HW3 Mengchu Li

2020年9月20日 星期日 14:44

§ 2.3

1. $f(x) = x^2 - 6 \quad p_0 = 1$

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad n \geq 1$$

$f'(x) = 2x$

$f(p_0) = 1^2 - 6 = -5$

$f'(p_0) = 2$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{-5}{2} = 3.5$$

$f(p_1) = f(3.5) = 3.5^2 - 6 = 6.25$

$f'(p_1) = 2 \times 3.5 = 7$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = 3.5 - \frac{6.25}{7} = \boxed{2.607142857}$$

3. (a) Secant Method $p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$

$f(x) = x^2 - 6 \quad p_0 = 3 \quad p_1 = 2$

$f(p_0) = 3^2 - 6 = 3$

$f(p_1) = 2^2 - 6 = -2$

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 2 - \frac{-2 \cdot (-1)}{-2-3} = 2 + \frac{2}{5} = 2.4$$

$f(p_2) = 2.4^2 - 6 = -0.24$

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)}$$

$$= 2.4 - \frac{-0.24(2.4-2)}{-0.24+2}$$

$$= 2.4 + \frac{0.4 \times 0.24}{1.76}$$

$$= 2.4 + 0.054545$$

$$= \boxed{2.454545}$$

6.

```
[59] def newton( x, f, df, E ):
         h = f(x) / df(x)
         n=0
         while abs(h) >= E:
             h = f(x)/df(x)
             x = x - h
             n+=1

         print("The value of the root is : ",
                                   "%.10f"% x)
         print("The number of iterations is : ", "%d"%n)
```

(b) $\ln(x-1) + \cos(x-1) = 0$ for $1.3 \leq x \leq 2$

```
[60] def f1(x):
       return math.log(x-1) + math.cos(x-1)
     def df1(x):
       return 1/(x-1) - math.sin(x-1)
     newton(1.3,f1,df1, 0.00001)

The value of the root is :  1.3977484760
The number of iterations is :  4
```

$x = 1.3977484760$ is the root

(d) $(x-2)^2 - \ln x = 0$ for $1 \leq x \leq 2$ and $e \leq x \leq 4$

```
def f2(x):
  return (x-2)*(x-2) - math.log(x)
def df2(x):
  return 2*(x-2) - 1/x
newton(1,f2,df2,0.00001)
newton(math.exp(1),f2,df2,0.00001)

The value of the root is :  1.4123911720
The number of iterations is :  4
The value of the root is :  3.0571035500
The number of iterations is :  5
```

i) $1 \leq x \leq 2$ $\quad x = 1.4123911720$ is the root

ii) $e \leq x \leq 4$ $\quad x = 3.0571035500$ is the root

8. Code for Secant Method

```
[65] def secant(f, p1, p2, E):
         n = 0;
         pm = 0;
         p0 = 0;
         c = 0;
         if (f(p1) * f(p2) < 0):
             while True:
                 p0 = ((p1 * f(p2) - p2 * f(p1)) /
                                 (f(p2) - f(p1)));
                 c = f(p1) * f(p0);
                 p1 = p2;
                 p2 = p0;
                 n += 1;
                 if (c == 0):
                     break;
                 pm = ((p1 * f(p2) - p2 * f(p1)) /
                                 (f(p2) - f(p1)));

                 if(abs(pm - p0) < E):
                     break;

             print("Root of the given equation =",
                                    round(p0, 6));
             print("Number of iterations = ", n);

         else:
             print("Can not find a root in ",
                           "the given inteval");
```

(b) $\ln(x-1) + \cos(x-1) = 0$ for $1.3 \leq x \leq 2$

```
def f1(x):
  return math.log(x-1) + math.cos(x-1)
secant(f1,1.3,2,0.00001)

Root of the given equation = 1.397749
Number of iterations =  7
```

$x = 1.397749$ is the root

(d) $(x-2)^2 - \ln x = 0$ for $1 \leq x \leq 2$ and $e \leq x \leq 4$

```
def f2(x):
  return (x-2)*(x-2) - math.log(x)
secant(f2, 1, 2, 0.00001)
secant(f2, math.exp(1), 4, 0.00001)

Root of the given equation = 1.412391
Number of iterations =  6
Root of the given equation = 3.057103
Number of iterations =  5
```

i) $x = 1.412391$ is the root

ii) $x = 3.057103$ is the root

15. Use newton(x, f, df, E) defined before

```
[70] def f3(x):
       return 4*x*x - math.exp(x) - math.exp(-x)
     def df3(x):
       return 8*x - math.exp(x) + math.exp(-x)
     #(a)
     newton(-10, f3, df3, 0.00001)

The value of the root is :  -4.3062452735
The number of iterations is :  11

[71] #(b)
     newton(-5, f3, df3, 0.00001)

The value of the root is :  -4.3062452735
The number of iterations is :  5

[72] #(c)
     newton(-3, f3, df3, 0.00001)

The value of the root is :  0.8244985853
The number of iterations is :  5

[73] #(d)
     newton(-1, f3, df3, 0.00001)

The value of the root is :  -0.8244985853
The number of iterations is :  4

[74] #(e)
     newton(0, f3, df3, 0.00001)
     # Fail since df3(0) = 0 Newton's method cannot implement

[75] #(f)
     newton(1, f3, df3, 0.00001)

The value of the root is :  0.8244985853
The number of iterations is :  4

[78] #(g)
     newton(3, f3, df3, 0.00001)

The value of the root is :  -0.8244985853
The number of iterations is :  5

[79] #(h)
     newton(5, f3, df3, 0.00001)

The value of the root is :  4.3062452735
The number of iterations is :  5

[80] #(i)
     newton(10, f3, df3, 0.00001)

The value of the root is :  4.3062452735
The number of iterations is :  11
```

§ 2.4

6. (a)

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{|\frac{1}{n+1} - 0|}{|\frac{1}{n} - 0|} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1 + \lim_{n\to\infty}\frac{1}{n}} = 1$$

$\therefore$ $p_n = \frac{1}{n}$ converges linearly to $p = 0$

$|p_n - p| \leq 5 \times 10^{-2}$

$|\frac{1}{n} - 0| \leq 5 \times 10^{-2}$

$\frac{1}{n} \leq 5 \times 10^{-2}$

$n \geq \frac{1}{5} \times 10^2 = 20$

$\therefore$ $n \geq 20$, $|p_n - p| \leq 5 \times 10^{-2}$

(b)

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{|\frac{1}{(n+1)^2} - 0|}{|\frac{1}{n^2} - 0|} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}}$$

$$= \frac{1}{1 + \lim_{n\to\infty}\frac{2}{n} + \lim_{n\to\infty}\frac{1}{n^2}} = \frac{1}{1+0+0} = 1$$

$\therefore$ $p_n = \frac{1}{n^2}$ converges linearly to $p = 0$

$|p_n - p| \leq 5 \times 10^{-2}$

$|\frac{1}{n^2} - 0| \leq 5 \times 10^{-2}$

$\frac{1}{n^2} \leq 5 \times 10^{-2}$

$n^2 \geq \frac{1}{5} \times 10^2$

$n \geq \sqrt{20} \simeq 5$

$\therefore$ when n is at least 5, $|p_n - p| \leq 5 \times 10^{-2}$

7. (a)

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = \lim_{n\to\infty} \frac{n^k}{(n+1)^k} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^k = 1^k = 1$$

$\therefore$ $p_n = \frac{1}{n^k}$ converges linearly to $p = 0$

(b) $\frac{1}{N^k} < 10^{-m}$

$\frac{1}{N^k} < \frac{1}{10^m}$

$N^k > 10^m$

$N > 10^{\frac{m}{k}}$

$\therefore$ integer N should satisfy $N > 10^{\frac{m}{k}}$

8. (a) $p_n = 10^{-2^n}$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}} - 0}{(10^{-2^n} - 0)^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^n \cdot 2}} = \lim_{n\to\infty} \frac{\cancel{10^{-2^{n+1}}}}{\cancel{10^{-2^{n+1}}}} = 1$$

$\therefore$ $p_n = 10^{-2^n}$ converges to $p = 0$ quadratically

(b) $p_n = 10^{-n^k}$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}} = \lim_{n\to\infty} 10^{2n^k - (n+1)^k}$$

$(n+1)^k = n^k + a_{k-1} n^{k-1} + \cdots + 1$

$= n^k + f_{k-1}(n)$

$$\therefore \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} 10^{2n^k - n^k - f_{k-1}(n)} = \lim_{n\to\infty} 10^{n^k - a_{k-1}n^{k-1} - a_{k-2}n^{k-2} - \cdots - 1} = \infty$$

$\therefore$ for any choice k, $p_n = 10^{-n^k}$ cannot converges to 0 quadratically

9. (a) like the example used in 8(a) $p_n = 10^{-2^n}$ converges to 0 quadratically

check $p_n = 10^{-3^n}$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^3} = \lim_{n\to\infty} \frac{10^{-3^{n+1}}}{(10^{-3^n})^3} = \lim_{n\to\infty} \frac{10^{-3^{n+1}}}{10^{-3^n \cdot 3}} = \lim_{n\to\infty} \frac{\cancel{10^{-3^{n+1}}}}{\cancel{10^{-3^{n+1}}}} = 1$$

$\therefore$ $p_n = 10^{-3^n}$ converges to 0 of order 3

(b) Same as 8(a), 9(a)

Assume $p_n = 10^{-\alpha^n}$ $(\alpha > 1)$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lim_{n\to\infty} \frac{10^{-\alpha^{n+1}}}{(10^{-\alpha^n})^\alpha} = \lim_{n\to\infty} \frac{10^{-\alpha^{n+1}}}{10^{-\alpha^n \cdot \alpha}} = \lim_{n\to\infty} \frac{\cancel{10^{-\alpha^{n+1}}}}{\cancel{10^{-\alpha^{n+1}}}} = 1$$

$\therefore$ $p_n = 10^{-\alpha^n}$ converges to 0 of order α.