HW 10 Mengchu Li

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§ 5.3

2. (a) $y' = e^{t-y}$, $0 \le t \le 1$, $y(0)=1$, with $h=0.5$

Taylor Method of order 2

$y_{i+1} = y_i + h \cdot f(t_i, y_i) + \frac{h^2}{2} f'(t_i, y_i)$

$y' = f(t,y) = e^{t-y}$

$y'' = f'(t,y) = (1-y')e^{t-y} = (1-e^{t-y})e^{t-y}$

$i=0$ $y(0) = 1 = y_0$

$y(0.5) \approx y_1 = y_0 + 0.5 \cdot f(t_0, y_0) + \frac{0.5^2}{2} f'(t_0, y_0)$
$= 1 + 0.5 \cdot f(0,1) + \frac{0.5^2}{2} f'(0,1)$
$= 1 + 0.5 \cdot e^{-1} + \frac{0.5^2}{2}(1-e^{-1})e^{-1}$
≈ 1.213008

$i=1$ $y_1 \approx 1.213008$

$y(1) \approx y_2 = y_1 + 0.5 \cdot f(0.5, 1.213008) + \frac{0.5^2}{2} f'(0.5, 1.213008)$
$= 1.213008 + 0.5 \cdot e^{0.5-1.213008} + \frac{0.5^2}{2} \cdot (1-e^{0.5-1.213008})e^{0.5-1.213008}$
≈ 1.48933

(b) $y' = \frac{1+t}{1+y}$, $1 \le t \le 2$, $y(1)=2$, with $h=0.5$

Taylor Method of order 2

$y_{i+1} = y_i + h \cdot f(t_i, y_i) + \frac{h^2}{2} \cdot f'(t_i, y_i)$

$y' = f(t,y) = \frac{1+t}{1+y}$

$y'' = f'(t,y) = \frac{(1+y)-(1+t)y'}{(1+y)^2}$
$= \frac{1+y-(1+t)\cdot\frac{1+t}{1+y}}{(1+y)^2}$
$= \frac{(1+y)^2-(1+t)^2}{(1+y)^3}$

$i=0$ $y(1) = y_0 = 2$

$y(1.5) \approx y_1 = y_0 + 0.5 \cdot f(t_0, y_0) + \frac{0.5^2}{2} f'(t_0, y_0)$
$= 2 + 0.5 \cdot f(1,2) + \frac{0.5^2}{2} f'(1,2)$
$= 2 + 0.5 \cdot \frac{1+1}{1+2} + \frac{0.5^2}{2} \cdot \frac{(1+2)^2-(1+1)^2}{(1+2)^3}$
$= 2 + 0.5 \cdot \frac{2}{3} + \frac{0.5^2}{2} \cdot \frac{5}{27}$
≈ 2.356481

$i=1$ $y_1 = 2.356481$

$y(2) \approx y_2 = y_1 + 0.5 f(t_1, y_1) + \frac{0.5^2}{2} f'(t_1, y_1)$
$= y_1 + 0.5 f(1.5, 2.356481) + \frac{0.5^2}{2} f'(1.5, 2.356481)$
$= 2.356481 + 0.5 \cdot \frac{1+1.5}{1+2.356481} + \frac{0.5^2}{2} \cdot \frac{(1+2.356481)^2-(1+1.5)^2}{(1+2.356481)^3}$
≈ 2.74548

4. (a) $y'(t) = f(t,y) = e^{t-y}$

$y''(t) = f'(t,y) = (1-e^{t-y})e^{t-y} = e^{t-y} - e^{2(t-y)}$

$y'''(t) = f''(t,y) = (1-e^{t-y})e^{t-y} - e^{2(t-y)} \cdot 2(1-y')$
$= (1-e^{t-y})e^{t-y} - e^{2(t-y)} \cdot 2(1-e^{t-y})$
$= e^{t-y} - e^{2(t-y)} - 2e^{2(t-y)} + 2e^{3(t-y)}$
$= e^{t-y} - 3e^{2(t-y)} + 2e^{3(t-y)}$
$= e^{(t-3y)}(e^{2y} - 3e^{(t+y)} + 2e^{2t})$

$y^{(4)}(t) = f'''(t,y) = e^{(t-3y)} \cdot (1-3y') \cdot (e^{2y} - 3e^{(t+y)} + 2e^{2t}) + e^{(t-3y)}(e^{2y} \cdot 2y' - 3e^{(t+y)} \cdot (1+y') + 4e^{2t})$
$= e^{(t-4y)}(-6e^{3t} + e^{3y} + 12e^{2t+y} - 7e^{t+2y})$

Taylor method of order 4:

$y_{i+1} = y_i + h \cdot T^{(4)}(t_i, y_i)$
$= y_i + h(f(t_i,y_i) + \frac{h}{2} f'(t_i,y_i) + \frac{h^2}{6} f''(t_i,y_i) + \frac{h^3}{24} f'''(t_i,y_i))$

$h=0.5$

$\therefore y_{i+1} = y_i + 0.5(e^{t_i-y_i} + \frac{0.5}{2}(1-e^{t_i-y_i})e^{t_i-y_i} + \frac{0.5^2}{6} e^{t_i-3y_i}(e^{2y_i} - 3e^{(t_i+y_i)} + 2e^{2t_i})$
$+ \frac{0.5^3}{24} \cdot e^{(t_i-4y_i)}(-6e^{3t_i} + e^{3y_i} + 12e^{2t_i+y_i} - 7e^{t_i+2y_i}))$

$i=0$ $y(0) = y_0 = 1$

$y(0.5) \approx y_1 = 1 + 0.5(e^{0-1} + \frac{0.5}{2}(1-e^{0-1})e^{0-1} + \frac{0.5^2}{6} e^{0-3}(e^2 - 3e^{0+1} + 2e^0)$
$+ \frac{0.5^3}{24} \cdot e^{0-4}(-6e^0 + e^3 + 12e^{0+1} - 7e^{0+2}))$
≈ 1.21405

$i=1$ $y(0.5) = y_1 = 1.21405$

$y(1) \approx y_2 = y_1 + 0.5(e^{0.5-y_1} + \frac{0.5}{2}(1-e^{0.5-y_1})e^{0.5-y_1} + \frac{0.5^2}{6} e^{0.5-3y_1}(e^{2y_1} - 3e^{(0.5+y_1)} + 2e^1)$
$+ \frac{0.5^3}{24} e^{(0.5-4y_1)}(-6e^{1.5} + e^{3y_1} + 12e^{1+y_1} - 7e^{0.5+2y_1}))$
≈ 1.4899

(b) $y'(t) = f(t,y) = \frac{1+t}{1+y}$

$y''(t) = f'(t,y) = \frac{(1+y)^2-(1+t)^2}{(1+y)^3}$

$y'''(t) = f''(t,y) = \frac{3(1+t)^3 - 3(1+t)(1+y)^2}{(1+y)^5}$

$y^{(4)}(t) = f'''(t,y) = \frac{-15(1+t)^4 + 18(1+t)^2(1+y)^2 - 3(1+y)^4}{(1+y)^7}$

Taylor's Method of order 4:

$y_{i+1} = y_i + h \cdot T^{(4)}(t_i, y_i)$
$= y_i + h(f(t_i,y_i) + \frac{h}{2} f'(t_i,y_i) + \frac{h^2}{6} f''(t_i,y_i) + \frac{h^3}{24} f'''(t_i,y_i))$

$i=0$ $h=0.5$ $y_0 = y(1) = 2$

$y(1.5) \approx y_1 = y_0 + 0.5(f(1,y_0) + \frac{0.5}{2} f'(1,y_0) + \frac{h^2}{6} f''(1,y_0) + \frac{h^3}{24} f'''(1,y_0))$
≈ 2.35411

$i=1$ $h=0.5$ $y(1.5) \approx y_1 = 2.35411$

$y(2) \approx y_2 = y_1 + 0.5(f(1.5,y_1) + \frac{0.5}{2} f'(1.5,y_1) + \frac{0.5^2}{6} f''(1.5,y_1) + \frac{0.5^3}{24} f'''(1.5,y_1))$
≈ 2.74167

9. $y' = \frac{2}{t}y + t^2e^t$ $1 \le t \le 2$ $y(1)=0$ exact solu: $y(t) = t^2(e^t - e)$

(a) $y' = f(t,y) = \frac{2}{t}y + t^2e^t$

$y''(t) = f'(t,y) = \frac{2}{t^2}y + 4t \cdot e^t + t^2e^t$

Use the code get the table:

t_i	y_i (Approx.)	$y(t_i)$ (exact)	Error $= \lvert y_i - y(t_i)\rvert$
1.1	0.339785	0.345920	0.006134648
1.2	0.852733	0.866643	0.01391
1.3	1.585067	1.607215	0.02214791
1.4	2.591293	2.620360	0.02906682
1.5	3.935496	3.967666	0.03217
1.6	5.692834	5.720962	0.02812795
1.7	7.951238	7.963873	0.012635
1.8	10.813374	10.793625	0.01975
1.9	14.398871	14.323082	0.07579
2.0	18.846883	18.683097	0.16378566

(b) i) Use interpolation of (1,0) & (1.1, 0.339785)

$y(1.04) \approx 0.135914$

error ≈ 0.0159265

ii) Use interpolation of (1.5, 3.935496) & (1.6, 5.692834)

$y(1.55) \approx 4.814165$

error ≈ 0.02553

iii) Use interpolation of (1.9, 14.398871) & (2.0, 18.846883)

$y(1.97) \approx 17.5124794$

error $= 0.233181$

(c) $y' = f(t,y) = \frac{2}{t}y + t^2e^t$

$y'' = f'(t,y) = \frac{2}{t^2}y + 4te^t + t^2e^t$

$y''' = f''(t,y) = e^t(6+6t+t^2)$

$y^{(4)} = f'''(t,y) = e^t(12+8t+t^2)$

Use the code get the table:

t_i	y_i (Approx)	$y(t_i)$ (exact)	Error $(\lvert y_i - y(t_i)\rvert)$
1.1	0.345913	0.345920	7.187694×10^{-6}
1.2	0.867226	0.866643	0.000583538
1.3	1.610540	1.607215	0.003325365
1.4	2.630781	2.620360	0.0104212132
1.5	3.992502	3.967666	0.024836
1.6	5.771386	5.720962	0.050424361
1.7	8.055951	7.963873	0.09207748
1.8	10.949519	10.793625	0.15589437
1.9	14.572458	14.323082	0.249376 24
2.0	19.064748	18.683097	0.38165

§5.4

3. (b) Modified - Euler method

$y_{i+1} = y_i + \frac{h}{2}(f(t_i,y_i) + f(t_{i+1}, y_i + h \cdot f(t_i,y_i)))$

Use the code to get the table:

t_i	y_i (Approx)	$y(t_i)$ (exact)	Error $(\lvert y_i - y(t_i)\rvert)$
1.2	0.219444	0.22124277	0.00179833
1.4	0.485049	0.48968166	0.00463219
1.6	0.80401161	0.81275274	0.00874113
1.8	1.8485597	1.19943864	0.01452867
2.0	1.63842289	1.66128176	0.02285886
2.2	2.1788721	2.21350181	0.0346246
2.4	2.82506509	2.87655142	0.05148633
2.6	3.60252472	3.67847533	0.07595061
2.8	4.54661355	4.65866506	0.11205151
3.0	5.70756991	5.87409998	0.16653007

(d) $y' = f(t,y) = -5y + 5t^2 + 2t$

$0 \le t \le 1$ $h=0.1$ $n=10$

Use the code get the table:

t_i	y_i (Approx)	$y(t_i)$ (exact)	Error $(\lvert y_i - y(t_i)\rvert)$
0.1	0.22083333	0.21217689	0.00865645
0.2	0.17427083	0.16262648	0.01164435
0.3	0.17641927	0.16437672	0.01204255
0.4	0.21651204	0.20511176	0.01140028
0.5	0.28782003	0.27736167	0.01045836
0.6	0.38613752	0.37659569	0.00954183
0.7	0.50883595	0.50006579	0.00877015
0.8	0.65427247	0.64610521	0.00816725
0.9	0.82142029	0.813703	0.00771729
1.0	1.00963768	1.00224598	0.0073917

7. (b) Mid-point method:

$y_{i+1} = y_i + h \cdot f(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i))$

Use the code get the table: $n=10$

t_i	y_i (Approx)	$y(t_i)$ (exact)	Error $(\lvert y_i - y(t_i)\rvert)$
1.2	0.21983471	0.22124277	0.00140806
1.4	0.48617705	0.48968166	0.00350462
1.6	0.80618489	0.81275274	0.00656785
1.8	1.18843926	1.19943864	0.01099938
2.0	1.64388892	1.66128176	0.01739283
2.2	2.18686089	2.21350181	0.02664092
2.4	2.83643571	2.87655142	0.04011571
2.6	3.61849255	3.67847533	0.05998278
2.8	4.56889438	4.65866506	0.08977067
3.0	5.73864747	5.87409998	0.13545251

(d) Use the code get the table: $n=10$

t_i	y_i (Approx)	$y(t_i)$ (exact)	Error $(\lvert y_i - y(t_i)\rvert)$
0.1	0.21958333	0.21217689	0.0074645
0.2	0.17223958	0.16262648	0.0096131
0.3	0.17389974	0.16437672	0.00952302
0.4	0.21368734	0.20511176	0.00857558
0.5	0.28480459	0.27736167	0.00744292
0.6	0.38300287	0.37659569	0.00640718
0.7	0.50562679	0.50006579	0.005561
0.8	0.65101674	0.64610521	0.00491153
0.9	0.81813547	0.813703	0.00443247
1.0	1.00633467	1.00224598	0.00408868

15. (b) Runge-Kutta Order Four

$y_0 = \alpha$

$k_1 = hf(t_i, y_i)$

$k_2 = hf(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1)$

$k_3 = hf(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_2)$

$k_4 = hf(t_{i+1}, y_i + k_3)$

$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Use the code get the table: $n=10$

t_i	y_i (Approx)	$y(t_i)$ (exact)	Error $(\lvert y_i - y(t_i)\rvert)$
1.2	0.22124571	0.22124277	$2.94353118 \times 10^{-6}$
1.4	0.48968417	0.48968166	$2.50269242 \times 10^{-6}$
1.6	0.81275216	0.81275274	$5.78537532 \times 10^{-7}$
1.8	1.19943202	1.19943864	6.6186870×10^{-6}
2.0	1.66126512	1.66128176	$1.66402729 \times 10^{-5}$
2.2	2.21346932	2.21350181	$3.24969043 \times 10^{-5}$
2.4	2.87649411	2.87655142	$5.73054533 \times 10^{-5}$
2.6	3.678379	3.67847533	$9.63347526 \times 10^{-5}$
2.8	4.65850628	4.65866506	$1.58774557 \times 10^{-4}$
3.0	5.87383857	5.87409998	$2.61408351 \times 10^{-4}$

(d) Use the code get the table

t_i	y_i (Approx)	$y(t_i)$ (exact)	Error $(\lvert y_i - y(t_i)\rvert)$
0.1	0.21228299	0.21217689	1.0609954×10^{-4}
0.2	0.16276546	0.16262648	$1.38977328 \times 10^{-4}$
0.3	0.16451654	0.16437672	$1.39820702 \times 10^{-4}$
0.4	0.20524051	0.20511176	$1.28744116 \times 10^{-4}$
0.5	0.27747666	0.27736167	$1.14994496 \times 10^{-4}$
0.6	0.37669808	0.37659569	$1.02388524 \times 10^{-4}$
0.7	0.50015795	0.50006579	$9.21538833 \times 10^{-5}$
0.8	0.64618959	0.64610521	$8.43754935 \times 10^{-5}$
0.9	0.8137817	0.813703	$7.87045663 \times 10^{-5}$
1.0	1.00232067	1.00224598	$7.46866646 \times 10^{-5}$

17. (b) i) $y(2.1)$

Use code and linear(2) interpolation of (2.0, 1.63842289) & (2.2, 2.17887721) to approximate $y(2.1)$

$y(2.1) \approx 1.90865$

error ≈ 0.0163116

ii) $y(2.75)$

Use code and linear interpolation of (2.6, 3.60252472) & (2.8, 4.54661355) to approximate $y(2.75)$

$y(2.75) \approx 4.31059$

error ≈ 0.0935784

(d) i) $y(0.54)$

Use code and linear interpolation of (0.5, 0.28782003) & (0.6, 0.38613752) to approximate $y(0.54)$

$y(0.54) \approx 0.327147026$

error ≈ 0.013145188

ii) $y(0.94)$

Use code and linear interpolation of (0.9, 0.82142029) & (1.0, 1.00963768) to approximate $y(0.94)$

$y(0.94) \approx 0.896707246$

error $\approx 0.0100754869 66$

29. $y' = f(t,y) = -y + t + 1$, $0 \le t \le 1$, $y(0)=1$

$f(t_i, y_i) = -y_i + t_i + 1$

Midpt: $y_{i+1} = y_i + h \cdot f(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i))$

Modified Euler: $y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, y_i + hf(t_i, y_i))]$

Assume that the right-side of these two equations are equal

$\cancel{y_i} + \cancel{h} f(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)) = \cancel{y_i} + \frac{\cancel{h}}{2}[f(t_i, y_i) + f(t_{i+1}, y_i + hf(t_i, y_i))]$

$f(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)) = \frac{1}{2}[f(t_i, y_i) + f(t_{i+1}, y_i + hf(t_i, y_i))]$

$-(y_i + \frac{h}{2} f(t_i, y_i)) + t_i + \frac{h}{2} + 1 = \frac{1}{2}(-y_i + t_i + 1) + \frac{1}{2}(-(y_i + hf(t_i, y_i)) + t_{i+1} + 1)$

$-(y_i + \frac{h}{2}(-y_i + t_i + 1)) + t_i + \frac{h}{2} + 1 = \frac{1}{2}(-y_i + t_i + 1) + \frac{1}{2}(-(y_i + h(-y_i + t_i + 1)) + t_{i+1} + 1)$

$\cancel{\frac{h}{2}y_i} - \cancel{\frac{h}{2}t_i} - \cancel{\frac{h}{2}} - \cancel{y_i} + t_i + \frac{h}{2} + \cancel{1} = -\cancel{\frac{1}{2}y_i} + \frac{1}{2}t_i + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}y_i} + \cancel{\frac{h}{2}y_i} - \cancel{\frac{h}{2}t_i} - \cancel{\frac{h}{2}} + \frac{1}{2}t_{i+1} + \cancel{\frac{1}{2}}$

$t_i + \frac{h}{2} = \frac{1}{2}t_i + \frac{1}{2}t_{i+1}$

$t_i = \frac{1}{2}t_i + \frac{1}{2}t_{i+1} + \frac{h}{2}$

$\frac{1}{2}t_i = \frac{1}{2}t_{i+1} + \frac{h}{2}$

$\frac{1}{2}t_i = \frac{1}{2}(t_{i+1} + h)$

$t_i = t_{i+1} + h$

$h = t_{i+1} - t_i$, which is always correct

Hence, for this initial value problem, for any h, the approximation of midpoint method and modified euler method should be the same.