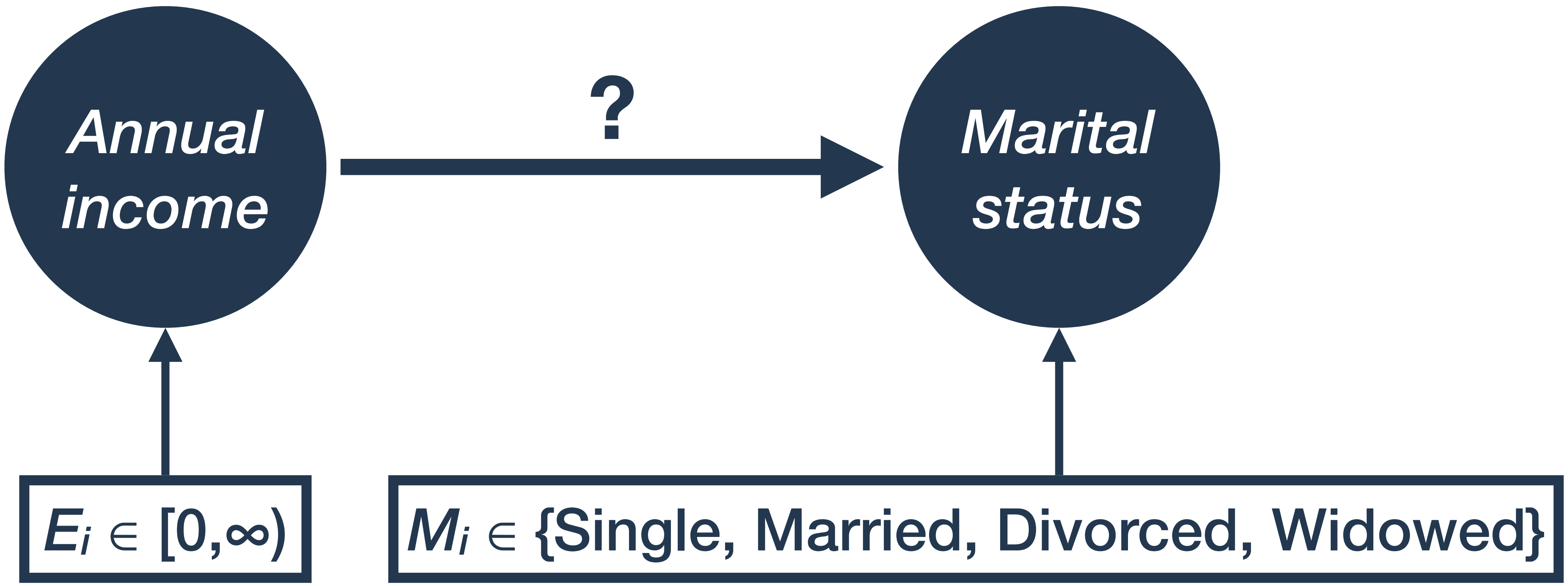
Agenda

1. Multinomial distribution
2. Categorical outcome variables
3. Softmax link function
4. Interpreting coefficients
5. Multinomial logistic in R

Income and marital status



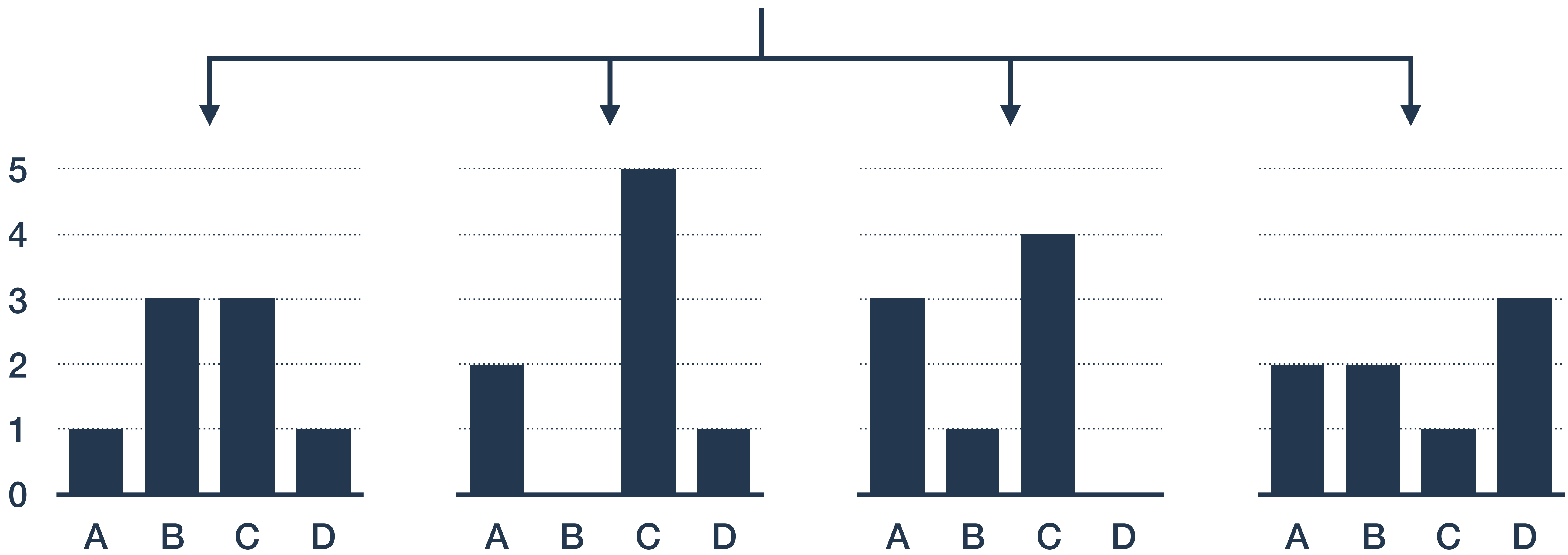
The problem	Outcome variable has multiple (>2) categories. Binomial and Poisson models won’t work.
The solution	Use a multinomial outcome distribution (and a new link function) to account for the data.

Multinomial distribution

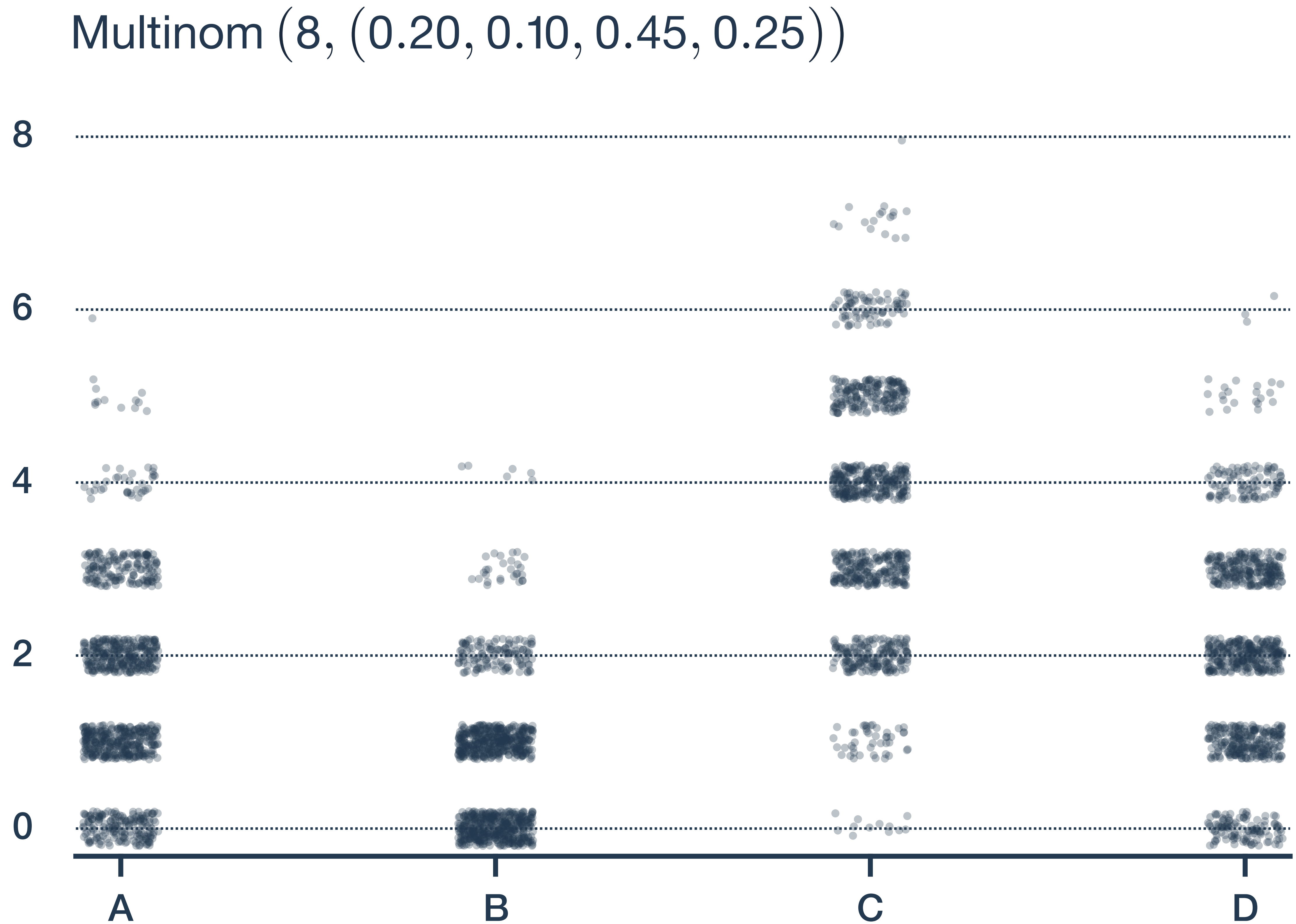
$$\text{Multinom}\left(n, \left(p_1, \ldots, p_k\right)\right)$$

Multinomial distribution | Result of *n* trials, each of which can result in one of *k* outcomes with probability p_1, p_2, …, p_k.

$$\text{Multinom}\left(8, \left(0.20, 0.10, 0.45, 0.25\right)\right)$$



Multinomial distribution



Multinomial distribution

Binomial, Bernoulli, and categorical distributions are special cases of the multinomial.

Binomial distribution | $\text{Bin}(n, p) = \text{Multinom}(n, (1-p, p))$

Bernoulli distribution | $\text{Bernoulli}(p) = \text{Multinom}(1, (1-p, p))$

Categorical distribution | $\text{Cat}(p_1, p_2, \ldots, p_k) = \text{Multinom}(1, (p_1, p_2, \ldots, p_k))$

	One trial	Multiple trials
Two categories	**Bernoulli**	**Binomial**
Multiple categories	**Categorical**	**Multinomial**

Categorical distribution

$$\text{Cat}\left(p_1, \ldots, p_k\right)$$

Categorical distribution | Multinomial distribution with just one trial

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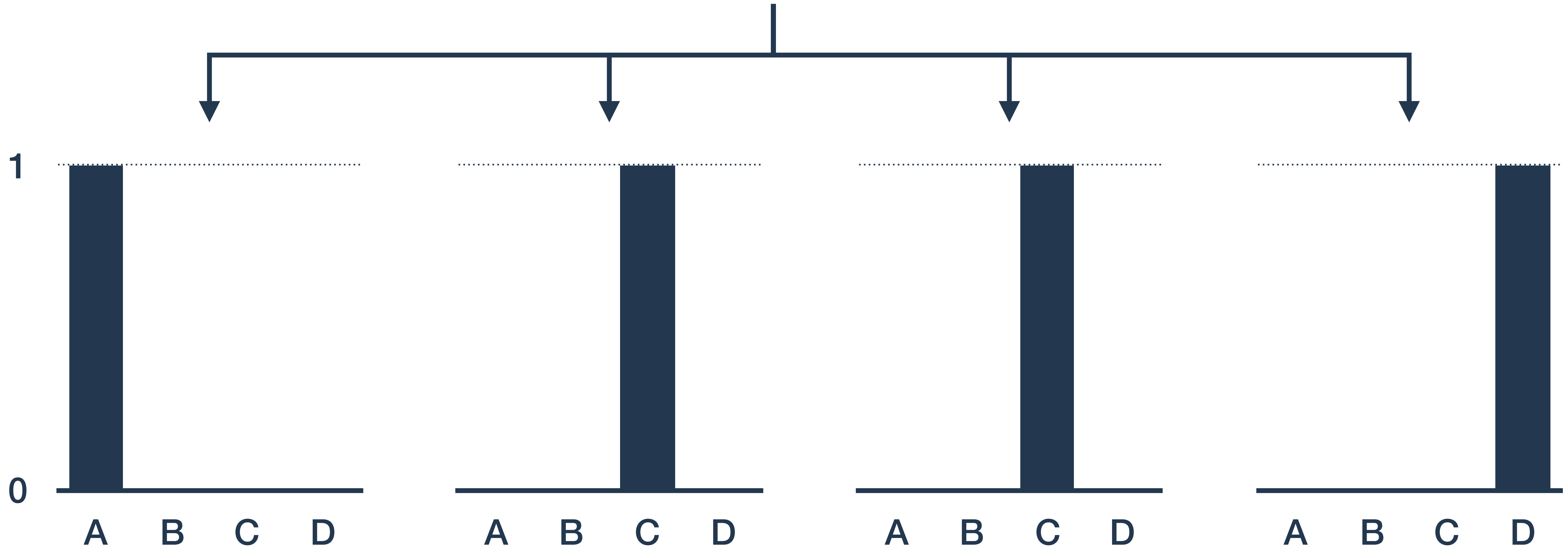
$$\text{Cat}\left(0.20, 0.10, 0.45, 0.25\right)$$



Categorical outcome

$M_i \in$ {Single, Married, Divorced, Widowed}

One category has to be the *reference* category.

$$s_m = 0$$

$$s_s = \alpha_s + \beta_s E_i$$

$$s_d = \alpha_d + \beta_d E_i$$

$$s_w = \alpha_w + \beta_w E_i$$

Each other category gets its own coefficient.

Softmax link function

$M_i \in$ {Single, Married, Divorced, Widowed}

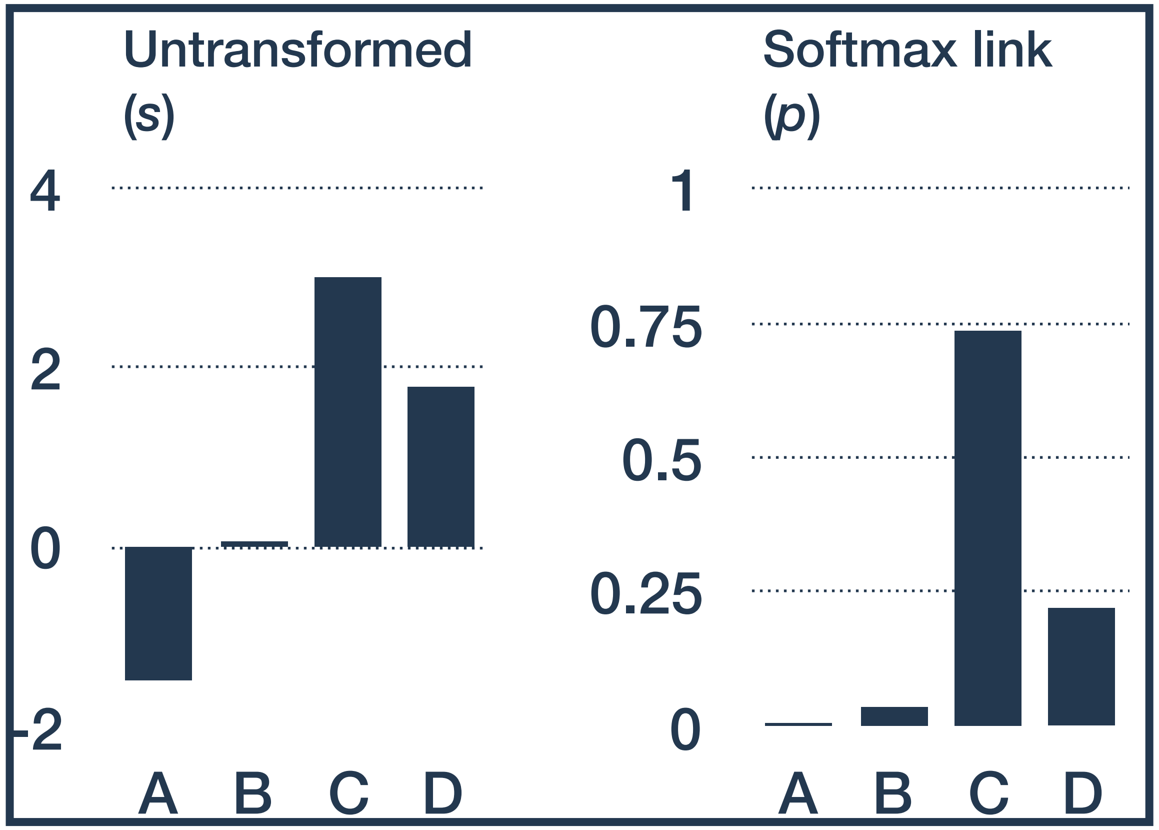
$$M_i \sim \text{Cat}\left(\text{softmax}(s_m, s_s, s_d, s_w)\right)$$

$$s_m = 0$$

$$s_s = a_s + \beta_s E_i$$

$$s_d = a_d + \beta_d E_i$$

$$s_w = a_w + \beta_w E_i$$



Softmax is a multivariate generalization of inverse logit.

$$p_s = \text{softmax}(s_s) = \frac{\exp(s_s)}{\exp(s_s) + \exp(s_m) + \exp(s_d) + \exp(s_w)}$$

Multinomial logistic regression

Multinomial logistic (or categorical) regression model.

$$
\begin{aligned}
M_i &\sim \mathrm{Cat}\left(\mathrm{softmax}(s_{mi}, s_{si}, s_{di}, s_{wi})\right) \\
s_{mi} &= 0 \\
s_{si} &= a_s + \beta_s E_i \\
s_{di} &= a_d + \beta_d E_i \\
s_{wi} &= a_w + \beta_w E_i \\
a_s, a_d, a_w &\sim \mathrm{Norm}(0, 2) \\
\beta_s, \beta_d, \beta_w &\sim \mathrm{Norm}(0, 3)
\end{aligned}
$$

Multinomial logistic regression

With two categories, the multinomial logistic model is the standard (binomial) logistic model.

$$\begin{aligned}
M_i &\sim \mathrm{Cat}\left(\mathrm{softmax}(s_{1i}, s_{2i})\right) \\
s_{1i} &= 0 \\
s_{2i} &= \alpha + \beta E_i \\
\alpha &\sim \mathrm{Norm}(0, 1) \\
\beta &\sim \mathrm{Norm}(0, 3)
\end{aligned}$$

$$\boxed{p_{2i} = \frac{\exp(s_{2i})}{1 + \exp(s_{2i})} = \mathrm{logit}^{-1}(s_{2i})}$$

Interpreting estimates

$$M_i \sim \text{Cat}\left(\text{softmax}(s_{mi}, s_{si}, s_{di}, s_{wi})\right)$$

$$s_{mi} = 0$$

$$s_{si} = a_s + \beta_s E_i$$

$$s_{di} = a_d + \beta_d E_i$$

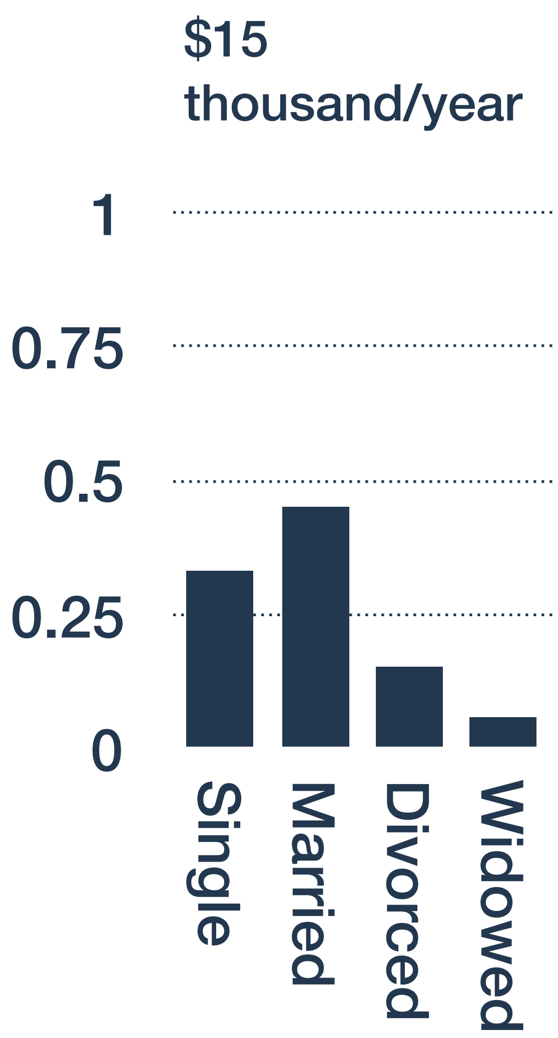
$$s_{wi} = a_w + \beta_w E_i$$

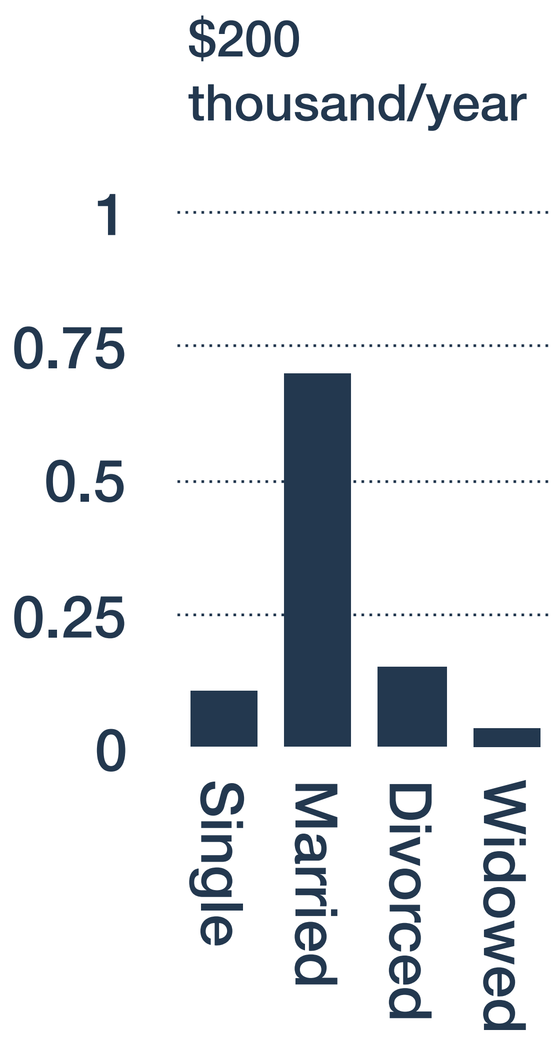
$$a_s, a_d, a_w \sim \text{Norm}(0, 2)$$

$$\beta_s, \beta_d, \beta_w \sim \text{Norm}(0, 3)$$

	Mean	*90% credible interval*	
$\boldsymbol{a_s}$	5.35	4.73	5.98
$\boldsymbol{\beta_s}$	-0.59	-0.65	-0.53
$\boldsymbol{a_d}$	0.57	-0.24	1.37
$\boldsymbol{\beta_d}$	-0.18	-0.25	-0.10
$\boldsymbol{a_w}$	1.94	0.89	2.98
$\boldsymbol{\beta_w}$	-0.40	-0.50	-0.30

Interpreting estimates





	Mean	*90% credible interval*	
α_s	5.35	4.73	5.98
β_s	-0.59	-0.65	-0.53
α_d	0.57	-0.24	1.37
β_d	-0.18	-0.25	-0.10
α_w	1.94	0.89	2.98
β_w	-0.40	-0.50	-0.30