Agenda

1. Precis feedback this week

2. A detailed, 2-level model of student achievement

3. Hierarchical versus expanded single-equation representations

4. Mixed effects in R:
   brms (and lme4)

Building a two-level model

Linear model of test scores

Scores depend on students' age, sex, race, and ethnicity.

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

Random intercept

Let each class have its own overall performance.

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$

Random slopes

Let the "effect" of students' characteristics differ from classroom to classroom

$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \eta_{3k}$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

Intercept predictors

Does each classes' average score depend on classroom features? (number of students and teacher's experience)

$$\beta_{0k} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \eta_{3k}$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} Age_i + \beta_{2k} Female_i + \beta_{3k} Black_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01} Size_k + \gamma_{02} Exp_i + \eta_{0k}$$

Slope predictors

Does the expected score difference between older and younger students depend on the size of the class?

$$\beta_{1k} = \gamma_{10} + \gamma_{11} Size_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \eta_{3k}$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

Group-level covariates on slope and intercept

In most cases, covariates on slope models should be included in intercept model as well

$$\beta_{0k} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i +$$
$$\gamma_{03}TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k +$$
$$\gamma_{05}PropBlack_k + \eta_{0k}$$
$$\beta_{1k} = \gamma_{10} + \gamma_{11}Size_k + \eta_{1k}$$
$$\beta_{2k} = \gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k}$$
$$\beta_{3k} = \gamma_{30} + \gamma_{31}TeacherBlack_k +$$
$$\gamma_{32}PropBlack_k + \eta_{3k}$$

Covariance model

Covariance structure

Covariance of the level-one coefficients modelled using variance Φ and correlations P.

$$\begin{bmatrix} \eta_{0k} \\ \eta_{1k} \\ \eta_{2k} \\ \eta_{3k} \end{bmatrix} \sim \text{MVNorm} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \Phi P \Phi \right)$$

$$\Phi = \begin{bmatrix} \phi_0 & 0 & 0 & 0 \\ 0 & \phi_1 & 0 & 0 \\ 0 & 0 & \phi_2 & 0 \\ 0 & 0 & 0 & \phi_3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{01} & 1 & \rho_{12} & \rho_{13} \\ \rho_{02} & \rho_{12} & 1 & \rho_{23} \\ \rho_{03} & \rho_{13} & \rho_{23} & 1 \end{bmatrix}$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} Age_i + \beta_{2k} Female_i + \beta_{3k} Black_i$$

Building a two-level model

Complete model

The full two-level model describes variation in test scores as a function of student- and class-level covariates.

Residual variation is attributed to within-class (σ) and between-class (ϕ) differences.

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i+$$
$$\gamma_{03}TeacherFemale_k+$$
$$\gamma_{04}TeacherBlack_k+$$
$$\gamma_{05}PropBlack_k + \eta_{0k}$$
$$\beta_{1k} = \gamma_{10} + \gamma_{11}Size_k + \eta_{1k}$$
$$\beta_{2k} = \gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k}$$
$$\beta_{3k} = \gamma_{30} + \gamma_{31}TeacherBlack_k+$$
$$\gamma_{32}PropBlack_k + \eta_{3k}$$

Two-level model: estimates

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

$$\beta_{0k} = \boxed{\gamma_{00}} + \boxed{\gamma_{01}}Size_k + \boxed{\gamma_{02}}Exp_i +$$
$$\gamma_{03}TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k +$$
$$\gamma_{05}PropBlack_k + \eta_{0k}$$

$$\beta_{1k} = \boxed{\gamma_{10}} + \boxed{\gamma_{11}}Size_k + \eta_{1k}$$
$$\beta_{2k} = \gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k}$$
$$\beta_{3k} = \gamma_{30} + \gamma_{31}TeacherBlack_k +$$
$$\gamma_{32}PropBlack_k + \eta_{3k}$$

		Mean	90% credible interval	
Expected score when all covariates are zero	γ_{00}	544.18	521.97	566.37
Baseline effect of class size	γ_{01}	–1.47	–2.02	–0.93
Baseline effect of teacher experience	γ_{02}	0.23	–0.01	0.47
Baseline effect of age	γ_{10}	–4.76	–5.87	–3.67
Effect of class size on age difference in scores	γ_{11}	0.10	–0.18	0.39

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i +$$
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$$\beta_{3k} = \gamma_{30} + \gamma_{31}TeacherBlack_k +$$
$$\gamma_{32}PropBlack_k + \eta_{3k}$$

	Mean	90% credible interval	
Baseline gender difference in teacher effectiveness γ_{03}	−12.73	−34.86	9.44
Expected score difference female versus male students with male teacher γ_{20}	10.61	−7.43	28.56
Expected score difference for female students with female versus male teachers γ_{21}	−0.61	−18.72	17.43

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i +$$
$$\gamma_{03}TeacherFemale_k +$$
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$$\beta_{1k} = \gamma_{10} + \gamma_{11}Size_k + \eta_{1k}$$
$$\beta_{2k} = \gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k}$$
$$\beta_{3k} = \boxed{\gamma_{30}} + \boxed{\gamma_{31}}TeacherBlack_k +$$
$$\boxed{\gamma_{32}}PropBlack_k + \eta_{3k}$$

		Mean	90% credible interval	
Race difference in teacher effectiveness (white students)	γ_{04}	−7.96	−17.80	1.74
Effect of class racial composition (white students)	γ_{05}	−25.99	−38.44	−13.41
Baseline race difference in student scores	γ_{30}	−23.73	−29.22	−18.09
Score difference for Black versus white students with Black teachers	γ_{31}	13.97	4.37	23.78
Effect of class racial composition for Black versus white students	γ_{32}	3.97	−10.53	18.72
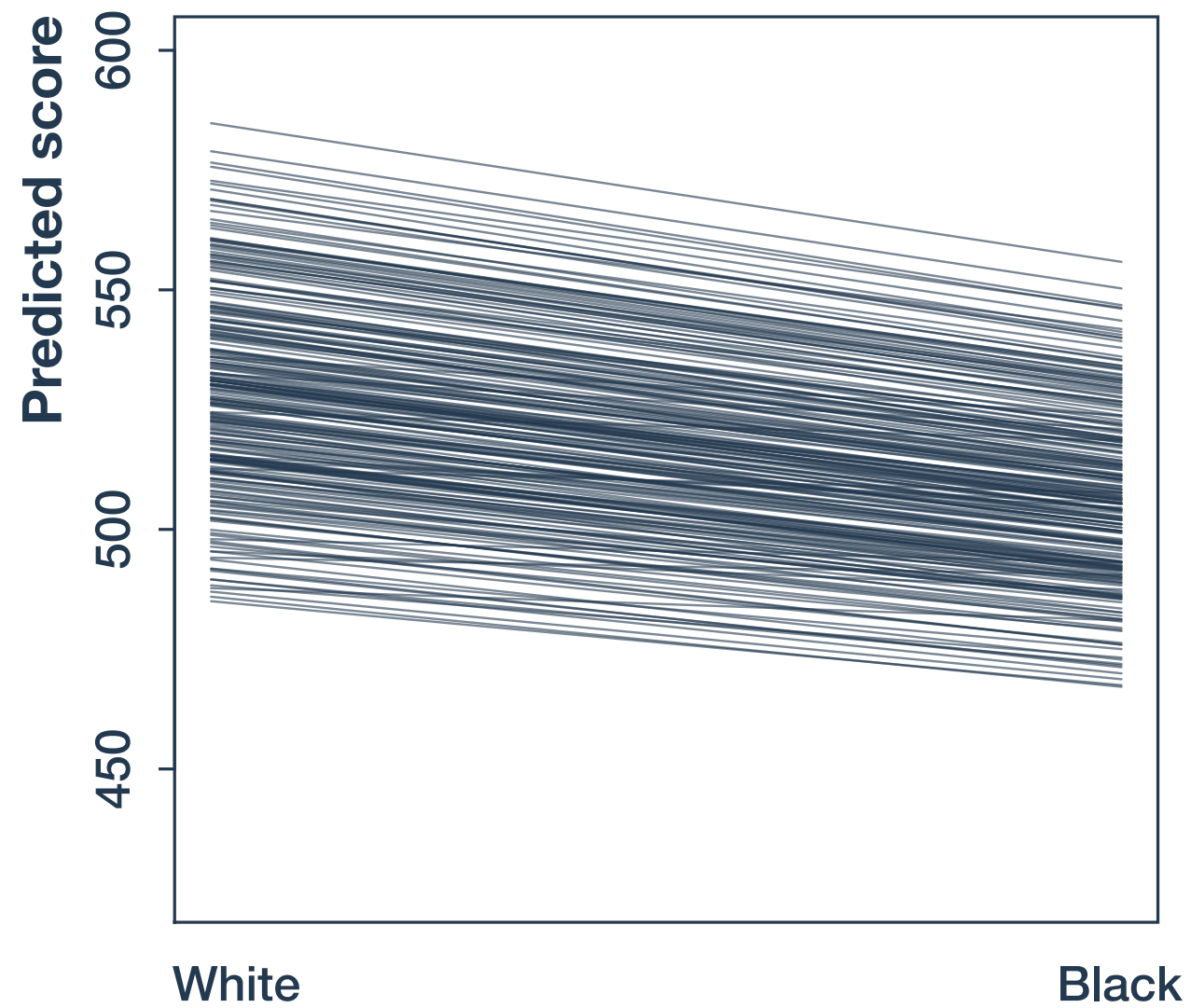
Two-level model: estimates
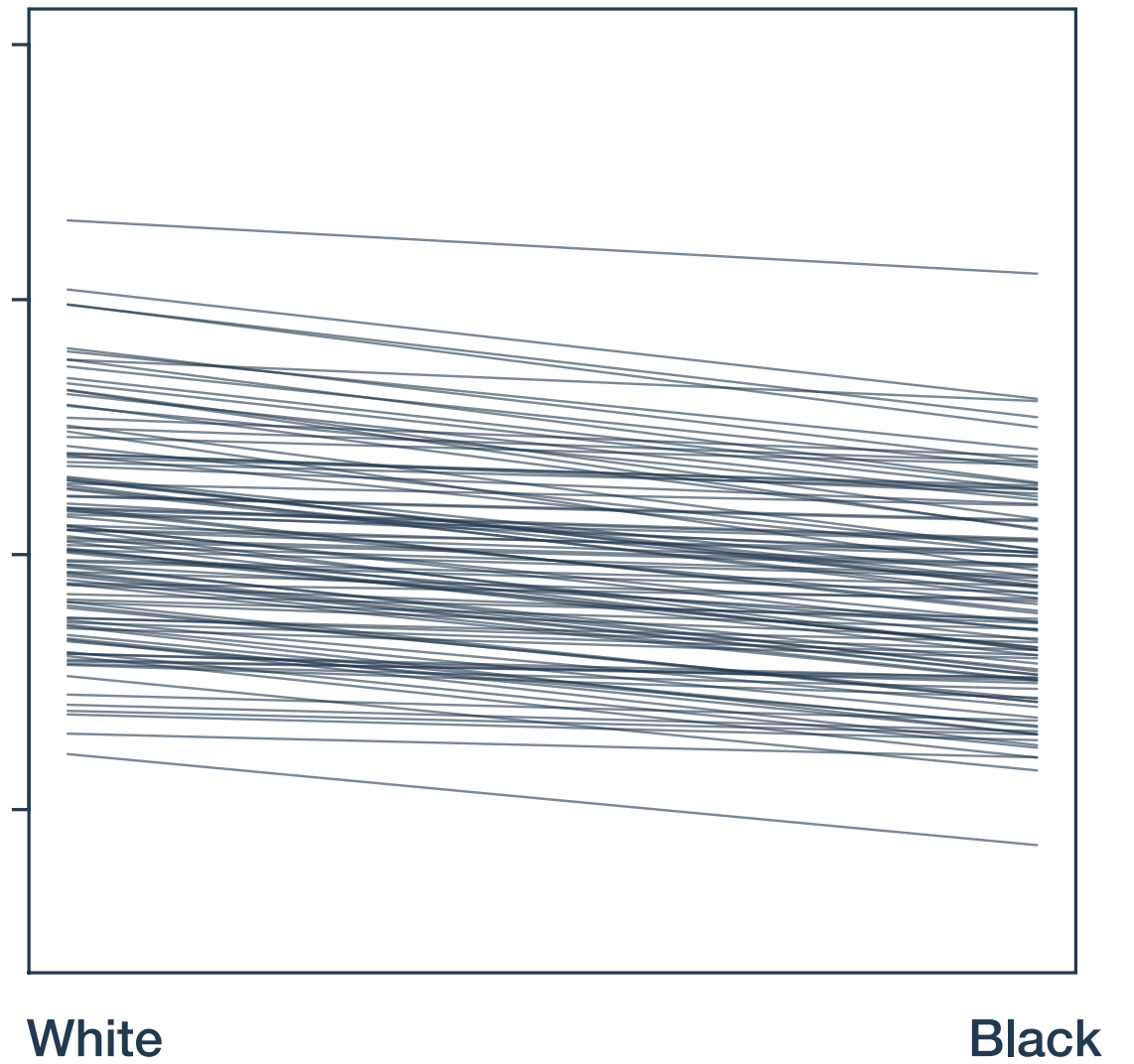
Expanded notation

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$

$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11}Size_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \gamma_{31}TeacherBlack_k + \gamma_{32}PropBlack_k + \eta_{3k}$$

$$\mu_{ik} = (\gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$

$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \eta_{0k})$$

$$(\gamma_{10} + \gamma_{11}Size_k + \eta_{1k})Age_i$$

$$(\gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k})Female_i$$

$$(\gamma_{30} + \gamma_{31}TeacherBlack_k + \gamma_{32}PropBlack_k + \eta_{3k})Black_i$$

Expanded notation

$$\mu_{ik} = (\gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \eta_{0k})$$
$$(\gamma_{10} + \gamma_{11}Size_k + \eta_{1k})Age_i$$
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$$(\gamma_{30} + \gamma_{31}TeacherBlack_k + \gamma_{32}PropBlack_k + \eta_{3k})Black_i$$

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$
$$\gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k +$$
$$\gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k +$$
$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Expanded notation

Fixed effects

Explained variation in outcome variable.

Describes the way that outcome and predictor variables co-vary.

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$
$$\gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k +$$
$$\gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k +$$
$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Random effects

Unexplained variation in outcome variable.

Described in terms of individual variability and different types of group variability.

Expanded notation

Hierarchical

$$\mu_{ik} = \beta_{0k} + \beta_{1k}Age_i + \beta_{2k}Female_i + \beta_{3k}Black_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \eta_{0k}$$
$$\beta_{1k} = \gamma_{10} + \gamma_{11}Size_k + \eta_{1k}$$
$$\beta_{2k} = \gamma_{20} + \gamma_{21}TeacherFemale_k + \eta_{2k}$$
$$\beta_{3k} = \gamma_{30} + \gamma_{31}TeacherBlack_k + \gamma_{32}PropBlack_k + \eta_{3k}$$

Expanded

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$
$$\gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k +$$
$$\gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k +$$
$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Building an R formula

R packages like brms (Bayesian) and lme4 (frequentist) use a "formula" notation to specify multilevel models.
(extension of syntax for standard regressions)

```
student_reading_score ~
        student_age_s*class_size_c + teacher_exper_c +
        student_female*teacher_female +
        student_re_black*teacher_re_black +
        student_re_black*class_prop_black +
        (1 + student_age_s + student_female +
         student_re_black | teacher_id)
```

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k+$$

$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$

$$\gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k+$$

$$\gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k+$$

$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Building an R formula

```
student_reading_score ~
        student_age_s*class_size_c + teacher_exper_c +
        student_female*teacher_female +
        student_re_black*teacher_re_black +
        student_re_black*class_prop_black +
        (1 + student_age_s + student_female +
         student_re_black | teacher_id)
```

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$
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$$\gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k +$$
$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Building an R formula

Global intercept (included automatically)

```
student_reading_score ~
        student_age_s*class_size_c + teacher_exper_c +
        student_female*teacher_female +
        student_re_black*teacher_re_black +
        student_re_black*class_prop_black +
        (1 + student_age_s + student_female +
         student_re_black | teacher_id)
```

$$S_{ik} = \boxed{\gamma_{00}} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$
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$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Building an R formula

Interactions (∗) automatically include standalone terms

```
student_reading_score ~
        student_age_s*class_size_c + teacher_exper_c +
        student_female*teacher_female +
        student_re_black*teacher_re_black +
        student_re_black*class_prop_black +
        (1 + student_age_s + student_female +
         student_re_black | teacher_id)
```

$$S_{ik} = \gamma_{00} + \boxed{\gamma_{01}Size_k} + \gamma_{02}Exp_i + TeacherFemale_k +$$

$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \boxed{\gamma_{10}Age_i}$$

$$\boxed{\gamma_{11}Age_iSize_k} + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k +$$

$$\gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k +$$

$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Building an R formula

```
student_reading_score ~
        student_age_s*class_size_c + teacher_exper_c +
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        student_re_black*teacher_re_black +
        student_re_black*class_prop_black +
        (1 + student_age_s + student_female +
          student_re_black | teacher_id)
```

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + \boxed{TeacherFemale_k} +$$

$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$

$$\gamma_{11}Age_iSize_k + \boxed{\gamma_{20}Female_i} + \boxed{\gamma_{21}Female_iTeacherFemale_k} +$$

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Building an R formula

Interactions (∗) automatically include standalone terms

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        student_re_black*teacher_re_black +
        student_re_black*class_prop_black +
        (1 + student_age_s + student_female +
         student_re_black | teacher_id)
```

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k+$$
$$\boxed{\gamma_{04}TeacherBlack_k} + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$
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$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Building an R formula

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        student_re_black*class_prop_black +
        (1 + student_age_s + student_female +
         student_re_black | teacher_id)
```

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k + \boxed{\gamma_{05}PropBlack_k} + \gamma_{10}Age_i$$
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$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Building an R formula

Random effects use pipe notation (|)

```
student_reading_score ~

        student_age_s*class_size_c + teacher_exper_c +

        student_female*teacher_female +

        student_re_black*teacher_re_black +

        student_re_black*class_prop_black +

        (1 + student_age_s + student_female +

         student_re_black │ teacher_id)
```

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$

$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$

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Building an R formula

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        student_re_black*teacher_re_black +
        student_re_black*class_prop_black +
        (1 + student_age_s + student_female +
         student_re_black | teacher_id)
```

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$
$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$
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$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Building an R formula

Random intercepts indicated with constant (1)

```
student_reading_score ~
        student_age_s*class_size_c + teacher_exper_c +
        student_female*teacher_female +
        student_re_black*teacher_re_black +
        student_re_black*class_prop_black +
        (1 + student_age_s + student_female +
         student_re_black | teacher_id)
```

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$
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$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

Building an R formula

```
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        student_female*teacher_female +
        student_re_black*teacher_re_black +
        student_re_black*class_prop_black +
        (1 + student_age_s + student_female +
        student_re_black | teacher_id)
```

$$S_{ik} = \gamma_{00} + \gamma_{01}Size_k + \gamma_{02}Exp_i + TeacherFemale_k +$$

$$\gamma_{04}TeacherBlack_k + \gamma_{05}PropBlack_k + \gamma_{10}Age_i$$

$$\gamma_{11}Age_iSize_k + \gamma_{20}Female_i + \gamma_{21}Female_iTeacherFemale_k +$$

$$\gamma_{30}Black_i + \gamma_{31}Black_iTeacherBlack_k + \gamma_{32}Black_iPropBlack_k +$$

$$\eta_{0k} + \eta_{1k}Age_i + \eta_{2k}Female_i + \eta_{3k}Black_i + \varepsilon_i$$

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