

# Agenda

- 1. Relevant upcoming talks**
- 2. Generalized multilevel linear models (very briefly)**
- 3. Specifying mixed effects in R: brms (and lme4)**
- 4. Using brms**

# Upcoming talks

**Andrew Gelman**

*Resolving the replication crisis  
using multilevel modelling*

Friday, March 22 (*tomorrow!*)

1:00–2:00pm

Education Building, Room 129 (3700 McTavish St)

**Peter McMahan**

*The structure and consequences  
of scholarly review articles*

Wednesday, March 27

12:00–1:00pm

Leacock, Room 429

**Jaclyn Wong**

*Negotiating competing desires:  
How young professionals make  
career and family decisions*

Friday, March 29

12:30–2:00pm

Leacock, Room 738

# Generalized multilevel models

## Generalized multilevel linear models

Simply add a link function and change the outcome distribution.

E.g. modelling whether a student did better on the math test than the reading test ( $M_{ik}$ ).

**All coefficients and parameters are affected by link function**

$$M_{ik} \sim \text{Binomial}(1, p_{ik})$$

$$\text{logit}(p_{ik}) = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01} TExp_k + \gamma_{02} TFemale_k + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

Interpretation requires careful thinking about the ways that coefficients affect  $p_{ik}$ .

Interpreting *direction* of effect (positive vs. negative) is still straightforward. E.g. a strong positive estimate on  $\gamma_{02}$  would suggest that female teachers do a relatively better job of teaching mathematics.

# Specifying models in R

## Linear model of test scores

Scores depend on students' age, sex, race, and ethnicity.

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

## R formula

```
student_reading_score ~ student_age_s +  
student_female + student_re_black
```

# Specifying models in R

## Random slope and intercepts

Each coefficient varies from classroom to classroom

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \eta_{3k}$$

### R formula

```
student_reading_score ~ student_age_s +  
student_female + student_re_black +  
(1 + student_age_s + student_female +  
student_re_black | teacher_id)
```

# Level-2 covariates

## Hierarchical

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11}Size_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \gamma_{31}TeacherBlack_k + \gamma_{32}PropBlack_k + \eta_{3k}$$

## Expanded

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

# Building an R formula

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
    student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

# Building an R formula

## Outcome variable

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$



# Building an R formula

Global intercept  
(included automatically)

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \boxed{\gamma_{00}} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

# Building an R formula

Interactions (\*) automatically include standalone terms

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i + \text{TeacherFemale}_k + \\ & \gamma_{04} \text{TeacherBlack}_k + \gamma_{05} \text{PropBlack}_k + \gamma_{10} \text{Age}_i \\ & \gamma_{11} \text{Age}_i \text{Size}_k + \gamma_{20} \text{Female}_i + \gamma_{21} \text{Female}_i \text{TeacherFemale}_k + \\ & \gamma_{30} \text{Black}_i + \gamma_{31} \text{Black}_i \text{TeacherBlack}_k + \gamma_{32} \text{Black}_i \text{PropBlack}_k + \\ & \eta_{0k} + \eta_{1k} \text{Age}_i + \eta_{2k} \text{Female}_i + \eta_{3k} \text{Black}_i + \varepsilon_i \end{aligned}$$

# Building an R formula

Interactions (\*) automatically include standalone terms

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
    student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

# Building an R formula

Interactions (\*) automatically include standalone terms

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
    student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

# Building an R formula

Redundant terms (`student_re_black`)  
are not added twice

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
    student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

# Building an R formula

Random effects use pipe notation (|)

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

# Building an R formula

## Grouping elements after the pipe

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$

# Building an R formula

Random intercepts indicated  
with constant (1)

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i \end{aligned}$$



# Building an R formula

Random-slope variables  
included in grouping expression

```
student_reading_score ~  
  student_age_s*class_size_c + teacher_exper_c +  
  student_female*teacher_female +  
  student_re_black*teacher_re_black +  
  student_re_black*class_prop_black +  
  (1 + student_age_s + student_female +  
   student_re_black | teacher_id)
```

$$\begin{aligned} S_{ik} = & \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k + \\ & \gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i \\ & \gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k + \\ & \gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k + \\ & \eta_{0k} + \boxed{\eta_{1k}Age_i} + \boxed{\eta_{2k}Female_i} + \boxed{\eta_{3k}Black_i} + \varepsilon_i \end{aligned}$$