GEOMETRIC MODELS AND ASYMPTOTIC DIMENSION FOR INFINITE-TYPE SURFACE MAPPING CLASS GROUPS

MICHAEL C. KOPRESKI, GEORGE SHAJI

Abstract. Let S be an infinite-type surface and let $G \leq \mathrm{Map}(S)$ be a locally bounded Polish subgroup. We construct a metric graph $\mathcal{M}$ of simple arcs and curves on S preserved by the action of G and for which the vertex orbit map $G \to V(\mathcal{M})$ is a coarse equivalence; if G is boundedly generated, then $\mathcal{M}$ is a Cayley–Abels–Rosendal graph for G and the orbit map is a quasi-isometry. In particular, if S contains a non-displaceable subsurface and $G \geq \mathrm{PMap}_c(S)$ is boundedly generated or $G \in \{\overline{\mathrm{PMap}_c(S)}, \mathrm{PMap}(S), \mathrm{Map}(S)\}$ and is locally bounded, then $\operatorname{asdim} \mathcal{M} = \operatorname{asdim} G = \infty$. This result completes the classification of the asymptotic dimension of stable boundedly generated infinite-type surface mapping class groups begun by Grant–Rafi–Verberne.

1. Introduction and main results

Let S be a surface of infinte topological type. A *(metric) arc and curve model* for $G \leq \mathrm{Map}(S)$ is a connected (metric) graph whose vertices are collections of (possibly intersecting) simple arcs and curves on S, with an isometric action of G induced by the permutation of its vertices.

Theorem 1.1. *Let S be an infinite-type surface and let $G \leq \mathrm{Map}(S)$ be a locally bounded Polish subgroup.*

(1) There exists a metric arc and curve model $\mathcal{M}$ for G for which the orbit map restricted to $V(\mathcal{M})$ is a continuous coarse equivalence.

(2) If additionally G is boundedly generated, then $\mathcal{M}$ is a Cayley–Abels–Rosendal graph for G and the orbit map is a continuous quasi-isometry.

In particular, the coarse equivalence and quasi-isometry types of $G \leq \mathrm{Map}(S)$ are described by a (metric) arc and curve model, whenever they are well-defined. A compact subsurface $\Delta \subset S$ is *non-displaceable by G* if there exists no $f \in G \leq \mathrm{Map}(S)$ such that $\Delta \cap f\Delta = \varnothing$. From Theorem 1.1 we obtain:

Theorem 1.2. *Let S be an infinite-type surface and $G \leq \mathrm{Map}(S)$ a Polish subgroup with a non-displaceable subsurface and containing $\mathrm{PMap}_c(S)$. If G is boundedly generated or $G \in \{\overline{\mathrm{PMap}_c(S)}, \mathrm{PMap}(S), \mathrm{Map}(S)\}$ and locally bounded, then $\operatorname{asdim} G = \infty$.*

Theorem 1.2 answers [GRV21, Qn. 1.8] of Grant–Rafi–Verberne and completes their characterization of the asymptotic dimension of stable boundedly generated infinite-type surface mapping class groups.

Corollary 1.3. *For stable S with boundedly generated* $\mathrm{Map}(S)$*,* $\operatorname{asdim}\mathrm{Map}(S) = \infty$ *if and only if S has a non-displaceable subsurface or an essential shift; otherwise,* $\mathrm{Map}(S)$ *is coarsely bounded and* $\operatorname{asdim}\mathrm{Map}(S) = 0$*.*

To our knowledge, our construction obtains the first examples of an arc and curve model admitting a geometric (Švarc–Milnor-type) action of the mapping class group of an arbitrary infinite-type surface; see [SC24] for a construction of curve graphs for translateable surfaces.

1.1. Outline.

In Section 2, we state some known results about the asymptotic dimension of mapping class groups of infinite-type surfaces, specifically from [GRV21], following which we introduce some background and relevant tools for the coarse geometry of Polish groups from [Ros21] and [BDHL25]. In Section 3, we describe witness-cocompactness, a key tool for computing asymptotic dimension, and sketch the main theorem in [Kop24]:

Theorem 1.4. *Let S be an infinite-type surface and let $\mathcal{M}$ be a witness-cocompact arc and curve model for* $\mathrm{PMap}_c(S)$*. Then* $\operatorname{asdim}\mathcal{M} = \infty$*.*

In Section 4, we introduce *coarse Cayley–Abels–Rosendal graphs*, which extend the Cayley–Abels–Rosendal graphs of [BDHL25] for locally bounded Polish groups; in particular, we prove a Švarc–Milnor-type result (Proposition 4.3). Section 5 constructs the model $\mathcal{M}$ satisfying Theorem 1.1. Finally, Section 6 proves Theorem 1.2.

1.2. Acknowledgements.

The authors would like to thank Thomas Hill for acquainting them with Cayley–Abels–Rosendal graphs and Robbie Lyman, George Domat and Brian Udall for helpful discussions. Additionally, the authors thank Mladen Bestvina and Priyam Patel for their suggestion to generalize the work to the locally bounded case. This work was supported by NSF grant no. 2304774 and no. 1840190: *RTG: Algebra, Geometry, and Topology at the University of Utah*.

2. Preliminaries

We first review Rosendal's work on Polish topological groups and introduce the necessary background on coarse structures from [Ros21]. We then recall Cayley–Abels–Rosendal graphs for topological groups [BDHL25] and several facts on the topology of boundedly generated mapping class groups. Lastly, we summarize the relevant results from [GRV21].

2.1. Coarse structure.

The Polish groups considered herein are typically not finitely or compactly generated. Nonetheless, following Rosendal we may associate to every topological group G a canonical left-invariant *coarse structure*, which generalizes the (quasi)geometric structure classically associated to a group. This coarse structure will permit a well-defined coarse equivalence and quasi-isometry type for *locally bounded* and *boundedly generated* Polish groups, respectively (Section 2.2).

Definition 2.1 ([Ros21, Defn. 2.2])**.** A *coarse structure* on a set X is a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$ satisfying the following:

- The diagonal $\{(x, x) \mid x \in X\}$ is in $\mathcal{E}$.
- If $F \in \mathcal{E}$ and $E \subseteq F$, then $E \in \mathcal{E}$

- if $E, F \in \mathcal{E}$, then $E \cup F, E^{-1}, E \circ F \in \mathcal{E}$, where $E^{-1} = \{(y,x) \mid (x,y) \in E\}$ and $E \circ F := \{(x,z) \mid \exists y \in X, (x,y) \in F, (y,z) \in E\}$.

Example 2.2 ([Ros21, Example 2.3])**.** The simplest examples of coarse structures arise from pseudometrics on a set X: Given a pseudometric d on X, we may define the coarse structure induced by d as follows:

$$\mathcal{E}_d := \{E \subset X \times X \mid E \subseteq E_\alpha \text{ for some } \alpha < \infty\}$$

where $E_\alpha := \{(x,y) \mid d(x,y) < \alpha\}$.

Definition 2.3 ([Ros21, Defn. 2.12])**.** A subset $A \subseteq X$ of a coarse space $(X, \mathcal{E})$ is said to be *coarsely bounded* if $A \times A \in \mathcal{E}$.

A topological group has a canonical left-invariant coarse structure:

Definition 2.4 ([Ros21, Defn. 2.10])**.** For a topological group G, the *left-coarse structure* $\mathcal{E}_L$ is defined by

$$\mathcal{E}_L := \bigcap \{\mathcal{E}_d \mid d \text{ is a continuous, left-invariant pseudometric on G}\}.$$

For a topological group G with its left-coarse structure $\mathcal{E}_L$, we denote by $\mathcal{CB}$ the collection of all coarsely bounded subsets of G. This collection is actually an ideal of sets additionally closed under the operations of topological closure, inversion and products.

Definition 2.5 ([Ros21, Defn. 2.10])**.** Given an ideal $\mathcal{A}$, we can define the coarse structure $\mathcal{E}_\mathcal{A}$ on the group G as follows:

$$\mathcal{E}_\mathcal{A} := \{E \mid E \subseteq E_A \text{ for some } A \in \mathcal{A}\}$$

where $E_A := \{(x,y) \in G \times G \mid x^{-1}y \in A\}$.

In particular, we can consider the coarse structure $\mathcal{E}_{\mathcal{CB}}$ on G, associated to the ideal $\mathcal{CB}$ of coarsely bounded subsets of G.

Lemma 2.6 ([Ros21, Cor. 2.23])**.** *For any topological group G, $\mathcal{E}_L = \mathcal{E}_{\mathcal{CB}}$.*

Henceforth, we will always endow a topological group with the left coarse structure $\mathcal{E}_L = \mathcal{E}_{\mathcal{CB}}$ and a pseudometric space (X,d) with the coarse structure $\mathcal{E}_d$.

Definition 2.7 ([Ros21, Defn. 2.43])**.** Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces.

- A map $\varphi : X \to Y$ is said to be *bornologous* if $(\varphi \times \varphi)(\mathcal{E}) \subseteq \mathcal{F}$
- A map $\varphi : X \to Y$ is said to be *expanding* if $(\varphi \times \varphi)^{-1}(\mathcal{F}) \subseteq \mathcal{E}$
- A map $\varphi : X \to Y$ is said to be a *coarse embedding* if it is both bornologous and expanding.
- Let Z be a set. Two maps $\alpha, \beta : Z \to X$ are said to be *close* if there exists $E \in \mathcal{E}$ such that $(\alpha(z), \beta(z)) \in E$ for all $z \in Z$.
- A bornologous map $\varphi : X \to Y$ is said to be a *coarse equivalence* if there exists a bornologous map $\psi : Y \to X$ such that $\psi \circ \varphi$ is close to Id_X and $\varphi \circ \psi$ is close to Id_Y.
- A subset $A \subseteq X$ is said to be *cobounded* if there exists $E \in \mathcal{E}$ such that

$$X = E[A] := \{x \in X \mid (x,y) \in E \text{ for some } y \in A\}$$

- A map $\varphi : X \to Y$ is *cobounded* if $\varphi(X)$ is cobounded in Y.

Remark 2.8. The terms in Definition 2.7 agree with their usual (metric) definitions when $\mathcal{E}$ and $\mathcal{F}$ are *metrizable* i.e. $\mathcal{E} = \mathcal{E}_d$ and $\mathcal{F} = \mathcal{E}_{d'}$ for metrics d, d' on X, Y respectively.

Lemma 2.9 ([Ros21, Lem. 2.45])**.** *Any cobounded coarse embedding is a coarse equivalence.*

Lemma 2.10. *Suppose that a topological group G acts continuously and isometrically on a metric space (X, d). Then the orbit map ω is bornologous.*

Proof. Since the action is continuous and isometric, the pullback metric $\omega^* d$ is a left-invariant continuous pseudometric on G and $(\omega \times \omega)[\mathcal{E}_{\omega^* d}] \subset \mathcal{E}_d$. By definition $\mathcal{E}_L \subset \mathcal{E}_{\omega^* d}$, hence ω is bornologous. □

Let $B_\alpha(x)$ denote the ball of radius $\alpha > 0$ centered at x.

Lemma 2.11. *Suppose that a topological group G acts continuously and isometrically on a metric space (X, d) and let ω be the orbit map based at $x_0 \in X$. Then ω is expanding if $A_\alpha := \omega^{-1}(B_\alpha(x_0))$ is coarsely bounded for all $\alpha > 0$.*

Proof. The orbit map ω is expanding if and only if $(\omega \times \omega)^{-1}(\mathcal{E}_d) \subseteq \mathcal{E}_L$. First consider $E_\alpha \in \mathcal{E}_d$ for $\alpha > 0$. Then

$$\begin{aligned}(\omega \times \omega)^{-1}(E_\alpha) &= \{(g, h) \in G \times G \mid (gx_0, hx_0) \in E_\alpha\} \\ &= \{(g, h) \in G \times G \mid d(gx_0, hx_0)) < \alpha\} \\ &= \{(g, h) \in G \times G \mid d(x_0, g^{-1}hx_0)) < \alpha\} \\ &= \{(g, h) \in G \times G \mid g^{-1}h \in A_\alpha\} \\ &= E_{A_\alpha}\end{aligned}$$

Since A_α is coarsely bounded, $E_{A_\alpha} \in \mathcal{E}_{\mathcal{CB}} = \mathcal{E}_L$ by Lemma 2.6. For general $E \in \mathcal{E}_d$, $E \subset E_\alpha$ for some $\alpha > 0$. Hence $(\omega \times \omega)^{-1}(E) \subset (\omega \times \omega)^{-1}(E_\alpha) \in \mathcal{E}_L$ and $(\omega \times \omega)^{-1}(E) \in \mathcal{E}_L$ as required. □

We conclude by stating a convenient criterion for coarse boundedness:

Proposition 2.12 ([Ros21, Prop. 2.15(5)])**.** *Let G be a Polish group. A subset $A \subset G$ is coarsely bounded if and only if for every identity neighborhood $U \subset G$, there exists a finite set F and $n \in \mathbb{N}$ such that $A \subset (FU)^n$.*

Corollary 2.13. *Let G be a Polish group and $H \leq G$ be coarsely bounded in G. If $H \leq H' \leq G$ such that $[H : H'] < \infty$ then H' is also coarsely bounded in G.*

Proof. Since H is coarsely bounded, for every open neighborhood $1 \in U \subset G$, there exists a finite set F and $n \in \mathbb{N}$ such that $H \subset (FU)^n$. If $H' = \bigcup_{i=1}^k h_i H$, let $F' := F \cup \{h_1, \dots h_k\}$. Clearly $H' \subset (F'U)^n$ and hence H' is coarsely bounded in G. □

2.2. **Local boundedness and bounded generation.** Analogously to locally compact and compactly generated groups, we introduce two classes of topological groups related to the metrizability of $\mathcal{E}_L$.

Definition 2.14. A topological group G is

(i) *locally bounded* if there is a coarsely bounded neighborhood of identity.

(ii) *boundedly generated* if it admits a coarsely bounded generating set.

Proposition 2.15 ([Ros21, Thm. 2.40])**.** *Any boundedly generated Polish group is locally bounded.*

Remark 2.16. The properties of local boundedness and bounded generation are not inherited by Polish subgroups. For example, consider the ladder surface S. Then $\mathrm{Map}(S)$ is boundedly generated and hence locally bounded as well [MR23] but $\mathrm{PMap}(S)$ is neither locally bounded nor boundedly generated [Hil25].

Proposition 2.17 ([Ros21, Cor. 3.26])**.** *Among Polish groups, the properties of being locally bounded and boundedly generated are both invariant under coarse equivalence. Moreover, every coarse equivalence between boundedly generated Polish groups is automatically a quasi-isometry.*

We recall that $\mathcal{E}_L$ is metrizable when it is induced by a (possibly discontinuous) metric on G, in which case $\mathcal{E}_L$ defines a coarse-equivalence type for G in the usual (metric) sense. Crucially:

Theorem 2.18 ([Ros21, Thm. 2.38])**.** *Let G be a Polish group. Then $\mathcal{E}_L$ is metrizable if and only if G is locally bounded if and only if $\mathcal{E}_L$ is induced by a continuous left-invariant pseudometric d on G.*

When $\mathcal{E}_L$ is metrizable, by definition, it is maximal among the set of coarse structures on G induced by continuous left-invariant pseudometrics, with respect to the partial ordering

$$\mathcal{E}_{d'} > \mathcal{E}_d \iff \mathcal{E}_{d'} \subset \mathcal{E}_d \iff \mathrm{Id} : (G, d') \to (G, d) \text{ is bornologous}.$$

In particular, $\mathcal{E}_L$ is the unique such coarse structure. Similarly, we may consider a (finer) partial ordering on the set of left-invariant continuous pseudometrics on G: let $d \gg d'$ whenever $\mathrm{Id} : (G, d) \to (G, d')$ is coarsely Lipschitz. We observe that any maximal d is unique up to quasi-isometry.

Theorem 2.19 ([Ros21, Prop. 2.72])**.** *Let G be a Polish group. Then G admits a continuous left-invariant pseudometric d maximal with respect to $\ll$ if and only if G is boundedly generated, if and only if d is quasi-isometric to the word metric on G with respect to a symmetric coarsely bounded generating set.*

It follows that the word metric on G with respect to any coarsely bounded generating set gives a well-defined quasi-isometry type whenever G is boundedly generated.

2.2.1. *Locally bounded subgroups of* $\mathrm{Map}(S)$. For a surface S, recall that

$$\mathrm{Map}(S) := \mathrm{Homeo}^+(S)/\,\mathrm{Homeo}_0(S)$$

where $\mathrm{Homeo}^+(S)$ is the group of orientation-preserving self-homeomorphisms of S, endowed with the compact-open topology, and $\mathrm{Homeo}_0(S)$ is its identity component. The induced (quotient) topology on $\mathrm{Map}(S)$ has a local (clopen) base at Id_S induced by the pointwise stabilizers $\tilde{U}_\Sigma := \{f \in \mathrm{Homeo}^+(S) : f|_\Sigma = \mathrm{Id}_\Sigma\}$ of compact, essential subsurfaces $\Sigma \subset S$; we denote the elements of this local base $U_\Sigma := \tilde{U}_\Sigma/(\tilde{U}_\Sigma \cap \mathrm{Homeo}_0(S)) < \mathrm{Map}(S)$.

Remark 2.20. Let $G \leq \mathrm{Map}(S)$ a Polish subgroup. Given an essential compact subsurface $\Sigma \subset S$ let ν_Σ denote the (pointwise) G-stabilizer for Σ, that is $\nu_\Sigma := U_\Sigma \cap G$. Since $\mathrm{Map}(S)$ has a local base $\{U_\Sigma\}_\Sigma$ at Id_S, likewise G has a local base $\{\nu_\Sigma\}_\Sigma$ at Id_S.

Remark 2.21. Since G has a local base of open subgroups at Id_S, G is non-Archimedean.

The following is immediate from Remark 2.20:

Lemma 2.22. *Let $G \leq \mathrm{Map}(S)$ be a locally bounded Polish subgroup. There exists a compact essential subsurface $\Sigma \subset S$ whose stabilizer ν_Σ is coarsely bounded in G.*

Some important subgroups. Let $\mathrm{Ends}(S)$ denote the (Freudenthal) endspace of S and $\mathrm{Ends}_g(S) \subset \mathrm{Ends}(S)$ the subspace of non-planar ends. By $\mathrm{PMap}(S) \leq \mathrm{Map}(S)$ we denote the *pure mapping class group of* S, which is the kernel of natural map

$$\pi : \mathrm{Map}(S) \to \mathrm{Homeo}(\mathrm{Ends}(S), \mathrm{Ends}_g(S))$$

obtained from the action of $\mathrm{Map}(S)$ on the endspace of S. Let $\mathrm{PMap}_c(S) \leq \mathrm{PMap}(S)$ denote the subgroup of compactly supported (necessarily pure) mapping classes. $\mathrm{PMap}(S)$ is closed in $\mathrm{Map}(S)$, hence it is a Polish subgroup. $\mathrm{PMap}_c(S)$ is not closed when S is infinite-type; let $\overline{\mathrm{PMap}_c(S)}$ denote its closure.

2.3. Cayley–Abels–Rosendal graphs.

Analogous to Cayley–Abels graphs for totally disconnected, locally compact groups, Branman–Domat–Hoganson–Lyman [BDHL25] define graphical models for boundedly generated Polish groups. We generalize these results to the locally bounded case in Section 4.

Definition 2.23 ([BDHL25, §3])**.** A connected, countable simplicial graph Γ is a *Cayley–Abels–Rosendal graph* for a topological group G if G admits a continuous, vertex-transitive, cocompact, and simplicial action with coarsely bounded vertex stabilizers.

Proposition 2.24 ([BDHL25, Prop. 8])**.** *Let G be a Polish group. Then G admits a Cayley–Abels–Rosendal graph if and only if G is boundedly generated. Moreover, the orbit map of G on any such graph is a quasi-isometry.*

2.4. Asymptotic dimension otherwise.

We now shift our focus to the asymptotic dimension of mapping class groups. Asymptotic dimension was introduced by Gromov and gives a 'large scale' notion of dimension; see [BD07] for a survey of results.

Definition 2.25. Let X be a metric space. Then $\mathrm{asdim}(X) \leq n$ if for every uniformly bounded open cover $\mathcal{U}$, there is a uniformly bounded open cover $\mathcal{V}$ of multiplicity $n+1$ such that $\mathcal{U}$ refines $\mathcal{V}$. We say that $\mathrm{asdim}(X) = n$ if $\mathrm{asdim}(X) \leq n$ but $\mathrm{asdim}(X) \not\leq n-1$.

Proposition 2.26 ([BD07, Prop. 22])**.** *Let X and Y be metric spaces with the standard coarse structure and $f : X \to Y$ a coarse embedding. Then asdim$(X) \leq$ asdim(Y).*

It follows that asymptotic dimension is a coarse invariant and hence well-defined in the setting of locally bounded Polish groups. In particular, we can look at the asymptotic dimension of locally bounded surface mapping class groups. When S is a finite-type surface, [BBF15] shows that the asymptotic dimension of $\mathrm{Map}(S)$ is finite. In the case of infinite type surfaces, the only result (as far as the authors know) appears in [GRV21]. We summarize the relevant details below.

Let S be an infinite-type surface. Suppose that there exists a countable family of homeomorphic subsurfaces $\Sigma_{i\in\mathbb{Z}} \subset S$, each with a single boundary component, and a simple path $\gamma \subset S \setminus \bigcup_i \mathring{\Sigma}_i$ intersecting each $\partial\Sigma_i$ sequentially and accumulating to two distinct ends. A *shift map* ω is a homeomorphism supported on a regular neighborhood of $\gamma \cup (\bigcup_i \Sigma_i)$, preserving γ set-wise and restricting to homeomorphisms $\Sigma_i \to \Sigma_{i+1}$. If in addition $\langle\omega\rangle$ is not coarsely bounded in $\mathrm{Map}(S)$, then it is an *essential shift* [GRV21, §1].

Theorem 2.27 ([GRV21, Thm. 1.1])**.** *If S is stable and $\mathrm{Map}(S)$ is boundedly generated and contains an essential shift, then $\operatorname{asdim}\mathrm{Map}(S) = \infty$.*

When S is stable, Theorem 1.2 and Theorem 2.27 fully classify the infinite asymptotic dimension cases:

Theorem 2.28 ([GRV21, Thm. 1.6])**.** *Let S be stable and $\mathrm{Map}(S)$ be boundedly generated. If S contains neither a non-displaceable subsurface nor an essential shift, then $\mathrm{Map}(S)$ is coarsely bounded.*

2.5. **Classification of local boundedness.** Since asymptotic dimension is well-defined for locally bounded mapping class groups, we recall results from [MR23] and [Hil25] that classify the infinite type surfaces whose mapping class groups and pure mapping class groups are locally bounded. Here, for $A \subset \mathrm{Ends}(S)$, $M(A)$ is the set of maximal ends in A with respect to the partial order on $\mathrm{Ends}(S)$ defined in [MR23].

Theorem 2.29 ([MR23, Thm. 1.4])**.** *Let S be an infinite type surface. Then $\mathrm{Map}(S)$ is locally bounded if and only if there is a finite type surface $\Sigma \subset S$ such that the complimentary regions of K each have infinite type and zero or infinite genus, and partition $\mathrm{Ends}(S)$ into finitely many clopen sets*

$$\mathrm{Ends}(S) = \Big(\bigsqcup_{A\in\mathcal{A}} A\Big) \sqcup \Big(\bigsqcup_{P\in\mathcal{P}} P\Big)$$

such that:

(1) Each $A \in \mathcal{A}$ is self-similar with $M(A) \subset M(\mathrm{Ends}(S))$ and $M(\mathrm{Ends}(S)) \subset \bigsqcup_{A\in\mathcal{A}} M(A)$.
(2) each $P \in \mathcal{P}$ is homeomorphic to a clopen subset of some $A \in \mathcal{A}$.
(3) for any $x_A \in M(A)$, and any neighborhood V of the end $x_A \in S$, there is $f_V \in \mathrm{Homeo}(S)$ so that $f_V(V)$ contains the complimentary region to K with end set A.

Moreover, in this case ν_Σ is a coarsely bounded neighborhood of the identity.

Theorem 2.30 ([Hil25, Thm. 1.1(b)])**.** *Let S be an infinite type surface. Then $\mathrm{PMap}(S)$ is locally bounded if and only if it is boundedly generated if and only if $|\mathrm{Ends}(S)| < \infty$ and S is not a Loch Ness monster with (non-zero) punctures.*

Remark. The authors are unaware of any work concerning the local boundedness of $\overline{\mathrm{PMap}_c(S)}$.

3. Witness-cocompactness

We discuss *cocompact* and *witness-cocompact* arc and curve models and sketch the proof of Theorem 1.4, which we will use in Section 6 to compute the asymptotic dimension of certain locally bounded surface mapping class groups. This section summarizes the results of [Kop24], to which we direct the reader for full detail; it is included here for convenience.

3.1. **Cocompact arc and curve models.** Let S be a surface of arbitrary topological type and let $\mathcal{K}(S) := K(V(\mathcal{AC}(S)))$ denote the set of finite collections of simple arcs and curves on S. Note the arcs and curves in $u \in \mathcal{K}(S)$ need not be pairwise disjoint.

Definition 3.1. A *(metric) arc and curve model for* $G \leq \mathrm{Map}(S)$ is a connected (metric) graph $\mathcal{G}$ with discrete $V(\mathcal{G}) \subset \mathcal{K}(S)$ that admits an action of G induced by the permutation of its vertices. $\mathcal{G}$ is *cocompact* if this action is cocompact.

Remark 3.2. Throughout Section 3, a (metric) arc and curve model *on* S will mean a (metric) arc and curve model for some $G \geq \mathrm{PMap}_c(S)$.

Remark 3.3. If S is finite-type, then *(i)* $\mathrm{PMap}_c(S) = \mathrm{PMap}(S)$ and *(ii)* $\mathcal{G}$ is cocompact if and only if $i(u,u)$ and $i(u,v)$ are uniformly bounded for $u \in V(\mathcal{G})$ and $(u,v) \in E(\mathcal{G})$.

Definition 3.4. Let $\mathcal{G}$ be an arc and curve model on S. A compact, essential (π_1-injective, non-peripheral) subsurface $W \subset S$ is a *witness* for $\mathcal{G}$ if W does not contain a pants component and every $u \in V(\mathcal{G})$ intersects every component of W.

We note that witnesses are not assumed to be connected. Let $\mathscr{X}^{\mathcal{G}}$ denote the set of witnesses of $\mathcal{G}$, and $\hat{\mathscr{X}}^{\mathcal{G}} \subset \mathscr{X}^{\mathcal{G}}$ the subset of connected witnesses. A *witness set* on S is any collection of compact, essential subsurfaces without pants components closed under enlargement and the action of $\mathrm{PMap}_c(S)$.

By [Kop23], the geometry of cocompact arc and curve models on finite-type surfaces is well understood. In particular:

Theorem 3.5 ([Kop24, Thm. 4.12])**.** *Let $\mathcal{G}$ be a cocompact arc and curve model on a finite-type surface Σ. Then $(\mathcal{G}, \mathscr{X}^{\mathcal{G}})$ is an asymphoric hierarchically hyperbolic space with respect to subsurface projection to witness curve graphs $\pi_W : \mathcal{G} \to 2^{\mathcal{C}W}$, $W \in \mathscr{X}^{\mathcal{G}}$.*

The $\mathrm{PMap}(\Sigma)$-equivariant geometry of $\mathcal{G}$ is uniquely determined by $\hat{\mathscr{X}}^{\mathcal{G}}$:

Theorem 3.6 ([Kop24, Thm. 4.13])**.** *The map $\mathcal{G} \mapsto \hat{\mathscr{X}}^{\mathcal{G}}$ induces a bijection between equivariant quasi-isometry types of cocompact arc and curve models on Σ and connected witness sets on Σ.*

Remark 3.7. The above is functorial in the following sense: whenever $\mathscr{X}^{\mathcal{G}'} \subset \mathscr{X}^{\mathcal{G}}$ (equivalently $\hat{\mathscr{X}}^{\mathcal{G}'} \subset \hat{\mathscr{X}}^{\mathcal{G}}$), there is a canonical equivariant coarsely surjective, coarse Lipschitz map $\iota : \mathcal{G} \to \mathcal{G}'$.

We note that Theorem 3.5 implies that cocompact $\mathcal{G}$ on a finite-type surface is δ-hyperbolic if and only if it has no pair of disjoint, connected witnesses. More broadly, $\mathcal{G}$ admits a distance formula in the sense of Masur–Minsky: there is some $K > 0$ such that for any $u, v \in V(\mathcal{G})$,

$$d_{\mathcal{G}}(u,v) \approx \sum_{W \in \mathscr{X}^{\mathcal{G}}} [d_{\mathcal{C}W}(\pi_W(a), \pi_W(b))]_K \, .$$

3.2. **Subsurface projection.** Given a compact, essential, connected, non-pants subsurface $\Sigma \subset S$, let $\mathcal{K}(S,\Sigma) \subset \mathcal{K}(S)$ denote the subset of collections containing an element that intersects Σ essentially. We construct a projection $\rho_\Sigma : \mathcal{K}(S,\Sigma) \to \mathcal{K}(\Sigma)$ as follows (see [Sch, §5.2]). Let $\iota : \Sigma \hookrightarrow S$ be the inclusion map, let $p : S_\Sigma \to S$ be the covering space associated to $\pi_1(\Sigma) \cong \operatorname{im} \iota_* < \pi_1(S)$, and let let $\tilde{\iota} : \Sigma \hookrightarrow S_\Sigma$ be the (unique) lift of ι into S_Σ. Fix any homeomorphism $\sigma : S_\Sigma \to \operatorname{int} \Sigma := \Sigma \setminus \partial\Sigma$ that is a homotopy inverse for $\tilde{\iota}|_{\operatorname{int}\Sigma}$; note that σ is unique up to homotopy, hence isotopy. Obtain $\tilde{\sigma}$ by composing σ with the inclusion $\operatorname{int}\Sigma \hookrightarrow \Sigma$.

$$\begin{tikzcd}
 & S_\Sigma \arrow[d, "p"] \arrow[dl, bend right, "\tilde{\sigma}"'] \\
\Sigma \arrow[ur, "\tilde{\iota}"] \arrow[r, hook, "\iota"'] & S
\end{tikzcd}$$

Given $u \in \mathcal{K}(S,\Sigma)$, let $\rho_\Sigma(u)$ be the closures of the non-peripheral components of $\tilde{\sigma}p^{-1}(u)$, up to isotopy.

One verifies that $\rho_\Sigma(u)$ is independent of the choice of representative for ω and σ. Likewise, ρ_Σ is independent of the choice of embedding of Σ: if $\iota' : \Sigma \hookrightarrow S$ is isotopic to ι, then the lift $\tilde{\iota}'$ is isotopic to $\tilde{\iota}$ and thus σ is likewise a homotopy inverse for $\tilde{\iota}'|_{\operatorname{int}\Sigma}$.

Remark 3.8. The definition here for ρ_Σ differs slightly from that in [Kop24], which instead passes to the Gromov closure of S_Σ; however, the definitions are consistent. We can likewise define $\rho_\Sigma(u)$ as the collection of essential intersections of u with Σ.

The natural action of $\operatorname{PMap}(\Sigma)$ on $\mathcal{K}(\Sigma)$ defines an action of $\operatorname{Map}(\Sigma, \partial\Sigma) \twoheadrightarrow \operatorname{PMap}(\Sigma)$. Similarly, $\operatorname{Map}(\Sigma,\partial\Sigma) \curvearrowright \mathcal{K}(S,\Sigma)$ via the homomorphism $\operatorname{Map}(\Sigma,\partial\Sigma) \to \operatorname{PMap}_c(S)$ obtained by extending by identity.

Lemma 3.9 ([Kop24, Lem. 4.14])**.** *$\rho_\Sigma : \mathcal{K}(S,\Sigma) \to \mathcal{K}(\Sigma)$ is $\operatorname{Map}(\Sigma,\partial\Sigma)$-equivariant.*

Corollary 3.10. *Let $\phi \in \operatorname{PMap}(\Sigma)$. Then there exists $\psi \in \operatorname{PMap}_c(S)$ preserving $\mathcal{K}(S,\Sigma)$ such that for any $\omega \in \mathcal{K}(S,\Sigma)$, $\phi\rho_\Sigma(\omega) = \rho_\Sigma(\psi\omega)$.*

3.2.1. *Witness-cocompactness.* Let $W \subset S$ be a connected witness for an arc and curve model $\mathcal{G}$ on S. Note that $V(\mathcal{G}) \subset \mathcal{K}(S,W)$ and obtain an arc and curve model $\mathcal{G}_W$ on W as follows: let $V(\mathcal{G}_W) = \rho_W(V(\mathcal{G}) \subset \mathcal{K}(W)$, and obtain $E(\mathcal{G}_W)$ as the push-forward of the edge relation on $\mathcal{G}$ by ρ_W. By Corollary 3.10 $\operatorname{PMap}(W)$ acts on $\mathcal{G}_W$ by permuting its vertices and the map $\rho_W : \mathcal{G} \to \mathcal{G}_W$ is $\operatorname{Map}(W,\partial W)$-equivariant; since $\mathcal{G}$ is connected, likewise is $\mathcal{G}_W$. If $\mathcal{G}$ is a metric graph, then likewise push forward the edge lengths on $\mathcal{G}$ to obtain a metric on $\mathcal{G}_W$; in either case, ρ_W is 1-Lipschitz.

Definition 3.11. Let $\mathcal{G}$ be a connected (metric) arc and curve model on S. Then $\mathcal{G}$ is *witness-cocompact* if

(1) $\mathcal{G}$ has a (compact) witness; and
(2) for every witness $W \subset S$, $\mathcal{G}_W$ is cocompact.

Remark 3.12. From Remark 3.3, it follows that $\mathcal{G}$ is witness-cocompact if and only if $\mathcal{G}$ has a witness and for any witness W there is a uniform bound on $i(\rho_W(u), \rho_W(u))$ and $i(\rho_W(u), \rho_W(v))$ for $u \in V(\mathcal{G})$ and $(u, v) \in E(\mathcal{G})$.

Lemma 3.13. *Let $\mathcal{G}$ be a witness-cocompact arc and curve model and let W be a witness. Then any* $\mathrm{Map}(W, \partial W)$*-equivariant section $\sigma_W : V(\mathcal{G}_W) \to V(\mathcal{G})$ is a quasi-isometric embedding.*

Proof. Since ρ_W is Lipschitz, it suffices to show that σ_W is likewise Lipschitz. Since $\mathcal{G}_W$ is cocompact, it has finitely many orbits of edges $(\bar{u}, \bar{v}) \in E(\mathcal{G}_W)$. Since σ_W is equivariant, there are likewise finitely many orbits of pairs $(\sigma_W(\bar{u}), \sigma_W(\bar{v}))$. Let L be the maximum of the distances $d_{\mathcal{G}}(\sigma_W(\bar{u}), \sigma_W(\bar{v}))$ for $(\bar{u}, \bar{v}) \in E(\mathcal{G}_W)$. Then σ_W is coarsely L-Lipschitz. ☐

Remark. If $\mathcal{G}$ is witness-cocompact, then for each witness W, $\mathcal{G}_W$ is cocompact: up to quasi-isometry, we may endow $\mathcal{G}_W$ with the usual simplicial metric.

3.3. **Asymptotic dimension lower bounds.** We sketch the arguments from [Kop24] to prove Theorem 1.4. We begin by computing lower bounds for the asymptotic dimension of cocompact arc and curve models on finite-type surfaces.

3.3.1. *For finite-type surfaces.* Let Σ be a finite-type surface with a cocompact arc and curve model $\mathcal{M}$. We aim to show the following:

Theorem 3.14 ([Kop24, Thm. 4.21])**.** *Let Σ be a genus g finite-type surface. If $\mathcal{M}$ is a (non-empty) δ-hyperbolic cocompact arc and curve model on Σ, then* $\operatorname{asdim} \mathcal{M} \geq g - \lceil \frac{1}{2}\chi(\Sigma) \rceil$.

Remark 3.15. In the complementary case, when $\mathcal{M}$ is not δ-hyperbolic or equivalently when $\mathcal{M}$ has $\nu > 1$ disjoint connected witnesses, it will suffice that $\operatorname{asdim} \mathcal{M} \geq \nu$. In particular, ν is exactly the HHS rank of $(\mathcal{M}, \mathscr{X}^{\mathcal{M}})$, which bounds $\operatorname{asdim} \mathcal{M}$ from below [BHS21, Thm. 1.15].

We prove Theorem 3.14 by finding a compact subspace $Z \subset \partial \mathcal{M}$ of known topological dimension. For proper δ-hyperbolic spaces, the topological dimension of the boundary gives bounds on the asymptotic dimension of the space [BL08, Prop. 6.2]; while $\mathcal{M}$ is typically non-proper, a minor adaptation of the lower bound suffices.

Proposition 3.16 ([Kop24, Prop. 2.5])**.** *Let X be a geodesic δ-hyperbolic space with $Z \subset \partial X$ compact. Then* $\operatorname{asdim} X \geq \dim Z + 1$.

We find Z as follows. Recall that, whenever $\mathcal{M}$ and $\mathcal{M}'$ are cocompact graph models on Σ and $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}^{\mathcal{M}'}$, there is a canonical coarsely surjective, coarsely Lipschitz map $\iota : \mathcal{M} \to \mathcal{M}'$. In particular, since $\mathscr{X}^{\mathcal{C}\Sigma} = \{\Sigma\}$, such a map $\iota : \mathcal{M} \to \mathcal{C}\Sigma$ exists for any cocompact graph model $\mathcal{M}$. We first prove that when $\mathcal{M}$ is δ-hyperbolic these maps are coarsely alignment preserving in the sense of Dowdall–Taylor [DT17]: there exists K for $\mathcal{M}, \mathcal{M}'$ such that for any aligned triple of vertices $(x, y, z) \in V(\mathcal{M})^3$, $d(\iota(x), \iota(y)) + d(\iota(y), \iota(z)) \leq d(\iota(x), \iota(z)) + K$.

Lemma 3.17 ([Kop24, Lem. 4.22]). *Let $\mathcal{M}$ and $\mathcal{M}'$ be arc and curve models on a finite-type surface such that $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}^{\mathcal{M}'}$, and let $\iota : \mathcal{M} \to \mathcal{M}'$ be the canonical coarse surjection. If $\mathcal{M}$ is δ-hyperbolic, then ι is coarsely alignment-preserving.*

The proof of Lemma 3.17 follows from the existence of hierarchy paths [BHS19, Thm. 4.4] in $\mathcal{M}, \mathcal{M}'$. Such paths are close to geodesics and have projections to witness curve graphs that are unparameterized quasi-geodesics. Applying the distance formulas for $\mathcal{M}, \mathcal{M}'$ derives the claim.

Crucially, the theory of alignment preserving maps implies an embedding of $\partial\mathcal{M}'$ into $\partial\mathcal{M}$, and in particular an embedding $\partial\mathcal{C}\Sigma \hookrightarrow \partial\mathcal{M}$ whenever $\mathcal{M}$ is δ-hyperbolic.

Theorem 3.18 ([DT17, Thm. 3.2]). *Let $f : X \to Y$ be a coarsely surjective, coarsely alignment preserving map between geodesic δ-hyperbolic spaces. Then f induces an embedding $\partial Y \hookrightarrow \partial X$.*

When Σ is a punctured sphere, we then conclude using a result of Gabai:

Theorem 3.19 ([Gab14, Thm. 1.2]). *Let Δ be the $(n + 4)$-times punctured sphere for $n \geq 0$. Then $\partial\mathcal{C}\Delta$ is homeomorphic to the n-dimensional Nöbeling space $\mathbb{R}^{2n+1}_n$.*

In particular, by the universal embedding property of Nöbeling spaces [Nöb30] any n-dimensional compactum Z embeds into $\partial\mathcal{C}\Delta \subset \partial\mathcal{M}$. For general Σ, we apply a result of Rafi–Schleimer [RS09, Thm. 7.1] to obtain an embedding of $\partial\mathcal{C}\Delta$ into $\partial\mathcal{C}\Sigma \subset \partial\mathcal{M}$, which completes the proof of Theorem 3.14.

Proposition 3.20 ([Kop24, Prop. 4.23]). *Let Σ be a finite-type hyperbolic surface of genus g and Δ the $(n+4)$-times punctured sphere, where $n = g - 1 - \lceil \frac{1}{2}\chi(\Sigma) \rceil$. Then $\partial\mathcal{C}\Delta$ embeds into $\partial\mathcal{C}\Sigma$.*

3.3.2. *For infinite-type surfaces.* We prove the following:

Theorem 3.21. *Let S be an infinite-type surface and let $\mathcal{M}$ be a witness-cocompact metric arc and curve model on S. Then* $\operatorname{asdim} V(\mathcal{M}) = \infty$.

In particular, since $V(\mathcal{M})$ is given the induced metric, it isometrically embeds into $\mathcal{M}$ and Theorem 1.4 follows. By Lemma 3.13 and the monotonicity of asymptotic dimension, it suffices to find for every $d \in \mathbb{N}$ some witness $W \subset S$ for which $\operatorname{asdim} \mathcal{M}_W \geq d$ (see [Kop24, §4.3.2]).

Given a witness-cocompact arc and curve model $\mathcal{M}$ on an infinite-type surface Ω, let $w_{\mathcal{M}} \in \mathbb{N} \cup \{\infty\}$ denote the least upper bound on cardinalities for a set of pairwise-disjoint connected witnesses for $\mathcal{M}$. If $w_{\mathcal{M}}$ is infinite, then for each d fix a compact subsurface Σ_d containing at least d pair-wise disjoint connected witnesses for $\mathcal{M}$. These witnesses are likewise witnesses for $\mathcal{M}_{\Sigma_d}$, hence $\mathcal{M}_{\Sigma_d}$ has rank $\nu \geq d$ and $\operatorname{asdim} \mathcal{M}_{\Sigma_d} \geq d$. If $w_{\mathcal{M}}$ is finite, then fix a set $\{\Delta_i\}$ of $w_{\mathcal{M}}$ pairwise disjoint witnesses, with Δ_0 a witness adjacent to an infinite-type component of $S \setminus \bigcup_i \Delta_i$. By enlarging Δ_0 disjointly from the remaining Δ_i, we obtain compact subsurfaces $\Sigma_d \subset S \setminus \bigcup_{i>0} \Delta_i$ such that $-\chi(\Sigma_d) > 2d$ and $\Delta_0 \subset \Sigma_d$. Since $\Sigma_d \supset \Delta_0$, it is a witness for $\mathcal{M}$, and each $\mathcal{M}_{\Sigma_d}$ must have rank $\nu = 1$ else we obtain a set of witnesses for $\mathcal{M}$ of cardinality greater than $w_{\mathcal{M}}$. It follows that $\mathcal{M}_{\Sigma_d}$ is δ-hyperbolic. Applying Theorem 3.14, we obtain $\operatorname{asdim} \mathcal{M}_{\Sigma_d} \geq d$ as required. Theorem 3.21 follows. //

4. A Švarc–Milnor lemma for locally bounded groups

Definition 4.1. The action of a group G on a metric graph Γ is *bounded-cocompact* if, for every closed bounded subgraph $\Lambda \subset \Gamma$, Λ/G is compact.

Definition 4.2. A connected metric graph Γ with a discrete vertex set $V(\Gamma)$ along with an isometric, isomorphic, and continuous action of a group G is a *coarse Cayley–Abels–Rosendal graph for G* if the action is vertex-transitive and bounded-cocompact with coarsely bounded vertex stabilizers.

Recall that a Polish group G is *non-Archimedean* if it has a (clopen) subgroup neighborhood basis at identity [Kec12]. The following extends Proposition 2.24.

Proposition 4.3. *Let G be a Polish group. If G admits a coarse Cayley–Abels–Rosendal graph then it is locally bounded; moreover, for any such graph Γ the orbit map to $V(\Gamma)$, with the induced metric, is a coarse equivalence. If G is non-Archimedean then the converse holds.*

Proof. Let $\omega : G \to V(\Gamma)$ denote the vertex orbit map; since the action is continuous, ω is continuous. Moreover, since $V(\Gamma)$ is discrete, the stabilizer of a vertex in Γ is thus a coarsely bounded neighborhood of identity and G is locally bounded. We must show that ω is bornologous, expanding, and cobounded, hence a coarse equivalence: the first follows from Lemma 2.10 and the last from vertex transitivity, hence we need only check that ω is expanding.

Let d denote the metric on Γ and fix a vertex $x \in V(\Gamma)$; we assume ω is the orbit map based at x. By Lemma 2.11, it suffices to show that $A_\alpha = \omega^{-1}(B_\alpha(x))$ is coarsely bounded. Fix a connected bounded subgraph $\Lambda_\alpha \subset \Gamma$ containing $B_\alpha(x) \cap \operatorname{im}\omega = B_\alpha(x) \cap V(\Gamma)$ and let $A'_\alpha = \omega^{-1}(\Lambda_\alpha) = \{g \in G : gx \in V(\Lambda_\alpha)\}$. Since $V(\Gamma)$ is discrete and G acts vertex-transitively, the infimum of edge lengths in Γ is nonzero. Hence by bounded-cocompactness, Λ_α/G is a finite graph, or equivalently Λ_α intersects finitely many G-orbits of edges: the midpoints of edges in Λ_α/G are discrete, hence must be finite by compactness. Let $\nu_x := \operatorname{stab}_G(x) \leq G$ denote the stabilizer of x and fix a finite set of elements $F_\alpha \subset G$ so that if $(gx, hx) \in E(\Lambda_\alpha)$, then $g^{-1}h \in \nu_x F_\alpha \nu_x$; additionally add some element $g_0 \in A'_\alpha$. Let $m = \operatorname{diam}\Lambda_\alpha$. Then $A'_\alpha \subset (F_\alpha \nu_x)^m$, hence $A'_\alpha \supset A_\alpha$ is coarsely bounded since likewise is ν_x.

The converse when G has small subgroups is shown in the following lemma. □

Lemma 4.4. *If G is a non-Archimedean locally bounded Polish group, then it admits a coarse Cayley–Abels–Rosendal graph.*

Proof. Fix a coarsely bounded clopen subgroup $H \leq G$ and a countable set $Z = \{z_i\} \subset G$ such that $G = \langle Z, H \rangle$. Such a Z always exists: for example, since G is separable and H open, H has a countable transversal in G. Construct a metric graph Γ on the vertex set G/H by attaching an edge of length i between gH and kH whenever $g^{-1}k \in Hz_iH$. The set Z generates G over H, hence Γ is connected. Since H is clopen, the left action of G on G/H is continuous; it induces a continuous, isometric, isomorphic, and vertex transitive action on Γ with coarsely bounded vertex stabilizer $\operatorname{stab}_G(H) = H$.

We verify bounded-cocompactness: Γ/G is the metric graph isomorphic to a bouquet of countably many circles e_i, each of length i. It suffices that if Λ is a subgraph of Γ for which Λ/G is not compact, then it is unbounded. In particular, $\Lambda/G \subset \Gamma/G$ must contain infinitely many edges and thus an edge of length at least i for every $i \in \mathbb{N}$, hence likewise does Λ. $\square$

Remark 4.5. If there exists a finite subset $Z \subset G$ and a coarsely bounded clopen subgroup H such that $G = \langle Z, H \rangle$, then clearly $H \cup Z$ is also coarsely bounded and hence G is boundedly generated. Conversely, if G is boundedly generated, then we may choose Z in Lemma 4.4 above to be finite by Proposition 2.12. Hence G is boundedly generated if and only if Z can be chosen to be finite.

Remark 4.6. For a Polish group G, let $H \leq G$ be a coarsely bounded open subgroup and $Z \subset G$ an enumerated countable set that generates G over H. Let $\mathcal{C}_{H,Z}(G)$ denote the coarse Cayley–Abels–Rosendal graph constructed as in the proof of Lemma 4.4; by Proposition 4.3, its vertex set is coarse equivalent to G.

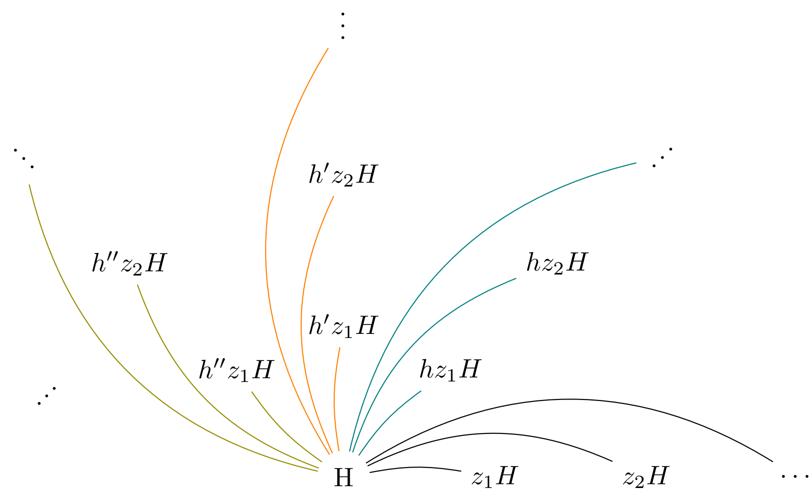

FIGURE 1. The neighborhood of the vertex H in $\mathcal{C}_{H,Z}(G)$. Here, $h, h', h'' \in H$ and $Z = \{z_i\}$

Remark 4.7. The construction of $\mathcal{C}_{H,Z}(G)$ exactly coincides with that in [BDHL25, §3] when Z is finite and generates G over H. In this case, $\mathcal{C}_{H,Z}(G)$ has only finitely many edge orbits and G acts *cocompactly* on $\mathcal{C}_{H,Z}(G)$, hence $\mathcal{C}_{H,Z}(G)$ (viewed as a simplicial graph) is a Cayley–Abels–Rosendal graph for G.

5. Arc and curve models for subgroups of the mapping class group

In this section, we adapt the construction in Lemma 4.4 to the context of subgroups of a mapping class group of an infinite type surface S. More specifically, if $G \leq \mathrm{Map}(S)$ is a locally bounded Polish subgroup, we construct an arc and curve model (Definition 3.1) that is also a coarse Cayley-Abels-Rosendal graph for G.

Let S be an infinite-type surface and let $G \leq \mathrm{Map}(S)$ be a locally bounded Polish subgroup. Suppose $\mu \in \mathcal{K}(S)$ is a finite collection of simple arcs and simple closed curves with a coarsely bounded (set-wise) G-stabilizer $\nu_\mu := \mathrm{stab}_G(\mu)$.

Then there exists a countable set $Z \subset G$ such that $G = \langle \nu_\mu, Z\rangle$. Define a graph $\mathcal{M}_{\mu,Z}(G)$ with vertex set $V(\mathcal{M}_{\mu,Z}(G)) := G\mu$. Consider the G-equivariant bijection $V(\mathcal{C}_{\nu_\mu,Z}(G)) = G/\nu_\mu \xrightarrow{\simeq} G\mu = V(\mathcal{M}_{\mu,Z}(G))$ defined by $g\nu_\mu \mapsto g\mu$ and obtain $\mathcal{M}_{\mu,Z}(G)$ by pushing forward the (metric) edge relation in $\mathcal{C}_{\nu_\mu,Z}(G)$ to $V(\mathcal{M}_{\mu,Z}(G))$. Note that $\mathcal{M}_{\mu,Z}(G)$ is G-equivariantly isometric to $\mathcal{C}_{\nu_\mu,Z}(G)$ and hence a coarse Cayley–Abels–Rosendal graph for G: if there exists such a μ, then G is coarsely equivalent to $\mathcal{M}_{\mu,Z}(G)$.

By Remark 4.5, G is boundedly generated if and only if Z can be chosen to be finite, which by Remark 4.7 occurs if and only if $\mathcal{C}_{\nu_\mu,Z}(G)$ (and hence $\mathcal{M}_{\mu,Z}(G)$) is a Cayley–Abels–Rosendal graph. The following lemma shows that there indeed exists a μ with coarsely bounded G-stabilizer ν_μ, whence Theorem 1.1 follows.

Lemma 5.1. *Let $G \leq \mathrm{Map}(S)$ be a locally bounded Polish subgroup. Then there exists $\mu \in \mathcal{K}(S)$ such that ν_μ is coarsely bounded in G.*

Proof. Since G is locally bounded, Lemma 2.22 tells us that there exists $\Sigma \subset S$ with a coarsely bounded G-stabilizer ν_Σ. Let $\mu_0 \in \mathcal{K}(S) \cap \mathcal{K}(\Sigma)$ be a filling collection of arcs and curves in Σ and $\mu := \mu_0 \cup \partial\Sigma$; clearly $\nu_\Sigma \leq \nu_\mu$. Let $S(\mu)$ denote the permutation group on μ and let K denote the kernel of the natural map $\nu_\mu \to S(\mu)$. In general $\nu_\Sigma \subseteq K$. Since μ_0 is filling, by Alexander's method [FM11, Prop. 2.8] we have in fact $\nu_\Sigma = K$. Finally since μ is a finite collection, so is $S(\mu)$ and consequently $[\nu_\Sigma : \nu_\mu] \leq |S(\mu)| < \infty$. By Corollary 2.13, ν_μ is also coarsely bounded in G, as required. □

6. Asymptotic dimension

Throughout this section, we assume S is an infinite-type surface with a non-displaceable subsurface for a Polish subgroup $\mathrm{PMap}_c(S) \leq G \leq \mathrm{Map}(S)$ such that G is boundedly generated or $G \in \{\overline{\mathrm{PMap}_c(S)}, \mathrm{PMap}(S), \mathrm{Map}(S)\}$ and locally bounded. To prove Theorem 1.2, it suffices to choose μ and Z so that $\mathcal{M} = \mathcal{M}_{\mu,Z}$ is witness-cocompact: by Theorem 3.21 $\operatorname{asdim} V(\mathcal{M}) = \infty$, hence likewise $\operatorname{asdim}\mathrm{Map}(S) = \infty$ by Theorem 1.1(1). Since $\mathrm{PMap}_c \leq G$ it suffices to ensure that

(i) $\mathcal{M}$ has a witness, and (by Remark 3.12)

(ii) that edge- and self-intersection numbers are uniformly bounded.

Only condition *(i)* uses non-displaceability; note also that it may be satisfied for arbitrary locally bounded Polish subgroups $G \leq \mathrm{Map}(S)$. We choose μ so that ν_μ is coarsely bounded and $V(\mathcal{M}) = G\mu$ has a witness. Let $\Delta \subset S$ to be a compact, essential, G-non-displaceable subsurface sufficiently large that ν_Δ is coarsely bounded. Fix $\mu_0 \in \mathcal{K}(\Delta) \cap \mathcal{K}(S)$ to be a filling collection of curves in Δ and let $\mu = \mu_0 \cup \partial\Delta$. By Lemma 5.1, ν_μ is coarsely bounded. For any $g \in G$, $g\mu_0$ is filling in $g\Delta$, hence since $g\Delta$ intersects Δ essentially likewise does $g\mu = g\mu_0 \cup \partial g\Delta$: Δ is a witness for $\mathcal{M}$.

Intersection numbers are invariant under $G \leq \mathrm{Map}(S)$, hence to ensure *(ii)* it suffices that

(ii') $i(\mu, z_i\mu)$ is uniformly bounded over $z_i \in Z$.

In particular, $i(g\mu, g\mu) = i(\mu,\mu) < \infty$ and if $(g\mu, k\mu) \in E(\mathcal{M})$ then $g^{-1}k = hz_ih'$ for some $z_i \in Z$ and $h, h' \in \nu_\mu$, thus $i(g\mu, k\mu) = i(h^{-1}\mu, z_ih'\mu) = i(\mu, z_i\mu)$. When G is boundedly generated we may choose Z to be finite, hence *(ii')* is immediate.

For the remainder of the section, we construct for each locally bounded case a (countable) $Z \subset G$ that generates G over ν_μ and satisfies *(ii')*.

6.1. **Enforcing small intersection.** We first produce topological generating sets for $G = \overline{\mathrm{PMap}_c(S)}, \mathrm{PMap}(S)$ satisfying *(ii')*. In particular, since ν_μ is open these generate G over ν_μ.

Lemma 6.1. *There exists a countable generating set T for $\mathrm{PMap}_c(S)$ such that $\{i(\mu, t\mu)\}_{t \in T}$ is finite and hence bounded above.*

Proof. Let $\Sigma_0 \subset \Sigma_1 \subset \Sigma_1 \subset \cdots$ be a compact exhaustion of S such that

(1) $\Sigma_0 \supset \Delta$.
(2) If C^i_j denotes the simple closed curves corresponding to the Dehn-Likorish generators for $\mathrm{Map}(\Sigma_i)$, then $\{C^i_j\}_j \cup \partial\Sigma_i \subset \{C^{i+1}_k\}_k \cup \partial\Sigma_{i+1}$.

Then the collection of Dehn twists $T := \{T_{ij}\}$ along the simple closed curves C^i_j generate $\mathrm{PMap}_c(S)$. Moreover, for sufficiently large i, $\{C^{i+1}_k\} \setminus \big(\{C^i_j\} \cup \partial\Sigma_i\big)$ only consists of simple closed curves outside of Δ and hence their corresponding Dehn twists fix Δ pointwise and therefore μ. As such, only finitely many of the Dehn twists T_{ij} act non-trivially on μ, hence $\{i(\mu, t\mu)\}$ is a finite collection bounded above by the maximum over these finitely many Dehn twists $t \in T$ that act non-trivially on μ. □

Recall that a *handle shift* is a shift map (see Section 2.4) with homeomorphic subsurfaces $\Sigma_i \cong \Sigma^1_1$, and let $h_\pm \in \mathrm{Ends}_g(S)$ denote the (forward and backward) accumulation points of the underlying path, which we will call the *endpoints* of h.

Lemma 6.2. *Let $H \subset \mathrm{PMap}(S)$ be a collection of handle shifts such that $\{(h_-, h_+) : h \in H\}$ is dense in $\mathrm{Ends}_g(S) \times \mathrm{Ends}_g(S)$. For any neighborhood $1 \in \nu \subset \mathrm{Map}(S)$, $\mathrm{PMap}(S) \leq \langle H, \mathrm{PMap}_c(S), \nu\rangle$.*

Proof. Let $\nu_P := \nu \cap \mathrm{PMap}(S)$ be a clopen subgroup of $\mathrm{PMap}(S)$. We know that $\mathrm{PMap}(S)$ is topologically generated by Dehn twists (which are compactly supported) and handle shifts [PV18, Thm. 4], [APV20, Cor. 6 and Section 2.3]. Since H is dense in $\mathrm{Ends}_g(S) \times \mathrm{Ends}_g(S)$, $\mathrm{PMap}(S)$ is in fact topologically generated by Dehn twists and H [AV20, Thm. 4.4]. If we consider translates of ν_P by Dehn twists and elements of H, we therefore get an open cover of $\mathrm{PMap}(S)$. Hence $\mathrm{PMap}(S) = \langle H, \mathrm{PMap}_c(S), \nu_P\rangle$ which implies $\mathrm{PMap}(S) \leq \langle H, \mathrm{PMap}_c(S), \nu\rangle$. □

Lemma 6.3. *There exists a countable set of handle shifts $H \subset \mathrm{PMap}(S)$ whose endpoints are dense in $\mathrm{Ends}_g(S) \times \mathrm{Ends}_g(S)$ and for which $i(\mu, h\mu)$ is uniformly bounded for $h \in H$.*

Proof. Let $S_1, \ldots S_b$ be the complementary components of Δ. Fix $k = \binom{b}{2}$ many pairwise disjoint strips $s_{\{i,j\}}$, connecting the i^{th} and j^{th} complementary components of Δ with $i \neq j$. For two distinct ends $x, y \in \mathrm{Ends}_g(S)$ accumulated by genus, consider handle shifts h_{xy} such that

(1) If $x, y \in S_i$, then $\mathrm{Domain}(h_{xy}) \cap \Delta = \varnothing$.
(2) If $x, x' \in S_i$, $y, y' \in S_j$ and $i \neq j$, then $\mathrm{Domain}(h_{xy}) \cap \Delta = s_{\{i,j\}}$ and $h_{xy}|_{s_{\{i,j\}}} = h_{x'y'}|_{s_{\{i,j\}}}$.
(3) For $x, y \in \mathrm{Ends}_g(S)$, $h_{yx} = h^{-1}_{xy}$.

Fix $E \subset \mathrm{Ends}_g(S) \times \mathrm{Ends}_g(S)$ a countable dense subset and set $H := \{h_{xy} \mid (x, y) \in E\}$. Let $h_{ij} := h|_{s_{\{i,j\}}}$ for $h \in H$ such that $h_- \in S_i,\ h_+ \in S_j$. Note that by (2), this is well-defined and for any $h \in H$, since $\mu \subset \Delta$, either h fixes μ pointwise or $i(\mu, h\mu) = i(\mu, h_{ij}\mu)$ for some $i \neq j$. Hence $i(\mu, h\mu)$ is uniformly bounded by $\max_{i,j} i(\mu, h_{ij}\mu)$ for any $h \in H$ as required. □

When $G = \overline{\mathrm{PMap}_c(S)}$, it suffices that $Z = T$, and when $G = \mathrm{PMap}(S)$, that $Z = T \cup H$. //

Remark 6.4. By [Hil25] (Theorem 2.30) $G = \mathrm{PMap}(S)$ is locally bounded if and only if it is boundedly generated, in which case Z may be chosen to be (in fact) finite. We include Lemma 6.3 so that our arguments are self-contained.

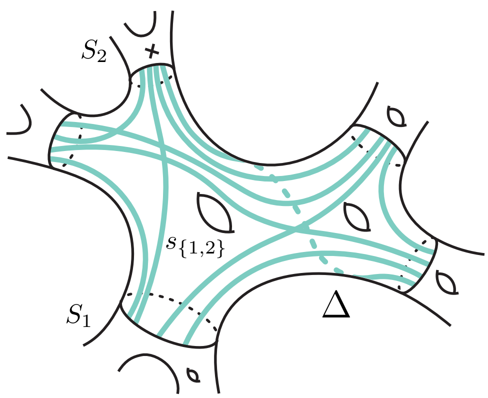

FIGURE 2. The strips $s_{\{i,j\}}$ in Δ

We conclude with the case when $G = \mathrm{Map}(S)$. We aim to construct a transversal I for $K := \langle T, H, \nu_\mu \rangle$ that satisfies the intersection condition *(ii')*; since K is open, I is countable and we may set $Z = T \cup H \cup I$. Consider the exact sequence

$$1 \to \mathrm{PMap}(S) \to \mathrm{Map}(S) \xrightarrow{\pi} \mathrm{Homeo}(E, E_g) \to 1$$

It suffices to construct a (set-theoretic) section σ for π whose whose image satisfies *(ii')*. Then $I_\sigma := \operatorname{im} \sigma$ is a transversal for $\mathrm{PMap}(S)$ (albeit possibly uncountable), and since $\mathrm{PMap}(S) \leq K$ by Lemma 6.2, I_σ contains a transversal $I \subset I_\sigma$ for K likewise satisfying *(ii')*. We construct σ below.

Lemma 6.5. *There exists a (set-theoretic) section* $\sigma : \mathrm{Homeo}(E, E_g) \to \mathrm{Map}(S)$ *such that* $i(\mu, \sigma(\varphi)\mu)$ *is uniformly bounded over* $\varphi \in \mathrm{Homeo}(E, E_g)$.

Remark. In the following, let $\Sigma^b_{g,p}$ denote the orientable surface with genus g, b boundary components, and p punctures.

Proof. Fix some connected, essential subsurface $\Pi \cong \Sigma^b_{g,0} \subset S$ such that $\Pi \supset \Delta$ and whose complementary components have either zero or infinite genus. Fix an embedding of Π into $\Sigma = \Sigma^{b^2}_{g,0}$ such that $\Sigma \setminus \Pi$ is the disjoint union of copies of $\Sigma^b_{0,1}$. Let $\sigma_0 : \mathrm{Sym}(\pi_0(\partial\Sigma)) \to \mathrm{Map}(\Sigma)$ be a choice of (set-theoretic) section; note that $i(\mu, \sigma_0(\alpha)\mu)$ is uniformly bounded over $\alpha \in \mathrm{Sym}(\pi_0(\partial\Sigma))$, a finite set.

Let $U_i \subset E$ be the clopen partition induced by Π, and let $U_{i,j} = U_i \cap \varphi(U_j)$; let $S_i \subset S \setminus \Pi$ denote the complementary component containing U_i and $C_i := \partial S_i$. Extend each component of $\partial\Pi$ by an embedded (but not necessarily essential) Σ_0^{b+1} to obtain a subsurface $\Pi_\varphi \cong \Sigma$ inducing the partition $U_{i,j}$. Note that some components of $S \setminus \Pi_\varphi$ may be disks and that every component has either zero or infinite genus. Let $S_{i,j} \subset S \setminus \Pi_\varphi$ denote the complementary component containing $U_{i,j}$ and $C_{i,j} := \partial S_{i,j}$.

Extend the embedding $\Pi \hookrightarrow \Pi_\varphi$ to a homeomorphism $\psi_\varphi : \Sigma \to \Pi_\varphi$. Fix a permutation $\alpha_\varphi \in \mathrm{Sym}(\pi_0(\partial\Sigma))$ and $\sigma_1(\varphi) := \sigma_0(\alpha_\varphi)^{\psi_\varphi} : \Pi_\varphi \to \Pi_\varphi$ such that $C'_j := \sigma_1(\varphi)C_j$ separates $C_{i,j}$ from $\sigma_1(\varphi)\Pi$ for all i. Let $\Pi' := \sigma_1(\varphi)\Pi$ and S'_j denote the component of $S \setminus \Pi'$ separated by C'_j. It follows that $\Pi' = \sigma_1(\varphi)\Pi$ induces the partition $\varphi(U_j)$, each of which is contained in S'_j. We note that C'_j is homeomorphic to C_j; likewise, since the S_j and S'_j have either zero or infinite genus, S_j has zero genus if and only if $U_j \cap \mathrm{Ends}_g(S) = \varnothing$ if and only if $\varphi(U_j) \cap \mathrm{Ends}_g(S) = \varnothing$ if and only if S'_j has zero genus, and otherwise both S_j, S'_j have infinite genus. Finally, we apply Richards' classification theorem [Ric63] to obtain $\sigma(\varphi)$ by extending $\sigma_1(\varphi)|_\Pi$ by homeomorphisms $\overline{S}_j \to \overline{S}'_j$ that induce φ on each U_j. Up to isotopy $\mu \subset \Delta \subset \Pi$, hence $i(\mu, \sigma(\varphi)\mu) = i(\mu, \sigma_0(\alpha_\varphi)\mu)$ is bounded independently of φ. $\square$

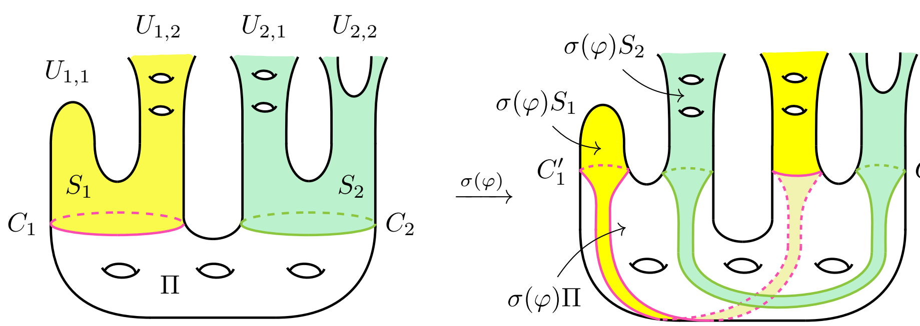

FIGURE 3. The map $\sigma(\varphi)$. Here, $U_{1,1} = \varnothing$

References

[APV20] Javier Aramayona, Priyam Patel, and Nicholas G. Vlamis, *The first integral cohomology of pure mapping class groups*, Int. Math. Res. Not. IMRN (2020), no. 22, 8973–8996. MR 4216709

[AV20] Javier Aramayona and Nicholas G. Vlamis, *Big mapping class groups: an overview*, arXiv preprint, https://arxiv.org/abs/2003.07950, 2020.

[BBF15] Mladen Bestvina, Ken Bromberg, and Koji Fujiwara, *Constructing group actions on quasi-trees and applications to mapping class groups*, Publications Mathématiques de l'IHÉS **122** (2015), 1–64 (en).

[BD07] G. Bell and A. Dranishnikov, *Asymptotic dimension*, arXiv preprint, https://arxiv.org/abs/math/0703766, 2007.

[BDHL25] Beth Branman, George Domat, Hannah Hoganson, and Robert Lyman, *Graphical models for topological groups: A case study on countable stone spaces*, Bulletin of the London Mathematical Society (2025).

[BHS19] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto, *Hierarchically hyperbolic spaces II: Combination theorems and the distance formula*, Pacific Journal of Mathematics **299** (2019), no. 2, 257–338.

[BHS21] ———, *Quasiflats in hierarchically hyperbolic spaces*, Duke Mathematical Journal **170** (2021), no. 5, 909 – 996.

[BL08] Sergei Buyalo and Nina Lebedeva, *Dimensions of locally and asymptotically self-similar spaces*, St. Petersburg Math. J. **19** (2008), 45–65.

[DT17] Spencer Dowdall and Samuel J. Taylor, *The co-surface graph and the geometry of hyperbolic free group extensions*, Journal of Topology **10** (2017), no. 2, 447–482.

[FM11] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical, Princeton University Press, 2011.

[Gab14] David Gabai, *On the topology of ending lamination space*, Geometry & Topology **18** (2014), no. 5, 2683 – 2745.

[GRV21] Curtis Grant, Kasra Rafi, and Yvon Verberne, *Asymptotic dimension of big mapping class groups*, arXiv preprint, https://arxiv.org/abs/2110.03087, 2021.

[Hil25] Thomas Hill, *Large-scale geometry of pure mapping class groups of infinite-type surfaces*, Proc. Amer. Math. Soc. **153** (2025), no. 6, 2667–2680. MR 4892635

[Kec12] Alexander S. Kechris, *Dynamics of non-archimedean polish groups*, p. 375–397, European Mathematical Society, Jul 2012.

[Kop23] Michael C. Kopreski, *Multiarc and curve graphs are hierarchically hyperbolic*, 2023.

[Kop24] ———, *The asymptotic dimension of the grand arc graph is infinite*, 2024.

[MR23] Kathryn Mann and Kasra Rafi, *Large-scale geometry of big mapping class groups*, Geom. Topol. **27** (2023), no. 6, 2237–2296. MR 4634747

[Nöb30] G. Nöbeling, *Über eine n-dimensionale Universalmenge im* $\mathbb{R}_{2n+1}$, Mathematische Annalen **104** (1930), 71–80.

[PV18] Priyam Patel and Nicholas G. Vlamis, *Algebraic and topological properties of big mapping class groups*, Algebr. Geom. Topol. **18** (2018), no. 7, 4109–4142. MR 3892241

[Ric63] Ian Richards, *On the classification of noncompact surfaces*, Transactions of the American Mathematical Society **106** (1963), no. 2, 259–269.

[Ros21] Christian Rosendal, *Coarse geometry of topological groups*, Cambridge Tracts in Mathematics, Cambridge University Press, 2021.

[RS09] Kasra Rafi and Saul Schleimer, *Covers and the curve complex*, Geometry & Topology **13** (2009), no. 4, 2141 – 2162.

[SC24] Anschel Schaffer-Cohen, *Graphs of curves and arcs quasi-isometric to big mapping class groups*, Groups Geom. Dyn. **18** (2024), no. 2, 705–735. MR 4729823

[Sch] Saul Schleimer, *Notes on the complex of curves*, https://sschleimer.warwick.ac.uk/Maths/notes2.pdf.