ABOUT GENERAL ORDER REGULAR FLEXAGONS

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ABSTRACT. The three and six face constructions of flexagons first appeared in the literature in 1956 and 1957 by Gardner and Oakely - Wisner respectively, and have been of great interest to mathematicians ever since. A recent article by Anderson et. al. at the European Journal of Combinatorics conducted a systematic study of the mathematics involved in regular flexagons. In particular, the authors defined equivalent pats and faces mathematically and then used recursion to count the pats and faces. For the number of pat classes with a given degree, we first make an observation that arranges "sub-pat numbers" at leaves of a binary tree. This observation, together with the previously achieved recursion, inspires a bijection between pat compositions and quartic trees. We hence provide an elementary approach to provide for the first time, the explicit formula for the number of equivalent pat classes. We point out that this formula can also be achieved using classic but technical combinatorial approaches as the binomial theorem and the Lagrange inversion formula. Based on the compositions of numbers, we obtain the explicit formula for the number of faces of a regular flexagon of any order. Then our focus shifts to achieving an even better understanding of how the faces/pats change under flexes. A novel labeling of the triangles in the construction of a flexagon is proposed. As a result, a detailed state diagram is shown to explain the transition between one face to another. The theoretical results of this flavor lead to algorithmic procedures to construct a regular hexaflexagon and reproduce the orientation of the top triangles of a flexagon. Consequently, a Java applet was produced for those who are interested in playing flexagons themselves.

1. Introduction and preliminaries

The three and six face construction were first published by Martin Gardner [4] and by Oakley - Wisner [11] respectively. One can find the nine face construction in Oakley - Wisner [11] and then again in McLean - McLean [10]. Martin Gardner [4] explains how to construct the twelve faced dodecahexaflexagon, which was further illustrated in [2].

Flexagons constructed from straight strips of paper are called regular. It is well known that all regular flexagons have order $3n$, $n > 0$, contain 9n equilateral triangles, have $3(3n-2)$ half-twists and $6n-3$ different mathematical faces if you restrict yourself to the pinch flex. To illustrate the concept, we construct a regular flexagon of order 6 (i.e. a hexahexaflexagon) as in [2]. Start with 18 ($= 6 \times 3$) equilateral triangles on a straight strip of paper with a small tab that we will apply tape to at the end, the triangles are numbered from 1 to 18 on the front as shown in Figure 1 and the same on the back, but underlined.

Begin by winding 2 under 1. Continue to wind the triangles in the same direction until you end up with a straight strip that has 9 triangular regions as shown in Figure 2. None of the visible triangles have underlined numbers.

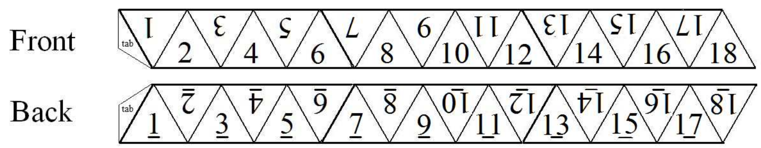

FIGURE 1. Numbered strip of paper for building a flexagon

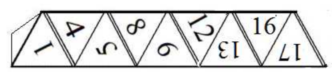

FIGURE 2. Initial foldings for building a hexahexaflexagon

Next twist 5 under the 4 hiding the 6 and the 3 as shown in Figure 3 (left). Finish by rolling the 11 under the 10, folding 17 behind the 16, and taping the tab to the buried 18.

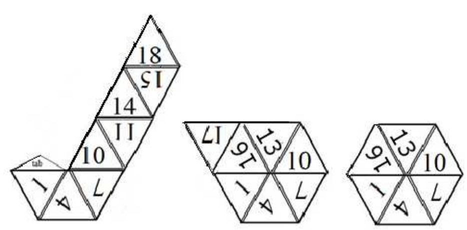

FIGURE 3. Finishing a hexahexaflexagon

In order to record this one flexagon face that appears in Figure 3 (right), we call it f and then place a dash between each of the 6 triangular adjacent pats and state

$$\begin{aligned} f = (p_i) &= (p_1 - p_2 - p_3 - p_4 - p_5 - p_6) \\ &= (1, 2 - 4\ 3, 6\ 5 - 7, 8 - 10\ 9, 12\ 11 - 13, 14 - 16\ 15, 18\ 17). \end{aligned}$$

Only the first number of each pat is visible on the top of the physical face. When a pat contains more than one triangle, a comma is placed where the pat splits naturally and this location is called a thumb hole. If p is a pat or a portion of a pat, $\sim p$ will refer to p turned upside down. If $p = 4\ 3, 6\ 5$ then $\sim p = 5\ 6, 3\ 4$. If a triangular region was made up of a multiple of 3 triangles, a hexagon could not be constructed so we will always have pats with thicknesses $3m + 1$ or $3m + 2$ for nonnegative m. These thicknesses are called degrees and we will say $D(p) = 4$ for our example when $p = 4\ 3, 6\ 5$, but only the 4 is visible in the face. Following this intuitive definition of a pat and its degree, we now define it inductively by giving you the base case, and then follow with the inductive case as stated in [2].

Definition 1 ([2])**.** If p is an ordered set of one natural number, then p is a pat of degree 1 and we will denote that fact by saying $D(p) = 1$.

That natural number is bounded by 18 in our example but can go as large as $9n$ for a flexagon of order $3n$ for any given n. If two pats share a common edge, we will call them adjacent if the numbers in the top left one are less than the numbers in the right one (modulo $9n$). Now that we have pats of degree one, we combine 2 adjacent ones into a pat of degree 2, and then 2 of those into one of degree 4. Degrees of 4, 5, 7, 8, 10, and 11 are possible for our example. Next is the inductive case for the definition of a pat.

Definition 2. If p and q are adjacent pats where $D(p) = 3m_1 + k$ and $D(q) = 3m_2 + k$, where each m_i is nonnegative, and $k = 1$ or 2, then $r = \sim p, \sim q$ is a pat of degree $D(p) + D(q) = 3(m_1 + m_2) + 2k$. Note that $\sim p$ is on top of $\sim q$ and the thumb hole separates r into 2 partial pats.

If $p = p_1p_2 \dots p_m$ and $q = q_1q_2 \dots q_m$ are pats of degree m, then p is a *translation* of q means that there exists a number t so that $p_k = q_k + t(\mod 9n)$ for all $1 \le k \le m$. If m is not a multiple of 3, we want to count how many different pats there are of degree m and call it $N(m)$. We will not count translations as being different. Because pats are built from smaller pats, neither number in the composition can be a multiple of 3. Thus we start with the compositions of m into distinct terms. If $m > 1$, we remove m and those compositions into distinct terms that contain a multiple of 3. Using the fundamental theorem of counting on the remaining compositions directs us to the number, $N(m)$, that we are seeking. These remaining compositions will be called partitions for the purpose of this paper. Table 1 is similar to those in [9] and [11].

$D(p)$	Partitions of $D(p)$	$N(D(p))$	Examples of p
1	1	1	1 2 3 4 5 6
2	$1+1$	1	1,2 2,3 3,4 4,5
4	$2+2$	1	21,43 32,54 43,65
5	$4+1=1+4$	$1+1=2$	3412,5 5634,7 1,4523 3,6745
7	$5+2=2+5$	$2+2=4$	52143,76 32541,76 21,74365 21,54763
8	$1+7=4+4=7+1$	4+1+4=9	6734125,8 6714523,8 3412,7856
10	$2+8=5+5=8+2$	$9+4+9=22$	5 2 1 4 3,10 7 6 9 8
11	$1+10=4+7=7+4=10+1$	$22+4+4+22=52$	8 9 6 7 10 3 4 1 2 5,11

Table 1. Examples

Here $p = 2\ 1, 7\ 4\ 3\ 6\ 5$ is a translation of $q = 8\ 7, 13\ 10\ 9\ 12\ 11$ where $t = 6$, so they are not counted as being different. They are both compositions of $2+5$ where the 5 was composed $1+4$ after it was reversed. In this example, $N(7) = 4$ because "2 1, 5 4 7 6 3", "7 3 4 6 5 , 9 8" and "5 4 7 6 3,9 8" are pats, but they are not translations to p nor to each other. Observing the two compositions, $5+2 = 2+5$, and using the fundamental theorem of counting,

$$N(7) = N(5)N(2) + N(2)N(5) = 2(1) + 1(2) = 4$$

An interesting as well as important observation about the "natural split" of the pat is, that any pat must uniquely split into two partitions such that all numbers in the left partition are smaller than all numbers in the right one (modulo $9n$). Along with Definition 2, this basically means that all numbers of a valid pat can be arranged as leaves of an "alternating binary tree", which is a binary tree with the following two properties:

(1) If the depth of a non-leaf node is even, then each number on the left hand side is smaller than any number on the right hand side.
(2) If the depth of a non-leaf node is odd, then each number on the left hand side is greater than any number on the right hand side.

If we take, for instance the pat "9 10 7 8, 13 14 11 12" of a hexahexaflexagon, then we can uniquely arrange these numbers as the following:

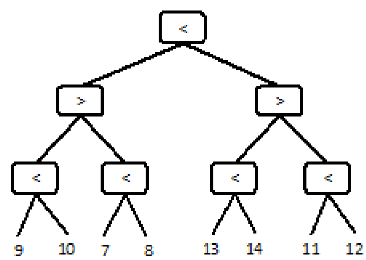

FIGURE 4. Triangles numbers arrangement in a pat

This observation may shed some light on the Catalan type recursions of the number of pat classes presented in [2]. It also plays an important role in the study of the algorithmic questions in this note.

The complexity of hexaflexagons is illustrated by how actions performed on a hexaflexagon (such as "pinch flex" and "V-flex" as described in [4, 5, 9, 10, 2, 11]) can shuffle the triangles in each pat and produce many different faces (top triangles). Every time one acts on a hexaflexagon, a new face is produced. Sometimes a certain sequence of actions will eventually produce a cycle, that is, one can arrive at the same face one started with.

It is known that both pinch flex and V-flex preserve pat partitioning. Therefore the value $N(D(p))$ for general values of $D(p)$ has been of great interest, a table and recursive formulas were provided in [2]. We will present explicit formula for $N(D(p))$ in this note. We also provide formula for equivalent initial faces. These are two main enumerative questions that remained open from previous studies.

2. ENUMERATIONS

In [2], a nice recursive relation for the number $N(D(p))$ of pat classes for $D(p)$ was presented:

Lemma 1 ([2]).

$$N(D(p)) = \sum_{i=0}^{\frac{D(p)-2}{3}} N(1+3i)N(D(p)-(1+3i))$$

where $3|D(p)+1$ *and*

$$N(D(p)) = \sum_{i=0}^{\frac{D(p)-4}{3}} N(2+3i)N(D(p)-(2+3i))$$

where $3|D(p)+2$.

As mentioned before, an interesting flavor of the Catalan numbers appear here. Although it seems to be difficult to find a direct connection between pat classes and other combinatorial concepts of this type, a generating function approach can provide more insights to this sequence of values.

Let $a_n = N(3n+1)$ and $b_n = N(3n+2)$ for $n = 0, 1, 2, \ldots$, the above recursions can be simplified to

$$b_n = \sum_{i=0}^{n} a_i a_{n-i} \text{ and } a_n = \sum_{i=0}^{n-1} b_i b_{n-i-1}. \tag{1}$$

The ordinary generating functions can be defined as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ and } g(x) = \sum_{n=0}^{\infty} b_n x^n$$

where the coefficients of $f(x)$ and $g(x)$ form the sequence of "pat numbers". Now (1) implies

$$f(x) = x(g(x))^2 + 1 \text{ and } g(x) = (f(x))^2,$$

or equivalently,

$$f(x) = x(f(x))^4 + 1. \tag{2}$$

One can solve (2) through a standard (but technical) approach in Enumerative Combinatorics, by using the binomial theorem and the Lagrange inversion formula. One can refer to, for example, [3] for details. In fact, the solution implies

Theorem 1.

$$N(3n+1) = a_n = \frac{1}{3n+1}\binom{4n}{n} \textit{ for } n = 0, 1, 2, \ldots.$$

The sequence $\{a_n\}$ appears at the On-Line Encyclopedia of Integer Sequences (A002293, [1]). Among many other interesting concepts that it is associated with, a_n counts the number of 4-ary or quartic trees with $3n+1$ leaves. These trees are defined as rooted *planar* (the order of the "sub-branches" matter) trees in which every vertex has exactly 1 or 4 children (Figure 5).

One of the most intriguing aspects of Combinatorics is to provide elementary combinatorial proofs from observations and results of analytical study. In this particular case, we provide a simple bijection between a pat class (or a partition of $D(p) = 3n+1$) and a quartic tree with $3n+1$ leaves, hence proving Theorem 1 with an elementary approach.

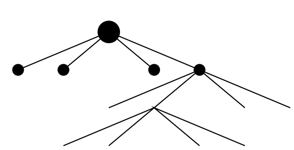

FIGURE 5. A quartic tree

(i) Start with the root and divide the tree in the middle (and duplicate the root), i.e., the left part that contains the two left most branches and the right part that contains the two right most branches;

(ii) For each part, divide the tree at the root (and remove the root) into two parts corresponding to the left and right branches of it;

(iii) Now we have 4 quartic trees (some of them might be a single vertex), continue until we have only single vertex trees left.

The first step is illustrated in Figure 6.

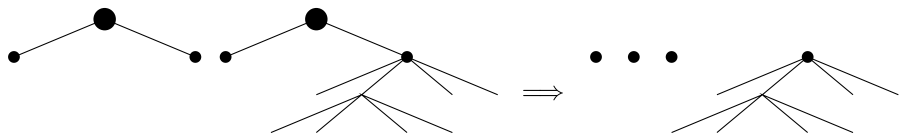

FIGURE 6. The partition of a quartic tree corresponding to the partition

As a result, the above quartic tree with 10 leaves corresponds to the partition of 10 as

$$\begin{aligned} 10 &= 2+8 = (1+1)+(1+7) = (1+1)+(1+(5+2)) \\ &= (1+1)+(1+((1+4)+(1+1))) \\ &= (1+1)+(1+((1+(2+2))+(1+1))) \\ &= (1+1)+(1+((1+((1+1)+(1+1)))+(1+1))) \end{aligned} .$$

The sequence b_n can be handled similarly through both generating functions and bijections, with

Theorem 2.

$$N(3n+2) = b_n = \frac{2}{3n+2}\binom{4n+1}{n} \text{ for } n = 0, 1, 2, \ldots .$$

Once again, $\{b_n\}$ can be found at the On-Line Encyclopedia of Integer Sequences (A069271, [1]).

Let $(x_1, y_1, x_2, y_2, x_3, y_3)$ denote a face with six pat degrees $x_1, y_1, x_2, y_2, x_3, y_3$ in this order (equivalent under rotation), another enumerative question raised in [2] is the number of such faces for flexagons of order $3n$ (hence $\sum x_i + \sum y_i = 9n$). It was established in [2], through an inclusion-exclusion approach, that

$$|C(n)| = \frac{|L(n) - K(n)|}{3} + |K(n)| = \frac{|L(n)| + 2n}{3}$$

where $|C(n)|$, the number of equivalent classes under rotation, is of interests. $K(n)$ is the set of faces where $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$, $L(n)$ is the set of faces where $x_i \equiv 1$ modulo 3 and $y_i \equiv 2$ modulo 3.

We provide the exact formula for $|L(n)|$ and $|C(n)|$ in the following:

Theorem 3.

$$|L(n)| = \binom{3(n-1)+5}{5} = \binom{3n+2}{5} \text{ and } |C(n)| = \frac{1}{3}\binom{3n+2}{5} + \frac{2n}{3}.$$

Proof. It is easy to see that $|K(n)| = n$. To find $|L(n)|$, consider

$$L(n) = \{(3z_1+1, 3z_2+2, 3z_3+1, 3z_4+2, 3z_5+1, 3z_6+2) \text{ such that}$$

$$9n = (3z_1+1)+(3z_2+2)+(3z_3+1)+(3z_4+2)+(3z_5+1)+(3z_6+2)\}$$

with $z_i \geq 0$ for $1 \leq i \leq 6$, or equivalently,

$$L(n) = \left\{ (3z_1+1, 3z_2+2, 3z_3+1, 3z_4+2, 3z_5+1, 3z_6+2) \text{ such that } 3(n-1) = \sum_{i=1}^{6} z_i \right\}.$$

Therefore, $|L(n)|$ is simply the number of compositions of $3(n-1)$ into a sum of 6 nonnegative integers.

Hence we have $|L(n)| = \binom{3(n-1)+5}{5} = \binom{3n+2}{5}$, and consequently $|C(n)| = \frac{1}{3}\binom{3n+2}{5} + \frac{2n}{3}$. □

3. Algorithmic studies

3.1. **Top triangle orientation.** While flexing a regular hexaflexagon one can not only achieve different faces, but also different orientations for top triangles (this fact was well known, as pointed out by Martin Gardner in [6]). We show that for any regular hexaflexagon, constructed from a straight strip of paper, one can always determine the orientation and face up/down properties of any triangle in the strip of paper. One can refer to Figure 1 for the notations. We will call the "free" edge of the triangle the triangle's edge that is not shared with the previous/next triangle in the sequence. Since the last triangle in the sequence will be glued together with the first triangle in the sequence, each triangle has only one "free" edge. When a regular hexaflexagon is constructed from such a strip of paper, the top triangles that are visible can have different numbers, face orientations (up or down), and triangle orientation (with the arrow pointing one of the three possible directions in the hexagon). For instance, Figure 7 shows a hexahexaflexagon with such information.

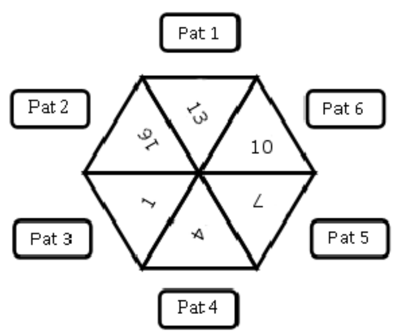

Figure 7. Counting flexagon pats

Table 2 defines the *state* of any triangle in the hexaflexagon as a triplet describing the face orientation, triangle number, and triangle's free edge orientation.

There are 12 distinct states. For instance, triangle number 4 on top of pat 4 in Figure 7 is in state UEP: face up, even numbered triangle, and arrow pointing to previous pat. With these notations, it is easy to observe that any triangle that follows a triangle in a DON state must be even numbered and in the same pat as the DON triangle it follows. The second triangle is folded (above or below) in the same pat, hence it must be facing up and with the arrow pointing outwards. The second triangle must be in a UEO state. Similarly, we can conclude that a triangle

State Identifier	Face orientation	Triangle number	Triangle orientation
Values	U - up	E - even	O - outside
	D - down	O - odd	N - next pat
			P - previous pat

TABLE 2. Flexagon's triangles states

following any triangle in one of the states DON, DOP, UEN, UEP, DEN, DEO, UON, and UOO must be in the same pat (hence facing the opposite direction) as the triangle in one of the above mentioned states. As for any triangle in one of the other 4 states (UEO, UON, DEP, and DOO), it can be followed by a triangle in the same pat (a folding) or a triangle in the next pat (a transition to the following pat). For instance, a triangle in UEO state can be followed by a triangle in DOP state (and the same pat) or a triangle in UON state (and in the following pat).

We can summarize this discussion by representing all 12 states as nodes of a graph, with an edge representing either a folding to the next triangle's state (when the following triangle is in the same pat) or a transition to the next pat (when the following triangle is in the next pat). More formally, we can represent the way a straight strip of paper can be folded into a regular hexaflexagon as a graph of Triangle State Transitions (TST graph) as follows (Figure 8).

Theorem 4 (Triangle State Transitions - TST graph)**.** *Transitions from one triangle to the next in a folded strip of paper (with triangles numbered* $1, 2, 3, \ldots$*; the first triangle's "free" edge points upward when face up) representing any regular hexaflexagon of any order can be represented as a directed graph of Triangle State Transitions such that:*

(1) $V = \{DON, DOP, UEN, UEP, DEN, DEO, UON, UOO, UEO, UON, DEP, DOO\}$
(2) $E = FT \cup PT$*, where* FT *represents the set of all folding transitions to the next triangle and* PT *represents the set of all pat transitions to the next triangle:*
$FT = \{(UEO, DOP), (DOP, UEN), (UEN, DOO), (DOO, UEP), (UEP, DON), (DON, UEO), (UOP, DEO), (DEO, UON), (UON, DEP), (DEP, UOO), (UOO, DEN), (DEN, UOP)\}$
$PT = \{(UOP, UEO), (UEO, UON), (DEP, DOO), (DOO, DEN)\}$

For instance, for the excerpt "1, 2 - 4 3, 6 5 - 7, 8 - 10 9, 12 11 - 13, 14 - 16 15, 18 17" of the hexahexaflexagon's face mentioned in Figure 7, we can see that triangle 1 is facing up, with the free edge pointing to the next pat. That is, triangle 1 is in UON. Since triangle 2 is in the same pat, we follow a FT transition from 1, which puts 2 in DEP, then we continue with a PT to triangle 3. This places triangle 3 in DOO state. Then, as we move to triangle 4 we must follow a FT edge (it is the same pat, hence a folding), which places triangle 4 in UEP state. We continue with FT, 5 in DON, FT then 6 in UEO, PT (we move to the next pat) then 7 in UON, FT then 8 in DEP, FT then 9 in UOO, FT then 10 in DEN, FT then 11 in UOP, FT then 12 in DEO, FT then 13 in UON. It is easy to see that, for any given face, once the state of any triangle has been determined then the states of all other triangles can be easily determined by following the TST graph edges. Hence

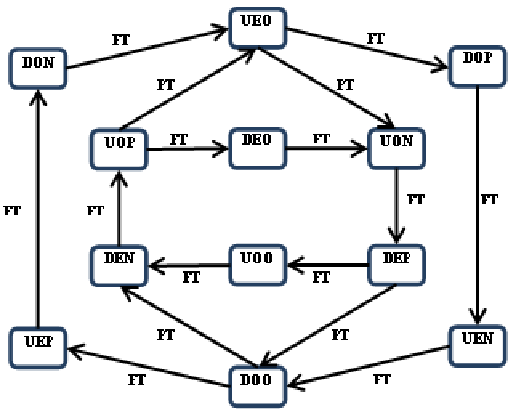

FIGURE 8. The Triangle State Transitions (TST) Graph

the natural question is, given a face, how to determine the state of a triangle. The following theorem provides the fundament for the answer.

Theorem 5. *For any pat p not containing the first or last triangle, the following statements hold:*

(1) *If $D(p)$ is $3k+1$, then the triangle with smallest number is in either UEO or DOO state.*
(2) *If $D(p)$ is $3k+2$, then the triangle with smallest number is in either UON or DEN state.*

Proof. It is easy to note that we must arrive to a new pat (to the triangle with smallest number in the pat) only through a PT edge in TST. Hence one of the states: UEO, UON, DOO, or DEN. Then we can leave the pat only through a PT edge in TST. Starting on a UEO or DOO state, we need $3k$ foldings to arrive at a new state that has an outgoing PT edge. This means $3k+1$ states (including the originating state), which proves claim (1). Similarly, starting on one of UON or DEN states, we need $3k+1$ foldings to arrive in a new state that has an outgoing PT edge. Which means $3k+2$ states, hence claim (2). □

NOTE: We leave aside what happens with the pats containing the first and/or last triangles. These triangles are glued together (hence not necessarily following TST) to connect the ends of the strip of paper. However, this does not affect the analysis we carry out here.

The following corollary is immediate.

Corollary 1. *For any pat p not containing the first or last triangle, let S be the smallest triangle number in the pat. Then:*

(1) *If S is odd and p is odd ($3k+1$ triangles), then the triangle is in DOO state.*
(2) *If S is odd and p is even($3k+2$ triangles), then the triangle is in UON state.*
(3) *If S is even and p is odd, then the triangle is in UEO state.*
(4) *If S is even and p is even, then the triangle is in DEN state.*

Given a face and any triangle number, we can now determine the state of the triangle, regardless whether the triangle appears on top or hidden beneath top triangles. Given a face configuration (list of triangle numbers in each pat, [2] defines precisely the face obtained after a pinch flex or V-flex action. Hence, after a pinch flex or V-flex, the algorithm below easily determines the state of each triangle and therefore the appearance of top triangles in any face.

Algorithm $TriangleState$ (**input** : hexaflexagon face F, triangle number N)
search F for the pat containing triangle number 1
move to next pat P
$S :=$ the smallest triangle number in P
$D :=$ degree of P
state := undefined

if S is odd and D is odd **then** *state* := DOO
else if S is odd and D is even **then** *state* := UON
else if S is even and D is odd **then** *state* := UEO
else if S is even and D is even **then** *state* := DEN
if $N > S$ **then**
follow the transitions in TST starting with *state*
change *state* accordingly and increment S until triangle N is reached
if $N < S$ **then**
follow backwards the transitions in TST starting with *state*
change *state* accordingly and decrement S until triangle N is reached
return *state*

3.2. **Folding algorithm for a regular hexaflexagon.** The problem we discuss here is how to fold, in general, a strip of paper with $9n$ triangles in order to construct a "valid" regular hexaflexagon of order 3n such that it can be pinch-flexed and V-flexed as described in [2]. More precisely, by determining an initial face to start with, one can construct a paper hexaflexagon or write a computer program to operate on hexaflexagon faces. The algorithm we present here would be able to produce any "mathematical" face that is valid for a hexaflexagon. However, an open question still remains: can one achieve all possible mathematical faces in practice, using the well known hexaflexagon transformations (pinch flex and V-flex)? A computer program would certainly help producing some constructive answers to this question.

For any positive integer D, with D non-divisible by 3, we can always construct a sequence of numbers from $1, 2, 3, \ldots, D$ that represents a valid pat. It is known that, in general, a pat of degree D is not unique. The algorithm we propose is flexible in the sense that it can produce sequences of different partitions.

Algorithm *InitialPat* (**input** : pat degree D, *depth*)
  if $D = 1$ **then return** $\{1\}$

  $\{LD, RD\}$:= partition D into a left-degree and right-degree

  $LeftPat := InitialPat(LD, depth + 1)$
  $RightPat := InitialPat(RD, depth + 1)$

  if *depth* is odd **then** exchange $LeftPat$ and $RightPat$ sequences

  if *depth* is odd **then** increment all numbers in $LeftPat$ by LD
  else increment all numbers in $RightPat$ by LD

  return $\{LeftPat, RightPat\}$

The algorithm proceeds recursively by producing an appropriate partitioning of D (such that each part is not a multiple of 3), then the sequence $\{1, 2\}$ is generated (in appropriate order depending of the depth) once it reaches partitions of degree 2, then proceeds bottom-up and joins adjacent sequences, reversing and translating the numbers as needed. Below is an example of producing a sequence for a pat of degree 4.

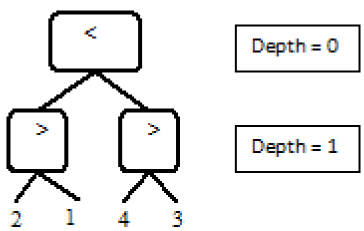

FIGURE 9. Generating a pat of 4 triangles

Note that we do not specify how to produce a partition for an integer D. We give implementer the liberty to choose how to partition, as different choices produce different sequences. Finally, the steps for producing an entire face for a hexaflexagon of order $3n$ (with $9n$ triangles, that is) are given in the following:

(1) Create a partition of $9n$, for instance a symmetric $\{x, y, x, y, x, y\}$, such that neither x, nor y are multiples of 3 and $x + y + x + y + x + y = 9n$
(2) Generate pat 1 as the result produced by $InitialPat(x)$
(3) Generate pat i, with $i = 2, 3, 4, 5, 6$ by using $InitialPat(x)$ or $InitialPat(y)$ as appropriate for the pat; then translate all numbers in the sequence by the sum of degrees of all preceding pats.

4. Summary

In this article we show a few results, of both enumerative and algorithmic nature, answering some of the previously raised questions as well as providing systematic ways of determining the state of a flexagon under flexes. There are still more that is unknown than known. One of the most interesting questions remained (as pointed out by Ralph Jones), which we would like to post as an open problem, is the following.

Conjecture: All regular flexagons can be translated by 1 (i.e., through a sequence of flexes, one can shift all the corresponding numbers by 1 under appropriate modular condition).

Remark: If the above statement is true, it would imply that all mathematical faces of any regular flexagon can really be achieved. Constructive proofs for this statement were found for the hexahexaflexagon and the nonahexaflexagon.

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