Planetary System Formation - Prof Luca Matra

Marc Lane

lanem2@tcd.ie

These notes reflect my own summary and understanding of lectures presented by Professor Luca Matra for the module "Planetary and Space Science" at Trinity College Dublin. In these notes, ω and Ω as angular velocity are used interchangeably.

1 Lecture 1

1.1 Solar System Budgets

Mass Budget of Solar System: Dominated by the star.

The general definition of spin angular momentum is given by:

$$L_s = I\Omega \tag{1}$$

where I is the momentum of inertia and Ω is the angular velocity. For a solid sphere $I = \frac{2}{5}M_sR_s^2$, subbing this into L_s gives $L_s \approx \frac{2}{5}M_sR_s^2\Omega$ where the $\frac{2}{5}$ factor can change due to the differential rotation of the sun.
Angular Momentum Budget: Dominated by the planets.

A planet's orbital angular momentum is given by $L \approx M_p\sqrt{GM_sa_p}$, where a_p is the planet's semi-major axis. We can derive where this formula for orbital angular momentum comes from. Starting from the definition of angular momentum:

$$L_p = r \times p \tag{2}$$

Subbing in $p = M_pv$ and $r = a_p$, and since r and p are orthogonal:

$$L = a_pM_pv \tag{3}$$

The planet will follow a Keplerian orbit, meaning it will obey Kelper's laws. We can find the Keplerian velocity by equating Newton's law of gravitation and the centripetal force. Solving for v gives

$$v_{kep} = \sqrt{\frac{GM_s}{a_p}} \tag{4}$$

Subbing v_{kep} into $L = a_pM_pv$ gives $L \approx M_p\sqrt{GM_sa_p}$.

1.2 Radial Distribution of Mass

We can examine the mass of material of Solar composition (mostly H+He) that would have been needed to condensate into known planet mass (mostly heavy elements) at given location. Then we uniformly spread this total mass of material necessary to make planet into area of annulus half-way to the next inner and outer planet. This leads to the concept of mass surface density:

$$\Sigma = \frac{dM}{dA} = \frac{M}{A} \tag{5}$$

We observe that in the solar system the mass surface density follows the power law with distance r from the sun:

$$\Sigma \propto r^{-\alpha} \tag{6}$$

where $\alpha \approx \frac{3}{2}$. Plotting this in log-log space, we get Figure 1.

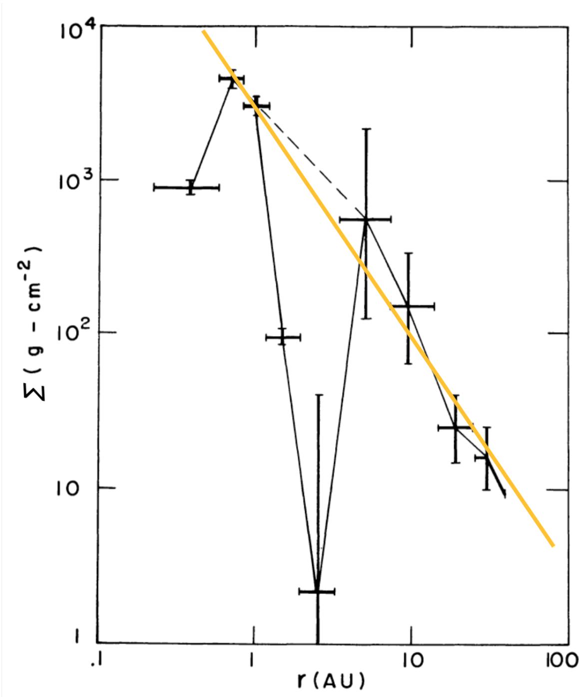

Figure 1: The Minimum Mass Solar Nebula (MMSN). The data points are the Solar system planets. Yellow line is $\Sigma \propto r^{-\alpha}$. Outlier points are Mars and asteroid belt. The yellow line is the minimum mass surface density at a given radius to produce the planet masses in the formation environment of the sun, i.e. the minimum mass solar nebula.

2 Lecture 2 - Cloud Core Collapse

2.1 Star Formation

Stars form in molecular clouds. The precursors to stars are cloud cores, which are high density compact regions with a molecular cloud. The full mechanism behind cloud core formation in a molecular cloud is not yet well understood. Molecular clouds are primarily dominated by H2, but CO is easier to observe despite the mass being dominated by H2. We can consider a cloud core to be roughly a sphere of mass M, radius R and uniform density ρ. The gravitational binding energy will be on the order of magnitude of:

$$E_{binding} \sim -\frac{GM^2}{R} \tag{7}$$

For uniform density sphere, we have $\rho \sim \frac{M}{R^3}$, so $M \sim \rho R^3$. Subbing in $E_{binding} \sim -\frac{G(\rho R^3)^2}{R}$, which gives $E_{binding} \propto R^5$.

The thermal kinetic energy of the particles in the core acts against the gravitational binding energy. The kinetic energy of a single mole of gas is $E_{kin} \sim \mu < v^2 >$, where v is the average velocity and μ is is the mean molecular weight ($\mu = Nm_p$ where on average $N = 2$ since the cloud is dominated by molecular hydrogen). The kinetic energy of the particles is equivalent to the thermal energy, which is given by:

$$E_{thermal} \sim Nk_BT = \frac{M}{\mu}k_BT \sim \frac{\rho R^3}{\mu}k_BT \tag{8}$$

so $E_{thermal} \propto R^3$. Therefore,

$$\frac{E_{binding}}{E_{thermal}} \propto R^2 \tag{9}$$

For this setup, a result from the virial theorem tells us that $E_{binding} = 2E_{thermal}$. So $E_{binding} \sim E_{thermal}$. When we set $E_{binding} = E_{thermal}$ and solve for R and M, we get the Jeans radius/length and the Jeans mass:

$$R_j \sim \sqrt{\frac{k_BT}{G\mu\rho}} \tag{10}$$

$$M_j \sim \sqrt{\frac{1}{\rho}\left(\frac{k_BT}{G\mu}\right)} \tag{11}$$

If the radius of the cloud is greater than R_j then gravity dominates and the core collapses. If the radius of the cloud is less than R_j then thermal pressure prevents the core from collapsing. The same logic can be applied to the mass M_j. It should be noted that the Jeans length can also be rewritten as

$$R_j \sim \sqrt{\frac{k_BT}{G\mu\rho}} = \frac{c_s}{t_{ff}} \tag{12}$$

where t_{ff} is the free-fall timescale (the time taken for an object to fall into the COM of an object) $t_{ff} = \sqrt{\frac{1}{G\rho}}$, and c_s is the sound speed given by:

$$c_s = \sqrt{\frac{k_BT}{\mu}} \tag{13}$$

Since the Jeans length and mass are dependant on temperature, we conclude that the gravitational collapse of a cloud core is a consequence of the cooling of a star. Assuming the cloud core isn't dispersed, the core will inevitably collapse by cooling.

2.2 Specific Angular Momentum of Cores

If the cloud has some initial rotation, the core will inherit angular momentum from the cloud. We can measure the magnitude of the rotation of a core by measuring the redshift/blueshift of either side of the core. Typically, we observe angular frequency of $\Omega = 10^{-14} - 10^{-13}$ [s^{-1}]. We can define the specific angular momentum of a core as the angular momentum per unit mass. Namely,

$$j = \frac{L}{M} \tag{14}$$

Since $L = r \times p = mvr$, and $v = \Omega r$, so $L = MR^2\Omega$. Subbing into j we get:

$$j = \frac{MR^2\Omega}{M} = R^2\Omega \tag{15}$$

If the core is just at the Jeans length, $j_{core} = R_j^2\Omega_{core} = 10^{21-22}$ [cm^2 s^{-1}]. This value for j will be a conserved process in the core collapse.

We can compare this value of j_{core} that the core will have when it collapses to the maximum j a core can have before it breaks up due to the centripetal force. This breakup will happen when

$$F_{gravity} = F_{centripetal} \tag{16}$$

or in terms of acceleration

$$a_{gravity} = a_{centripetal} \tag{17}$$

$$\frac{GM}{R^2} = \frac{mv^2}{R} = \Omega^2 R \tag{18}$$

Solving for Ω, we get

$$\Omega_{breakup} = \sqrt{\frac{GM}{R^3}} \tag{19}$$

For a sun-like star, $\Omega_{breakup} \sim 10^{-3}$ [s^{-1}]. Subbing this into $j = R^2\Omega$, we get $j_{breakup} \sim 10^{18-19}$[cm^2 s^{-1}]. Comparing the values of j_{core} and $j_{breakup}$, we can see that j_{core} is larger than the breakup specific angular momentum. To prevent breakup, the cloud core must lose angular momentum. This means that j cannot be conserved in the collapsing process. We have already seen that most of the angular momentum in a system is not in the star, but in the planets.

3 Lecture 3 - Disk Formation

3.1 Disk Formation Derivation

Figure 2 shows a parcel of gas, initially at the edge of a cloud core, and its motion into the core. When the parcel starts at the edge of the core, the only force it feels is the gravitational attraction towards the centre of the core. The parcel doesn't move radially straight towards the COM since it has an initial angular momentum rotation. This angular momentum applies a centripetal acceleration on the parcel of gas. As the parcel gets closer to the COM, the gravitational acceleration increases and, due to conservation of angular momentum, the centripetal acceleration also increases. This gives the parcel a parabolic trajectory. The length of closest approach to the COM is r_{min} and distance from the COM to the point where the parcel intersects the equator of the core is the equatorial radius r_{eq}. Geometrically, we have radial distance r of the parcel in terms of the angle Ψ as

$$r = \frac{r_{eq}}{1 + \cos\Psi}. \tag{20}$$

Additionally, we can relate r_{min} and r_{eq} by:

$$r_{min} = \frac{r_{eq}}{2}. \tag{21}$$

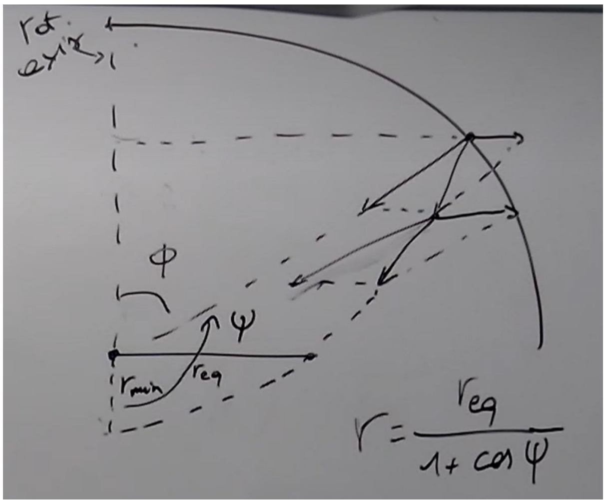

Figure 2: Trajectory of a gas parcel starting at the edge of a cloud core. Here, the parcel is rotating into the page.

We can describe this relationship more mathematically. Using angular momentum conservation, we can write

$$\omega_2 r_2^2 \sin^2(\phi_2) = \omega_1 r_1^2 \sin^2(\phi_1). \tag{22}$$

Rearranging, we get

$$\omega_2 = \omega_1 \frac{r_1^2 \sin^2(\phi_1)}{r_2^2 \sin^2(\phi_2)} \tag{23}$$

The formula for centripetal acceleration is $a = r\omega^2$. Therefore, at points r_1 and r_2 the centripetal acceleration will be

$$a_{r1} = \omega_1^2 r_1 \sin(\phi_1). \tag{24}$$

$$a_{r2} = \omega_2^2 r_2 \sin(\phi_2). \tag{25}$$

Subbing in the formula for ω_2 into a_{r2} we get

$$a_{r2} = r_2 \sin(\phi_2)\omega_1^2 \frac{r_1^4 \sin^4 \phi_1}{r_2^4 \sin^4 \phi_2} \tag{26}$$

Factoring out a_{r1} from this expression,

$$a_{r2} = a_{r1} \frac{r_1^3 \sin^3 \phi_1}{r_2^3 \sin^3 \phi_2} \tag{27}$$

This describes how the centripetal acceleration a_r evolves over the course of the trajectory. This shows that $a_r \propto \omega^2 r \sin\phi \propto r^{-3}(\sin\phi)^{-3}$. We conclude that $a_r \propto r^{-3}$. Therefore, as we get closer to the COM, a_r increases. Comparing this to the gravitational acceleration which scales as $g \propto r^{-2}$, we see that the centripetal acceleration dominates for small r. We can see this by calculating the ratio:

$$\frac{g}{a_r} \propto \frac{r^{-2}}{r^{-3}\sin(\phi)^{-3}} = r^{-1}\sin(\phi)^{-3} \tag{28}$$

This means the parcel doesn't end up in the COM, but instead only reaches the point r_{min}. At r_{min}, we have $g = a_r$. By equating $g = a_r$, we end up with Keplerian angular velocity:

$$\Omega_k = \sqrt{\frac{GM}{r^3}} \tag{29}$$

Since specific angular momentum $j = \frac{L}{M} = \Omega r^2$, we have

$$j = \Omega_k r_{min}^2 = \sqrt{\frac{GM}{r_{min}^3}} r_{min}^2 \sim \sqrt{GM r_{min}} \tag{30}$$

Solving for r_{min},

$$r_{min} = \frac{j^2}{GM} \tag{31}$$

Realistically, parcels of gas don't make it as far as r_{min}, because usually the parcel will collide with another parcel by the time it reaches r_{eq}. But since $r_{min} = \frac{r_{eq}}{2}$, we can roughly expect $r_{min} \sim r_{eq}$, so:

$$r_{min} \sim r_{eq} \sim \frac{j^2}{GM} \tag{32}$$

Therefore, r_{eq} crucially depends of the specific angular momentum j. For this reason, r_{eq} is also called the centrifugal radius. This means that if the parcel starts closer to the equator of the core, it will have a larger j and therefore a larger r_{eq}. Similarly, if the parcel starts close to the pole (far from equator), j will be smaller so r_{eq} will be small. In this case, the parcel falls straight down towards the COM without feeling much centripetal acceleration. This formula for r_{eq} agrees with observations of the size of protoplanetary disks.

We conclude that because the core has an initial rotation and a specific angular momentum j, the parcels follow a parabolic trajectory, and thus a disk of material will form. If the star hasn't formed yet, this disk is called a proto-stellar disk. Once the star forms, the disk is called a circumstellar disk. This disk is the same disk that planetary systems form in, and this explain why planets are found in a single plane.

Since stars don't form in isolation, there are a variety of different conditions that proto-stellar disks form in. There are a diverse variety of disk initial conditions such as binary systems.

3.2 General Disk Observations

Since disks are formed from molecular cloud material, the disk mass is dominated by H2. Since H2 has few rotational transitions, it is hard to observe. CO is much easier to observe, but is much rarer. The ratio of their number densities is $\frac{n_{CO}}{n_{H2}} \sim 10^{-4} - 10^{-6}$. Dust grains make up only 1% of the mass, but it dominates the opacity, making it easy to observe.

When making observations of disks in molecular clouds, the proto-stars are called Young Stellar Objects (YSO). They are characterised by the shape of their Spectral Energy Distribution (SED). For SEDs, instead of plotting a regular spectrum (flux on the y-axis), energy is plotted on the y-axis (flux times λ). However SED is a lose term, so sometimes flux can also be plotted on the y-axis. Typically, these plots are on a log-log scale.

Generally we observe a modified blackbody distribution with a small UV excess and high IR excess. The IR emission comes from cold dust grains which emit at higher wavelengths. The UV excess is produced by hot regions on the surface of the protostar, typically where material from the disk is being accreted onto the protostar.

4 Lecture 4 - Protoplanetary Disk Observations

4.1 IR Spectral Index

We can distinguish between SEDs of protoplanetary disks using the IR spectrum index. We measure the IR spectral slope of a SED, which is the slope of the SED in log-log space between near-IR and mid-IR wavelengths. This section of the spectrum is a straight line with negative slope given by:

$$\alpha_{IR} = \frac{\Delta \log(\lambda F_\lambda)}{\Delta \log(\lambda)} \tag{33}$$

The value of this slope gives us a measure of the amount of dust (which is emitting in the IR) in the disk. It is also a measure of the temperature of the dust, since colder dust will emit further in the IR.

4.2 Disk Classifications

Class 0

Figure 3 shows the SED for a class 0 disk. This is the earliest phase of disk formation, and the core is still undergoing gravitational collapse. If the protostar has formed yet, it is deeply embedded in the material from the core. The SED is dominated by the far-IR, due to cold dust emission. The V-shape feature is an absorption spectral feature of silicate dust grains. The spectral slope α_{IR} is not defined for class 0 objects.

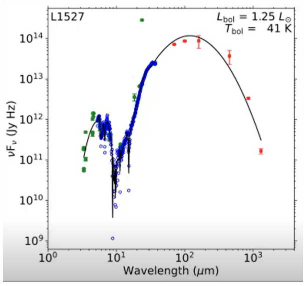

Figure 3: Class 0 SED

Class 1

Figure 4 shows the SED for a class 1 disk. By this point the disk has definitely formed. There is still material accreting on to the protostar from the core, however the protostar is not longer enshrouded in the core. As the protostar accretes, there is a bipolar flow jet shooting out from the centre of the disk. The peak of the SED is a shorter wavelength, therefore the material has collapsed more and has warmed up. There is still an envelope around the protostar, so we still excess IR due to dust grain emission. For class 1, $\alpha_{IR} >\sim 0$.

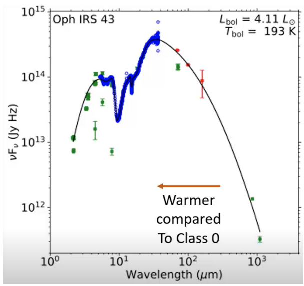

Figure 4: Class 1 SED

Class 2

Figure 5 shows the SED for a class 2 disk. At this point, there is no envelope left, so accretion of material onto the central star stops. The young, pre-main sequence star has now ignited, and it is optically visible. We begin to observe substructures in the disk. In figure 5, the blue line is the stellar spectrum in the absence of a disk, and the red and black dots is the observed spectrum. For class 2,$\alpha_{IR} >\sim -1$.

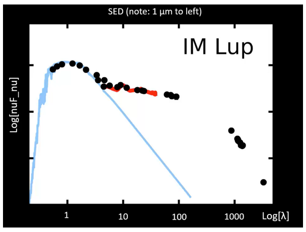

Figure 5: Class 2 SED

Class 3
Figure 6 shows the SED for a class 3 disk. The young star is clearly optically visible. There is little remaining dust emission is contrast to stellar emission, which dominates the SED. The is no remaining emission from the disk. This means we observe the expected Rayleigh-Jeans law for a stellar spectrum for high wavelengths. For class 3,$\alpha_{IR} >\sim -2$.

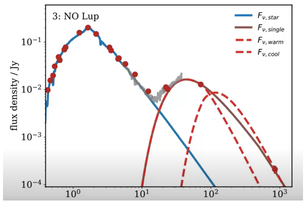

Figure 6: Class 3 SED

Figure 7 shows a summary of the classifications for young stellar objects.

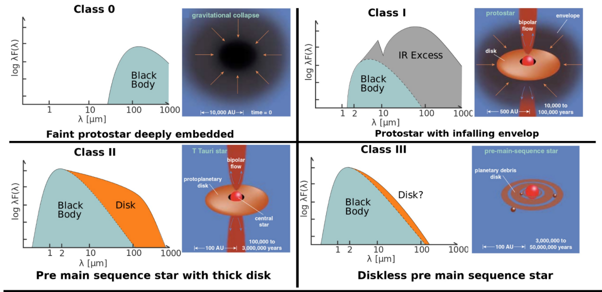

Figure 7: Summary of young stellar object classifications

Further Classifications
Another way to classify young stellar objects is from the accretion onto the protostar. This is done for Class 2 objects, where accretion is still happening but the star is also visible. Young low mass stars (less than solar mass) are known as T Tauri stars, and their higher mass counter parts (higher then solar mass) are known as Herbig Ae/Be stars. Due to the initial mass function, there are far fewer Herbig Ae/Be stars than T Tauri stars. Sometimes there is excess UV emission for T Tauri stars, due to high temperature accretion collisions onto the star. This allows us to further classify T Tauri stars into classical T Tauri stars (signatures of accretion, larger H alpha line) and weak-lined T-Tauri stars (no accretion, look like stellar photosphere, smaller H alpha line).

4.3 Class 2 Dust Emission

Class 2 disks are easier to study since since we can visibly observe the star but also study the disk. It was traditionally thought that class 2 disks are where planet formation takes place, and thus they take the name protoplanetary disks. If the dust was a certain distance away from the star, we would expect the SED to be a superposition of the dust blackbody spectrum and the stellar blackbody spectrum. This is not observed since the dust in the disk extends over a wide range of radii, so it has a variety of different temperatures. Inner-most disk emits at near IR and is very optically thick, further out dust emits at mid IR, and the outer disk emits at far IR and is optically thin.

We can measure the dust mass of a class 2 protoplanetary disk from its SED using the equation of radiative transfer. The solution of the equation of radiative transfer for a homogeneous medium is

$$I_\nu = I_{\nu B} e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu}) \tag{34}$$

where $I_{\nu B}$ is the incoming background intensity, τ_ν is the optical depth, and S_ν is the source function. The star is faint at the edge of the disk, so we can ignore the background radiation

$$I_\nu = S_\nu (1 - e^{-\tau_\nu}) \tag{35}$$

Since the edge of the disk is optically thin, $\tau_\nu \to 0$, so

$$I_\nu = S_\nu \tau_\nu \tag{36}$$

The source function will be a blackbody since the disk is in local thermodynamic equilibrium:

$$I_\nu = B_\nu \tau_\nu \tag{37}$$

We can write the optical depth for dust as $\tau_\nu = N\sigma_\nu = \Sigma\kappa_\nu$, where N is the number density, σ_ν is the cross section, Σ is the mean surface density and κ_ν is the opacity. So

$$I_\nu = B_\nu \Sigma \kappa_\nu \tag{38}$$

We can convert from intensity to flux by integrating over the solid angle:

$$F_\nu = \int I_\nu d\Omega = I_\nu \Delta\Omega = I_\nu \frac{\Delta A}{d^2} \tag{39}$$

where A is the area of the disk, and d is the distance to the disk.

$$F_\nu = B_\nu \Sigma \kappa_\nu \frac{\Delta A}{d^2} \tag{40}$$

But the surface density is defined as $\Sigma = \frac{M}{\Delta A}$, so

$$F_\nu = \frac{B_\nu M \kappa_\nu}{d^2} \tag{41}$$

We can measure F_ν at a ν that is optically thin. Then with an estimate of κ_ν and an estimate for the temperature to input into the blackbody function, this formula will then allow us to calculate the mass of the disk.

We can plot all the total dust masses for a number of disks to see the number of disks at certain masses. We can see that the class 0 disks are more massive than class 1 disks which are more massive than class 2 disks. Therefore, dust mass is being lost as a function of time. Observationally, we find that there is not enough mass in class 2 disks to form exoplanets. This leads to the conclusion that the planet formation process must begin earlier in the class 0 and class 1 stage. The mass of disks for the earlier stages of the evolution are about 1% of the stellar mass, which agrees with the mass needed for the minimum mass solar nebula from Lecture 1.

5 Lecture 5 - Protoplanetary Disk Structure 1

5.1 Disk Lifetimes

It is observed that protoplanetary disks lose their mass with time. We can examine how long they last before all of the mass is accreted onto the central star or used to form exoplanets. For class 2 stages, the accretion rate of the star is typically about $\dot{M} = \frac{dM}{dt} = 10^{-7} \sim 10^{-9}$ [$\mathrm{M}_\odot$ yr^{-1}]. We can write a timescale for the disk as

$$timescale = \frac{M_{disk}}{\dot{M}} \tag{42}$$

which for a disk of 1% solar mass and $\dot{M} = \frac{dM}{dt} = 10^{-7} \sim 10^{-9}$ [$\mathrm{M}_\odot$ yr^{-1}] gives a timescale of about 1-10 Myr. We can observationally test this figure by counting the ratio of the number of stars in a star-forming molecular cloud that have disks. Then the ratio is plotted as a function of the age of that cloud. We find that the fraction of stars with disks decays exponentially with time like

$$Fraction \sim Ae^{\frac{t}{\tau}} \tag{43}$$

At time time $t = 0$, the fraction is given by A. It is observed that $A \approx 100\%$. Since most stars start with a disk, this means that most stars have an opportunity to form planets. For this model, $\tau \approx$ a few Myr. This value gives the upper limit of the time it takes for planets to form. The most giant planets most from within this given timescale.

5.2 Disk Physics Basics

Now we begin a physical description of the structure of protoplanetary disks. We have seen that disks are primarily made of gas (H2), with some amount of dust (dust grains). This means that protoplanetary disks can be described as a fluid.

A parcel of gas in the mid plane of protoplanetary disk will be subject to radial gravitational acceleration (toward the star) $a_g = \frac{GM_\star}{r^2}$ and centripetal acceleration (away from the star) $a_c = \frac{v_\phi^2}{r}$, where v_ϕ is the tangential velocity.

In addition to this, there is a pressure gradient force from the other gas particles. We can calculate this contribution by imagining a cubic parcel of fluid with area dA, height dr and density ρ. Therefore, it will have mass $m = \rho drdA$. The pressure difference dP between the top and bottom of the parcel results in a buoyant force. The magnitude of this force is

$$F = --dPdA = ma = \rho drdAa_{pres\_grad} \tag{44}$$

Solving for a_{pres_grad} gives

$$a_{pres\_grad} = -\frac{1}{\rho}\frac{dP}{dr} \tag{45}$$

This pressure gradient force is always opposing the increase in pressure. In the disk, the pressure is higher closer to the star, due to the higher number density and higher temperature. This is due to the ideal gas law which applies to disks, given as

$$P = nRT \tag{46}$$

where $n = \frac{N}{V}$. This means that the pressure gradient force will point away from the star (in the same direction as centripetal force).

We can now write the acceleration balance for the gravitational, centripetal and pressure gradient accelerations as

$$\frac{v_\phi^2}{r} = \frac{1}{\rho}\frac{dP}{dr} + \frac{GM_\star}{r^2} \tag{47}$$

If $\frac{dP}{dr} = 0$, then we can solve for v_ϕ to get the Keplerian velocity

$$v_k = \sqrt{\frac{GM_\star}{r}} \tag{48}$$

This can also be written as an angular velocity using the relation $\Omega_k = \frac{v_k}{r}$ which gives

$$\Omega_k = \sqrt{\frac{GM_\star}{r^3}} \tag{49}$$

Dust grains (or planets), as opposed to gas parcels, do not feel the pressure gradient force. This means they will orbit at Keplerian velocity. On the other hand, gas parcels orbit at sub-Keplerian velocities due to the pressure gradient force. Therefore, solving for v_ϕ using the acceleration balance equation will give sub-Keplerian v_ϕ for gas. Typically we get that v_ϕ is 0.5% below Keplerian velocity. This is important later on for planet formation.

5.3 Hydrostatic Balance

Now we consider the force balance for a gas parcel above the mid plane of the disk. Figure 8 shows the forces acting on gas parcel at the edge of the mid plane. For this parcel, we can balance the vertical component of the gravitation acceleration with the pressure gradient force. The pressure gradient force will oppose the vertical component of gravity, since the pressure will be higher closer to the mid plane of the disk.

Typically, disks are vertically thin, so $z << r$, and $\sin\theta \approx \theta$. The balance between gravity and pressure is called hydrostatic balance, which is given as

$$-\frac{1}{\rho}\frac{dP}{dz} = g_z \tag{50}$$

Rearranging to get

$$\frac{dP}{dz} = -\rho g_z \tag{51}$$

So far, we have implicitly neglected that the disk has mass, since the star dominates the mass and therefore the gravitational attraction. Once again neglecting the disk mass, we get

$$g_z = \frac{GM_\star}{d^2}\sin\theta \tag{52}$$

Geometrically, $z = d\sin\theta$, so

$$g_z = \frac{GM_\star}{d^3}z \tag{53}$$

Recognizing that $z << r$, and so $d \approx r$ gives

$$g_z = \frac{GM_\star}{r^3}z \tag{54}$$

Earlier, we saw that the Keplerian angular velocity $\Omega_k = \sqrt{\frac{GM_\star}{r^3}}$, so

$$g_z = \Omega_k^2 z \tag{55}$$

Subbing this into the Hydrostatic balance equation,

$$\frac{dP}{dz} = -\rho\Omega_k^2 z \tag{56}$$

This equation describes the Hydrostatic balance, but it doesn't directly tell us about the structure of the disk. We can find out about the structure of the disk by using an equation of state to get from pressure to density. Assuming the disk is an ideal gas, we have

$$P = nk_BT \tag{57}$$

Multiplying and dividing by the mean molecular weight, this can be rewritten as

$$P = n\mu\frac{k_BT}{\mu} \tag{58}$$

The definition of the mass density is $n\mu = \rho$, and the speed of sound $\sqrt{\frac{k_BT}{\mu}} = c_s$, so the ideal gas law can be rewritten as

$$P = \rho c_s^2 \tag{59}$$

Since the speed of sound is constant with z, we can plug this equation for P into the hydrostatic equilibrium equation to get

$$c_s^2\frac{d\rho}{dz} = -\rho\Omega_k^2 z \tag{60}$$

or rearranging as

$$c_s^2\frac{1}{\rho}\frac{d\rho}{dz} = -\Omega_k^2 z \tag{61}$$

Using the mathematical trick (chain rule) that $\frac{1}{\rho}\frac{d\rho}{dz} = \frac{d\log(\rho)}{dz}$, we have

$$\frac{d\log(\rho)}{dz} = -z\frac{\Omega_k^2}{c_s^2} \tag{62}$$

We define the scale height of the disk as $H = \frac{c_s}{\Omega_k}$, so

$$\frac{d\log(\rho)}{dz} = -\frac{z}{H^2} \tag{63}$$

Solving this differential equation by integrating both sides,

$$\rho(z) = \rho_0 e^{-\frac{z^2}{2H^2}} \tag{64}$$

This is a Gaussian function with standard deviation $\sigma = H$. This means that the disk has a Gaussian vertical structure, with highest density at the mid plane.

If we increase the temperature, c_s will increase, so H will increase, and the vertical distribution of the disk will be wider. This can be viewed as the pressure gradient force dominating over gravity in the Hydrostatic equilibrium equation. On the other hand, if we increase the mean molecular weight μ, c_s will decrease, so H will decrease and the Gaussian distribution will be narrower. This is the situation where gravity dominates over the pressure gradient force.

6 Lecture 6 - Protoplanetary Disk Structure 2

6.1 Verticle Density of the Disk

To discover the meaning of the constant ρ_0 from the equation for the density of the disk $\rho(z) = \rho_0 e^{-\frac{z^2}{2H^2}}$, we think about how this relates to the surface mass density $\Sigma = \frac{M}{A}$. The surface density for an annulus around the disk will be the integral of the volumn density over z, namely

$$\Sigma = \int \rho(z) dz \tag{65}$$

Integrating the Gaussian function to find the surface mass density

$$\Sigma = \int_{-\infty}^{+\infty} \rho_0 e^{-\frac{z^2}{2H^2}} dz \tag{66}$$

This is a standard integral which evaluates to

$$\Sigma = \sqrt{2\pi} H \rho_0 \tag{67}$$

Rearranging for ρ_0,

$$\rho_0 = \frac{\Sigma}{H\sqrt{2\pi}} \tag{68}$$

Therefore, the verticle density distribution becomes

$$\rho(z) = \frac{\Sigma}{H\sqrt{2\pi}} e^{-\frac{z^2}{2H^2}} \tag{69}$$

So far, this equation has assumed that there is hydrostatic balance and that the temperature is constant vertically (disk is vertically isothermal).

Additionally, we can rewrite $\rho(z)$ in terms of the number density as

$$n(z) = \frac{N}{H\sqrt{2\pi}} e^{-\frac{z^2}{2H^2}} \tag{70}$$

where N is the column density.

6.2 Verticle Density as a function of r

We obviously don't expect $\rho(z)$ to be the same at all radii. This is because the scale height H changes with Ω which depends on r, and c_s which depends of temperature. We can define the unit-less disk aspect ratio as

$$h = \frac{H}{r} \tag{71}$$

Since $H = \frac{c_s}{\Omega}$, then h can be rewritten as

$$h = \frac{c_s}{\Omega r} = \frac{c_s}{v_\phi} = \frac{1}{\#M} \tag{72}$$

because the definition of angular velocity is $v_\phi = \Omega r$. The ratio of sound speed to the velocity is also the Mach number $\#M$ of the flow of the gas. If the aspect ratio h is constant with r, then $H \propto r$. This means that the disk would look like a wedge, with the scale height (or standard deviation) H scaling linearly for larger radii.

Instead, we expect c_s will scale with radius as

$$c_s \propto r^{-\beta} \tag{73}$$

where β is a constant. This implies that the aspect ratio scales as

$$h \propto r^{-\beta+\frac{1}{2}} \tag{74}$$

Therefore, h will increase with r if $\beta < \frac{1}{2}$. If this condition is satisfied, then the disk will be flared.

6.3 Dependence of Temperature of Radius

Clearly if we want to see how h actually scales with r, we need the temperature dependence on radius. We can start from assuming that the gas and the dust are at the same temperature $T_{gas} \sim T_{dust}$. Assuming the temperature is only due to the temperature of the central star (i.e. flux received is dominated by the radiation from the star), we can set the flux into the disk equal to the flux out

$$F_{disk} = F_{received} \tag{75}$$

Assuming a very thin disk, and using the Stefan-Boltzmann law, it can be shown that $T_{disk} \propto r^{-\frac{3}{4}}$. This is called a passively irradiated disk. This means the sound speed scales as

$$c_s \propto r^{-\frac{3}{4}\frac{1}{2}} = r^{-\frac{3}{8}} \tag{76}$$

so from the previous equation, $\beta = \frac{3}{8}$. Thus the scale height scales as

$$h \propto r^{-\frac{3}{8}+\frac{1}{2}} = r^{-\frac{1}{8}} \tag{77}$$

This confirms that the disk is flared since $\beta < \frac{1}{2}$, however since β is closer to 0 the flaring is quite shallow. The problem with this derivation is that for deriving $T_{disk} \propto r^{-\frac{3}{4}}$, we assumed an infinitely thin disk, but ended up with a vertically flared disk. So this solution is not self-consistent, since our assumption is contradicted. In reality, the radial dependence on the temperature is expected to be shallower than $T_{disk} \propto r^{-\frac{3}{4}}$, since there is heating further out due to the flared parts of the disk are directly facing the star. Therefore, we expect the disk to be slightly more flared than $h \propto r^{-\frac{1}{8}}$.

Since the temperature will effect the emission of the disk, we can measure the SEDs of class 2 disks to investigate the temperature dependence on radius. We have seen previously that class 2 SED are stellar blackbodies with IR excess due to the disk. The flux of the of the system is

$$F_\lambda = \int I_\lambda d\Omega \tag{78}$$

with $d\Omega = \frac{dA}{r^2}$. An annulus of the disk will have $d\Omega = \frac{rdrd\phi}{r^2}$. So we get

$$F_\lambda = \frac{1}{d^2}\int I_\lambda rdrd\phi = \frac{1}{d^2}\int_0^{2\pi} d\phi \int_{r_{in}}^{r_{out}} I_\lambda rdr \tag{79}$$

From the solution to the equation of radiative transfer:

$$I_\lambda = I_{\lambda B}e^{-\tau_\lambda} + S_\lambda(1-e^{-\tau_\lambda}) \tag{80}$$

For the optically thick disk, $\tau \to +\infty$, so

$$I_\lambda = S_\lambda \tag{81}$$

In thermal equilibrium $S_\lambda = B_\lambda$. So $I_\lambda = S_\lambda = B_\lambda$. Solving the integral for F_λ, we get

$$F_\lambda \propto \int_{r_{in}}^{r_{out}} 2\pi B_\lambda rdr \tag{82}$$

where B_λ is a function of T, which is a function of r.

For the coldest part of the disk (r_{out}), we have long wavelengths ($\lambda >> \frac{hc}{k_BT}$), so the Planck law becomes the Rayleigh-Jeans law. This gives

$$F_\lambda \propto B_\lambda \propto \lambda^{-4} \tag{83}$$

But in the SED we plot λF_λ, which scales as

$$\lambda F_\lambda \propto \lambda^{-3} \tag{84}$$

For the hottest part of the disk (r_{in}), we have short wavelengths ($\lambda << \frac{hc}{k_B T}$), there is an exponential cut-off,

$$F_\lambda \propto \lambda^{-5} e^{-\frac{hc}{\lambda k_B T}} \tag{85}$$

so for the SED we have

$$\lambda F_\lambda \propto \lambda^{-4} e^{-\frac{hc}{\lambda k_B T}} \tag{86}$$

For the intermediate regime, we have medium wavelengths of $\frac{hc}{k_B T_{r_{in}}} \leq \lambda \leq \frac{hc}{k_B T_{r_{out}}}$. If we assume $T_{disk} \propto r^{-\frac{3}{4}}$, then we can write

$$\frac{T(r)}{T(r_{in})} = \left(\frac{r}{r_{in}}\right)^{-\frac{3}{4}} \tag{87}$$

This implies that

$$\frac{hc}{\lambda k_B T} = \frac{hc}{\lambda k_B T_{r_{in}}} \left(\frac{r}{r_{in}}\right)^{-\frac{3}{4}} \tag{88}$$

We define the quantity $\frac{hc}{\lambda k_B T_{r_{in}}} \left(\frac{r}{r_{in}}\right)^{-\frac{3}{4}} = x$. For this regime, we end up with

$$F_\lambda \propto \lambda^{-\frac{7}{3}} \int_0^{+\infty} \frac{x^{\frac{5}{3}}}{e^x - 1} \tag{89}$$

The solution to this integral is a constant. So we get the final result as

$$F_\lambda \propto \lambda^{-\frac{7}{3}} \tag{90}$$

So

$$\lambda F_\lambda \propto \lambda^{-\frac{4}{3}} \tag{91}$$

These regimes describe the SEDs of class 2 disks. This is a good model, but it sometimes fails due to the assumption of a passively irradiated disk (i.e. $T_{disk} \propto r^{-\frac{3}{4}}$). As discussed previously, since the disk is usually even more flared than predicted by $T_{disk} \propto r^{-\frac{3}{4}}$, there is usually a slightly flatter section of the spectrum for the intermediate regime than predicted by $\lambda F_\lambda \propto \lambda^{-\frac{4}{3}}$.

Another assumption that went into this model was that the temperature was the same for each annulus of the disk. This is not exactly true since in reality since not all of the grains emit as blackbodies.

7 Lecture 7 - Protoplanetary Disk Evolution

7.1 Acceleration Balance Equation Consequences

We start our analysis of the evolution of a disk from the radial acceleration balance equation

$$\frac{v_\phi^2}{r} = \frac{1}{\rho}\frac{dP}{dr} + \frac{GM_\star}{r^2} \tag{92}$$

The pressure gradient term scales roughly as $\frac{1}{\rho}\frac{dP}{dr} \sim \frac{1}{\rho}\frac{P}{r}$. Using the ideal gas law ($P = \rho c^2$), we can rewrite the scaling as $\frac{1}{\rho}\frac{dP}{dr} \sim \frac{1}{\rho}\frac{\rho c_s^2}{r}$. Considering that $H = \frac{c_s}{\Omega}$, the scaling becomes

$$\frac{1}{\rho}\frac{dP}{dr} \sim \frac{\Omega^2 H^2}{r} \sim \frac{GM_\star}{r^2}\left(\frac{H}{r}\right)^2 \propto h^2 \tag{93}$$

Physically, this means that the magnitude of the decrease in velocity in the orbit of the gas than Keplerian velocity scales with the aspect ratio squared. We can write

$$v_\phi^2 \sim v_k^2(1 - O(h^2)) \tag{94}$$

where $O(h^2)$ means of the order of h^2. Typically $h << 1$, so the gas is moving at a very small amount less than sub-Keplerian velocity. Despite moving only slightly less than Keplerian, this difference becomes important later.

As a result of this, the angular momentum of the gas is roughly that of a Keplerian orbit. Therefore, we have specific angular momentum

$$j = \frac{L}{M} \sim \Omega_k r^2 \tag{95}$$

Since $\phi_k = \sqrt{\frac{GM}{r^3}}$, we get

$$j \sim \sqrt{GM_\star r} \tag{96}$$

This tell us that the angular momentum of the gas increases with distance. We know that mass from the disk is accreting onto the star. However, as a gas parcel moves inward, this formula for j implies that the gas must lose angular momentum. To enable the disk to accrete, the gas must be able to lose angular momentum. Either the angular momentum is being redistributed within the disk, or it is being lost out of the system (through winds).

7.2 Gas Accretion - Mass Continuity Equation

Accretion if the movement of gas radially inwards toward the central star with velocity v_r. When the disk is accreting the convention is that v_r is negative. The accretion velocity is very small compared to the sound speed $v_r << c_s$ and compared to the Keplerian angular velocity $v_r << v_\phi$.

Consider two annuli around the face-on view of the disk. There will be a difference in the rate of mass inflow from the inner edge $\dot{M_{in}}$ to the outer edge $\dot{M_{out}}$. Using the chain rule and remembering that $dM = \Sigma dA = d\Sigma 2\pi r dr$,

$$\dot{M_{in}} = \frac{dM}{dt} = \frac{dM}{dr}\frac{dr}{dt} = \frac{d\Sigma 2\pi r dr}{dr} v_r \tag{97}$$

$$\dot{M_{in}} = 2\pi r \Sigma(r) v_r(r) \tag{98}$$

where $\Sigma(r)$ and $v_r(r)$ are the values of the surface mass density and radial (accretion) velocity at the radius r. For the outer edge of the annulus, we replace r with $r + dr$ to get

$$\dot{M_{out}} = 2\pi(r + dr)\Sigma(r + dr)v_r(r + dr) \tag{99}$$

Overall, the mass change rate of the annulus will be

$$\dot{M} = 2\pi dr \frac{\partial \Sigma}{\partial t} = \dot{M_{in}} - \dot{M_{out}} \tag{100}$$

Subbing in $\dot{M_{in}}$ and $\dot{M_{out}}$ and rearranging, we get

$$r\frac{\partial \Sigma}{\partial t} = -\frac{(r+dr)\Sigma(r+dr)v_r(r+dr) - r\Sigma(r)v_r(r)}{\partial r} \tag{101}$$

The nominator is just the rate of change of $r\Sigma v_r$. Recognizing this gives us the mass continuity equation, given by

$$r\frac{\partial \Sigma}{\partial t} + \frac{\partial (r\Sigma v_r)}{\partial r} = 0 \tag{102}$$

This equation is a statement of mass conservation for radial movement in the disk.

7.3 Gas Accretion - Angular Momentum

We can derive a similar equation to the mass continuity equation for angular momentum remembering that angular momentum is given by $L = r^2\Omega M$, so

$$\frac{L}{A} = r^2\Omega\Sigma \tag{103}$$

The same exact derivation can be done as for mass for angular momentum, where Σ is replaced by $\frac{L}{A}$. The angular momentum continuity equation is then given as

$$r\frac{\partial (r^2\Omega\Sigma)}{\partial t} + \frac{\partial}{\partial r}(r^2\Omega r\Sigma v_r) = ? \tag{104}$$

If angular momentum was conserved, this equation would be equal to 0 like the mass continuity equation. But if this was true, the gas would not lose angular momentum and would not accrete. If angular momentum is not conserved, then there must be an external torque on the system.

The definition of a torque (G) is the cross product of the radius with force, namely

$$G = r \times F \tag{105}$$

where is force is the time derivative of linear momentum $F = \frac{dp}{dt}$ and so torque is the time derivative of angular momentum $G = \frac{dL}{dt}$.

For the mass continuity equation, we had $\dot{M} = 2\pi dr\frac{\partial\Sigma}{\partial t} = \dot{M_{in}} - \dot{M_{out}}$, so similarly for angular momentum we have

$$2\pi dr \frac{\partial (L/A)}{\partial t} = \dot{L_{in}} - \dot{L_{out}} + \dot{L_{torque_{in}}} - \dot{L_{torque_{out}}} \tag{106}$$

with a change in the torque between the inner and outer edges. So the angular momentum conservation equation becomes

$$r\frac{\partial (r^2\Omega\Sigma)}{\partial t} + \frac{\partial}{\partial r}(r^2\Omega r\Sigma v_r) = \frac{1}{2\pi}\frac{\partial G}{\partial r} \tag{107}$$

Therefore, to achieve accretion, not only do we need a torque G, but a torque that changes across the annulus $\frac{\partial G}{\partial r}$. The origin of this torque is that the disk doesn't rotate as a rigid disk. If we split the disk into a number of annuli with different angular momenta, the differential rotation creates sheer at the interface between different annuli. This sheer results in a viscous torque.

(Don't need to learn off equation for viscous force, just understand)

The viscous force will oppose the flow of the fluid. The viscous force can be written as

$$F_{visc} = A_{interface}\mu r\frac{d\Omega}{dr} \tag{108}$$

where μ is the viscosity and $\frac{d\Omega}{dr}$ is the differential change in angular velocity. The area of the interface is the area of a hoop (circumference of the hoop multiplied by the height). Additionally, the viscosity can be written in terms of the kinematic viscosity ($\mu = \nu\rho$), so we have

$$F_{visc} = 2\pi r^2 dz \nu\rho \frac{d\Omega}{dr} \tag{109}$$

We can write the surface mass density as $\Sigma = \frac{M}{A} = \rho dz$, so we have

$$F_{visc} = 2\pi r^2 \nu\Sigma \frac{d\Omega}{dr} \tag{110}$$

Our viscous torque is then

$$G_{visc} = r \times F_{visc} = 2\pi r^3 \nu\Sigma \frac{d\Omega}{dr} \tag{111}$$

The conclusion of this equation is that if we have a differential rotation $\frac{d\Omega}{dr}$, then there will be viscous torque. Since the angular velocity Ω decreases with radius, $\frac{d\Omega}{dr}$ will be negative so G_{visc} will be negative.

Subbing this equation for the viscous torque into the angular momentum equation $r\frac{\partial(r^2\Omega\Sigma)}{\partial t} + \frac{\partial}{\partial r}(r^2\Omega r\Sigma v_r) = \frac{1}{2\pi}\frac{\partial G}{\partial r}$ and rearranging we get

$$\frac{\partial(r^2\Omega\Sigma)}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r^2\Omega r\Sigma v_r) = \frac{1}{r}\frac{\partial}{\partial r}(r^3\Sigma\nu\frac{d\Omega}{dr}) \tag{112}$$

This is the evolution equation for disk surface density in an accretion disk. This is a non-linear partial differential equation that can be solved for certain assumptions. (Don't need to remember this equation for the exam)

8 Lecture 8 - Planetesimal Formation 1: Gas Drag on Grains

8.1 Solutions to the Evolution Equation

Before moving the planetesimal formation, we finish the analysis of the evolution equation. The evolution equation for disk surface density in an accretion disk derived in the last lecture looks similar to a diffusion equation. Rearranging it we get

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}[r^{\frac{1}{2}}\frac{\partial}{\partial r}(\nu \Sigma r^{\frac{1}{2}})] \tag{113}$$

If we have $x = 2r^{\frac{1}{2}}$ and $\rho = \frac{3}{2}\Sigma x$, we can clearly see this is a diffusion equation given by

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} \tag{114}$$

with diffusion coefficient $D = \frac{12\nu}{x^2}$. We can define the viscous timescale as

$$t_\nu = \frac{x^2}{D} = \frac{r^2}{\nu} \tag{115}$$

The physical meaning of the viscous timescale is the time required to locally (at a certain r) alter the surface density Σ by a factor of order unity. It can be interpreted as the timescale for an accretion disk to change significantly.

We examine 2 analytical solutions to the evolution equation that are relevant for planet formation.

Solution 1: Spreading Ring
This solution assumes that 1. ν is constant and 2. at $t = 0$ all the gas lies in an infinitely thin ring of mass m and radius r_0. The solution is given in terms of modified Bessel functions. In this solution, the surface density starts as a Dirac delta function at r_0, and spreads out. Figure 8 shows the evolution of the surface mass density as a function of radius for this solution.

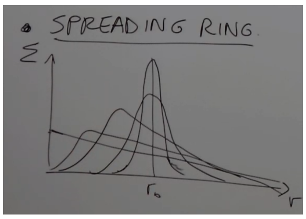

Figure 8: Spreading ring solution to the evolution equation

The key takeaways of this solution is that 1. the mass flows towards $r = 0$ (into the star) and 2. the angular momentum flows to $r \to \infty$ (outwards). Even if only a small fraction of the mass is found at a large radius, it will carry most of the angular momentum. The difference of behaviour of the mass and angular momentum is known as the segregation of mass and angular momentum. This solves the issue from lecture 2 where if all the mass went into the star, the star would be spinning so rapidly it would breakup. This solution is physically unrealistic since the surface mass density will not begin distributed as a Dirac delta function.

Solution 2: Self-similar Solution of Lyndon-Bell-Pringle 1974
This solution assumes 1. that viscosity scales as $\nu \propto r^{-\gamma}$, and 2. at $t = 0$ Σ is a steady states solution to the evolution equation. The analytical solution is (don't need to remember, but know graphs)

$$\Sigma(r,t) = \frac{M(2-\gamma)}{2\pi r_0^2 r^\gamma} e^{-\frac{\frac{5}{2}-\gamma}{2-\gamma}} e^{-\frac{r^2\gamma}{\tau}} \tag{116}$$

where r_0 is the initial disk size and $\tau = \frac{t}{t_\nu} + 1$ where t_ν is the viscous timescale. The first two terms are a power law radially. The third term is an exponential truncation. Figure 9 shows this solution plotted in log-log space. We can see that at later times some mass is accreting onto the star and some is spreading out carrying angular momentum.

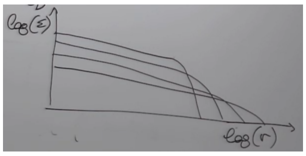

Figure 9: Self-similar solution to the evolution equation. The power law section of the equation is essentially a straight line in log-log space, until for large radii there is an exponential truncation.

8.2 Growth of Solids in the Disk

For the next sections of the course, disks are left behind and we instead study how solids go from dust to planets. Planet growth can be separated into 3 regimes.

1. Dust: sub-μm to 100's m

The dynamics of the dust is dominated by interactions with the gas (aerodynamic drag of gas on the dust). The growth happens through collisions of dust grains, leading to conglomeration.

2. Planetesimals: km to just below terrestrial planet size

The dynamics is no longer dominated by interaction with gas, but instead by gravity (mass becomes sufficiently large). The dynamics of the planetesimals is therefore decoupled from the gas. We need to study N-body dynamics.

2. Terrestrial Planets/Planet Size: 1 earth mass and above

Mass is so large that the planet becomes coupled with gas again by gravity (not by aerodynamics). This can lead to planet migration. Can also lead to accretion of gas onto planet and gas giant formation.

8.3 Dust Grain Dynamics

Dust grains interact with the gas in the disk through aerodynamic drag. The grains will feel the gravitational force, centripetal force and the drag force. Since the dust grains don't experience the pressure gradient force, they will move at Keplerian velocity. On the other hand, the gas moves at sub-Keplerian velocities due to the pressure gradient force. Therefore, the relative velocity between the gas and dust leads to a drag force on the grains given by

$$F_D = -\frac{1}{2} C_D \pi a^2 \rho_d v^2 \tag{117}$$

where v is the relative velocity between gas and dust, C_D is the drag coefficient, and the grains have a radius $r = a$ and a mass density ρ_d. The cross section of the grains is given by πa^2. We can interpret the term $\rho_d v^2$ to be the ram pressure the gas exerts on the dust grains. The complication with this formula is that usually C_D depends on the relative velocity v, and also on the size of the grains relative to the mean free path of the gas molecules (collisions). In general, the mean free path of the gas molecules is defined as

$$\lambda_g = \frac{1}{\sigma_g n_g} \tag{118}$$

where σ_g is the cross section of the gas molecules and n_g is the number density of the gas.

We can define two regimes for describing the drag force on the dust grains

1. Small particles: $a < \frac{9}{4}\lambda_g$

In this regime, the drag originates from collisions between the dust and the individual gas molecules. The drag coefficient is therefore given by

$$C_D = \frac{8}{3}\frac{v_{th}}{v} \tag{119}$$

where v is the relative velocity and v_{th} is the thermal velocity of the molecules in the gas. The mean thermal velocity of the gas molecules from the Maxwell-Boltzmann distribution is given by

$$v_{th} = \sqrt{\frac{8}{\pi}} c_s = \sqrt{\frac{8}{\pi}\frac{k_B T}{m}} \tag{120}$$

This regime for small dust grains is called Epstein drag. Substituting the value for C_D into the general drag force formula gives the Epstein drag force as

$$F_D = -\frac{4}{3}\pi a^2 v_{th} \rho_g v^2 \tag{121}$$

2. Larger Particles: $a > \frac{9}{4}\lambda_g$
Larger particles interact with the gas as a fluid. Therefore, the drag coefficient depends on the Reynold's number of the fluid. In this regime, the drag force is called the Stokes drag.

8.4 Stopping Timescale

The stopping timescale (t_s) of a grain defines the dynamics of a dust grain in the disk. The stopping timescale is the time in which a grain will feel an change of order unity in linear momentum mv, where v is the relative velocity. It can also be interpreted as the time it takes a grain in the disk to stop due to the drag force. The stopping timescale (in units of seconds) is defined as

$$t_s = \frac{mv}{|F_D|} \tag{122}$$

Assuming spherical dust grains, we can write the mass of the dust grain as $m = \rho_d V = \frac{4}{3}\pi a^3 \rho_d$. In the Epstein regime (smallest grains), subbing the the Epstein drag gives

$$t_s = \frac{\rho_d}{\rho_g}\frac{a}{v_{th}} \tag{123}$$

Therefore, the shorter the stopping timescale, the stronger the drag force of the gas on the dust. If the gas has a higher thermal velocity, then the collision probability will be higher, and then the stopping timescale will be shorter. Also, the larger the grain, the larger the stopping timescale. Effectively, the stopping timescale gives us a sense of the strength of the coupling of the dust and gas. For a short t_s, the gas and dust are well coupled. This means the dust will quickly be brought down to the same velocity as the gas. For longer t_s, the dust and gas are not well coupled. For planetesimals, the stopping timescale becomes much larger then the length of the evolution of the disk, so therefore planetesimals are decoupled from the gas. The equation for t_s changes in the Stokes regime, however we find still that $t_s \propto a$.

We can determine how coupled the dust is to the gas by comparing the stopping timescale to the period of the disk. If $t_s << P$, they are strongly coupled. If $t_s \sim P$, they are moderately coupled. If $t_s >> P$, they are weakly coupled.

9 Lecture 9 - Planetesimal Formation 2: Dust settling, Radial drift

9.1 Verticle Dynamics of Dust

We have defined the quantities describing the drag force on the dust grain in the gas, and now we'll see how these quantities effect the dynamics of the grains. Consider a dust grain just above the mid plane in the disk. This dust grain feels the z component of the gravitational force pulling down to the mid plane, and the drag force opposing the gravitation force. The drag force is large so it quickly balances with the gravitational force, causing the dust grain to move at terminal velocity. To find the terminal velocity, we can set the z component of the gravitational force ($F_{G_z} = m\Omega_k^2 z$) equal to the Epstein drag force $|F_D| = |F_{G_z}|$. Solving for the velocity towards the mid plane gives

$$|v_{settling}| = \frac{\Omega_k}{v_{th}} \frac{\rho_d}{\rho_g} az \tag{124}$$

This is the dust verticle settling velocity. We know from earlier lectures that the density of the gas is a function of z, $\rho_g(z)$. This means the settling velocity will be higher for higher z (further above the mid plane of the disk). We can use the settling velocity to find the settling timescale, given as

$$t_{settling} = \frac{z}{|v_{settling}|} = \frac{v_{th}}{\Omega_k^2} \frac{\rho_g}{\rho_d} \frac{1}{a} \tag{125}$$

This formula implies that the smallest grains will have the longest settling timescales, and that the largest grains settle quickly. If the size of the grains a continues to increase, the Epstein regime is no longer valid and the formulas for $t_{settling}$ and $v_{settling}$ break down. At ~ 1 AU radii, for μm size grains, we get that $v_{settling} \sim 0.1$ cm s^{-1} and $t_{settling} \sim 10^5$ years = 0.1 Myrs. For mm size grains, $t_{settling} \sim$100 years.

Observationally, we see that mm size grains are mostly in the mid plane, we agrees with theoretical calculation for the short settling timescale. However, μm size grains are observed to be vertically thicker than mm size grains. Since the disks we observe are much older than $t_{settling}$ for μm size grains also, we would expect to see that the these grains should have settled in the mid plane. Instead we see that μm size grains have a verticle distribution. This leads to the conclusion that there must be other factors influencing the verticle distribution of the dust grains. One factor that could effect this is that in our derivations we assumed that the gas is laminar. In reality, turbulent flows in the gas may cause the gas to move vertically.

9.2 Radial Dynamics of Dust

In previous lectures on disk structure, we saw the radial force balance equation for a gas parcel on the mid plane of the disk. The radial force balance equation balances the gravitational force with the centripetal and pressure gradient forces as

$$\frac{v_{\phi_g}^2}{r} = \frac{GM_\star}{r^2} + \frac{1}{\rho_g} \frac{dP}{dr} \tag{126}$$

We found that if the pressure gradient force was neglected, v_{ϕ_g} becomes the Keplerian velocity given by $v_k = \sqrt{\frac{GM_\star}{r}}$. However, if we include the pressure gradient force, the gas moves at sub-Keplerian velocity. Including the pressure gradient force, the gas moves at

$$v_{\phi_g}^2 = v_k^2(1 - \eta) \tag{127}$$

where η determines how much slower than Keplerian the gas moves. We found that $\eta \propto \frac{c_s^2}{v_k^2} = h^2$, where h is the aspect ratio of the disk. Taking a typical value of aspect ratio of $h \sim 0.05$, we get that v_{ϕ_g} moves 10^{-3} times less than Keplerian velocity. This means that the gas is only moving slightly sub-Keplerian.

We have also seen that solids don't feel the pressure gradient force of the gas, so solids will orbit with Keplerian velocity. The velocity difference between the solids and the gas leads to a head wind (drag force) for dust grains. If the gas moves 10^{-3} times less than Keplerian velocity, this head wind will be very strong for ~ 1 AU at around ~ 100 ms^{-1} = 360 km hr^{-1}.

Previously, we compared the orbital period of the disk to the stopping timescale to determine how coupled the gas was to the dust. We can define the Stokes number (also called dimensionless stopping time) as

$$S_t = \frac{t_s}{P} \tag{128}$$

If the Stokes number is high, it means the grains are weakly coupled to the gas. If the Stokes number is ~ 1, it means the grains are moderately coupled to the gas. If the Stokes number is small, it means the grains are strongly coupled to the gas.

The headwind drag force will apply a torque to the dust grains, causing them to lose angular momentum and drift inwards. We can write radial and azimuthal equations of motion for the dust grains (don't need to remember off by heart). The radial acceleration equation of motion for the dust grains is

$$\frac{dv_(r_d)}{dt} = \frac{v_{\phi_d}}{r} - \Omega_k^2 r - \frac{1}{t_s}(v_{r_d} - v_{r_g}) \tag{129}$$

The azimuthal equation of motion is

$$\frac{d(v_{\phi_d} r)}{dt} = -\frac{r}{t_s}(v_{\phi_d} - v_{\phi_g}) \tag{130}$$

This equation can be viewed as the rate of change of specific angular momentum. The right hand side is the torque divided by mass. Since there is a difference between v_{ϕ_d} and v_{ϕ_g}, there is a torque on the dust which causes a change in angular momentum. As a result of this, the dust grains will radially drift inwards. The dust grains follow a spiral path as they drift radially inwards towards the central star.

Taking some approximations, we can solve the radial and azimuthal equations of motion as a system of equations to get the radial velocity (inward drift) as (don't need to remember exact equation, only graph):

$$v_{r_d} \sim \frac{\eta v_k}{S_t + S_t^{-1}} \tag{131}$$

Figure 10 shows this equation plotted in log-log space. Remember that the Stokes number tells us how coupled the dust is to the gas, we can see that the radial velocity peaks for Stokes number ~ 1. This is because if the dust is strongly coupled to the gas, the dust is moving at the same velocity as the gas, so there is no inward drift. If the dust is weakly coupled to the gas ($S_t >> 1$), there is no gas drag and therefore no radial drift.

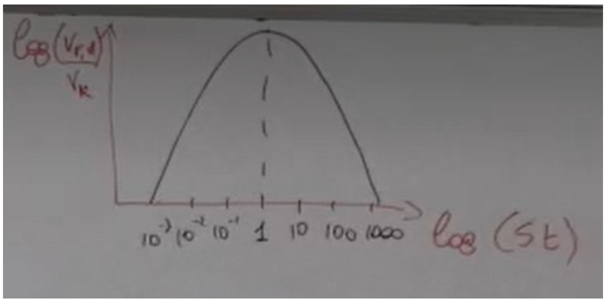

Figure 10: Radial velocity (inward drift) of the dust grains as a function of Stokes number.

We can further interpret Figure 10 by remembering that, for Epstein drag, $S_t \propto a$. Therefore, for small sized dust grains there is no radial drift. Moderately sized grains will feel the strongest radial drift. Large dust grains will return to not feeling any radial drift. Since the Stokes number depends on the stopping timescale which depends on the gas density, the stokes number changes with location within the disk. We can see this by plugging in the definition of orbital period and the stopping timescale for Epstein drag, to get

$$S_t = \frac{t_s}{P} = t_s \frac{v_k}{r} = t_s \Omega_k = \frac{\rho_d}{\rho_g} \frac{a}{v_{th}} \Omega_k \tag{132}$$

For a Stokes number $S_t = 1$, with $a \sim 10$ cm - 1 m, into $v_{r_d} \sim \frac{\eta v_k}{S_t + S_t^{-1}}$ we get $v_{r_d} \sim 10$ ms^{-1}. We have the radial velocity, so for 1 AU we can get a timescale $\sim \frac{1AU}{v_{r_d}} \sim$ 100's of years. So for this model, it takes the dust grains 100's of years to move into the central star. This radial drift therefore poses a problem for planet formation, because once the dust grows to about $a \sim 10$ cm - 1 m, the dust grains will fall into the central star in only 100's of years.

9.3 Overcoming Radial Drift

To form the population of exoplanets, we need an modified model to overcome the radial drift problem. We know that the pressure in the disk scales as a power law with radius (negative slope in log-log space). Because of this relationship, the gas moves at sub-Keplerian velocity, which causes a drag between the dust grains and the gas which causes the radial drift problem

There are two potential solutions to the radial drift problem. The first solution is to propose that the dust grains can grow quickly enough in mass to avoid staying at the critical Stokes number $S_t \sim 1$ long enough for them to radially drift into the central star. We investigate this proposal later to see if the dust can grow quickly enough to avoid drifting too far radially inwards.

The other solution is to propose that there are pressure bumps at certain radii along the disk. This pressure bump is shown in Figure 11. Because the pressure gradient force doesn't always point away from the star, but instead to the region of lower pressure, the dust grains will drift radially towards the region of lower pressure. For this reason, these pressure bumps are known as dust traps.

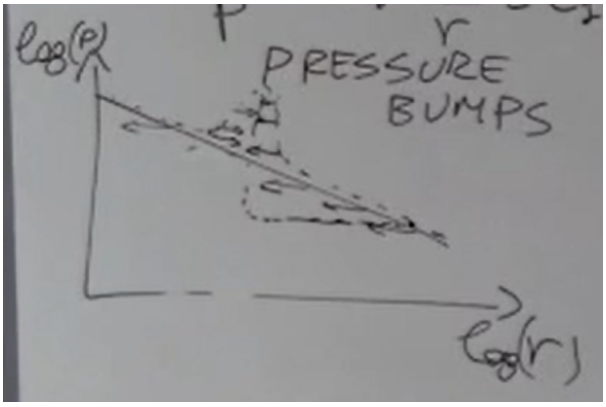

Figure 11: Pressure bumps inside the disk that prevent the dust grains from drifting radially inwards.

10 Lecture 10 - Planet Formation 1: Grain and Planetesimal Growth, Gravitational Focusing

10.1 Dust Grain Growth

Dust grains grow gradually through collision physics. To determine the growth, we are interested in the mass rate $\frac{dm}{dt}$. Assuming 100% sticking for each collision (which is reasonable for small grains), the growth rate is the rate of collisions (R) with other grains times the mass of the other grains, written as

$$\frac{dm}{dt} = R \times m \tag{133}$$

Assuming the disk is a sea of spherical grains of equal mass, the collision rate is given as

$$R = \pi a^2 \frac{\sigma}{V} \tag{134}$$

where πa^2 is the geometric cross section of collision, σ is the relative velocity (or velocity dispersion) of the grains and V is the volumn occupied by grains.

For small grains (drag dominated), the velocity dispersion can be approximated by Brownian motion. This means that the velocity dispersion becomes $\sigma \sim \sqrt{\frac{m_H}{m}} c_s$. Subbing the collision rate and the velocity dispersion in $\frac{dm}{dt}$, gives

$$\frac{dm}{dt} = \pi a^2 \frac{\sigma}{V} m = \pi a^2 \rho_g Z \sigma \tag{135}$$

where Z is the dust to gas ratio. We write this because $\rho_g Z = \rho_d$. For spherical grains, $m = \frac{4}{3}\pi a^3 \rho_d$ where ρ_d is the density of the individual dust grains (not the density of grains in the volumn in the entire disk).

$$\frac{da}{dt} = \frac{\rho_g}{4\rho_d} Z\sigma \tag{136}$$

Typically $Z \sim 0.01$, so for 1 AU, $\frac{da}{dt} = 10^{-4}$ cm yr^{-1}. In reality, the rate of growth in even faster. This allows grains to grow from sub μm to mm-cm sizes efficiently. These fast rates are confirmed for young protoplanetary disks.

Once we get to mm-cm sizes, the assumption of 100% sticking no longer holds. This is called the bouncing barrier, where when dust grains collide they compact and fragment instead of sticking. This poses another problem for planetesimal formation.

The bouncing barrier problem is significant for mm-cm size dust grains, whilst the radial drift problem is significant for cm-m ($S_t \sim 1$) size grains. One possible mechanism to avoid both of these barriers is through streaming instabilities (and other instabilities). This is when there are certain portions of the disk where there are pressure maxima of grains. Figure 12 shows the barriers facing dust grains to form planetesimals.

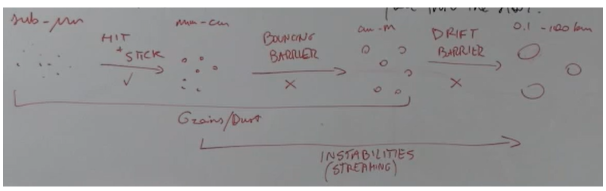

Figure 12: Summary of barriers facing the dust to planetesimals.

10.2 Planetesimal Growth

Once the grains have overcome the bouncing barrier and radial drift barrier, they will be size 0.1-100 km and are considered planetesimals. The Stokes number for these sized objects is very low, meaning that the planetesimals are decoupled from the gas and do not feel drag. Instead, gravity becomes dominate since planetesimals have considerably more mass than dust grains. Because of the gravitational attraction between the planetesimals, their cross section of collision is larger than usually expected.

To derive the cross section for collision, consider two planetesimals of equal mass moving towards each other at an offset with velocity $\frac{\sigma}{2}$, and therefore relative velocity of σ. as shown in Figure 13. We are interested in the distance of closest approach (ΔR) between the two planetesimals. At the distance of closest approach, the velocity of each of the planetesimals will be at a maximum, and will be perpendicular to the distance of closest approach ΔR.

We can use the conservation of energy to equate the energy before the interaction with the energy at the distance of closest approach. Before the interaction, the planetesimals will only have kinetic energy and no gravitational energy. We write the conservation of energy as

$$2(\frac{1}{2}m(\frac{\sigma_v}{2})^2) = 2(\frac{1}{2}mv_{max}^2) - \frac{Gm^2}{\Delta R} \tag{137}$$

We can write a similar equation except for specific angular momentum conservation ($\frac{L}{M} = r \times v$). The angular momentum conservation becomes

$$\frac{1}{2}\Delta R v_{max} = \frac{1}{2}\sigma\frac{1}{2}b \tag{138}$$

where b is the impact parameter (distance between the trajectory of the incoming particle with the other particle). The LHS of this equation is just the angular momentum at the distance of closest approach, and the RHS of the equation comes from geometric considerations. Solving for v_{max}, we get

$$v_{max} = \frac{b}{2\Delta R}\sigma \tag{139}$$

Plugging this value for v_{max} into the energy conservation equation, we get can solve to get a value for b. For a collision to occur, ΔR must be less than the size of the planetesimals ($\Delta R < a$). We can set $\Delta R = a$ to find the largest impact parameter b that will lead to a collision. Setting $v_{max} = \frac{b}{2\Delta R}\sigma$ and $\Delta R = a$, the energy conservation equation becomes

$$m\frac{\sigma^2}{4} = m\frac{\sigma^2}{4}\frac{b^2}{a^2} - \frac{Gm^2}{a} \tag{140}$$

Solving for b gives

$$b^2 = a^2 + \frac{4Gma}{\sigma^2} \tag{141}$$

This is the largest impact parameter for a collision to occur. If b is larger, then the planetesimals scatter. At the point of contact between the planetesimals, the escape velocity will be given by

$$v_{esc} = \sqrt{\frac{4Gm}{a}} \tag{142}$$

where the gravitation potential energy is twice as much than for the usual formula because the two planetesimals attract each other. This leads to the cross section of the gravity dominated collision between planetesimals. Generally, the cross section of a collision is πb^2. For the gravity dominated collision, we can sub in the definition of v_{esc} to get

$$\pi b^2 = \pi a^2(1 + \frac{v_{esc}^2}{\sigma^2}) \tag{143}$$

This equation shows how the cross section is increased by the effect of gravity. We label the escape velocity squared term as the gravitational focusing factor (or the Safronov number) $\Theta = \frac{v_{esc}^2}{\sigma^2}$. The gravitational focusing factor determines how much the cross section is increased by gravity. If the initial velocities of the planetesimals σ is lower, then there is a higher chance of collision. Since the escape velocity depends on the mass which can be written in terms of the density, it will scale with$v_{esc} \propto a^2$. This means that the cross section will increase for larger planetesimals.

11 Lecture 11 - Planet Formation 2: Runaway Growth, Isolation Mass and Terrestrial Planet Formation

11.1 Runaway Growth

Using the cross section for collision between planetesimals from the previous lecture, we can calculate the collision rate to find the rate of growth of planetesimals. For spherical dust grains, we had that $\frac{dm}{dt} = R \times m$ and $R = \pi a^2 \frac{\sigma}{V}$. We can apply these equations again to planetesimals, so $\frac{dm}{dt} = \pi a^2 \frac{\sigma}{V} m$. Using the increased cross section due to the gravitational attraction between the planetesimals from the previous lecture, we get that the rate of growth is

$$\frac{dm}{dt} = \pi b^2 \sigma \rho_p \tag{144}$$

where ρ_p is the density of planetesimals in the disk. We have seen before we can write the surface mass density of the planetesimals as an integral of the volumn density over z. If the density doesn't change much with height, we have

$$\Sigma_p = \int \rho_p dz \sim 2H_p \rho_p \tag{145}$$

where H_p is the scale height of the planetesimals. Scale height of the planetesimals is not necessarily equal to the scale height of the grains, which is also not not necessarily equal to the scale height of the gas $H_p \neq H_{grains} \neq H_{gas}$. The scale height of the planetesimals is given as

$$H_p \sim \frac{\sigma}{\Omega_k} \tag{146}$$

where σ is the velocity dispersion of the planetesimals. The scale height depends on the velocity dispersion since if planetesimals have large velocities their orbits will likely have higher inclination above the mid plane. For all these considerations, subbing into the rate of growth equation and rearranging gives

$$\frac{dm}{dt} = \frac{1}{2} \Sigma_p \Omega_k \pi a^2 (1 + \frac{v_{esc}^2}{\sigma^2}) \tag{147}$$

From this equations, we can see that the velocity dispersion only enters the equation through the gravitational focusing factor. We can also see that the growth rate scales linearly with the surface density of the planetesimals (the more material there is the faster the growth). Additionally, the growth is slower at larger radii since for larger radii the Keplerian velocity will be slower (interpret as there is less material further out). If the velocity dispersion is low, this implies that $\frac{v_{esc}^2}{\sigma^2} = \Theta >> 1$. In this regime, (and subbing in the escape velocity) the growth rate becomes

$$\frac{dm}{dt} = \frac{1}{2} \Sigma_p \Omega_k \pi a^2 \frac{4GM}{a\sigma^2} = \frac{1}{2} \Sigma_p \Omega_k \pi a \frac{4Gm}{\sigma^2} \tag{148}$$

For this regime, we can see that $\frac{dm}{dt} \propto ma$. If we assume spherical planetesimals, $a \propto m^{\frac{1}{3}}$. Then we get the scaling relationship

$$\frac{dm}{dt} \propto m^{\frac{4}{3}} \tag{149}$$

This scaling is good for planet formation, since it implies that the growth scales faster the bigger the planetesimals get. This is called runaway growth. For runaway growth to continue the velocity dispersion must remain low since that was assumed when deriving this result. This is a problem since near-miss close encounter collisions increase the velocity dispersion of the planetesimals. Therefore, gravitational kicks provided by the near collisions slow down the planetesimal growth down, and the runaway growth scaling relationship no longer holds. In reality, the runaway growth relationship doesn't hold anyway since, as well as constructive collisions, there can also be catastrophic collisions that shatter planetesimals.

11.2 Isolation Mass

The growth of planetesimals eventually stops altogether once the reservoir of solid material in that section of the disk has been accreted onto the planetesimal. The sphere of gravitational influence of the planet is defined as the Hill sphere. It is defined as the radius where the planet's gravitational influence dominates over the star. It is given as

$$r_H = r\left(\frac{m_p}{M_s}\right)^{\frac{1}{3}} \tag{150}$$

where r_H is the distance from the planet and r is the distance from the star. The radius multiplied by 2 will be the length of the annulus of influence surrounding the orbit of the planet around the central star. The area of this annulus is known as the feeding zone of the planet. Since the surface density of planetesimals at that radius can be written as $\Sigma_p = \frac{M}{A_{annulus}}$, the total mass of planetesimals that can be accreted from the feeding zone is

$$M = \Sigma_p 2\pi r 2r_H = 4\pi r^2\left(\frac{m}{M_\star}\right)^{\frac{1}{3}}\Sigma_p \tag{151}$$

Cancelling the Σ_p from both sides, the total mass is given by

$$M = 4\pi r^2\left(\frac{m_{iso}}{M_\star}\right)^{\frac{1}{3}}\Sigma_p \tag{152}$$

Solving for m_{iso}, we find that $m_{iso} \propto \Sigma_p^{\frac{3}{2}} r^3 M_\star^{-\frac{1}{2}}$. This isolation mass is the maximum mass a planet can achieve by feeding over its entire feeding zone (defined from r_H). We can see that generally the further away from r the planet is, the larger the isolation mass. This means that generally we expect larger planets further away from the central star. Plugging in typical parameters for the terrestrial region of the solar system, we get $m_{iso} \sim 0.1M_\oplus$ (roughly Mars mass). For the giant planet region of the solar system, we get $m_{iso} \sim 10M_\oplus$ (roughly the expect cores of Jupiter and Saturn).

These are good predictions, however this model is a simplification since we are neglecting the accretion of dust grains that add to the total mass the of planets. These dust grains will radially drift inwards into the feeding zone of the planet. This phenomena is termed pebble accretion, and it is most efficient for grains with Stokes number $S_t \sim 1$ (fastest radial drift). This process allows the planets to get larger faster than thought previously.

11.3 Terrestrial Planet Formation Overview

The first step in terrestrial planet formation is the formation of planetesimals (sub-μm size to km size). This is done initially by hit and stick collisions, which works until the bouncing fragmentation barrier is reached. After the bouncing barrier, the radial drift barrier must be overcome. These processes hamper growth, but they may be overcome by streaming instabilities or pressure bumps. Once the planetesimals form, there is a period of runaway growth enhanced by gravitational focusing for low dispersion velocity. Gravitational focusing is more effective for larger size and mass, however, large masses results in higher dispersion velocities (through close encounters) which hampers runaway growth. The growth of the planets is aided also by pebble accretion depending on the number of dust grains left in the system. A few "lucky" planets in the system grow rapidly at the expense of other bodies. This phase is called oligarchic growth. These larger planets will excite the velocities of the surrounding planetesimals and lead to a higher dispersion velocity, hampering growth. But the growth of the oligarchic bodies is still faster than the planetesimals. The growth process ends when the isolation mass is reached. After this point, the only way for terrestrial planets to grow is growth by giant impacts with other large planets. Once Mars mass is reached, the planets are called planetary embryos, and they have exhausted their feeding zone, and typically the protoplanetary disk has been dissipated. In the giant impact phase, planetary embryos grow by stochastic collisions with each other.

12 Lecture 12 - Planet Formation 3: Gas Giant Formation: Core Accretion vs Gravitational Instability

12.1 Core Accretion Model

Giant planets need to already be massive enough to accrete the surrounding gas. The first model for giant planet formation is the core accretion model. In the first stages of evolution of this model, the planets grow the same way as terrestrial planets until they reach the isolation mass. The isolation mass must be enough so that the planet can accrete gas and retain it the gas atmosphere. The particles in an atmosphere are retained if their average velocity is less than the escape velocity for that planet. The escape velocity is given by

$$v_{esc} = \sqrt{\frac{2Gm}{a}} \tag{153}$$

where $m = \rho_p \frac{4}{3}\pi a^3$. We can compare this to the sound speed of the gas, given as $c_s = \sqrt{\frac{k_B T}{m}}$, where m is the mass of the molecules of gas (mass of H2). Earlier in the course, we derived that

$$c_s = h v_k \tag{154}$$

where h is the aspect ratio and v_k is the Keplerian velocity. Equating the sound speed with the escape velocity and solving for the mass of the planet gives

$$m >\sim \sqrt{\frac{3}{32\pi}} h^3 \frac{M_\star^{\frac{3}{2}}}{\rho_p^{\frac{1}{2}} a^{\frac{3}{2}}} \tag{155}$$

At Jupiter radius ($r = 5$ AU), we get that $m >\sim 4 \cdot 10^{-3} M_\oplus$. This mass is realistically too low. This mass should actually be interpreted as the mass required to maintain a thin envelope atmosphere (Mars atmosphere). More complicated models are needed to take into account the hydrostatic balance in the atmosphere.

There are three phases in the core accretion model of giant planets. The first phase is the core formation phase (previously discussed up to isolation mass). This must happen before the gas has dissipated (within the protoplanetary disk timescale). The second phase is the stage of hydrostatic growth. This is the stage where the core accretes gas while maintaining hydrostatic equilibrium. If the planet grows radially while accreting gas, the mass of the planet will become greater than the minimum mass from the mass formula obtained by equating sound speed and escape velocity. To prevent this from happening, the planet must contract. For this to happen, the planet must lose energy by radiative cooling (emission of IR radiation). Because only the envelope can contract, the hydrostatic growth phase is slow. As the planet mass is gradually increasing, the size of the feeding zone also increases, so the planet can increase by more solid mass. This phase ends once the mass of the gas envelope is roughly equal to the mass of the core (takes Myr timescales). Once the envelope is this size, then the envelope can contract much more efficiently because of the small density of the gas. This leads to the third phase which is a phase of runaway growth because the gas atmosphere can contract by a large amount. This phase stops once either torques/tidal forces between the planet and the disk form a gap in the gas, or when the disk disperses (gas runs out) after about a few 10 Myrs.

12.2 Gravitational Instability Model

The gravitational instability model is almost the opposite to the core accretion model. In this model, the giant planet forms via the gravitational fragmentation/collapse within the disk. Previously we saw that protostars form in cloud cores when the gravitational potential energy of the gas overcame the pressure force, leading to collapse. The gravitational instability model of giant planet formation is a similar idea. If a sufficient amount of gas in the disk has its own gravitational energy that can overcome the pressure force, a giant planet may directly form. The mass of a square patch of gas of size l in the disk will be given by

$$\Delta M_{patch} = \Sigma l^2 \tag{156}$$

The gravitational potential due to self-gravity will be given by

$$U_G = -\frac{G\Delta M^2}{l} = G\Sigma^2 l^3 \tag{157}$$

This energy needs to overcome the thermal energy of the gas given by

$$U_T \sim \frac{1}{2}k_B T \sim \frac{1}{2}\Delta M c_s^2 = \frac{1}{2}\Sigma l^2 c_s^2 \tag{158}$$

Equating the thermal energy to the gravitational energy is an over simplification for this patch of gas. This is because the patch of gas has differential rotation around the disk. Therefore, the patch of gas will experience a sheer that stretches the gas preventing gravitational collapse. We can describe this as the rotational kinetic energy of the patch, given as

$$U_R \sim \frac{1}{2}I\omega^2 \tag{159}$$

where I is the moment of inertia and ω is the angular velocity. For the patch of gas $I = \Delta M l^2$, so

$$U_R \sim \frac{1}{2}\Delta M l^2 \omega^2 \sim \frac{1}{2}\Sigma\Omega^2 l^4 \tag{160}$$

If gravity is greater than the sum of the thermal and rotational potential energies, the patch of gas can collapse. Balancing $U_G + U_R + U_T < 0$, we get

$$-\Sigma^2 l^3 + \Sigma\Omega^2 l^4 + \Sigma l^2 c_s^2 < 0 \tag{161}$$

which becomes

$$\Sigma l^2(-2G\Sigma l + \Omega^2 l^2 + c_s^2) < 0 \tag{162}$$

Since Σl^2 is always positive, the solution is found by setting the terms in the brackets equal to 0. This is a quadratic equation, which can be solved to get

$$\frac{c_s\Omega}{G\Sigma} < 1 \tag{163}$$

In reality, a more complex analysis leads to an extra factor of π, so

$$Q = \frac{c_s\Omega}{\pi G\Sigma} < 1 \tag{164}$$

Q is called the Toomre Q parameter/condition. If this condition is satisfied, the patch of gas will be gravitationally unstable and a gas giant may form. If the temperature of the patch increases, c_s increases and the patch becomes more stable due to the pressure support. If the patch is closer to the star (increase Ω), the rotational energy will be larger which will prevent collapse. If instead we increase the surface density of the gas, it will have more self-gravity and will be less stable.