HW8 Mengchu Li

2020年11月10日 星期二 22:04

§4.4

1. (a) $\int_1^2 x\ln x\,dx$, $n=4$

$a=1$ $b=2$ $h=\frac{2-1}{4}=0.25$

$f(x)=x\ln x$

$f(1)=1\cdot\ln 1=0$

$f(1.25)=1.25\cdot\ln(1.25)\approx 0.2789294$

$f(1.5)=1.5\cdot\ln(1.5)\approx 0.60819766$

$f(1.75)=1.75\cdot\ln(1.75)\approx 0.979327628$

$f(2)=2\ln 2\approx 1.386294361$

$$\int_1^2 x\ln x\,dx=\frac{h}{2}[f(1)+2(f(1.25)+f(1.5)+f(1.75))+f(2)]$$

$$=\frac{1}{8}[f(1)+2(f(1.25)+f(1.5)+f(1.75))+f(2)]$$

$$\approx 0.6399$$

(d) $\int_0^{\pi} x^2\cos x\,dx$ $n=6$

$a=0$ $b=\pi$ $h=\frac{b-a}{n}=\frac{\pi}{6}$

$f(x)=x^2\cos x$

$f(0)=0$

$f(\frac{\pi}{6})=(\frac{\pi}{6})^2\cos(\frac{\pi}{6})\simeq 0.237185$

$f(\frac{\pi}{3})=(\frac{\pi}{3})^2\cos(\frac{\pi}{3})\approx 0.13693887$

$f(\frac{\pi}{2})=(\frac{\pi}{2})^2\cos(\frac{\pi}{2})=0$

$f(\frac{2}{3}\pi)=(\frac{2}{3}\pi)^2\cos(\frac{2}{3}\pi)\approx -2.191$

$f(\frac{5}{6}\pi)=(\frac{5}{6}\pi)^2\cos(\frac{5}{6}\pi)\approx -5.9296$

$f(\pi)=(\pi)^2\cos(\pi)\simeq -9.8596$

$$\int_0^{\pi} x^2\cos x\,dx=\frac{\pi}{12}[f(0)+2(f(\tfrac{\pi}{6})+f(\tfrac{\pi}{3})+f(\tfrac{\pi}{2})+f(\tfrac{2}{3}\pi)+f(\tfrac{5}{6}\pi))+f(\pi)]$$

$$\approx -6.634$$

3. (a) $\int_1^2 x\ln x\,dx$ $n=4$

$a=1$ $b=2$ $h=\frac{2-1}{4}=\frac{1}{4}=0.25$

$f(x_0)=f(1)=0$

$f(x_1)=f(1.25)\approx 0.2789294$

$f(x_2)=f(1.5)\approx 0.60819766$

$f(x_3)=f(1.75)\approx 0.979327628$

$f(x_4)=f(2)\approx 1.386294361$

$$\int_1^2 x\ln x\,dx=\frac{h}{3}[f(x_0)+4[f(x_1)+f(x_3)]+2f(x_2)+f(x_4)]$$

$$=\frac{1}{12}[f(1)+4(f(1.25)+f(1.75))+2f(1.5)+f(2)]$$

$$\approx 0.63631$$

(d) $\int_0^{\pi} x^2\cos x\,dx$

$a=0$ $b=\pi$ $h=\frac{b-a}{n}=\frac{\pi}{6}$

$f(x)=x^2\cos x$

$f(x_0)=f(0)=0$

$f(x_1)=f(\frac{\pi}{6})=(\frac{\pi}{6})^2\cos(\frac{\pi}{6})\simeq 0.237185$

$f(x_2)=f(\frac{\pi}{3})=(\frac{\pi}{3})^2\cos(\frac{\pi}{3})\approx 0.13693887$

$f(x_3)=f(\frac{\pi}{2})=(\frac{\pi}{2})^2\cos(\frac{\pi}{2})=0$

$f(x_4)=f(\frac{2}{3}\pi)=(\frac{2}{3}\pi)^2\cos(\frac{2}{3}\pi)\approx -2.191$

$f(x_5)=f(\frac{5}{6}\pi)=(\frac{5}{6}\pi)^2\cos(\frac{5}{6}\pi)\approx -5.9296$

$f(x_6)=f(\pi)=(\pi)^2\cos(\pi)\simeq -9.8596$

$$\int_0^{\pi} x^2\cos x\,dx=\frac{\pi}{18}[f(x_0)+4(f(x_1)+f(x_3)+f(x_5))+2(f(x_2)+f(x_4))+f(x_6)]$$

$$=\frac{\pi}{18}[f(0)+4(f(\tfrac{\pi}{6})+f(\tfrac{\pi}{2})+f(\tfrac{5}{6}\pi))+2(f(\tfrac{\pi}{3})+f(\tfrac{2}{3}\pi))+f(\pi)]$$

$$\approx -6.41188$$

8. $\int_0^2 x^2e^{-x^2}\,dx$ $h=0.25$

$a=0$ $b=2$ $n=\frac{2-0}{0.25}=8$

$f(x)=x^2e^{-x^2}$

$f(x_0)=f(0)=0$

$f(x_1)=f(0.25)=(0.25)^2\cdot e^{-0.25^2}\approx 0.05871331643$

$f(x_2)=f(0.5)=(0.5)^2\cdot e^{-0.5^2}\approx 0.1947001958$

$f(x_3)=f(0.75)=(0.75)^2\cdot e^{-0.75^2}\approx 0.3205028389$

$f(x_4)=f(1)=1\cdot e^{-1}\approx 0.3678794412$

$f(x_5)=f(1.25)=(1.25)^2e^{-1.25^2}\approx 0.3275177924$

$f(x_6)=f(1.5)=(1.5)^2\cdot e^{-1.5^2}\approx 0.2371482553$

$f(x_7)=f(1.75)=(1.75)^2\cdot e^{-1.75^2}\approx 0.1432350311$

$f(x_8)=f(2)=4\cdot e^{-4}\approx 0.0732625555$

a) composite T.R. Rule

$$\int_0^2 x^2e^{-x^2}dx=\frac{1}{8}[f(0)+2(f(0.25)+f(0.5)+f(0.75)+f(1)+f(1.25)+f(1.5)+f(1.75))+f(2)]$$

$$\approx 0.4215820372$$

b) composite Simpson's Rule

$$\int_0^2 x^2e^{-x^2}dx=\frac{1}{12}[f(x_0)+f(x_8)+4(f(x_1)+f(x_3)+f(x_5)+f(x_7))+2(f(x_2)+f(x_4)+f(x_6))]$$

$$=\frac{1}{12}[f(0)+f(2)+4(f(0.25)+f(0.75)+f(1.25)+f(1.75))+2(f(0.5)+f(1)+f(1.5))]$$

$$\approx 0.4288214009$$

11. $\int_0^2 e^{2x}\sin 3x\,dx$

$f(x)=e^{2x}\sin 3x$

$f'(x)=2e^{2x}\sin 3x+3e^{2x}\cos 3x$

$=e^{2x}(2\sin 3x+3\cos 3x)$

$f''(x)=2e^{2x}(2\sin 3x+3\cos 3x)+e^{2x}(6\cos 3x-9\sin 3x)$

$=e^{2x}(12\cos 3x-5\sin 3x)$

$f'''(x)=e^{2x}(9\cos 3x-46\sin 3x)$

$f^{(4)}(x)=-e^{2x}(120\cos 3x+119\sin 3x)$

(a) error $=|\frac{b-a}{12}h^2f''(\xi)|$ $\xi\in[0,2]$

$f''(x)=e^{2x}(2\sin 3x+3\cos 3x)$

The maximum of $f''(x)$ when $x\in[0,2]$ with the plots on demos web.

$f''(2)=705.360102876$

$\therefore f''(2)\leq 705.36$

error $=|\frac{2}{12}\cdot h^2\cdot f''(\xi)|\leq|\frac{h^2}{6}\cdot 705.36|<0.0001$

$$h<\sqrt{\frac{0.0006}{705.36}}\approx 0.000922295$$

$$n=\frac{b-a}{h}=\frac{2}{0.0000922295}=2168.501787$$

$\therefore h<0.000922295$, $n\geq 2168$

(b) error $=|\frac{b-a}{180}h^4f^{(4)}(\xi)|$ $\xi\in[0,2]$

$f^{(4)}(x)=-e^{2x}(120\cos 3x+119\sin 3x)$

$f^{(4)}(0)=0$ $|f^{(4)}(2)|=4475.40981837.$

The max of $f^{(4)}(x)$ is $f^{(4)}(2)$ when $x\in[0,2]$

$f^{(4)}(x)\leq 4475.41$ when $x\in[0,2]$

error $=|\frac{2}{180}h^4f^{(4)}(\xi)|\leq|\frac{h^4}{90}\cdot 4475.41|<0.0001$

$$h<\sqrt[4]{\frac{0.0001\times 90}{4475.41}}=0.03765758$$

$$n>\frac{b-a}{h}=\frac{2}{0.03765758}=53.11015737$$

$\therefore h<0.03765758$, $n\geq 54$

§4.6

1. (b) $\int_0^1 x^2e^{-x}\,dx$

$h=\frac{b-a}{2}=\frac{1-0}{2}=0.5$

$S(a,b)=\frac{h}{3}[f(a)+4f(a+h)+f(b)]$

$S(0,1)=\frac{0.5}{3}[0\cdot e^{-0}+4\cdot(0.5)^2\cdot e^{-0.5}+1\cdot e^{-1}]$

$=0.16240168$

$S(a,\frac{a+b}{2})=\frac{h}{6}[f(a)+4f(a+\frac{h}{2})+f(a+h)]$

$\overset{\parallel}{S(0,0.5)}=\frac{0.5}{6}[0\cdot e^{-0}+4\cdot 0.25^2\cdot e^{-0.25}+0.5^2\cdot e^{-0.5}]$

$\simeq 0.02886107$

$S(\frac{a+b}{2},b)=\frac{h}{6}[f(a+h)+4f(a+\frac{3}{2}h)+f(b)]$

$\overset{\parallel}{S(0.5,1)}=\frac{0.5}{6}[0.5^2\cdot e^{-0.5}+4\cdot 0.75^2\cdot e^{-0.75}+1\cdot e^{-1}]$

$\simeq 0.1318614041$

$|S(0,1)-S(0,0.5)-S(0.5,1)|=|0.16240168-0.02886107-0.1318614041|$

$=0.0016792$

3. (b) $\int_0^1 x^2e^{-x}dx\approx S(a,\frac{a+b}{2})+S(\frac{a+b}{2},b)=S(0,0.5)+S(0.5,1)$

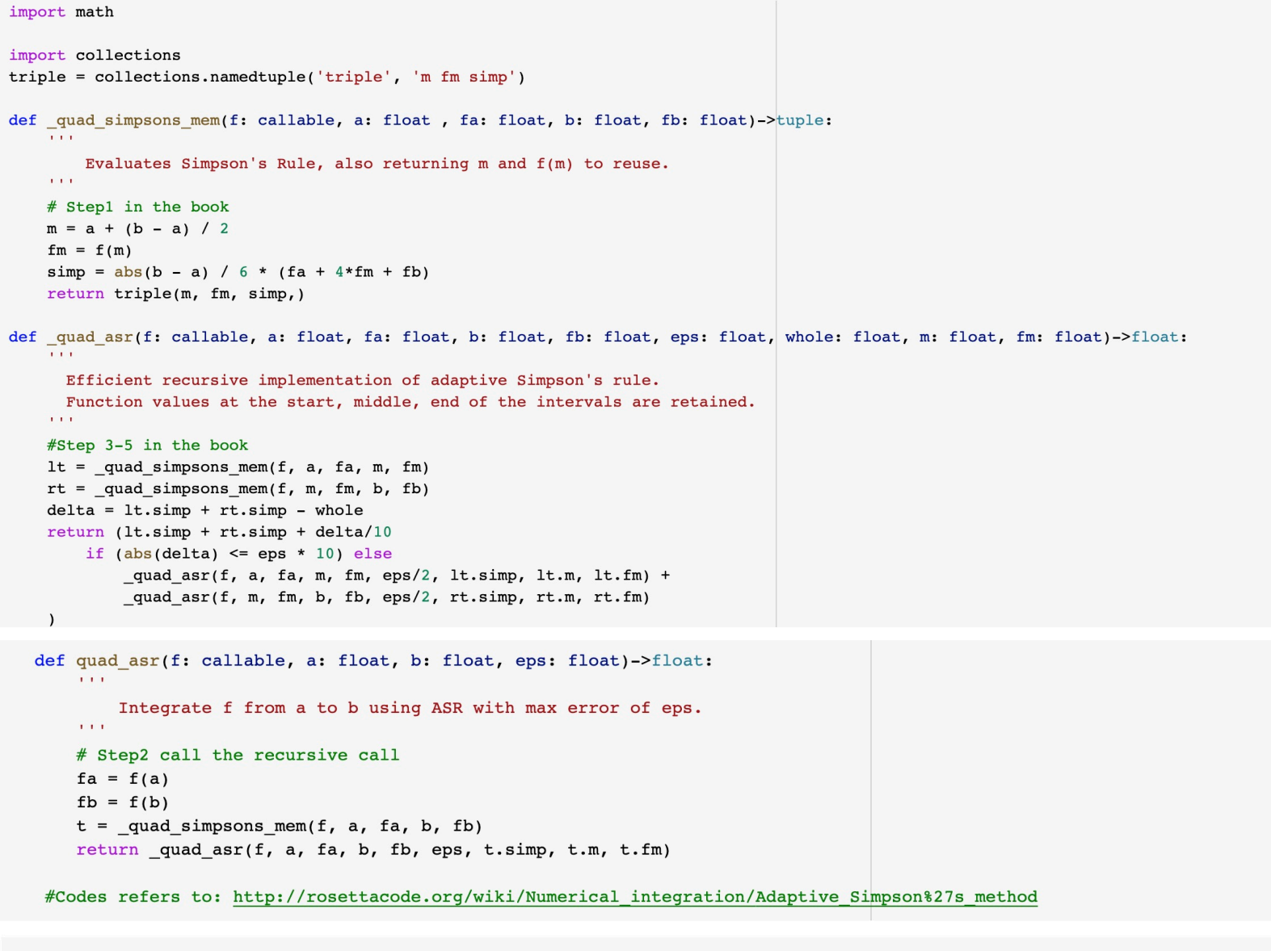
$=0.02886107+0.1318614041$

$=0.1607224741$

≈ 0.161

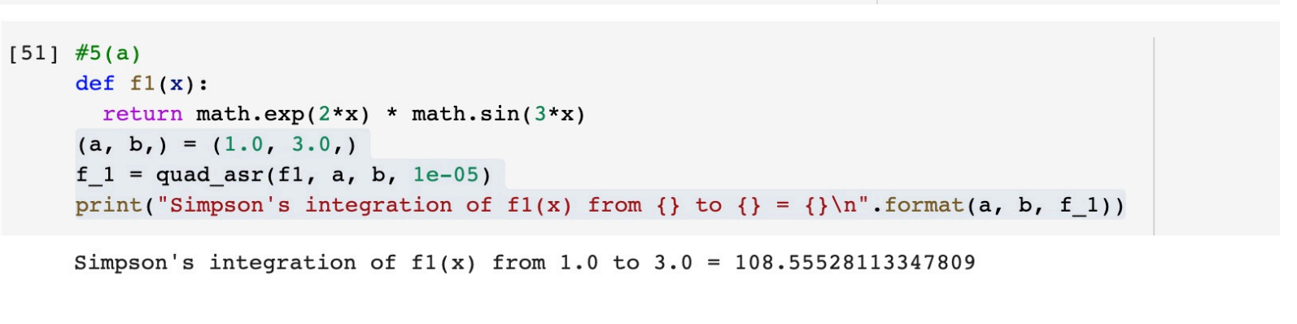
5. (a) $\int_1^3 e^{2x}\sin 3x\,dx$

Use the code get $\int_1^3 e^{2x}\sin 3x\,dx\approx 108.55528$



```
import math

import collections
triple = collections.namedtuple('triple', 'm fm simp')

def _quad_simpsons_mem(f: callable, a: float , fa: float, b: float, fb: float)->tuple:
    '''
        Evaluates Simpson's Rule, also returning m and f(m) to reuse.
    '''
    # Step1 in the book
    m = a + (b - a) / 2
    fm = f(m)
    simp = abs(b - a) / 6 * (fa + 4*fm + fb)
    return triple(m, fm, simp,)

def _quad_asr(f: callable, a: float, fa: float, b: float, fb: float, eps: float, whole: float, m: float, fm: float)->float:
    '''
      Efficient recursive implementation of adaptive Simpson's rule.
      Function values at the start, middle, end of the intervals are retained.
    '''
    #Step 3-5 in the book
    lt = _quad_simpsons_mem(f, a, fa, m, fm)
    rt = _quad_simpsons_mem(f, m, fm, b, fb)
    delta = lt.simp + rt.simp - whole
    return (lt.simp + rt.simp + delta/10
        if (abs(delta) <= eps * 10) else
            _quad_asr(f, a, fa, m, fm, eps/2, lt.simp, lt.m, lt.fm) +
            _quad_asr(f, m, fm, b, fb, eps/2, rt.simp, rt.m, rt.fm)
    )
```

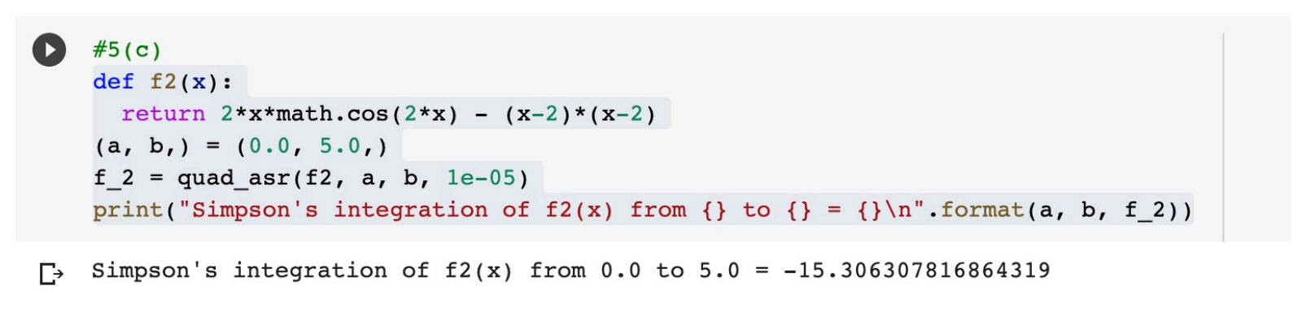
```
def quad_asr(f: callable, a: float, b: float, eps: float)->float:
    '''
        Integrate f from a to b using ASR with max error of eps.
    '''
    # Step2 call the recursive call
    fa = f(a)
    fb = f(b)
    t = _quad_simpsons_mem(f, a, fa, b, fb)
    return _quad_asr(f, a, fa, b, fb, eps, t.simp, t.m, t.fm)

#Codes refers to: http://rosettacode.org/wiki/Numerical_integration/Adaptive_Simpson%27s_method
```



```
[51] #5(a)
     def f1(x):
       return math.exp(2*x) * math.sin(3*x)
     (a, b,) = (1.0, 3.0,)
     f_1 = quad_asr(f1, a, b, 1e-05)
     print("Simpson's integration of f1(x) from {} to {} = {}\n".format(a, b, f_1))

     Simpson's integration of f1(x) from 1.0 to 3.0 = 108.55528113347809
```

(c) $\int_0^5 (2x\cos(2x)-(x-2)^2)\,dx$

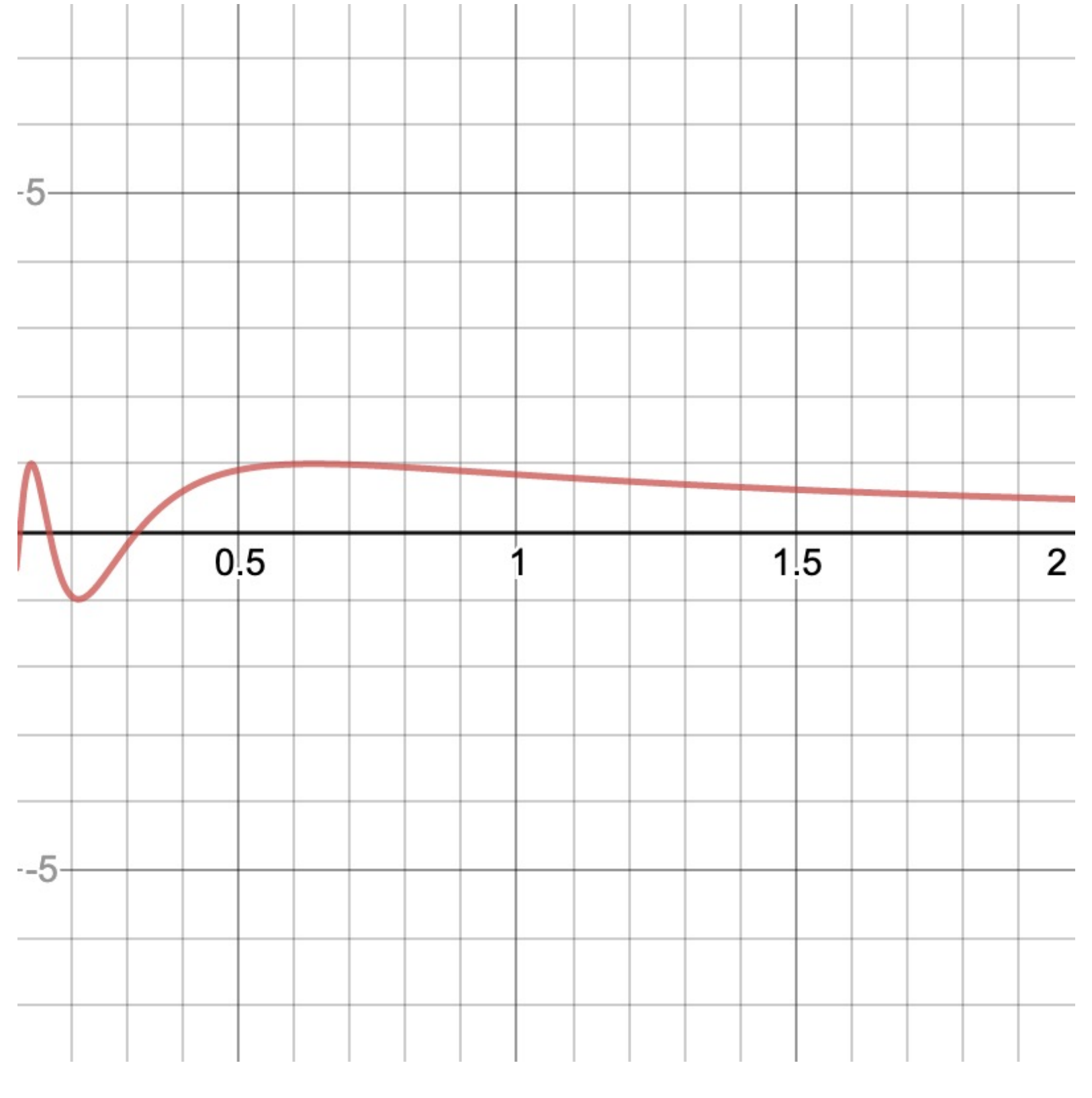
Use the codes to get $\int_0^5 (2x\cos(2x)-(x-2)^2)\,dx\approx -15.30631$



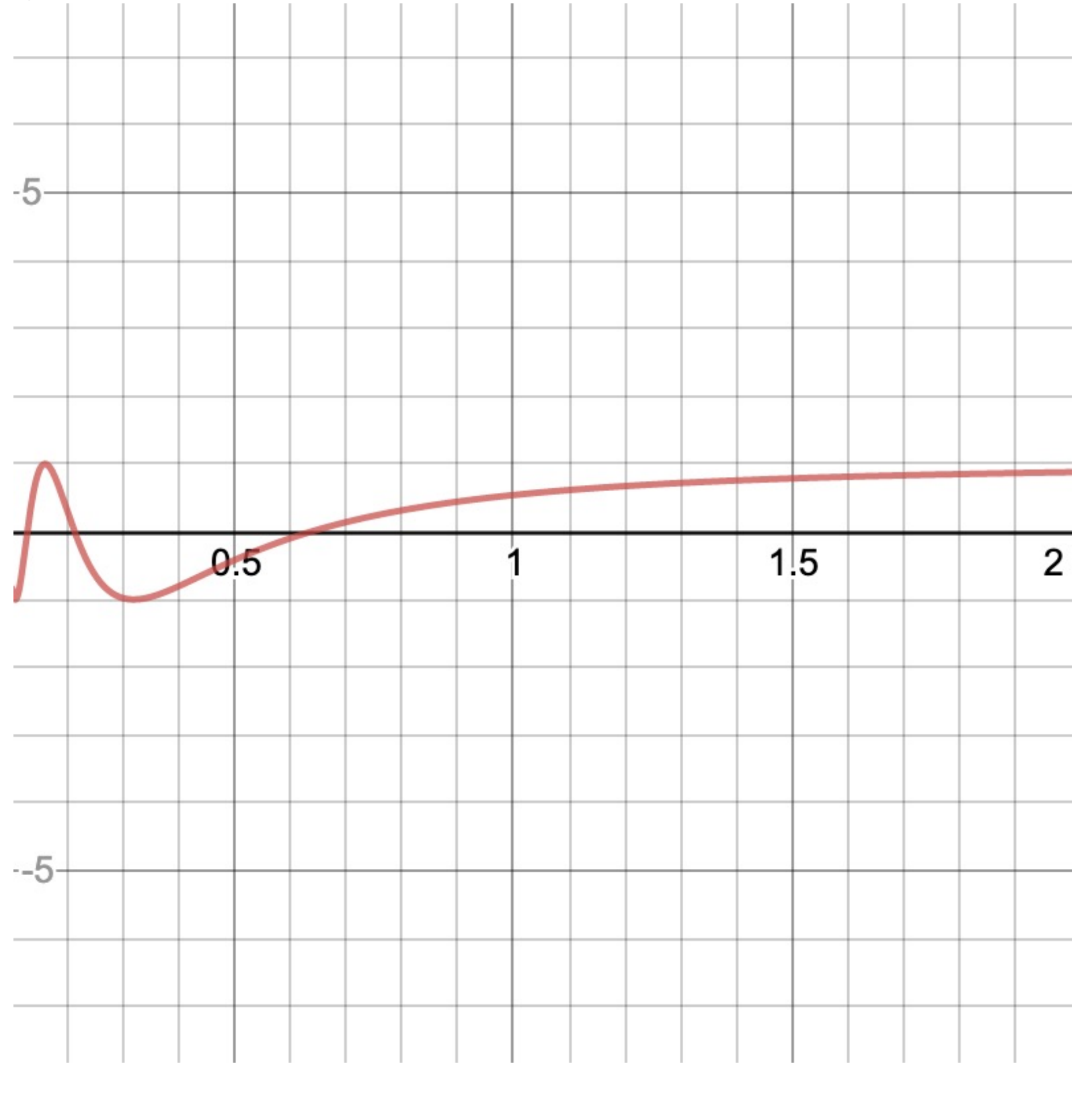
```
#5(c)
def f2(x):
  return 2*x*math.cos(2*x) - (x-2)*(x-2)
(a, b,) = (0.0, 5.0,)
f_2 = quad_asr(f2, a, b, 1e-05)
print("Simpson's integration of f2(x) from {} to {} = {}\n".format(a, b, f_2))

Simpson's integration of f2(x) from 0.0 to 5.0 = -15.306307816864319
```

9. Graphs

$y=\sin(\frac{1}{x})$ $[0.1,2]$

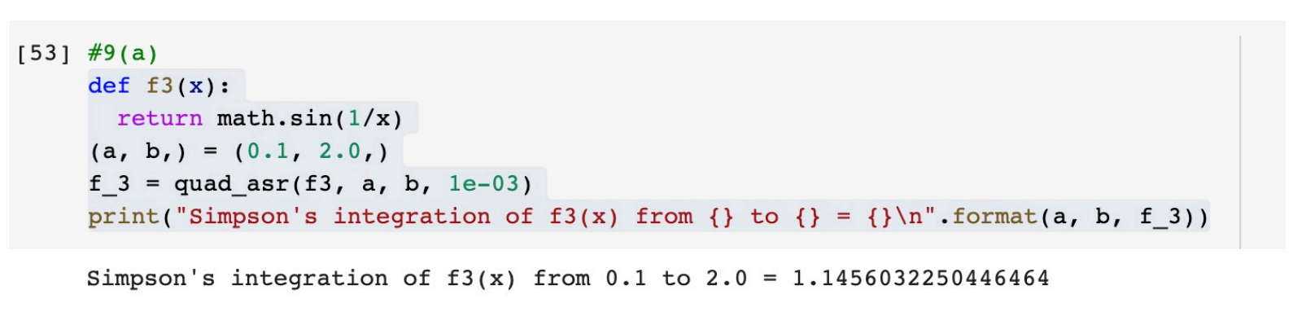


$y=\cos(\frac{1}{x})$ on $[0.1,2]$



a) $\int_{0.1}^2 \sin(\frac{1}{x})\,dx$

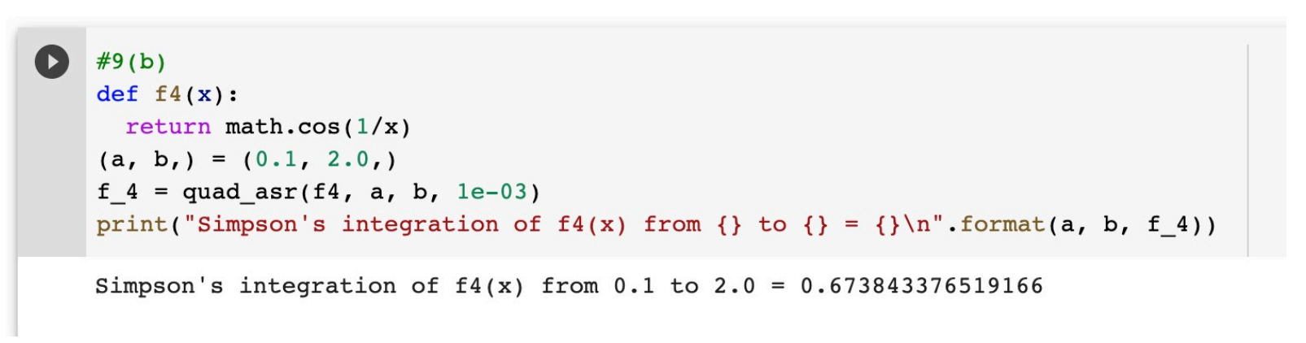
Use the codes get: $\int_{0.1}^2 \sin(\frac{1}{x})\,dx\simeq 1.1456$



```
[53] #9(a)
     def f3(x):
       return math.sin(1/x)
     (a, b,) = (0.1, 2.0,)
     f_3 = quad_asr(f3, a, b, 1e-03)
     print("Simpson's integration of f3(x) from {} to {} = {}\n".format(a, b, f_3))

     Simpson's integration of f3(x) from 0.1 to 2.0 = 1.1456032250446464
```

(b) $\int_{0.1}^2 \cos(\frac{1}{x})\,dx$

Use the codes get: $\int_{0.1}^2 \cos(\frac{1}{x})\,dx\simeq 0.67384$



```
#9(b)
def f4(x):
  return math.cos(1/x)
(a, b,) = (0.1, 2.0,)
f_4 = quad_asr(f4, a, b, 1e-03)
print("Simpson's integration of f4(x) from {} to {} = {}\n".format(a, b, f_4))

Simpson's integration of f4(x) from 0.1 to 2.0 = 0.673843376519166
```

13. By T.R. $\int_a^b f(x)\,dx=\frac{b-a}{2}(f(a)+f(b))-\frac{(b-a)^3}{12}f''(\xi)$ $\xi\in(a,b)$

$T(a,b)$ is the single application of T.R.

$\int_a^b f(x)\,dx=T(a,b)-\frac{(b-a)^3}{12}f''(\xi)$

$\therefore \int_a^b f(x)\,dx-T(a,b)=-\frac{(b-a)^3}{12}f''(\xi)$ ①

$b\to\frac{a+b}{2}$ in ①:

$$\int_a^{\frac{a+b}{2}} f(x)\,dx-T(a,\tfrac{a+b}{2})=-\frac{(\frac{a+b}{2}-a)^3}{12}f''(\xi_1)\qquad \xi_1\in(a,\tfrac{a+b}{2})$$

$$=-\frac{(a+b-2a)^3}{8\times 12}\cdot f''(\xi_1)$$

$$=-\frac{(b-a)^3}{96}f''(\xi_1)$$

$\therefore \int_a^{\frac{a+b}{2}} f(x)\,dx=T(a,\frac{a+b}{2})-\frac{(b-a)^3}{96}f''(\xi_1)$ $\xi_1\in(a,\frac{a+b}{2})$ ②

$a\to\frac{a+b}{2}$ in ①:

$$\int_{\frac{a+b}{2}}^b f(x)\,dx-T(\tfrac{a+b}{2},b)=-\frac{(b-\frac{a+b}{2})^3}{12}f''(\xi_2)\qquad \xi_2\in(\tfrac{a+b}{2},b)$$

$$=-\frac{(2b-a-b)^3}{12\cdot 8}f''(\xi_2)$$

$$=-\frac{(b-a)^3}{96}f''(\xi_2)$$

$\therefore \int_{\frac{a+b}{2}}^b f(x)\,dx=T(\frac{a+b}{2},b)-\frac{(b-a)^3}{96}f''(\xi_2)$ $\xi_2\in(\frac{a+b}{2},b)$ ③

$$|T(a,b)-T(a,\tfrac{a+b}{2})-T(\tfrac{a+b}{2},b)|$$

$$=\left|\int_a^b f(x)dx+\frac{(b-a)^3}{12}f''(\xi)-\int_a^{\frac{a+b}{2}}f(x)dx-\frac{(b-a)^3}{96}f''(\xi_1)-\int_{\frac{a+b}{2}}^b f(x)dx-\frac{(b-a)^3}{96}f''(\xi_2)\right|$$

$$\approx\left|\cancel{\int_a^{\frac{a+b}{2}}f(x)dx}+\cancel{\int_{\frac{a+b}{2}}^b f(x)dx}+\frac{(b-a)^3}{12}f''(\xi)-\cancel{\int_a^{\frac{a+b}{2}}f(x)dx}-\frac{(b-a)^3}{96}f''(\xi_1)-\cancel{\int_{\frac{a+b}{2}}^b f(x)dx}-\frac{(b-a)^3}{96}f''(\xi_2)\right|$$

$$=\left|\frac{(b-a)^3}{12}f''(\xi)-\frac{(b-a)^3}{96}f''(\xi_1)-\frac{(b-a)^3}{96}f''(\xi_2)\right|$$

$$=\left|\frac{(b-a)^3}{12}f''(\xi)-\frac{(b-a)^3}{96}(f''(\xi_1)+f''(\xi_2))\right|$$

$$\approx\left|\frac{(b-a)^3}{12}f''(\xi)-\frac{(b-a)^3}{96}\cdot 2f''(\xi)\right|$$

$$\simeq\frac{(b-a)^3}{16}f''(\xi)\approx\frac{h^3}{16}f''(\xi)$$

Hence, $|T(a,b)-T(a,\frac{a+b}{2})-T(\frac{a+b}{2},b)|\approx\frac{h^3}{16}f''(\xi)$

$$\int_a^b f(x)\,dx=\int_a^{\frac{a+b}{2}}f(x)\,dx+\int_{\frac{a+b}{2}}^b f(x)\,dx$$

From ②,③:

$$\int_a^b f(x)\,dx=T(a,\tfrac{a+b}{2})-\frac{(b-a)^3}{96}f''(\xi_1)+T(\tfrac{a+b}{2},b)-\frac{(b-a)^3}{96}f''(\xi_2)$$

$$=T(a,\tfrac{a+b}{2})+T(\tfrac{a+b}{2},b)-\frac{(b-a)^3}{96}(f''(\xi_1)+f''(\xi_2))$$

f'' is continuous, $f''(\xi_1)+f''(\xi_2)=2f''(\xi_3)$ $\xi_3\in[a,b)$

$$\therefore \int_a^b f(x)\,dx=T(a,\tfrac{a+b}{2})+T(\tfrac{a+b}{2},b)-\frac{(b-a)^3}{48}f''(\xi_3)$$

$$\int_a^b f(x)\,dx-T(a,\tfrac{a+b}{2})-T(\tfrac{a+b}{2},b)=-\frac{(b-a)^3}{48}f''(\xi_3)$$

$$\left|\int_a^b f(x)dx-T(a,\tfrac{a+b}{2})-T(\tfrac{a+b}{2},b)\right|=\frac{(b-a)^3}{48}f''(\xi_3)\approx\frac{h^3}{48}f''(\xi_3)=\frac{1}{3}\cdot\frac{h^3}{16}f''(\xi_3)$$

$$=\frac{1}{3}|T(a,b)-T(a,\tfrac{a+b}{2})-T(\tfrac{a+b}{2},b)|$$

Therefore:

$$\left|\int_a^b f(x)dx-T(a,\tfrac{a+b}{2})-T(\tfrac{a+b}{2},b)\right|=\frac{1}{3}\cdot|T(a,b)-T(a,\tfrac{a+b}{2})-T(\tfrac{a+b}{2},b)|$$