MATH1090/1098
(Honours) Introduction to Set Theory

The Chinese University of Hong Kong

2025-2026 Term 2

Contents

Chapter 1

Advice to the student

- Bring a paper and pencil / pens to class, and leave your phone / laptop / tablet in your bag.
- During the in-class exercises, work with the people next to you. Ask lots of questions! You will learn better if you say your thoughts out loud.
- This class will have graded assignments and optional exercises. However, most of your grade comes from the midterm and final marks. The assignments are mainly there to help you practice. We suggest you spend 10 minutes every day doing a few optional exercises; this is the best way to internalize the subject.

Chapter 2

Logic

2.1 The general shape of a mathematical argument

This first chapter is about the different kind of arguments used in mathematics. Very roughly, these look like

$$\begin{array}{c} \text{Hypothesis A} \\ \text{Hypothesis B} \\ \hline \text{Conclusion C} \end{array}$$

or

$$\begin{array}{c} A \\ B \\ \hline \therefore C \end{array}$$

or in plain English: 'If A and B are true, then C is also true'. More concisely: 'C follows from A and B'.

We want to know:

Question 1. *What kind of statements can we use as hypotheses or conclusions?*

Question 2. *How do we deduce the conclusion from the hypotheses?*

Of course, in life there is no single correct answer to these questions. In mathematics, however, there is a more or less universal answer, which you will learn in the next few sections.

2.2 Propositional logic.

Suppose you are investigating a murder. A, B, C, D, E, F, G, H represent your hypotheses, each of which can be either true or false.

- A: the murder happened during the night.

- B: the murder happened during the day.
- C: Omar was asleep during the night.
- D: Michael was asleep during the day.
- E: Mykola was asleep between 3am and 11am.
- F: Omar is the murderer.
- G: Michael is the murderer.
- H: Mykola is the murderer.

These hypotheses are not independent. For instance:

1. If A is false, then B is true.
2. If B and D are true, then G is false.

We would like a good way to think about these dependencies. Our tool will be *propositional logic*.[1]

2.3 The symbolic language of propositional logic

Propositional logic is a kind of language, a bit like English or Chinese. It uses a very small alphabet of symbols. Here they are, labeled by their English names.

- $\wedge$ (AND)
- $\vee$ (OR)
- $\neg$ (NOT)
- $\leftrightarrow$ (IF AND ONLY IF or IS EQUIVALENT TO)
- $\rightarrow$ (IMPLIES)

A *Boolean formula* is a formula constructed using our propositions $A, B, C, ..., H$ together with the above symbols. For example, we can write

If A is false, then B is true

as the Boolean formula

$$\neg A \rightarrow B. \tag{2.1}$$

We can write

[1] Also called zero-th order or Boolean logic.

$$\text{If } B \text{ and } D \text{ are true, then } G \text{ is false}$$

as

$$(B \wedge D) \to \neg G. \tag{2.2}$$

Exercise 1. *Can you write down some more logical dependencies among A, B, C, D, E, F, G, H as Boolean formulas?*

Boolean grammar We do not allow arbitrary strings of Boolean symbols when making Boolean formulas. For example,

$$A \vee \vee \vee \neg$$

is not a Boolean formula: it is just nonsense. There is a list of rules for making well-formed Boolean formulas, a bit like a the grammar rules of a language. However, we will not spell out explicitly these rules explicitly. Instead, we trust that you will learn to do this intuitively through practice. If you find this confusing, ask us for some more examples.

Order of operations. Consider the formula

$$A \wedge B \vee C.$$

This formula is ambiguous! Does it mean 'Either A and B are both true, or C is true'? Or does does it mean 'A is true and either B or C' is true'?

To make it unambiguous, we add parenthesis. We write

$$(A \wedge B) \vee C$$

when we mean 'Either A and B are both true, or C is true', and we write

$$A \wedge (B \vee C)$$

when we mean 'A is true and either B or C' is true'. Now consider the formula

$$\neg A \wedge B.$$

Does it mean 'A is false and B is true' or 'It is not true that both A and B are true'? Here there is no ambiguity: by convention, we first apply $\neg$, then $\wedge$, so that the first reading is correct.

More generally, the conventional order in which to read a Boolean formula is: first $\neg$, then $\wedge$ and $\vee$, and finally $\to$ and $\leftrightarrow$. Any remaining ambiguities must be resolved by parentheses.

2.4 Truth tables

In Boolean logic, we forget all the irrelevant details of a proposition like A or D: *Who was murdered? Who is Michael? Why was he sleeping during the day?*. All we care about is whether A or D are TRUE or FALSE.

We write $A = T$ to indicate 'A is true' and $A = F$ to indicate 'A is false'. We call this the *truth value*, or simply the *value*, of the proposition.

Boolean formulas like

$$\neg A, \quad A \wedge B \quad A \vee B$$

can also be either true or false, depending on whether A and B are true or false.

We can describe the conditions under which they are true or false using 'truth tables'. The left-hand columns below indicate possible values of A, B, whereas the right-hand columns indicate the resulting value of the formula.

A	$\neg A$
F	T
T	F

A	B	$A \wedge B$	$A \vee B$
F	F	F	F
F	T	F	T
T	F	F	T
T	T	T	T

A	B	$A \to B$	$A \leftrightarrow B$
F	F	T	T
F	T	T	F
T	F	F	F
T	T	T	T

Every Boolean formula thus determines a *truth function* whose input is a collection of values (either T or F) for each proposition appearing in the formula, and whose output is either T or F.

Equivalent formulas Just as in English and Chinese, we may formulate the same sentence in multiple ways, there can be many different Boolean formulas which express the same statement. More precisely, we will say two Boolean formulas A and B are *logically equivalent* if they determine the same truth function. In this case, we will write

$$A \equiv B$$

The most immediate example of this is something common in natural language too: the double negative. If I say I am "not not cold" it means I am cold. Likewise, for any proposition A, we have

$$\neg(\neg A) \equiv A.$$

As a slightly less trivial example, we may observe that we can write $A \to B$ by referring only to "simpler" symbols $\neg$ and $\vee$:

$$A \to B \equiv (\neg A) \vee B.$$

That is, A implying B is the same thing as saying that either A is false or B is true (or both).

The following equivalences are common and important enough that we record them here:

1. Symmetry:
   - $A \wedge B \equiv B \wedge A$
   - $A \vee B \equiv B \vee A$
2. Associativity:
   - $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$
   - $A \vee (B \vee C) \equiv (A \vee B) \vee C$
3. Distributivity:
   - $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$
   - $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$
4. de Morgan's laws:
   - $\neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$
   - $\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$
5. Contrapositive: $A \to B \equiv (\neg B) \to (\neg A)$

Exercise 2. *Check that the above Boolean formulas are logically equivalent by inspecting their truth tables.*

Warning 1. *In general we need to be careful to distinguish between two forms of equivalence:*

$$A \leftrightarrow B \tag{2.3}$$

and

$$\phi \equiv \psi \tag{2.4}$$

The expression (2.3) *is a Boolean formula stating that the propositions A and B are equivalent. The expression* (2.4)*, on the other hand, is not a Boolean formula (the symbol $\equiv$ is not part of the Boolean alphabet!). It is a proposition which states that the Boolean formulae ϕ and ψ are logically equivalent.*