Agenda

1. **Course evaluations**
   https://horizon.mcgill.ca/pban1/twbkwbis.P_WWWLogin?ret_code=f
   (Minerva > Mercury)
2. **Models of time**
3. **Nesting time within students**
4. **Example in R**

Models of time

Common models of time

Autoregression models

Model outcome at time t as a function of covariates and outcome at time t–1.
$y_t = y_{t-1} + \beta X_{t-1} + \varepsilon_t$
$y_{\Delta t} = \beta X_{t-1} + \varepsilon_t$

Survival / event-history models

Model the timing of a one-time event (graduation, job acquisition, death).
$\lambda(t \mid X) = \lambda_0 \exp(\beta X)$

Age–period–cohort models

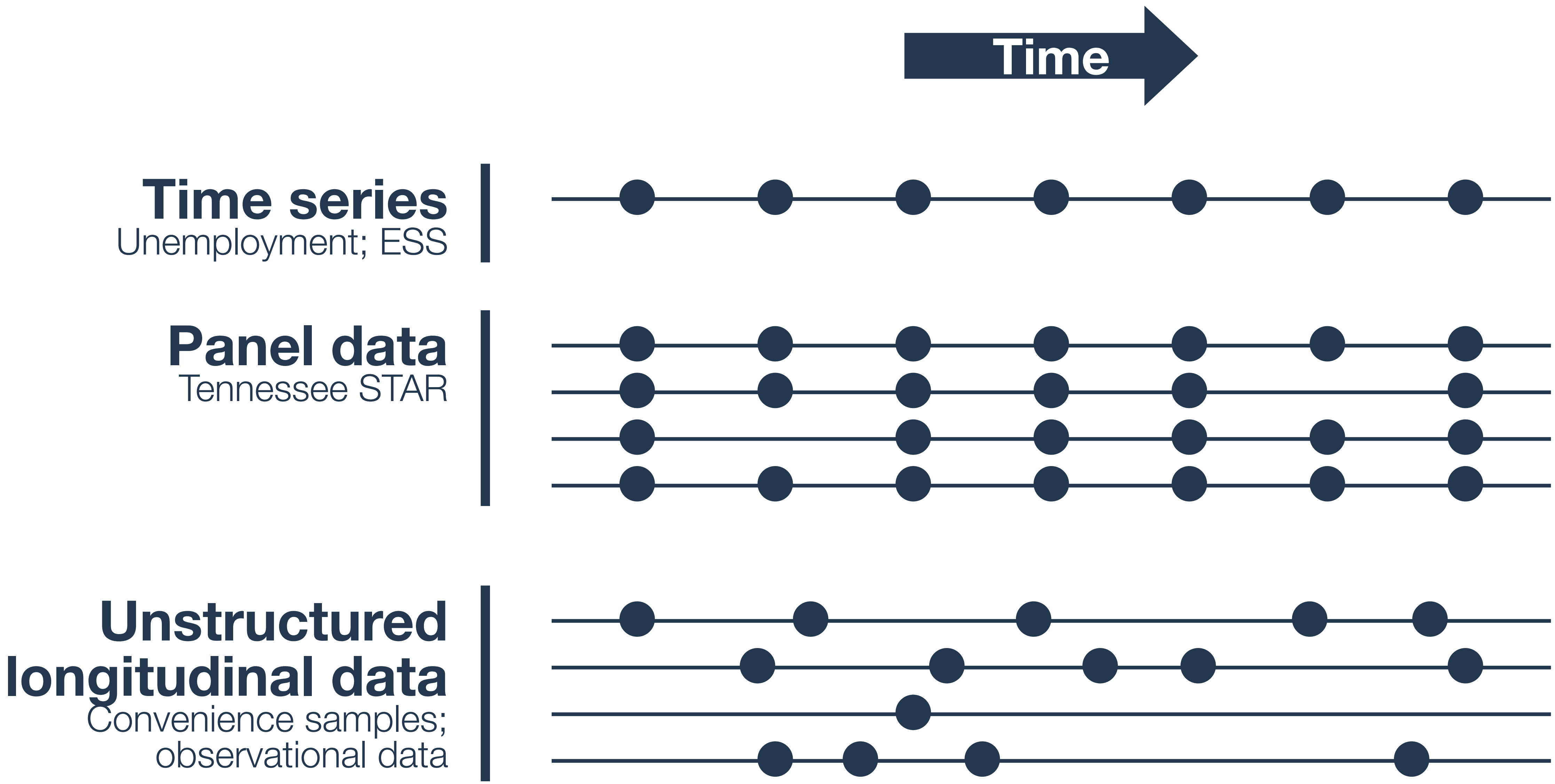
Demographic models aiming to differentiate between effects of individuals' *age*, the date of measurement (*period*), and birth *cohort*.

***Ad hoc* models**

Countless context-specific ways to model a randomly varying or functionally defined effect of time on outcomes.

Models of time

Common temporal data structures



Student scores over time

Correlation between scores from the same student.

	1985	1986	1987	1988
Student 1	540	531	564	563
Student 2	479	487	505	510
Student 3	503	505	501	503
Student 4	461	471	—	—
Student 5	525	—	506	541
	…	…	…	…

Student scores over time

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	…	…	…	…

Scores tend to increase over time.

Student scores over time

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Student 1	540	531	564	563
Student 2	479	487	505	510
Student 3	503	505	501	503
Student 4	461	471	—	—
Student 5	525	—	506	541
	…	…	…	…

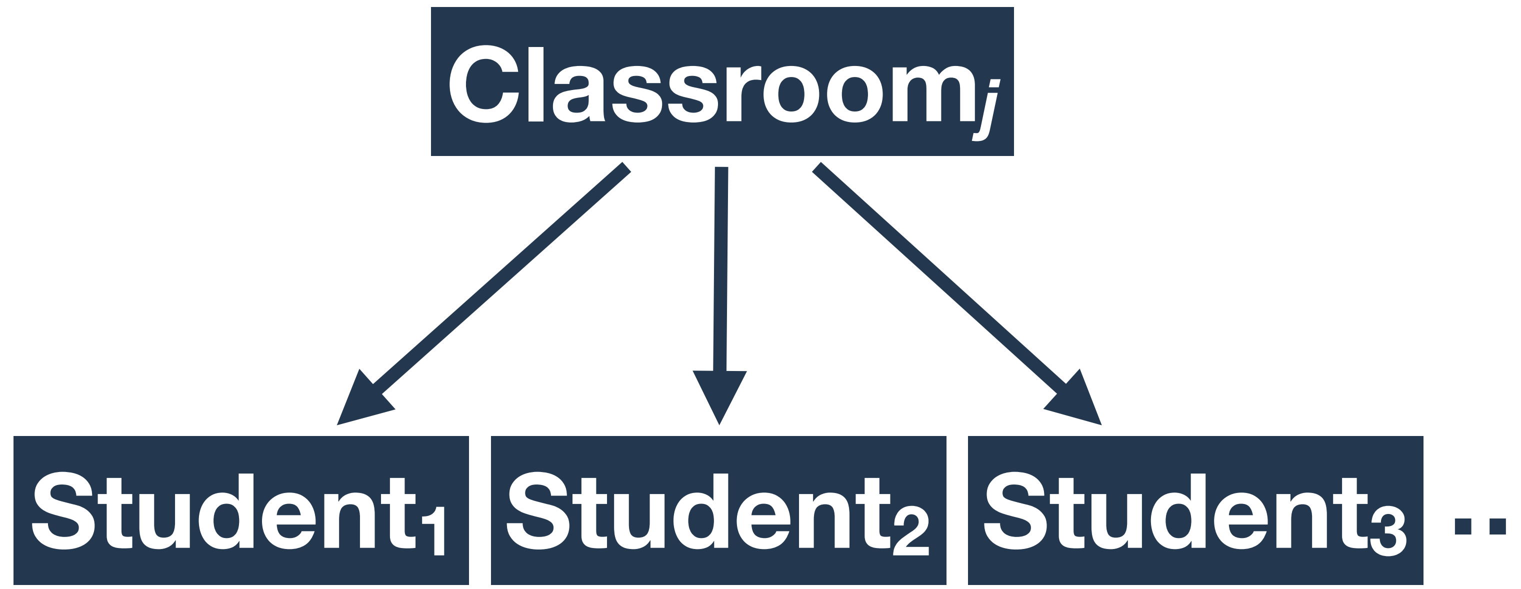
→

Student	Year	Score
1	1985	540
1	1986	531
1	1987	564
1	1988	563
2	1985	479
2	1986	487
2	1987	505
2	1988	510
3	1985	503
…	…	…

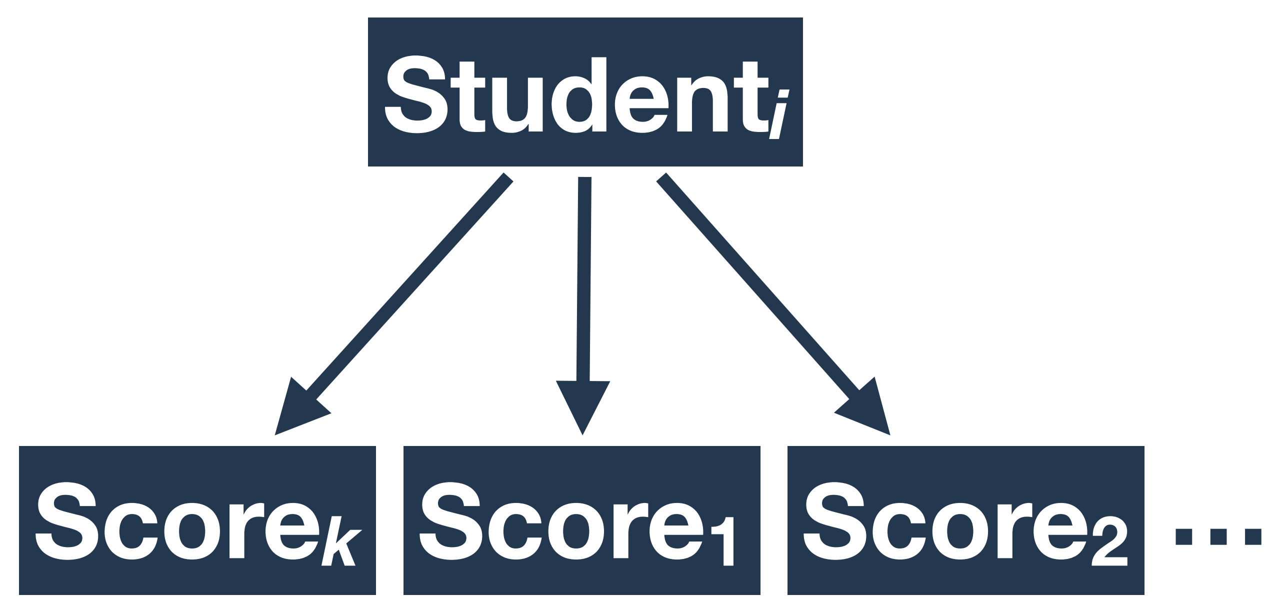
Student random effects

Students groups into classrooms

Single year



Test scores grouped by student



Student random effects

Score for student i at time t.

$$\boxed{S_{ti}} \sim \mathrm{Norm}(\mu_{ti}, \sigma)$$

Average score for student i.

$$\mu_{ti} = \boxed{\beta_{0i}} + \beta_1 CSize_{ti}$$

$$\beta_{0i} = \boxed{\gamma_{00}} + \eta_{0i}$$

Average score across all students.

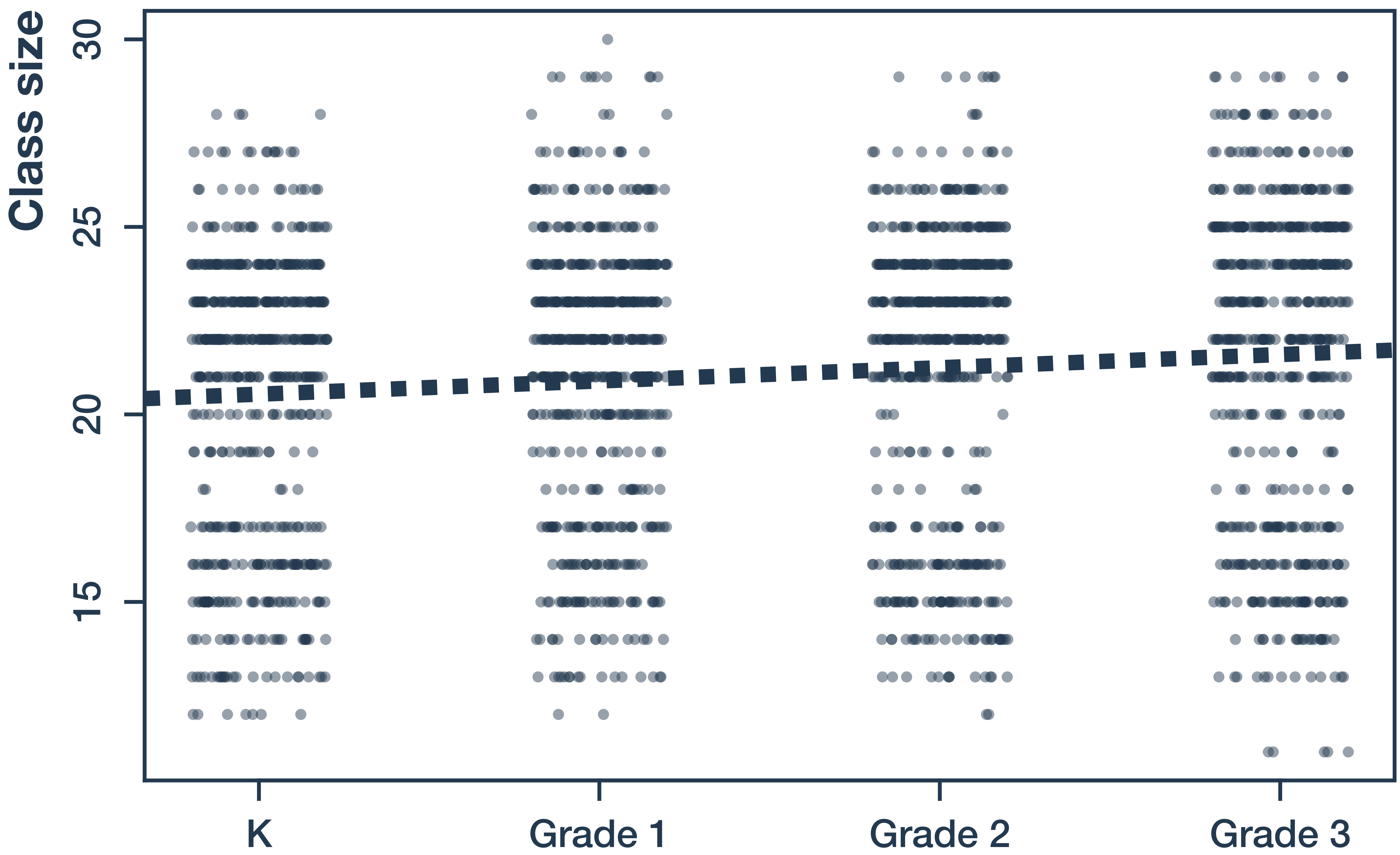
$$\eta_{0i} \sim \mathrm{Norm}(0, \phi_0)$$

Student random effects

$$
\begin{aligned}
S_{ti} &\sim \text{Norm}(\mu_{ti}, \sigma) \\
\mu_{ti} &= \beta_{0i} + \beta_1 CSize_{ti} \\
\beta_{0i} &= \gamma_{00} + \eta_{0i} \\
\\
\eta_{0i} &\sim \text{Norm}(0, \phi_0) \\
\\
\gamma_{00} &\sim \text{Norm}(500, 100) \\
\beta_1 &\sim \text{Norm}(0, 50) \\
\sigma &\sim \text{HalfCauchy}(0, 50) \\
\phi_0 &\sim \text{HalfCauchy}(0, 50)
\end{aligned}
$$

	Mean	***90% credible interval***	
$\boldsymbol{\gamma_{00}}$	547.06	544.05	549.94
$\boldsymbol{\beta_1}$	1.23	0.58	1.87
$\boldsymbol{\sigma}$	56.94	54.97	58.93
$\boldsymbol{\phi_0}$	33.91	30.16	37.60

Student random effects



Linear time trend

$$
\begin{aligned}
S_{ti} &\sim \mathrm{Norm}(\mu_{ti}, \sigma) \\
\mu_{ti} &= \beta_{0i} + \beta_{1i} Year_{ti} + \beta_2 CSize_{ti} \\
\beta_{0i} &= \gamma_{00} + \eta_{0i} \\
\beta_{1i} &= \gamma_{10} + \eta_{1i}
\end{aligned}
$$

$$
[\eta_{0i}, \eta_{1i}] \sim \mathrm{MVNorm}([0, 0], \Phi, R)
$$

Linear time trend

$$S_{ti} \sim \mathrm{Norm}(\mu_{ti}, \sigma)$$
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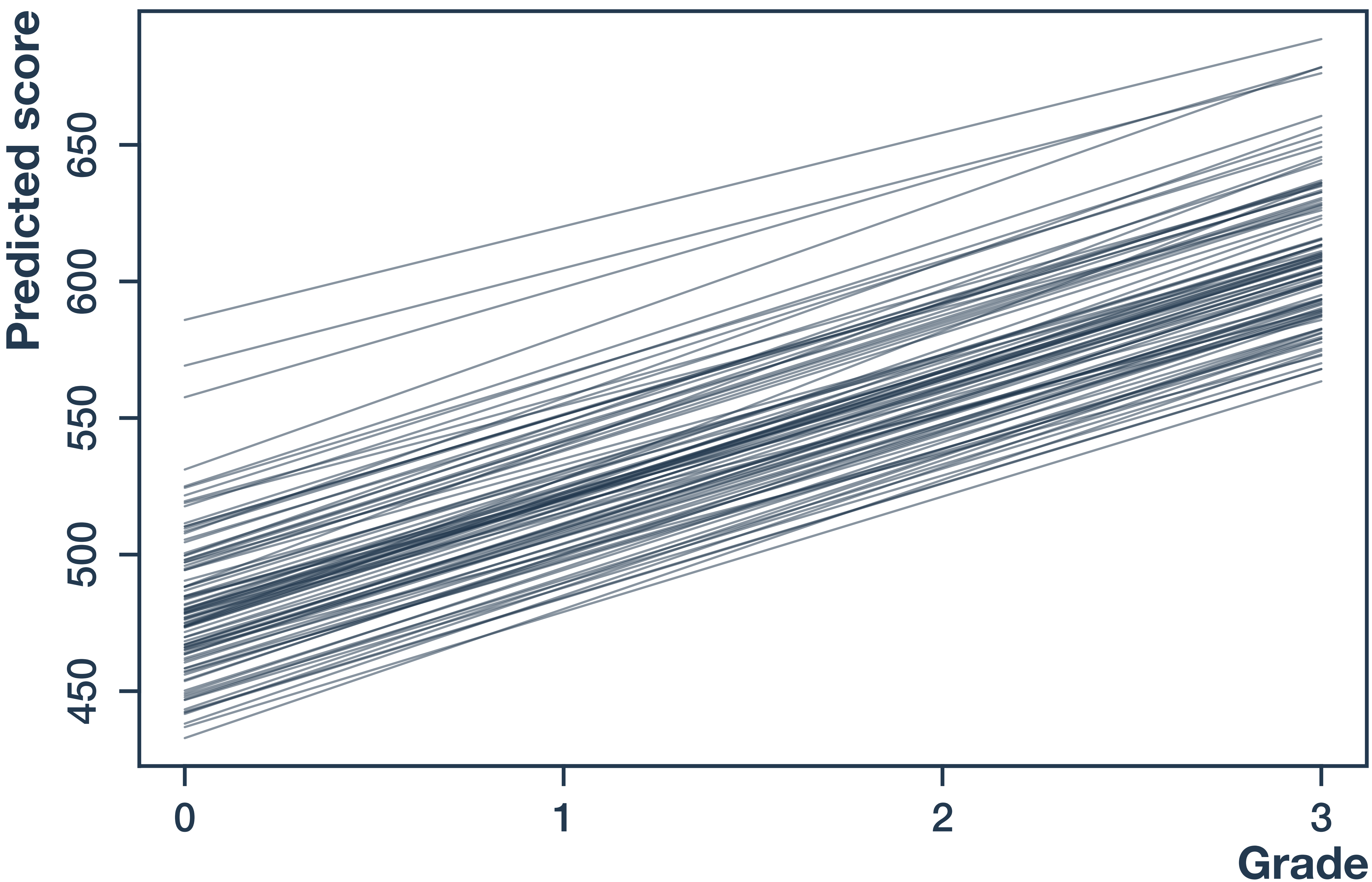
$$[\eta_{0i}, \eta_{1i}] \sim \mathrm{MVNorm}([0, 0], \Phi, R)$$

$$\gamma_{00} \sim \mathrm{Norm}(500, 100)$$
$$\gamma_{10} \sim \mathrm{Norm}(0, 50)$$
$$\beta_2 \sim \mathrm{Norm}(0, 50)$$
$$\sigma \sim \mathrm{HalfCauchy}(0, 50)$$
$$\phi_0, \phi_1 \sim \mathrm{HalfCauchy}(0, 50)$$
$$R \sim LKJ(2, 2)$$

	Mean	***90% credible interval***	
$\boldsymbol{\gamma_{00}}$	484.64	481.84	487.39
$\boldsymbol{\gamma_{10}}$	42.88	41.75	44.02
$\boldsymbol{\beta_2}$	–0.33	–0.73	0.070
$\boldsymbol{\sigma}$	24.27	23.27	25.32
$\boldsymbol{\phi_0}$	38.25	35.84	40.65
$\boldsymbol{\phi_1}$	8.89	7.17	10.56
$\boldsymbol{\rho_{01}}$	–0.37	–0.48	–0.23

Linear time trend



Quadratic time trend

$$
\begin{aligned}
S_{ti} &\sim \text{Norm}(\mu_{ti}, \sigma) \\
\mu_{ti} &= \beta_{0i} + \beta_{1i} Year_{ti} + \beta_{2i} Year_{ti}^2 + \beta_3 CSize_{ti} \\
\beta_{0i} &= \gamma_{00} + \eta_{0i} \\
\beta_{1i} &= \gamma_{10} + \eta_{1i} \\
\beta_{2i} &= \gamma_{20} + \eta_{2i}
\end{aligned}
$$

$$[\eta_{0i}, \eta_{1i}, \eta_{2i}] \sim \text{MVNorm}([0, 0, 0], \Phi, R)$$

Quadratic time trend

$$
\begin{aligned}
S_{ti} &\sim \text{Norm}(\mu_{ti}, \sigma) \\
\mu_{ti} &= \beta_{0i} + \beta_{1i} Year_{ti} + \beta_{2i} Year_{ti}^2 + \beta_3 CSize_{ti} \\
\beta_{0i} &= \gamma_{00} + \eta_{0i} \\
\beta_{1i} &= \gamma_{10} + \eta_{1i} \\
\beta_{2i} &= \gamma_{20} + \eta_{2i}
\end{aligned}
$$

$$
[\eta_{0i}, \eta_{1i}, \eta_{2i}] \sim \text{MVNorm}([0, 0, 0], \Phi, R)
$$

$$
\begin{aligned}
\gamma_{00} &\sim \text{Norm}(500, 100) \\
\gamma_{10} &\sim \text{Norm}(0, 50) \\
\gamma_{20} &\sim \text{Norm}(0, 50) \\
\beta_3 &\sim \text{Norm}(0, 50) \\
\sigma &\sim \text{HalfCauchy}(0, 50) \\
\phi_0, \phi_1, \phi_2 &\sim \text{HalfCauchy}(0, 50) \\
R &\sim LKJ(2, 3)
\end{aligned}
$$

	Mean	*90% credible interval*	
γ_{00}	482.32	479.37	485.30
γ_{10}	49.28	45.77	52.88
γ_{20}	–2.10	–3.14	–1.09
β_3	–0.36	–0.74	0.02
σ	21.91	20.76	23.04
ϕ_0	40.70	38.16	43.43
ϕ_1	31.48	26.35	36.38
ϕ_2	7.29	5.62	8.96
ρ_{01}	–0.45	–0.55	–0.35
ρ_{02}	0.40	0.25	0.53
ρ_{12}	–0.98	–1.00	–0.96

Quadratic time trend