HW9 Mengchu Li

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§5.1

4. (i) (a) $f(t,y) = e^{t-y}$

$$|f(t,y_2) - f(t,y_1)| = |e^{t-y_2} - e^{t-y_1}|$$
$$= |e^t| \cdot |e^{-y_2} - e^{-y_1}|$$
$$= e^t |e^{-y_2} - e^{-y_1}|$$
$$\not\leq L|y_2 - y_1|$$

$\therefore f(t,y) = e^{t-y}$ does not satisfy the Lipschitz condition on $D = \{(t,y) \mid 0 \leq t \leq 1, -\infty < y < \infty\}$.

(b) $f(t,y) = \frac{1+y}{1+t}$

$$|f(t,y_2) - f(t,y_1)| = \left|\frac{1+y_2}{1+t} - \frac{1+y_1}{1+t}\right|$$
$$= \left|\frac{y_2 - y_1}{1+t}\right|$$
$$= \left|\frac{1}{1+t}\right| \cdot |y_2 - y_1|$$
$$\leq |y_2 - y_1| \qquad t \in [0,1]$$

$\therefore f$ satisfies the Lipschitz condition on D with Lipschitz constant $L = 1$

(c) $f(t,y) = \cos(yt)$

$$|f(t,y_2) - f(t,y_1)| = |\cos(y_2 t) - \cos(y_1 t)|$$

By M.V.T: $\frac{f(y_2) - f(y_1)}{y_2 - y_1} = f'(\xi)$

$$\therefore \frac{|f(t,y_2) - f(t,y_1)|}{y_2 - y_1} = \frac{\partial}{\partial y}(f(t,\xi))$$
$$= -t\sin(t\xi)$$
$$|f(t,y_2) - f(t,y_1)| = |t\sin(t\xi)|\,|y_2 - y_1|$$
$$\leq |y_2 - y_1| \qquad t \in [0,1]$$

$\therefore f(t,y) = \cos(yt)$ satisfies the Lipschitz condition on D with Lipschitz constant $L = 1$

(d) $f(t,y) = \frac{y^2}{1+t}$

$$|f(t,y_2) - f(t,y_1)| = \left|\frac{y_2^2}{1+t} - \frac{y_1^2}{1+t}\right|$$
$$= \left|\frac{y_2^2 - y_1^2}{1+t}\right|$$
$$= \left|\frac{1}{1+t}\right| |y_2^2 - y_1^2|$$
$$= \left|\frac{1}{1+t}\right| |y_1 + y_2|\,|y_2 - y_1|$$
$$\not\leq L|y_2 - y_1|$$

$f(t,y) = \frac{y^2}{1+t}$ does not satisify the Lipschitz condition

ii) (a) $f(t,y) = e^{t-y}$ does not satisify Lip. condition

Theorem 5.6 cannot be used to show that the initial-value problem is well posed.

(b) $f(t,y) = \frac{1+y}{1+t}$ satisfies the Lip. Cond. in y on D

$f(t,y) = \frac{1+y}{1+t}$ is conutinuous in D

By Theorem 5.6, the initial-value problem

$$y' = f(t,y) = \frac{1+y}{1+t}, \quad 0 \leq t \leq 1, \quad y(0) = 1$$

is well posed.

(c) $f(t,y) = \cos(yt)$ satisfies Lip. Cond. in y on D

$f(t,y) = \cos(yt)$ is continuous on D

By Theorem 5.6, the initial-value problem

$$y' = \cos(yt), \quad 0 \leq t \leq 1, \quad y(0) = 1$$

is well posed.

(d) $f(t,y) = \frac{y^2}{1+t}$ does not satisify Lip. Cond.

Theorem 5.6 cannot applied to show whether a initial-value problem is well posed.

§5.2

1. (c) $y' = 1 + \frac{y}{t}$, $1 \leq t \leq 2$, $y(1) = 2$, with $h = 0.25$

$$n = \frac{b-a}{h} = \frac{2-1}{0.25} = 4$$

$$\begin{cases} y_0 = 2 \\ y_{i+1} = y_i + hf(t_i, y_i) \end{cases}$$

$i=0$ $\quad y_1 = y_0 + \frac{1}{4} \cdot f(t_0, y_0)$
$$= 2 + \frac{1}{4}\left(1 + \frac{y_0}{t_0}\right)$$
$$= 2 + \frac{1}{4}\left(1 + \frac{2}{1}\right) = 2.75$$

$\therefore y_1 = y(1.25) = 2.75$

$i=1$ $\quad y_2 = y_1 + \frac{1}{4}f(t_1, y_1)$
$$= 2.75 + \frac{1}{4}\left(1 + \frac{2.75}{1.25}\right)$$
$$= 3.55$$

$\therefore y_2 = y(1.5) = 3.55$

$i=2$ $\quad y_3 = y_2 + \frac{1}{4}f(t_2, y_2)$
$$= 3.55 + \frac{1}{4}\left(1 + \frac{3.55}{1.5}\right)$$
$$= 4.3916667$$

$\therefore y_3 = y(1.75) = 4.3916667$

$i=3$ $\quad y_4 = y_3 + \frac{1}{4}f(t_3, y_3)$
$$= 4.3916667 + \frac{1}{4}(1 + 4.3916667/1.75)$$
$$= 5.2690476$$

$\therefore y_4 = y(2) = 5.2690476$

(d) $y' = \cos 2t + \sin 3t$, $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.25$

$$n = \frac{b-a}{h} = \frac{1-0}{0.25} = 4$$

$$\begin{cases} y(0) = 1 \\ y_{i+1} = y_i + h \cdot f(t_i, y_i) \end{cases}$$

$i=0$ $\quad y_1 = y_0 + \frac{1}{4}(\cos 2t_0 + \sin 3t_0)$
$$= y_0 + \frac{1}{4}(\cos 0 + \sin 0)$$
$$= 1.25$$

$\therefore y_1 = y(0.25) = 1.25$

$i=1$ $\quad y_2 = y_1 + \frac{1}{4}(\cos(0.5) + \sin(0.75))$
$$= 1.6398053$$

$\therefore y_2 = y(0.5) = 1.6398053$

$i=2$ $\quad y_3 = y_2 + \frac{1}{4}(\cos(1) + \sin(1.5))$
$$= 2.0242547$$

$\therefore y_3 = y(0.75) = 2.0242547$

$i=3$ $\quad y_4 = y_3 + \frac{1}{4}(\cos(1.5) + \sin(2.25))$
$$= 2.2364573$$

$\therefore y_4 = y(1) = 2.2364573$

3. (c) $y(t) = t\ln t + 2t$

$$y' = f(t,y) = 1 + \frac{y}{t}$$
$$\frac{\partial}{\partial y}(f(t,y)) = \frac{1}{t}$$
$$\left|\frac{\partial}{\partial y}(f(t,y))\right| = \left|\frac{1}{t}\right| \leq 1 = L \qquad t \in [1,2]$$

$$|y''(t)| = \left|\frac{y't - y}{t^2}\right|$$
$$= \left|\frac{(1+\frac{y}{t})t - y}{t^2}\right|$$
$$= \left|\frac{1}{t}\right|$$

$\therefore |y''(t)| \leq 1 \qquad \therefore M = 1$

error bound $\quad t_1 = 1.25 \quad h = 0.25 \quad M = 1$

$$|y(t_1) - y_1| \leq \frac{hM}{2L}[e^{L(t_1-a)} - 1]$$
$$= \frac{0.25}{2}(e^{0.25} - 1)$$
$$\approx 0.0355032$$

Actual error

$$y(1.25) = 1.25 \cdot \ln 1.25 + 2 \cdot 1.25 = 2.77893$$
$$|y(1.25) - y_1| = |2.77893 - 2.75| = 0.02893$$

$\therefore$ Actuall error is smaller than error bound

$t_2 = 1.5 \quad h = 0.25 \quad M = 1$

$$|y(t_2) - y_2| \leq \frac{hM}{2L}[e^{L(t_2-a)} - 1]$$
$$= \frac{0.25}{2}[e^{0.5} - 1]$$
$$\approx 0.0810902$$

$$y(1.5) = 1.5\ln 1.5 + 2 \cdot 1.5 = 3.6082$$
$$|y(1.5) - y_2| = |3.6082 - 3.55| = 0.0582$$

$\therefore$ Actual error is smaller than error found

$t_3 = 1.75 \quad h = 0.25 \quad M = 1$

$$|y(t_3) - y_3| \leq \frac{hM}{2L}[e^{L(t_3-a)} - 1]$$
$$= \frac{0.25}{2}(e^{0.75} - 1)$$
$$\approx 0.139625$$

$$y(1.75) = 1.75\ln 1.75 + 2 \cdot 1.75 = 4.47933$$
$$|y(1.75) - y_3| = |4.47933 - 4.3916667|$$
$$= 0.0876633$$

$\therefore$ Actuall error is smaller than error bound

$t_4 = 2 \quad h = 0.25 \quad M = 1$

$$|y(t_4) - y_4| \leq \frac{hM}{2L}[e^{L(t_4-a)} - 1]$$
$$= \frac{0.25}{2}(e^1 - 1)$$
$$\approx 0.214785$$

$$y(2) = 2 \cdot \ln 2 + 2 \cdot 2 = 5.38629$$
$$|y(2) - y_4| = |5.38629 - 5.2690476|$$
$$\approx 0.1172424$$

$\therefore$ Actuall error is smaller than error bound.

(d) $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$

$$f(t,y) = \cos(2t) + \sin 3t$$
$$\frac{\partial}{\partial y}(f(t,y)) = 0$$

$\therefore L = 0$

$\because L = 0$ the definition of error bound

$$|y(t_i) - y_i| \leq \frac{hM}{2L}[e^{L(t_i-a)} - 1]$$

is undefined.

9. $y' = \frac{2}{t}y + t^2 e^t \qquad 1 \leq t \leq 2 \qquad y(1) = 0$

exact solu: $y(t) = t^2(e^t - e)$

(a) Use the codes get:

t_i	y_i (Approx)	$y(t_i)$ (Exact)	Error
1.1	0.271828	0.345920	0.07409169
1.2	0.684756	0.866643	0.18186958
1.3	1.276978	1.607215	0.330236734
1.4	2.093548	2.620360	0.5268118634
1.5	3.187445	3.967666	0.78022172
1.6	4.620818	5.720962	1.10014368
1.7	6.466396	7.963873	1.497477
1.8	8.809120	10.793625	1.98450497
1.9	11.747997	14.323082	2.575085
2	15.398236	18.683097	3.28486143

(b) (i) Use the code get:

$y(1.04) \approx 0.1087312 \qquad$ error ≈ 0.011256297

using the interpolation of $(1, 0)$ & $(1.1, 0.271828)$

(ii) Using the interpolation of $(1.5, 3.187445)$ & $(1.6, 4.620818)$ to approximate $y(1.55)$

Using the code get:

$y(1.55) \approx 3.9041315 \qquad$ error ≈ 0.88450352

(iii) Using interpolation of $(1.9, 11.747997)$ & $(2, 15.398236)$ to approximate $y(1.97)$

Using the code get:

$y(1.97) \approx 14.3031643 \qquad$ error ≈ 2.97613135

(c) error bound

$$|y(t_i) - y_i| \leq \frac{hM}{2L}[e^{L(t_i-a)} - 1]$$
$$\frac{\partial}{\partial y}(f(t,y)) = \left|\frac{2}{t}\right| \leq 2 = L \qquad t \in [1,2]$$

$$|y''(t)| = |4te^t + t^2e^t + 2e^t - 2e| \leftarrow \text{increasing}$$
$$\leq |4 \cdot 2 \cdot e^2 + 4 \cdot e^2 + 2e^2 - 2e|$$
$$\approx 98.0102$$

$\therefore M = 98.0102$

From the problem, we need $|y(t_i) - y_i| = \frac{hM}{2L}[e^{L(t_i-a)} - 1] = 0.1$

$L = 2 \quad M = 98.0102 \quad t_i = 2 \quad a = 1$

$$\therefore \frac{h \cdot 98.0102}{2 \cdot 2}(e^2 - 1) = 0.1$$

$$h \approx 0.00063878.$$

11. $y' = -y + t + 1$, $0 \leq t \leq 5$, $y(0) = 1$

exact solu: $y(t) = e^{-t} + t$

(a) $h = 0.2$ Using Euler's method (code)

$y(5) \approx 5.003778 \qquad$ error $= 0.00296$

$h = 0.1$ using Euler's method (code)

$y(5) \approx 5.104638 \qquad$ error $= 0.0979$

$h = 0.05$ using Euler's method (code)

$y(5) \approx 5.055625 \qquad$ error $= 0.048887$

(b) $|y''(t)| = |e^{-t}| \leq 1 \qquad t \in [0,5]$

$\therefore M = 1$

$M = 1 \qquad \delta = 10^{-6}$

Eq. 5.14 $\quad h = \sqrt{\frac{2\delta}{M}}$
$$= \sqrt{\frac{2 \cdot 10^{-6}}{1}}$$
$$= 0.0014142$$