemma 2.9 ([\[Ros21, Lem. 2.45\]](#)). *Any cobounded coarse embedding is a coarse equivalence.*

Lemma 2.10. *Suppose that a topological group G acts continuously and isometrically on a metric space (X, d) . Then the orbit map ω is bornologous.*

Proof. Since the action is continuous and isometric, the pullback metric ω^*d is a left-invariant continuous pseudometric on G and $(\omega \times \omega)[\mathcal{E}_{\omega^*d}] \subset \mathcal{E}_d$. By definition $\mathcal{E}_L \subset \mathcal{E}_{\omega^*d}$, hence ω is bornologous. \square

Let $B_\alpha(x)$ denote the ball of radius $\alpha > 0$ centered at x .

Lemma 2.11. *Suppose that a topological group G acts continuously and isometrically on a metric space (X, d) and let ω be the orbit map based at $x_0 \in X$. Then ω is expanding if $A_\alpha := \omega^{-1}(B_\alpha(x_0))$ is coarsely bounded for all $\alpha > 0$.*

Proof. The orbit map ω is expanding if and only if $(\omega \times \omega)^{-1}(\mathcal{E}_d) \subseteq \mathcal{E}_L$. First consider $E_\alpha \in \mathcal{E}_d$ for $\alpha > 0$. Then

$$\begin{aligned} (\omega \times \omega)^{-1}(E_\alpha) &= \{(g, h) \in G \times G \mid (gx_0, hx_0) \in E_\alpha\} \\ &= \{(g, h) \in G \times G \mid d(gx_0, hx_0) < \alpha\} \\ &= \{(g, h) \in G \times G \mid d(x_0, g^{-1}hx_0) < \alpha\} \\ &= \{(g, h) \in G \times G \mid g^{-1}h \in A_\alpha\} \\ &= E_{A_\alpha} \end{aligned}$$

Since A_α is coarsely bounded, $E_{A_\alpha} \in \mathcal{E}_{CB} = \mathcal{E}_L$ by [Lemma 2.6](#). For general $E \in \mathcal{E}_d$, $E \subset E_\alpha$ for some $\alpha > 0$. Hence $(\omega \times \omega)^{-1}(E) \subset (\omega \times \omega)^{-1}(E_\alpha) \in \mathcal{E}_L$ and $(\omega \times \omega)^{-1}(E) \in \mathcal{E}_L$ as required. \square

We conclude by stating a convenient criterion for coarse boundedness:

Proposition 2.12 ([\[Ros21, Prop. 2.15\(5\)\]](#)). *Let G be a Polish group. A subset $A \subset G$ is coarsely bounded if and only if for every identity neighborhood $U \subset G$, there exists a finite set F and $n \in \mathbb{N}$ such that $A \subset (FU)^n$.*

Corollary 2.13. *Let G be a Polish group and $H \leq G$ be coarsely bounded in G . If $H \leq H' \leq G$ such that $[H : H'] < \infty$ then H' is also coarsely bounded in G .*

Proof. Since H is coarsely bounded, for every open neighborhood $1 \in U \subset G$, there exists a finite set F and $n \in \mathbb{N}$ such that $H \subset (FU)^n$. If $H' = \bigcup_{i=1}^k h_i H$, let $F' := F \cup \{h_1, \dots, h_k\}$. Clearly $H' \subset (F'U)^n$ and hence H' is coarsely bounded in G . \square

2.2. Local boundedness and bounded generation. Analogously to locally compact and compactly generated groups, we introduce two classes of topological groups related to the metrizability of \mathcal{E}_L .

Definition 2.14. A topological group G is

- (i) *locally bounded* if there is a coarsely bounded neighborhood of identity.

(ii) *boundedly generated* if it admits a coarsely bounded generating set.

Proposition 2.15 ([Ros21, Thm. 2.40]). *Any boundedly generated Polish group is locally bounded.*

Remark 2.16. The properties of local boundedness and bounded generation are not inherited by Polish subgroups. For example, consider the ladder surface S . Then $\text{Map}(S)$ is boundedly generated and hence locally bounded as well [MR23] but $\text{PMap}(S)$ is neither locally bounded nor boundedly generated [Hil25].

Proposition 2.17 ([Ros21, Cor. 3.26]). *Among Polish groups, the properties of being locally bounded and boundedly generated are both invariant under coarse equivalence. Moreover, every coarse equivalence between boundedly generated Polish groups is automatically a quasi-isometry.*

We recall that \mathcal{E}_L is metrizable when it is induced by a (possibly discontinuous) metric on G , in which case \mathcal{E}_L defines a coarse-equivalence type for G in the usual (metric) sense. Crucially:

Theorem 2.18 ([Ros21, Thm. 2.38]). *Let G be a Polish group. Then \mathcal{E}_L is metrizable if and only if G is locally bounded if and only if \mathcal{E}_L is induced by a continuous left-invariant pseudometric d on G .*

When \mathcal{E}_L is metrizable, by definition, it is maximal among the set of coarse structures on G induced by continuous left-invariant pseudometrics, with respect to the partial ordering

$$\mathcal{E}_{d'} > \mathcal{E}_d \iff \mathcal{E}_{d'} \subset \mathcal{E}_d \iff \text{Id} : (G, d') \rightarrow (G, d) \text{ is bornologous}.$$

In particular, \mathcal{E}_L is the unique such coarse structure. Similarly, we may consider a (finer) partial ordering on the set of left-invariant continuous pseudometrics on G : let $d \gg d'$ whenever $\text{Id} : (G, d) \rightarrow (G, d')$ is coarsely Lipschitz. We observe that any maximal d is unique up to quasi-isometry.

Theorem 2.19 ([Ros21, Prop. 2.72]). *Let G be a Polish group. Then G admits a continuous left-invariant pseudometric d maximal with respect to \ll if and only if G is boundedly generated, if and only if d is quasi-isometric to the word metric on G with respect to a symmetric coarsely bounded generating set.*

It follows that the word metric on G with respect to any coarsely bounded generating set gives a well-defined quasi-isometry type whenever G is boundedly generated.

2.2.1. *Locally bounded subgroups of $\text{Map}(S)$.* For a surface S , recall that

$$\text{Map}(S) := \text{Homeo}^+(S) / \text{Homeo}_0(S)$$

where $\text{Homeo}^+(S)$ is the group of orientation-preserving self-homeomorphisms of S , endowed with the compact-open topology, and $\text{Homeo}_0(S)$ is its identity component. The induced (quotient) topology on $\text{Map}(S)$ has a local (clopen) base at Id_S induced by the pointwise stabilizers $\tilde{U}_\Sigma := \{f \in \text{Homeo}^+(S) : f|_\Sigma = \text{Id}_\Sigma\}$ of compact, essential subsurfaces $\Sigma \subset S$; we denote the elements of this local base $U_\Sigma := \tilde{U}_\Sigma / (\tilde{U}_\Sigma \cap \text{Homeo}_0(S)) < \text{Map}(S)$.

Remark 2.20. Let $G \leq \text{Map}(S)$ a Polish subgroup. Given an essential compact subsurface $\Sigma \subset S$ let ν_Σ denote the (pointwise) G -stabilizer for Σ , that is $\nu_\Sigma := U_\Sigma \cap G$. Since $\text{Map}(S)$ has a local base $\{U_\Sigma\}_\Sigma$ at Id_S , likewise G has a local base $\{\nu_\Sigma\}_\Sigma$ at Id_S .

Remark 2.21. Since G has a local base of open subgroups at Id_S , G is non-Archimedean.

The following is immediate from [Remark 2.20](#):

Lemma 2.22. *Let $G \leq \text{Map}(S)$ be a locally bounded Polish subgroup. There exists a compact essential subsurface $\Sigma \subset S$ whose stabilizer ν_Σ is coarsely bounded in G .*

Some important subgroups. Let $\text{Ends}(S)$ denote the (Freudenthal) endspace of S and $\text{Ends}_g(S) \subset \text{Ends}(S)$ the subspace of non-planar ends. By $\text{PMap}(S) \leq \text{Map}(S)$ we denote the *pure mapping class group* of S , which is the kernel of natural map

$$\pi : \text{Map}(S) \rightarrow \text{Homeo}(\text{Ends}(S), \text{Ends}_g(S))$$

obtained from the action of $\text{Map}(S)$ on the endspace of S . Let $\text{PMap}_c(S) \leq \text{PMap}(S)$ denote the subgroup of compactly supported (necessarily pure) mapping classes. $\text{PMap}(S)$ is closed in $\text{Map}(S)$, hence it is a Polish subgroup. $\text{PMap}_c(S)$ is not closed when S is infinite-type; let $\overline{\text{PMap}_c(S)}$ denote its closure.

2.3. Cayley–Abels–Rosendal graphs. Analogous to Cayley–Abels graphs for totally disconnected, locally compact groups, Branman–Domat–Hoganson–Lyman [\[BDHL25\]](#) define graphical models for boundedly generated Polish groups. We generalize these results to the locally bounded case in [Section 4](#).

Definition 2.23 ([\[BDHL25, §3\]](#)). A connected, countable simplicial graph Γ is a *Cayley–Abels–Rosendal graph* for a topological group G if G admits a continuous, vertex-transitive, cocompact, and simplicial action with coarsely bounded vertex stabilizers.

Proposition 2.24 ([\[BDHL25, Prop. 8\]](#)). *Let G be a Polish group. Then G admits a Cayley–Abels–Rosendal graph if and only if G is boundedly generated. Moreover, the orbit map of G on any such graph is a quasi-isometry.*

2.4. Asymptotic dimension otherwise. We now shift our focus to the asymptotic dimension of mapping class groups. Asymptotic dimension was introduced by Gromov and gives a ‘large scale’ notion of dimension; see [\[BD07\]](#) for a survey of results.

Definition 2.25. Let X be a metric space. Then $\text{asdim}(X) \leq n$ if for every uniformly bounded open cover \mathcal{U} , there is a uniformly bounded open cover \mathcal{V} of multiplicity $n+1$ such that \mathcal{U} refines \mathcal{V} . We say that $\text{asdim}(X) = n$ if $\text{asdim}(X) \leq n$ but $\text{asdim}(X) \not\leq n-1$.

Proposition 2.26 ([\[BD07, Prop. 22\]](#)). *Let X and Y be metric spaces with the standard coarse structure and $f : X \rightarrow Y$ a coarse embedding. Then $\text{asdim}(X) \leq \text{asdim}(Y)$.*

It follows that asymptotic dimension is a coarse invariant and hence well-defined in the setting of locally bounded Polish groups. In particular, we can look at the asymptotic dimension of locally bounded surface mapping class groups. When S is a finite-type surface, [BBF15] shows that the asymptotic dimension of $\text{Map}(S)$ is finite. In the case of infinite type surfaces, the only result (as far as the authors know) appears in [GRV21]. We summarize the relevant details below.

Let S be an infinite-type surface. Suppose that there exists a countable family of homeomorphic subsurfaces $\Sigma_{i \in \mathbb{Z}} \subset S$, each with a single boundary component, and a simple path $\gamma \subset S \setminus \bigcup_i \Sigma_i$ intersecting each $\partial \Sigma_i$ sequentially and accumulating to two distinct ends. A *shift map* ω is a homeomorphism supported on a regular neighborhood of $\gamma \cup (\bigcup_i \Sigma_i)$, preserving γ set-wise and restricting to homeomorphisms $\Sigma_i \rightarrow \Sigma_{i+1}$. If in addition $\langle \omega \rangle$ is not coarsely bounded in $\text{Map}(S)$, then it is an *essential shift* [GRV21, §1].

Theorem 2.27 ([GRV21, Thm. 1.1]). *If S is stable and $\text{Map}(S)$ is boundedly generated and contains an essential shift, then $\text{asdim Map}(S) = \infty$.*

When S is stable, Theorem 1.2 and Theorem 2.27 fully classify the infinite asymptotic dimension cases:

Theorem 2.28 ([GRV21, Thm. 1.6]). *Let S be stable and $\text{Map}(S)$ be boundedly generated. If S contains neither a non-displaceable subsurface nor an essential shift, then $\text{Map}(S)$ is coarsely bounded.*

2.5. Classification of local boundedness. Since asymptotic dimension is well-defined for locally bounded mapping class groups, we recall results from [MR23] and [Hil25] that classify the infinite type surfaces whose mapping class groups and pure mapping class groups are locally bounded. Here, for $A \subset \text{Ends}(S)$, $M(A)$ is the set of maximal ends in A with respect to the partial order on $\text{Ends}(S)$ defined in [MR23].

Theorem 2.29 ([MR23, Thm. 1.4]). *Let S be an infinite type surface. Then $\text{Map}(S)$ is locally bounded if and only if there is a finite type surface $\Sigma \subset S$ such that the complimentary regions of K each have infinite type and zero or infinite genus, and partition $\text{Ends}(S)$ into finitely many clopen sets*

$$\text{Ends}(S) = \left(\bigsqcup_{A \in \mathcal{A}} A \right) \sqcup \left(\bigsqcup_{P \in \mathcal{P}} P \right)$$

such that:

- (1) Each $A \in \mathcal{A}$ is self-similar with $M(A) \subset M(\text{Ends}(S))$ and $M(\text{Ends}(S)) \subset \bigsqcup_{A \in \mathcal{A}} M(A)$.
- (2) each $P \in \mathcal{P}$ is homeomorphic to a clopen subset of some $A \in \mathcal{A}$.
- (3) for any $x_A \in M(A)$, and any neighborhood V of the end $x_A \in S$, there is $f_V \in \text{Homeo}(S)$ so that $f_V(V)$ contains the complimentary region to K with end set A .

Moreover, in this case ν_Σ is a coarsely bounded neighborhood of the identity.

Theorem 2.30 ([Hil25, Thm. 1.1(b)]). *Let S be an infinite type surface. Then $\text{PMap}(S)$ is locally bounded if and only if it is boundedly generated if and only if $|\text{Ends}(S)| < \infty$ and S is not a Loch Ness monster with (non-zero) punctures.*

Remark. The authors are unaware of any work concerning the local boundedness of $\overline{\text{PMap}_c(S)}$.

3. WITNESS-COCOMPACTNESS

We discuss *cocompact* and *witness-cocompact* arc and curve models and sketch the proof of [Theorem 1.4](#), which we will use in [Section 6](#) to compute the asymptotic dimension of certain locally bounded surface mapping class groups. This section summarizes the results of [\[Kop24\]](#), to which we direct the reader for full detail; it is included here for convenience.

3.1. Cocompact arc and curve models. Let S be a surface of arbitrary topological type and let $\mathcal{K}(S) := K(V(\mathcal{AC}(S)))$ denote the set of finite collections of simple arcs and curves on S . Note the arcs and curves in $u \in \mathcal{K}(S)$ need not be pairwise disjoint.

Definition 3.1. A (metric) arc and curve model for $G \leq \text{Map}(S)$ is a connected (metric) graph \mathcal{G} with discrete $V(\mathcal{G}) \subset \mathcal{K}(S)$ that admits an action of G induced by the permutation of its vertices. \mathcal{G} is *cocompact* if this action is cocompact.

Remark 3.2. Throughout [Section 3](#), a (metric) arc and curve model on S will mean a (metric) arc and curve model for some $G \geq \text{PMap}_c(S)$.

Remark 3.3. If S is finite-type, then (i) $\text{PMap}_c(S) = \text{PMap}(S)$ and (ii) \mathcal{G} is cocompact if and only if $i(u, u)$ and $i(u, v)$ are uniformly bounded for $u \in V(\mathcal{G})$ and $(u, v) \in E(\mathcal{G})$.

Definition 3.4. Let \mathcal{G} be an arc and curve model on S . A compact, essential (π_1 -injective, non-peripheral) subsurface $W \subset S$ is a *witness* for \mathcal{G} if W does not contain a pants component and every $u \in V(\mathcal{G})$ intersects every component of W .

We note that witnesses are not assumed to be connected. Let $\mathcal{X}^{\mathcal{G}}$ denote the set of witnesses of \mathcal{G} , and $\hat{\mathcal{X}}^{\mathcal{G}} \subset \mathcal{X}^{\mathcal{G}}$ the subset of connected witnesses. A *witness set* on S is any collection of compact, essential subsurfaces without pants components closed under enlargement and the action of $\text{PMap}_c(S)$.

By [\[Kop23\]](#), the geometry of cocompact arc and curve models on finite-type surfaces is well understood. In particular:

Theorem 3.5 ([\[Kop24, Thm. 4.12\]](#)). *Let \mathcal{G} be a cocompact arc and curve model on a finite-type surface Σ . Then $(\mathcal{G}, \mathcal{X}^{\mathcal{G}})$ is an asymphoric hierarchically hyperbolic space with respect to subsurface projection to witness curve graphs $\pi_W : \mathcal{G} \rightarrow 2^{CW}$, $W \in \mathcal{X}^{\mathcal{G}}$.*

The $\text{PMap}(\Sigma)$ -equivariant geometry of \mathcal{G} is uniquely determined by $\hat{\mathcal{X}}^{\mathcal{G}}$:

Theorem 3.6 ([\[Kop24, Thm. 4.13\]](#)). *The map $\mathcal{G} \mapsto \hat{\mathcal{X}}^{\mathcal{G}}$ induces a bijection between equivariant quasi-isometry types of cocompact arc and curve models on Σ and connected witness sets on Σ .*

Remark 3.7. The above is functorial in the following sense: whenever $\mathcal{X}^{\mathcal{G}'} \subset \mathcal{X}^{\mathcal{G}}$ (equivalently $\hat{\mathcal{X}}^{\mathcal{G}'} \subset \hat{\mathcal{X}}^{\mathcal{G}}$), there is a canonical equivariant coarsely surjective, coarse Lipschitz map $\iota : \mathcal{G} \rightarrow \mathcal{G}'$.

We note that [Theorem 3.5](#) implies that cocompact \mathcal{G} on a finite-type surface is δ -hyperbolic if and only if it has no pair of disjoint, connected witnesses. More broadly, \mathcal{G} admits a distance formula in the sense of Masur–Minsky: there is some $K > 0$ such that for any $u, v \in V(\mathcal{G})$,

$$d_{\mathcal{G}}(u, v) \approx \sum_{W \in \mathcal{X}^{\mathcal{G}}} [d_{CW}(\pi_W(a), \pi_W(b))]_K.$$

3.2. Subsurface projection. Given a compact, essential, connected, non-pants subsurface $\Sigma \subset S$, let $\mathcal{K}(S, \Sigma) \subset \mathcal{K}(S)$ denote the subset of collections containing an element that intersects Σ essentially. We construct a projection $\rho_{\Sigma} : \mathcal{K}(S, \Sigma) \rightarrow \mathcal{K}(\Sigma)$ as follows (see [\[Sch, §5.2\]](#)). Let $\iota : \Sigma \hookrightarrow S$ be the inclusion map, let $p : S_{\Sigma} \rightarrow S$ be the covering space associated to $\pi_1(\Sigma) \cong \text{im } \iota_* < \pi_1(S)$, and let $\tilde{\iota} : \Sigma \hookrightarrow S_{\Sigma}$ be the (unique) lift of ι into S_{Σ} . Fix any homeomorphism $\sigma : S_{\Sigma} \rightarrow \text{int } \Sigma := \Sigma \setminus \partial \Sigma$ that is a homotopy inverse for $\tilde{\iota}|_{\text{int } \Sigma}$; note that σ is unique up to homotopy, hence isotopy. Obtain $\tilde{\sigma}$ by composing σ with the inclusion $\text{int } \Sigma \hookrightarrow \Sigma$.

$$\begin{array}{ccc} & & S_{\Sigma} \\ & \nearrow \tilde{\iota} & \downarrow p \\ \Sigma & \xrightarrow{\iota} & S \end{array}$$

Given $u \in \mathcal{K}(S, \Sigma)$, let $\rho_{\Sigma}(u)$ be the closures of the non-peripheral components of $\tilde{\sigma}p^{-1}(u)$, up to isotopy.

One verifies that $\rho_{\Sigma}(u)$ is independent of the choice of representative for ω and σ . Likewise, ρ_{Σ} is independent of the choice of embedding of Σ : if $\iota' : \Sigma \hookrightarrow S$ is isotopic to ι , then the lift $\tilde{\iota}'$ is isotopic to $\tilde{\iota}$ and thus σ is likewise a homotopy inverse for $\tilde{\iota}'|_{\text{int } \Sigma}$.

Remark 3.8. The definition here for ρ_{Σ} differs slightly from that in [\[Kop24\]](#), which instead passes to the Gromov closure of S_{Σ} ; however, the definitions are consistent. We can likewise define $\rho_{\Sigma}(u)$ as the collection of essential intersections of u with Σ .

The natural action of $\text{PMap}(\Sigma)$ on $\mathcal{K}(\Sigma)$ defines an action of $\text{Map}(\Sigma, \partial \Sigma) \twoheadrightarrow \text{PMap}(\Sigma)$. Similarly, $\text{Map}(\Sigma, \partial \Sigma) \curvearrowright \mathcal{K}(S, \Sigma)$ via the homomorphism $\text{Map}(\Sigma, \partial \Sigma) \rightarrow \text{PMap}_c(S)$ obtained by extending by identity.

Lemma 3.9 ([\[Kop24, Lem. 4.14\]](#)). $\rho_{\Sigma} : \mathcal{K}(S, \Sigma) \rightarrow \mathcal{K}(\Sigma)$ is $\text{Map}(\Sigma, \partial \Sigma)$ -equivariant.

Corollary 3.10. Let $\phi \in \text{PMap}(\Sigma)$. Then there exists $\psi \in \text{PMap}_c(S)$ preserving $\mathcal{K}(S, \Sigma)$ such that for any $\omega \in \mathcal{K}(S, \Sigma)$, $\phi \rho_{\Sigma}(\omega) = \rho_{\Sigma}(\psi \omega)$.

3.2.1. Witness-cocompactness. Let $W \subset S$ be a connected witness for an arc and curve model \mathcal{G} on S . Note that $V(\mathcal{G}) \subset \mathcal{K}(S, W)$ and obtain an arc and curve model \mathcal{G}_W on W as follows: let $V(\mathcal{G}_W) = \rho_W(V(\mathcal{G})) \subset \mathcal{K}(W)$, and obtain $E(\mathcal{G}_W)$ as the push-forward of the edge relation on \mathcal{G} by ρ_W . By [Corollary 3.10](#) $\text{PMap}(W)$ acts on \mathcal{G}_W by permuting its vertices and the map $\rho_W : \mathcal{G} \rightarrow \mathcal{G}_W$ is $\text{Map}(W, \partial W)$ -equivariant; since \mathcal{G} is connected, likewise is \mathcal{G}_W . If \mathcal{G} is a metric graph, then likewise push forward the edge lengths on \mathcal{G} to obtain a metric on \mathcal{G}_W ; in either case, ρ_W is 1-Lipschitz.

Definition 3.11. Let \mathcal{G} be a connected (metric) arc and curve model on S . Then \mathcal{G} is *witness-cocompact* if

- (1) \mathcal{G} has a (compact) witness; and
- (2) for every witness $W \subset S$, \mathcal{G}_W is cocompact.

Remark 3.12. From [Remark 3.3](#), it follows that \mathcal{G} is witness-cocompact if and only if \mathcal{G} has a witness and for any witness W there is a uniform bound on $i(\rho_W(u), \rho_W(u))$ and $i(\rho_W(u), \rho_W(v))$ for $u \in V(\mathcal{G})$ and $(u, v) \in E(\mathcal{G})$.

Lemma 3.13. *Let \mathcal{G} be a witness-cocompact arc and curve model and let W be a witness. Then any $\text{Map}(W, \partial W)$ -equivariant section $\sigma_W : V(\mathcal{G}_W) \rightarrow V(\mathcal{G})$ is a quasi-isometric embedding.*

Proof. Since ρ_W is Lipschitz, it suffices to show that σ_W is likewise Lipschitz. Since \mathcal{G}_W is cocompact, it has finitely many orbits of edges $(\bar{u}, \bar{v}) \in E(\mathcal{G}_W)$. Since σ_W is equivariant, there are likewise finitely many orbits of pairs $(\sigma_W(\bar{u}), \sigma_W(\bar{v}))$. Let L be the maximum of the distances $d_{\mathcal{G}}(\sigma_W(\bar{u}), \sigma_W(\bar{v}))$ for $(\bar{u}, \bar{v}) \in E(\mathcal{G}_W)$. Then σ_W is coarsely L -Lipschitz. \square

Remark. If \mathcal{G} is witness-cocompact, then for each witness W , \mathcal{G}_W is cocompact: up to quasi-isometry, we may endow \mathcal{G}_W with the usual simplicial metric.

3.3. Asymptotic dimension lower bounds. We sketch the arguments from [\[Kop24\]](#) to prove [Theorem 1.4](#). We begin by computing lower bounds for the asymptotic dimension of cocompact arc and curve models on finite-type surfaces.

3.3.1. For finite-type surfaces. Let Σ be a finite-type surface with a cocompact arc and curve model \mathcal{M} . We aim to show the following:

Theorem 3.14 ([\[Kop24, Thm. 4.21\]](#)). *Let Σ be a genus g finite-type surface. If \mathcal{M} is a (non-empty) δ -hyperbolic cocompact arc and curve model on Σ , then $\text{asdim } \mathcal{M} \geq g - \lceil \frac{1}{2}\chi(\Sigma) \rceil$.*

Remark 3.15. In the complementary case, when \mathcal{M} is not δ -hyperbolic or equivalently when \mathcal{M} has $\nu > 1$ disjoint connected witnesses, it will suffice that $\text{asdim } \mathcal{M} \geq \nu$. In particular, ν is exactly the HHS rank of $(\mathcal{M}, \mathcal{X}^{\mathcal{M}})$, which bounds $\text{asdim } \mathcal{M}$ from below [\[BHS21, Thm. 1.15\]](#).

We prove [Theorem 3.14](#) by finding a compact subspace $Z \subset \partial\mathcal{M}$ of known topological dimension. For proper δ -hyperbolic spaces, the topological dimension of the boundary gives bounds on the asymptotic dimension of the space [\[BL08, Prop. 6.2\]](#); while \mathcal{M} is typically non-proper, a minor adaptation of the lower bound suffices.

Proposition 3.16 ([\[Kop24, Prop. 2.5\]](#)). *Let X be a geodesic δ -hyperbolic space with $Z \subset \partial X$ compact. Then $\text{asdim } X \geq \dim Z + 1$.*

We find Z as follows. Recall that, whenever \mathcal{M} and \mathcal{M}' are cocompact graph models on Σ and $\mathcal{X}^{\mathcal{M}} \supset \mathcal{X}^{\mathcal{M}'}$, there is a canonical coarsely surjective, coarsely Lipschitz map $\iota : \mathcal{M} \rightarrow \mathcal{M}'$. In particular, since $\mathcal{X}^{C\Sigma} = \{\Sigma\}$, such a map $\iota : \mathcal{M} \rightarrow C\Sigma$ exists for any cocompact graph model \mathcal{M} . We first prove that when \mathcal{M} is δ -hyperbolic these maps are coarsely alignment preserving in the sense of Dowdall–Taylor [\[DT17\]](#): there exists K for $\mathcal{M}, \mathcal{M}'$ such that for any aligned triple of vertices $(x, y, z) \in V(\mathcal{M})^3$, $d(\iota(x), \iota(y)) + d(\iota(y), \iota(z)) \leq d(\iota(x), \iota(z)) + K$.

Lemma 3.17 ([Kop24, Lem. 4.22]). *Let \mathcal{M} and \mathcal{M}' be arc and curve models on a finite-type surface such that $\mathcal{X}^{\mathcal{M}} \supset \mathcal{X}^{\mathcal{M}'}$, and let $\iota : \mathcal{M} \rightarrow \mathcal{M}'$ be the canonical coarse surjection. If \mathcal{M} is δ -hyperbolic, then ι is coarsely alignment-preserving.*

The proof of [Lemma 3.17](#) follows from the existence of hierarchy paths [BHS19, Thm. 4.4] in $\mathcal{M}, \mathcal{M}'$. Such paths are close to geodesics and have projections to witness curve graphs that are unparameterized quasi-geodesics. Applying the distance formulas for $\mathcal{M}, \mathcal{M}'$ derives the claim.

Crucially, the theory of alignment preserving maps implies an embedding of $\partial\mathcal{M}'$ into $\partial\mathcal{M}$, and in particular an embedding $\partial\mathcal{C}\Sigma \hookrightarrow \partial\mathcal{M}$ whenever \mathcal{M} is δ -hyperbolic.

Theorem 3.18 ([DT17, Thm. 3.2]). *Let $f : X \rightarrow Y$ be a coarsely surjective, coarsely alignment preserving map between geodesic δ -hyperbolic spaces. Then f induces an embedding $\partial Y \hookrightarrow \partial X$.*

When Σ is a punctured sphere, we then conclude using a result of Gabai:

Theorem 3.19 ([Gab14, Thm. 1.2]). *Let Δ be the $(n+4)$ -times punctured sphere for $n \geq 0$. Then $\partial\mathcal{C}\Delta$ is homeomorphic to the n -dimensional Nöbeling space \mathbb{R}_n^{2n+1} .*

In particular, by the universal embedding property of Nöbeling spaces [Nöb30] any n -dimensional compactum Z embeds into $\partial\mathcal{C}\Delta \subset \partial\mathcal{M}$. For general Σ , we apply a result of Rafi–Schleimer [RS09, Thm. 7.1] to obtain an embedding of $\partial\mathcal{C}\Delta$ into $\partial\mathcal{C}\Sigma \subset \partial\mathcal{M}$, which completes the proof of [Theorem 3.14](#).

Proposition 3.20 ([Kop24, Prop. 4.23]). *Let Σ be a finite-type hyperbolic surface of genus g and Δ the $(n+4)$ -times punctured sphere, where $n = g - 1 - \lceil \frac{1}{2}\chi(\Sigma) \rceil$. Then $\partial\mathcal{C}\Delta$ embeds into $\partial\mathcal{C}\Sigma$.*

3.3.2. *For infinite-type surfaces.* We prove the following:

Theorem 3.21. *Let S be an infinite-type surface and let \mathcal{M} be a witness-cocompact metric arc and curve model on S . Then $\text{asdim } V(\mathcal{M}) = \infty$.*

In particular, since $V(\mathcal{M})$ is given the induced metric, it isometrically embeds into \mathcal{M} and [Theorem 1.4](#) follows. By [Lemma 3.13](#) and the monotonicity of asymptotic dimension, it suffices to find for every $d \in \mathbb{N}$ some witness $W \subset S$ for which $\text{asdim } \mathcal{M}_W \geq d$ (see [Kop24, §4.3.2]).

Given a witness-cocompact arc and curve model \mathcal{M} on an infinite-type surface Ω , let $w_{\mathcal{M}} \in \mathbb{N} \cup \{\infty\}$ denote the least upper bound on cardinalities for a set of pairwise-disjoint connected witnesses for \mathcal{M} . If $w_{\mathcal{M}}$ is infinite, then for each d fix a compact subsurface Σ_d containing at least d pair-wise disjoint connected witnesses for \mathcal{M} . These witnesses are likewise witnesses for \mathcal{M}_{Σ_d} , hence \mathcal{M}_{Σ_d} has rank $\nu \geq d$ and $\text{asdim } \mathcal{M}_{\Sigma_d} \geq d$. If $w_{\mathcal{M}}$ is finite, then fix a set $\{\Delta_i\}$ of $w_{\mathcal{M}}$ pairwise disjoint witnesses, with Δ_0 a witness adjacent to an infinite-type component of $S \setminus \bigcup_i \Delta_i$. By enlarging Δ_0 disjointly from the remaining Δ_i , we obtain compact subsurfaces $\Sigma_d \subset S \setminus \bigcup_{i>0} \Delta_i$ such that $-\chi(\Sigma_d) > 2d$ and $\Delta_0 \subset \Sigma_d$. Since $\Sigma_d \supset \Delta_0$, it is a witness for \mathcal{M} , and each \mathcal{M}_{Σ_d} must have rank $\nu = 1$ else we obtain a set of witnesses for \mathcal{M} of cardinality greater than $w_{\mathcal{M}}$. It follows that \mathcal{M}_{Σ_d} is δ -hyperbolic. Applying [Theorem 3.14](#), we obtain $\text{asdim } \mathcal{M}_{\Sigma_d} \geq d$ as required. [Theorem 3.21](#) follows. //

4. A ŠVARC–MILNOR LEMMA FOR LOCALLY BOUNDED GROUPS

Definition 4.1. The action of a group G on a metric graph Γ is *bounded-cocompact* if, for every closed bounded subgraph $\Lambda \subset \Gamma$, Λ/G is compact.

Definition 4.2. A connected metric graph Γ with a discrete vertex set $V(\Gamma)$ along with an isometric, isomorphic, and continuous action of a group G is a *coarse Cayley–Abels–Rosendal graph for G* if the action is vertex-transitive and bounded-cocompact with coarsely bounded vertex stabilizers.

Recall that a Polish group G is *non-Archimedean* if it has a (clopen) subgroup neighborhood basis at identity [Kec12]. The following extends Proposition 2.24.

Proposition 4.3. *Let G be a Polish group. If G admits a coarse Cayley–Abels–Rosendal graph then it is locally bounded; moreover, for any such graph Γ the orbit map to $V(\Gamma)$, with the induced metric, is a coarse equivalence. If G is non-Archimedean then the converse holds.*

Proof. Let $\omega : G \rightarrow V(\Gamma)$ denote the vertex orbit map; since the action is continuous, ω is continuous. Moreover, since $V(\Gamma)$ is discrete, the stabilizer of a vertex in Γ is thus a coarsely bounded neighborhood of identity and G is locally bounded. We must show that ω is bornologous, expanding, and cobounded, hence a coarse equivalence: the first follows from Lemma 2.10 and the last from vertex transitivity, hence we need only check that ω is expanding.

Let d denote the metric on Γ and fix a vertex $x \in V(\Gamma)$; we assume ω is the orbit map based at x . By Lemma 2.11, it suffices to show that $A_\alpha = \omega^{-1}(B_\alpha(x))$ is coarsely bounded. Fix a connected bounded subgraph $\Lambda_\alpha \subset \Gamma$ containing $B_\alpha(x) \cap \text{im } \omega = B_\alpha(x) \cap V(\Gamma)$ and let $A'_\alpha = \omega^{-1}(\Lambda_\alpha) = \{g \in G : gx \in V(\Lambda_\alpha)\}$. Since $V(\Gamma)$ is discrete and G acts vertex-transitively, the infimum of edge lengths in Γ is non-zero. Hence by bounded-cocompactness, Λ_α/G is a finite graph, or equivalently Λ_α intersects finitely many G -orbits of edges: the midpoints of edges in Λ_α/G are discrete, hence must be finite by compactness. Let $\nu_x := \text{stab}_G(x) \leq G$ denote the stabilizer of x and fix a finite set of elements $F_\alpha \subset G$ so that if $(gx, hx) \in E(\Lambda_\alpha)$, then $g^{-1}h \in \nu_x F_\alpha \nu_x$; additionally add some element $g_0 \in A'_\alpha$. Let $m = \text{diam } \Lambda_\alpha$. Then $A'_\alpha \subset (F_\alpha \nu_x)^m$, hence $A'_\alpha \supset A_\alpha$ is coarsely bounded since likewise is ν_x .

The converse when G has small subgroups is shown in the following lemma. \square

Lemma 4.4. *If G is a non-Archimedean locally bounded Polish group, then it admits a coarse Cayley–Abels–Rosendal graph.*

Proof. Fix a coarsely bounded clopen subgroup $H \leq G$ and a countable set $Z = \{z_i\} \subset G$ such that $G = \langle Z, H \rangle$. Such a Z always exists: for example, since G is separable and H open, H has a countable transversal in G . Construct a metric graph Γ on the vertex set G/H by attaching an edge of length i between gH and kH whenever $g^{-1}k \in Hz_iH$. The set Z generates G over H , hence Γ is connected. Since H is clopen, the left action of G on G/H is continuous; it induces a continuous, isometric, isomorphic, and vertex transitive action on Γ with coarsely bounded vertex stabilizer $\text{stab}_G(H) = H$.

We verify bounded-cocompactness: Γ/G is the metric graph isomorphic to a bouquet of countably many circles e_i , each of length i . It suffices that if Λ is a

subgraph of Γ for which Λ/G is not compact, then it is unbounded. In particular, $\Lambda/G \subset \Gamma/G$ must contain infinitely many edges and thus an edge of length at least i for every $i \in \mathbb{N}$, hence likewise does Λ . \square

Remark 4.5. If there exists a finite subset $Z \subset G$ and a coarsely bounded clopen subgroup H such that $G = \langle Z, H \rangle$, then clearly $H \cup Z$ is also coarsely bounded and hence G is boundedly generated. Conversely, if G is boundedly generated, then we may choose Z in [Lemma 4.4](#) above to be finite by [Proposition 2.12](#). Hence G is boundedly generated if and only if Z can be chosen to be finite.

Remark 4.6. For a Polish group G , let $H \leq G$ be a coarsely bounded open subgroup and $Z \subset G$ an enumerated countable set that generates G over H . Let $\mathcal{C}_{H,Z}(G)$ denote the coarse Cayley–Abels–Rosendal graph constructed as in the proof of [Lemma 4.4](#); by [Proposition 4.3](#), its vertex set is coarse equivalent to G .

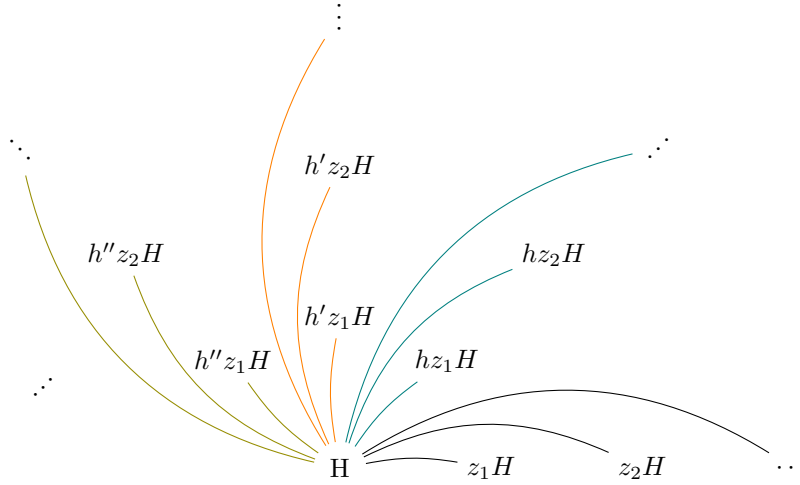


FIGURE 1. The neighborhood of the vertex H in $\mathcal{C}_{H,Z}(G)$. Here, $h, h', h'' \in H$ and $Z = \{z_i\}$

Remark 4.7. The construction of $\mathcal{C}_{H,Z}(G)$ exactly coincides with that in [\[BDHL25, §3\]](#) when Z is finite and generates G over H . In this case, $\mathcal{C}_{H,Z}(G)$ has only finitely many edge orbits and G acts *cocompactly* on $\mathcal{C}_{H,Z}(G)$, hence $\mathcal{C}_{H,Z}(G)$ (viewed as a simplicial graph) is a Cayley–Abels–Rosendal graph for G .

5. ARC AND CURVE MODELS FOR SUBGROUPS OF THE MAPPING CLASS GROUP

In this section, we adapt the construction in [Lemma 4.4](#) to the context of subgroups of a mapping class group of an infinite type surface S . More specifically, if $G \leq \text{Map}(S)$ is a locally bounded Polish subgroup, we construct an arc and curve model ([Definition 3.1](#)) that is also a coarse Cayley–Abels–Rosendal graph for G .

Let S be an infinite-type surface and let $G \leq \text{Map}(S)$ be a locally bounded Polish subgroup. Suppose $\mu \in \mathcal{K}(S)$ is a finite collection of simple arcs and simple closed curves with a coarsely bounded (set-wise) G -stabilizer $\nu_\mu := \text{stab}_G(\mu)$.

Then there exists a countable set $Z \subset G$ such that $G = \langle \nu_\mu, Z \rangle$. Define a graph $\mathcal{M}_{\mu,Z}(G)$ with vertex set $V(\mathcal{M}_{\mu,Z}(G)) := G\mu$. Consider the G -equivariant bijection $V(\mathcal{C}_{\nu_\mu,Z}(G)) = G/\nu_\mu \xrightarrow{\cong} G\mu = V(\mathcal{M}_{\mu,Z}(G))$ defined by $g\nu_\mu \mapsto g\mu$ and obtain $\mathcal{M}_{\mu,Z}(G)$ by pushing forward the (metric) edge relation in $\mathcal{C}_{\nu_\mu,Z}(G)$ to $V(\mathcal{M}_{\mu,Z}(G))$. Note that $\mathcal{M}_{\mu,Z}(G)$ is G -equivariantly isometric to $\mathcal{C}_{\nu_\mu,Z}(G)$ and hence a coarse Cayley–Abels–Rosendal graph for G : if there exists such a μ , then G is coarsely equivalent to $\mathcal{M}_{\mu,Z}(G)$.

By [Remark 4.5](#), G is boundedly generated if and only if Z can be chosen to be finite, which by [Remark 4.7](#) occurs if and only if $\mathcal{C}_{\nu_\mu,Z}(G)$ (and hence $\mathcal{M}_{\mu,Z}(G)$) is a Cayley–Abels–Rosendal graph. The following lemma shows that there indeed exists a μ with coarsely bounded G -stabilizer ν_μ , whence [Theorem 1.1](#) follows.

Lemma 5.1. *Let $G \leq \text{Map}(S)$ be a locally bounded Polish subgroup. Then there exists $\mu \in \mathcal{K}(S)$ such that ν_μ is coarsely bounded in G .*

Proof. Since G is locally bounded, [Lemma 2.22](#) tells us that there exists $\Sigma \subset S$ with a coarsely bounded G -stabilizer ν_Σ . Let $\mu_0 \in \mathcal{K}(S) \cap \mathcal{K}(\Sigma)$ be a filling collection of arcs and curves in Σ and $\mu := \mu_0 \cup \partial\Sigma$; clearly $\nu_\Sigma \leq \nu_\mu$. Let $S(\mu)$ denote the permutation group on μ and let K denote the kernel of the natural map $\nu_\mu \rightarrow S(\mu)$. In general $\nu_\Sigma \subseteq K$. Since μ_0 is filling, by Alexander’s method [[FM11](#), Prop. 2.8] we have in fact $\nu_\Sigma = K$. Finally since μ is a finite collection, so is $S(\mu)$ and consequently $[\nu_\Sigma : \nu_\mu] \leq |S(\mu)| < \infty$. By [Corollary 2.13](#), ν_μ is also coarsely bounded in G , as required. \square

6. ASYMPTOTIC DIMENSION

Throughout this section, we assume S is an infinite-type surface with a non-displaceable subsurface for a Polish subgroup $\text{PMap}_c(S) \leq G \leq \text{Map}(S)$ such that G is boundedly generated or $G \in \{\text{PMap}_c(S), \text{PMap}(S), \text{Map}(S)\}$ and locally bounded. To prove [Theorem 1.2](#), it suffices to choose μ and Z so that $\mathcal{M} = \mathcal{M}_{\mu,Z}$ is witness-cocompact: by [Theorem 3.21](#) $\text{asdim } V(\mathcal{M}) = \infty$, hence likewise $\text{asdim } \text{Map}(S) = \infty$ by [Theorem 1.1\(1\)](#). Since $\text{PMap}_c \leq G$ it suffices to ensure that

- (i) \mathcal{M} has a witness, and (by [Remark 3.12](#))
- (ii) that edge- and self-intersection numbers are uniformly bounded.

Only condition (i) uses non-displaceability; note also that it may be satisfied for arbitrary locally bounded Polish subgroups $G \leq \text{Map}(S)$. We choose μ so that ν_μ is coarsely bounded and $V(\mathcal{M}) = G\mu$ has a witness. Let $\Delta \subset S$ to be a compact, essential, G -non-displaceable subsurface sufficiently large that ν_Δ is coarsely bounded. Fix $\mu_0 \in \mathcal{K}(\Delta) \cap \mathcal{K}(S)$ to be a filling collection of curves in Δ and let $\mu = \mu_0 \cup \partial\Delta$. By [Lemma 5.1](#), ν_μ is coarsely bounded. For any $g \in G$, $g\mu_0$ is filling in $g\Delta$, hence since $g\Delta$ intersects Δ essentially likewise does $g\mu = g\mu_0 \cup \partial g\Delta$: Δ is a witness for \mathcal{M} .

Intersection numbers are invariant under $G \leq \text{Map}(S)$, hence to ensure (ii) it suffices that

- (ii') $i(\mu, z_i\mu)$ is uniformly bounded over $z_i \in Z$.

In particular, $i(g\mu, g\mu) = i(\mu, \mu) < \infty$ and if $(g\mu, k\mu) \in E(\mathcal{M})$ then $g^{-1}k = hz_ih'$ for some $z_i \in Z$ and $h, h' \in \nu_\mu$, thus $i(g\mu, k\mu) = i(h^{-1}\mu, z_ih'\mu) = i(\mu, z_i\mu)$. When G is boundedly generated we may choose Z to be finite, hence (ii') is immediate.

For the remainder of the section, we construct for each locally bounded case a (countable) $Z \subset G$ that generates G over ν_μ and satisfies (ii').

6.1. Enforcing small intersection. We first produce topological generating sets for $G = \overline{\text{PMap}_c(S)}, \text{PMap}(S)$ satisfying (ii'). In particular, since ν_μ is open these generate G over ν_μ .

Lemma 6.1. *There exists a countable generating set T for $\text{PMap}_c(S)$ such that $\{i(\mu, t\mu)\}_{t \in T}$ is finite and hence bounded above.*

Proof. Let $\Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \dots$ be a compact exhaustion of S such that

- (1) $\Sigma_0 \supset \Delta$.
- (2) If C_j^i denotes the simple closed curves corresponding to the Dehn-Likorish generators for $\text{Map}(\Sigma_i)$, then $\{C_j^i\}_j \cup \partial\Sigma_i \subset \{C_k^{i+1}\}_k \cup \partial\Sigma_{i+1}$.

Then the collection of Dehn twists $T := \{T_{ij}\}$ along the simple closed curves C_j^i generate $\text{PMap}_c(S)$. Moreover, for sufficiently large i , $\{C_k^{i+1}\} \setminus (\{C_j^i\} \cup \partial\Sigma_i)$ only consists of simple closed curves outside of Δ and hence their corresponding Dehn twists fix Δ pointwise and therefore μ . As such, only finitely many of the Dehn twists T_{ij} act non-trivially on μ , hence $\{i(\mu, t\mu)\}$ is a finite collection bounded above by the maximum over these finitely many Dehn twists $t \in T$ that act non-trivially on μ . \square

Recall that a *handle shift* is a shift map (see [Section 2.4](#)) with homeomorphic subsurfaces $\Sigma_i \cong \Sigma_1^1$, and let $h_\pm \in \text{Ends}_g(S)$ denote the (forward and backward) accumulation points of the underlying path, which we will call the *endpoints* of h .

Lemma 6.2. *Let $H \subset \text{PMap}(S)$ be a collection of handle shifts such that $\{(h_-, h_+) : h \in H\}$ is dense in $\text{Ends}_g(S) \times \text{Ends}_g(S)$. For any neighborhood $1 \in \nu \subset \text{Map}(S)$, $\text{PMap}(S) \leq \langle H, \text{PMap}_c(S), \nu \rangle$.*

Proof. Let $\nu_P := \nu \cap \text{PMap}(S)$ be a clopen subgroup of $\text{PMap}(S)$. We know that $\text{PMap}(S)$ is topologically generated by Dehn twists (which are compactly supported) and handle shifts [[PV18](#), Thm. 4], [[APV20](#), Cor. 6 and Section 2.3]. Since H is dense in $\text{Ends}_g(S) \times \text{Ends}_g(S)$, $\text{PMap}(S)$ is in fact topologically generated by Dehn twists and H [[AV20](#), Thm. 4.4]. If we consider translates of ν_P by Dehn twists and elements of H , we therefore get an open cover of $\text{PMap}(S)$. Hence $\text{PMap}(S) = \langle H, \text{PMap}_c(S), \nu_P \rangle$ which implies $\text{PMap}(S) \leq \langle H, \text{PMap}_c(S), \nu \rangle$. \square

Lemma 6.3. *There exists a countable set of handle shifts $H \subset \text{PMap}(S)$ whose endpoints are dense in $\text{Ends}_g(S) \times \text{Ends}_g(S)$ and for which $i(\mu, h\mu)$ is uniformly bounded for $h \in H$.*

Proof. Let S_1, \dots, S_b be the complementary components of Δ . Fix $k = \binom{b}{2}$ many pairwise disjoint strips $s_{\{i,j\}}$, connecting the i^{th} and j^{th} complementary components of Δ with $i \neq j$. For two distinct ends $x, y \in \text{Ends}_g(S)$ accumulated by genus, consider handle shifts h_{xy} such that

- (1) If $x, y \in S_i$, then $\text{Domain}(h_{xy}) \cap \Delta = \emptyset$.
- (2) If $x, x' \in S_i$, $y, y' \in S_j$ and $i \neq j$, then $\text{Domain}(h_{xy}) \cap \Delta = s_{\{i,j\}}$ and $h_{xy}|_{s_{\{i,j\}}} = h_{x'y'}|_{s_{\{i,j\}}}$.
- (3) For $x, y \in \text{Ends}_g(S)$, $h_{yx} = h_{xy}^{-1}$.

Fix $E \subset \text{Ends}_g(S) \times \text{Ends}_g(S)$ a countable dense subset and set $H := \{h_{xy} \mid (x, y) \in E\}$. Let $h_{ij} := h|_{s_{\{i,j\}}}$ for $h \in H$ such that $h_- \in S_i$, $h_+ \in S_j$. Note that by (2), this is well-defined and for any $h \in H$, since $\mu \subset \Delta$, either h fixes μ pointwise or $i(\mu, h\mu) = i(\mu, h_{ij}\mu)$ for some $i \neq j$. Hence $i(\mu, h\mu)$ is uniformly bounded by $\max_{i,j} i(\mu, h_{ij}\mu)$ for any $h \in H$ as required. \square

When $G = \overline{\text{PMap}_c(S)}$, it suffices that $Z = T$, and when $G = \text{PMap}(S)$, that $Z = T \cup H$. \parallel

Remark 6.4. By [Hil25] (Theorem 2.30) $G = \text{PMap}(S)$ is locally bounded if and only if it is boundedly generated, in which case Z may be chosen to be (in fact) finite. We include Lemma 6.3 so that our arguments are self-contained.

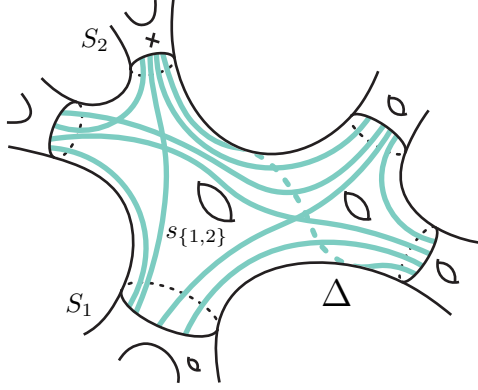


FIGURE 2. The strips $s_{\{i,j\}}$ in Δ

We conclude with the case when $G = \text{Map}(S)$. We aim to construct a transversal I for $K := \langle T, H, \nu_\mu \rangle$ that satisfies the intersection condition (ii'); since K is open, I is countable and we may set $Z = T \cup H \cup I$. Consider the exact sequence

$$1 \rightarrow \text{PMap}(S) \rightarrow \text{Map}(S) \xrightarrow{\pi} \text{Homeo}(E, E_g) \rightarrow 1$$

It suffices to construct a (set-theoretic) section σ for π whose image satisfies (ii'). Then $I_\sigma := \text{im } \sigma$ is a transversal for $\text{PMap}(S)$ (albeit possibly uncountable), and since $\text{PMap}(S) \leq K$ by Lemma 6.2, I_σ contains a transversal $I \subset I_\sigma$ for K likewise satisfying (ii'). We construct σ below.

Lemma 6.5. *There exists a (set-theoretic) section $\sigma : \text{Homeo}(E, E_g) \rightarrow \text{Map}(S)$ such that $i(\mu, \sigma(\varphi)\mu)$ is uniformly bounded over $\varphi \in \text{Homeo}(E, E_g)$.*

Remark. In the following, let $\Sigma_{g,p}^b$ denote the orientable surface with genus g , b boundary components, and p punctures.

Proof. Fix some connected, essential subsurface $\Pi \cong \Sigma_{g,0}^b \subset S$ such that $\Pi \supset \Delta$ and whose complementary components have either zero or infinite genus. Fix an embedding of Π into $\Sigma = \Sigma_{g,0}^{b^2}$ such that $\Sigma \setminus \Pi$ is the disjoint union of copies of $\Sigma_{0,1}^b$. Let $\sigma_0 : \text{Sym}(\pi_0(\partial\Sigma)) \rightarrow \text{Map}(\Sigma)$ be a choice of (set-theoretic) section; note that $i(\mu, \sigma_0(\alpha)\mu)$ is uniformly bounded over $\alpha \in \text{Sym}(\pi_0(\partial\Sigma))$, a finite set.

Let $U_i \subset E$ be the clopen partition induced by Π , and let $U_{i,j} = U_i \cap \varphi(U_j)$; let $S_i \subset S \setminus \Pi$ denote the complementary component containing U_i and $C_i := \partial S_i$. Extend each component of $\partial \Pi$ by an embedded (but not necessarily essential) $\Sigma_{0,0}^{b+1}$ to obtain a subsurface $\Pi_\varphi \cong \Sigma$ inducing the partition $U_{i,j}$. Note that some components of $S \setminus \Pi_\varphi$ may be disks and that every component has either zero or infinite genus. Let $S_{i,j} \subset S \setminus \Pi_\varphi$ denote the complementary component containing $U_{i,j}$ and $C_{i,j} := \partial S_{i,j}$.

Extend the embedding $\Pi \hookrightarrow \Pi_\varphi$ to a homeomorphism $\psi_\varphi : \Sigma \rightarrow \Pi_\varphi$. Fix a permutation $\alpha_\varphi \in \text{Sym}(\pi_0(\partial \Sigma))$ and $\sigma_1(\varphi) := \sigma_0(\alpha_\varphi)^{\psi_\varphi} : \Pi_\varphi \rightarrow \Pi_\varphi$ such that $C'_j := \sigma_1(\varphi)C_j$ separates $C_{i,j}$ from $\sigma_1(\varphi)\Pi$ for all i . Let $\Pi' := \sigma_1(\varphi)\Pi$ and S'_j denote the component of $S \setminus \Pi'$ separated by C'_j . It follows that $\Pi' = \sigma_1(\varphi)\Pi$ induces the partition $\varphi(U_j)$, each of which is contained in S'_j . We note that C'_j is homeomorphic to C_j ; likewise, since the S_j and S'_j have either zero or infinite genus, S_j has zero genus if and only if $U_j \cap \text{Ends}_g(S) = \emptyset$ if and only if $\varphi(U_j) \cap \text{Ends}_g(S) = \emptyset$ if and only if S'_j has zero genus, and otherwise both S_j, S'_j have infinite genus. Finally, we apply Richards' classification theorem [Ric63] to obtain $\sigma(\varphi)$ by extending $\sigma_1(\varphi)|_\Pi$ by homeomorphisms $\bar{S}_j \rightarrow \bar{S}'_j$ that induce φ on each U_j . Up to isotopy $\mu \subset \Delta \subset \Pi$, hence $i(\mu, \sigma(\varphi)\mu) = i(\mu, \sigma_0(\alpha_\varphi)\mu)$ is bounded independently of φ . \square

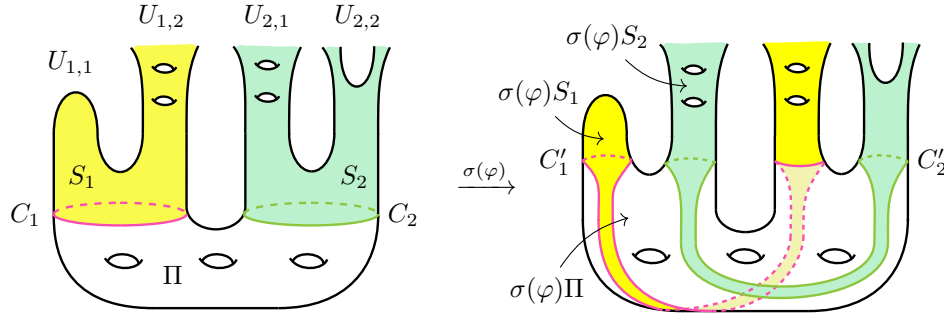


FIGURE 3. The map $\sigma(\varphi)$. Here, $U_{1,1} = \emptyset$

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