Agenda

1. Priors on covariance matrices
2. Comparing correlated and independent models
3. Comparing to unpooled model
4. Estimating random-slopes models in R

Priors on covariance

Predicting reading score with random intercepts and random slopes

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} Age_i$$

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\begin{bmatrix} \eta_{0k} \\ \eta_{1k} \end{bmatrix} \sim \text{MVNorm}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Phi \right)$$

Φ is a matrix of variance and covariance terms

$$\Phi = \begin{bmatrix} \phi_0^2 & \phi_{01} \\ \phi_{01} & \phi_1^2 \end{bmatrix}$$

Priors on covariance

What is a reasonable prior for Φ?

Not every matrix is a covariance matrix.

$$\Phi = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \Rightarrow \mathrm{Cor}(\eta_0, \eta_1) = 1.5$$

Seemingly uninformative priors can be surprisingly restrictive.
E.g. using an inverse-Wishart distribution induces a dependency between correlations and standard deviations.

Priors on covariance

LKJ correlation prior

Decompose the covariance matrix into matrices of standard deviations and correlations.

Lewandowski, Daniel, Dorota Kurowicka, and Harry Joe. "Generating Random Correlation Matrices Based on Vines and Extended Onion Method." *Journal of Multivariate Analysis* 100, no. 9 (October 1, 2009)

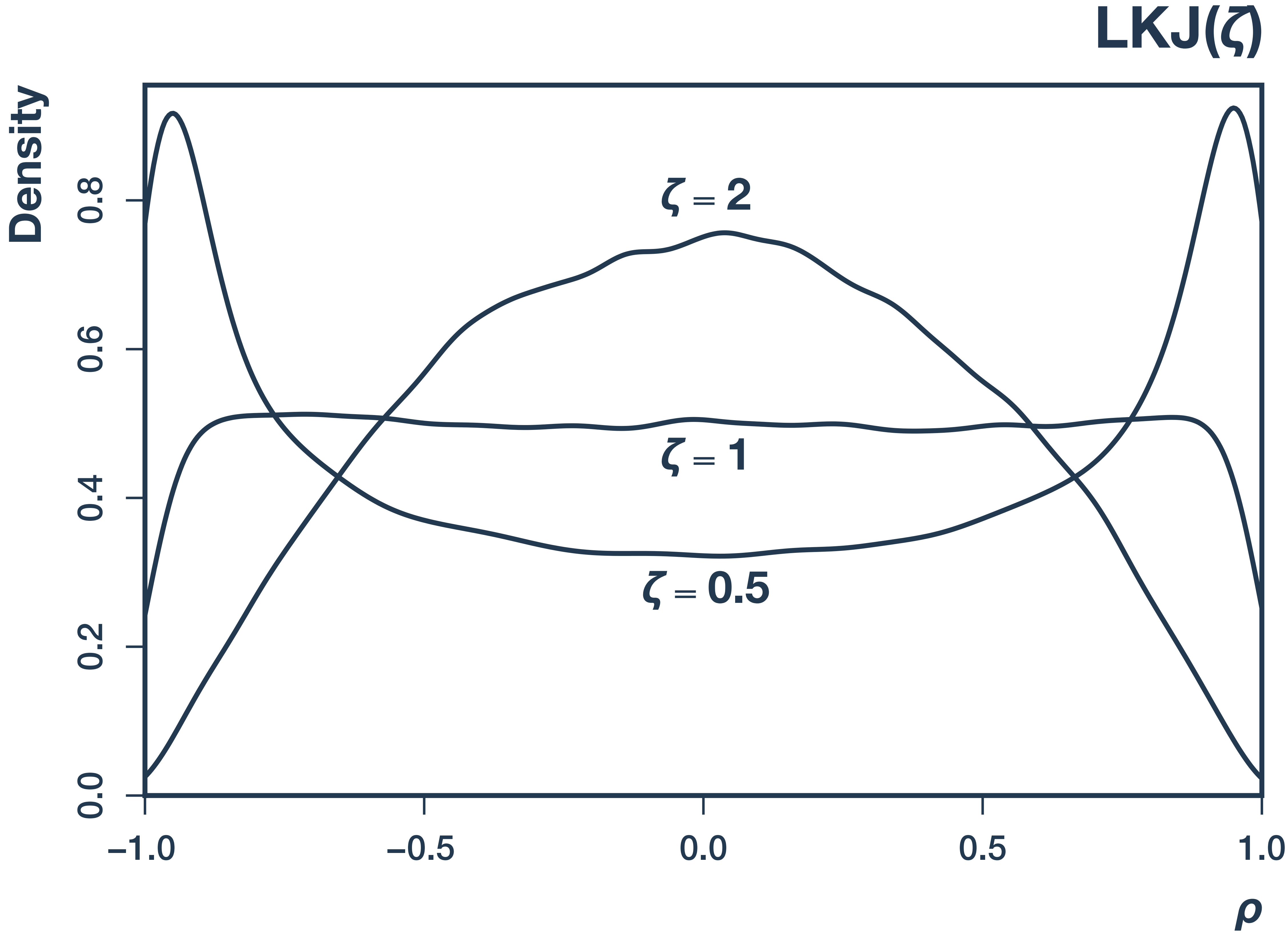
$$\begin{bmatrix} \phi_0^2 & \phi_{01} \\ \phi_{01} & \phi_1^2 \end{bmatrix} = \begin{bmatrix} \phi_0^2 & \phi_0\phi_1\rho_{01} \\ \phi_0\phi_1\rho_{01} & \phi_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{bmatrix} \begin{bmatrix} 1 & \rho_{01} \\ \rho_{01} & 1 \end{bmatrix} \begin{bmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{bmatrix}$$

ϕ_0 and ϕ_1 are the standard deviations of η_0 and η_1, respectively (priors for these are straightforward).

The correlation matrix describes correlations for every pair of variables (in this case only ρ_{01}).

Priors on covariance



Full model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$
$$\mu_{ik} = \beta_{0k} + \beta_{1k} Age_i$$

Student-level linear model

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$
$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

Class-level linear models

$$\begin{bmatrix} \eta_{0k} \\ \eta_{1k} \end{bmatrix} \sim \text{MVNorm}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Phi \right)$$

Joint distribution of class-level random effects

$$\Phi = \begin{bmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{bmatrix} R \begin{bmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{bmatrix}$$

Covariance decomposition
(standard deviation and correlation)

$$\sigma \sim \text{Unif}(0, 100)$$
$$\gamma_{00} \sim \text{Norm}(500, 100)$$
$$\gamma_{10} \sim \text{Norm}(0, 20)$$
$$\phi_0 \sim \text{Unif}(0, 100)$$
$$\phi_1 \sim \text{Unif}(0, 100)$$
$$R \sim \text{LKJ}(2, 2)$$

Priors

Full model

$$
\boxed{
\begin{aligned}
S_{ik} &\sim \text{Norm}(\mu_{ik}, \sigma) \\
\mu_{ik} &= \beta_{0k} + \beta_{1k} Age_i \\
\\
\beta_{0k} &= \gamma_{00} + \eta_{0k} \\
\beta_{1k} &= \gamma_{10} + \eta_{1k}
\end{aligned}
}
$$

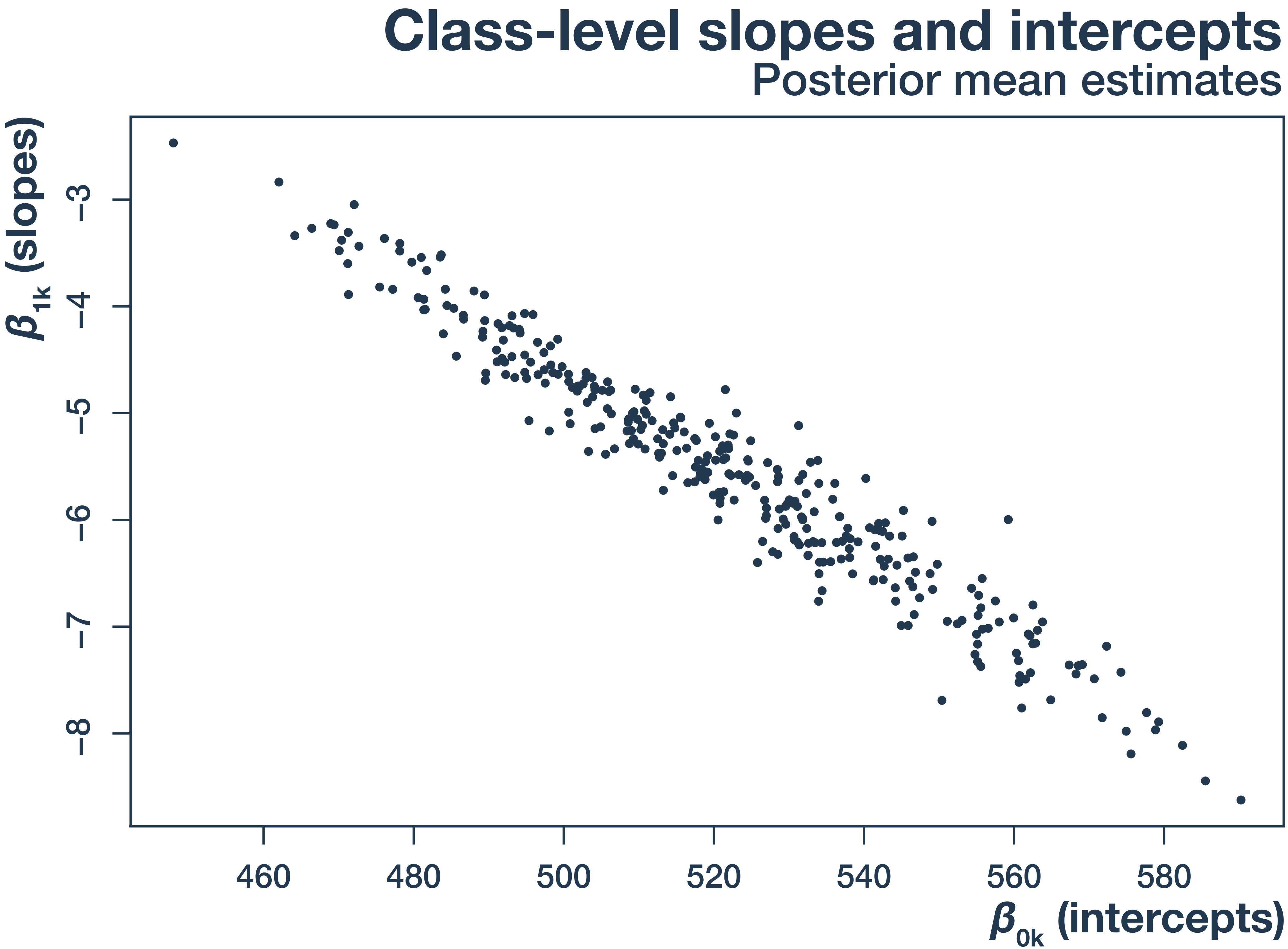
The linear model is the part that social scientists care most about.

$$
\begin{bmatrix} \eta_{0k} \\ \eta_{1k} \end{bmatrix} \sim \text{MVNorm}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Phi \right)
$$

$$
\Phi = \begin{bmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{bmatrix} R \begin{bmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{bmatrix}
$$

$$
\begin{aligned}
\sigma &\sim \text{Unif}(0, 100) \\
\gamma_{00} &\sim \text{Norm}(500, 100) \\
\gamma_{10} &\sim \text{Norm}(0, 20) \\
\phi_0 &\sim \text{Unif}(0, 100) \\
\phi_1 &\sim \text{Unif}(0, 100) \\
R &\sim \text{LKJ}(2, 2)
\end{aligned}
$$

Correlation of coefficients



Correlated versus independent

Correlated random effects

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$
$$\mu_{ik} = \beta_{0k} + \beta_{1k} Age_i$$

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$
$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\begin{bmatrix} \eta_{0k} \\ \eta_{1k} \end{bmatrix} \sim \text{MVNorm}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Phi \right)$$

Independent random effects

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$
$$\mu_{ik} = \beta_{0k} + \beta_{1k} Age_i$$

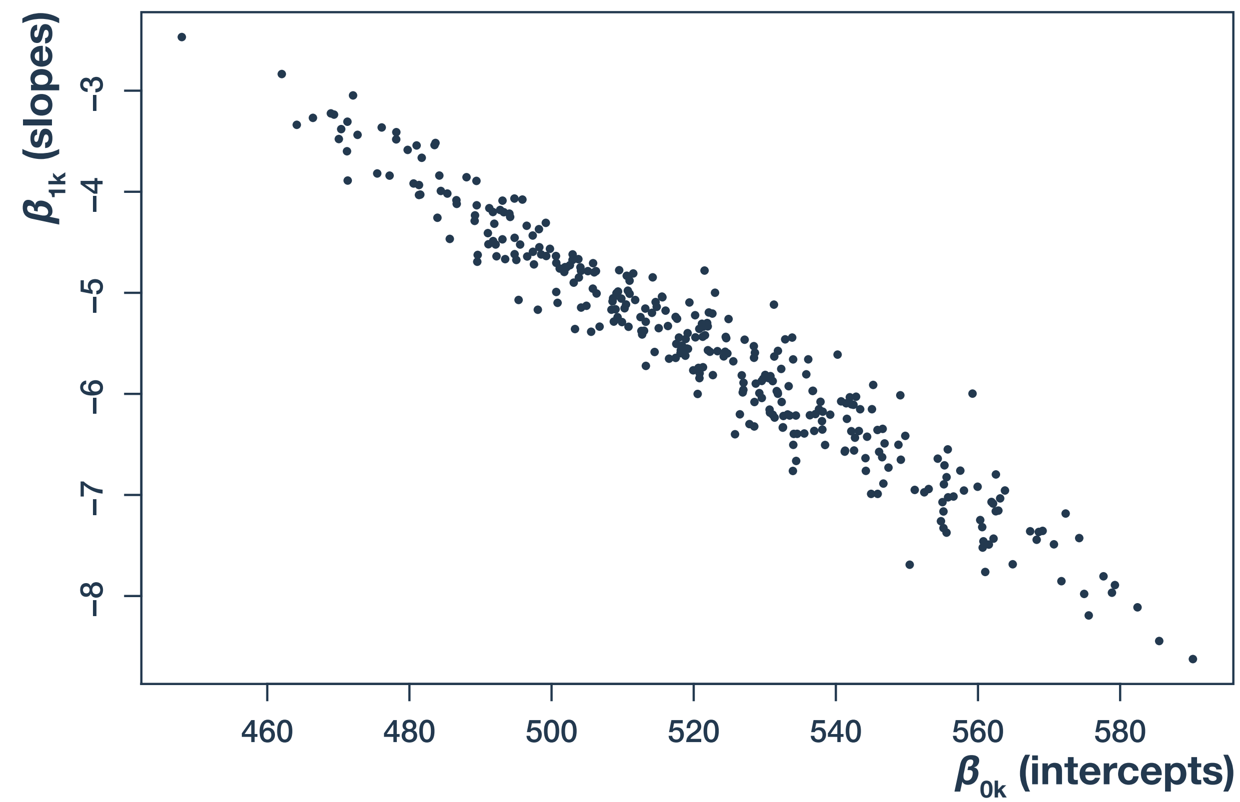
$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$
$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\eta_{0k} \sim \text{Norm}(0, \phi_0)$$
$$\eta_{1k} \sim \text{Norm}(0, \phi_1)$$

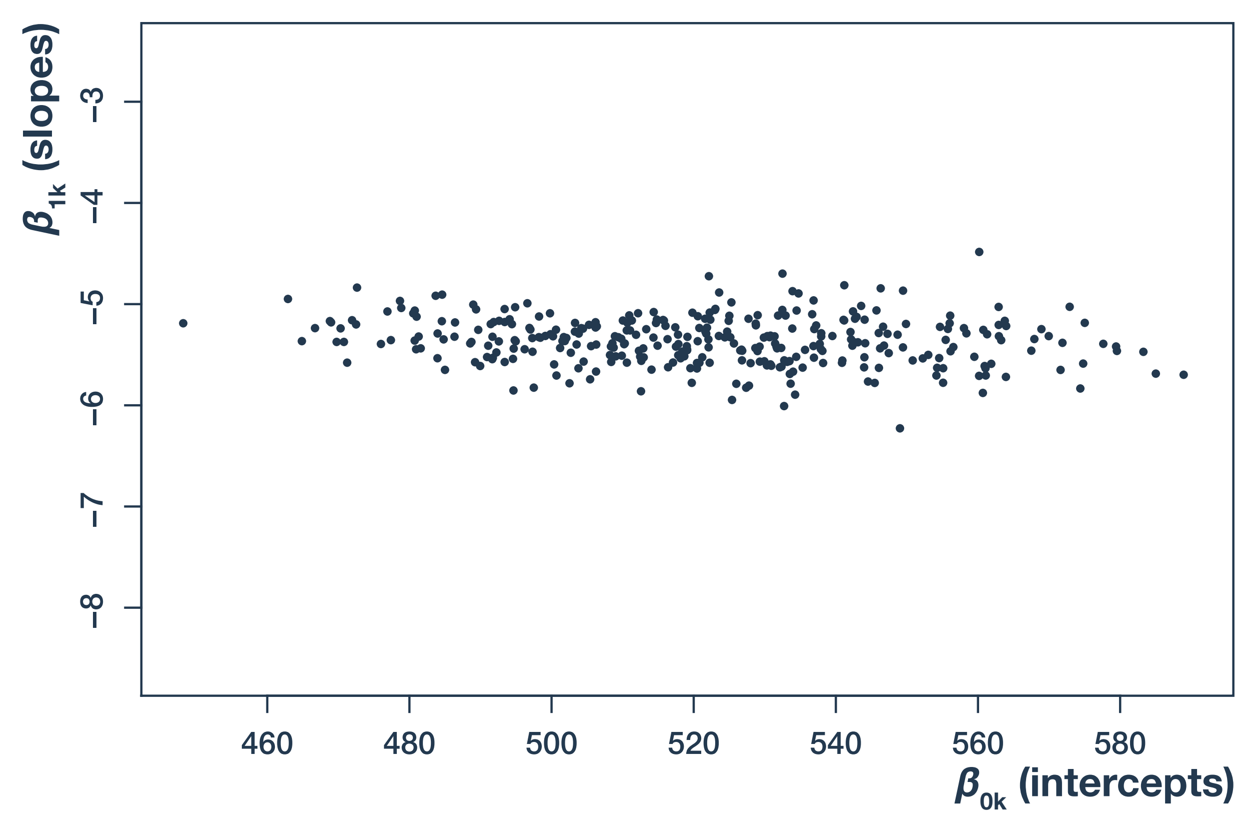
(Fixed priors omitted)

Correlation of coefficients

Correlated coefficients



Independent coefficients



Partial versus total pooling

Partial pooling (random effects)

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} Age_i$$

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\begin{bmatrix} \eta_{0k} \\ \eta_{1k} \end{bmatrix} \sim \text{MVNorm}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Phi \right)$$

Total pooling (fixed effects)

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} Age_i$$

$$\beta_{0k} \sim \text{Norm}(500, 100)$$

$$\beta_{1k} \sim \text{Norm}(0, 20)$$

(Fixed priors omitted)

Partial versus total pooling