CHAPTER 2

APRIORITY AND APPLICATION: PHILOSOPHY OF MATHEMATICS IN THE MODERN PERIOD

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In the modern period[1]—which might be thought to have begun with a new conception of the natural world as uniquely quantifiable—the term "science" was used to denote a systematic body of knowledge based on a set of self-evident first principles. Mathematics was understood to be the science that systematized our knowledge of magnitude, or quantity. But the mathematical notion of magnitude, and

This material is based upon work generously supported by the National Science Foundation under grant no. SES-0135441. I am grateful to Gary Hatfield and Stewart Shapiro for helpful comments and suggestions, and to Sukjae Lee for ongoing discussion.

[1] For the purposes of this chapter, I am construing this period in the standard way, as ranging from Descartes to Kant. I do not intend to offer an encyclopedic account of the issues in the philosophy of mathematics that faced the modern philosophers, nor even a survey of each major philosopher's account of mathematical cognition. I hope, rather, to tell a story about some key issues in the philosophy of mathematics in the modern period.

the methods used to investigate it, underwent a period of radical transformation during the early modern period: Descartes's new analytic methods for solving geometric problems, Vieta's and Fermat's systems of "specious arithmetic," and Newton's and Leibniz's independent discoveries of the calculus are just a few of the developments in mathematical practice that witness this transformation. These innovations in mathematical practice inspired similarly radical innovations in the philosophy of mathematics: philosophers were confronted with a changing mathematical landscape, and their assessment of the ontological and epistemological terrain reveals that both the basic mathematical notions and the philosophers' tools for comprehending and explaining those notions were in transition.

While mathematical methods were becoming decidedly more analytical and abstract, they nevertheless remained anchored by a concrete conception of magnitude, the object of mathematics. Ultimately, mathematics was meant to provide a quantitative description of *any* quantifiable entity: the number of cups on the table, the volume and surface area of a particular cup, the amount of liquid the cup contains, the temperature of the liquid, and so on. The term "magnitude" was used to describe both the quantifiable entity and the quantity it was determined to have. That is, for the moderns, magnitudes have magnitude.[2] In the seventeenth and eighteenth centuries, modern mathematical practices had not yet become sufficiently abstract so as to fully detach the latter notion from the former: mathematics was not yet about number or shape (much less set or manifold) conceived independently from numbered or shaped *things*. This is not to suggest that the modern mathematician lacked the resources to represent such quantifiables in abstract mathematical form; indeed, the period was rich in such representational innovation. The point is, rather, that modern mathematical ontology included both abstract mathematical representations and their concrete referents. Accordingly, modern mathematical epistemology could not rest with an account of our cognitive ability to manipulate mathematical abstractions but had also to explain the way in which these abstractions made contact with the natural world.

The state of modern mathematical practice called for a modern *philosopher* of mathematics to answer two interrelated questions. Given that mathematical ontology includes quantifiable *empirical* objects, how to explain the paradigmatic features of pure mathematical reasoning: universality, certainty, necessity. And, without giving up the special status of pure mathematical reasoning, how to explain the ability of pure mathematics to come into contact with and describe the

[2] Thus, for example, the pens on my table are a discrete magnitude having, say, a magnitude of four units where the unit is one pen; the surface of my desk is a continuous magnitude having, say, a magnitude of two units where the unit is square meters. For a discussion of Newton's conception of magnitude, and of his distinction between quantum and quantity, see McGuire (1983), p. 75. For a discussion of Wolff and Kant on the same distinction, see Shabel (2003), pp. 124–126.

empirically accessible natural world.[3] The first question comes to a demand for *apriority*: a viable philosophical account of early modern mathematics must explain the apriority of mathematical reasoning. The second question comes to a demand for *applicability*: a viable philosophical account of early modern mathematics must explain the *applicability* of mathematical reasoning. Ultimately, then, the early modern philosopher of mathematics sought to provide an explanation of the relation between the mathematical features of the objects of the natural world and our paradigmatically a priori cognition thereof, thereby satisfying both demands.

At the end of the modern period, Kant attempts to meet these demands with his doctrine of Transcendental Idealism, including arguments for the synthetic a priori status of mathematical cognition. In the course of defending his own theory of how we come to cognize the mathematical world, Kant distinguishes two strains of thought in his predecessors' competing accounts of mathematical cognition, which he determines to be inadequate and misguided.[4] Taking the science of Euclidean geometry, an exemplar of pure mathematical reasoning, to provide substantive cognition of space, spatial relations, and empirically real spatial objects, Kant claims that prior attempts to account for our cognitive grasp of geometry and its objects "come into conflict with the principles of experience."[5] On Kant's view, both the "mathematical investigators of nature" (who suppose that the spatial domain of geometric investigation is an eternal and infinite subsisting real entity) and the "metaphysicians of nature" (who suppose that the spatial domain of geometric investigation comprises relations among confused representations of real entities that are themselves ultimately nonspatial) fail to provide a viable account of early modern mathematics in the sense described above. In particular, the "mathematicians" fail to meet the applicability demand, and the "metaphysicians" fail to meet the apriority demand.[6] Kant's claim, of course, is that his own theory of synthetic a priori mathematical cognition meets with greater success.

[3] The closest analog to our pure/applied distinction in the modern period is captured by a distinction between pure and "mixed" mathematics. According to Christian Wolff, the mixed mathematical disciplines are those that "consider and measure the particular magnitude of things found in nature," while the pure or unmixed mathematical disciplines "consider only the magnitude as magnitude" (Wolff (1965), pp. 866, 868).

[4] It is obviously problematic to assess Kant's early modern predecessors from Kant's own perspective. But Kant's way of articulating his own position in contrast to competing positions does provide us with a nice framework for understanding the entire period. I will do my best to present the views of Kant's predecessors objectively, with the caveat that I am telling a story in which Kant gets the final word.

[5] Kant (1998), B56.

[6] These failures are not straightforward since, as we will see below, one can explicate the view of a "mathematician" like Descartes in a way that seems clearly to satisfy both demands. The sense in which, according to Kant, the apriority and applicability demands are not met by any view prior to his own will be explained in section 3.

In what follows, I will discuss these three major attempts to provide a viable philosophical account of early modern mathematical practice. Descartes and Newton stand as examples of the "mathematicians"; Leibniz exemplifies the "metaphysicians"; and Kant sees himself as correcting the common error that, on his view, leads both "mathematician" and "metaphysician" astray. I will begin by providing a brief account of a relevant aspect of early modern mathematical practice, in order to situate our philosophers in their historical and mathematical context.

1. Representational Methods in Mathematical Practice

As noted above, the modern mathematician's task included systematizing the science of quantity. This required, first and foremost, a systematic method for representing real, quantifiable objects mathematically, as well as a systematic method for manipulating such representations. The real, quantifiable objects were conceived to include both discrete magnitudes, or those that could be represented numerically and manipulated arithmetically, and continuous magnitudes, or those that could be represented spatially and manipulated geometrically. These categories, however, were neither fully determinate nor mutually exclusive; they did not serve to demarcate two distinct sets of real, quantifiable objects because, for example, a given magnitude might be considered with respect to both shape *and* number, and thus might be treated geometrically for one purpose and arithmetically for another. Mathematical progress required a representational system that was adequate to symbolize any and all quantifiable features of the natural world in such a way that comparisons could be easily made among them. Moreover, mathematical progress required that such a representational system be as abstract and general as possible while nevertheless retaining its purchase on the concrete and particular quantifiables of the natural world.

For a simple example, consider a bag of marbles. A geometric diagram of a sphere can be taken to represent the shape of any individual marble and, indeed, any spherically shaped object. This sort of representation can be mathematically manipulated using classical techniques of Euclidean solid geometry, but its intermathematical utility is limited: while a diagram of a sphere can be useful in plainly geometric reasoning about spheres (and cross sections thereof, tangents thereto, etc.), it is impotent for the purpose of more abstract reasoning, such as might be required mathematically to compare the sphere to nonspatial magnitudes. A qualitative geometric diagram can represent a magnitude qua spatial object, but cannot aid us in abstracting from spatiality in order to represent it as a discrete quantity.

Moreover, while a number can be taken to represent the determinate ratio of all of the marbles in the bag to a single marble, it can do so only upon specification of a marble as the unit of measure. On the modern conception, a number "arises" upon consideration of a group of things of a particular kind in relation to a single thing of that kind and thus is, in a sense, context-sensitive: the use of number as a quantitative tool is inextricably tied to the choice of unit, which is not fixed. On this conception, it follows that numbers are representationally inflexible: it is difficult to see how to make quantitative comparisons among magnitudes of different *kinds*. More generally, on this conception the possibility of a notion of number that ranges beyond the positive integers appears remote.

Early modern innovations in representational flexibility began with Descartes's observation that we can represent and compare magnitudes uniformly and in the simplest possible terms by formulating ratios and proportions between and among their "dimensions." "By 'dimension,' " Descartes writes, "we mean simply a mode or aspect in respect of which some subject is considered to be measurable."[7] According to Descartes, dimensionality comprises countless quantitative features which include length, breadth, and depth, but also weight, speed, the order of parts to whole (counting), and the division of whole into parts (measuring). More importantly, a finite straight line segment is identified as the most versatile tool for representing any dimension of magnitude; any and all quantitative dimensions can be represented simply and uniformly via configurations and ratios of straight line segments.[8] Notably, in Descartes's system the unit segment, to which any other representative segment stands in relation, is problem-specific: for solution of a particular problem "we may adopt as unit either one of the magnitudes already given or any other magnitude, and this will be the common measure of all the others."[9] Once the unit magnitude is chosen and represented as a particular finite line segment, representations of all other relevant magnitudes can be constructed in relation to that unit.[10]

[7] Descartes (1985), *Rules*, AT 10:447.

[8] Descartes discusses this procedure in rules 14 through 18 of his *Rules for the Direction of the Mind*. Descartes (1985) AT 10: 438–468. For further discussion see Shabel (2003), pp. 65–67.

[9] Descartes (1985), *Rules*, AT 10:450.

[10] Thus, Descartes's unit segment cannot be identified with the real unit interval [0, 1], nor with sections of the orthogonal number lines that we use to construct what we anachronistically call the "Cartesian" coordinate system. Descartes's unit segment is a line segment of arbitrary length that stands for whatever particular magnitude functions as the unit in a particular problem. Even if the representational system were generalized and a fixed unit segment were chosen to represent the unit of magnitude functioning in any mathematical context, nevertheless the Cartesian unit segment would still serve as unit by virtue of the ratio in which it stands to the other magnitudes of the problem, as evidenced by their relative lengths, but not by virtue of its structure as a dense linear ordering.

To solve a mathematical problem, one thus begins by identifying and representing the magnitudes involved, no matter whether they be determinate or indeterminate, known or unknown, geometric or arithmetic, extended or multitudinous, continuous or discrete. One chooses a unit of measure and conceives each relevant magnitude in general terms—in abstraction from the object or group of objects that is ordered or measured but in relation to the chosen unit—by following Descartes's technical constructive procedures. Algebraic symbols can then be used as a heuristic aid for manipulating the line segments, and algebraic equations formulated to express the mathematical relations in which the segments stand to one another.[11] Famously, the Cartesian representational system liberates formerly heterogeneous magnitudes: on the Cartesian system, the multiplication of two line segments yields another line segment, rather than the area of a rectangle. It follows that products, quotients, and roots of linear magnitudes can be construed to stand in proportion to the magnitudes themselves; likewise, the degree of algebraic variables need not be taken to indicate strictly *geometric* dimensionality.

Descartes's own use of his representational system was primarily directed at solving problems in geometry,[12] though he certainly thought it adequate for other mathematical manipulations. Modern mathematicians who adopted the Cartesian system emphasized its utility across mathematical disciplines, and used straight lines to represent numbers for arithmetic and number-theoretic purposes. Once a single straight line is designated as unity, any positive integer can be straightforwardly represented as a simple concatenation of units. One advantage of the Cartesian representational system is that it allows the notion of number to expand beyond the positive integers so constructed: numbers can now be conceived as ratios between line segments of any arbitrary length and a chosen unit. Rational numbers are identified as those segments that are (geometrically) commensurable with unity, and irrational numbers those that are incommensurable. These representational innovations witness a number concept in transition: the moderns are able to treat rational and irrational magnitudes numerically, but the notion of number remains tied to the geometric concept of commensurability. Importantly, the moderns are *unable* to use the Cartesian representational system to conceive negative quantities, a disadvantage of the system. Consequently, negative quantities are variously deemed "absurd," "privative of true," "wanting reality," and "not real."

Actual mathematical progress in the modern period might be said to outpace the fundamental and foundational tools and concepts available to mathematicians

[11] For a discussion of the details of Descartes's technical constructive procedures, see Shabel (2003), pp. 65–69.

[12] His *Géometrie* opens with the claim "Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction" (Descartes (1954), p. 2).

at the time. Geometric problems far more sophisticated than the ancients could have imagined are solved on ancient geometric foundations; algebraic methods are developed and fruitfully applied long before algebra is conceived to be a mathematical discipline in its own right, with its own domain of problems distinct from those of arithmetic and geometry; number-theoretic investigations and the discovery of the calculus proceed despite the foundational inadequacy of a number concept that cannot handle negative or infinitesimal quantities with logical rigor.[13] In this sense, early modern mathematical practice awaits deep philosophical response in the nineteenth century, when logical and mathematical foundations are reassessed in the face of the striking developments in mathematical practice in the seventeenth and eighteenth centuries. The early modern philosopher remains occupied by the more basic philosophical problems raised by a newly mathematized natural world.[14] It is to those problems, and their solutions, that we now turn.

2. "Mathematicians," "Metaphysicians," and the "Natural Light"

In the Fifth Meditation of his *Meditations on First Philosophy*, Descartes famously argues that we have a clear and distinct perception of the essence of material substance, and that that essence is pure extension. According to Descartes, our clear and distinct perception of the immutable and eternal natures of all material objects, which we find represented in our idea of extension, is the basis for sure and certain knowledge of those objects. This knowledge is systematized by our pure mathematics, in particular by geometry, which thus provides us with an a priori

[13] The calculus is based on a nonnumerical notion of the infinitesimally small differentials between magnitudes conceived as finite line segments, called *differences* by Leibniz and *moments* by Newton. See Struik (1986), pp. 272, 300–301.

[14] Berkeley's critique of the techniques of the calculus is an obvious exception to this generalization, though I believe a case could be made that Berkeley's criticisms prefigured and anticipated issues that did not become fully transparent until the nineteenth century. For this reason, and also due to the scope of this chapter, I have chosen not to address Berkeley's philosophy of mathematics here. For discussion of Berkeley's philosophy of mathematics, see Jesseph (1993). For discussion of the modern debates that are the precursors to the nineteenth-century debates on the foundations of the calculus, see Mancosu (1996).

science of the nature of the corporeal world. For Descartes, then, the essential nature of the material world is perfectly mathematical and perfectly knowable.[15]

Descartes begins his Fifth (and penultimate) Meditation, "The Essence of Material Things, and the Existence of God Considered a Second Time," by proposing to investigate "whether any certainty can be achieved regarding material objects."[16] Before demonstrating that we have certain knowledge of the real existence of material things, as he will do in the Sixth (and final) Meditation, Descartes must first investigate his *ideas* of material things: "But before I inquire whether any such things exist outside me, I must consider the ideas of these things, in so far as they exist in my thought, and see which of them are distinct, and which confused."[17] He proceeds to show that his clear and distinct ideas of extension provide insight into "that corporeal nature which is the subject-matter of pure mathematics."[18]

First, Descartes confirms the clarity and distinctness of his idea of quantity, in particular "the extension of the quantity (or rather of the thing which is quantified) in length, breadth, and depth." He continues, "I also enumerate various parts of the thing, and to these parts I assign various sizes, shapes, positions and local motions; and to the motions I assign various durations."[19] The clarity and distinctness of our general idea of quantity is best understood by considering the mathematically fundamental representations discussed above, in section 1. For Descartes, particular quantities are imaginable via line segments; lengths, breadths, and depths, as well as enumerations of parts, sizes, shapes, positions, motions and durations, are, qua quantities, all mathematically represented with straight line segments, relations among straight line segments, and algebraic equations expressing those relations. These precise and perspicuous mathematical symbols witness the

[15] On its surface, it should appear that Descartes's account of our mathematical cognition easily satisfies *both* the applicability and the apriority demands. Below, we will discuss the specific sense in which—according to Kant—it fails to satisfy the applicability demand.

[16] Descartes (1985), AT 7:63.

[17] Descartes (1985), AT 7:63.

[18] Descartes (1985), AT 7:71. I take it that whereas the Sixth Meditation demonstrates the *existence* of material things, the Fifth Meditation shows that *if* material things exist, *then* they have the *properties* that I clearly and distinctly perceive them to have, that is, all the properties which "viewed in general terms, are comprised within the subject-matter of pure mathematics." See passages at AT 7:65 and AT 7:80.

[19] Descartes (1985), AT 7:63. In this passage Descartes actually speaks of quantity as that which he distinctly "imagines." Throughout the Fifth Meditation he moves back and forth between his imaginings and his ideas, but his conclusions are formulated with respect to the clear and distinct deliverances of the intellect. For a discussion of the relation between the imagination and the intellect, and their roles in our knowledge of geometry, see Hatfield (1986), pp. 62–63.

quantitative properties they symbolize. But our perception of both these symbols and the real quantitative properties they symbolize depends on our having clear and distinct perception of pure extension and its modes—that is, on an idea of quantity in general that is accessible to our intellect.

Descartes claims further that his perception of particular features of quantity is in harmony with his very own nature, "like noticing for the first time things which were long present within me although I had never turned my mental gaze on them before."[20] This is a claim that is buttressed by the "wax argument" of the Second Meditation. There Descartes identifies extension, or extendedness, as that feature of the wax that makes it a material thing, its nature or essence. More important, perhaps, he identifies his own intellect as that cognitive tool that allows him to perceive the extendedness of all material things: Descartes's perception of the essential feature of material substance is due neither to sensation nor to imagination, but to a "purely mental scrutiny" which enables his clear and distinct perception of pure extension.

Descartes combines these results with his more general (and independently demonstrated) conclusions that what he clearly and distinctly perceives is *true*; that his idea of God is innate; that God exists; and that God is no deceiver. Since Descartes has claimed that he clearly and distinctly perceives the quantitative features of material objects, it follows that these quantitative properties are "true"—that is, that material objects (if they exist) really do have the quantitative properties Descartes perceives them to have. The existence of a nondeceiving God underwrites Descartes's perception in a strong sense: his intellectual capacity for clear and distinct perception is a God-given "natural light" providing him with the ability to access mathematical truths, God's own "free creations."[21]

Upon demonstrating in the Sixth Meditation that material substance exists and is really distinct from immaterial substance, Descartes completes his Meditative project. In the process of pursuing his broad metaphysical and epistemological goals, Descartes can be understood to have articulated a philosophy of mathematics according to which we have a priori knowledge of mathematical truths about real, extramental material substance. In particular, our intellect affords us clear and distinct perception of the quantitative features of mind-independent entities, which we represent to ourselves via ideas of extension. Moreover, we know that our ideas of extension perfectly describe the really extended material world—that is, that our ideas of extension are in harmony with the really extended matter that is their object.[22] Descartes's epistemological and metaphysical conclusions

[20] Descartes (1985), AT 7:64.

[21] More specifically, the existence of a good God underwrites Descartes's ability to retain mathematical knowledge that is based on clear and distinct perception. See the argument at AT 7:70.

[22] Descartes uses the harmony metaphor at AT 7:64.

vis-à-vis mathematical objects are thus entwined: he has identified a mental faculty, the rational intellect, to be a cognitive tool for accessing mind-independent material natures. Since, on Descartes's view, those natures are quantitative and mathematically describable, he conceives the "natural light of reason" to illuminate the metaphysically essential features of the mind-independent world and to provide us with "full and certain knowledge . . . concerning the whole of that corporeal nature which is the subject-matter of pure mathematics."[23]

It appears that if Descartes's arguments are accepted, his rationalist philosophy of mathematics satisfies both of the demands on a viable account of early modern mathematical practice, identified above. He accounts for the apriority of mathematics with his theory that the intellect, a faculty of mind independent of the bodily faculties of imagination and sensation, provides direct access to innate mathematical ideas of eternal and immutable natures. He accounts for the applicability of mathematical reasoning by identifying the essence of the natural material world with pure extension, the very object of our innate mathematical ideas: an explanation of how a priori mathematics applies to the natural world is easily forthcoming from a theory that directly identifies the essential features of the natural world with the subject matter of pure mathematics. Since, on Descartes's view, our a priori mathematical knowledge systematizes mathematical truths about a mind-independent and really extended natural world, he thus appears to have satisfied both the apriority and the applicability demands. It will follow from Kant's critique, however, that a theory like Descartes's goes too far in satisfying the applicability demand: the mathematical sciences as Descartes understands them apply far beyond the bounds of what Kant takes to be the limits of our "possible experience." To see why this is so, it will be helpful to consider the views of another "mathematician," and Kant's real target, Isaac Newton.

Newton did not concede all (or even many) of Descartes's metaphysical and physical conclusions; indeed, he spends much of his famous essay "On the Gravity and Equilibrium of Fluids"[24] disputing Descartes's theory of space. On Descartes's view, space is not distinct from the extended material world of spatial things:

> There is no real distinction between space, or internal place, and the corporeal substance contained in it; the only difference lies in the way in which we are accustomed to conceive of them. For in reality the extension in length, breadth and depth which constitutes a space is exactly the same as that which constitutes a body.[25]

Thus, for Descartes, any knowledge we acquire via geometrical cognition of space is, likewise, knowledge of bodily extension. By contrast, Newton conceives space

[23] Descartes (1985), AT 7:71.
[24] Newton (1962).
[25] Descartes (1985), *Principles*, AT 8:45.

to be distinct from bodies, arguing that we can clearly conceive of extension independent of bodies: "... we have an exceptionally clear idea of extension, abstracting the dispositions and properties of a body so that there remains only the uniform and unlimited stretching out of space in length, breadth and depth."[26] It follows that, for Newton, our geometrical cognition of space affords us knowledge of an entity that is infinitely extended, continuous, motionless, eternal, and immutable, but that is not itself corporeal and that can be conceived as empty of bodies. Moreover, space is a unified whole of strictly contiguous parts: the single infinite space encompasses every possible spatial figure and position that a bodily object might "materially delineate."[27] Bodies, for Newton, are the movable and impenetrable entities that *occupy* space, and thus provide us with corporeal instances of spatial parts.

Thus, Descartes and Newton disagree in a deep sense about how to understand the ontology of space and matter, about whether extension can be distinguished from corporeal reality, and thus about whether we have claims to mathematical knowledge of space that are distinct from our claims to mathematical knowledge of spatial things.[28] For Descartes, our knowledge of quantifiable things just is knowledge of the nature of quantity in general, whereas for Newton our knowledge of quantifiable things (which occupy *parts* of space) is acquired via knowledge of the general nature of extension, knowledge of how parts of space relate to space conceived as an infinite whole. But Descartes and Newton nevertheless agree in their accounts of how we come to acquire knowledge of what they both conceive to be an extramental reality that is the ultimate subject matter of pure mathematics. Newton follows Descartes in positing a faculty of understanding as the real source of mathematical cognition, a tool with which we can comprehend the eternal and immutable nature of extension, which he conceives as infinite space.[29] While sensation allows us to represent "materially delineated" bits of extension (i.e., bodies), and imagination allows us to represent *indefinitely* great extension, according to Newton only the faculty of understanding can clearly represent the true and general nature of space/extension.[30]

[26] Newton (1962), p. 132

[27] Newton (1962), p. 133.

[28] They also disagree about infinite versus indefinite extension, and about how to understand the connection between created extension and the mind of God. These are issues that I cannot pursue here.

[29] Newton (1962), p. 134.

[30] McGuire makes this point: "In this sense, then, the understanding possess [*sic*] a non-sensuous representation of infinite distance which has its ultimate ground in the real but uncreated nature of extension itself" (1983, p. 107). According to Newton, the understanding can likewise represent the finite but infinitesimally small quantities that are the basis of his calculus: "fluxions are finite quantities and real, and consequently ought to have their own symbols; and each time it can conveniently so be done, it is preferable to

Newton, like Descartes, thus accounts for our applicable a priori mathematical cognition by claiming that we have a clear perception of the mathematical features of an extramental natural world. Both Descartes and Newton conceive this clear perception of the mathematical features of an extramental natural world to be illuminated by the *natural light* of reason, a metaphor for the sense in which our faculty of understanding is, on their views, an acute mental vision bestowed by a nondeceiving God.[31] As before, it appears that Newton's account of mathematical cognition satisfies both the apriority and the applicability demands: according to Newton, we have the necessary cognitive tools to acquire a priori knowledge of the mathematical features of the natural world. As noted, however, Kant considered Newton's account of mathematical cognition—representative of what he took to be the "mathematician's" position—to be deeply inadequate. He considered Leibniz's alternative account—representative of what he took to be the "metaphysician's" position—just as problematic, as we will see in detail below.

Famously, Leibniz claims that the ultimate constituents of reality, the true substances, are monads: partless, simple, sizeless entities that have perceptual states, like minds. It follows that, for Leibniz, no extended thing is an ultimate constituent of reality and the extended natural world is merely a "phenomenon": our perceptions of a spatial material world are confused representations of a metaphysically fundamental realm of nonspatial monads. Space, time, and "the other entities of pure mathematics" are "always mere abstractions." Their perfect uniformity indicates that they are not intrinsic denominations, internal properties of things, but extrinsic denominations, orderings or relations among things: "[Space] is a relationship: an order, not only among existents, but also among possibles as though they existed. But its truth and reality are grounded in God, like all eternal truths."[32] While it might be accurate to say that, on Leibniz's view, mathematics describes relations among material entities, it is important to remember that for Leibniz material entities are exhausted by our confused phenomenal representations of a perceptually inaccessible underlying reality. This underlying monadic realm is populated by entities that are *not* mathematically describable; monads as Leibniz understands them *cannot* be identified with mathematical points, either

express them by finite lines visible to the eye rather than by infinitely small ones" (1982, p. 107). Of course, that these finite lines be literally "visible to the eye" is, on his own view, irrelevant. What matters is that the finite lines be mathematically manipulable symbols of infinitely small but nevertheless *real* quantities, the ultimate subject matter of the calculus.

31 As mentioned above, God plays different roles in the epistemological systems of Descartes and Newton; this difference cannot be explored here. For discussion, see McGuire (1983) and Stein (2002).

32 Leibniz (1996), II.xiii.149.

geometric or numeric. Given his monadology, Leibniz thus owes us an account of the status of mathematical claims to knowledge: Since our *metaphysical* knowledge of reality does not include *mathematical* knowledge, how do we acquire mathematical knowledge, and what is it about?

Leibniz expresses an important component of his mature philosophy of mathematics in a letter to Queen Sophie Charlotte of Prussia,[33] his student, friend, and epistolary interlocutor. There he explains his position that an internal "common sense" allows us to form those clear and distinct notions that are for Leibniz the ultimate subject matter of pure mathematics, such as number and shape. These are ideas of qualities more "manifest" than those sensible and "occult" qualities we access via the clear but confused notions of a single sense, such as colors, sounds, and odors. The notions we acquire through the "common sense" describe qualities accessible to more than one external sense: "Such is the idea of *number*, which is found equally in sounds, colors, and tactile qualities. It is in this way that we also perceive *shapes* which are common to colors and tactile qualities. . . ."[34] Leibniz further identifies the imagination as the faculty of mind that operates on the clear and distinct notions of the common sense, the ideas that arise from the perception of qualities common to more than a single external sense, to produce our conceptions of mathematical objects:

> . . . these clear and distinct ideas [of the common sense], subject to imagination, are the objects of the mathematical sciences, namely arithmetic and geometry, which are pure mathematical sciences, and the objects of these sciences as they are applied to nature, which make up applied (mixtes) mathematics.[35]

He adds, however, that sense and imagination must be augmented by understanding in order to attain a full conception of mathematical objects and "to build sciences from them." It is, according to Leibniz, the understanding or reason that assures that the mathematical sciences are demonstrative and that mathematical truths are *universally* true, for without the application of such a "higher" faculty of reasoning, the mathematical sciences would consist merely of observations and inductive generalizations therefrom. Indeed, Leibniz claims that despite the fact that our notions of mathematical objects originate in sensible experience, nevertheless a demonstrative mathematical truth is fully "independent of the truth or the existence of sensible and material things outside of us." Thus, for Leibniz, mathematical truths describe features of our sensible experience despite finding their justification in the understanding alone.

[33] Leibniz (1989), pp. 186–192. This letter is also known as "On What Is Independent of Sense and Matter." For more discussion of this letter and Leibniz's philosophy of mathematics, see McCrae (1995).

[34] Leibniz (1989), p. 187.

[35] Leibniz (1989), pp. 187–188.

Leibniz resolves this apparent tension—between the sensible source of our notions of mathematical objects and the intelligible source of our justification of mathematical truths—with recourse to the "natural light" of reason. The "natural light" of reason is solely responsible for our recognition of the necessary truth of the axioms of mathematics and for the force of demonstrations based on such axioms. Mathematical demonstration thus depends solely on "intelligible notions and truths, which alone are capable of allowing us to judge what is necessary."[36] For Leibniz, then, our pure mathematical knowledge is formal knowledge of the logic of mathematical relations, which are not directly dependent on any sensible data. This pure mathematical knowledge might be described as *verified* by our sensible experience, but ultimately our ideas of the mathematical features of sensible things conform to the mathematical necessities that we understand by the natural light:

> [Experience is] useful for verifying our reasonings as by a kind of proof. . . . But, to return to *necessary truths*, it is generally true that we know them only by this natural light, and not at all by the experiences of the senses.[37]

According to Leibniz, we have intelligible knowledge by the "natural light" of "what must be" and "what cannot be otherwise"—that is to say, of necessary (mathematical) truths such as the law of noncontradiction and the axioms of geometry. We can apply these laws to particular sensible qualities with our "common sense," which represents for us particular ideas of magnitude and multitude, thereby instancing universal mathematical truths. But, on this picture, we can discover necessary truths about many more things than we can possibly imagine, in Leibniz's sense: there are mathematical objects about which we can deduce mathematical truths but which our "common sense" cannot access. For example, even though "one finds ordinarily that two lines that continually approach finally intersect," the geometer nevertheless makes use of asymptotes, "extraordinary lines . . . that when extended to infinity approach continually and yet never intersect."[38] Likewise, the arithmetician operates with negative magnitudes, the analyst with infinitesimally small magnitudes, the algebraist with imaginary roots of magnitudes. Leibniz calls such notions "useful fictions," that is, notions which cannot be found in nature and which may seem even to contradict our sensible experience, but which nevertheless have obvious mathematical utility. He is in a unique position to account for them, despite the general inadequacies of modern symbolic systems to do so, because of another aspect of his approach to mathematics: his formalism.

[36] Leibniz (1989), p. 189.
[37] Leibniz (1989), p. 189.
[38] Leibniz (1989), p. 191.

Leibniz's formalist attitude toward mathematical reasoning allows him to conceive the symbols of arithmetic, algebra, and analysis independently from the geometric figures they were originally devised to stand for, thus allowing those symbols to be manipulated formally with only "intermittent attention" paid to the figures that are their referent.[39] The use of "fictional" mathematical notions is thus warranted rather than deemed absurd, since, on Leibniz's view, mathematical formalisms are detachable from their domain of application. In this respect, Leibniz can construe the objects of mathematical reasoning more abstractly than could either Descartes or Newton, both of whom used formal symbolic manipulation only as an aid to fundamentally geometric reasoning. Since Leibniz, by contrast, identifies the systematization of logical reasoning as the primary aim of the mathematical sciences, and separates his claims to mathematical knowledge from his claims to metaphysical knowledge, he can disregard the suggestion that metaphysical absurdities arise out of the use of effective mathematical symbolisms, thus initiating a transition to a more abstract and formal conception of the mathematical sciences.[40]

How, then, does Leibniz's account of mathematical cognition fare in the face of our two demands? His account seems to satisfy the apriority demand, since the role of the understanding in a mathematical context is specifically to provide us with knowledge of the mathematical axioms and the laws of logic. On Leibniz's view, then, our claims to mathematical knowledge are a priori because they are all founded on and deduced from universal and necessary truths known to us by the "natural light." Moreover, according to Leibniz, these claims to mathematical knowledge find application in the natural world: the clear and distinct deliverances of the "common sense" demarcate the domain of mathematical applicability. That our "common sense" notions are not notions of a metaphysically fundamental monadic realm is of no bother to Leibniz's account of mathematical cognition: despite the fact that what is metaphysically fundamental is for Leibniz not mathematically describable, we nevertheless have substantive mathematical knowledge of relations among objects of the *phenomenal* natural world.

We are now in a position to understand and appreciate Kant's critique of the philosophies articulated by the "mathematician" and the "metaphysician" who preceded him, and to explicate his alternative account of mathematical cognition.

[39] Leibniz (1996), II.xxi.186.

[40] Leibniz was skeptical that algebra could fully systematize this more formal conception of mathematics, stating in the *New Essays* that "algebra falls far short of being the art of invention, since even it needs the assistance of a more general art" (1996, IV.xvii. 489). He implies in the same passage that the "general procedures" of his infinitesimal calculus are more promising, though he speaks elsewhere of an even more general mathematical art.

3. Kant's Response

A coherent philosophy of mathematics—including an account of prevailing mathematical practice and an articulation of the epistemological and metaphysical conditions on the success of such practice—is a vital component of Kant's critical project. In an important portion of the "Transcendental Aesthetic,"[41] Kant provides the general outlines of his philosophy of mathematics, and touts its virtues, by drawing a contrast between his view and those held by his predecessors. Kant's claims in this section are that the "metaphysician's" account of mathematical cognition has failed the apriority demand; that the "mathematician's" account has failed the applicability demand; and that his own view satisfies both.

Famously, Kant's own view includes the doctrine of Transcendental Idealism, according to which space and time are the pure forms of our sensible intuition, and the sources of synthetic a priori cognitions.[42] Mathematics, geometry in particular, provides a "splendid example" of such cognitions: according to Kant, geometry is the science of our pure cognition of space and its relations, and provides us with a priori knowledge of the spatial form of the objects of our possible experience, those sensible things about which we can make objectively valid judgments.[43] As we will see in more detail below, Kant claims that his own account fully satisfies both the apriority and the applicability demands. According to Kant, mathematical cognition is cognition of our own intuitive capacities, of our own pure intuitions. Since Kant argues that our intuition of space is prior to and independent of our experience of empirical spatial objects, geometric cognition is paradigmatically a priori; thus his account satisfies the apriority demand. But, on Kant's view, mathematical cognition is also cognition of the empirical objects that we represent as having spatiotemporal form, that is, of the objects of our possible experience. Inasmuch as we have a priori geometric cognition of space, we have a priori geometric cognition of the spatial *form* of real spatial objects. The domain of our a priori mathematical cognition extends beyond the pure intuition of space to the formal conditions under which we represent empirical objects as being in space, thus allowing a priori mathematical cognition to find application in the realm of real empirical objects. Importantly, for Kant our a priori mathematical cognition extends to, but not beyond, the bounds of possible experience: a priori

[41] Kant (1998), A39/B56–A41/B58.

[42] See Shabel (2004) for a discussion of how this particular claim relates to the general doctrine of Transcendental Idealism.

[43] For the sake of simplicity, and because of the direct connection between space and mathematical (geometric) cognition, I will concentrate here on Kant's views of space and geometry, and will not discuss his account of time or other mathematical disciplines. For discussion of Kant's theory of algebra, see Shabel (1998).

mathematical cognition applies to all and only the spatiotemporal objects of a possible experience.

Kant defends his own account of mathematical cognition with positive arguments in its favor (some of which we will examine below) as well as with his critique of competing accounts. Having proclaimed the transcendental ideality of space and time, and offered an associated account of the apriority and applicability of our mathematical cognition, he objects that "Those, however, who assert the absolute reality of space and time, whether they assume it to be subsisting or merely inhering, must themselves come into conflict with the principles of experience."[44] Kant here takes both of his opponents to defend the *absolute reality* of space; on this basis, he judges that both opponents fail to solve the problem of identifying the a priori principles of our experience of the natural world.[45] But the details of Kant's critique, and the contrasting ways in which his opponents fail where he succeeds, emerge only upon discussion of what distinguishes those who "assume [space] to be subsisting," the "mathematical investigators of nature," from those who "assume [space] to be ... merely inhering," the "metaphysicians of nature." We will begin by examining Kant's objections to the "metaphysicians," proceed to his objections to the "mathematicians," and, finally, rehearse his own view.

Kant claims that Leibniz, Wolff, and other "metaphysicians of nature" hold "space and time to be relations of appearances ... that are abstracted from experience though confusedly represented in this abstraction. ..."[46] On this view, according to Kant, our notions of space and time, the alleged source of our mathematical cognition, are "only creatures of the imagination, the origin of which must really be sought in experience." The representations that lie at the basis of mathematical reasoning and cognition are abstracted by the imagination from or out of our sensible contact with appearances, and thus have an empirical origin. On this view, space and time are "only inhering" because they are relations among objects and not self-subsisting entities; our representations of space and time are constructed by abstraction from spatiotemporal relata. Note that from Kant's perspective, space and time are on this relationist view nevertheless "absolutely real": the objects or appearances that stand in such spatiotemporal relations are conceived to be the source of our notions of space and time and, thus, to be "absolutely real," that is, independent of what Kant takes to be transcendental conditions on experience. It follows from the metaphysicians' view that the subject matter of mathematics is derived from our experience of the natural world, and is not a factor in our own construction of that experience.

[44] Kant (1998), A39/B56.

[45] This should make clear the sense in which Kant sees his opponents' project from the perspective of his own. His criticism of their accounts of mathematical cognition is not wholly objective, since they would surely have formulated the "problem" quite differently.

[46] Kant (1998), A40/B57. Subsequent quotations are from this same passage.

Kant admits that on such an account, mathematical cognition does not exceed what he takes to be the limits of possible experience: if mathematical cognition derives directly *from* our engagement with the realm of appearances, then our application of mathematical cognition to all and only appearances seems guaranteed. This is a virtue of the metaphysicians' account and explains the sense in which Kant accepts it as satisfying the applicability demand. But the metaphysicians' account has, according to Kant, the fatal defect of not satisfying the apriority demand. As Kant puts it, the metaphysicians "must dispute the validity or at least the apodictic certainty of a priori mathematical doctrines in regard to real things (e.g., in space), since this certainty does not occur *a posteriori*...." Here Kant claims that because the original source of mathematical cognition is, on the metaphysicians' view, experiential, our geometric cognition of the spatial features of empirical objects must be a posteriori. This might seem not to be a problem for the metaphysician; he could reply to Kant's criticism by reiterating his account of our a priori knowledge of the laws and axioms of formal mathematics, available to us via the understanding. Thus the metaphysician would claim to satisfy the apriority demand by having given an account according to which some mathematical cognition is a priori (e.g., knowledge of laws, axioms, and theorems derivable therefrom) while some is a posteriori (e.g., knowledge of the mathematical features of real objects to which those theorems might be thought to apply).

But this sort of response misses the real thrust of Kant's criticism. Kant claims that the metaphysicians "can neither offer any ground for the possibility of a priori mathematical cognitions ... nor can they bring the propositions of experience into necessary accord with those assertions." According to Kant, the metaphysicians' account serves to detach the two primary features of mathematical cognition—apriority and applicability—in such a way that there is no explanatory or meaningful harmony between the universal and a priori mathematical laws known by the understanding and the substantive but a posteriori mathematical cognition of the natural world acquired via the "common sense." Kant's charge is that formal a priori mathematical cognition as the metaphysician understands it is altogether isolated from the domain of objects taken to be mathematically and scientifically describable; universal and a priori mathematical truths can *only* be about "useful fictions" and not "real things." If our account of a priori mathematical cognition does not give us a priori knowledge of the objects of our possible experience, then our account has failed the apriority demand, at least as Kant conceives it.

Proceeding to the "mathematicians," Kant claims that "they must assume two eternal and infinite self-subsisting non-entities (space and time) which exist (yet without there being anything real) only in order to comprehend everything real within themselves." Kant here describes Newton's absolutist view that space is a container existing independently of the real spatial objects it contains, though is not itself a real empirical entity. Kant's assessment of this view is brief; he writes that "[The mathematicians] succeed in opening the field of appearances for mathematical assertions. However, they themselves become very confused through precisely these conditions if the understanding would go beyond this field." In the first sentence, Kant makes clear that he understands the strength of the mathematicians' account to be its defense of our ability to achieve a priori insight into the mathematical features of the empirical natural world. Whereas the metaphysician could only explain a posteriori knowledge of such features, the mathematician proffers apriority via our perfect understanding of extension. Thus, on the mathematicians' account we have a priori mathematical knowledge of the field of appearances, of *all* of the objects that are contained within the domain of mathematical applicability. In the second sentence, however, Kant identifies the defect in the mathematicians' account: on this view, our a priori mathematical knowledge is applicable to all—*but not only*—appearances. It attempts to extend our a priori mathematical knowledge beyond the domain of appearances without explanation or justification of that extension. That is, in positing our special knowledge of an absolutely real and self-subsisting space or extension, the mathematician makes a claim to valid mathematical knowledge of entities that cannot themselves be described as within the realm of our possible experience. Given the mathematicians' absolutist conception of space, our achieving a priori knowledge of the mathematical features of the spatiotemporal natural world (e.g., knowledge of the geometry of spatial objects) requires that we also achieve a priori knowledge of the features of the *supra*natural world (e.g., knowledge of space itself, conceived independently from both our understanding and the objects it is thought to contain). Kant considers this to be a kind of "confusion": the "mathematician" achieves apriority only by extending the domain of applicability beyond the bounds of our possible experience. For Kant, the cost of apriority cannot (and need not) be that high. Thus, Kant takes the mathematicians' account to fail the applicability demand in a special sense: on the mathematicians' account, a priori mathematical cognition is applicable, but it is applicable beyond acceptable limits, that is, beyond the limits of our possible experience and knowledge of nature.

Kant thus charges that both of his opponents "come into conflict with the principles of experience," albeit in different ways. The metaphysician conflicts with the alleged apriority of those principles as applied to experience; the mathematician with the limits of their domain of applicability. While Kant's predecessors each satisfied only one of the two demands on a successful account of mathematical cognition, Kant considers his own theory to satisfy both the apriority and the applicability demands without conflicting with the principles of experience. As noted above, his theory hinges on his claim that space and time are pure forms of sensible intuition and sources of synthetic a priori cognitions. In the case of space, our pure intuition of space is the source of our claims to geometric knowledge: we have an a priori representation of space that is the ground for geometric reasoning and cognition. Kant's defense of this notorious claim is complex and beyond the scope of this chapter;[47] it will suffice for our purposes to consider the sense in which Kant's theory of space provides him with an account of mathematical cognition that satisfies both the apriority and the applicability demands.

Kant's argument runs roughly as follows. Because space and time are forms of sensibility and cognitive sources of a priori mathematical principles, space and time "determine their own boundaries" and "apply to objects only so far as they are considered as appearances.... Those alone are the field of their validity, beyond which no further objective use of them takes place." Kant is making a claim about the connection between our way of intuiting, representing, and knowing the structure of space and our way of intuiting, representing, and knowing the features of the objects we experience to be in space: the former *determines* the latter. For this reason, Kant claims that our a priori representation of space determines its own domain of applicability, its "field of validity." Insofar as our a priori mathematical cognition has its source in our *sensible* faculty, including our a priori representation of space, such cognition can be objectively valid with respect to all and only those objects we can *sense*, the realm of appearances or objects of our "possible experience." Thus, for Kant, mathematics is a body of synthetic a priori cognition providing us with knowledge of *both* (1) the pure conditions on our sensible representations *and*, by this means, (2) the objects that appear to us under those conditions. As Kant says, geometry (which is available to us via our a priori representation of space) provides the paradigm example. In the first sense, geometry is the science of a cognitive capacity, in particular, a capacity to represent spatial relations; geometry is the unique science that describes the various ways in which that capacity is both warranted and constrained. In the second sense, as explained in Kant's transcendental philosophy, geometry is the science of the natural or empirical objects that we conceive to stand in such spatial relations as we are conditioned to represent. Thus, geometry can inform us as to the proper function of our cognitive capacity for pure spatial intuition while also providing us with knowledge of the spatial form of those objects we intuit as empirically real spatial things.

Because, for Kant, mathematics provides us with knowledge of the pure conditions on our sensible representations, mathematical cognition is a priori. Being knowledge of the cognitive conditions on our *having* sensible experience, mathematics is cognition that is necessarily acquired prior to and independent from such experience. Because, for Kant, mathematics provides us with knowledge of the objects that appear to us under such cognitive conditions, mathematical cognition is applicable to the natural empirical world. Being knowledge of the formal features of sensible spatiotemporal objects, mathematics is cognition that is necessarily about the things that inhabit the natural world, at least insofar

[47] I take these claims to be defended in the "Metaphysical Exposition" and the "Transcendental Exposition of the Concept of Space" see Shabel (2004).

as we represent them. If we are willing to accept this last caveat—that the natural world comprises all and only those things that we have the cognitive capacity to represent—then we can secure the "certainty of experiential cognition": our a priori mathematical claims have direct and complete purchase on (our experience of) the natural world. Kant thus claims to have provided an account of mathematical cognition that satisfies both the apriority and the applicability demands with which we began—and that, moreover, does not *conflict* with, but rather helps to establish, the "principles of experience."

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