Combinatorial Morse Theory

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1 Introduction

The 1996 paper, *Morse theory and finiteness properties of groups*, by Mladen Bestvina and Noel Brady deploys the then-novel techniques of "combinatorial" Morse theory applied to affine CW complexes to determine the distance in resolution between homotopic (classical) and homological criteria for group finiteness properties. Results concerning such questions, regarding the gap in capability between homological tools and underlying homotopy type are seminal; moreover, the diversity and concreteness of the methods used to achieve these results make the paper attractive from an academic perspective.

This essay attempts to keep faithfully to the overall argument and structure of the above, while expanding substantially on the stated proofs and including pertinent background material when necessary. Hence (although the original work has been revisited in a somewhat more modern light since its publication), the content of this essay is at least spiritually familiar to the original work, with one exception: in the original paper, the crucial objects of the investigation, *sheets*, are defined with respect to the notion of a *flat* in a CAT(0) cubical complex. In practice, the author found this definition vague and difficult to deploy. Hence this essay instead regards sheets as the embedded universal covers of n-tori, proceeding with the subsequent arguments from the perspective of covering theory.

In addition, there are three analogous results whose proofs make use of a certain convexity condition on all right spherical simplicial complexes. The author was unable to complete a rigorous justification of this condition before submitting their essay, hence Lemma 8.6 and the related results Lemma 6.11 and Corollary 8.4 remain without proof. The corresponding result may be found as Lemma 8.3 in the original paper.

1.1 Conventions

Let every deformation retraction be strong unless otherwise noted. Homotopy equivalences are denoted with $\simeq$.

Given subspaces $U, V \subset X$ and a map $f : U \to Y$, then $f|_V$ is understood to denote the restriction of f to $U \cap V$.

Let all CW complexes be finite dimensional, and let cells in such complexes be *closed* cells, *i.e.* the images of closed balls under their characteristic maps. In contrast, the *interior* of such a cell is the corresponding *open* cell, *i.e.* the image of the interior of the closed ball under the characteristic map of the cell.

In general, we will assume groups are given the discrete topology.

2 Morse theory on affine cell complexes

We begin by introducing affine cell complexes, which generalize the notion of polyhedral cell complexes to include the attachment of polyhedra via affine maps, rather than just isometries. Although we will be more interested in polyhedral complexes (in fact cubical complexes) for the remainder of the paper, here we introduce Morse functions in the affine case and develop some related results in greater generality.

Throughout, we will refer to the restriction of an affine homeomorphism as a *partial affine homeomorphism.*

Definition 2.1. A CW complex X is an **affine cell-complex** if it is given the following structure. Let $m \in \mathbb{Z}$ be fixed. Then for each cell $e \subset X$, we are given

- a convex polyhedral cell $C_e \subset \mathbb{R}^m$, and
- a characteristic map $\chi_e : C_e \to X$ such that the restriction of χ_e to any face of C_e is the characteristic map for another cell, up to precomposition by a partial affine homeomorphism of $\mathbb{R}^m$.

An **admissible characteristic map** for e is any function obtained from χ_e by precomposition by a partial affine homeomorphism of $\mathbb{R}^m$.

Remark. By the above, the restriction of an admissible characteristic map to a proper face is the admissible characteristic map of another cell.

Notation. We will denote the k-skeleton of a cell complex X as $X_{(k)}$.

2.1 Deformation retractions of affine cell-complexes

In the following, it will be useful to define deformation retractions of an affine cell complex in terms of deformation retractions on its cells. We establish below sufficient conditions on such a collection of cell-wise deformation retractions to extend to a deformation retraction of the complex.

Lemma 2.2. *Let X an affine cell complex, and suppose $A \subset B \subset X$ such that for every cell $e \subset X$ and choice of admissible characteristic map $\chi_e : C_e \to X$, there exists a deformation retraction $H_t^{\chi_e}$ of $\chi_e^{-1}(B)$ to $\chi_e^{-1}(A)$ fulfilling the following naturality conditions:*

(i) for any admissible characteristic map $\chi'_e = \chi_e h$, where $h : C'_e \to C_e$ is a partial affine homeomorphism, then

$$H_t^{\chi'_e} = H_t^{\chi_e h} = h^{-1} H_t^{\chi_e} h \tag{1}$$

(ii) for any face $F \subset C_e$, the restriction $H_t^{\chi_e}|_F$ is the deformation retraction associated to that face, i.e.

$$H_t^{\chi_e}|_F = H_t^{\chi_e|_F} \tag{2}$$

Then the $H_t^{\chi_e}$ induce a deformation retraction of B to A.

Proof. We induce on k-skeleta $X_{(k)}$ of X. We note that for any $j \le k \in \mathbb{Z}$, X_j embeds into $X_{(k)}$, which in turn embeds in X. Let $\hat{\pi}_k : X_{(k-1)} \sqcup (\bigsqcup_e C_e) \to X_{(k)}$ denote the quotient map associated with $X_{(k)}$, regarded as a quotient space, with k-cells $e \subset X_{(k)} \subset X$. For a k-cell $d \subset X_{(k)}$, let χ_d denote (by abuse of notation) the range restriction of the characteristic map to $X_{(k)}$, *i.e.*, $\chi_d : C_d \hookrightarrow X_{(k-1)} \sqcup (\bigsqcup_e C_e) \xrightarrow{\hat{\pi}_k} X_{(k)}$, or equivalently, $\chi_d := \hat{\pi}_k|_{C_d}$. Let $\tilde{X}_{(k)}$ denote the intersection $X_{(k)} \cap B$, and for any k-cell e, let $\tilde{C}_e = \chi_e^{-1}(B) \subset C_e$. In particular, let $\pi_k : \tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e) \to \tilde{X}_{(k)}$ denote the restriction of the quotient map $\hat{\pi}_k$, and observe that since $\hat{\pi}_k^{-1}(\tilde{X}_{(k)}) = \tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)$, π_k is also a quotient map; let $\overline{\pi}_k := (\pi_k, \mathrm{id}_I) : (\tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)) \times I \to \tilde{X}_{(k)} \times I$ denote the corresponding quotient map induced on the product space. Fix some $k \in \mathbb{Z}$. For any j-cell d with $j \le k$, define the restriction of the characteristic map $\tilde{\chi}_d : \tilde{C}_d \hookrightarrow \tilde{X}_{(j-1)} \sqcup (\bigsqcup_e \tilde{C}_e) \xrightarrow{\pi_j} \tilde{X}_{(j)} \hookrightarrow \tilde{X}_{(k)}$, and let $\overline{\chi}_d : \tilde{C}_d \times I \hookrightarrow (\tilde{X}_{(j-1)} \sqcup (\bigsqcup_e \tilde{C}_e)) \times I \xrightarrow{\overline{\pi}_j} \tilde{X}_{(j)} \times I \hookrightarrow \tilde{X}_{(k)} \times I$ denote the corresponding map on the product spaces; finally, note that $\tilde{X}_{(k-1)} \hookrightarrow \tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e) \xrightarrow{\pi_k} \tilde{X}_{(k)}$ and the corresponding map on the product spaces are embeddings. Then there exists a unique continuous map $H^{(k)} : \tilde{X}_{(k)} \times I \to \tilde{X}_{(k)}$ such that (i) $H^{(k)}|_{\tilde{X}_{(k-1)} \times I} = H^{(k-1)}$ and (ii) for any cell e such that $\dim e \le k$ and any choice of admissible characteristic map χ_e, $H^{(k)}\overline{\chi}_e = \tilde{\chi}_e H^{\chi_e}$, or equivalently, that the following set of diagrams commute:

$$\begin{tikzcd} \tilde{X}_{(k-1)} \times I \arrow[r, "H^{(k-1)}"] \arrow[d, hook] & \tilde{X}_{(k-1)} \arrow[d, hook] \\ \tilde{X}_{(k)} \times I \arrow[r, "H^{(k)}"'] & \tilde{X}_{(k)} \end{tikzcd} \quad \text{and} \quad \forall e : \dim e \le k, \quad \begin{tikzcd} \tilde{C}_e \times I \arrow[r, "H^{\chi_e}"] \arrow[d, "\overline{\chi}_e"'] & \tilde{C}_e \arrow[d, "\tilde{\chi}_e"] \\ \tilde{X}_{(k)} \times I \arrow[r, "H^{(k)}"'] & \tilde{X}_{(k)} \end{tikzcd} . \tag{3}$$

(Note that if $\tilde{X}_{(k-1)} = \varnothing$ or $\tilde{C}_e = \varnothing$, then the respective condition is met trivially.)

For $X_{(0)}$, observe that $X_{(0)} \cap (B \setminus A) = \varnothing$. Else, suppose $q \in X_{(0)} \cap (B \setminus A)$ and let $e = \{q\} \subset X_{(0)}$ and $\chi_e : p \mapsto q$ its characteristic map. Then $\{p\} = \chi_e^{-1}(B)$ is homotopy equivalent to $\varnothing = \chi_e^{-1}(A)$, a contradiction. Hence, for any $e = \{w\} \subset \tilde{X}_{(0)}$, $w \in A$, and in particular, if $\chi_e : v \mapsto w$ then $H_t^{\chi_e}$ must be the constant homotopy $(\mathrm{id}_{\{v\}})_t$. Hence the constant homotopy $H^{(0)} := (\mathrm{id}_{\tilde{X}_{(0)}})_t$ suffices.

Assume that there exists such a map $H^{(k-1)} : \tilde{X}_{(k-1)} \times I \to \tilde{X}_{(k-1)}$. We note that for a collection of spaces $\{U_\alpha\}$, $\bigsqcup_\alpha (U_\alpha \times I) \cong (\bigsqcup_\alpha U_\alpha) \times I$. Hence given a choice of admissible charateristic maps χ_e for k-cells e, there exists a unique continuous function $\tilde{H} : (\tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)) \times I \to \tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)$ such that $\tilde{H}|_{\tilde{X}_{(k-1)} \times I} = H^{(k-1)}$ and for every k-cell d, $\tilde{H}|_{\tilde{C}_d \times I} = H^{\chi_d}$, or equivalently, that the following set of diagrams commute:

$$\forall d : \dim d = k, \quad \begin{tikzcd} \tilde{C}_d \times I \arrow[r, "H^{\chi_d}"] \arrow[d, hook] & \tilde{C}_d \arrow[d, hook] \\ (\tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)) \times I \arrow[r, "\tilde{H}"'] & \tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e) \end{tikzcd}$$

$$\text{and} \quad \begin{tikzcd} \tilde{X}_{(k-1)} \times I \arrow[r, "H^{(k-1)}"] \arrow[d, hook] & \tilde{X}_{(k-1)} \arrow[d, hook] \\ (\tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)) \times I \arrow[r, "\tilde{H}"'] & \tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e) \end{tikzcd} . \tag{4}$$

Let $\pi, \overline{\pi}$ denote $\pi_k, \overline{\pi}_k$ respectively. We claim that $\pi\tilde{H}$ is constant on the fibers of $\overline{\pi}$ over $X_{(k)} \times I$, or equivalently, that for fixed $t \in I$, $\pi\tilde{H}_t$ is constant on the fibers of π over $\tilde{X}_{(k)}$. Let $x \in \tilde{X}_{(k)}$. Then let d be the unique cell such that $x \in \mathring{d}$. If d is a k-cell, then $\pi^{-1}(x) = \{w\}$ for some $w \in X_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)$, hence $\pi\tilde{H}_t$ is constant over $\pi^{-1}(x)$. Else, $x \in d \subset X_{(k-1)}$ and $\pi^{-1}(x) \cap \tilde{X}_{(k-1)} = x \in \tilde{X}_{(k-1)} \subset \tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)$. Hence, $\tilde{H}_t(x) = \tilde{H}_t|_{X_{(k-1)}}(x) = \iota H_t^{(k-1)}$ where ι is the inclusion $\tilde{X}_{(k-1)} \hookrightarrow \tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)$. But then $\pi\tilde{H}_t(x) = \pi\iota H_t^{(k-1)}(x) = H_t^{(k-1)}(x) \in \tilde{X}_{(k-1)} \subset \tilde{X}_{(k)}$, since $\tilde{X}_{(k-1)} \overset{\iota}{\hookrightarrow} \tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e) \overset{\pi}{\hookrightarrow} \tilde{X}_{(k)}$ is an embedding. Let $y = H_t^{(k-1)}(x) \in \tilde{X}_{(k)}$ and let $z \in \pi^{-1}(x) \cap C_e$ for some k-cell e, hence $z \in F \subsetneq C_e$ for some proper face F. Then

$$\pi\tilde{H}_t(z) = (\pi\tilde{H}_t|_{C_e})|_F(z) = \chi_e H_t^{\chi_e}|_F = \chi_e|_F H_t^{\chi_e|_F}(z) = \chi_{e'} H_t^{\chi_{e'}}(z)$$

where $e' \subset X_{(k-1)}$ is the cell associated with F, and we make use of (4) and property (ii) of H^{χ_e}. Finally, from (3), we have that by restricting the range of $\chi_{e'}$ to $\tilde{X}_{(k-1)}$,

$$\tilde{\chi}_{e'} H_t^{\chi_{e'}} = H_t^{(k-1)} \tilde{\chi}_{e'}(z) = H^{(k-1)}(x) = y \in \tilde{X}_{(k-1)} \subset \tilde{X}_{(k)}.$$

Hence $\pi\tilde{H}_t(z) = y$ as desired.

Since $\pi\tilde{H}$ is constant on the fibers of $\overline{\pi}$, there exists a unique continuous map $H^{(k)} : \tilde{X}_{(k)} \times I \to \tilde{X}_{(k)}$ such that the following diagram commutes:

$$\begin{tikzcd} (\tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)) \times I \arrow[d, "\overline{\pi}"'] \arrow[dr, "\pi\tilde{H}"] & \\ \tilde{X}_{(k)} \times I \arrow[r, "H^{(k)}"'] & \tilde{X}_{(k)} \end{tikzcd} \tag{5}$$

Composing the diagrams in (4) and (5), we have that $H^{(k)}\overline{\chi}_e = \tilde{\chi}_e H^{\chi_e}$ for all k-cells e, and likewise $H^{(k)}|_{X_{(k-1)} \times I} = H^{(k-1)}$. In fact, given a choice of characteristic maps χ_e, $H^{(k)}$ is the unique map with this property. Suppose

$G : \tilde{X}_{(k)} \times I \to \tilde{X}_{(k)}$ such that the following commute:

$$\begin{tikzcd} \tilde{X}_{(k-1)} \times I \arrow[r, "H^{(k-1)}"] \arrow[d, hook] & \tilde{X}_{(k-1)} \arrow[d, hook] \\ \tilde{X}_{(k)} \times I \arrow[r, "G"'] & \tilde{X}_{(k)} \end{tikzcd} \quad \text{and} \quad \forall e : \dim e = k, \quad \begin{tikzcd} \tilde{C}_e \times I \arrow[r, "H^{\chi_e}"] \arrow[d, "\overline{\chi}_e"'] & \tilde{C}_e \arrow[d, "\tilde{\chi}_e"] \\ \tilde{X}_{(k)} \times I \arrow[r, "G"'] & \tilde{X}_{(k)} \end{tikzcd} . \tag{6}$$

Then we can decompose (6) as

$$\begin{tikzcd} \tilde{C}_e \times I \arrow[r, "H^{\chi_e}"] \arrow[d, hook] & \tilde{C}_e \arrow[d, hook] \\ (\tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e)) \times I \arrow[r, "\tilde{H}"] \arrow[d, "\overline{\pi}"'] \arrow[dr, shift left, "\pi\tilde{H}"] \arrow[dr, shift right, "G\overline{\pi}"'] & \tilde{X}_{(k-1)} \sqcup (\bigsqcup_e \tilde{C}_e) \arrow[d, "\pi"] \\ \tilde{X}_{(k)} \times I \arrow[r, "G"'] & \tilde{X}_{(k)} \end{tikzcd} \tag{7}$$

for all k-cells e, and similarly for the $\tilde{X}_{(k-1)}$ diagram. Hence it remains only to check that $G\overline{\pi} = \pi\tilde{H}$. But from (4) and (6) we have

$$G\overline{\chi}_e = \tilde{\chi}_e H^{\chi_e}$$
$$G\overline{\pi}\overline{\iota} = \pi\iota H^{\chi_e} = \pi\tilde{H}\overline{\iota}$$

where $\iota, \overline{\iota}$ are the appropriate inclusions. Hence $G\overline{\pi}|_{\tilde{C}_e \times I} = \pi\tilde{H}|_{\tilde{C}_e \times I}$ for all k-cells e and likewise over $\tilde{X}_{(k-1)} \times I$. $G\overline{\pi} = \pi\tilde{H}$ and $G = H^{(k)}$ by the uniqueness of $H^{(k)}$ in (5).

Finally, we show that $H^{(k)}$ is independent of our choice of characteristic maps. Suppose we are given another collection of admissible maps for the cells $e \subset X_{(k)}$, such that $\tilde{\chi}'_e = \tilde{\chi}_e h_e$ with h_e a partial affine homeomorphism, and let $H^{(k)'} : \tilde{X}_{(k)} \times I \to \tilde{X}_{(k)}$ be the induced homotopy. Then, fixing some $t \in I$, for every k-cell e

$$\tilde{\chi}'_e H_t^{\chi'_e} = H_t^{(k)'} \tilde{\chi}'_e$$
$$\tilde{\chi}_e h h^{-1} H_t^{\chi_e} h = H_t^{(k)'} \tilde{\chi}_e h$$
$$\tilde{\chi}_e H_t^{\chi_e} = H_t^{(k)'} \tilde{\chi}_e$$

where $h := h_e$, and $H^{(k)'}|_{\tilde{X}_{(k-1)}} = H^{(k-1)'} = H^{(k-1)}$ by assumption. Then by uniqueness, $H^{(k)'} = H^{(k)}$.

We note that $H^{(k)}$ is uniquely determined by $H^{(k-1)}$ and the deformation retractions H^{χ_e} (up to conjugation by partial affine homeomorphisms) associated with k-cells e. Nonetheless, the following property holds for any cell $d \subset X_{(k-1)}$. By assumption, for any admissible characteristic map $\chi_d : C_d \to X_{(k-1)}$, $H^{(k-1)}\overline{\chi}_d = \tilde{\chi}_d H^{\chi_d}$, which we can compose with (3) to write the following commuting diagram:

$$\begin{tikzcd}
\tilde{C}_d \times I \arrow[r, "H^{\chi_d}"] \arrow[d, "\overline{\chi}_d"'] & \tilde{C}_d \arrow[d, "\tilde{\chi}_d"] \\
\tilde{X}_{(k-1)} \times I \arrow[r, "H^{(k-1)}"] \arrow[d, hook] & \tilde{X}_{(k-1)} \arrow[d, hook] \\
\tilde{X}_{(k)} \times I \arrow[r, "H^{(k)}"'] & \tilde{X}_{(k)}
\end{tikzcd} \tag{8}$$

Thus (denoting $\tilde{C}_d \xrightarrow{\chi_d} \tilde{X}_{(k-1)} \hookrightarrow \tilde{X}_k$ also by $\tilde{\chi}_d$, and likewise for $\overline{\chi}_d$) we have $H^{(k)}\overline{\chi}_d = \tilde{\chi}_d H^{\chi_d}$, completing our characterization of $H^{(k)}$.

X is finite dimensional; suppose $m = \dim X$. Then let $H := H^{(m)} : B \times I \to B$. We have that for every cell $e \subset X$ and any admissible characteristic map $\chi_e : C_e \to X$, the following diagram commutes:

$$\begin{tikzcd}
\tilde{C}_e \times I \arrow[r, "H^{\chi_e}"] \arrow[d, "\overline{\chi}_e"'] & \tilde{C}_e \arrow[d, "\tilde{\chi}_e"] \\
B \times I \arrow[r, "H"'] & B
\end{tikzcd} \tag{9}$$

Thus since the H^{χ_e} are deformation retractions of $\chi_e^{-1}(B)$ to $\chi_e^{-1}(A)$ and, as a set, $X = \bigsqcup_e \mathring{e}$ with χ_e bijective over $\mathring{e}$, H is a deformation retraction of B to A as desired. $\square$

2.2 Deformation retractions of sublevel sets

We now introduce Morse functions in the context of affine cell-complexes.

Definition 2.3. Given an affine cell-complex X, a map $f : X \to \mathbb{R}$ is a **Morse function** if

- for every cell $e \subset X$, $f\chi_e : C_e \to \mathbb{R}$ extends to an affine map $\mathbb{R}^m \to \mathbb{R}$ and $f\chi_e$ is constant only when e is a vertex, and
- the image of the 0-skeleton is discrete and closed in $\mathbb{R}$.

Principally, we will concern ourselves with the f-preimages of closed, connected intervals in $\mathbb{R}$. We use the following notation and terminology:

Notation. For a non-empty, closed set $J \subset \mathbb{R}$, denote $X_J := f^{-1}(J)$ and let $X_t = X_{\{t\}}$. Sets X_t are *level sets* with respect to f; sets $X_{(-\infty,t]}$, resp. $X_{[t,\infty)}$ are *sublevel sets*, resp. *superlevel sets* corresponding to X_t.

We are interested in the deformation retractions of sets $X_J \hookrightarrow X_{J'}$ for closed, connected subsets $J \subset J' \subset \mathbb{R}$. Using Lemma 2.2 above, we can construct deformation retractions for J, J' that are *close* with respect to f; in particular, for which $J' \setminus J$ contains no more than one point of $f(X_{(0)})$. First we establish some useful lemmas.

Lemma 2.4. *For any convex polygonal cell C with a proper top-dimensional face K, then any deformation retraction of $\overline{\partial C \setminus K}$ to $L \subset \overline{\partial C \setminus K}$ extends to a deformation retraction of C to L.*

Proof. Suppose K is a vertex. Then C is a closed line segment and $\overline{\partial C \setminus K}$ is the disjoint union of two points, which cannot deformation retract to either. Hence K is not a vertex. Let F_t be a deformation retraction of $\overline{\partial C \setminus K}$ to L. Choose any point p in the interior of K. Then by the convexity of C, as a set, $C = \{p\} \sqcup (\bigsqcup_x (L_x \setminus \{p\})$ where L_x is the line segment between $x \in \overline{\partial C \setminus K}$ and p. Let $h_{x,t}$ be the partial affine homeomorphism $L_x \to L_{F_t(x)}$ such that $h_{x,t}(p) = p$ and let $s_{x,t}$ be the t-linear interpolation of L_x and $\{x\}$. Then define $H_t : C \to C$ to be the map such that $H_t|_{L_x \setminus \{p\}} = h_{x,t} s_{x,t}|_{L_x \setminus \{p\}}$ for all $x \in \overline{\partial C \setminus K}$, and $H_t(p) = h_{q,t} s_{q,t}(p)$ for a choice of $q \in \overline{\partial C \setminus K}$. Then H_t is the desired deformation retraction. □

Corollary 2.5. *For any convex polygonal cell C with proper disjoint faces Σ_a, Σ_b where Σ_a is top-dimensional or a vertex and Σ_b is top-dimensional, then any deformation retraction of $\overline{\partial C \setminus \Sigma_b}$ to Σ_a extends to a deformation retraction of C to Σ_a.*

Lemma 2.6. *Let e be an affine cell and $\chi_e : C_e \to X$ a characteristic map, and let $A \subset B \subset X$. If $H_t^{\chi_e}$ is a deformation retraction of $\chi_e^{-1}(B)$ to $\chi_e^{-1}(A)$, then for any admissible characteristic map $\chi'_e = \chi_e h$ for $h : C'_e \to C_e$ a partial affine homeomorphism, $H_t^{\chi'_e} := h^{-1} H_t^{\chi_e} h$ is a deformation retraction of $\chi_e'^{-1}(B)$ to $\chi_e'^{-1}(A)$.*

Proof. We note that for any $U \subset X$, $h^{-1}\chi_e^{-1}(U) = (\chi_e h)^{-1}(U) = (\chi'_e)^{-1}(U)$. Hence,

$$H_0^{\chi'_e} = h^{-1} \operatorname{id}_{\chi_e^{-1}(B)} h = \operatorname{id}_{h^{-1}\chi_e^{-1}(B)} = \operatorname{id}_{\chi_e'^{-1}(B)}$$

as desired; similarly

$$H_t^{\chi'_e}|_{\chi_e'^{-1}(A)} = \operatorname{id}_{\chi_e'^{-1}(A)} .$$

Finally,

$$H_1^{\chi'_e}(\chi_e'^{-1}(B)) = h^{-1} H_1^{\chi_e} h h^{-1} \chi_e(B) = h^{-1}\chi_e^{-1}(A) = \chi_e'^{-1}(A),$$

hence $H_t^{\chi'_e}$ is the desired deformation retraction. □

First we consider the case for which $J' \setminus J$ contains no points of $f(X_{(0)})$.

Lemma 2.7. *Given a Morse function $f : X \to \mathbb{R}$ on an affine cell complex X, if $J \subset J' \subset \mathbb{R}$ are connected, closed and non-empty, and $X_{J'} \setminus X_J$ contains no vertices of X, then $X_J \hookrightarrow X_{J'}$ is a homotopy equivalence.*

Proof. For each cell $e \subset X$ and choice of admissible characteristic map $\chi_e : C_e \to X$, we construct a deformation retraction $H_t^{\chi_e}$ of $(f\chi_e)^{-1}(J') = \chi_e^{-1}(X_{J'})$ to $(f\chi_e)^{-1}(J) = \chi_e^{-1}(X_J)$ fulfilling the naturality conditions in Lemma 2.2. Hence, the induced deformation retraction of $X_{J'}$ to X_J gives the desired homotopy equivalence.

We construct $H_t^{\chi_e}$ by induction on the dimension of e. Suppose $\dim e = 0$. Then $e = \{q\} \subset X$, hence let $C_e = \{p\} \subset \mathbb{R}^m$ and $\chi_e : p \mapsto q$. q is a vertex, hence either $f(q) \in J$ or $f(q) \notin J'$: we have that $(f\chi_e)^{-1}(J') = (f\chi_e)^{-1}(J) = \{p\}$ or $\varnothing$ respectively. Hence the constant map $H_t^{\chi_e} = \mathrm{id}_{\{p\}}$ suffices. In particular, for any admissible characteristic function $\chi'_e = \chi_e h : p' \mapsto q$ for $h : p' \mapsto p$, define

$$H_t^{\chi'_e} := h^{-1} H_t^{\chi_e} h = \mathrm{id}_{\{p'\}},$$

fulfilling (i). C_e has no proper faces, hence (ii) is met trivially.

Construct $H_t^{\chi_e}$ for all e and admissible χ_e for $\dim e \leq k$. Suppose $\dim e = k+1$. $J, J' \subset \mathbb{R}$ are closed intervals, hence $J' \setminus J = U_+ \cup U_-$, where $U_\pm$ are disjoint and each either a half-open interval or empty. Suppose $U_+ = (a, b]$, with $a < b \in \mathbb{R} \cup \{\pm\infty\}$, hence $U_+ = (a, \infty) \cap (-\infty, b]$. Let $\Sigma_a \subset C_e$ denote the fiber of a when $a \neq -\infty$, and $\Sigma_b \subset C_e$ denote the fiber of b when $b \neq \infty$. $f\chi_e : C_e \to \mathbb{R}$ is the restriction of an affine map $\Phi : \mathbb{R}^m \to \mathbb{R}$; let $\varphi := d(\Phi) : \mathbb{R}^m \to \mathbb{R}$ denote the linear map induced on tangent spaces. The tangent space $\mathbb{R}^m$ acts transitively on C_e, hence if $\varphi = 0$ then $f\chi_e$ is constant. Hence, $\dim \ker \varphi = m - 1$ and $\Sigma_{a,b} = C_e \cap \mathcal{H}_{a,b}$ respectively (when defined), where $\mathcal{H}_{a,b}$ are disjoint $(m-1)$-hyperplanes. Let $H_a := \Phi^{-1}([a, \infty))$ and $H_b := \Phi^{-1}((-\infty, b])$ denote, for $a, b \neq \pm\infty$, the (closed) half-spaces bounded by $\mathcal{H}_{a,b}$ respectively, else $\mathbb{R}^m$. Then $C_e \cap H_a = (f\chi_e)^{-1}([a, \infty))$ and $C_e \cap H_b = (f\chi_e)^{-1}((-\infty, b])$ and

$$(f\chi_e)^{-1}(\overline{U}_+) = (f\chi_e)^{-1}([a, b]) = (f\chi_e)^{-1}([a, \infty)) \cap (f\chi_e)^{-1}((-\infty, b]) = C_e \cap H_a \cap H_b.$$

Let $C_e^+ := (f\chi_e)^{-1}(\overline{U}_+)$. Since $X_{J'} \setminus X_J \supset \chi_e(C_e^+ \setminus \Sigma_a)$ contains no vertices and $C_e \cap H_{a,b}$ contains at least one vertex or is empty, then if $C_e \subset H_{a,b}$, we have that $C_e^+ = C_e \cap H_a \cap H_b = \Sigma_a$ or $\varnothing$. Hence if $C_e^+ \not\subset \Sigma_a$, then $\Sigma_{a,b} \neq \varnothing$. Moreover, $\Sigma_{a,b}$ cannot coincide with any (non-vertex) face of C_e, else the restriction of $f\chi_e$ to that face would be constant; finally, since $X_{J'} \setminus X_J \supset \chi_e(\Sigma_b)$ contains no vertices, neither does Σ_b. Thus if $C_e^+ \not\subset \Sigma_a$ then it is a convex polyhedron with boundary composed of subsets of proper faces of C_e and the disjoint proper faces $\Sigma_{a,b}$ where Σ_b is top-dimensional and Σ_a is top-dimensional or a vertex.

Suppose that $C_e^+ \not\subset \Sigma_a$. For each proper face $F \subsetneq C_e \subset \mathbb{R}^m$ corresponding to a proper face $e' \subsetneq e \subset X$, $\chi_e|_F$ is an admissible characteristic map for e'. Thus by assumption, we have a strong deformation retraction of $(f\chi_e|_F)^{-1}(J') = F \cap (f\chi_e)^{-1}(J')$ to $(f\chi_e|_F)^{-1}(J) = F \cap (f\chi_e)^{-1}(J)$. Noting that by (ii) these homotopies agree on the (closed) intersections of proper faces, they patch together to a deformation retraction of $\partial C_e \cap (f\chi_e)^{-1}(J')$ to $\partial C_e \cap (f\chi_e)^{-1}(J)$. In particular, this homotopy restricts over ∂C_e^+ to a deformation retraction of $\partial C_e^+ \setminus (\Sigma_a \cup \Sigma_b)$ to $\partial \Sigma_a$. Extending by the constant homotopy $(\mathrm{id}_{\Sigma_a})_t := \mathrm{id}_{\Sigma_a}$ on Σ_a, we have a strong deformation retraction of $\overline{\partial C_e^+ \setminus \Sigma_b}$ to Σ_a; by Corollary 2.5, this in turn extends to a deformation retraction of C_e^+ to Σ_a, which we shall denote as Σ^+.

If $U_- = [a, b)$ with $a < b \in \mathbb{R} \cup \{\pm\infty\}$, then an identical argument applies to $C_e^- := (f\chi_e)^{-1}(\overline{U}_-)$ by exchanging the roles of a, b. (For $U_\pm = \varnothing$, note that $C_e^\pm = \varnothing$ respectively.) For simplicity, let $(C_e)_K := C_e \cap (f\chi_e)^{-1}(K)$ for $K \subset \mathbb{R}$. We can extend the strong deformation retractions of $C_e^\pm$ to $\Sigma^\pm$ (where defined, *i.e.* for $C_e^{\pm} \not\subset \Sigma^{\pm}$) to a deformation retraction $H_t^{\chi_e}$ of $(C_e)_{J'} = (C_e)_J \cup C_e^+ \cup C_e^-$ to $(C_e)_J$ by patching with the constant homotopy $(\mathrm{id}_{(C_e)_J})_t$ over $(C_e)_J$.

For any choice of admissible characteristic map $\chi'_e = \chi_e h$, where $h : C'_e \to C_e$ is a partial affine homeomorphism, define

$$H_t^{\chi'_e} := h^{-1} H_t^{\chi_e} h, \tag{10}$$

fulfilling (i). It remains only to check that $H_t^{\chi'_e}$ is the desired deformation retraction and that (ii) holds. We observe that $(C_e)_{J'} = \chi_e^{-1}(X_J)$ and $(C'_e)_{J'} = \chi_e'^{-1}(X_{J'})$, and similarly for $(C_e)_J, (C'_e)_J$, hence by Lemma 2.6, $H_t^{\chi'_e}$ is a deformation retraction of $(C'_e)_{J'}$ to $(C'_e)_J$. Finally, for any face $F' \subset C'_e$, let $\tilde{h} : F' \to F$ denote the restriction of h to F'. Then $F = \tilde{h}F'$ is a face of C_e and $H_t^{\chi'_e}|_{F'} = \tilde{h}^{-1}(H_t^{\chi_e}|_F)\tilde{h}$. $H_t^{\chi_e}$ extends the deformation retractions of its proper faces, hence by construction $H_t^{\chi_e}|_F = H_t^{\chi_e|_F}$ and

$$H_t^{\chi'_e}|_{F'} = \tilde{h}^{-1} H_t^{\chi_e|_F} \tilde{h} = H_t^{\chi'_e|_{F'}} \tag{11}$$

by (i) and the fact that $(\chi_e|_F)\tilde{h} = \chi'_e|_{F'}$, demonstrating (ii). $\square$

Unsurprisingly, the presence of vertices in $X_{J'} \setminus X_J$ presents a complication in the construction of our deformation retraction: in such cases in general, $X_{J'} \setminus X_J$ is not a homotopy equivalence. To resolve this case, we must develop a picture of the local structure about a vertex.

2.2.1 Ascending and descending links

Definition 2.8. The **link** $\mathrm{Lk}(p, C)$ of a point p in a convex polyhedron C is the *space of directions* at p in C, defined as the set of initial vectors of geodesic segments joining p to the points of C.

In an affine cell-complex, there in general does not exist a tangent space in which to define "inital vectors"; however, for cells e with characteristic maps χ_e the χ_e-images of geodesic segments in C_e induce a natural notion of the space of directions. In particular, let $p \in X$ an affine cell-complex, and let $e \subset X$ be a cell containing x with characteristic map χ_e. Let the χ_e-images of geodesic segments in C_e be called *segments in* e. Then define the *direction* or *germ* of a segment issuing from p to be the equivalence class of all segments in e who pairwise contain a common initial segment at p; define $\mathrm{Lk}(p, e)$ to be the space of directions at p in e.

Hence we may define

Definition 2.9. For any $p \in X$ an affine cell complex, let the **link** of p in X be defined as

$$\mathrm{Lk}(p, X) := \bigcup \{\mathrm{Lk}(p, e) : e \text{ a cell containing } p\}.$$

We note that for any cell e an admissible characteristic map χ_e induces a well defined map

$$\chi_{e*} : \mathrm{Lk}(x, C_e) \to \mathrm{Lk}(\chi_e(x), e)$$

for all vertices $x \in C_e$.

Remark. For vertices $v \in X$, we can view $\mathrm{Lk}(v, X)$ as a quotient of the disjoint union of links $\mathrm{Lk}(w, C_e)$ for cells $e \ni v$ with characteristic map χ_e and $w \in \chi_e^{-1}(v)$.

Moreover, we define the following:

Definition 2.10. Let X an affine cell-complex and $f : X \to \mathbb{R}$ a Morse function. Then for any vertex $v \in X$, the **ascending link** ($\uparrow$-link) of v is defined as

$$\mathrm{Lk}_\uparrow(v, X) := \bigcup\{\chi_{e*}(\mathrm{Lk}(w, C_e)) : \chi_e(w) = v \text{ and } f\chi_e \text{ has a minimum at } w\} \subset \mathrm{Lk}(v, X)$$

and the the **descending link** ($\downarrow$-link) of v is defined as

$$\mathrm{Lk}_\downarrow(v, X) := \bigcup\{\chi_{e*}(\mathrm{Lk}(w, C_e)) : \chi_e(w) = v \text{ and } f\chi_e \text{ has a maximum at } w\} \subset \mathrm{Lk}(v, X).$$

Lemma 2.11. *Suppose $f : X \to \mathbb{R}$ is a Morse function on an affine cell complex X, and $J \subset J' \subset \mathbb{R}$ are connected, closed and non-empty such that* $\inf J = \inf J'$ *and $\overline{J' \setminus J}$ contains exactly one point r of $f(X_{(0)})$ such that $r \notin J$. Then there exists a subspace $\hat{X} \subset X_{J'}$ such that $X_{J'}$ is homotopy equivalent to $X_J \cup \hat{X}$ and for each component $\hat{X}_\alpha$ of $\hat{X}$, $\hat{X}_\alpha$ is contractible and $\hat{X}_\alpha \cap X_J$ is homeomorphic to* $\mathrm{Lk}_\downarrow(u, X)$ *for some vertex $u \in f^{-1}(r) \cap X_{(0)}$.*

Remark. We may think of $X_J \cup \hat{X}$ as the space formed from X_J by "coning off" the copies of $\mathrm{Lk}_\downarrow(v, X)$ for vertices $v \in f^{-1}(r) \cap X_{(0)}$, though we defer the precise definition of a cone until Section 6.

Proof. We note that $J' \cap (-\infty, r] \subset J' \subset \mathbb{R}$ meet the conditions of Lemma 2.7, hence $X_{J' \cap (-\infty, r]} \hookrightarrow X_{J'}$ is a homotopy equivalence. Without loss of generality, assume $\sup J' = r$ and let $r - \epsilon = \sup J$. Since $\inf J = \inf J'$, any deformation retraction of $X_{[r-\epsilon, r]}$ to $Y \supset X_{r-\epsilon}$ extends to a deformation retraction of $X_{J'}$ to $X_J \cup Y$. Let $K = (r - \epsilon, r]$ and let

$$\hat{X} := X_{\overline{K}} \cap \bigcup\{e : \max f|_e = r\}.$$

For any cells e, d and characteristic map χ_d, we note that $\chi_d^{-1}(e)$ is a face of C_d, since $e \cap d \subset d$ is a cell and χ_d induces a bijective correspondence between the cells contained in d and the faces of C_d. Hence:

$$\chi_d^{-1}(X_{r-\epsilon} \cup \hat{X}) = (C_d)_{r-\epsilon} \cup \left((C_d)_{\overline{K}} \cap \bigcup\{F : F \text{ is a face of } C_d \text{ s.t. } f\chi_d(F) \subset (-\infty, r]\}\right). \tag{12}$$

For any face $F \subset C_d$ such that $f\chi_d(F) \subset (-\infty, r]$ and $(C_d)_K \cap F \neq \varnothing$, we show that F has exactly one vertex $v \in (C_d)_K$ such that $f\chi_d|_F$ is maximum at v, and in particular, such that $f\chi_d(v) = r$. Clearly for any vertex $w \in F$, $f\chi_d(w) \leq r$; moreover, for any such vertex in $(C_d)_K$, equality is achieved. Hence by the convexity of F, if there exist more than one vertex of F in $(C_d)_K$, then there exists a face $L \subset F \subset C_d$ such that $f\chi_d|_L$ is constant, a contradiction. Finally, suppose that F has no vertices in $(C_d)_K$. Then for every vertex $w \in F$, $f\chi_e(w) \leq r - \epsilon$, hence w lies in the closed half-space H of $\mathbb{R}^m$ such that

$H \cap C_d = (C_d)_{(-\infty, r-\epsilon]}$. Thus, since F is the convex hull of its vertices, $F \subset H$ and $(C_d)_K \cap F = \varnothing$, a contradiction. We can now express:

$$\chi_d^{-1}(X_{r-\epsilon} \cup \hat{X}) = (C_d)_{r-\epsilon} \cup \Big((C_d)_{\overline{K}} \cap \bigcup \{F : F \text{ is a face of } C_d \text{ s.t. } \max f\chi_d|_F = r\} \Big)$$

or equivalently,

$$\chi_d^{-1}(X_{r-\epsilon} \cup \hat{X}) = (C_d)_{r-\epsilon} \cup \bigcup \{(C_d)_{\overline{K}} \cap \mathrm{St}_\downarrow(v, C_d) : v \text{ a vertex of } C_d \text{ s.t. } f\chi_d(v) = r\},$$

where $\mathrm{St}_\downarrow(v, C_d) := \bigcup \{F : F \text{ is a face of } C_d \text{ s.t. } f\chi_d|F \text{ is maximum at } v\}$, defined for vertices v such that $f\chi_d(v) = r$.

Let $\hat{X}_u := X_{\overline{K}} \cap \bigcup \{e : f|_e \text{ is max. at } u \in e\}$ for $u \in f^{-1}(r) \cap X_{(0)}$ and note that $\hat{X} = \bigcup_u \hat{X}_u$. Define the affine subcomplex $\tilde{X}_u := \bigcup \{e \subset X : f|_e \text{ is max. at } u \in e\}$. Then $\hat{X}_v = (\tilde{X}_u)_{\overline{K}}$ and the proceeding arguments give $(\tilde{X}_u)_r = \{u\}$. Since $(\tilde{X}_u)_{\overline{K}} \setminus (\tilde{X}_u)_r$ contains no vertices, by Lemma 2.7, $\hat{X}_u$ deformation retracts to $\{u\}$, hence is contractible. Moreover, since distinct $\hat{X}_u, \hat{X}_{u'}$ deformation retract to discrete points $\{u, u'\}$, they are disconnected: $\{\hat{X}_u\}$ comprise the connected components of $\hat{X}$. Finally, we claim that $\hat{X}_u \cap X_J = \hat{X}_u \cap X_{r-\epsilon} \cong \mathrm{Lk}_\downarrow(u, X)$.

Suppose d is a cell such that $f|_d$ is maximum at u, and let χ_d be an admissible characteristic map. Then for the unique maximum vertex $v \in C_d$, we have $\chi_d(v) = u$. Hence $\mathrm{St}_\downarrow(v, C_d) = C_d$ and $\chi_d^{-1}(\hat{X}_u) = (C_d)_{\overline{K}}$. There is a natural homeomorphism via radial projection between $\mathrm{Lk}(v, C_d)$ and the union of faces $F' \subset C_d$ such that $v \notin F'$. Moreover, $(C_d)_{\overline{K}}$ is a convex polyhedron locally identical to $\mathrm{St}_\downarrow(v, C_d)$ at v, and $(C_d)_{\overline{K}}$ meets only faces of C_d containing v, hence $(C_d)_{r-\epsilon}$ is the unique face of $(C_d)_{\overline{K}}$ not containing v. Hence, again by radial projection, we have

$$\mathrm{Lk}(v, C_d) \cong (C_d)_{r-\epsilon} \tag{13}$$

for any such cell d. These homeomorphisms descend to a corresponding homeomorphism $\mathrm{Lk}_\downarrow(u, X) \cong X_{r-\epsilon} \cap \hat{X}_u$. In particular, for any point in $X_{r-\epsilon} \cap \hat{X}_u$, resp. initial segment in $\mathrm{Lk}_\downarrow(u, X)$, these lie in, resp. point into, the interior of a unique (minimal) cell e such that $f|_e$ is maximum at u. Hence, by the affine structure of X, we have a well defined bijection (hence homeomorphism) induced by (13) in the quotient spaces $\mathrm{Lk}_\downarrow(u, X)$, $X_{r-\epsilon} \cap \hat{X}_u$.

For each cell $e \subset X$ and choice of admissible characteristic map $\chi_e : C_e \to X$, we construct a deformation retraction $H_t^{\chi_e}$ of $\chi_e^{-1}(X_{\overline{K}})$ to $\chi_e^{-1}(X_{r-\epsilon} \cup \hat{X})$ fulfilling the naturality properties in Lemma 2.2. Hence, the $H_t^{\chi_e}$ induce the desired deformation retraction of $X_{\overline{K}}$ to $X_{r-\epsilon} \cup \hat{X}$, and hence a homotopy equivalence between $X_{J'}$ and $\hat{X} \cup X_J$.

Again, we construct $H_t^{\chi_e}$ by induction on the dimension of e. Suppose $\dim e = 0$. Then $e = \{q\} \subset X$, hence let $C_e = \{p\} \subset \mathbb{R}^m$ and $\chi_e : p \mapsto q$. q is a vertex, hence either $q \notin X_{\overline{K}}$, or $q \in X_{r-\epsilon}$ or $f(q) = r$, hence $q \in \hat{X}$. Then $\chi_e^{-1}(X_{\overline{K}}) = \chi_e^{-1}(X_{r-\epsilon} \cup \hat{X})$, hence the constant map $H_t^{\chi_e} = \mathrm{id}_{\{p\}}$ suffices, and for any admissible characteristic function $\chi'_e = \chi_e h : p' \mapsto q$ for $h : p' \mapsto p$, define

$$H_t^{\chi'_e} := h^{-1} H_t^{\chi_e} h = \mathrm{id}_{\{p'\}},$$

fulfilling (i). C_e has no proper faces, hence (ii) is met trivially.

Construct $H_t^{\chi_e}$ for all e and admissible χ_e for $\dim e \le k$. Suppose $\dim e = k+1 > 0$. Following the argument used for Lemma 2.7, let $H_{r-\epsilon}$ be the half-space of $\mathbb{R}^m$ such that $H_{r-\epsilon} \cap C_e = (C_e)_{[r-\epsilon,\infty)}$ and H_r be the half-space such that $H_r \cap C_e = (C_e)_{(-\infty,r]}$. Then $(C_e)_{\overline{K}} = C_e \cap H_r \cap H_{r-\epsilon}$. Let $\Sigma_r := (C_e)_r = \partial H_r \cap C_e$ and similarly for $\Sigma_{r-\epsilon}$. We note that $\Sigma_{r,r-\epsilon}$ are either top-dimensional convex polyhedra (*i.e.* $\dim \Sigma_{r,r-\epsilon} = k$) or vertices, respectively: $\Sigma_{r,r-\epsilon}$ cannot contain a non-vertex face of C_e, else $f\chi_e$ is constant on that face, a contradiction. We consider the following cases:

- Suppose $(C_e)_{\overline{K}} = \Sigma_r = \{v\}$, a vertex of C_e. If e' is the cell associated with $\{v\}$ (regarded as a proper face of C_e), then it suffices to let $H_t^{\chi_e} = H_t^{\chi_{e'}}$.

- Suppose $(C_e)_{\overline{K}} = \Sigma_{r-\epsilon}$ or $\Sigma_r = \{v\}$, a vertex of C_e, and $(C_e)_{\overline{K}} \neq \Sigma_r$. Then $(C_e)_{\overline{K}} \subset \chi_e^{-1}(X_{r-\epsilon} \cup \hat{X})$: in the latter case, note that if Σ_r contains a proper face of C_e and $\mathring{H}_r \cap C_e \neq \varnothing$, then $f\chi_e(v) = r$ is a maximum. Hence necessarily $H_t^{\chi_e} = \mathrm{id}_{(C_e)_{\overline{K}}}$.

Finally, suppose Σ_r, $\Sigma_{r-\epsilon}$ are disjoint and non-empty, and Σ_r is top-dimensional. Then $(C_e)_{\overline{K}}$ is a convex polyhedron with boundary composed of subsets of proper faces of C_e and the disjoint proper faces Σ_r and $\Sigma_{r-\epsilon}$, and $C_e \not\subset H_r$, hence $f\chi_e(C_e) \not\subset (-\infty, r]$. Then from (12), $\chi_e^{-1}(X_{r-\epsilon} \cup \hat{X}) \subset \overline{\partial(C_e)_{\overline{K}} \setminus \Sigma_r}$. For each proper face $F \subsetneq C_e \subset \mathbb{R}^m$ corresponding to a proper face $e' \subsetneq e \subset X$, $\chi_e|_F$ is an admissible characteristic map for e'. Thus by assumption, we have a strong deformation retraction of $(\chi_e|_F)^{-1}(X_{\overline{K}}) = F \cap \chi_e^{-1}(X_{\overline{K}})$ to $(\chi_e|_F)^{-1}(X_{r-\epsilon} \cup \hat{X}) = F \cap \chi_e^{-1}(X_{r-\epsilon} \cup \hat{X})$. Noting that by (ii) these homotopies agree on the (closed) intersections of proper faces, they patch together to a deformation retraction of $\partial C_e \cap \chi_e^{-1}(X_{\overline{K}}) = \overline{\partial(C_e)_{\overline{K}} \setminus (\Sigma_r \cup \Sigma_{r-\epsilon})}$ to $\partial C_e \cap \chi_e^{-1}(X_{r-\epsilon} \cup \hat{X}) \supset \partial\Sigma_{r-\epsilon}$. Extending by the constant homotopy $(\mathrm{id}_{\Sigma_{r-\epsilon}})_t := \mathrm{id}_{\Sigma_{r-\epsilon}}$ on $\Sigma_{r-\epsilon}$, we have a strong deformation retraction of $\overline{\partial(C_e)_{\overline{K}} \setminus \Sigma_r}$ to $\overline{\partial(C_e)_{\overline{K}} \setminus \Sigma_r} \cap \chi_e^{-1}(X_{r-\epsilon} \cup \hat{X}) = \chi_e^{-1}(X_{r-\epsilon} \cup \hat{X})$; by Lemma 2.4, this in turn extends to a deformation retraction of $(C_e)_{\overline{K}} = \chi_e^{-1}(X_{\overline{K}})$ to $\chi_e^{-1}(X_{r-\epsilon} \cup \hat{X})$, as desired. Define $H_t^{\chi_e}$ to be this deformation retraction.

For any choice of admissible characteristic map $\chi_e' = \chi_e h$, where $h : C_e' \to C_e$ is a partial affine homeomorphism, define

$$H_t^{\chi_e'} := h^{-1} H_t^{\chi_e} h, \tag{14}$$

fulfilling (i). We verify that $H_t^{\chi_e'}$ is the desired deformation retraction by Lemma 2.6; noting that (ii) holds for χ_e by construction, then from the argument presented in Lemma 2.7, (ii) holds for χ_e' as well. □

Remark. An analogous result holds for appropriate sets $J \subset J' \subset \mathbb{R}$ such that $\sup J = \sup J'$; in this case, the space $\hat{X}$ is defined identically except replacing $\mathrm{Lk}_\downarrow$ with $\mathrm{Lk}_\uparrow$.

2.2.2 Colimits and pushouts

Given some restrictions on the (homological, homotopic) connectedness of the $\uparrow$-links and $\downarrow$-links, Lemmas 2.7 and 2.11 in fact extend to arbitrary closed, connected, non-empty $J \subset J'$. However, before we continue to the next claim, we will need some elementary (homotopy, category theoretic) claims.

Lemma 2.12. *Suppose that A and B are spaces such that for each component $A_\alpha \subset A$, A_α and $A_\alpha \cap B$ are contractible. Then $A \cup B$ is homotopy equivalent to B.*

Proof. $A_\alpha \cup B/(A_\alpha \cap B) \simeq A_\alpha \cup B$, $A_\alpha/(A_\alpha \cap B) \simeq A_\alpha$ and $B/(A_\alpha \cap B) \simeq B$, hence $A_\alpha \cup B \simeq A_\alpha \cup B/(A_\alpha \cap B) \simeq A_\alpha \vee B \simeq B$. The direct product of these homotopy equivalences gives $A \cup B \simeq B$. □

Notation. Let $\langle A_\alpha, \psi_{\alpha\beta} \rangle$ denote a direct system of objects A_α and morphisms $\psi_{\alpha\beta}$ for all $\alpha \leq \beta$.

Definition 2.13. Given a direct system $\langle A_\alpha, \psi_{\alpha\beta} \rangle$, let a **target** $\langle A', \phi'_\alpha \rangle$ of $\langle A_\alpha, \psi_{\alpha\beta} \rangle$ be an object A' and a collection of morphisms $\phi_\alpha : A_\alpha \to A'$ such that the following diagram commutes for all $\alpha \leq \beta$:

$$\begin{tikzcd} A_\alpha \arrow[rr, "\psi_{\alpha\beta}"] \arrow[dr, "\phi'_\alpha"'] & & A_\beta \arrow[dl, "\phi'_\beta"] \\ & A' & \end{tikzcd}$$

Remark. If $A = \varinjlim A_\alpha$ and projection maps $\phi_\alpha : A_\alpha \to A$ comprise a direct limit of $\langle A_\alpha, \psi_{\alpha\beta} \rangle$, then $\langle A, \phi_\alpha \rangle$ is the target with the universal property that for any target $\langle A', \phi'_\alpha \rangle$, there exists a unique map $u : A \to A'$ such that $u\phi_\alpha = \phi'_\alpha$ for all α.

Lemma 2.14. *Let $\langle A_\alpha, \psi_{\alpha\beta} \rangle$ be a direct system in the category of groups with direct limit $\langle A, \phi_\alpha \rangle$. Then:*

- *(i) If every morphism $\psi_{\alpha\beta}$ is an epimorphism, then so are the projections ϕ_α.*
- *(ii) If every morphism $\psi_{\alpha\beta}$ is an isomorphism, then so are the projections ϕ_α.*
- *(iii) Suppose $\langle B_\alpha, \zeta_{\alpha\beta} \rangle$ is a direct system with $\varinjlim B_\alpha = B$ and with isomorphisms $\gamma_\alpha : A_\alpha \to B_\alpha$. Then $A \cong B$.*

Proof. Let $\langle B, \xi_\alpha \rangle$ denote the direct limit for $\langle B_\alpha, \zeta_{\alpha\beta} \rangle$ as in the Lemma. Then $\langle B, \xi_\alpha \gamma_\alpha \rangle$ is also a direct limit for $\langle A_\alpha, \psi_{\alpha\beta} \rangle$, hence $A \cong B$ and (iii) follows.

Suppose that every map $\psi_{\alpha\beta}$ is an epimorphism and suppose that $g, h : A \to A'$ are morphisms such that $g\phi_\alpha = h\phi_\alpha$ for some fixed α. Then we have the following diagram:

$$\begin{tikzcd}
A_\alpha \arrow[rr, "\psi_{\alpha\beta}"] \arrow[dr, "\phi_\alpha"'] & & A_\beta \arrow[dl, "\phi_\beta"] \\
& A \arrow[dl, "g"] \arrow[dr, "h"'] & \\
A' & & A'
\end{tikzcd}$$

Hence $g\phi_\beta\psi_{\alpha\beta} = h\phi_\beta\psi_{\alpha\beta}$; since $\psi_{\alpha\beta}$ is epic, this implies $g\phi_\beta = h\phi_\beta$. Define ϕ'_β to be $g\phi_\beta = h\phi_\beta$ for all $\beta \geq \alpha$ and the trivial map otherwise. Hence $\langle A, \phi'_\beta\rangle$ comprise a target of $\langle A_\alpha, \psi_{\alpha\beta}\rangle$; thus by the universal property of direct limits, there exists a unique morphism $u : A \to A'$ such that $\phi'_\alpha = u\phi_\alpha$ for all α. Hence $g = u = h$ and ϕ_α is an epimorphism, showing (i).

To prove (ii), suppose instead that every $\psi_{\alpha\beta}$ is an isomorphism, hence an epimorphism as well. Fix α. Then we have the target $\langle A_\alpha, \psi_{\alpha\gamma}\rangle$ (ranging over γ), where we define $\psi_{\alpha\gamma} := \psi_{\gamma\alpha}^{-1}$ for $\gamma < \alpha$. Hence there exists $u : A \to A_\alpha$ such that $u\phi_\alpha = \psi_{\alpha\alpha} = \mathrm{id}$. Hence ϕ_α is monic; by (i), it is also an epimorphism, hence an isomorphism. $\square$

Lemma 2.15. *Consider the following pushout P in the category of groupoids over a set A:*

$$P = \begin{tikzcd}
W \arrow[r] \arrow[d] & X \arrow[d, "\alpha"] \\
Y \arrow[r] & Z
\end{tikzcd}$$

where all morphisms act as the identity on A and W, Y are totally disconnected. Let $\mathcal{D}_A$ denote the discrete groupoid over A. Then:

(i) If $W = \mathcal{D}_A$, then α is a monomorphism.

(ii) If $Y = \mathcal{D}_A$, then α is an epimorphism.

Proof. Suppose $W = \mathcal{D}_A$. Let 0_A denote the groupoid homomorphism acting by identity on A and for each $a \in A$, mapping $Y(a) \to \{0\} \subset X(a)$. Then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}_A \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{id}"] \\
Y \arrow[r, "0_A"] & X
\end{tikzcd}$$

Hence the universal property of P gives the existence of a groupoid homomorphism $\gamma : Z \to X$ such that $\gamma\alpha = \mathrm{id}$. Hence for any morphisms ρ, σ such that $\alpha\rho = \alpha\sigma$, we have $\gamma\alpha\rho = \gamma\alpha\sigma$ hence $\rho = \sigma$, and (i) is shown.

For (ii), suppose instead that $Y = \mathcal{D}_A$. Then suppose that $\mu, \nu : Z \to G$ are morphisms such that $\mu\alpha = \nu\alpha$, and let $\eta : \mathcal{D}_A \to G$ be the morphism coinciding with $\mu\alpha$ on A. Then the following diagrams commute:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "\mu\alpha"] \\ \mathcal{D}_A \arrow[r, "\eta"'] & G \end{tikzcd} \quad ; \quad \begin{tikzcd} & & X \arrow[dl, "\alpha"'] \arrow[dd, "\mu\alpha"] \\ & Z \arrow[dr, shift left, "\mu"] \arrow[dr, shift right, "\nu"'] & \\ \mathcal{D}_A \arrow[ur, "0_A"] \arrow[rr, "\eta"'] & & G \end{tikzcd}$$

where we note that 0_A is the only possible morphism $\mathcal{D}_A \to Z$ and α and 0_A both act on A by identity. Hence, by the uniqueness of the pushout map, $\mu = \nu$. □

2.2.3 The general case

Finally, we are ready to show the following:

Theorem 2.16. *Let $f : X \to \mathbb{R}$ be a Morse function on an affine cell complex X. Suppose that $J \subset J' \subset \mathbb{R}$ are closed, nonempty, and connected and let a $\uparrow$-link or $\downarrow$-link be called **pertinent** if it belongs to $X_{J'} \setminus X_J$. Then:*

(i) If every pertinent $\uparrow$-link and $\downarrow$-link is homologically n-connected, then the inclusion $X_J \hookrightarrow X_{J'}$ induces an isomorphism in $\tilde{H}_i$ for $i \leq n$ and an epimorphism in $\tilde{H}_{i+1}$.

(ii) If every pertinent $\uparrow$-link and $\downarrow$-link is simply connected, then the inclusion $X_J \hookrightarrow X_{J'}$ induces an isomorphism in π_1.

(iii) If every pertinent $\uparrow$-link and $\downarrow$-link is connected, then the inclusion $X_J \hookrightarrow X_{J'}$ induces an epimorphism in π_1.

(iv) If every pertinent $\uparrow$-link and $\downarrow$-link is contractible, then the inclusion $X_J \hookrightarrow X_{J'}$ is a homotopy equivalence.

Proof. Without loss of generality, we may assume either $\inf J = \inf J'$ or $\sup J = \sup J'$; else, let $K \subset \mathbb{R}$ be the closed, connected subset such that $J \subset K \subset J'$ and $\sup K = \sup J$ and $\inf K = \inf J'$, observing that if the results hold for $J \subset K$ and $K \subset J'$, they likewise hold for $J \subset J'$. For now we will assume $\inf J = \inf J'$, since an analogous proof suffices for the case $\sup J = \sup J'$.

In this case, it suffices to prove the Theorem for subsets J' that are bounded from above. Suppose J' is not bounded from above. If J is also not bounded from above, then $J = J'$ and the results follow trivially; assume J is bounded from above. Then define for every $r \in \mathbb{Z}$ the closed, connected, and bounded from above set $L_r := J' \cap (-\infty, r]$, noting that $L_r \subset L_s$ for $r \leq s$. Hence the f-preimages $\{X_{L_r}\}_{r\in\mathbb{Z}}$ comprise a directed set of spaces (ordered by inclusion) such that $X_{J'} = \bigcup_{r\in\mathbb{Z}} X_{L_r}$ and every compact subspace C of $X_{J'}$ lies in some preimage X_{L_s}: $f(C) \subset \mathbb{R}$ is compact, hence bounded. Thus we have:

$$\varinjlim \tilde{H}_i(X_{L_r}) \cong \tilde{H}_i(X_{J'}) \quad ; \quad \varinjlim \pi_1(X_{L_r}) \cong \pi_1(X_{J'})$$

where the maps $\varphi_{r,s} : \tilde{H}_i(X_{L_r}) \to \tilde{H}_i(X_{L_s})$, $r \leq s$, are induced by the inclusions $X_{L_r} \hookrightarrow X_{L_s}$ and the canonical maps $\varphi_t : \tilde{H}_i(X_{L_t}) \to \varinjlim \tilde{H}_i(X_{L_r}) \cong \tilde{H}_i(X_{J'})$ are induced by the inclusions $X_{L_t} \hookrightarrow X_{J'}$, and similarly for π_1. In particular, Lemma 2.14 states that if every $\varphi_{r,s}$ is an isomorphism, resp. an epimorphism, then so are the φ_t. J is bounded from above, hence choose some L_t such that $J \subset L_t$. Let $\iota : X_J \hookrightarrow X_{L_t}$. Then the inclusion $X_J \hookrightarrow X_{L_t} \hookrightarrow X_{J'}$ induces a map $\varphi_t \circ \iota_* : \tilde{H}_i(X_J) \to \tilde{H}_i(X_{J'})$, which is an isomorphism, resp. epimorphism, if its factors are as well, and similarly for π_1. Finally, by constructing an appropriate mapping telescope T from the inclusions of the subspaces X_{L_r}, we have that $X_{L_r} \hookrightarrow T$ is a homotopy equivalence and that $X_{J'} \simeq T$. Hence if $X_J \hookrightarrow X_{L_t}$ is a homotopy equivalence, then so is $X_J \hookrightarrow X_{J'}$. Hence if our claims hold for subsets bounded above, then they likewise hold for J' not bounded.

Assume J' is bounded from above. Then since $f(X_{(0)})$ is closed, $f(X_{(0)}) \cap \overline{J' \setminus J}$ is closed and bounded, hence compact; since $f(X_{(0)})$ is discrete, $f(X_{(0)}) \cap \overline{J' \setminus J}$ is finite. In particular, $J' \setminus J$ contains finitely many points of $f(X_{(0)})$. We proceed by induction on the cardinality of $f(X_{(0)}) \cap J' \setminus J$ and note that the base case is proven by Lemma 2.7.

Assume that the claims hold for appropriate J', J such that $J' \setminus J$ contains fewer than m points of $f(X_{(0)})$. Suppose instead that $f(X_{(0)}) \cap J' \setminus J$ contains m points. $f(X_{(0)})$ is discrete, hence choose a closed, connected subset $K \subset \mathbb{R}$ such that $J \subset K \subset J'$ (hence $\inf K = \inf J = \inf J'$) and $K \setminus J$ and $\overline{J' \setminus K}$ contain $m-1$ points and a single point of $f(X_{(0)})$ respectively. By assumption, claims (i), (ii), (iii), and (iv) hold for $X_J \hookrightarrow X_K$; it suffices to show the same for $X_K \hookrightarrow X_{J'}$.

For (i), by Lemma 2.11 we have that $X_{J'}$ is homotopy equivalent to $X_K \cup \hat{X}$, where $\hat{X}$ is homotopy equivalent to a discrete set of points and $X_K \cap \hat{X}$ is homeomorphic to a disjoint union of descending links, and note that we can choose open neighborhoods of $X_K, \hat{X}$ in $X_K \cup \hat{X}$ that deformation retract to $X_K, \hat{X}$ respectively. Hence, we have the following reduced Mayer-Vietoris (exact) sequence:

$$\cdots \to \tilde{H}_{i+1}(X_{J'}) \longrightarrow \bigoplus_v \tilde{H}_i(\mathrm{Lk}_\downarrow(v, X)) \longrightarrow \tilde{H}_i(\hat{X}) \oplus \tilde{H}_i(X_K) \longrightarrow \tilde{H}_i(X_{J'}) \to \cdots$$

Noting that $\tilde{H}_i(\hat{X}) = 0$ and $\tilde{H}_i(X_K \cap \hat{X}) \cong \bigoplus_v \tilde{H}_i(\mathrm{Lk}_\downarrow(v, X)) = 0$ for $i \leq n$, the relevant sequence segments reduce to

$$0 \to \tilde{H}_i(X_K) \to \tilde{H}_i(X_{J'}) \to 0 \quad \text{for } 0 \leq i \leq n \quad ; \quad \tilde{H}_{n+1}(X_K) \to \tilde{H}_n(X_{J'}) \to 0$$

and the claim is shown.

To show (ii) and (iii), we again consider $\hat{X}$ from Lemma 2.11 and use the Siefert-van Kampen theorem for fundamental groupoids. Suppose that every pertinent $\downarrow$-link in X is connected. Then for any subspace Y let a set $A \subset Y$ be *representative* in Y if A meets every component of Y. $X_K \cap \hat{X}$ is homeomorphic to a disjoint union of (connected) $\downarrow$-links, hence choose a point from (the image of) each such $\downarrow$-link to form the discrete set A, and note that A is representative in $X_K, \hat{X}$, and $X_K \cap \hat{X}$. We note that $\Pi(X_K \cap \hat{X}, A)$ is totally disconnected and that each component $\hat{X}_\alpha$ of $\hat{X}$ is contractible, hence by deformation retracting each $\hat{X}_\alpha$ to the (unique) $x_\alpha \in A \cap \hat{X}_\alpha$, we have $\Pi(\hat{X}, A) \cong \mathcal{D}_A$.

By the Siefert-van Kampen theorem, we have the following pushout in the category of groupoids, induced by the appropriate inclusions:

$$\begin{tikzcd} \Pi(X_K \cap \hat{X}, A) \arrow[r] \arrow[d] & \Pi(X_K, A) \arrow[d, "\alpha"] \\ \mathcal{D}_A \arrow[r] & \Pi(X_{J'}, A) \end{tikzcd} \tag{15}$$

In particular, by Lemma 2.15 we have that α is an epimorphism; if every $\downarrow$-link is simply connected, *i.e.* if $\Pi(X_K \cap \hat{X}, A) \cong \mathcal{D}_A$, then α is a monomorphism as well, hence a groupoid isomorphism. A is representative in X_K, hence for any $x \in X_K$, x is path connected to some $x_0 \in A$. Then up to conjugation, it suffices to consider the subgroupoids of $\Pi(X_K, A)$ and $\Pi(X_{J'}, A)$ over $\{x_0\}$ and note that $\pi_1(X_K, x_0) = \Pi(X_K, \{x_0\})$ and similarly for $X_{J'}$. Hence (ii) and (iii) are shown.

Finally, suppose that every pertinent $\downarrow$-link is contractible. Then by Lemmas 2.11 and 2.12 it is clear that $X_K \hookrightarrow X_{J'}$ is a homotopy equivalence, showing (iv). $\square$

3 Finiteness properties of groups

We introduce a number of classical finiteness properties for groups. In the following, let H be a discrete group and R a ring with $1 \neq 0$.

Definition 3.1. Let M be a module over a ring R. Then a **left resolution** of M is an exact sequence of R-modules N_i such that

$$\cdots \to N_m \to N_{m-1} \to \cdots \to N_0 \to M \to 0.$$

A resolution is **finite** if cofinitely many modules N_i are zero.

Definition 3.2. A group H is of **type** $\mathrm{FP}_n(R)$ if there exists a partial left resolution

$$P_n \to P_{n-1} \to \cdots \to P_0 \to R \to 0$$

of the trivial RH-module R by finitely generated projective RH-modules P_i, $0 \leq i \leq n$.

Remark. The property $\mathrm{FP}_n(R)$ is a natural extension of more common algebraic finiteness properties: $\mathrm{FP}_0(R)$ is equivalent to finite generation; $\mathrm{FP}_1(R)$ is equivalent to finite presentedness.

Definition 3.3. A group H is of **type** $\mathrm{FP}(R)$ if there exists a finite left resolution

$$0 \to P_n \to \cdots \to P_0 \to R \to 0$$

of R by finitely generated projective RH-modules P_i.

Remark. Clearly if $H \in \mathrm{FP}(R)$, then $H \in \mathrm{FP}_n(R)$ for all n.

Definition 3.4. A group H is of **type** F_n if there exists a $\mathrm{K}(H,1)$ with a finite n-skeleton. If H has a finite Eilenberg-Mac Lane space, then H is of **finite type**, denoted F.

The following is an application of Schanuel's Lemma from homological algebra, and will provide us with a method for demonstrating that a group H is not $\mathrm{FP}_{n+1}(R)$.

Proposition 3.5. *If there exists a resolution*

$$0 \to Z_n \to P_n \to \cdots \to P_0 \to R \to 0$$

such that the P_i are finitely generated projective RH-modules and Z_n is a non-finitely generated RH-module, then H is of type $\mathrm{FP}_n(R)$ but not of type $\mathrm{FP}_{n+1}(R)$.

3.1 Homological finiteness conditions

In Lemma 3.8 below, we demonstrate a method for producing projective (in fact, free) resolutions of R over RH via techniques in algebraic topology.

Definition 3.6. A group H is of **type** $\mathrm{FH}_n(R)$ if it acts freely, faithfully, properly discontinuously, cellularly, and cocompactly on a cell complex X for which the reduced homology $\tilde{H}_i(X, R)$ vanishes for all $i \leq n-1$.

Definition 3.7. A group H is of **type** $\mathrm{FH}(R)$ if it acts freely, faithfully, properly discontinuously, cellularly, and cocompactly on a R-acyclic cell complex X (*i.e.* for which $\tilde{H}_i(X, R) = 0$ for all i).

Lemma 3.8. *Let H be a group and R a ring with $1 \neq 0$. Then the following implications hold:*

(i) $H \in \mathrm{FH}_n(R) \implies H \in \mathrm{FP}_n(R)$.
(ii) $H \in \mathrm{FH}(R) \implies H \in \mathrm{FP}(R)$.
(iii) If H acts cellularly, cocompactly, and properly discontinuously on a $(n-1)$-connected cell complex X, $n \geq 2$, then H is of type F_n.

Proof. Suppose H is of type $\mathrm{FH}_n(R)$ or $\mathrm{FH}(R)$. Then let H act cellularly and cocompactly on a cell complex X. Since the action is cellular, we have that X/H is a cell complex with k-skeleta X_k/H; by cocompactness, X/H is compact. Then X/H is a finite cell complex, and in particular, the orbit space X_k/H is finite.

We recall that the cellular chain complex $C_k(X, R) := H_k(X_k, X_{k-1}; R)$ is a R-module freely generated by the set of k-cells e_α. We claim that $C_k(X, R)$ is in fact a free finitely-generated RH-module. Observe that H acts on the k-skeleta of X, therefore via pairwise maps $(X_k, X_{k-1}) \to (X_k, X_{k-1})$. Thus H induces an action on $C_*(X, R)$ for which, by naturality, the differentials $d_k : C_k(X, R) \to C_{k-1}(X, R)$ are equivariant maps. Since H acts cellularly on X, this action permutes the k-cell basis of $C_k(X, R)$, and the bijection between this basis and the set of k-cells of X is H-equivariant. Let $\overline{e}_\alpha \in C_k(X, R)$ denote the basis element corresponding to a k-cell e_α, where $\alpha \in A$ indexes the (perhaps infinte) set of k-cells, and choose a representative $e_j, j \in J$ for each H-orbit of k-cells, where J is a finite subset of A. Hence for any element $\sum_{\alpha \in A} r_\alpha \overline{e}_\alpha \in C_k(X, R)$, we have

$$\sum_{\alpha \in A} r_\alpha \overline{e}_\alpha = \sum_{j \in J} \left[\sum_{e_\beta = h_\beta e_j} r_\beta h_\beta \right] \overline{e}_j$$

for $h_\beta \in H$. Since $J \subset A$, the $\overline{e}_j$ are R-linearly independent; since the orbits of k-cells are disjoint, they are RH-linearly independent, hence a RH-basis, completing the claim.

We consider the augmented chain complex

$$\cdots \to C_n(X, R) \to C_{n-1}(X, R) \to \cdots \to C_0(X, R) \xrightarrow{\epsilon} R \to 0 \tag{16}$$

where ϵ denotes the augmentation map $\sum_{\alpha \in A} r_\alpha \overline{e}_\alpha \mapsto \sum_\alpha r_\alpha$. Regarding R as the trivial RH-module, it is clear that ϵ is H-equivariant. Then H-equivariance of the differentials $d_k : C_k(X, R) \to C_{k-1}(X, R)$ gives that the maps in (16) are RH-linear. Hence the conditions on homology along with (16) give long exact sequences of free (hence projective) RH-modules: in particular, if $H \in \mathrm{FH}_n(R)$, then let $\tilde{H}_i(X, R) = 0$ for $0 \leq i \leq n-1$, hence

$$C_n(X, R) \to C_{n-1}(X, R) \to \cdots \to C_0(X, R) \xrightarrow{\epsilon} R \to 0$$

is a left resolution of R by projective RH-modules; if $H \in \mathrm{FH}(R)$, then let X be R-acyclic, hence if $\dim X = m$, we have the finite resolution

$$0 \to C_m(X, R) \to C_{m-1}(X, R) \to \cdots \to C_0(X, R) \xrightarrow{\epsilon} R \to 0.$$

Thus the implications (i) and (ii) are shown.

For (iii), since H acts cellularly and cocompactly on X, X/H is a finite cell complex; since the action is properly discontinuous and X is simply connected, X/H is path-connected and locally path-connected and the quotient map $X \to X/H$ is a universal cover, with $\pi_1(X/H) \cong H$ and $\pi_k(X/H) \cong \pi_k(X)$ for $k \geq 2$. Hence for $2 \leq k \leq n-1$, $\pi_k(X/H) = 0$; to construct a $\mathrm{K}(H, 1)$, it suffices to attach to X/H cells of dimension $n+1$ or higher, corresponding to the generators of $\pi_k(X/H)$ with $k \geq n$. We observe that the resulting space has a finite n-skeleton, as desired. $\square$

In light of the arguments in Lemma 3.8, we observe that we may use Proposition 3.5 to show sufficient topological conditions for a group to be of type $\mathrm{FP}_n(R)$ but not $\mathrm{FP}_{n+1}(R)$.

Corollary 3.9. *Suppose that a group H acts freely, faithfully, properly,*[1] *cellularly, and cocompactly on a cell complex X such that*

(i) $\tilde{H}_i(X, R) = 0$ for $0 \leq i \leq n-1$, and
(ii) $\tilde{H}_n(X, R)$ is not finitely generated as an RH-module.

Then H is of type $\mathrm{FH}_n(R)$ but not of type $\mathrm{FP}_{n+1}(R)$.

Proof. $H \in \mathrm{FH}_n(R)$ by definition. We note that H acts on the k-skeleta of X, hence induces an action on homology: $\tilde{H}_n(X, R)$ is in fact an RH-module, as stated in the Lemma. By the H-equivariance of the differential $d_k : C_k(X, R) \to C_{k-1}(X, R)$, $\ker d_k$ is an RH-submodule of $C_k(X, R)$. Since $\tilde{H}_n(X, R)$ is a non-finitely generated quotient of $\ker d_k$, $\ker d_k$ cannot be a finitely generated RH-module. Hence, from the argument presented in Lemma 3.8, we have the following exact sequence of RH-modules,

$$0 \to \ker d_n \xrightarrow{\iota} C_n(X, R) \xrightarrow{d_n} C_{n-1}(X, R) \to \cdots \to C_0(X, R) \to R \to 0$$

where ι is the inclusion map and the $C_k(X, R)$ are free (hence projective) and finitely generated. By Proposition 3.5, $H \notin \mathrm{FP}_{n+1}(R)$. $\square$

We will use the following proof in Section 8.

Proposition 3.10. *Suppose the H is a finitely presented group that acts freely, faithfully, properly discontinuously, cocompactly, and cellularly on a connected cell complex Y. Then it is possible to attach to Y finitely many H-orbits of 2-cells so that the resulting complex is simply connected.*

[1] Since H has the discrete topology, a free and proper action is also properly discontinuous.

Proof. All spaces considered are (path) connected; we will suppress basepoints. Since H acts properly discontinuously on Y, the quotient map $p : Y \to Y/H$ is a normal cover such that $H \cong \pi_1(Y/H)/p_*(\pi_1(Y))$. Let $Z = Y/H$. By cocompactness, Z is compact, hence a finite cell complex: $\pi_1(Z)$ is finitely presented. Let $\pi_1(Z) = \langle a_i : r_j \rangle$ be a finite presentation of $\pi_1(Z)$, and let $p_*(\pi_1(Y)) = \langle t_\alpha \rangle$. Then $H \cong \langle a_i : r_j, t_\alpha \rangle$, hence choose a finite subpresentation $\langle a_i : r_j, t_{\alpha_k} \rangle$: we observe that $p_*(\pi_1(Y)) = \langle t_{\alpha_k} \rangle$ is finitely generated. Hence attach finitely many 2-cells to Z, or equivalently, H-orbits of 2-cells to Y, to yield Z', Y' such that $\pi_1(Y)$ vanishes under the map induced by the inclusion $j : Z \hookrightarrow Z' = Y'/H$. We have the following diagram of induced maps:

$$\begin{array}{ccc} \pi_1(Y) & \xrightarrow{i_*} & \pi_1(Y') \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p'_*} \\ \pi_1(Z) & \xrightarrow{j_*} & \pi_1(Z') \end{array}$$

where i_* is the map induced by the inclusion $Y \hookrightarrow Y'$ and p'_* is the map induced by the quotient $p' : Y' \to Z'$. Hence $\pi_1(Y) \subset \ker p'_* \circ i_*$; since p'_* is injective, $\pi_1(Y) \subset \ker i_*$. Thus every representative loop in Y is nulhomotopic in Y'; in particular, since Y and Y' have identical 1-skeleta, Y' is simply connected as desired. $\square$

4 Kernels of homomorphisms to $\mathbb{Z}$

Let G be a discrete group that acts freely, faithfully, properly,[2] cocompactly, and cellularly on a contractible affine cell complex X such that the action of G agrees with the affine structure, *i.e.* for any cell $e \subset X$ with an admissible characteristic map χ_e, and for any $g \in G$, $g\chi_e$ is an admissible characteristic map for the cell $g(e)$. Let $\phi : G \to \mathbb{Z}$ be a surjective homomorphism, and let $f : X \to \mathbb{R}$ be a ϕ-equivariant Morse function with respect to the usual translation action of $\mathbb{Z}$ on $\mathbb{R}$, *i.e.*

$$\forall g \in G, x \in X : f(gx) = f(x) + \phi(g).$$

Finally, let $H = \ker \phi$. Then we claim the following:

Proposition 4.1. *H acts freely, faithfully, properly discontinuously, cellularly, and cocompactly on level sets $X_t = f^{-1}(t)$, for any $t \in \mathbb{R}$.*

Proof. For any $h \in H$, $x \in X_t$, $f(hx) = f(x) + \phi(h) = f(x) = t$, hence H acts on X_t. Since G acts freely on X, the action of H on X_t is free and faithful; since G acts properly discontinuously on X, so does H on X_t (as a subspace of X). It remains only to check that H acts on X_t cellularly and cocompactly as well.

Cellular. We must verify first that X_t is a cell complex. Define for each cell $e \subset X$ the subspace $e_t = e \cap X_t$. Recall (*e.g.* from the proof of Lemma 2.7) that $(C_e)_t := \chi_e^{-1}(X_t) = C_e \cap \mathcal{H}_t$, where $\mathcal{H}_t \subset \mathbb{R}_m$ is a (top-dimensional) hyperplane. Then $(C_e)_t$ is a convex polyhedron and $\chi_{e_t} := \chi_e|_{(C_e)_t} : (C_e)_t \to X_t$ is the characteristic map of e_t as a cell in X_t, viewed as a cell complex with k-skeleta $X^k \cap X_t$.

Suppose e is a (closed) cell in X and $\tilde{e} := \chi_e(\mathring{C}_e)$ the corresponding open cell. Note that $\mathring{C}_e \cap (C_e)_t = \mathring{C}_e \cap \mathcal{H}_t = \operatorname{int}((C_e)_t)$, hence $\tilde{e}_t := \tilde{e} \cap X_t = \chi_e(\mathring{C}_e \cap (C_e)_t)$ is the open cell corresponding to the cell e_t in X_t. G acts cellularly on X, hence for any $h \in H$, let e, e' be closed cells in X such that $h(\tilde{e}) = \tilde{e}'$. Then $h(\tilde{e}_t) \subset \tilde{e}'_t$ and since h is invertible, we have equality. Moreover, if $h(\tilde{e}_t) = \tilde{e}_t$, then let d be the minimal cell in X such that $\tilde{e}_t \subset \tilde{d}$. Then $h(d) = d$, else if $h(d) = d'$, then let $d'' = d \cap d' \subsetneq d$ and note that $\tilde{d}''$ contains $\tilde{e}_t$. Hence $h(\tilde{d}) = \tilde{d}$ and h acts as identity on $\tilde{d}$ and therefore $\tilde{e}_t$.

Cocompact. From the ϕ-equivariance of f, we observe that G acts on $\bigsqcup_{i \in \mathbb{Z}} X_{t+i}$, the latter given the disjoint topology. $\bigsqcup_i X_{t+i}$ is closed, since it is the f-preimage of the closed set $\mathbb{Z} \subset \mathbb{R}$. Hence $(\bigsqcup_i X_{t+i})/G$ is a closed subset of X/G, a compact space, hence also compact. But ϕ is surjective onto $\mathbb{Z}$, hence $X_t/H = (\bigsqcup_i X_{t+i})/G$, thus a compact space. $\square$

Theorem 4.2. *Let $f : X \to \mathbb{R}$ be a ϕ-equivariant Morse function for X, G, and ϕ defined as above. Then if $H = \ker \phi$, the following implications hold:*

(i) If every $\uparrow$-link and $\downarrow$-link is homologically n-connected over R, then $H \in \mathrm{FH}_{n+1}(R)$.

[2] Hence properly discontinuous.

(ii) If every $\uparrow$-link and $\downarrow$-link is R-acyclic, then $H \in \mathrm{FH}(R)$.

(iii) If every $\uparrow$-link and $\downarrow$-link is simply connected, then H is finitely presented.

Proof. Assume that homologies are computed over the coefficient ring R. Suppose that every $\uparrow$-link and $\downarrow$-link is homologically n-connected. Then by Theorem 2.16, we have that for any $s \leq t \in \mathbb{R}$, the inclusion of sublevel sets $\iota : X_{(-\infty,s]} \hookrightarrow X_{(-\infty,t]}$ induces an isomorphism $\iota_* : \tilde{H}_i(X_{(-\infty,s]}) \to \tilde{H}_i(X_{(-\infty,t]})$ for all $0 \leq i \leq n$. We note that the integer sublevel sets $\{X_{(-\infty,s]}\}_{s\in\mathbb{Z}}$ form a directed set of spaces (ordered by inclusion) such that $X = \bigcup_{s\in\mathbb{Z}} X_{(-\infty,s]}$ and every compact subspace $C \subset X$ lies in some sublevel set: $f(C) \subset \mathbb{R}$ is compact, hence bounded. Recalling that X is contractible by assumption, we have that

$$\varinjlim \tilde{H}_i(X_{(-\infty,s]}) \cong \tilde{H}_i(X) = 0.$$

However, for $0 \leq i \leq n$ and any $q, r \in \mathbb{Z}$, $\tilde{H}_i(X_{(-\infty,q]}) \cong \tilde{H}_i(X_{(-\infty,r]})$, hence by Lemma 2.14

$$\tilde{H}_i(X_{(-\infty,r]}) \cong \varinjlim \tilde{H}_i(X_{(-\infty,s]}) \cong 0$$

is vanishing as well. Thus, for any $t \in \mathbb{R}$ and $0 \leq i \leq n$, $\tilde{H}_i(X_{(-\infty,t]}) = 0$ and a similar argument yields the equivalent statement for the superlevel set, $\tilde{H}_i(X_{[t,\infty)}) = 0$. Finally, we have the (exact) Mayer-Vietoris sequence

$$0 = \tilde{H}_{i+1}(X) \longrightarrow \tilde{H}_i(X_t) \longrightarrow \tilde{H}_i(X_{(-\infty,r]}) \oplus \tilde{H}_i(X_{[r,\infty)}) = 0$$

for $0 \leq i \leq n$, hence $\tilde{H}_i(X_t) = 0$. Then by Proposition 4.1, $H \in \mathrm{FH}_{n+1}(R)$, proving (i).

A similar argument (again using the appropriate claim of Theorem 2.16) shows (ii). For (iii), note that if every $\uparrow$-link and $\downarrow$-link is simply connected, then they are homologically 1-connected as well, hence (i) implies that for each $t \in \mathbb{R}$, $\tilde{H}_0(X_t) = 0$ and X_t is connected. By Theorem 2.16, the inclusion $X_t \hookrightarrow X$ induces an isomorphism in fundamental groups, hence X_t is simply connected as well. Then by Lemma 3.8, H is of type F_2, *i.e.* there exists a $\mathrm{K}(H, 1)$ with a finite 2-skeleton. Hence H is finitely presented. $\square$

5 Right-angled Artin groups

Definition 5.1. Let L be a simplicial complex. If every finite set of pairwise adjacent vertices of L spans a simplex in L, then L is a **flag complex**.

Remark. Flag complexes are completely determined by their 1-skeleta.

Definition 5.2. Let L be a finite flag complex with vertex set $L_0 = \{v_1, \dots v_N\}$. Then the **right-angled Artin group** G_L associated to L is the group with finite presentation

$$G_L := \langle v_1, \dots, v_N : [v_i, v_j] \text{ for all edges } \{v_i, v_j\} \text{ in } L_1 \rangle.$$

We note that G_L is a *subpresentation* of the standard presentation of $\mathbb{Z}^N$,

$$\mathbb{Z}^N = \langle v_1, \dots, v_N : [v_i, v_j] \text{ for all } i, j \rangle,$$

i.e., the specified relators of G_L form a subset of those for $\mathbb{Z}^N$. Hence there is a natural epimorphism (abelianization) $G_L \to \mathbb{Z}^N$ mapping the generators of G_L to the standard basis elements of $\mathbb{Z}^N$, which, composed with the augmentation epimorphism $\epsilon : \mathbb{Z}^N \to \mathbb{Z}$ gives an epimorphism $\phi : G_L \to \mathbb{Z}$.

Hence there exists the following short exact sequence of groups:

$$1 \to H_L \to G_L \xrightarrow{\phi} \mathbb{Z} \to 1$$

We are interested in the finiteness properties of $H_L = \ker \phi$, and proceed by constructing an Eilenberg-Mac Lane space Q_L for G_L. For this purpose, we will need some familiarity with piecewise Euclidean cubical complexes.

5.1 Piecewise Euclidean cubical complexes

Let $\square^n$ denote the *regular n-cube* in $\mathbb{R}^n$ with vertex at the origin and edges defined by the unit basis vectors. Given $x \in \square^n$, let $\operatorname{supp}(x)$ be the unique face of $\square^n$ containing x in its interior. Then we define the following:

Definition 5.3. Let $\{Q_\lambda\}_\lambda$ be a family of regular n-cubes with $n \leq m$ for some fixed m and let $X = \bigsqcup_\lambda Q_\lambda$ be their disjoint union. Let K be a quotient space of X and let $p : X \to K$ be the quotient map, with p_λ the restriction to Q_λ. Then K is a **piecewise Euclidean cubical complex**[3] if:

- for every λ, the restriction of p_λ to a face of Q_λ is injective, and
- for all λ_1, λ_2 and $x_1 \in Q_{\lambda_1}, x_2 \in Q_{\lambda_2}$, if $p_{\lambda_1}(x_1) = p_{\lambda_2}(x_2)$ then there is an isometry $h : \operatorname{supp}(x_1) \to \operatorname{supp}(x_2)$ such that $p_{\lambda_1}(y) = p_{\lambda_2}(h(y))$ for all $y \in \operatorname{supp}(x_1)$.

Remark. We may regard PE cubical complexes as cell complexes composed of disjoint regular cubes glued along faces by isometries.

[3]Note that this definition is more general than the one offered in ; there, the above would be a *cube* complex.

Viewing a cubical complex K as a special case of affine cell complex, we may analogously define the *link* of a vertex in K. However, in this case we have a well defined quotient metric on both K and $\mathrm{Lk}(x, K)$ for $x \in K$. Then vertex link of a cubical complex is an *all right simplicial complex*:

Definition 5.4. A simplex endowed with a spherical metric for which all dihedral angles are right is a **all right simplex**. A simplicial complex composed of such simplices is an **all right** piecewise spherical complex.

We give an (equivalent) combinatorial definition of nonpositive curvature:

Definition 5.5. A PE cubical complex is nonpositively curved if the link of each vertex is a flag complex.

In addition, we collect several relevant results concerning the geometry of such complexes, the first of which is a lemma of Gromov.

Proposition 5.6. .

(i) If L is an all right, piecewise sphereical complex, then L is a CAT(1) space if and only if it is a flag complex.

(ii) Let K be a PE cubical complex whose links all all CAT(1) spaces. Then K is locally CAT(0), hence by the Cartan-Hadamard Theorem, its universal cover is a CAT(0) (hence contractible) metric space.

5.2 Properties of flag complexes

Definition 5.7. Let L be a simplicial complex and let $\{S^0_\alpha\}_{\alpha \in L_0}$ be a collection of 0-spheres indexed by the set of vertices in L. Then the **spherical complex associated to** L, denoted $S(L)$, is the union[4] of the joins of subcollections of 0-spheres whose vertices span simplices in L:

$$S(L) := \bigcup \{S^0_{\alpha_0} * \cdots * S^0_{\alpha_m} : \langle \alpha_0, \dots, \alpha_m \rangle \text{ an } m\text{-simplex of } L\}$$

Lemma 5.8. *Suppose L is a flag complex. Then $S(L)$ is also a flag complex.*

Proof. Let $\pi : S(L)_0 \to L_0$ be the 2-to-1 map such that $\pi(S^0_\alpha) = \{\alpha\}$ for all $\alpha \in L_0$. From the construction of $S(L)$, we observe that the pair of vertices comprising any 0-sphere S^0_α is not adjacent, and moreover, for any adjacent vertices $u, v \in S(L)_0$, $u \in S^0_\alpha$, $v \in S^0_\beta$ for $\alpha = \pi(u), \beta = \pi(v)$ distinct, hence α, β belong to a simplex in L, or equivalently, α, β are adjacent. Hence, for any set U of pairwise adjacent vertices, π is injective over U and $\pi(U)$ is pairwise adjacent; in particular, we consider a finite pairwise adjacent set $\{v_0, \dots, v_k\} \subset S(L)_0$. Then $\{\pi(v_0), \dots, \pi(v_k)\}$ spans a k-simplex in L and $\{v_0, \dots, v_k\}$ belong to a k-sphere Z (triangulated as the join of $(k+1)$ 0-spheres $S^0_{\pi(v_i)}$) in $S(L)$. Moreover, since the v_i belong to distinct 0-spheres, they span the join of $k+1$ points in Z, *i.e.* a k-simplex. $\square$

[4] More technically, $S(L)$ is the quotient space of such joins, where common 0-spheres are identified and this identification is extended simplicially.

Definition 5.9. Let L be a simplicial complex and σ a simplex of L. The the **closed star** $\mathrm{St}(\sigma, L)$ is the subcomplex of L consisting of faces of simplices for which σ is *also a face.* Let $\mathrm{St}'(\sigma, L)$ denote the subcomplex consisting of the faces of simplices *sharing* a face with σ. Define the **frontier** of $\mathrm{St}'(\sigma, L)$, denoted $\mathrm{Fr}(\mathrm{St}'(\sigma, L))$, to be the subcomplex consisting of the faces of $\mathrm{St}'(\sigma, L)$ disjoint from σ.

Remark. We note that if ρ is a face of σ, then $\mathrm{St}(\sigma, L) \subset \mathrm{St}(\rho, L)$. Hence $\mathrm{St}'(\sigma, L) = \bigcup_v \mathrm{St}(v, L)$ for vertices v of σ.

Proposition 5.10. *Let L be a flag complex equipped with the all right spherical metric and let σ be a simplex of L. Then:*

(i) $\mathrm{St}'(\sigma, L)$ *is contractible.*

(ii) Suppose that τ is another simplex of L such that there exist points $a \in \tau$ and $b \in \sigma$ with $d(a, b) < \pi/2$. Then $\sigma \cap \tau \neq \varnothing$.

We will use the following general claim:

Lemma 5.11. *Let $\mathcal{U}$ be the collection of n sets A with a property P, such that the intersection of any subcollection of sets in $\mathcal{U}$ also has property P. Suppose the following is a property of P:*

$$\forall A, B \in P, \quad (A \cap B \in P) \implies (A \cup B \in P) \tag{17}$$

Then the union of all sets in $\mathcal{U}$ has P.

Proof. The claim clearly holds for $n = 0$. Assume true for $n = k$ and suppose $n = k + 1$. Then choose a k-subcollection $\mathcal{U}' \subset \mathcal{U}$. Clearly $\mathcal{U}'$ meets the conditions of the Lemma, hence its union $\bigcup \mathcal{U}'$ has property P. Let A be the unique element in $\mathcal{U} \setminus \mathcal{U}'$. Then $A \cap \mathcal{U}'$ is an intersection of elements in $\mathcal{U}$, hence also has property P. But $\bigcup \mathcal{U} = \bigcup \mathcal{U}' \cup A$, hence by (17), $\bigcup \mathcal{U}$ has property P. □

This result readily extends to contractible subcomplexes:

Corollary 5.12. *Suppose $\mathcal{U}$ is a finite collection of contractible subcomplexes such that the intersection of any subcollection is contractible. Then $\bigcup \mathcal{U}$ is contractible.*

Proof. By Lemma 5.11, it suffices to show that for any two contractible subcomplexes whose intersection is contractible, so is their union.

Suppose A, B are such subcomplexes and let Z be their intersection; note that Z is hence a (closed) contractible subcomplex, as are the products $A \times I$, $B \times I$, and $Z \times I$. Choose a point $p \in Z$ and let $F : A \times I \to A$ and $G : B \times I \to B$ be deformation retractions to p. Let $\tilde{F}, \tilde{G} : Z \times I \to Z$ coincide with F, resp. G, over $Z \times I$. Since $Z \times I$ contractible, we may homotope $\tilde{F}$ to $\tilde{G}$; since $Z \times I$ is a subcomplex of $A \times I$, this extends to a homotopy $\Phi : A \times I \times I \to A$ of $F = \Phi|_{A \times I \times \{0\}}$ to some homotopy on A, $F' = \Phi|_{A \times I \times \{1\}}$ which coincides with

G over $Z \times I$. Hence let $F'' : A \times I \to A$ be the deformation retraction of A to p defined as follows:

$$F''(a,t) = \begin{cases} \Phi(a,0,2t) : t \in [0,1/2] \\ F'(a,2t-1) = \Phi(a,2t-1,1) : t \in [1/2,1] \end{cases}$$

Define the deformation retraction $G' : B \times I \to B$ of B to p such that:

$$G'(b,t) = \begin{cases} G(b,0) : t \in [0,1/2] \\ G(b,2t-1) : t \in [1/2,1] \end{cases}$$

F'' and G' coincide over over the closed subspace $Z \times I$ and hence paste together to construct a deformation retraction of $A \cup B$ to p. □

Proof of Proposition 5.10. For any simplex $\rho \subset L$, $\mathrm{St}(\rho, L)$ is the union of finitely many closed cells (simplices and their faces) such that any intersection thereof is a closed cell (a common face) containing ρ. Hence any finite intersection is contractible, and thus so is $\mathrm{St}(\rho, L)$.

We note that any collection $\{v_1, \ldots, v_k\}$ of vertices of σ spans a face $\langle v_1, \ldots, v_k\rangle$ of σ. We claim that

$$\mathrm{St}(v_1, L) \cap \ldots \cap \mathrm{St}(v_k, L) = \mathrm{St}(\langle v_1, \ldots, v_k\rangle, L),$$

hence is contractible. Recalling that $St'(\sigma, L)$ is the union of the closed stars of the vertices of σ, this suffices to show that $\mathrm{St}'(\sigma, L)$ is contractible, proving (i). Clearly, $\mathrm{St}(\langle v_i\rangle_i, L) \subset \bigcap_i \mathrm{St}(v_i, L)$. Conversely, suppose $\rho \in \mathrm{St}(v_i, L)$ for all i. Then ρ is the face of a simplex containing v_i for all i. Hence the vertices of ρ are pairwise adjacent to the v_i: since L is flag, together they span a simplex τ with $\langle v_i\rangle_i$ a face, hence (since ρ is a face of τ) $\rho \in \mathrm{St}(\langle v_i\rangle_i, L)$.

For (ii), let d denote the metric in $\mathrm{St}'(\sigma, L)$. Let η be a 1-simplex equipped with the all right spherical metric (hence of length $\pi/2$) and let $\varphi : \mathrm{St}'(\sigma, L) \to \eta$ be the simplicial map constructed by sending σ and $\mathrm{Fr}(\mathrm{St}'(\sigma, L))$ to distinct vertices of η and extending simplicially; since the vertices of $\mathrm{St}'(\sigma, L)$ are partitioned by the vertex sets of σ and $\mathrm{Fr}(\mathrm{St}'(\sigma, L))$, this map is well defined. We claim that φ is length non-increasing. In fact, it suffices to show that φ restricted to any simplex τ in $\mathrm{St}'(\sigma, L)$ is length non-increasing, since then for any points $p, q \in \mathrm{St}'(\sigma, L)$ and any geodesic m-string[5] γ connecting p, q in $\mathrm{St}'(\sigma, L)$, the length of $\varphi(\gamma)$ is not greater than $d(p,q)$, hence $d_\eta(p,q) \le d(p,q)$.

Let $\omega = \tau \cap \sigma$ and $\omega' = \tau \cap \mathrm{Fr}(\mathrm{St}'(\sigma, L))$, and note that $\tau = \omega * \omega'$. Then for any two points $s, t \in \tau$, let Γ_s, resp, Γ_t be the unique geodesic segments between ω and ω' containing s, resp. t. Since $\mathrm{St}'(\sigma, L)$ is an all right complex (hence dihedral angles are identically $\pi/2$), φ is an isometry on Γ_s and Γ_t. Hence, let t' be the preimage of $\varphi(t)$ on Γ_s and let $w \in \omega$, $w' \in \omega'$ be the endpoints of Γ_s and $u' \in \omega'$ an endpoint of Γ_t. Note that $d(t', w') = d_\eta(\varphi(t), \varphi(w')) = d(t, u')$, hence if $d(s,t) < d_\eta(\varphi(s), \varphi(t))$, then the

[5] We note that all right spherical simplicial complexes are geodesic; moreover, every geodesic is an m-string.

path formed of segments $[w,s],[s,t],[t,u']$ is strictly shorter than Γ_s, formed of segments $[w,s],[s,t'],[t',w']$. Hence $d(w,u') < \pi/2$, a contradiction since ω,ω' are disjoint faces of τ, an all right spherical simplex.

Finally, suppose ρ is a simplex of L such that there exist points $a \in \rho$ and $b \in \sigma$ with $d(a,b) < \pi/2$. If $\rho \cap \sigma = \varnothing$, then the geodesic γ between a, b intersects $\mathrm{Fr}(\mathrm{St}'(\sigma, L))$ (note that a may be a point in $\mathrm{Fr}(\mathrm{St}'(\sigma, L))$). φ maps γ sujectively onto η, hence $d(a,b) \geq \pi/2$, a contradiction. $\square$

5.3 The cubical complex Q_L

In Theorem 5.14 below we construct Q_L. We first need the following technical lemma regarding (double) liftings of maps to universal covers.

Lemma 5.13. *Suppose that X, Y are spaces with a continuous map $\varphi : X \to Y$; let $p : \tilde{X} \to X$, resp. $p' : \tilde{Y} \to Y$ be the universal covers of X, resp. Y. If $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(y_0))$ is the map induced by φ, then there exists a φ_*-equivariant double lift $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ of φ.*

Proof. $\tilde{X}$ is simply connected, hence define $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ to be a lift of $\varphi p : \tilde{X} \to Y$ to $\tilde{Y}$, *i.e.* such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

Since $\tilde{\varphi}$ is a lift of covering spaces, it is uniquely determined by a single point, *e.g.* $\tilde{x}_0 \in p^{-1}(x_0)$. Let $\tilde{y}_0 = \tilde{\varphi}(\tilde{x}_0)$ and note that $p'(\tilde{y}_0) = y_0 := \varphi(x_0)$.

For any $g \in \pi_1(X, x_0)$, let $\gamma : I \to \tilde{X}$ denote the unique lift of a representative curve in the homotopy class for g starting at $\tilde{x}_0$. Then $\tilde{\varphi}\gamma$ is the unique lift of $\varphi p \gamma$ to $\tilde{Y}$ starting at $\tilde{y}_0$. We note that $\varphi p \gamma$ belongs to the equivalence class $\varphi_*(g) \in \pi_1(Y, y_0)$, hence any lift starting at $\tilde{y}_0$ must end at $\varphi_*(g)\tilde{y}_0$. Hence $\tilde{\varphi}(g\tilde{x}_0) = \varphi_*(g)\tilde{\varphi}(\tilde{x}_0)$, or equivalently $\varphi_*(g)^{-1}\tilde{\varphi}(g\tilde{x}_0) = \tilde{\varphi}(\tilde{x}_0)$, hence by uniqueness $\varphi_*(g)^{-1}\tilde{\varphi}(g\tilde{x}) = \tilde{\varphi}(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$ and $\tilde{\varphi}$ is φ_*-equivariant. $\square$

Theorem 5.14. *Let L be a finite flag complex with G_L the associated right-angled Artin group. Let $\phi : G_L \to \mathbb{Z}$ be the epimorphism defined above. Then there exists a nonpositively curved PE cubical complex Q_L and a map $l : Q_L \to S^1$ such that*

(i) $\pi_1(Q_L) \cong G_L$ and l induces the epimorphism $\phi : G_L \to \mathbb{Z}$.
(ii) Q_L has one vertex, whose link is isomorphic to $S(L)$.
(iii) Let X be the universal cover of Q_L and $\mathbb{R}$ the universal cover of S^1. Then the lift of l to $X, \mathbb{R}$ is a ϕ-equivariant Morse function $f : X \to \mathbb{R}$.
(iv) All $\uparrow$-links and $\downarrow$-links of X with respect to f above are isomorphic to L.

Remark. Since Q_L is nonpositively curved, it is a $K(G_L, 1)$. In particular, by the Cartan-Hadamard Theorem, the universal cover X is a simply connected CAT(0) space, hence contractible. Then $\pi_k(X) \cong \pi_k(Q_L) = 0$ for all $k \geq 2$.

Proof. We first construct the standard affine realization of L in $\mathbb{R}^N$, where N is the number of vertices of L. In particular, choose a bijection between the vertices v_i of L and the (endpoints of the) basis vectors e_{v_i} of $\mathbb{R}^N$. Then identify each simplex σ of L with the convex hull of the vectors $\{e_{v_{ij}}\}_{v_{ij}\in\sigma}$ corresponding to the vertices of σ. For each m-simplex σ in L, define $\square_\sigma$ to be the regular $(m+1)$-cube based at the origin in $\mathbb{R}^N$ whose edges are defined by the basis vectors $\{e_{v_{ij}} : v_{ij} \in \sigma\}$.

Define Q_L to be the image of the union of cubes

$$\overline{Q}_L := \bigcup\{\square_\sigma : \sigma \text{ a simplex of } L\}$$

under the projection $q : \mathbb{R}^N \to \mathbb{R}^N/\mathbb{Z}^N = T^N$ with respect to the usual action of $\mathbb{Z}^N$ on $\mathbb{R}^N$ by translations, hence T^N is the standard N-torus.

Let $\square^N$ denote the regular N-cube in $\mathbb{R}^N$ with edges defined by the full set of basis vectors $\{e_{v_i} : v_i \in L\}$. Clearly $\overline{Q}_L \subset \square^N$ is a PE cubical subcomplex of $\square^N$, *i.e.* each $\square_\sigma$ is a face of $\square^N$. Since the image $q(\square^N) = T^N$ is a PE cubical complex, then so is Q_L. Moreover, since the vertices of $\overline{Q}_L$ have integer coordinates, Q_L has just one vertex v_0 and each $(m+1)$-cube $\square_\sigma$ in $\overline{Q}_L$ descends to a distinct torus in Q_L. The space of directions $\Omega_{v_0}T^N$ is an $(N-1)$-sphere homeomorphic to $\mathrm{Lk}(v_0, T^N)$, which may be naturally triangulated as the join of N 0-spheres corresponding to the unit vectors e_{v_i}, or equivalently, the vertices $v_i \in L_0$. Hence for each m-simplex σ in L, the m-torus $q(\square_\sigma)$ contributes an $(m-1)$-sphere to $\mathrm{Lk}(v_0, Q_L)$ (viewed as a subcomplex of $\mathrm{Lk}(v_0, T^N)$) that may be triangulated as above. From Definition 5.7, we have $\mathrm{Lk}(v_0, Q_L) \cong S(L)$, proving (ii).

Note that by Lemma 5.8, $S(L)$ is also a flag complex, hence Q_L is nonpositively curved.

We claim that the augmentation map $\epsilon : \mathbb{R}^N \to \mathbb{R} : (x_i) \mapsto \sum_i x_i$ descends to a continuous map $\overline{\epsilon} : \mathbb{R}^N/\mathbb{Z}^N \to S^1$. It suffices to check that the map $\mathbb{R}^N \xrightarrow{\epsilon} \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ is constant over the $\mathbb{Z}^N$ orbits (lattices) in $\mathbb{R}^N$; noting that the ϵ-image of such lattices is a $\mathbb{Z}$-lattice in $\mathbb{R}$, the claim is shown. Hence denote the restriction of $\overline{\epsilon}$ to Q_L by

$$l : Q_L \to S^1.$$

To show (i), we first compute the fundamental group of Q_L from its 2-skeleton. Q_L has a single vertex, hence the 1-cells of Q_L generate $\pi_1(Q_L)$; these correspond bijectively with the vertices in L_0, hence label the generators of $\pi_1(Q_L)$ by their corresponding vertices $v_i \in L_0$. Similarly, the 2-cells in Q_L correspond bijectively with the edges in L; each 2-cell σ is a 2-torus in Q_L with 1-cells corresponding to two adjacent vertices $v_i, v_j \in L_0$. Thus σ contributes the relation $v_i v_j v_i^{-1} v_j^{-1} = [v_i, v_j] = 1$ and G_L and $\pi_1(Q_L)$ have identical presentations. Finally, we note that ϵ maps the 1-cubes $\{\square_{v_i} : v_i \in L_0\}$ into the unit interval in $\mathbb{R}$, hence l_* maps the generators of $\pi_1(Q_L)$ into $1 \in \mathbb{Z} = \pi_1(S^1)$; since $\mathbb{Z}$ is Abelian, l_* factors through the abelianization map, hence l_* is exactly $\phi : G_L \to \mathbb{Z}$ as defined above.

Let $p : X \to Q_L$ be the universal cover of Q_L, and note that X is a PE cubical complex such that p is a cellular map. Define the ϕ-equivariant (double) lift of l, as given in Lemma 5.13, to be

$$f : X \to \mathbb{R}.$$

Hence, to show (iii) it remains only to check that f is a Morse function on X. We first explicitly describe $f\chi_e$ for any m-cell $e \subset X$ with admissible characteristic map $\chi_e : \square^m \to X$. In particular, note that since Q_L is a subcomplex of $\mathbb{R}^N/\mathbb{Z}^N$, the map $\square^m \xrightarrow{\chi_e} Q_L \hookrightarrow \mathbb{R}^N/\mathbb{Z}^N$ defines a characteristic map for $q(\square_\rho) \subset \mathbb{R}^N/\mathbb{Z}^N$, hence (up to precomposition by an isometry $\xi : \square^m \to \square^m$) coincides with the characteristic map $\square^m \xrightarrow{\chi_\rho} \mathbb{R}^N \xrightarrow{q} \mathbb{R}^N/\mathbb{Z}^N$, where $q : \mathbb{R}^N \to \mathbb{R}^N/\mathbb{Z}^N$ is the quotient map as above and χ_ρ is a characteristic map for $\square_\rho$ such that χ_ρ is a partial affine map and $\epsilon\chi_\rho$ coincides with the restriction of ϵ to $\square^m$. Hence we have the following commuting diagram:

$$\begin{tikzcd}
\square^m \arrow[r, "\chi_e"] \arrow[dddd, "\xi"'] & X \arrow[r, "f"] \arrow[d, "p"] & \mathbb{R} \arrow[d] \\
 & Q_L \arrow[r, "l"] \arrow[d, hook, "i"] & S^1 \arrow[d, "\mathrm{id}"] \\
 & \mathbb{R}^N/\mathbb{Z}^N \arrow[r, "\overline{\epsilon}"] & S^1 \\
 & & \\
\square^m \arrow[r, "\chi_\rho"] & \mathbb{R}^N \arrow[uu, "q"'] \arrow[r, "\epsilon"'] & \mathbb{R} \arrow[uu]
\end{tikzcd} \tag{18}$$

where i is the appropriate inclusion. In particular, we note that $f\chi_e$ and $\epsilon\chi_\rho\xi$ are both lifts of $\overline{\epsilon}ip\chi_e = \overline{\epsilon}q\chi_\rho\xi$, hence differ only by postcomposition by a deck transformation of $\mathbb{R}$ (viewed as the universal cover of S^1), *i.e.* by an integer translation τ. Hence we have

$$f\chi_e = \tau\epsilon\chi_\rho\xi.$$

If τ is a translation by $z \in \mathbb{Z}$, then by our choice of χ_ρ, $f\chi_e\xi^{-1}$ is the restriction to $\square^m$ of the affine map

$$\mathbb{R}^m \to \mathbb{R} : (x_i) \mapsto \sum_i x_i + z, \tag{19}$$

which we note to be constant only when e is a vertex; precomposing the above by ξ, we have that $f\chi_e$ is a partial affine homeomorphism and non-constant for non-vertices. Finally, the f-image of X_0 is exactly $\mathbb{Z} \subset \mathbb{R}$, which is discrete and closed in $\mathbb{R}$. Thus f is a Morse function of X and (iii) is shown.

Finally, fix a vertex $x \in X$ and let $\tilde{\imath} : X \to \mathbb{R}^N$ be the (cellular) double lift of $i : Q_L \hookrightarrow \mathbb{R}^N/\mathbb{Z}^N$ such that $\tilde{\imath}(x) = 0$ and the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\tilde{\imath}"] \arrow[d, "p"'] & \mathbb{R}^N \arrow[d, "q"] \\
Q_L \arrow[r, hook, "i"] & \mathbb{R}^N/\mathbb{Z}^N
\end{tikzcd}$$

Note that since i is an embedding, $\tilde{\imath}$ is locally an embedding. Hence, since the link of a vertex is a local property, $\tilde{\imath}$ induces a simplicial embedding

$$\eta : \mathrm{Lk}(x, X) \hookrightarrow \mathrm{Lk}(0, \mathbb{R}^N).$$

We first claim that $\eta(\mathrm{Lk}_\uparrow(x, X)) = \mathrm{Lk}(0, \overline{Q}_L)$. For an m-simplex $\sigma \subset \eta(\mathrm{Lk}(x, X))$, let h denote the m-cube in $\mathbb{R}^N$ such that $\sigma = \mathrm{Lk}(0, h)$. Let σ also denote $\eta^{-1}(\sigma) \subset \mathrm{Lk}(x, X)$ and let e be the m-cube in X such that $\sigma = \mathrm{Lk}(x, e)$, hence $\tilde{\imath}(e) = h$. Since $\tilde{\imath}$ is a lift of i, both e and h descend to the same cell $q(\square_\rho) \subset \mathbb{R}^N/\mathbb{Z}^N$, hence $\tilde{\imath}(e) = h = \delta(\square_\rho)$ for $\delta \in \mathbb{Z}^N$ a deck transformation and $\square_\rho \subset \overline{Q}_L$. Noting that $q\delta = q$ (since δ is a deck transformation), we have that (18) and the subsequent claims remain valid if we replace χ_ρ with $\delta\chi_\rho$. That is, for χ_e the characteristic map of e,

$$f\chi_e = \tau\epsilon\delta\chi_\rho\xi = \tau'\epsilon\chi_\rho\xi \tag{20}$$

where τ' is an integer translation. We also note that $\tilde{\imath}\chi_e$ and $\delta\chi_\rho$ are both characteristic maps of $\square_\rho$, hence differ by precomposition by an isometry $\xi' : \square^m \to \square^m$. Moreover, since $\tilde{\imath}$ is a lift of ip,

$$q\delta\chi_\rho\xi' = q\tilde{\imath}\chi_e = ip\chi_e = q\delta\chi_\rho\xi.$$

Hence, since $q\delta\chi_\rho$ is injective on the interior of $\square^m$ and ξ, ξ' are isometries, $\xi = \xi'$. Choose χ_e such that $\xi' = \xi$ are identity. Then (19) and (20) imply that $f\chi_e$ is minimum at $0 \in \square^m$ and hence $f|_e$ is minimum at x if and only if $\chi_e(0) = x$. Finally, we will show below that δ acts by identity if and only if $\chi_e(0) = x$. Note that this suffices to prove our claim: $\sigma \subset \mathrm{Lk}_\uparrow(x, X)$ is equivalent to $f|_e$ achieving a minimum at x. Hence we have that $\sigma \subset \eta(\mathrm{Lk}_\uparrow(x, X))$ if and only if $h = \delta(\square_\rho) \subset \overline{Q}_L$, or equivalently, $\sigma \subset \mathrm{Lk}(x, \overline{Q}_L)$.

Suppose that $\chi_e(0) = x$. Then $\tilde{\imath}\chi_e(0) = \delta\chi_\rho(0) = 0$; by definition, $\chi_\rho(0) = 0$, hence $\delta(0) = 0$ and δ acts by identity. Conversely, if $\delta = \mathrm{id}$, then $\delta(0) = 0$, hence by the above $\tilde{\imath}\chi_e(0) = 0$. We claim that x is the unique vertex[6] in e such that $\tilde{\imath}(x) = 0$, hence $\chi_e(0) = x$. Suppose that $x' \in e$ is a vertex such that $\tilde{\imath}(x') = 0$, hence a lift of v_0. X is a normal cover of Q_L, hence let $g \in \pi_1(Q_L) \cong G_L$ map x to x'; since $\tilde{\imath}$ is i_*-equivariant, $g \in \ker i_*$. However, since x, x' are vertices of e, g has word length at most 2. Clearly $i_*(G_L)$ is a subpresentation of $\pi_1(\mathbb{R}^N/\mathbb{Z}^N) \cong \mathbb{Z}^N$, hence g must be an element of length at most 2 of the commutator subgroup C of the free group F_N; however, the only elements of C with length less than 4 are trivial. Hence $g = 1$ and $x = x'$.

An analogous argument shows that $\eta(\mathrm{Lk}_\downarrow(x, X)) = \mathrm{Lk}(0, -\overline{Q}_L)$, where $-\overline{Q}_L$ denotes the reflection of $\overline{Q}_L$ through 0. We proceed as above, but choose χ_e such that ξ is the antipodal map about the barycenter of $\square^m$. Then (19) and (20) imply that $f\chi_e$ is maximum at $0 \in \square^m$ and hence $f|_e$ is maximum at x if and only if $\chi_e(0) = x$. We claim that $\chi_e(0) = x$ if and only if $\delta\chi_\rho\xi = -\chi_\rho$. We note that $\xi(0) = \mathbb{1} := (1, \ldots, 1) \in \square^m$, and that by definition, $\chi_\rho(\mathbb{1}) = \chi_\rho\xi(0) = \mathbb{1}_\rho := (e_{v_{i_1}}, \ldots, e_{v_{i_m}})$ for $\{v_{i_j}\}$ the vertices of ρ. As above, if $\chi_e(0) = x$, then

[6] It suffices to consider vertices: $\tilde{\imath}$ is cellular.

$\tilde{\imath}\chi_e(0) = \delta\chi_\rho\xi(0) = 0$ and $\delta(\mathbb{1}_\rho) = 0$, hence δ is the integer coordinate translation $-\mathbb{1}_\rho$. Since χ_ρ is a partial affine map, the implication follows. Conversely, if $\delta\chi_\rho\xi = -\chi_\rho$, then $\delta = -\mathbb{1}_\rho$ and $\tilde{\imath}\chi_e(0) = 0$, hence $\chi_e(0) = x$. Following the previous argument, this suffices to prove $\eta(\mathrm{Lk}_\downarrow(x, X)) = \mathrm{Lk}(0, -\tilde{Q}_L)$. In particular, $\mathrm{Lk}(0, \overline{Q}_L) \cong \mathrm{Lk}(0, -\overline{Q}_L) \cong L$, showing (iv). □

Remark. Viewing $\mathrm{Lk}_\uparrow(x, X)$ as $\mathrm{Lk}(0, \overline{Q}_L)$ and $\mathrm{Lk}_\downarrow(x, X)$ as $\mathrm{Lk}(0, -\overline{Q}_L)$, the reflection $\overline{Q}_L \to -\overline{Q}_L$ induces a natural simplicial isomorphism $\tilde{r}_x : \mathrm{Lk}_\uparrow(x, X) \to \mathrm{Lk}_\downarrow(x, X)$. Hence, by fixing the vertices of $\mathrm{Lk}_\downarrow(x, X)$, mapping the vertices of $\mathrm{Lk}_\uparrow(x, X)$ into $\mathrm{Lk}_\downarrow(x, X)$ by the above, and extending simplicially, we construct a retraction[7]

$$r_x := \mathrm{Lk}(x, X) \to \mathrm{Lk}_\downarrow(x, X). \tag{21}$$

[7]This map corresponds with the canonical 2-to-1 map $\pi : S(L) \to L$, as stated in Lemma 5.8. It therefore follows that it acts by isometries on each simplex, thus is distance non-increasing (*c.f.* Proposition 5.10(ii)).

5.4 The Main Result

The results in Sections 3, 4 and 5 suffice to prove one direction of implications for our main result, which we state below. The remainder of the essay will be devoted to proving the converse statements.

Main Theorem 5.15. *Let L be a finite flag complex, $G = G_L$ the associated right-angled Artin group, and $\phi : G \to \mathbb{Z}$ the homomorphism with kernel $H = H_L$ as stated in Section 5. Let R be the ring of coefficients with $0 \neq 1$. Then:*

(i) $H \in \mathrm{FP}_{n+1}(R)$ if and only if L is homologically n-connected.

(ii) $H \in \mathrm{FP}(R)$ if and only if L is acyclic.

(iii) H is finitely presented if and only if L is simply connected.

Proof. Theorem 5.14(iv) states that every $\uparrow$-link and $\downarrow$-link is isomorphic to L. Hence if L is homologically n-connected, then Theorem 4.2(i) implies that $H \in \mathrm{FH}_{n+1}(R)$, which by Lemma 3.8(i) gives the ($\Leftarrow$) direction of (i). Similarly, the ($\Leftarrow$) direction of (ii) follows from Theorem 4.2(ii) and Lemma 3.8(ii). Finally, Theorem 4.2(iii) shows (iii)($\Leftarrow$). $\square$

6 Sheets

Define L, Q_L, and $\overline{Q}_L$ as in Section 5, and let $p : X \to Q_L$ denote the universal cover and $q : \overline{Q}_L \to Q_L$ the restriction of the appropriate covering map. Recall that each simplex σ in L defines a unique cell $\square_\sigma$ in $\overline{Q}_L$, which descends to a distinct cell (torus) $q(\square_\sigma)$ in Q_L. Let $T_\sigma := q(\square_\sigma)$ and note that Q_L is the union of such tori.

We now introduce *sheets* and enumerate some of their properties. Sheets are particular subsets that we will use to deconstruct the structure of X, and thereby determine the homology and homotopy type of (sub)level sets in Sections **??** and **??**.

Definition 6.1. A **sheet** is a connected component of the p-preimage in X of the cell $T_\sigma \subset Q_L$ for some simplex σ in L.

Lemma 6.2. *Let $p : (\tilde{Y}, \tilde{y}_0) \to (Y, y_0)$ be the universal cover of a pointed space Y, and let $\iota : (U, y_0) \hookrightarrow (Y, y_0)$ be the inclusion map for a path-connected subspace $U \subset Y$ containing y_0, with induced map ι_*. Suppose that $(\overline{U}, \tilde{y}_0)$ is a path-component of $p^{-1}((U, y_0))$. Then the restriction $\overline{p} : (\overline{U}, \tilde{y}_0) \to (U, y_0)$ is a normal cover such that*

$$\overline{p}_*(\pi_1(\overline{U}, \tilde{y}_0)) = \ker \iota_*.$$

Proof. We verify that $\overline{p}$ is in fact a covering map. Since $\overline{p}$ is a restriction of p, it is clear that it evenly covers U; we need only check that it is surjective. This follows from the path-connectedness of $\overline{U}, U$: for any point $x \in U$, any path from y_0 to x lifts to a path from $\tilde{y}_0$ to a lift $\tilde{x}$; the lifted path must be contained in $\overline{U}$, hence so is $\tilde{x}$.

Let $\bar{\iota}$ denote the inclusion $\overline{U} \hookrightarrow \tilde{Y}$. Then we have the following diagram of induced maps:

$$\begin{tikzcd} \pi_1(\overline{U}, \tilde{y}_0) \arrow[r, "\bar{\iota}_*"] \arrow[d, hook, "\overline{p}_*"'] & \pi_1(\tilde{Y}, \tilde{y}_0) \arrow[d, hook, "p_*"] \\ \pi_1(U, y_0) \arrow[r, "\iota_*"'] & \pi_1(Y, y_0) \end{tikzcd}$$

In particular, $\iota_* \overline{p}_*$ factors through $\pi_1(\tilde{Y}, \tilde{y}_0) \cong 1$, hence is the trivial map: $\operatorname{im} \overline{p}_* \subset \ker \iota_*$. Conversely, suppose $\gamma : I \to U$ is a loop with endpoints y_0 in the equivalence class $[\gamma] \in \ker \iota_*$. Then γ is contractible in Y, hence by homotopy lifting, lifts to a (contractible) loop $\tilde{\gamma}$ at $\tilde{y}_0 \in \tilde{Y}$. Instead, let $\overline{\gamma}$ be the lift of γ to $\overline{U}$ at $\tilde{y}_0$. Then note that $\tilde{\gamma}$ and $\overline{\iota\gamma}$ are both lifts of $\iota\overline{p\gamma}$ at $\tilde{y}_0$, hence are identical. Hence $\overline{\gamma}$ defines an element $[\overline{\gamma}] \in \pi_1(\overline{U}, \tilde{y}_0)$ which descends to $[\gamma] \in \ker \iota_*$. $\overline{p}_*(\pi_1(\overline{U}, \tilde{y}_0)) = \ker \iota_*$ as desired. $\square$

Remark. We note that since $\overline{p}$ is a covering map, $\overline{p}_*$ is injective. Hence $\overline{p}$ is a universal cover (or equivalently, $\overline{U}$ is simply connected) if and only if ι_* is injective.

Proposition 6.3. *Let K be a sheet in the p-preimage of the m-torus T_σ. Then the restriction $p : K \to T_\sigma$ is a universal cover.*

Proof. By Lemma 6.2, it suffices to show that $\iota : T_\sigma \hookrightarrow Q_L$ induces an injective homomorphism in π_1. By considering the 2-skeleton of T_σ as a subcomplex of that of Q_L, it is clear that ι_* maps $\pi_1(T_\sigma, v_0)$ to a subpresentation of $\pi_1(Q_L, v_0) \cong G_L$. In particular, since the vertices of σ are pairwise adjacent, the (sub)presentation of $\iota_*(\pi_1(T_\sigma, v_0))$ contains the full set of relators $[v_i, v_j]$ in G_L for v_i, v_j vertices of σ. Hence $\iota_*(\pi_1(T_\sigma, v_0)) \cong \pi_1(T_\sigma, v_0)$ and ι_* is injective. □

Corollary 6.4. *Every sheet is contractible.*

Proof. Let K be a sheet corresponding to the m-torus T_σ. Then by the Proposition, $K \cong \mathbb{R}^m$, hence K is contractible. □

6.1 Cones and cone neighborhoods

Before proceeding, we will need a precise notion of the *cone* of a space; in addition, we take this opportunity to develop a useful picture of the local structure of cubical complexes.

Definition 6.5. For any space Y, let

$$CY := [0, \infty) \times Y / \{0\} \times Y$$

be the **cone over** Y, with a **cone point** the equivalence class of $(0, y)$. Denote the class of (t, y) as ty and that of $(0, y)$ as 0. If Y is a metric space with metric d_Y, then CY may be endowed with a metric d defined such that, for any $x = ty, x' = t'y'$ in CY,

(i) $d(x, x') = t$ if $x' = 0$, and

(ii) for $x, x' \neq 0$, the following equality holds:

$$d(x, x')^2 = t^2 + t'^2 - 2tt' \cos(d_\pi(y, y'))$$

where $d_\pi(y, y') = \min\{\pi, d_Y(y, y')\}$.

One can verify that d defined above is in fact a metric . Moreover, the reader may note that condition (ii) is in fact the Law of Cosines for $d_\pi(y, y')$ viewed as the angle between x, x' about the cone point.

We also have the following construction. Given a map $\varphi : Y \to Z$, then the map

$$\mathrm{id} \times \varphi : ([0, \infty) \times Y, \{0\} \times Y) \to ([0, \infty) \times Z, \{0\} \times Z)$$

is a map of pairs, hence descends to a map $CY \to CZ$. Hence:

Definition 6.6. For any map $\varphi : Y \to Z$, let $C\varphi : CY \to CZ$ denote the **cone of** φ, defined to be the map induced on CY by the map of pairs $\mathrm{id} \times \varphi$.

We have the following local property of cubical complexes:

Proposition 6.7. *Let Q be a cubical complex and let $x \in Q$. Then there exists some $\delta > 0$ such that $B(x, \delta)$ is naturally isometric to the open ball of radius δ about the cone point in $C(\mathrm{Lk}(x, Q))$.*

Proof. We refer to , from which it suffices to prove that the quantity $\epsilon(x) > 0$, where

$$\epsilon(x) := \inf\{\epsilon(x, C) : C \text{ is a cube containing } x\}$$

and $\epsilon(x, C) := \inf\{d_C(x, F) : F \text{ is a face of } C \text{ s.t. } x \notin F\}$ (assume $\epsilon(x, \{x\}) = \infty$). If x is a vertex, then let $\epsilon' = 1$. If x is not a vertex, then let C be the unique (non-vertex) cube containing x in its interior[8] and let $\epsilon' = \epsilon(x, C) > 0$. Then it follows that for any cube C' such that $x \in C'$, $\epsilon' \leq \epsilon(x, C')$. □

Remark. Explicitly, the isometry above maps points $y \in B(x, \delta)$ to elements $tu \in C(\mathrm{Lk}(x, Q))$, where $t = d(y, x)$ and u is the initial segment issuing from x toward y. One can verify that u is unique for $d(y, x) < \epsilon(x)$, hence the map is well defined.

6.1.1 Cone neighborhoods of sheets

Noting that any union of sheets is a cubical subcomplex of X, we may define the following:

Definition 6.8. Given a vertex $v \in X$ and $A \subset X$ a union of sheets, let $\mathrm{St}(v, A)$ be a *closed*[9] δ-neighborhood of 0 in $C(\mathrm{Lk}(v, A))$, viewed as a δ-neighborhood of v in A, for sufficiently small $\delta > 0$. Define $\mathrm{St}_\downarrow(x, A)$ as the isometrically embedded closed δ-neighborhood of 0 in $C(\mathrm{Lk}_\downarrow(v, A)) \subset C(\mathrm{Lk}(v, A))$.

Since Q_L is covered by the tori T_σ, it follows that X is covered by sheets: for any point $x \in X$, x lies in the interior of a unique cell e that descends to some T_σ. Hence for a vertex $v \in X$, the neighborhoods[10] $\mathrm{St}(v, X)$ and $\mathrm{St}_\downarrow(v, X)$ are defined as above. Consider the cone Cr_v of the natural retraction $r_v : \mathrm{Lk}(v, X) \to \mathrm{Lk}_\downarrow(v, X)$ introducted at the end of Section 5. Then observe that by the above definitions, the restriction of Cr_v to the closed neighborhood $\mathrm{St}(v, X)$ defines the retraction of neighborhoods $\mathrm{St}(v, X) \to \mathrm{St}_\downarrow(v, X)$, which we will also denote by r_v.

6.2 Properties of sheets

We continue to characterize sheets.

Proposition 6.9. *We have the following properties of sheets:*

(i) X is covered by sheets.

(ii) Suppose $J \subset \mathbb{R}$ is closed and connected. Then for any sheet $K \subset X$, $K_J = K \cap X_J$ is contractible. All $\uparrow$-links and $\downarrow$-links of the restriction are single simplices.

[8] Here we refer to interior in the cellular sense, *i.e.* the open cell corresponding to the cube in question.

[9] *E.g.*, given the isometry in Proposition 6.7, choose a closed δ'-neighborhood contained within $B(x, \delta)$.

[10] Technically, $\mathrm{St}_\downarrow(v, X)$ is a subspace of the δ-neighborhood $\mathrm{St}(v, X) \subset X$, or, fixing δ for $\mathrm{St}(v, X)$, the δ-neighborhood of x in a subcomplex $U \subset X$ such that $\mathrm{Lk}_\downarrow(x, X) = \mathrm{Lk}(x, U)$.

(iii) If $A \subset X$ is a union of sheets and $v \in A$ is a vertex, then the restriction of r_v induces retractions $\mathrm{Lk}(v, A) \to \mathrm{Lk}_{\downarrow}(v, A)$ *and hence* $\mathrm{St}(v, A) \to \mathrm{St}_{\downarrow}(v, A)$*. Moreover,* $r_v^{-1}(\mathrm{Lk}_{\downarrow}(v, A)) = \mathrm{Lk}(v, A)$.

(iv) The intersection of any collection of sheets is either empty, a vertex, or a sheet.

We will need the following Lemma, whose proof we defer until a later section
:

Definition 6.10. A subcomplex $M \subset N$ is **full** if for every simplex $\sigma \subset N$ with vertices in M, $\sigma \subset M$.

Remark. Every simplex is full.

Lemma 6.11. *Suppose that K is the union of a collection of sheets containing a vertex v. Then if* $\mathrm{Lk}_{\downarrow}(v, K) \subset \mathrm{Lk}_{\downarrow}(v, X)$ *is full, then K is convex in X.* $\square$

Proof of Proposition 6.9. We have shown (i) above. To show (ii), we first calculate explicitly the restriction of f to a sheet $K \subset X$. Let K descend to the m-torus T_μ. Since the N-simplex Δ and its faces comprise a flag complex, by Proposition 6.3 the universal cover $\mathbb{R}^m$ of $T_\mu \subset T^N$ embeds as a sheet in $\mathbb{R}^N$, viewed as the universal cover of the cube complex $Q_\Delta = T^N$. Thus we have the following diagram:

$$\begin{tikzcd}
 & & K \arrow[r, hook] \arrow[d, "\overline{p}"] \arrow[ddll, "\xi"'] & X \arrow[d, "p"] \\
 & & T_\mu \arrow[r, hook] \arrow[dl, "\mathrm{id}"] & Q_L \arrow[ddll, hook] \\
\mathbb{R}^m \arrow[r, "\overline{q}"'] \arrow[d, hook] & T_\mu \arrow[d, hook] & & \\
\mathbb{R}^N \arrow[r, "q"'] & T^N & &
\end{tikzcd}$$

where ξ is the double lift of $\mathrm{id} : T_\mu \to T_\mu$, which is $\pi_1(T_\mu) \cong \mathbb{Z}^m$-equivariant by Lemma 5.13. Composing with the right hand side of (18) in the preceding section, we have the (familiar) diagram

$$\begin{tikzcd}
K \arrow[r, hook, "s"] \arrow[ddd, "\xi"'] & X \arrow[r, "f"] \arrow[d, "p"] & \mathbb{R} \arrow[d] \\
 & Q_L \arrow[r, "l"] \arrow[d, hook] & S^1 \arrow[d, "\mathrm{id}"] \\
 & \mathbb{R}^N/\mathbb{Z}^N \arrow[r, "\overline{\epsilon}"] & S^1 \\
\mathbb{R}^m \arrow[r, hook, "t"] & \mathbb{R}^N \arrow[r, "\epsilon"'] \arrow[u, "q"'] & \mathbb{R} \arrow[u]
\end{tikzcd}$$

where s and t are the appropriate inclusions and the maps $\mathbb{R} \to S^1$ are universal covers. Hence fs and $\epsilon t \xi$ are lifts of the same map $K \to S^1$, hence differ only by postcomposition by an translation τ by $z \in \mathbb{Z}$, and in particular,

$$f|_K = \tau\tilde{\epsilon}\xi$$

where $\tilde{\epsilon}$ is the restriction of ϵ to $\mathbb{R}^m$. Regarding K as homeomorphic to $\mathbb{R}^m$, we claim that ξ is an affine homeomorphism: since ξ lifts identity, the restriction to any cell of K (a lift of a cell in T_μ) is a partial affine homeomorphism; by $\mathbb{Z}^m$-equivariance, the claim follows. Hence up to an affine change of coordinates on K, $f|_K$ is the following affine map:

$$\mathbb{R}^m \to \mathbb{R} : (x_i) \mapsto \sum_i x_i + z$$

Hence for $J \subset \mathbb{R}$ closed and connected (hence convex), $K_J = K \cap X_J$ is convex, hence contractible. Finally, K is the universal cover of $Q_\mu = T_\mu$, hence by Theoreom 5.14, the $\uparrow$-links and $\downarrow$-links in K are isomorphic to μ, completing the proof of (ii).

To prove (iii), it therefore suffices to show that the isomorphism

$$\tilde{r}_v : \mathrm{Lk}_\uparrow(v, X) \to \mathrm{Lk}_\downarrow(v, X)$$

induced by the reflection $\overline{Q}_L \to -\overline{Q}_L$ (as described at the end of Section 5) restricts to the corresponding isomorphism

$$\tilde{r}_{v,K} : \mathrm{Lk}_\uparrow(v, K) \to \mathrm{Lk}_\downarrow(v, K)$$

for any vertex v and sheet K, where, if K corresponds to a simplex $\mu \subset L$, $\tilde{r}_{v,K}$ is induced by the reflection $\overline{Q}_\mu \to -\overline{Q}_\mu$ for $\overline{Q}_\mu$ as above. Viewing $\overline{Q}_\mu$ as $\square_\mu \subset \overline{Q}_L$, the claim follows immediately from the arguments in Theorem 5.14(iv).

Finally, let $\mathcal{S}$ be an arbitrary collection of sheets. If the subcomplex $S = \bigcap \mathcal{S}$ is not empty, then $\mathcal{S}$ contains at most one distinct sheet for each simplex in L, a finite simplicial complex. Hence without loss of generality, we may assume $\mathcal{S}$ to be finite; else, consider a finite subcollection whose intersection covers S. We prove by induction on the cardinality of $\mathcal{S}$ that the intersection of any finite collection of sheets is empty, a vertex, or a sheet, proving (iv).

It suffices to show the above for the intersection of two sheets. Let K, L be sheets in X whose intersection is nonempty. Then let σ, resp. ρ denote the simplices in L corresponding to K, resp. L and let $T_\mu = T_\sigma \cap T_\rho$ for $\mu = \sigma \cap \rho$, where $T_\mu = \{v_0\}$ if $\mu = \varnothing$. Hence we have that $K \cap L \subset p^{-1}(T_\mu) = p^{-1}(T_\sigma) \cap p^{-1}(T_\rho)$ and it suffices to show that $S = K \cap L$ is a connected component of $p^{-1}(T_\mu)$. Let S' be a path-component of $p^{-1}(T_\mu)$ that meets S. S' is connected, hence either $S' \cap K = \varnothing$ or $S' \subset K$, and likewise with L, hence $S' \subset S$. Conversely, by (ii) and Lemma 6.11, K, L are convex; since X is a CAT(0) space, it is uniquely geodesic, hence convex subsets have path-connected intersection. Thus S is path-connected, hence $S \subset S'$ and $S = S'$. $\square$

Corollary 6.12. *For any collection $\mathcal{S}$ of sheets containing a common vertex v, $\bigcup \mathcal{S}$ is contractible. Let $J \subset \mathbb{R}$ be closed and connected. If $v \in X_J$, then $X_J \cap \bigcup \mathcal{S}$ is contractible.*

Proof. Since L is a finite complex, there exist finitely many distinct sheets containing v. Then without loss of generality we may assume $\mathcal{S}$ is finite; else, choose a finite subcollection that covers $\bigcup \mathcal{S}$. By Proposition 6.9, the intersection of any subcollection of $\mathcal{S}$ is a sheet, hence contractible; by Corollary 5.12, $\bigcup \mathcal{S}$ is contractible as well.

Since Proposition 6.9 states that for any sheet S, S_J is contractible, the above argument also shows that $X_J \cap \bigcup \mathcal{S}$ is contractible. $\square$

7 Homology of (sub)level sets

Let L be a finite flag complex with associated right-angled Artin group $G = G_L$. Define $Q_L, X, \phi : G \to \mathbb{Z}$, and $f : X \to \mathbb{R}$ as in Section 5 and let $H = H_L = \ker \phi$. Finally, let J be a non-empty, closed, and connected subset of $\mathbb{R}$. We wish to compute the homology groups for the set X_J.

Theorem 7.1. *Let A be an arbitrary union of sheets and vertices in X and denote $A_J := A \cap X_J$. Then*

$$H_*(A, A_J) \cong \bigoplus_{v \notin A_J} H_*(\mathrm{St}_\downarrow(v, A), \mathrm{Lk}_\downarrow(v, A))$$

where the sum ranges over vertices in $A \setminus A_J$.

Remark. Since $H_*(\varnothing, \varnothing) = 0$, we could equivalently sum over all vertices in $X \setminus X_J$.

Proof. We will explicity construct an isomorphism

$$\Psi_A : H_*(A, A_J) \to \bigoplus H_*(\mathrm{St}_\downarrow(v, A), \mathrm{Lk}_\downarrow(v, A))$$

as follows. For each $v \in A \setminus A_J$, note that we may choose $\mathrm{St}(v, A)$ such that $\mathrm{St}(v, A) \cap A_J = \varnothing$. Moreover, $\mathrm{St}(v, A)$ is the closed δ-neighborhood of v in A for some $\delta > 0$. Hence by the isometry given in Proposition 6.7, $\mathrm{Int}(\mathrm{St}(v, A))$ is the open δ-neighborhood of v and $\mathrm{St}(v, A) \setminus \mathrm{Int}(\mathrm{St}(v, A)) \cong \mathrm{Lk}(v, A)$ by restricting the same isometry. $A \setminus \mathrm{Int}\,\mathrm{St}(v, A)$ and $\mathrm{St}(v, A)$ cover A, hence by excision[11] the inclusion

$$(\mathrm{St}(v, A), \mathrm{Lk}(v, A)) \hookrightarrow (A, A \setminus \mathrm{Int}\,\mathrm{St}(v, A))$$

induces the isomorphism

$$H_*(\mathrm{St}(v, A), \mathrm{Lk}(v, A)) \cong H_*(A, A \setminus \mathrm{Int}\,\mathrm{St}(v, A)).$$

Let $\iota : (A, A_J) \hookrightarrow (A, A \setminus \mathrm{Int}\,\mathrm{St}(v, A))$ and define the coordinate of Ψ_A corresponding to v by the composition

$$\begin{aligned} H_*(A, A_J) \xrightarrow{\iota_*} H_*(A, A \setminus \mathrm{Int}\,\mathrm{St}(v, A)) &\cong H_*(\mathrm{St}(v, A), \mathrm{Lk}(v, A)) \\ &\xrightarrow{r_{v*}} H_*(\mathrm{St}_\downarrow(v, A), \mathrm{Lk}_\downarrow(v, A)). \end{aligned} \tag{22}$$

To show that Ψ_A is a homomorphism, we need only verify that it is pointwise finitely supported, *i.e.*, for every element $x \in H_*(A, A_J)$, $\Psi_A(x)$ has cofinitely many vanishing components. But x may be represented as a relative cycle, *i.e.* a map with (compact) image contained within a finite subcomplex of A: the coordinates of $\Psi_A(x)$ corresponding to vertices outside of this subcomplex vanish. Finally, Ψ_A is *natural*: given $A \subset A'$ and the corresponding inclusions

[11] We would actually like the *interiors* of $A \setminus \mathrm{Int}\,\mathrm{St}(v, A)$ and $\mathrm{St}(v, A)$ to cover A; however, it is clear from the definition of $\mathrm{St}(v, A)$ that we can choose an open neighborhood N for which $\mathrm{St}(v, A)$ is a deformation retract, and for which the intersection $U \cap (A \setminus \mathrm{Int}\,\mathrm{St}(v, A))$ has the homotopy type of $\mathrm{Lk}(v, A)$.

$i : (A, A_J) \hookrightarrow (A', A'_J)$ and $j : (\mathrm{St}_\downarrow(v, A), \mathrm{Lk}_\downarrow(v, A)) \hookrightarrow (\mathrm{St}_\downarrow(v, A'), \mathrm{Lk}_\downarrow(v, A))$, the following diagram commutes:

$$\begin{array}{ccc} H_*(A, A_J) & \xrightarrow{\Psi_A} & \bigoplus H_*(\mathrm{St}_\downarrow(v, A), \mathrm{Lk}_\downarrow(v, A)) \\ \big\downarrow{\scriptstyle i_*} & & \big\downarrow{\scriptstyle j_*} \\ H_*(A', A'_J) & \xrightarrow{\Psi_{A'}} & \bigoplus H_*(\mathrm{St}_\downarrow(v, A'), \mathrm{Lk}_\downarrow(v, A')) \end{array} \tag{23}$$

To show this claim, we take the above sums to range over all vertices in $X \setminus X_J$ and note that for each component of Ψ_A, the factors in the composition (22) are induced by continuous maps. Hence by the functorality of H_*, the square corresponding to each component commutes.

It remains to verify that Ψ_A is an isomorphism. We claim that it suffices to consider A a finite union of sheets. Let $\mathcal{A}$ be the collection of all unions of finite subcollections of sheets in A, which is a direct system under inclusion. Then $\mathcal{A}$ contains every sheet and vertex in A, hence covers A. Moreover, any compact subset of A lies in a finite subcomplex, hence within a finite union of sheets and vertices in A. Similarly, $\mathcal{A}$ indexes direct systems $\{K_J = K \cap X_J\}_{K \in \mathcal{A}}, \{\mathrm{St}_\downarrow(v, K)\}_{K \in \mathcal{A}}$, and $\{\mathrm{Lk}_\downarrow(v, K)\}_{K \in \mathcal{A}}$, that cover A_J, $\mathrm{St}_\downarrow(v, A)$, and $\mathrm{Lk}_\downarrow(v, A)$ respectively and, by inclusion into A, fulfill the same compactness property. Hence we have that

$$\varinjlim H_*(K, K_J) \cong H_*(A, A_J) \quad \text{and}$$
$$\varinjlim H_*(\mathrm{St}_\downarrow(v, K), \mathrm{Lk}_\downarrow(v, K)) \cong H_*(\mathrm{St}_\downarrow(v, A), \mathrm{Lk}_\downarrow(v, A)).$$

Moreover, since direct sums commute with direct limits (both are in fact colimits), we have

$$\varinjlim \bigoplus H_*(\mathrm{St}_\downarrow(v, K), \mathrm{Lk}_\downarrow(v, K)) \cong \bigoplus H_*(\mathrm{St}_\downarrow(v, A), \mathrm{Lk}_\downarrow(v, A)).$$

Thus given isomorphisms $\{\Psi_K\}_{K \in \mathcal{A}}$, by Lemma 2.14 we induce the desired isomorphism over A.

We assume that A is the union of k many sheets and vertices and proceed by induction on k. If $A = \varnothing$ or a union of vertices in A_J, then $H_*(A, A_J)$ and $H_*(\mathrm{St}_\downarrow(v, A), \mathrm{Lk}_\downarrow(v, A))$ (for any $v \in A \setminus A_J$) vanish, hence are trivially isomorphic. Similarly, if A is a single sheet (*e.g.* corresponding to a simplex $\sigma \subset L$), then by Proposition 6.9(ii) A, A_J, $\mathrm{St}_\downarrow(v, A)$ and $\mathrm{Lk}_\downarrow(v, A) \cong \sigma$ are contractible, hence all groups vanish. Finally, if A is composed of a collection of m vertices $V \in A \setminus A_J$, then $A_J = \varnothing = \mathrm{Lk}_\downarrow(v, A)$ and the non-vanishing groups are concentrated in dimension 0. In particular, if R is the coefficient group, then $H_*(V, \varnothing) \cong R^{\oplus m} \cong \bigoplus H_*(\mathrm{St}(v, V), \varnothing)$, where it is clear that Ψ_A defines an isomorphism.

Assume Ψ_A is an isomorphism whenever A is the union of at most k sheets and vertices and let A be the union of $k+1$ sheets and vertices such that A contains at least one sheet S. Then let $A = A' \cup S$, where A' is the union of at most k sheets and vertices. Proposition 6.9 implies that $A'' = A' \cap S$ is the union of at most k sheets or vertices, hence $\Psi_{A''}$, $\Psi_{A'}$, and Ψ_S are isomorphisms. Since

S is a sheet, we note that $H_*(S, S_J)$ and $H_*(\mathrm{St}_\downarrow(v, S), \mathrm{Lk}_\downarrow(v, S))$ vanish. Finally, we note that Ψ_A is composed of homomorphisms induced by continuous maps, hence it is natural with respect to the Mayer-Vietoris boundary map. Since the other Mayer-Vietoris maps reduce to maps induced by inclusions, the relative Mayer-Vietoris theorem and the naturality condition in (23) gives the following diagram:

$$\begin{array}{ccc}
H_i(A'', A''_J) & \xrightarrow{\Psi_{A''}} & \bigoplus H_i(\mathrm{St}_\downarrow(v, A''), \mathrm{Lk}_\downarrow(v, A'')) \\
\downarrow & & \downarrow \\
H_i(A', A'_J) & \xrightarrow{\Psi_{A'}} & \bigoplus H_i(\mathrm{St}_\downarrow(v, A'), \mathrm{Lk}_\downarrow(v, A')) \\
\downarrow & & \downarrow \\
H_i(A, A_J) & \xrightarrow{\Psi_{A}} & \bigoplus H_i(\mathrm{St}_\downarrow(v, A), \mathrm{Lk}_\downarrow(v, A)) \\
\downarrow & & \downarrow \\
H_{i-1}(A'', A''_J) & \xrightarrow{\Psi_{A''}} & \bigoplus H_{i-1}(\mathrm{St}_\downarrow(v, A''), \mathrm{Lk}_\downarrow(v, A'')) \\
\downarrow & & \downarrow \\
H_{i-1}(A', A'_J) & \xrightarrow{\Psi_{A'}} & \bigoplus H_{i-1}(\mathrm{St}_\downarrow(v, A'), \mathrm{Lk}_\downarrow(v, A'))
\end{array}$$

where the vertical arrows are the usual maps and the above holds for any i. Hence Ψ_A is an isomorphism by the 5-lemma. □

Corollary 7.2. *Let R be the ring of coefficients and $H = H_L = \ker \phi$. Then we have the following isomorphisms of RH-modules:*

(i) $\tilde{H}_*(X_{(-\infty,t]}) \cong \bigoplus_{v \in X_{(t,\infty)}} \tilde{H}_*(L)$.

(ii) $\tilde{H}_*(X_t) \cong \bigoplus_{v \notin X_t} \tilde{H}_*(L)$.

where the sums range over vertices v.

Proof. We first recall that X is the union of sheets. From the definition of $\mathrm{St}_\downarrow(v, X)$, it is clear that $\mathrm{Lk}_\downarrow(v, X)$ is a neighborhood retract in $\mathrm{St}_\downarrow(v, X)$. Similarly, from Lemma 2.7 and the discreteness of $f(X_0)$, for any $J \subset \mathbb{R}$ closed and connected, X_J is a neighborhood retract in X. Hence $(\mathrm{St}_\downarrow(v, X), \mathrm{Lk}_\downarrow(v, X))$ and (X, X_J) for $J = \{t\}, (-\infty, t]$ are good pairs and thus

$$H_*(\mathrm{St}_\downarrow(v, X), \mathrm{Lk}_\downarrow(v, X)) \cong \tilde{H}_*(\mathrm{St}_\downarrow(v, X)/\mathrm{Lk}_\downarrow(v, X))$$
$$H_*(X, X_J) \cong \tilde{H}_*(X/X_J).$$

We observe that $\mathrm{St}_\downarrow(v, X)/\mathrm{Lk}_\downarrow(v, X) \cong \Sigma\,\mathrm{Lk}_\downarrow(v, X)$, the (unreduced) suspension of $\mathrm{Lk}_\downarrow(v, X)$, hence

$$\tilde{H}_{i+1}(\mathrm{St}_\downarrow(v, X)/\mathrm{Lk}_\downarrow(v, X)) \cong \tilde{H}_{i+1}(\Sigma\,\mathrm{Lk}_\downarrow(v, X)) \cong \tilde{H}_i(\mathrm{Lk}_\downarrow(v, X)).$$

We also note that we have the following (reduced) long exact sequence of pairs:

$$\cdots \to \tilde{H}_{i+1}(X) \to \tilde{H}_{i+1}(X/X_J) \to \tilde{H}_i(X_J) \to \tilde{H}_i(X) \to \cdots$$

Since X is a (contractible) CAT(0) space, $\tilde{H}_*(X) = 0$. Hence the above gives

$$\tilde{H}_{i+1}(X/X_J) \cong \tilde{H}_i(X_J)$$

and Ψ_X is the desired isomorphisms of groups, hence R-modules. We recall from Proposition 4.1 that H acts on (sub)level sets, hence X_J. Hence $\tilde{H}_i(X_J)$ is an RH-module. Similarly, H acts on the complements $X \setminus X_J$ and we may endow the sums

$$\bigoplus_{v \notin X_J} \tilde{H}_*(L)$$

with an H-action by acting on the vertices $v \in X \setminus X_J$. It remains only to check that Ψ_X is H-equivariant. In particular, every element $h \in H$ acts on X by an automorphism that permutes the vertices and their local neighborhoods, which are themselves homeomorphic and whose links are likewise isomorphic. Hence, observing that the homomorphism $\iota_* : H_*(X, X_J) \to H_*(X, X \setminus \operatorname{Int}\operatorname{St}(v, X))$ maps (classes of) relative cycles into the local homology of the vertex v, the claim follows immediately from (22). $\square$

Proof of (i) and (ii) of Theorem 5.15. Let n be the smallest integer such that $\tilde{H}_n(L) \neq 0$. Then by Theorem 7.1, $\tilde{H}_n(X_t)$ is not a finitely generated RH-module, hence $H \notin \mathrm{FP}_{n+1}(R)$ by Corollary 3.9 and the claims are shown. $\square$

8 Homotopy type of (sub)level sets

As before, let L be a finite flag complex with associated right-angled Artin group $G = G_L$. Define $Q_L, X, \phi : G \to \mathbb{Z}$, and $f : X \to \mathbb{R}$ as in Section 5 and let $H = H_L = \ker \phi$. Let J be a non-empty, closed, and connected subset of $\mathbb{R}$. We continue to investigate the homotopy type of sets X_J below, by constructing X from successive unions of sheets.

8.1 Sheets through a vertex

Lemma 8.1. *Let w be a vertex in X and let K be the union of a collection of sheets containing w. Let $J \subset \mathbb{R}$ be closed and connected. Then:*

(i) K is contractible.

(ii) All $\uparrow$-links and $\downarrow$-links of K not at w are contractible. For any vertex v, $\mathrm{Lk}_{\downarrow}(v, K) \cong \mathrm{Lk}_{\uparrow}(v, K)$.

(iii) $K_J = K \cap X_J$ is homotopy equivalent to $\mathrm{Lk}_{\downarrow}(w, K)$ when $w \notin X_J$, else contractible.

Proof. (i) was proven in Corollary 6.12, from which we also recall that there are finitely many distinct sheets containing w. Suppose $v \neq w$ is a vertex in K. Hence let $S \subset K$ be the minimal sheet in K containing both v, w and let $\sigma = \mathrm{Lk}_{\downarrow}(v, S)$. Thus $\mathrm{Lk}_{\downarrow}(v, K)$ is the union of (finitely many) simplices of the form $\mathrm{Lk}_{\downarrow}(v, S')$, where S' is a sheet containing v, w, such that any intersection thereof is a common face containing σ, hence contractible. By Corollary 5.12, this suffices to show that $\mathrm{Lk}_{\downarrow}(v, K)$ is contractible as well. Finally, the arguments in Proposition 6.9(iii) show that the restriction of $\tilde{r}_v$ gives the desired isomorphism $\mathrm{Lk}_{\uparrow}(v, K) \to \mathrm{Lk}_{\downarrow}(v, K)$, hence completing (ii).

For (iii), if $w \in X_J$, then the result follows from 6.12. Suppose $w \notin X_J$, and without loss of generality, assume $f(w) > \sup J$. Choose $\epsilon \in (0, 1)$ such that $f(w) - \sup J > \epsilon$. Since every $\uparrow$-link and $\downarrow$-link not at w is contractible, by Theorem 2.16(iv) the following inclusions are homotopy equivalences:

$$K_J \hookrightarrow K_{(-\infty, f(w)-\epsilon]} \hookleftarrow K_{f(w)-\epsilon}$$

Define closed covers $\{\mathrm{Lk}_{\downarrow}(w, S)\}$, resp. $\{S \cap K_{f(w)-\epsilon}\}$ over $\mathrm{Lk}_{\downarrow}(w, K)$, resp. $K_{f(w)-\epsilon}$, indexed by sheets $S \subset K$, whose elements are clearly contractible and closed under intersections. We demonstrate a bijection

$$\mathrm{Lk}_{\downarrow}(w, S) \mapsto S \cap K_{f(w)-\epsilon}, \tag{24}$$

hence both covers have identical nerves, N, whose geometric realization $|N|$ is the homotopy type of $\mathrm{Lk}_{\downarrow}(w, K)$, resp. $K_{f(w)-\epsilon}$, completing our claim. First observe that the map $S \mapsto \mathrm{Lk}_{\downarrow}(w, S)$ for sheets $S \subset K$ is a bijection: there exists at most one distinct sheet S through w for each simplex $\sigma \subset L$, with $\mathrm{Lk}_{\downarrow}(w, S) \cong \sigma$. Hence (24) is a well defined surjection. Let e be a cube in $K \subset X$ with w as a vertex, and χ_e its characteristic map. Then from our calculation of $f\chi_e$ (19) we observe that $d_e(w, t) < 1$ for any $t \in e \cap K_{f(w)-\epsilon}$.

Hence there is a natural embedding by radial projection $\mathrm{Lk}_{\downarrow}(w, K) \hookrightarrow K_{f(w)-\epsilon}$ (*c.f.* the argument given in Lemma 2.11), thus (24) is an injection, hence a bijection as desired. $\square$

Definition 8.2. Given a cube Q in a cubical complex X, let the **closed star** $\mathrm{St}(Q, X)$ be the subcomplex consisting of the faces of cubes for which Q is also a face.

Lemma 8.3. *Let X be a CAT(0) cubical complex with $Q \subset X$ a cube. Then Q is convex in X and has a metric product structure.*

Proof. For any $x, y \in Q$, let $\gamma \subset Q$ be a geodesic path between x, y with respect to the metric d_Q. For each $z \in \gamma$, d_Q and d, the metric on X, coincide on the neighborhood $B_Q(z, \epsilon(z))$ by . Hence γ is locally geodesic; by it is geodesic as well. Moreover, the above implies that $d = d_Q$ over Q, and hence Q has a metric product structure inherited from $\square^m$. $\square$

Corollary 8.4. *If X is a CAT(0) cubical complex with $Q \subset X$ a cube, then* $\mathrm{St}(Q, X)$ *is convex in X.* $\square$

Lemma 8.5. *Let X be a CAT(0) cubical complex, $Q \subset X$ a cube, and $c \in X$ a vertex. Then the minimal distance from c to Q is obtained at a unique vertex of Q.*

Proof. Suppose $x, y \in Q$ both realize the minimum distance to c and are distinct. Since Q is convex, $[x, y] \subset Q$; hence by the CAT(0) inequality applied to the geodesic triangle xyc, all interior points of $[x, y]$ are closer to c than x, y, a contradiction. Hence it remains only to show that the unique distance-minizing point in Q is a vertex.

Suppose c is a vertex of minimum distance from Q such that there exists a distance minimizing geodesic γ from c to Q such that γ does not terminate on a vertex of Q. Then γ meets Q in the interior of some (unique) face T with an edge e. Let C be the minimal subcomplex containing γ, *i.e.*, comprised of the minimal cubes Q_p containing $p \in \gamma$ for all such p; since γ is an m-string, we note that C is finite. We claim there is a cellular map from C to a 1-simplex which maps γ to an interior point, a contradiction since c is a vertex, which proves the Lemma.

Let $Q_0 \subset C$ be the minimal cube which contains the initial segment of γ, and note that $e \subset T \subset Q_0$. Consider Q_0 with the metric product structure of $e \times F_0$, where F_0 is a codimension-1 face of Q_0, as stated in Lemma 8.3. Since γ is distance minimizing, $\gamma \cap Q_0$ must be perpendicular to T, hence the e-coordinate.

Let Q_i denote the ith cube in C along the path of γ; note that this labelling is well defined by the convexity of cubes Q_i, the minimality of γ, and the fact that X is uniquely geodesic.[12] By the minimality of our choice of c, no $Q_i \neq c$ is a vertex (since γ contains no vertices except for c); moreover, either $Q_i \subset Q_{i+1}$

[12] It may not be unique, however; *e.g.* if $Q_k \subset Q_{k+1} \subset Q_{k+2}$.

or $Q_{i+1} \subset Q_i$ for all i. Let $C_k := \bigcup_{i=0}^k Q_i$. Then we claim that $Q_{k+1} \cap C_k \subset Q_k$. If $Q_k \supset Q_{k+1}$ then the claim is clear, hence suppose $Q_k \subset Q_{k+1}$ and that let Q_l with $l \leq k$ be a cube with a face $F = Q_l \cap Q_{k+1} \neq \varnothing$. Then by the convexity of $\mathrm{St}(F, X)$, between any two points $x, y \in \gamma$ that are interior to Q_l, Q_{k+1} respectively, there exists a geodesic γ' strictly contained in $\mathrm{St}(F, X)$, hence cubes with F as a face. By the uniqueness of γ, $\gamma' \subset \gamma$ and in particular, $Q_k \subset \mathrm{St}(F, X)$, hence $F \subset Q_k$ as desired.

Suppose that C_k has a metric product structure $e \times \bigcup_{i=0}^k F_i$ for codimension-1 faces $F_i \subset Q_i$. Then the e-coordinate of $Q_k \subset Q_{k+1}$ defines a metric product structure $e \times F_{k+1}$ on Q_{k+1}, which agrees with C_k on $Q_{k+1} \cap C_k \subset Q_k$; hence we have the metric product structure $C_{k+1} = e \times \bigcup_{i=0}^{k+1} F_i$. Thus we may iteratively extend the metric product structure from Q_0 to $C = e \times \bigcup F_i$ and note that γ remains perpendicular to the e-coordinate throughout. Mapping the e-coordinate onto a 1-simplex and collapsing the coordinate corresponding to $\bigcup F_i$ to a point gives the desired cellular map: since γ is perpendicular to the e-coordinate, it maps to an interior point of the 1-simplex, a contradiction. □

We reproduce the following important result below; regrettably, we fail to include its proof.

Lemma 8.6. *Let $x \in X$ be a vertex and K_x the union of all sheets through x. Then K_x is a convex subset of X.* □

Proof. □

8.2 Constructing X_J

G_L is finitely generated and acts transitively on the vertices of X, hence X_0 is countable. Choose a base vertex $v \in X$ and let $\{v_i\}_{i \in \mathbb{N}}$ be an ordering of vertices such that $v_1 = v$ and $d(v, v_i) \leq d(v, v_j)$ whenever $i \leq j$. Denote by $K_i = K_{v_i}$ the union of all sheets through v_i. We wish to study the homotopy type of $K_1 \cup K_2 \cup \cdots \cup K_n$ inductively. Hence, fix $n > 1$ and let $w = v_n$ and $K = K_n \cap (K_1 \cup \cdots \cup K_{n-1})$.

Lemma 8.7. *Let $K = K_n \cap (K_1 \cup \cdots \cup K_{n-1})$ as above. Then K is the union of all sheets that contain w and at least one v_j with $j < n$.*

Proof. The union is contained in K by definition. To prove the reverse inclusion, it suffices to show that for any $x \in K$, there exists a sheet $S \subset X$ through $w = v_n$ which contains x and some v_j with $j < n$.

We first consider vertices in K. Let $v \in K$ be a vertex, and in particular, note that $v \in K_n \cap K_i$ for some $i < n$. Hence there exist sheets through v containing v_i, v_n respectively, hence $v_i, v_n \in K_v$, the union of all sheets through v. By Lemma 8.6, K_v is convex, hence contains the geodesic $\gamma := [v_i, v_n]$. Let Q be the minimal cube containing v_n and the end segment of γ. $Q \cap \gamma \subset K_v$, hence by minimality $Q \subset K_v$. Let $S \subset K_v$ be a sheet containing Q and v. We claim that Q contains some vertex v_j for $j < n$, hence S contains v, v_n and v_j $(j < n)$ as required.

Since Q contains a segment of γ, Q contains an interior point of γ. By our ordering of the vertices, $[v_1, v_i]$ is not longer than $[v_1, v_n]$, hence the CAT(0) inequality applied to $v_1 v_i v_n$ implies that the distance from v_1 to any interior point of $[v_i, v_n] = \gamma$ is strictly less than $d(v_1, v_n)$. Thus v_n is not the point of Q nearest to v_1: by Lemma 8.5 there exists a vertex v_j in Q strictly nearer to v_1, *i.e.* such that $j < n$.

Finally, suppose that $x \in K$ is not a vertex. Then let $S' \subset K_n$ be a sheet containing x and v_n and let S' correspond to an m-simplex $\mu \subset L$, and hence $S' \cong \mathbb{R}^m$. Label the vertices of S' by their coordinates in the lattice $\mathbb{Z}^m \subset \mathbb{R}^m$, fixing v_n at the origin. Let a set of vertices $w_i \in S'$ be in *general position with respect to* v_n if the deck transformations $\omega_i : v_n \mapsto w_i$ generate $\pi_1(T_\mu)$, or equivalently, if their coordinates generate $\mathbb{Z}^m$. We note that any finite partition of $\mathbb{Z}^m$ contains a general position subset and every vertex of S' is contained in a sheet through v_n and some v_j for $j < n$, of which there exist finitely many, since finitely many distinct sheets intersect v_n. Hence let S be one such sheet containing a general position vertex subset of S'. S is also the universal cover of some torus T_σ, hence $\pi_1(T_\sigma)$ acts transitively on its vertices; in particular, $\{\omega_i\}_i \subset \pi_1(T_\sigma)$ and hence $S' \subset S$ as desired. □

Lemma 8.8. $\mathrm{Lk}_\downarrow(w, K) \cong \mathrm{Lk}_\uparrow(w, K)$ *is contractible.*

Proof. Let $a \in \mathrm{Lk}(w, X)$ be the point determined by the geodesic $[w, v]$, where $v = v_1$ is the fixed base vertex. Let $b = r_w(a) \in \mathrm{Lk}_\downarrow(w, X)$ for the retraction r_w as defined in Proposition 6.9(iii) and let σ denote the unique (minimal) simplex of $\mathrm{Lk}_\downarrow(w, X)$ containing b in its interior.

Let $\mathscr{T}$ denote the collection of simplices $\tau \subset \mathrm{Lk}_\downarrow(w, X)$ such that for each τ the corresponding sheet through w contains one of the v_i $(i < n)$. Since X is the union of sheets, Lemma 8.7 implies that $\mathrm{Lk}_\downarrow(w, K) = \bigcup \mathscr{T} \subset \mathrm{Lk}_\downarrow(w, X)$. Let $T := \bigcup \mathscr{T}$. We will show that T is exactly $\mathrm{St}'(\sigma, L)$, hence $\mathrm{Lk}_\downarrow(w, X) \cong \mathrm{Lk}_\uparrow(w, X)$ is contractible by Proposition 5.10(i).

First, we prove that every face of σ belongs to $\mathscr{T}$. Let S_0 be the (unique) sheet through w such that $\mathrm{Lk}_\downarrow(w, S_0) = \sigma$, which must exist by Proposition 6.9(ii) and the fact that X is the union of sheets. By Prop. 6.9(iii), $\mathrm{Lk}(w, S_0)$ contains a. Let Q be the minimal cube containing w such that $\mathrm{Lk}(w, Q) \subset \mathrm{Lk}(w, X)$ contains a; by minimality, $Q \subset S_0$ and hence $r_w(\mathrm{Lk}(w, Q)) = \sigma$. We claim that every vertex in Q that is distance 1 from w is a vertex v_i for $i < n$, hence every face of σ belongs to $\mathscr{T}$.

Let $v' \in Q$ be distance 1 from w. Since a is interior to $\mathrm{Lk}(w, Q)$ by minimality, the angle at w between $[w, v]$ and $[w, v']$ is strictly less than $\pi/2$. Given Proposition 6.7, choose a δ-neighborhood U of w isometric to the corresponding neighborhood of $C\,\mathrm{Lk}(w, X)$ and points $s, t \in U$ that lie on the interiors of $[w, v]$ and $[w, v']$ respectively. Then by Definition 6.5(ii), $d(s, t) < d(w, s)$, hence $d(t, v)$ is strictly less than $d(w, v)$. Hence w is not the nearest point of $[w, v']$ to v and Lemma 8.5 implies that v' is strictly closer to v than w, hence $v' = v_i$ for some $i < n$.

Finally, we show that $\mathscr{T} = \{\tau \text{ a simplex of } L : \tau \cap \sigma \neq \varnothing\}$. Suppose $\tau \in \mathscr{T}$, with the corresponding sheet $S \subset K$ and $v_i \in S$ for some $i < n$. Thus in the geodesic triangle wvv_i, $d(v,w) \geq d(v,v_i)$, hence since X is CAT(0) the angle at w is strictly less than $\pi/2$. Hence $\mathrm{Lk}(w,S)$ contains a point strictly closer than $\pi/2$ from a in $\mathrm{Lk}(w,X)$. Since r_w is distance non-increasing, $\tau = \mathrm{Lk}_\downarrow(w,S) = r_w(\mathrm{Lk}(w,S))$ contains a point strictly within distance $\pi/2$ from $b = r_w(a) \in \sigma$. Thus it must intersect σ by Proposition 5.10(ii).

For the reverse inclusion, note that if $S_1 \subset S_2$ are sheets and $\mathrm{Lk}_\downarrow(w,S_1) \in \mathscr{T}$, then $\mathrm{Lk}_\downarrow(w,S_2) \in \mathscr{T}$. In particular, if a face μ of $\tau \subset L$ is in $\mathscr{T}$, then let S_1, resp. S_2 be the sheets corresponding to μ, resp. τ, hence viewed as components of p-preimages of $T_\mu \subset T_\sigma$, it is clear that $S_1 \subset S_2$ and hence $\tau \in \mathscr{T}$. Thus if $\tau \cap \sigma \neq \varnothing$, then $\tau \in \mathscr{T}$: τ contains some face of σ, which we have shown to be an element of $\mathscr{T}$. □

Theorem 8.9. *X_J is homotopy equivalent to the wedge of copies of L*

$$X_J \simeq \bigvee_{v \notin X_J} L \tag{25}$$

indexed by vertices in $X \setminus X_J$.

Proof. Define increasing chain of unions

$$X(n) := X_J \cap (K_1 \cup K_2 \cup \cdots \cup K_n)$$

where we note that $X(n) = X(n-1) \cup (X_J \cap K_n)$. From Lemma 8.1(iii) and Lemma 8.8, we see that the intersection

$$X(n-1) \cap (X_J \cap K_n) = X_J \cap (K_n \cap (K_1 \cup \cdots \cup K_{n-1}))$$

is contractible. Moreover, Lemma 8.1(iii) implies that $X_J \cap K_n$ is contractible if $v_n \in X_J$, else homotopy equivalent to L. Hence it follows by induction on n that

$$X(n) \simeq \bigvee_{v_i \notin X_J, i \leq n} L$$

where the restriction of the above homotopy equivalence to $X(n-1) \hookrightarrow X(n)$ agrees with that of $X(n-1)$. Hence, constructing the appropriate mapping telescope, the theorem follows. □

Remark. If L is connected, then the choice of basepoints in (25) does not affect homotopy type. When L is disconnected, then the copies of L may have basepoints in different components, as determined by the relevant vertex $v \in X \setminus X_J$.

Proof of (iii) of Theorem 5.15. If L is not connected, then it is not homologically 0-connected. Hence Theorem 5.15(i) implies that $H \notin \mathrm{FP}_1(R)$, *i.e.* it is not finitely presented.

If L is connected but not simply connected, then $\pi_1(X_t)$ is the free product of copies of $\pi_1(L)$ indexed by vertices $v \notin X_t$. Suppose that H is finitely presented. Then by Proposition 3.10, $\pi_1(X_t)$ is generated by the H translates (orbits) of finitely many loops. X is contractible, hence each such loop is nulhomotopic in X, and in particular (since $I^2 = I \times I$ is compact), in some finite subcomplex of

X. Hence choose $T \in \mathbb{R}$ such that every such loop is nulhomotopic in $X_{[t-T,t+T]}$ and $X_{[t-T,t+T]} \setminus X_t$ contains at least one vertex. We observe that the inclusion $\iota : X_t \hookrightarrow X_{[t-T,t+T]}$ induces the trivial map on π_1; however, by Theorem 2.16(iii) ι_* must be an epimorphism, hence $X_{[t-T,t+T]}$ is simply connected, a contradiction with Theorem 8.9 above. $\square$