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3 Kant's philosophy of mathematics

In his *Critique of Pure Reason*, Kant proposes to investigate the sources and boundaries of pure reason by, in particular, uncovering the ground of the possibility of synthetic *a priori* judgments: "The real problem of pure reason is now contained in the question: **How are synthetic judgments *a priori* possible**?" (*Pure Reason*, B 19). In the course of answering this guiding question, Kant defends the claim that all properly mathematical judgments are synthetic *a priori*, the central thesis of his account of mathematical cognition, and provides an explanation for the possibility of such mathematical judgments.

In what follows I aim to explicate Kant's account of mathematical cognition, which will require taking up two distinct issues. First, in sections 2 and 3, I will articulate Kant's philosophy of mathematics. That is, I will identify the conception of mathematical reasoning and practice that provides Kant with evidence for his claim that all mathematical judgments are synthetic *a priori*, and I will examine in detail the philosophical arguments he gives in support of this claim. Second, in section 4, I will explain the role that Kant's philosophy of mathematics – and, in particular, his claim that mathematical judgments are synthetic *a priori* – plays in his critical (transcendental) philosophy. That is, I will identify the way in which Kant's philosophy of mathematics informs his arguments for transcendental idealism, and thus serves his larger philosophical goals.

It will be helpful to begin in section 1 with some historical background.

1. BACKGROUND

Despite his commitment to the importance of the synthetic *a priori* to both mathematical and metaphysical inquiry, Kant conceives the mathematical method to be distinct and different from the philosophical. In holding this view (which I will discuss in section 2 and 4) Kant departs from the received wisdom of his predecessors and contemporaries, many of whom were actively engaged in considering the question of the relation between mathematical and philosophical demonstration. In particular, Leibniz, Wolff, and Mendelssohn all contributed to the debate on this question and their views, taken together, constitute a rationalist philosophy of mathematics that dominated in the period prior to and contemporary with Kant. Because Kant conceives his own view to displace that of the "dogmatic metaphysicians," theirs must briefly be considered before we turn to Kant.

Christian Wolff argues that because both mathematics and philosophy seek certitude, their ideal methods are identical: "both philosophy and mathematics derive their methods from true logic."[1] The method so derived depends upon the accurate determination of the subject and predicate of demonstrable propositions, which are "rigorously demonstrated from previously established definitions and propositions" in a proper order.[2] Certitude is the result of following such a method: one is guaranteed that a mathematical proposition demonstrated in this manner can be known with certainty, in part because our access to mathematical concepts is via clear and distinct ideas. Wolff thus articulates a philosophy of mathematics according to which the rigorous logical analysis of mathematical concepts and propositions is sufficient to account for mathematical truth. Moreover, philosophical inquiry is to be modeled on the prototype of mathematical analysis.

Wolff holds this view with respect to all mathematical inquiry, including geometry, despite his use of diagrams to support geometric proof in his own mathematical work. That is, Wolff takes every step of a mathematical demonstration to rest on conceptual analysis and syllogistic inference, and thus conceives of diagrammatic evidence as reducible to logical evidence.[3] Wolff here follows Leibniz, who conceives every mathematical proposition to express an identity,[4] every step in the demonstration of which depends on the Law of Non-Contradiction. For Leibniz, even the propositions of geometry rest on the general principles of logic, and not on the singular evidence provided by geometric diagrams:

> You must understand that geometers do not derive their proofs from diagrams, although the expository approach makes it seem so. The cogency of the demonstration is independent of the diagram, whose only role is to make it easier to understand what is meant and to fix one's attention. It is universal propositions, i.e. definitions and axioms and theorems which have already been demonstrated, that make up the reasoning, and they would sustain it even if there were no diagram.[5]

The view evinced here makes clear why Leibniz's philosophy of mathematics, as well as that of his follower Wolff, might aptly be called a formalist and logicist account of mathematical reasoning.

Moses Mendelssohn, a contemporary of Kant's, follows in the Leibniz-Wolffian tradition and presents perhaps the clearest statement of the rationalist philosophy of mathematics in his so-called "prize essay."[6] Despite the fact that his account of mathematical reasoning is more subtle than Wolff's, and also that his acceptance of the substantive use of diagrams, or signs *in concreto*, in a mathematical context is a departure from Leibniz, nevertheless Mendelssohn is committed to the rationalist tenet that mathematical truth and metaphysical truth are equally certain due to their common method of reasoning, namely, conceptual analysis. The evidence for mathematical truth is obtained by "unpacking" and thereby making distinct the content of our mathematical concepts. Once mathematical concepts are sufficiently "unpacked," their contents can be compared, causing underlying identities to surface:

> The certainty of mathematics is based upon the general axiom that nothing can be and not be at the same time. In this science each proposition such as, for example, "A is B," is proven in one of two ways. Either one unpacks the concepts of A and shows "A is B," or one unpacks the concepts of B and infers from this that not-B must also be not-A. Both types of proof are thus based upon the principle of contradiction, and since the object of mathematics in general is magnitude and that of geometry in particular extension, one can say that in mathematics in general our concepts of magnitude are unpacked and analyzed, while in geometry in particular our concepts of extension are unpacked and analyzed.[7]

About the "unpacking" process Mendelssohn claims that the mathematician examines the "real and essential signs" of given concepts to reveal the order of our thoughts and the necessary connection between the subject and predicate of a mathematical proposition. The technique of conceptual analysis that one employs to accomplish this process is

> for the understanding nothing more than what the magnifying glass is for sight.... [The analysis of concepts] makes the parts and members of these concepts, which were previously obscure and unnoticed, distinct and recognizable, but it does not introduce anything into the concepts that was not already to be found in them.[8]

He claims further that this process is similar to that famously described by Plato in the *Meno*, without the "mystical aspect." So, for Mendelssohn, the natural unfolding of concepts in the human soul is the source of our ability to achieve mathematical certainty.[9]

I will show that Kant is concerned with the same issues about the mathematical method and the certainty of mathematical propositions as are his predecessors. But he is concerned to show, contrary to the views of his predecessors, that the method that yields mathematical certainty is unique and cannot be assimilated to the conceptual analysis that occupies philosophy. In the course of so distinguishing the mathematical from the philosophical method, Kant articulates a coherent and compelling philosophy of mathematics that engages with the mathematical practice of his time and that moreover serves his own metaphysical and epistemological purposes in a variety of ways. It is to Kant's view that I now turn.

2. THE SYNTHETICITY OF MATHEMATICS

Kant agrees with his rationalist predecessors that mathematical propositions are expressed as judgments that relate a subject concept to a predicate concept. For instance, Proposition I.32 in Euclid's *Elements* says that the three interior angles of any triangle are equal to two right angles.[10] In this case, the concept of being equal to two right angles is predicated of the subject concept, the interior angle sum of any triangle. But Kant disagrees that such propositions can be understood by virtue of conceptual analyses of the subject and predicate concepts. That is, Kant rejects the idea that mathematical judgments are *analytic*.[11] For Kant, mathematical propositions involve conceptual *syntheses*: a predicate concept not already contained in the subject concept is shown to "belong to" the subject concept nonetheless, thus issuing in a true mathematical judgment. In order to defend this view, Kant must provide a complete account of such mathematical syntheses by identifying the cognitive grounds for nonanalytic mathematical judgments.

Central to this account is Kant's claim that the mathematical method is distinguished from the philosophical by virtue of its dependence on the *construction* – and not the analysis – of concepts: "Philosophical cognition is rational cognition from concepts, mathematical cognition that from the construction of concepts" (A 713/B 741). To understand this claim, and Kant's thesis that mathematical concepts and propositions are constructible, we must first understand Kant's taxonomy for pure concepts:

> Now an *a priori* concept (a non-empirical concept) either already contains a pure intuition in itself, in which case it can be constructed; or else it contains nothing but the synthesis of possible intuitions, which are not given *a priori*, in which case one can well judge synthetically and *a priori* by its means but only discursively, in accordance with concepts, and never intuitively through the construction of the concept. (A 719–20/B 747–8)

Here Kant conceives an exhaustive division between those pure concepts that contain pure intuitions in themselves and are thereby constructible, and those that are not. What this comes to becomes clear given what he says next, namely, that the pure sensible concepts that provide the form of appearances are constructible: "space and time, and a concept of these, as *quanta*, can be exhibited *a priori* in pure intuition, i.e., constructed, together with either its quality (its shape) or else merely its quantity (the mere synthesis of the homogeneous manifold) through number" (A 720/B 748). Because mathematical concepts are derived from the combination of the categories of quantity with space and time, "the *modis* of sensibility" (A 82/B 108), mathematical concepts are precisely those concepts that Kant conceives to be constructible. The constructibility of mathematical concepts, and the nonconstructibility of the categories, thus provides the basis for Kant's distinction between mathematical and philosophical cognition.

Now we must consider more precisely what it means to construct a mathematical concept. Early in the Preface to the second edition of the *Critique*, Kant characterizes mathematics as having found the "royal path" to the secure course of a science as the result of an ancient geometer's realization: the key to mathematical demonstration is the mathematician's ability to produce figures via construction according to *a priori* concepts (B xii). Later, in the Discipline of Pure Reason, he says that "to construct a concept means to exhibit *a priori* the intuition corresponding to it" (A 713/B 741) and, further, that only mathematics has the means to so construct, and thereby define, its concepts (A 729/B 757). Taking these comments together suggests that Kant conceives mathematics to have a unique ability to define its concepts by constructing them, which amounts to exhibiting their content in the form of a singular representation, or intuition. In producing a figure in intuition, the mathematician defines a mathematical concept by constructing an individual figure to correspond to that concept.

For example, to attempt to define the concept *triangle* one considers the possibility of constructing a three-sided rectilinear figure. Kant thinks of this concept as "arbitrary" in the following sense: in considering such a concept, one knows precisely what its content is since one "deliberately made it up," and, moreover, the concept was not "given through the nature of the understanding or through experience" (A 729/B 757). Mathematical concepts thus contain an "arbitrary synthesis": in the case of a triangle, one considers the concept *figure* (that which is contained by any boundary or boundaries) together with the concepts *straight line* and *three*, and then proceeds to effect the synthesis of these concepts by exhibiting an object[12] corresponding to this new concept, namely, by constructing a triangular figure, either in imagination or by rendering a drawn diagram. In either case, the triangle so constructed and exhibited is presented intuitively, that is, as a singular and immediate mental representation. Mathematical concepts are thus given through synthetic definitions, which prescribe a rule or pattern for constructing a corresponding intuition.[13] Geometric concepts in particular provide us with the rule or pattern for constructing sensible intuitions of the spatial magnitudes of objects of outer sense.

Even in the case of nongeometric concepts, such as the numeric concept *five*, one must still "**make** an abstract concept **sensible**," that is, "display the object that corresponds to it in intuition, since without this the concept would remain (as one says) without **sense**, i.e., without significance" (A 240/B 299). Numeric concepts provide us with the rule or pattern for constructing sensible intuitions of the magnitudes of objects in general, that is, of the quantitative measures of objects of both inner and outer sense. These "seek their standing and sense" in "the fingers, in the beads of an abacus, or in strokes and points that are placed before the eyes" (A 240/B 299). Thus, the concept *five* can be constructed by representing five discrete units in the following way: |||||.

Both geometric and arithmetic concepts are exhibited via "ostensive" constructions, which show or display the content of the concepts to which they correspond. The geometer's triangular figure and the arithmetician's five strokes serve to make manifest the sensible content of the concepts *triangle* and *five*, respectively, and to connect abstract mathematical concepts to the sensible intuitions of space and time. The arithmetician's strokes differ from the geometer's figure in that the former use spatial distinctness not to represent qualitative spatial magnitudes, such as shapes, but only to represent discrete quantitative units. Thus the stroke, despite being ostensive, is nevertheless a more abstract mathematical tool than might appear from its sensible rendering.[14] Arithmetic construction represents features of our temporal intuition by displaying number as the result of a (temporal) counting process, but this process includes the use of spatial intuition: the construction of a numeric magnitude as a temporal sequence requires the use of spatial intuition to exhibit discrete and countable objects. Likewise, geometric construction represents features of our spatial intuition by displaying shapes as the result of a (spatial) drawing or mapping process, but this process includes the use of temporal intuition: the construction of a geometric magnitude as a spatial figure requires the use of temporal intuition to exhibit continuous and extended objects.[15]

Kant contrasts such ostensive constructions with the "symbolic" constructions of algebra:

Mathematics does not merely construct magnitudes (*quanta*) as in geometry, but also mere magnitude (*quantitatem*), as in algebra, where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude.... [Algebra] thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves), which discursive cognition could never achieve by means of mere concepts. (A 717/B 745)

Kant does not mean to draw a strict distinction here between two *kinds* of mathematical construction. For Kant, the algebraist's "symbolic" construction is that which symbolizes an ostensive construction, as described above. So, an algebraic symbol, such as the variable "x," can be used to represent a concretely constructible entity, such as a line segment. In such a case, the variable symbolically constructs the concrete object by symbolizing the ostensive construction of that object. Thus, Kant does not use "symbolic construction" to designate a category of mathematical constructions that are constructed out of symbols, and thereby distinct from ostensive constructions. He rather uses "symbolic construction" to designate that which symbolizes ostensive constructions. If we fail to regard symbolic construction as a species of ostensive construction, it is difficult to see how a symbolic construction of, say, an algebraic variable could be the exhibition of an intuition in Kant's sense, for the display of an algebraic variable does not in itself reveal anything about the spatio-temporal forms of objects.[16] Thus, the procedure and result of *all* mathematical construction is, for Kant, fundamentally ostensive: to construct a mathematical concept one necessarily exhibits an intuition that displays its features manifestly.[17]

According to Kant's account, then, one defines a mathematical concept by constructing it, that is, by exhibiting its content ostensively in intuition. One might suspect that such an ability to construct definitions for our mathematical concepts would render our mathematical propositions analytic: because the precise and determinate content of our mathematical concepts is available to us via construction, one might suppose that we can determine the truth of a mathematical proposition by analyzing the relation between perfectly well-defined subject and predicate concepts. Of course, Kant rejects this inference: Kant's theory of the constructibility of mathematical concepts is the basis for his claim to the contrary that mathematical propositions are *synthetic*, and is thus the ground for his rejection of his predecessors' views. I will now consider his arguments for the syntheticity thesis, and see how, in particular, these arguments depend on the constructibility thesis.

In the second-edition Introduction, Kant claims that "Mathematical judgments are all synthetic" (B 14).[18] He begins his defense of this claim by dismissing unnamed opponents, the "analysts of human reason," who argued that because the certainty of any particular mathematical inference is assured only if it proceeds in accordance with the principle of contradiction, the principles and propositions of mathematics could be "cognized from the principle of contradiction," and thus are analytic. Kant concedes the analysts' claim that mathematical propositions are deduced in accordance with the law of contradiction but denies that this shows that mathematical propositions are analytic. He proceeds to an argument in favor of his contrary view, namely, that mathematical propositions are synthetic.[19]

Kant argues for the general claim that mathematical propositions are synthetic in two cases, the arithmetic case and the geometric case.[20] This strategy reflects his understanding of the elementary mathematics of his day, which took mathematics to be the science of discrete and continuous magnitudes (number and extension, respectively). Arithmetic and geometry, the most basic mathematical sciences, are thus those to which Kant here directs his philosophical attention. Beginning with the arithmetic case, Kant asks us to consider the proposition "$7 + 5 = 12$" and argues first that the proposition is not analytic. He claims that in thinking the subject concept, the sum of 7 and 5, one thinks "nothing more than the unification of both numbers in a single one," but does not think what this single number is (B 15). That is, the concept of a sum of two numbers contains only the concepts of each of the two numbers, together with the concept of summing them, but does not contain the number that is their sum: "no matter how long I analyze my concept of such a possible sum I will still not find twelve in it" (B 15). Here Kant argues against the analyticity of all arithmetic propositions by arguing against the analyticity of a representative numerical formula. The basis for his argument is a challenge to the opponent: if one could provide an analysis of the concept of the sum of 7 and 5 that yields the concept of equal to 12, then one would have to grant that the proposition is analytic. But, no such analysis is possible.[21] Therefore, by definition of analyticity, the proposition is not analytic.

On its own this argument is clearly insufficient to defend Kant's claim that mathematical propositions are synthetic since he has not examined any candidate analyses of the relevant concepts but has merely declared such analyses to be impossible. He thus needs a positive argument in favor of syntheticity to support his denial of analyticity. The positive argument for the syntheticity of arithmetical propositions begins with a claim that depends on Kant's constructibility thesis, discussed earlier: to grasp the relation between the subject and predicate concepts of an arithmetic proposition, one must "go beyond" the subject concept to the intuition that corresponds to it and identify properties that are not analytically contained in the concept, yet still belong to it (B 15; A 718/B 746). Kant holds therefore that *construction* of the concept of the sum of 7 and 5 is necessary if we are seeking grounds for judging whether the proposition "$7 + 5 = 12$" is true or false. By constructing the concept of the sum, we are able to judge that the concept of the sum of 7 and 5 has the property of being equal to 12, even though that property is not analytically contained in the concept of the sum of 7 and 5:

> For I take first the number 7, and, as I take the fingers of my hand as an intuition for assistance with the concept of 5, to that image of mine I now add the units that I have previously taken together in order to constitute the number 5 one after another to the number 7, and thus see the number 12 arise. (B 16)

Whether one uses fingers, strokes, points, or the beads of an abacus to represent the content of the number concepts 7 and 5, one must put the intuition of 7 together with the intuition of 5 to represent their sum to determine that, taken together, they come to 12:

$$|||||||+|||||=||||||||||||$$

Importantly, Kant takes the content of the concept 12 to "arise" from this intuitive computation: the construction and summing of the concepts comprising the subject concept is a process that generates the properties of that very concept, expressed in the predicate concept. Kant concludes from this that "The arithmetical proposition is therefore always synthetic" (B 16) since it is a judgment whose predicate concept is not "covertly contained" in its subject concept but rather "lies entirely outside" the subject concept while still standing "in connection with it" (A 6/B 10). In proceeding "outside" of the subject concept to discover the properties that are connected with it, we are constrained by the general conditions of sensible intuition: it is due to features of our sensible faculty and its original *a priori* representations of space and time that the sum should be determined in exactly the way that it is.[22]

Before proceeding to the geometric case, Kant adds one final consideration to support his argument for the syntheticity of arithmetical propositions. He claims that one is made more "distinctly aware" of the syntheticity of arithmetical propositions upon considering large number calculations, "for it is then clear that, twist and turn our concepts as we will, without getting help from our intuition we could never find the sum by means of the mere analysis of our concepts" (B 16). Kant cannot mean to suggest that we actually mark off strokes or points in intuition in order to calculate the sum of, say, 7,007 and 5,005 and determine the truth of the proposition "7,007 + 5,005 = 12,012." His idea is rather that such strokes or points are, ultimately, the justification for the truth of the large number proposition because our methods for performing large number calculations depend on our methods for performing small number calculations; that is, they depend on the use of intuition to display or exhibit the content of our small number concepts. Kant's point is that, even if we have a shortcut (perhaps symbolic) method for performing large number arithmetic calculations, the relations among large number concepts must be justified on intuitive, and thus synthetic, grounds.[23]

Kant's next move is to argue for syntheticity in the geometric case. In the second-edition Introduction, as noted, he considers geometric principles, or axioms, as examples of synthetic propositions. Elsewhere, he argues in favor of the syntheticity of geometric theorems. Taken together with his arguments in favor of the syntheticity of arithmetic just discussed, these arguments complete his defense of the syntheticity of mathematics.

Beginning with the geometric principles, Kant considers as an example the proposition that the straight line is the shortest line between two points (B 16).[24] He says that because the "concept of the straight contains nothing of quantity, but only a quality," so "the concept of the shortest is therefore entirely additional to it, and cannot be extracted out of the concept of the straight line by any analysis" (B 16). In other words, the concept of the straight line between two points A and B does not analytically contain the concept of the shortest line between A and B, since it speaks only of the shape of the line between them and not the measure of the line between them; it follows that the straight line between A and B cannot be judged to be the shortest line between A and B merely by conceptual analysis. In order to judge the identity between the straight line and the shortest line between two points, one must synthesize the concept of the straight line with the concept of the shortest line by seeking "help" from intuition (B 16). Kant takes this particular judgment to be axiomatic because the synthesis between the concepts is immediately evident: upon constructing a straight line between two points, and thereby exhibiting the content of the subject concept of the proposition, one judges immediately, without any mediating inferences, that the straight line so constructed likewise exhibits the content of the predicate concept. Again, features of our sensible faculty determine this identity: on Kant's view, were we to connect A to B by constructing a line longer than the straight line between A and B, our line so constructed would either be curved or bent. Kant reiterates this point in the Axioms of Intuition where he writes that geometry and its axioms are "grounded" on the "successive synthesis of the productive imagination, in the generation of shapes" and "express the conditions of sensible intuition" (A 163/B 204).[25] The synthetic activity of shape construction is our means for displaying the features of our original spatial intuition. It thus makes evident the spatial forms that we are warranted and constrained to represent.[26]

Finally, Kant provides an argument in favor of the syntheticity of geometric theorems, an argument that makes especially clear the role that constructibility plays in his account of syntheticity and the understanding of mathematical proof. This argument occurs in the Discipline of Pure Reason, where Kant considers Euclid's proposition I.32[27] in the context of his comparison of the mathematical to the philosophical method.[28] Here Kant contrasts the fortunes of the philosopher and the mathematician when faced with the task of determining the relation between the sum of the interior angles of a triangle and a right angle. This contrast is meant to emphasize that the analytic tools of the philosopher are inadequate to the task, whereas the synthetic and constructive tools of the mathematician are adequate. To make this point, Kant notes first the weakness of the philosopher's position, who faces the task armed only with the technique of conceptual analysis and the concept of a figure that is both tri-lateral and tri-angular: "[The philosopher] can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts" (A 716/B 744). As before, Kant declares the impossibility of using conceptual analysis to extract from a mathematical concept any properties that are not given discursively in its definition, and as before, he must supplement this declaration with an argument in favor of the role of intuitive synthesis in the identification of such properties.

The geometer, by contrast to the philosopher, "begins at once to construct a triangle" (A 716/B 745). To construct the concept *triangle*, the geometer displays an intuitively accessible three-sided rectilinear figure: Δ. The proof can then be effected in several simple steps, as Kant describes:

> Since [the geometer] knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and *obtains* two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here *there arises* an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question. (A 716/B 744, *emphasis added*)

We will take up the question of what makes this solution general in section 3. Here I must emphasize Kant's insight that the construction of the triangle, and the auxiliary constructions of the lines and angles adjacent to the triangle, provide information to the geometer that was not contained within the concepts that compose the proposition to be proved, namely, that the three interior angles of a triangle are equal to two right angles. That this proposition *cannot* be deduced by analysis of the concepts of angle, triangle, etc., but *can* be deduced by construction of those same concepts, Kant gleans from the geometer's own practice. In particular, Kant observes that from the geometer's construction of two lines auxiliary to the original triangle there "arise," first, one new angle, exterior and adjacent to the triangle, and second, the two angles that are its parts. The geometer thus "obtains" these new angles as intuitive representations that are connected to but not contained in the concept of the original triangle. Moreover, the geometer's technique of displaying the intuition of the triangle and its adjacent angles makes available diagrammatic information that is indispensable for the ensuing demonstration. In particular, the diagram so constructed witnesses the part-whole relations among the triangle and its adjacent angles, and so testifies to the relevant spatial containments, namely, that the exterior adjacent angle is equivalent to the two opposite interior angles of the triangle and thus, together with the adjacent interior angle, sums to two right angles.

The key point here is Kant's recognition that the geometer's proof cannot succeed without information about the relations among the spatial regions delimited by the triangle and its external angles, information that is unavailable to the philosopher examining the bare concept of a triangle. Kant understood that the constructed figure in Euclid's proof is not a heuristic aid to understanding, but rather an essential and ineliminable component of the reasoning that leads the geometer from the interior angles of a triangle, "through a chain of inferences that is always guided by intuition," to the sum of two right angles. The syntheticity of the resulting theorem is due to the fact that the relation between its subject and predicate concepts (the three interior angles of a triangle and two right angles, respectively) is discovered via a deduction that depends on the construction of each concept and the additional intuitive information that each concept thereby reveals.[29]

The syntheticity of geometry and arithmetic is ultimately due to the fact that their propositions codify and describe the content of our original *a priori* representations of space and time, which are presented in intuition and not through mere concepts. The construction of mathematical concepts in intuition thus serves more generally to reveal or exhibit the sensible conditions that warrant and constrain mathematical judgment. Because Kant takes mathematics to be built upon the basic propositions of arithmetic and geometry, he takes his arguments for the syntheticity of the propositions of arithmetic and geometry to constitute an argument for the syntheticity of all mathematical propositions. Having rehearsed those arguments, I must now consider the second part of the central thesis of Kant's philosophy of mathematics, namely, that all mathematical cognition is *a priori*.

3. THE APRIORITY OF MATHEMATICS

Just prior to presenting his arguments for the syntheticity of mathematics in the second-edition Introduction, Kant offers a brief argument in support of what appears to be a background assumption, namely, that math is *a priori*:

Properly mathematical judgments are always *a priori* judgments and are never empirical, because they carry necessity with them, which cannot be derived from experience. But if one does not want to concede this, well then, I will restrict my proposition to pure mathematics, the concept of which already implies that it does not contain empirical but merely pure *a priori* cognition. (B 14–15)

Here Kant claims that the apparent necessity of mathematical propositions is sufficient evidence of their apriority; this follows from his equation, offered earlier, that "Necessity and strict universality are therefore secure indications of an *a priori* cognition, and also belong together inseparably" (B 4). But his subsequent arguments for the syntheticity of mathematical propositions make it difficult to accept such a terse defense of apriority since Kant's conception of the construction of mathematical concepts, on which the syntheticity arguments rest, suggests that mathematical reasoning depends on *singular* and *concrete* representations. This leads to the worry that mathematical constructions cannot possibly support reasoning that is fully general or universal, and further, that mathematical judgments justified with such reasoning are neither necessary nor *a priori*. For Kant to provide a coherent philosophy of mathematics and defend his central thesis that mathematical cognition is synthetic and *a priori*, he owes us an account of how the mathematician's constructive practices can provide evidence for and support arguments that lead to fully general and universal mathematical propositions.

Kant's main argument in support of the claim that mathematical propositions are fully general and universal is that the concept constructions on which they rest, despite producing singular and concrete intuitions, are *themselves* fully general and universal processes resulting in fully general and universal representations. It will follow that mathematical propositions relating such constructible concepts are fully general and universal. The question is: What makes concept construction a fully general and universal process resulting in fully general and universal representations?

An important passage that is relevant to Kant's answer to this question is worth quoting in full:

For the construction of a concept, therefore, a **non-empirical** intuition is required, which consequently, as intuition, is an individual object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept. Thus I construct a triangle by exhibiting an object corresponding to this concept, either through mere imagination, in pure intuition, or on paper, in empirical intuition, but in both cases completely *a priori*, without having had to borrow the pattern for it from any experience. The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle. (A 713–4/B 741–2)

Here Kant addresses how a single intuition can represent all possible intuitions that fall under the same concept – for example, how a particular triangular figure can serve to represent all triangles and so be thought to construct the general concept of triangle. Kant's first point is that whether the triangle is constructed in empirical intuition, by sketching it on paper or with a stick in the sand, or in pure intuition using only the imagination, the triangle so constructed is exhibited *a priori*. This is because its pattern is not borrowed from experience. That is, the shapes we construct in a mathematical context are not abstracted from our sensible impressions of shaped objects, such as plates or tables. Rather, on Kant's view, our empirical intuitions of shaped objects borrow *their* patterns from our pure intuitions of shapes in space.[30] So, an empirical intuition of a triangle can function in a mathematical context because it itself relies on a prior ability to construct shapes *a priori* with the productive imagination, and thus on a pure intuition of space. Constructed figures thus need not (but can) be rendered empirically to serve their indispensable role in mathematical reasoning: if such figures *are* rendered empirically, the apriority of the reasoning they support is not surrendered.

There remains the question how an intuition, pure *or* empirical, can represent the general content of a concept. This is addressed in the second point Kant makes in the passage above, where he distinguishes the act of construction from the constructed object: in constructing the intuition that corresponds to a mathematical concept, we attend not to the particular features of the resultant figure, but to the act that produced it. So, in constructing the concept *triangle* one might produce a scalene or an equilateral figure; either way, one has produced a representation of all possible triangles by producing a single paradigm triangle. That one figure has unequal and another equal sides is irrelevant: one abstracts from the particular magnitudes of the sides and angles in order to recognize the relevant feature of the figure, namely, three-sidedness. And this recognition is effected by taking account of the act of constructing a three-sided rectilinear figure "to which many determinations . . . are entirely indifferent."

As Kant sees it, then, the act of construction is the ultimate source of the full generality and universality of intuitive mathematical representations. In the next passage, he points us to the Schematism for an explanation of the universality of the act of construction:

> Mathematical cognition considers the universal in the particular, indeed even in the individual, yet nonetheless *a priori* and by means of reason, so that just as this individual is determined under certain general conditions of construction, the object of the concept, to which this individual corresponds only as its schema, must likewise be thought as universally determined. (A 714/B 742)

And later:

> By means of [geometrical construction] I put together in a pure intuition, just as in an empirical one, the manifold that belongs to the schema of a triangle in general and thus to its concept, through which general synthetic propositions must be constructed. (A 718/B 746)

In these passages, Kant suggests that in constructing the concept *triangle* we produce an individual triangle that, because it is determined under certain general conditions, provides the pattern for triangles in general, and thus provides the pattern for *all* triangular objects of sensation. The "general conditions of construction" that determine the features of our pure intuition of a triangle include the general features of our pure intuition (the "infinite given magnitudes" of space and time described in the Metaphysical Exposition) together with the general features of our pure sensible concepts, in this case triangularity, as given by the definition of triangle.[31] That mathematical concepts can be synthetically defined and constructed makes it possible for us to have direct cognitive access to such general mathematical patterns; to construct a triangle by acting in accordance with general conditions of construction while ignoring the particular determinations of the constructed figure is to cognize the general pattern for *any* three-sided rectilinear figure.

Kant fills out this picture in the Schematism, where he argues that schemata are the mediating representations that are needed to link pure concepts to appearances.[32] In the case of the concept *triangle*, the schema – a product of pure imagination – is the determinate procedure for constructing a three-sided rectilinear figure, a procedure which must itself be consistent with universal spatiotemporal conditions of construction. Because Kant construes mathematical concepts like *triangle* to *contain* such determinate construction procedures, mathematical concepts like *triangle* provide us with rules for representing the objects that instantiate them. Thus, there is no heterogeneity between mathematical concepts and the intuitions that directly correspond to them via construction: the pure mathematical concept *triangle* is homogeneous with all pure and empirical intuitions of triangles, and so with all triangular objects of experience, since the concept *triangle* provides us with the rule for representing *any* three-sided rectilinear object.[33] In the case of mathematical concepts, then, schemata are strictly redundant: no "third thing" is needed to mediate between a mathematical concept and the objects that instantiate it since mathematical concepts come equipped with determinate conditions on and procedures for their construction.[34]

Mathematical schemata thus have the generality of a concept (since they represent the general content of a mathematical concept) but the particularity of an intuition (since they issue in a concrete display of that content). Kant denies, however, that such schemata are *images*:

> In fact it is not images of objects but schemata that ground our pure sensible concepts. No image of a triangle would ever be adequate to the concept of it. For it would not attain the generality of the concept, which makes this valid for all triangles, right or acute, etc., but would always be limited to one part of this sphere. The schema of the triangle can never exist anywhere except in thought, and signifies a rule of the synthesis of the imagination with regard to pure shapes in space. (B 180)

Ultimately, then, the generality and universal applicability of a concept like *triangle* is due not to the individual triangle that is produced in constructing that concept, but to the awareness that the production of such a concrete representation of the general concept *triangle* depends on a general "rule of synthesis" for the production of any such figure, that is, its schema. This rule of synthesis has its source in our own cognitive faculties and explains why Kant holds that in performing acts of mathematical construction, we must take account of the act itself, noting what we contribute to the constructed figure:

> [The geometer] found that what he had to do was . . . to produce the [properties of the figure] from what he himself thought into the object and presented (through construction) according to *a priori* concepts, and that in order to know something securely *a priori* he had to ascribe to the thing nothing except what followed necessarily from what he himself had put into it in accordance with its concept. (B xii)

To "take account only of the action of constructing" a shape, and thereby to ascribe features to the shape not only as it is given in its general *a priori* concept but also as it is determined by more general features of the spatial mode of construction, is to display the *general* content of a mathematical concept in a particular concrete entity. The general cognitive conditions that govern mathematical thought and ground mathematical reasoning are thus accessible via the performance of mental acts that produce singular and concrete representations.

The generality of mathematical construction becomes a bit more clear, perhaps, in the arithmetical case. A number concept can be constructed ostensively with strokes or points, but what allows any number concept so constructed to represent that number universally, in abstraction from some particular set of numbered things, is the mental act we perform in exhibiting the strokes or points. The schema of any number concept includes the representation of a general counting procedure: "if I only think a number in general, which could be five or a hundred, this thinking is more the representation of a method for representing a multitude (e.g., a thousand) in accordance with a certain concept than the image itself" (A 140/B 179). Moreover, such a procedure requires the generation of "time itself" (A 142/B 182) and so provides insight into the general temporal conditions that govern numeric operations.

As Kant sees it, the generality of mathematical representations is due to the fact that both mathematical concepts and the pure intuitions that correspond to those concepts depend on universal and necessary features of our pure cognitive faculties of sensibility, imagination, and understanding.[35] A mental act of mathematical construction must accord with a rule of synthesis prescribed by a pure concept of understanding, though under the constraints imposed by the pure intuitions of sensibility. For example, the act of constructing a triangle accords with the geometric concept of a triangle given in its definition by synthesizing (or "putting together the manifold" of) three straight lines in the space of pure intuition, of which the resulting triangle is merely a limitation. Thus, the act of constructing a singular and concrete triangle serves to exhibit the general features of any three-sided rectilinear figure, the spatial relations among the parts of any such figure, and the general features of the space in which it is constructed. These features and relations include, for example, that any three-sided rectilinear figure has also three interior angles, that the three sides of a triangle bound a region of space, that there is an inside and an outside of the region so bounded, and so on. Likewise, the act of constructing the concept *five* by representing "the successive addition of one (homogeneous) unit to another" (A 142/B 182) accords with both the general concept of magnitude and the pure intuition of time, as well as with the arithmetic concept *five*: the act of constructing the number five exhibits not only the features of five-unit quanta but also the general features of magnitude and the temporal conditions under which such magnitudes can be counted or measured.

On Kant's view, then, the act of construction in accordance with a rule transmits generality and universality to mathematical representations. It follows that the mathematical judgments that relate such general and universal mathematical representations themselves hold generally and apply universally. For the same reasons, mathematical judgments are known with apodictic certainty and so are *a priori*: because Kant takes the pure spatiotemporal intuition on which mathematical propositions are grounded to be both a "subjective condition regarding form" and a "universal *a priori* condition" of experience, mathematical propositions are necessarily true. Thus, on Kant's view, we can be apodictically certain that the synthetic propositions of pure mathematics are generally true of and apply universally to all spatiotemporal objects of experience.[36]

4. MATHEMATICS IN SERVICE OF THE CRITICAL PHILOSOPHY

As we have seen, Kant denies that mathematics and philosophy share a method; this is the thrust of his rejection of the views of his predecessors, who argued that both mathematics and philosophy proceed by the analysis of concepts. Kant has argued, on the contrary, that mathematics proceeds by the construction of concepts, which is a synthetic and not an analytic exercise. Despite this methodological difference, mathematics and its distinctive style of reasoning nevertheless play the role of a paradigm for Kant's philosophical investigations and are pivotal elements of his arguments for transcendental idealism. In this final section, I will discuss, albeit briefly, the role that the synthetic apriority of mathematics plays in Kant's critical philosophy.[37]

In the Transcendental Aesthetic Kant argues, first, that space and time are *a priori* intuitions. By this he means that we possess original nonconceptual representations of space and time that have their source in pure sensibility, that mental faculty that enables our cognitive receptivity of objects. He argues further that these representations provide the pure form for all sensible intuition, that is, that they provide us with a structure for cognizing empirical objects. The synthetic *a priori* propositions of mathematics, he claims, are "derived from" these *a priori* intuitions of space and time and so are grounded by our pure sensible faculty.[38] That geometry is the science of space thus means for Kant that geometry, and mathematics more generally, at the very least codifies and describes the original intuitive representations afforded by pure sensibility.

This conception of the relation between pure sensibility and the science of mathematics, which science (as we have seen) Kant takes to comprise a set of synthetic *a priori* propositions, provides Kant with an argument for transcendental idealism, according to which the pure intuitions of space and time and the pure concepts of the understanding apply to all – but only – appearances, and not to things as they are in themselves.[39] In the particular case of space, transcendental idealism amounts to the claim that space is itself nothing over and above the original sensible representation described and codified by geometry, that is, that space is not a property inhering in objects independent of our cognitive contact therewith. So, Kant takes the transcendental ideality of space to follow from the previously defended premises that space is a pure intuition and that mathematical cognition is synthetic *a priori*. His argument is a *reductio* of the supposition that space is *not* transcendentally ideal: suppose that "space and time are in themselves objective and conditions of the possibility of things in themselves"; then one cannot account for the "large number of *a priori* apodictic and synthetic propositions about [space]" (A 46/B 64). That is, the doctrine of transcendental realism contradicts the synthetic apriority of mathematics, which itself rests on the *a priori* intuitivity of space and time. Therefore, transcendental realism must be rejected in favor of transcendental idealism.[40]

Kant fills out the argument by sketching a possible account of the synthetic apriority of the propositions of mathematics on the assumption that space and time are transcendentally real. He first shows with an argument from elimination that realist and idealist alike must concede that mathematical cognition is attained via pure intuition. Mathematical cognition, like any cognition, is attained via either concepts or intuitions, both of which are either pure or empirical. But mathematics cannot be based on empirical concepts or empirical intuitions, for such representations "cannot yield any synthetic proposition except one that is also merely empirical" and so "can never contain necessity and absolute universality of the sort that is nevertheless characteristic of all propositions of geometry" (A 47/B 64). There remains the possibility that mathematics be based on pure concepts or intuitions. Pure concepts are ruled out on the grounds that "from mere concepts no synthetic cognition but only merely analytic cognition can be attained" (A 47/B 65). The mathematician must therefore "take refuge in intuition . . . give your object *a priori* in intuition, and ground your synthetic proposition on this" (A 47–8/B 65).[41]

Kant next introduces the consequences of pairing transcendental realism about space with this account of mathematics: given the *reductio* assumption that space is transcendentally real, it follows that the object represented in pure intuition for the purposes of geometric reasoning, such as a triangle, is "something in itself without relation to your subject," and moreover that the latter is "given prior to" the former, and not "through it" (A 48/B 65). If this is the case – if the triangle in itself is cognized independent of the triangle constructed in pure intuition for the purposes of mathematical reasoning – then what mathematics shows with necessity to lie in the subjective conditions for constructing a triangle cannot be shown with necessity to apply to the triangle in itself. That is, the mathematical method of construction required for mathematical proof cannot be brought to bear on the triangle in itself. Thus, on the assumption that space is transcendentally real, one can "make out absolutely nothing synthetic and *a priori* about outer objects" (A 48/B 66) since outer objects are taken to be objects in themselves.

But this result directly contradicts the premises of the argument, which take mathematics (geometry) to be the synthetic *a priori* science of space, and space to be an original representation of "outer sense." In short, transcendental realism entails that the science of space cannot yield synthetic *a priori* propositions about outer objects, but mathematics apparently provides us with just such propositions. So it is inconsistent to suppose both that mathematics is synthetic *a priori* cognition of space *and* that space is transcendentally real. Thus, the *reductio* assumption that space is transcendentally real is rejected in favor of the view that space is transcendentally ideal:

> It is therefore indubitably certain and not merely possible or even probable that space and time, as the necessary conditions of all (outer and inner) experience, are merely subjective conditions of all our intuition, in relation to which therefore all objects are mere appearances and not things given for themselves in this way; about these appearances, further, much may be said *a priori* that concerns their form but nothing whatsoever about the things in themselves that may ground them. (A 48–9/B 66)

Here Kant concludes that space and time are *merely* the forms of intuition, that is, that space and time are nothing over and above the way we represent them in pure intuition, and so are not properties of things as they are in themselves, as the transcendental realist had supposed.[42] He has thus used the synthetic apriority of mathematics to defend the broad doctrine of the transcendental ideality of space and time.[43]

Kant's account of mathematics and its relation to the pure intuitions of space and time plays an equally important role in a variety of arguments that occur after the Transcendental Aesthetic. In the first section of the Deduction, Kant states that all pure concepts require a transcendental deduction, that is, an explanation of the way in which such concepts relate to objects *a priori* (A 85/B 117). In the case of the pure cognitions of space and time, however, it is enough to have shown in the Transcendental Aesthetic that these are pure intuitions that "contain *a priori* the conditions of the possibility of objects as appearances," which exercise has already served to explain and determine their *a priori* objective validity (A 89/B 121). Because space and time are forms of sensibility, they "determine their own boundaries" and "apply to objects only so far as they are considered as appearances, but do not present things in themselves. Those alone are the field of their validity, beyond which no further objective use of them takes place" (A 39/B 56). Space, time, and the mathematical cognition that is grounded thereon, are thus guaranteed to apply to all and only objects of experience, or appearances, for these are first given through the pure forms of sensibility. In other words, the mathematical propositions that derive from the pure forms of sensibility are necessarily applicable to all and only those objects that appear to us by means of space and time.

Space, time, and the mathematical concepts thereof thus provide a point of contrast as well as a sort of paradigm for the deduction of the pure concepts of understanding, or categories. Because mathematical concepts "speak of objects" through "predicates of intuition," sensibility itself is the source of their relation to objects in general; by contrast, the categories "speak of objects not through predicates of intuition and sensibility but through those of pure *a priori* thinking" and so must "relate to objects generally without any conditions of sensibility" (A 88/B 120). Accordingly, the categories cannot use pure sensibility to "ground their synthesis prior to any experience" (A 88/B 120), as can mathematical concepts, and so "do not represent to us the conditions under which objects are given in intuition at all" (A 89/B 122). It follows that, despite the fact that objects *cannot* appear to us without necessarily having to be related to the forms of sensibility, and so cannot be objects for us without having mathematical properties, "objects can indeed appear to us without necessarily having to be related to functions of the understanding" (A 89/B 122). Thus, although the objects of experience necessarily possess the sensible features we represent them to have, and so are necessarily mathematically describable, such objects do not necessarily possess the conceptual or categorical features we represent them to have, at least without further argument. That is, though the results of the Transcendental Aesthetic assure us that our sensible concepts are objectively valid, we have no such assurance in the case of pure concepts of understanding: "Thus a difficulty is revealed here [in pure understanding] that we did not encounter in the field of sensibility, namely how **subjective conditions of thinking** should have **objective validity**, i.e., yield conditions of the possibility of all cognition of objects" (A 90/B 122). Kant of course resolves this difficulty and demonstrates the transcendental ideality of the categories with the Transcendental Deduction. In some sense, then, the project of demonstrating the objective validity of the categories can be seen as modeled on, or at least as motivated by, the successful prior demonstration of the objective validity of space, time, and mathematics.

Kant revisits the objective validity of mathematical concepts and propositions in the Axioms of Intuition, where he identifies and defends the synthetic *a priori* judgment that he claims is the principle of the possibility of all mathematical principles, including the axioms of geometry.[44] Here he answers for the specific case of mathematics the guiding question with which this essay began, namely, "How are synthetic judgments *a priori* possible?" Synthetic judgments are *a priori* possible in mathematics only given the prior synthetic *a priori* principle that "All intuitions are extensive magnitudes"[45] (B 202). The sense in which mathematical judgments are thereby made possible is quite specific: only the principle that all intuitions are extensive magnitudes can make it possible for each and every mathematical judgment to apply to – and thereby provide synthetic and *a priori* cognition of – the objects of experience, or appearances.[46] Kant claims that the principle that all intuitions are extensive magnitudes, what he calls the "transcendental principle of the mathematics of appearances,"

> yields a great expansion of our *a priori* cognition. For it is this alone that makes pure mathematics in its complete precision applicable to objects of experience, which without this principle would not be so obvious, and has indeed caused much contradiction.[47] (A 165/B 206)

Kant's argument in support of this claim begins with a restatement of the central thesis of transcendental idealism, defended earlier: "Appearances are not things in themselves. Empirical intuition is possible only through the pure intuition (of space and time)" (A 165/B 206). He then notes that if objects of the senses – which are given in empirical intuition – were not in complete agreement with the mathematical rules of ostensive construction – which are given in pure intuition – then mathematics would not be objectively valid. That is, the objective validity of mathematics depends on such agreement between pure and empirical intuition, and on the fact that "what geometry says about the latter is therefore undeniably valid of the former" (A 165/B 206). But this agreement is precisely what is expressed by the principle that "All intuitions are extensive magnitudes," which means that as intuitions all appearances "must be represented through the same synthesis as that through which space and time in general are determined" (B 203). Therefore, the objective validity of mathematics, and the possibility that the synthetic *a priori* propositions of mathematics are applicable to the appearances, is explained by the transcendental "axiom" of intuition.

This "axiom" clarifies Kant's reasons for denying, contra his predecessors, that philosophy and mathematics can share a methodology. Mathematics is distinguished from philosophy by virtue of its constructive procedure, which is the cause of its "pertaining solely to quanta": because mathematics constructs its object *a priori* in intuition, and because the only concept that can be so constructed is the concept of magnitude, mathematics necessarily takes quantity as its object (A 714/B 742). But, according to the Axioms of Intuition, such constructed quanta make possible the apprehension of appearances and the cognition of outer objects. Thus, "what mathematics in its pure use proves about the former is also necessarily valid for the latter" (A 166/B 207). That is, our mathematical cognition of purely constructed quanta is likewise cognition of the quantitative form of empirical objects. Philosophical cognition, by contrast, cannot construct and exhibit qualities in an analogous way and so cannot hope to achieve rational cognition of objects of experience via a mathematical method.

5. CONCLUSION

Kant, a long-time teacher and student of mathematics, developed his theory of mathematics in the context of the actual mathematical practices of his predecessors and contemporaries, and he produced thereby a coherent and compelling account of early modern mathematics.[48] As is well known, however, mathematical practice underwent a significant revolution in the nineteenth century, when developments in analysis, non-Euclidean geometry, and logical rigor forced mathematicians and philosophers to reassess the theories that Kant and the moderns used to account for mathematical cognition. Nevertheless, the basic theses of Kant's view played an important role in subsequent discussions of the philosophy of mathematics. Frege defended Kant's philosophy of geometry, which he took to be consistent with logicism about arithmetic;[49] Brouwer and the Intuitionists embraced Kant's idea that mathematical cognition is constructive and based on mental intuition;[50] and Husserl's attempt to provide a psychological foundation for arithmetic owes a debt to Kant's characterization of mathematics as providing knowledge of the formal features of the empirical world.[51]

In the later twentieth century, by contrast, most philosophers accepted some version of Bertrand Russell's withering criticism of Kant's account, which he based on his own logicist program for mathematics.[52] But now it is clearly time to reassess the relevance of Kant's philosophy of mathematics to our own philosophical debates. For just a few examples, contemporary work in diagrammatic reasoning and mereotopology raise issues that engage with Kant's philosophy of mathematics;[53] Lakatos-style antiformalism is arguably a descendant of Kant's constructivism;[54] and our contemporary understanding of the relation between pure and applied mathematics, especially in the case of geometry, is illuminated by Kant's conception of the sources of mathematical knowledge. More generally, because we persist in considering mathematics to be a sort of epistemic paradigm, our current investigations into the possibility of substantive *a priori* knowledge would surely benefit from reflection on Kant's own subtle and insightful account of mathematics.

I hope to have shown that Kant's account is not an isolated philosophy of mathematics, developed only to make sense of early modern practices and as a tangent to his primary purposes, but is rather a crucial component of his broader philosophical project. It is impossible to appreciate fully Kant's thesis that all mathematical cognition is synthetic and *a priori*, and the arguments that he offers in its support, in isolation from his theory of pure sensibility, doctrine of transcendental idealism, and views on appropriate and successful methods of reasoning. Likewise, the general aims of Kant's broad and deep critical project are themselves much easier to appreciate given the insights afforded by his philosophy of mathematics. Further, I would argue that understanding Kant's philosophy of mathematics, despite its association with his own mathematical and historical context, speaks directly to our own views about the relation between philosophical and mathematical reasoning.[55]

NOTES

1. Christian Wolff, *Preliminary Discourse on Philosophy in General* (Indianapolis: Bobbs-Merrill Co, Inc., 1963 [1779]), p. 77.
2. Wolff, *Preliminary Discourse*, p. 76.
3. For a detailed discussion of Wolff's philosophy of mathematics and, in particular, his conception of analysis, see Lanier Anderson, "The Wolffian Paradigm and its Discontents: Kant's Containment Definition of Analyticity in Historical Context," *Archiv für Geschichte der Philosophie* (forthcoming).
4. That, for Leibniz, every mathematical proposition expresses a demonstrable identity is no surprise, given his more general predicate containment theory of truth. For discussion, see G. H. R. Parkinson, "Philosophy and Logic," in Nicholas Jolley, ed., *The Cambridge Companion to Leibniz* (Cambridge: Cambridge University Press, 1995).
5. G. W. Leibniz, *New Essays in Human Understanding*, Peter Remnant and Jonathan Bennett, eds. and trans. (Cambridge: Cambridge University Press, 1996 [1704]), IV, I, pp. 360–1.
6. Moses Mendelssohn, "On Evidence in Metaphysical Sciences," in Daniel Dahlstrom, ed., *Philosophical Writings* (Cambridge: Cambridge University Press, 1997 [1763]). Mendelssohn's was the prize winning essay in a contest held by the Royal Academy on the question of the relation between mathematical and metaphysical truth. The second place essay was Kant's *Inquiry concerning the Distinctness of the Principles of Natural Theology and Morals*. For a thorough and helpful discussion of both essays, see Paul Guyer, "Mendelssohn and Kant: One Source of the Critical Philosophy," *Philosophical Topics*, 19:1 (1991), reprinted in his *Kant on Freedom, Law, and Happiness* (Cambridge: Cambridge University Press, 2000).
7. Mendelssohn, "On Evidence," p. 257.
8. Mendelssohn, "On Evidence," p. 258.
9. Because of space limitations, I have not discussed the relation between mathematical concepts and the sensible world on the rationalist view. For some further discussion of these issues see Lisa Shabel, "Apriority and Application: Philosophy of Mathematics in the Modern Period," in

Stewart Shapiro, ed., *The Oxford Handbook of Philosophy of Math and Logic* (Oxford: Oxford University Press, 2005).

10. Euclid, *The Elements*, T. L. Heath, ed. and trans. (New York: Dover, 1956), p. 316.
11. For a fascinating discussion of Kant's notion of analyticity, and the related notion of concept containment, see Anderson, "Wolffian Paradigm," and Lanier Anderson, "It Adds Up After All: Kant's Philosophy of Arithmetic in Light of the Traditional Logic," *Philosophy and Phenomenological Research*, LXIX:3 (2004).
12. Kant's use of "object" to describe a geometric figure is in tension with his commitment to the idea that a pure geometric figure is the form of an empirical object of the same shape. Technically, a pure geometric figure is not itself an "object" in Kant's sense.
13. For a discussion of Kant on definition, see Lewis White Beck, "Kant's Theory of Definition," in Hoke Robinson, ed., *Selected Essays on Kant* (Rochester: University of Rochester Press, 2002). For a discussion of Kant on mathematical definitions in particular, see Emily Carson, "Kant on the Method of Mathematics," *Journal of the History of Philosophy*, 37:4 (1999).
14. This makes plain another difference between our construction of geometric and arithmetic concepts. In constructing the concept *triangle*, one could produce a right or scalene triangle, an equilateral or isosceles. That is, there are multiple distinct and different three-sided rectilineal figures that count as triangles, in the relevant sense. However, there is only one way to construct the concept five, by counting out five discrete units of some uniform kind. Thus our intuitive representation of any particular number concept is unique in a way that our intuitive representation of any particular shape concept is not. Whether the concept *five* is represented with strokes, points, or fingers, there is only one way to count to five; this fully general procedure is captured by an intuitive representation of number. Kant owes us an explanation, then, of how a particular geometric figure can attain the generality necessary to adequately represent a general spatial concept. I will return to this in the next section.
15. Kant discusses this in the A-Deduction where he writes: "Now it is obvious that if I draw a line in thought, or think of the time from one noon to the next, or even want to represent a certain number to myself, I must necessarily first grasp one of these manifold representations after another in my thoughts. But if I were always to lose the preceding representations (the first parts of the line, the preceding parts of time, or the successively represented units) from my thoughts and not reproduce them when I proceed to the following ones, then no whole

representation and none of the previously mentioned thoughts, not even the purest and most fundamental representations of space and time, could ever arise" (A 102).

16. The same follows for numerals.
17. For the details of this interpretation of symbolic construction see Lisa Shabel, "Kant on the 'Symbolic Construction' of Mathematical Concepts," *Studies in History and Philosophy of Science*, 29:4 (1998). For alternative interpretations of Kant on symbolic construction, see Michael Friedman, *Kant and the Exact Sciences*. (Cambridge, MA: Harvard University Press, 1992), Chapter 2, and articles by Gordon Brittan, Jaako Hintikka, Philip Kitcher, Charles Parsons, Manley Thompson, and J. Michael Young collected in Carl Posy, ed., *Kant's Philosophy of Mathematics: Modern Essays* (Dordrecht: Kluwer Academic Publishers, 1992).
18. The corresponding argument in the *Prolegomena* occurs in the Preamble (at 4:269–71).
19. Kant moves back and forth inconsistently between discussing "principles" ("Grundsätze") and "propositions" ("Sätze") in this passage. In the succeeding passages, his argument for the syntheticity of geometry is directed at principles, or axioms, whereas his argument for the syntheticity of arithmetic is directed at propositions. Though he has further arguments in support of the syntheticity of geometric propositions or theorems, which I will discuss later in this section, he does not have further arguments in support of the syntheticity of arithmetic principles, for he denies that arithmetic has principles, or axioms (A 164/B 205). The reason for this denial is that the "numerical formulas" of arithmetic are synthetic but singular, and thus not general like the synthetic principles of geometry. Kant takes the singularity of a numerical formula such as "$7 + 5 = 12$" to be captured by the fact (mentioned above in note 14) that "the synthesis here can take place only in a single way, even though the subsequent use of these numbers is general" (A 164/B 205).
20. Before moving to the arguments for syntheticity, Kant makes a very brief remark about the apriority of mathematical judgments. I come back to this claim, and his arguments in support of it, in the next section.
21. The impossibility of such an analysis may be best understood as a function of the traditional logic. See Anderson, "It Adds Up After All."
22. I will say more about these general conditions of sensible intuition later.
23. Kant's philosophy of arithmetic has been discussed by many commentators. See, in particular, Charles Parsons's "Kant's Philosophy of Arithmetic" and "Arithmetic and the Categories," both reprinted in Posy, *Kant's Philosophy of Mathematics*, and Béatrice Longuenesse, *Kant and*

the Capacity to Judge (Princeton: Princeton University Press, 1998), Chapter 9.

24. This is not an axiom in Euclid's *Elements*, but it was included as an axiom in many early modern treatments of Euclidean geometry.
25. In this passage, Kant gives two more examples of synthetic geometric axioms: between two points only one straight line is possible, and two straight lines do not enclose a space. It is harder to make out Kant's claim for these two examples since one would have to take intuition to display not merely the possibility of constructing lines, but rather the *impossibility* of constructing a second straight line between two points, as well as the *impossibility* of a space enclosed by two straight lines. Despite the obvious difficulty, I think that Kant holds that the conditions of sensible intuition make these impossibilities apparent: one would fail were one to attempt construction of multiple straight lines between two points, or of a figure bounded by two straight lines. The failure would be the result of contradicting one's own prior definitions. In the first case, if one were to construct multiple lines between two points, all but one of them would fail to be straight, by definition of straight; in the second case, if one were to construct a figure bounded by two lines, one of the boundary lines would fail to be straight, by definition of straight. He discusses mathematical axioms further at A 47/B 65 and A 732/B 760.
26. Kant follows this discussion of the syntheticity of the axioms of geometry with a caveat that would seem to defeat his claim that *all* mathematical propositions are synthetic: "To be sure, a few principles that the geometers presuppose are actually analytic and rest on the principle of contradiction" (B 16). Here he mentions identities that Euclid took to be "common notions," or logical principles that apply to any scientific discipline (e.g., the whole is greater than the part). Kant proceeds to defend the idea that these are not true principles and that, in any case, one must exhibit their concepts in intuition actually to think them.
27. "In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles." Euclid, *Elements*, p. 316.
28. Because of space limitations, I will not be able here to discuss all of the reasons why philosophy can never appropriate the mathematical method. The bulk of the argument is at A 727–35/B 755–63. The discussion concludes with Kant's proclamation that "it is not suited to the nature of philosophy, especially in the field of pure reason, to strut about with a dogmatic gait and to decorate itself with the titles and ribbons of

mathematics, to whose ranks philosophy does not belong, although it has every cause to hope for a sisterly union with it" (A 735/B 763). For discussion, see Carson, "Kant on the Method of Mathematics."

29. There is a great literature on the role of intuition in Kant's theory of mathematics. Roughly speaking, commentators have divided on whether that role is primarily logical, or primarily phenomenological. I intend for my explication of the syntheticity of mathematical propositions to suggest that constructed geometric diagrams play both roles: they provide phenomenological evidence that warrants the logical inferences of a deductive proof. For further discussion of the role of the Euclidean diagram in Euclidean proof, see Lisa Shabel, *Mathematics in Kant's Critical Philosophy: Reflections on Mathematical Practice* (New York: Routledge, 2003), Part I. For further discussion of Kant's interpretation of Euclid I.32, see Lisa Shabel, "Kant's 'Argument from Geometry,'" *Journal of the History of Philosophy*, 42:2 (2004). See also Longuenesse, *Kant and the Capacity to Judge*, pp. 287–291, and Friedman, *Kant and the Exact Sciences*, Chapter 1. For sources relevant to the debate between the logical and the phenomenological interpretation of the role of intuition in Kant's theory of mathematics, see articles by Hintikka, Parsons, and Friedman in Posy, *Kant's Philosophy of Mathematics*, as well as Emily Carson, "Kant on Intuition in Geometry," *Canadian Journal of Philosophy*, 27:4 (1997), and Michael Friedman, "Geometry, Construction and Intuition in Kant and his Successors," in Gila Sher and Richard Tieszen, eds., *Between Logic and Intuition: Essays in Honor of Charles Parsons* (Cambridge: Cambridge University Press, 2000).
30. This step requires a result from the Transcendental Aesthetic, namely, that pure intuition is the form of empirical intuition. I will discuss this briefly in section 4.
31. Since "the concept is first given through the definition, it contains just that which the definition would think through it" (A 731/B 759). As I mentioned earlier, spatial concepts derive from the combination of a pure concept of quantity with space, a mode of sensibility.
32. For discussion of the Schematism and its particular relation to Kant's philosophy of mathematics, see J. Michael Young's "Construction, Schematism and Imagination," in Posy, *Kant's Philosophy of Mathematics*, and Longuenesse, *Kant and the Capacity to Judge*, Chapters 8 and 9.
33. Kant uses the example of a pure concept of a circle and a plate. See (A 137/B 176).
34. In other words, mathematical concepts are unique among pure concepts for being, strictly speaking, identical to their schemata. On this point I

concur with Paul Guyer, who writes that "[Kant's] view of [pure sensible] concepts is that they basically *are* rules for applying predicates to particular objects or their images, and thus virtually identical to schemata." Paul Guyer, *Kant and the Claims of Knowledge* (Cambridge: Cambridge University Press, 1987), p. 159.

35. For a detailed discussion of these and related issues, see Longuenesse, *Kant and the Capacity to Judge*, especially Chapters 8 and 9.
36. This last move requires additional arguments in favor of Kant's theory of transcendental idealism, the structure of which I will discuss briefly in section 4.
37. Many commentators, of course, have addressed the question of the role of mathematics in Kant's critical philosophy. See, in particular, Friedman, *Kant and the Exact Sciences*, especially Chapter 1, and Longuenesse, *Kant and the Capacity to Judge*, Chapters 8 and 9. Also, on the particular topic of the problem of incongruent counterparts (which I have not addressed here) and its role in Kant's arguments for transcendental idealism, see Jill Vance Buroker, *Space and Incongruence: The Origin of Kant's Idealism* (Dordrecht: D. Reidel, 1981). For a less recent but classic discussion of the relation between Kant's philosophy of mathematics and his doctrine of transcendental idealism, see P. F. Strawson, *The Bounds of Sense* (London: Routledge, 1995 [1966]), Part 5.
38. At (A 29/B 44) Kant says explicitly that the synthetic *a priori* propositions of mathematics "derive" from the intuition of space. There he refers back to the Transcendental Exposition of the Concept of Space, where he uses the metaphor that the synthetic *a priori* cognitions of mathematics "flow from" the representation of space (A 25/B 40). Later he uses another metaphor, saying that the synthetic cognitions of mathematics "can be drawn *a priori*" from the representation of space (A 39/B 55). For discussion of the relation between the original representation of space and the cognitions of geometry, see Shabel, "Kant's 'Argument from Geometry'."
39. Kant has, of course, a variety of arguments in defense of transcendental idealism. For a helpful introductory discussion of his different argumentative strategies, see Sebastian Gardner, *Routledge Philosophy Guidebook to Kant and the Critique of Pure Reason* (New York: Routledge, 1999), Chapter 5. For further discussion, see Henry Allison, *Kant's Transcendental Idealism* (New Haven: Yale University Press, 1983), and Guyer, *Kant and the Claims of Knowledge*, Part V. The exegesis of Kant's arguments that I offer in what follows is similar in spirit, if not in detail, to Guyer's analysis in *Kant and the Claims of Knowledge*, pp. 354 ff.

40. Of course, this only follows given Kant's further supposition that transcendental realism and transcendental idealism are the only two possible philosophical positions to take with respect to the status of space and time.
41. Up to this point in the argument, Kant basically has reiterated support for his premises that mathematics (geometry) is the synthetic *a priori* science of space, a pure intuition. What he must do next is show that the *reductio* supposition contradicts these premises.
42. In the course of the argument, Kant offers various formulations of the doctrine of transcendental idealism, not identical to the one just quoted. For example: the "subjective condition regarding form" is "at the same time the universal *a priori* condition under which alone the object of this (outer) intuition is itself possible"; and, "space (and time as well)" is "a mere form of your intuition that contains *a priori* conditions under which alone things could be outer objects for you, which are nothing in themselves without these subjective conditions" (A 48/B 66). Note too that his argument is specifically focused on the case of space, yet he draws his conclusion with respect to the ideality of both space and time.
43. The structure of Kant's arguments in the *Prolegomena* of course differ from those in the *Critique*, and the way in which Kant uses the thesis that mathematical cognition is synthetic *a priori* to support the doctrine of transcendental idealism is likewise different. A comparison of these texts is unfortunately beyond the scope of this paper.
44. Kant explains that mathematical principles, such as the axioms of geometry, are not themselves included in the "analytic of principles" that includes the Axioms of Intuition, and "do not constitute any part of this system, since they are drawn only from intuition, not from the pure concept of the understanding." Nevertheless, it is necessary to identify the *principle* of these principles, that is, the synthetic *a priori* judgment that makes possible the synthetic *a priori* mathematical principles, such as the axioms of geometry (A 149/B 188–9).
45. In the A-edition the principle reads "All appearances are, as regards their intuition, **extensive magnitudes**" (A 162).
46. In a sense, the A-edition version of the principle makes this move more perspicuous than does the B-edition version, since the A-edition version confirms that it is our intuitions of *appearances* that are extensive magnitudes.
47. Kant takes contradiction to arise from the failure to identify the proper bounds of cognition. In particular, the failure to identify the proper "field of validity" for mathematical cognition has, according to Kant, led his predecessors to develop accounts of mathematical cognition that are

in direct conflict with the "principles of experience." I explore Kant's account of this conflict, and his proposed resolution thereof, in Shabel, "Apriority and Application."

48. For further discussion, see Shabel, "Kant on the 'Symbolic Construction'" and *Mathematics in Kant's Critical Philosophy*.
49. See Gottlob Frege, *On the Foundations of Geometry and Formal Theories of Arithmetic* (New Haven, CT: Yale University Press, 1971 [1903–1906]).
50. See L. E. J. Brouwer, *Collected Works 1. Philosophy and Foundations of Mathematics*, A. Heyting, ed. (Amsterdam: North Holland Publishing Company, 1975) and L. E. J. Brouwer, *Brouwer's Cambridge Lectures on Intuitionism*, D. van Dalen, ed. (Cambridge: Cambridge University Press, 1981).
51. See Edmund Husserl, *Philosophy of Arithmetic: Psychological and Logical Investigations with supplementary texts from 1887–1901*, Dallas Willard, ed. and trans. (Dordrecht: Kluwer Academic Publishers, 2003).
52. See Bertrand Russell, *An Essay on the Foundations of Geometry* (New York: Dover, 1956), Chapter 2, and Bertrand Russell, *Principles of Mathematics* (New York: Norton, 1938), Chapter LII.
53. For a comprehensive bibliography of sources, see http://www.hcrc.ed.ac.uk/gal/Diagrams/biblio.html.
54. See Imre Lakatos, *Proofs and Refutations: The Logic of Mathematical Discovery* (Cambridge: Cambridge University Press, 1976).
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