

Agenda

- 1. Precis feedback this week**
- 2. A detailed, 2-level model of student achievement**
- 3. Hierarchical versus expanded single-equation representations**
- 4. Mixed effects in R: brms (and lme4)**

Building a two-level model

Linear model of test scores

Scores depend on students' age, sex, race, and ethnicity.

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \textit{Age}_i + \beta_{2k} \textit{Female}_i + \beta_{3k} \textit{Black}_i$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

Random intercept

Let each class have its own overall performance.

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

Random slopes

Let the “effect” of students’ characteristics differ from classroom to classroom

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \eta_{3k}$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \textit{Age}_i + \beta_{2k} \textit{Female}_i + \beta_{3k} \textit{Black}_i$$

Intercept predictors

Does each classes' average score depend on classroom features?
(number of students and teacher's experience)

$$\beta_{0k} = \gamma_{00} + \gamma_{01} \textit{Size}_k + \gamma_{02} \textit{Exp}_i + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \eta_{3k}$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \eta_{3k}$$

Slope predictors

Does the expected score difference between older and younger students depend on the size of the class?

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

Group-level covariates on slope and intercept

In most cases, covariates on slope models should be included in intercept model as well

$$\begin{aligned} \beta_{0k} = & \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i + \\ & \gamma_{03} \text{TeacherFemale}_k + \\ & \gamma_{04} \text{TeacherBlack}_k + \\ & \gamma_{05} \text{PropBlack}_k + \eta_{0k} \end{aligned}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21} \text{TeacherFemale}_k + \eta_{2k}$$

$$\begin{aligned} \beta_{3k} = & \gamma_{30} + \gamma_{31} \text{TeacherBlack}_k + \\ & \gamma_{32} \text{PropBlack}_k + \eta_{3k} \end{aligned}$$

Covariance model

Covariance structure

Covariance of the level-one coefficients modelled using variance Φ and correlations P .

$$\begin{bmatrix} \eta_{0k} \\ \eta_{1k} \\ \eta_{2k} \\ \eta_{3k} \end{bmatrix} \sim \text{MVNorm} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \Phi P \Phi \right)$$

$$\Phi = \begin{bmatrix} \phi_0 & 0 & 0 & 0 \\ 0 & \phi_1 & 0 & 0 \\ 0 & 0 & \phi_2 & 0 \\ 0 & 0 & 0 & \phi_3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{01} & 1 & \rho_{12} & \rho_{13} \\ \rho_{02} & \rho_{12} & 1 & \rho_{23} \\ \rho_{03} & \rho_{13} & \rho_{23} & 1 \end{bmatrix}$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

Building a two-level model

Complete model

The full two-level model describes variation in test scores as a function of student- and class-level covariates.

Residual variation is attributed to within-class (σ) and between-class (ϕ) differences.

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \textit{Age}_i + \beta_{2k} \textit{Female}_i + \beta_{3k} \textit{Black}_i$$

$$\begin{aligned} \beta_{0k} = & \gamma_{00} + \gamma_{01} \textit{Size}_k + \gamma_{02} \textit{Exp}_i + \\ & \gamma_{03} \textit{TeacherFemale}_k + \\ & \gamma_{04} \textit{TeacherBlack}_k + \\ & \gamma_{05} \textit{PropBlack}_k + \eta_{0k} \end{aligned}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \textit{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21} \textit{TeacherFemale}_k + \eta_{2k}$$

$$\begin{aligned} \beta_{3k} = & \gamma_{30} + \gamma_{31} \textit{TeacherBlack}_k + \\ & \gamma_{32} \textit{PropBlack}_k + \eta_{3k} \end{aligned}$$

Two-level model: estimates

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\begin{aligned} \beta_{0k} = & \boxed{\gamma_{00}} + \boxed{\gamma_{01}} \text{Size}_k + \boxed{\gamma_{02}} \text{Exp}_i + \\ & \gamma_{03} \text{TeacherFemale}_k + \\ & \gamma_{04} \text{TeacherBlack}_k + \\ & \gamma_{05} \text{PropBlack}_k + \eta_{0k} \end{aligned}$$

$$\beta_{1k} = \boxed{\gamma_{10}} + \boxed{\gamma_{11}} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21} \text{TeacherFemale}_k + \eta_{2k}$$

$$\begin{aligned} \beta_{3k} = & \gamma_{30} + \gamma_{31} \text{TeacherBlack}_k + \\ & \gamma_{32} \text{PropBlack}_k + \eta_{3k} \end{aligned}$$

Expected score when
all covariates are zero

Baseline effect of class size

Baseline effect of
teacher experience

Baseline effect of age

Effect of class size on age
difference in scores

**90% credible
interval**

	Mean	90% credible interval	
γ_{00}	544.18	521.97	566.37
γ_{01}	-1.47	-2.02	-0.93
γ_{02}	0.23	-0.01	0.47
γ_{10}	-4.76	-5.87	-3.67
γ_{11}	0.10	-0.18	0.39

Two-level model: estimates

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\begin{aligned} \beta_{0k} = & \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i + \\ & \boxed{\gamma_{03}} \text{TeacherFemale}_k + \\ & \gamma_{04} \text{TeacherBlack}_k + \\ & \gamma_{05} \text{PropBlack}_k + \eta_{0k} \end{aligned}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \boxed{\gamma_{20}} + \boxed{\gamma_{21}} \text{TeacherFemale}_k + \eta_{2k}$$

$$\begin{aligned} \beta_{3k} = & \gamma_{30} + \gamma_{31} \text{TeacherBlack}_k + \\ & \gamma_{32} \text{PropBlack}_k + \eta_{3k} \end{aligned}$$

Baseline gender difference
in teacher effectiveness

Expected score difference female
versus male students with male teacher

Expected score difference for female
students with female versus male teachers

	Mean	90% credible interval	
γ_{03}	-12.73	-34.86	9.44
γ_{20}	10.61	-7.43	28.56
γ_{21}	-0.61	-18.72	17.43

Two-level model: estimates

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i +$$

$$\gamma_{03} \text{TeacherFemale}_k +$$

$$\boxed{\gamma_{04}} \text{TeacherBlack}_k +$$

$$\boxed{\gamma_{05}} \text{PropBlack}_k + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21} \text{TeacherFemale}_k + \eta_{2k}$$

$$\beta_{3k} = \boxed{\gamma_{30}} + \boxed{\gamma_{31}} \text{TeacherBlack}_k + \boxed{\gamma_{32}} \text{PropBlack}_k + \eta_{3k}$$

Race difference in teacher effectiveness (white students)

Effect of class racial composition (white students)

Baseline race difference in student scores

Effect of having a Black teacher for Black (versus white) students

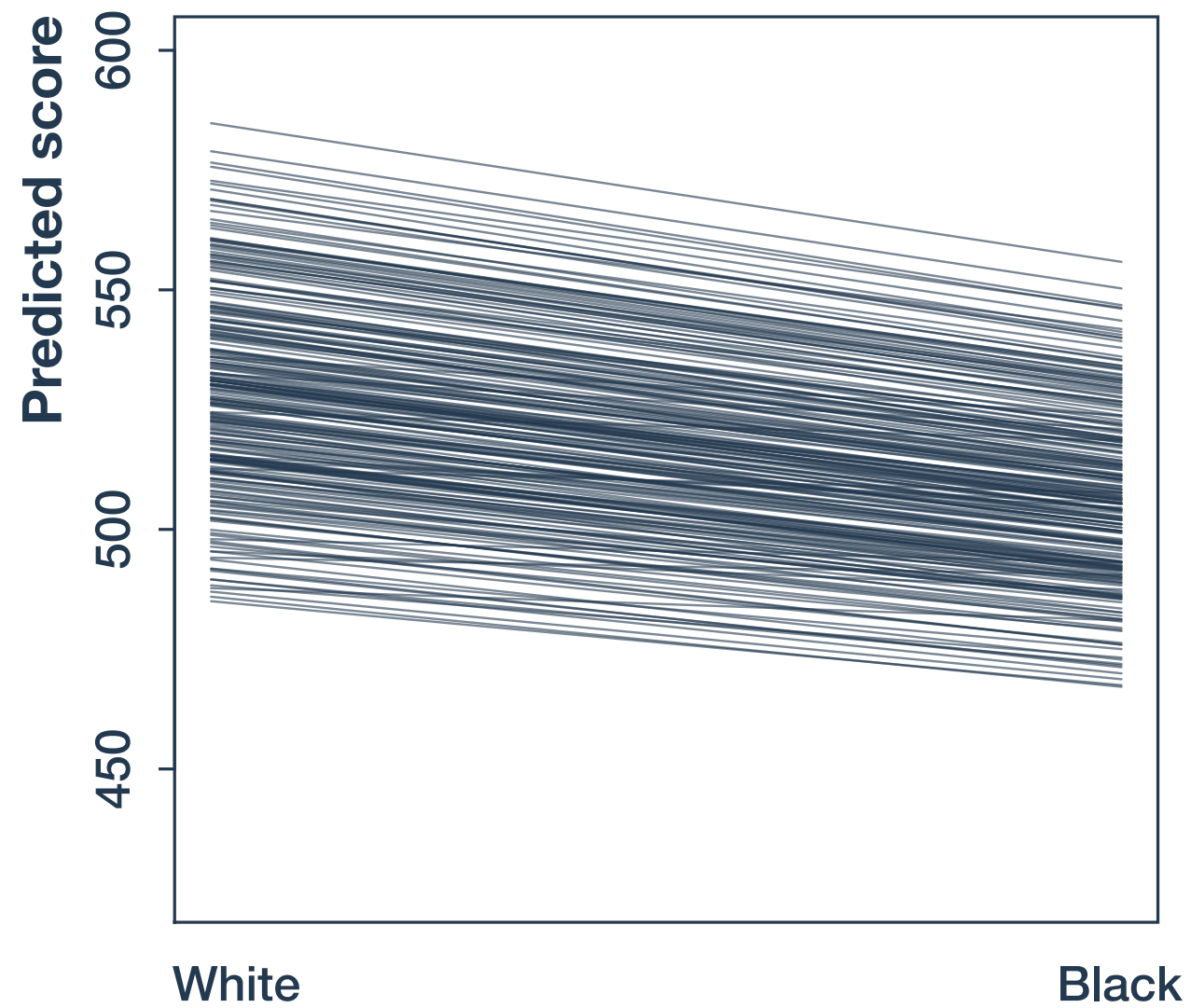
Effect of class racial composition for Black (versus white) students

90% credible interval

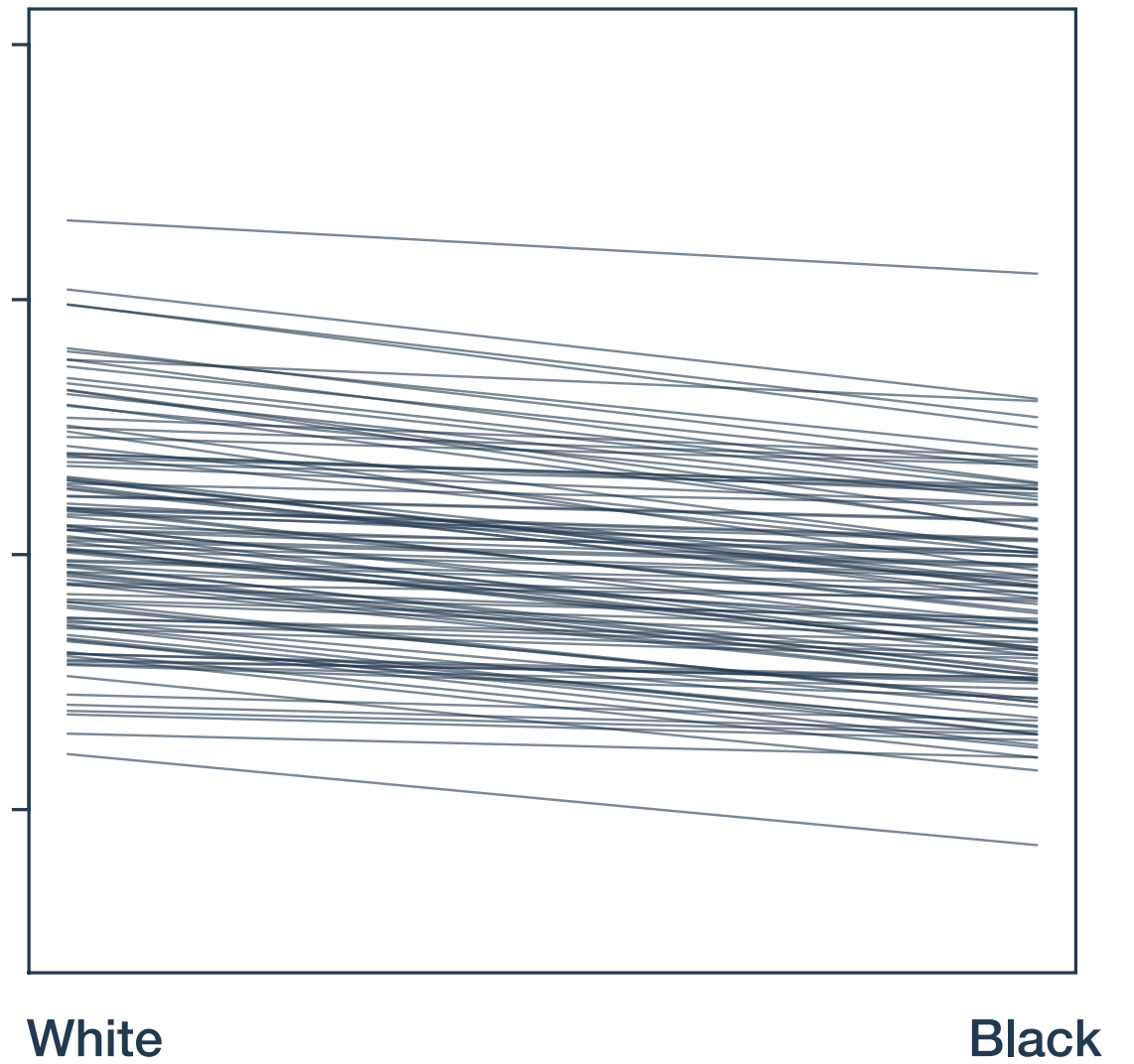
	Mean	90% credible interval	
γ_{04}	-7.96	-17.80	1.74
γ_{05}	-25.99	-38.44	-13.41
γ_{30}	-23.73	-29.22	-18.09
γ_{31}	13.97	4.37	23.78
γ_{32}	3.97	-10.53	18.72

Two-level model: estimates

White teachers



Black teachers



Expanded notation

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11}Size_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \gamma_{31}TeacherBlack_k + \gamma_{32}PropBlack_k + \eta_{3k}$$

$$\begin{aligned} \mu_{ik} = & (\gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \eta_{0k}) \\ & (\gamma_{10} + \gamma_{11}Size_k + \eta_{1k})Age_i \\ & (\gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k})Female_i \\ & (\gamma_{30} + \gamma_{31}TeacherBlack_k + \gamma_{32}PropBlack_k + \eta_{3k})Black_i \end{aligned}$$

Expanded notation

$$\begin{aligned}\mu_{ik} = & (\gamma_{00} + \gamma_{01}\text{Size}_k + \gamma_{02}\text{Exp}_i + \text{TeacherFemale}_k + \\ & \gamma_{04}\text{TeacherBlack}_k + \gamma_{05}\text{PropBlack}_k + \eta_{0k}) \\ & (\gamma_{10} + \gamma_{11}\text{Size}_k + \eta_{1k})\text{Age}_i \\ & (\gamma_{20} + \gamma_{21}\text{TeacherFemale}_k + \eta_{2k})\text{Female}_i \\ & (\gamma_{30} + \gamma_{31}\text{TeacherBlack}_k + \gamma_{32}\text{PropBlack}_k + \eta_{3k})\text{Black}_i\end{aligned}$$

$$\begin{aligned}S_{ik} = & \gamma_{00} + \gamma_{01}\text{Size}_k + \gamma_{02}\text{Exp}_i + \text{TeacherFemale}_k + \\ & \gamma_{04}\text{TeacherBlack}_k + \gamma_{05}\text{PropBlack}_k + \gamma_{10}\text{Age}_i \\ & \gamma_{11}\text{Age}_i\text{Size}_k + \gamma_{20}\text{Female}_i + \gamma_{21}\text{Female}_i\text{TeacherFemale}_k + \\ & \gamma_{30}\text{Black}_i + \gamma_{31}\text{Black}_i\text{TeacherBlack}_k + \gamma_{32}\text{Black}_i\text{PropBlack}_k + \\ & \eta_{0k} + \eta_{1k}\text{Age}_i + \eta_{2k}\text{Female}_i + \eta_{3k}\text{Black}_i + \varepsilon_i\end{aligned}$$

Expanded notation

Fixed effects

Explained variation in outcome variable.

Describes the way that outcome and predictor variables co-vary.

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i + \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Random effects

Unexplained variation in outcome variable.

Described in terms of individual variability and different types of group variability.

Expanded notation

Hierarchical

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11}Size_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \gamma_{31}TeacherBlack_k + \gamma_{32}PropBlack_k + \eta_{3k}$$

Expanded

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

Building an R formula

R packages like brms (Bayesian) and lme4 (frequentist) use a “formula” notation to specify multilevel models.

(extension of syntax for standard regressions)

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
    student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

Building an R formula

Outcome variable

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

Building an R formula

Global intercept
(included automatically)

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \boxed{\gamma_{00}} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

Building an R formula

Interactions (*) automatically include standalone terms

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
    student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i + \text{TeacherFemale}_k + \\ & \gamma_{04} \text{TeacherBlack}_k + \gamma_{05} \text{PropBlack}_k + \gamma_{10} \text{Age}_i \\ & \gamma_{11} \text{Age}_i \text{Size}_k + \gamma_{20} \text{Female}_i + \gamma_{21} \text{Female}_i \text{TeacherFemale}_k + \\ & \gamma_{30} \text{Black}_i + \gamma_{31} \text{Black}_i \text{TeacherBlack}_k + \gamma_{32} \text{Black}_i \text{PropBlack}_k + \\ & \eta_{0k} + \eta_{1k} \text{Age}_i + \eta_{2k} \text{Female}_i + \eta_{3k} \text{Black}_i + \varepsilon_i \end{aligned}$$

Building an R formula

Interactions (*) automatically include standalone terms

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
    student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

Building an R formula

Interactions (*) automatically include standalone terms

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
    student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

Building an R formula

Redundant terms (`student_re_black`)
are not added twice

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
    student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

Building an R formula

Random effects use pipe notation (|)

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

Building an R formula

Grouping elements after the pipe

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

Building an R formula

Random intercepts indicated
with constant (1)

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

Building an R formula

Random-slope variables
included in grouping expression

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$