Agenda

1. Roadmap
2. Learning from categories using pooled and unpooled models
3. Partial pooling
4. Specifying nested models
5. Indexing in models in R

Roadmap—the course so far

1. Redefining linear regression in a Bayesian framework

Outcome as draws from a probability distribution

$$Y_i \sim \text{Norm}(\mu_i, \sigma)$$

$$\mu_i = X_i \beta$$

Center of distribution modelled as linear combination of predictors

Roadmap—the course so far

1. Redefining linear regression in a Bayesian framework
2. Dealing with discrete outcome variables using GLM

Changing the outcome distribution to fit the data

$$Y_i \sim \text{Pois}(\lambda_i)$$

$$\log(\lambda_i) = X_i \beta$$

Using link functions to map linear combination of predictors to an appropriate scale

Roadmap—the course so far

1. Redefining linear regression in a Bayesian framework
2. Dealing with discrete outcome variables using GLM
3. **Dealing with structured predictors using hierarchical linear models**

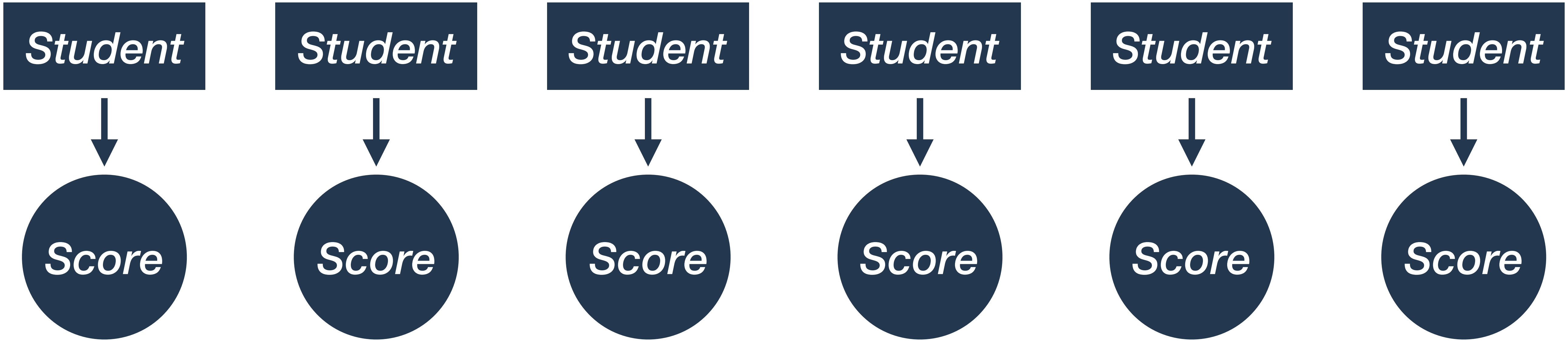
$$Y_i \sim \mathrm{Norm}(\mu_i, \sigma)$$
$$\mu_i = \boxed{X_i \beta}$$

Adding structure to the linear model to account for relationships between observations

Framing the problem

Student performance on a standardized test

Each student has a score, which we can model using characteristics of the student, their school, etc.



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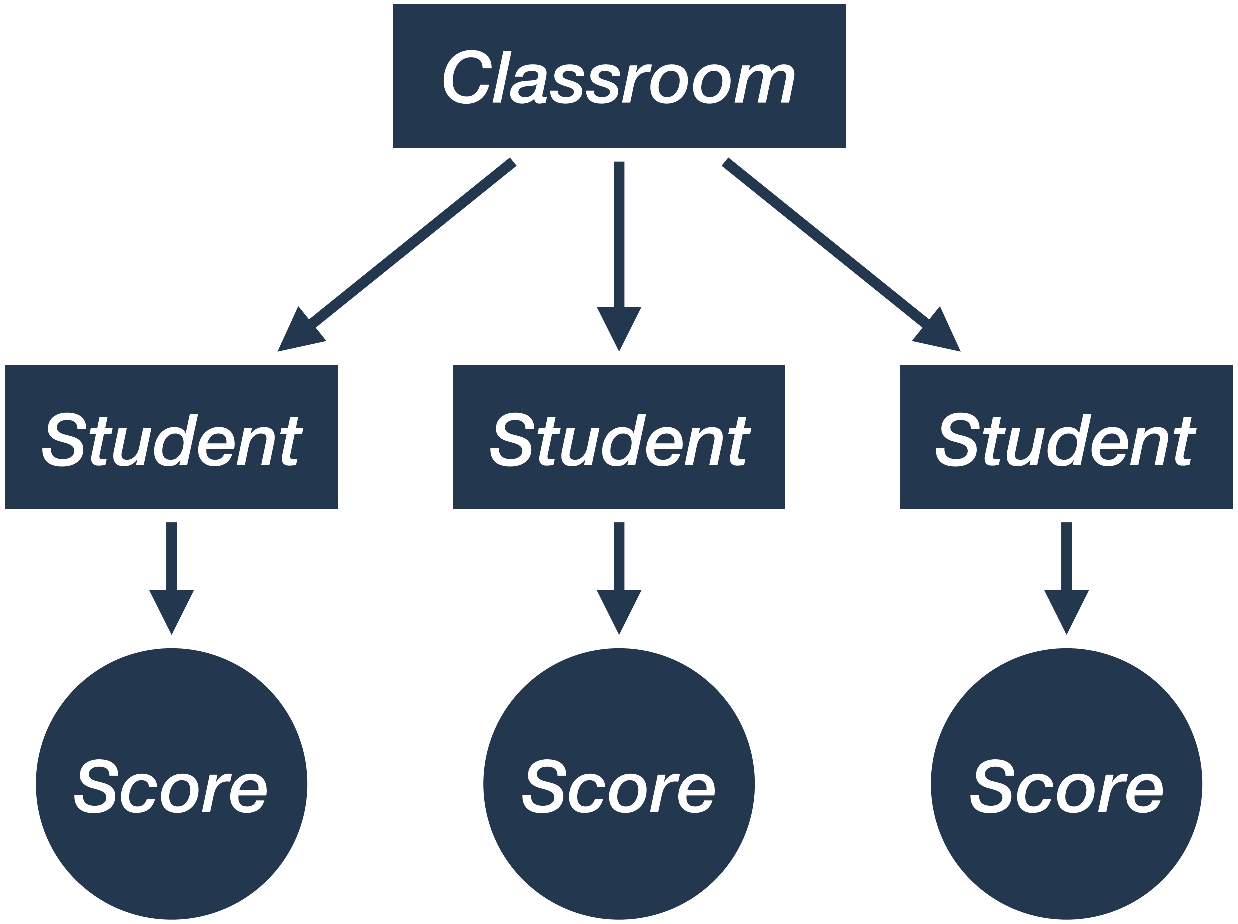
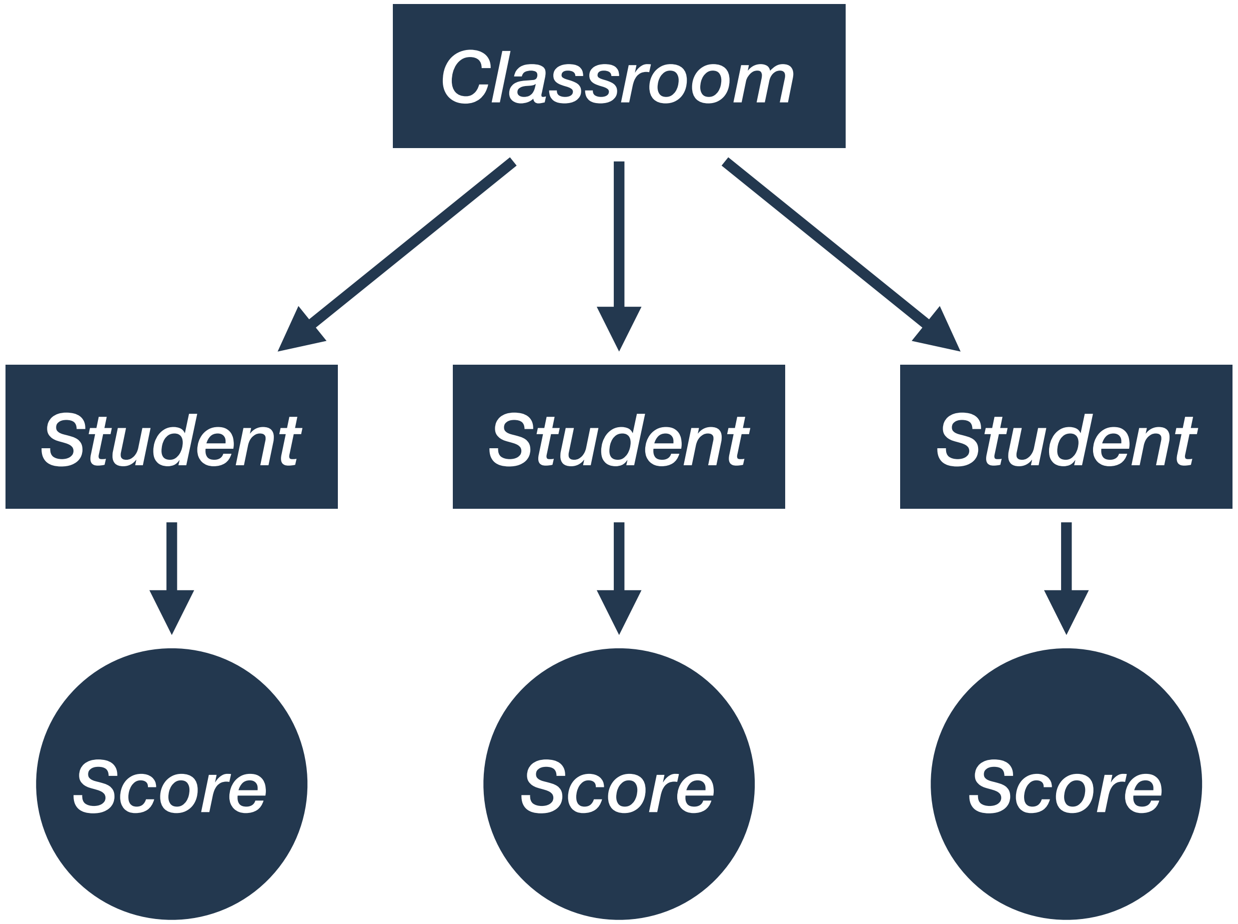


Students' performance is not independent

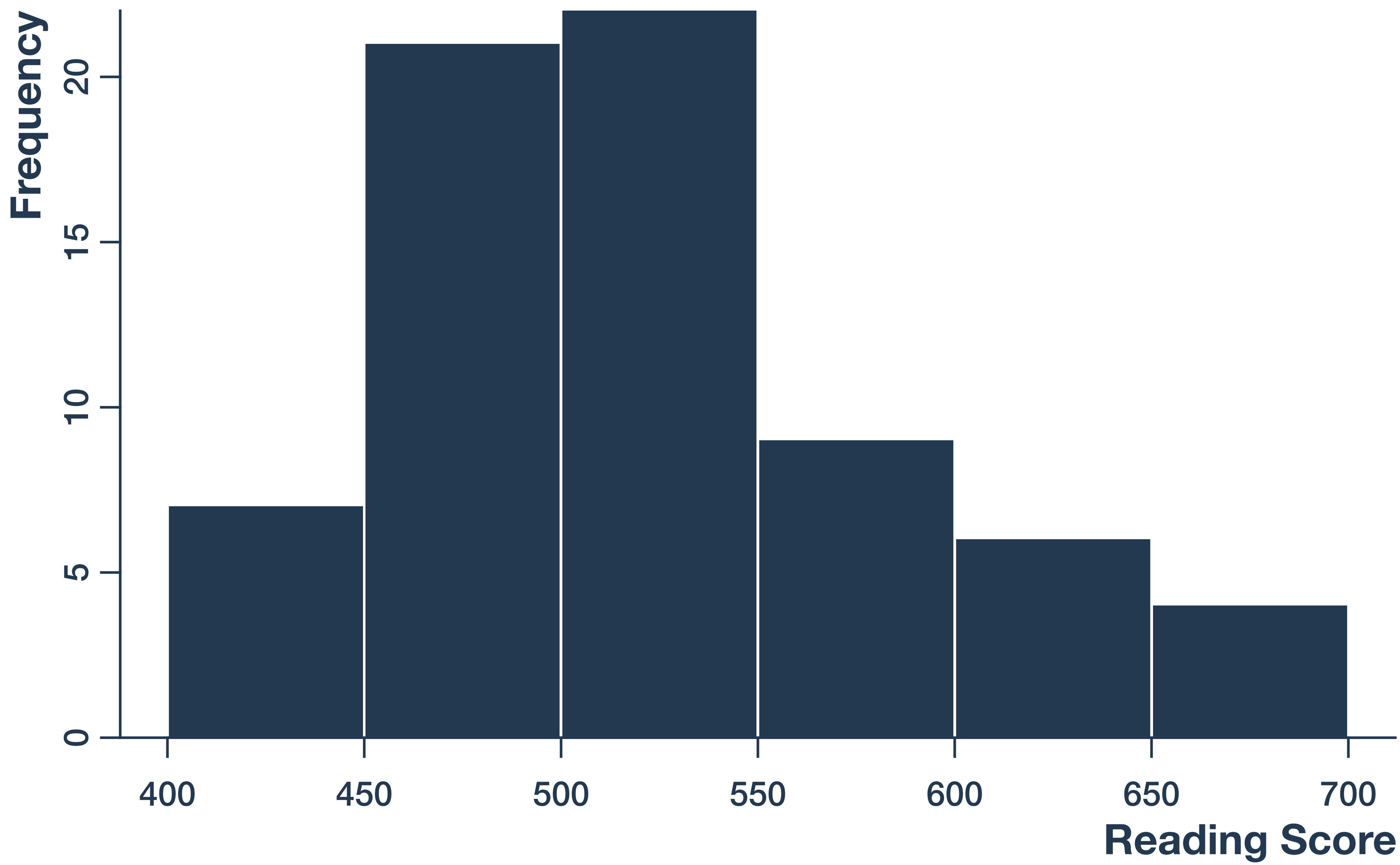
Students in the same classroom share many relevant characteristics (teacher, school, funding, classroom environment, etc.)



Tennessee STAR study

Sample | 69 students in 16 grade-one classrooms from Tennessee schools in 1986

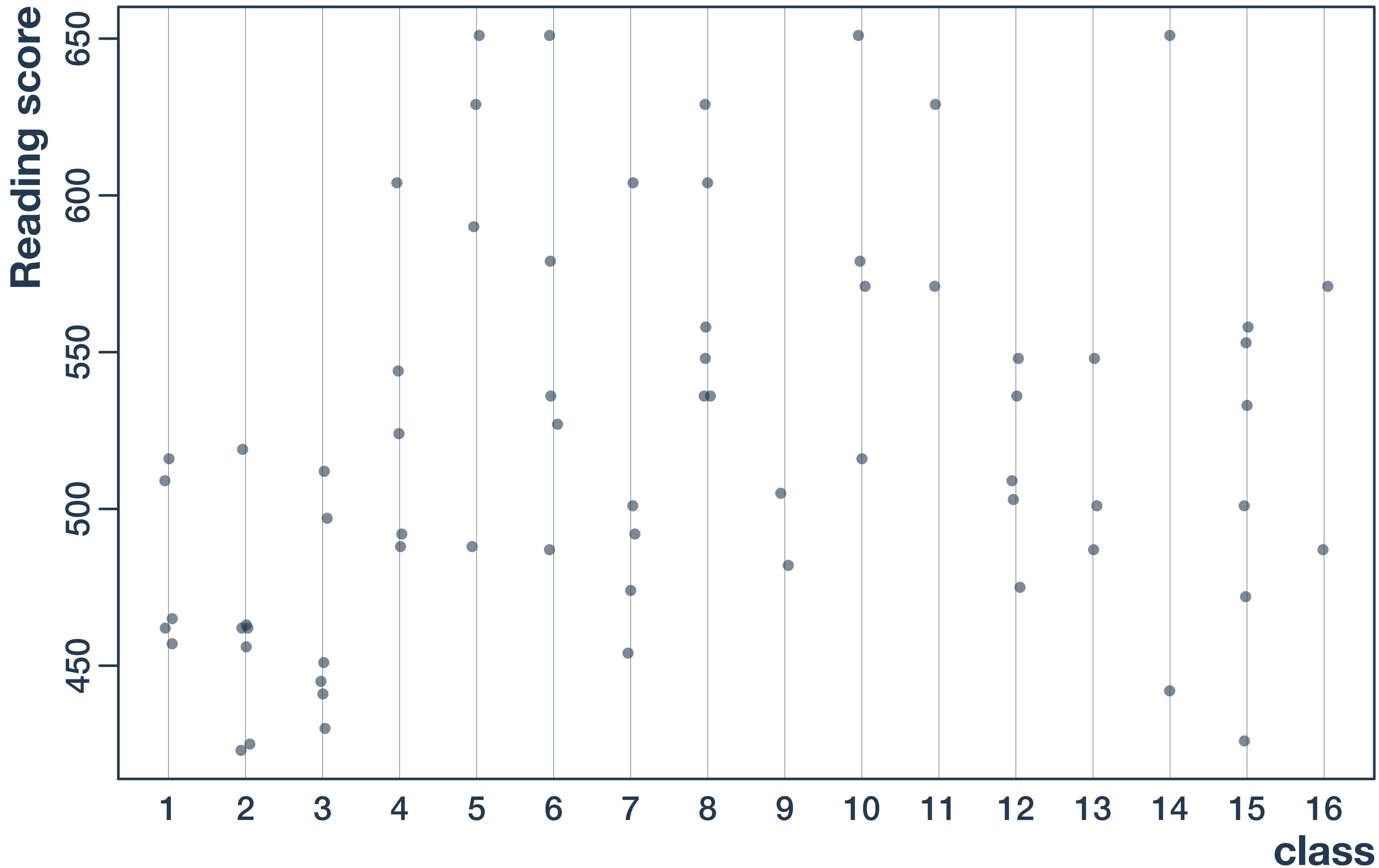
Outcome | Standardized reading score on Stanford Achievement Test (SAT)



Data

Sample | 69 students in 16 grade-one classrooms from Tennessee schools in 1986

Outcome | Standardized reading score on Stanford Achievement Test (SAT)



Complete pooling

All variation attributed to global standard deviation σ

Strategy 1 (pooling)

Ignore hierarchical structure of data and pool all students' scores together.

$$S_i \sim \text{Norm}(\mu_i, \sigma)$$

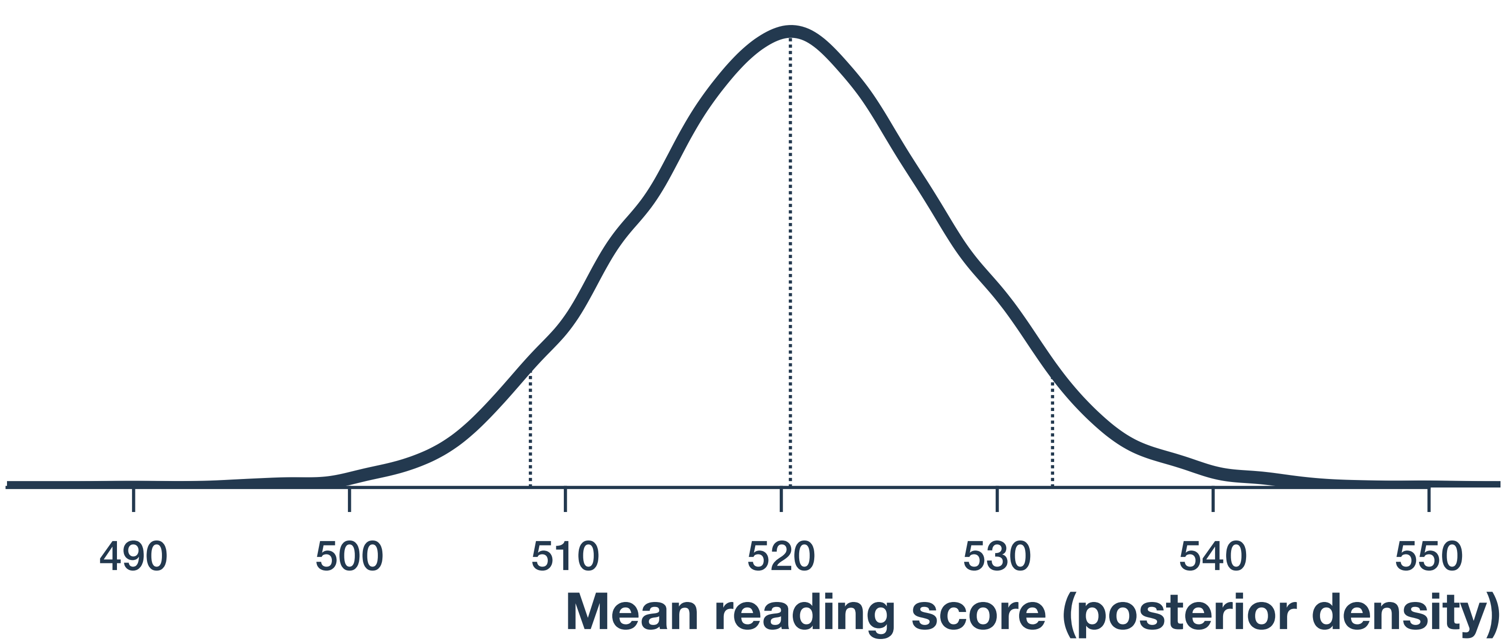
$$\mu_i = \alpha$$

Single average score α for all students

$$\alpha \sim \text{Norm}(500, 100)$$

$$\sigma \sim \text{Unif}(0, 100)$$

	Mean	*90% credible interval*	
α	520.54	508.46	532.56
σ	61.02	52.49	69.55



Incorporating categories

Indicator variables

$$Y_i \sim \text{Norm}(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta X_i$$

$$E(Y_i | X_i = 0) = \alpha$$

$$E(Y_i | X_i = 1) = \alpha + \beta$$

Pick a reference category and construct indicator variables for all other categories.

Intercept α captures reference mean, other means measured as difference from reference.

Can be fit with OLS in standard matrix specification of linear regression.

Fixed effects

$$Y_i \sim \text{Norm}(\mu_i, \sigma)$$

$$\mu_i = \begin{cases} \alpha_0 & \text{if } X_i = 0 \\ \alpha_1 & \text{if } X_i = 1 \end{cases}$$

$$E(Y_i | X_i = 0) = \alpha_0$$

$$E(Y_i | X_i = 1) = \alpha_1$$

Omit global mean α and give each category its own mean α_k.

Numerically identical to indicator variables, but harder to specify statistically.

No pooling

Standard deviation σ explains variation around each α_k

Strategy 2 (no pooling)

Include a separate parameter for each classroom's average score (fixed effects).

$$S_{ik} \sim \mathrm{Norm}(\mu_{ik}, \sigma)$$

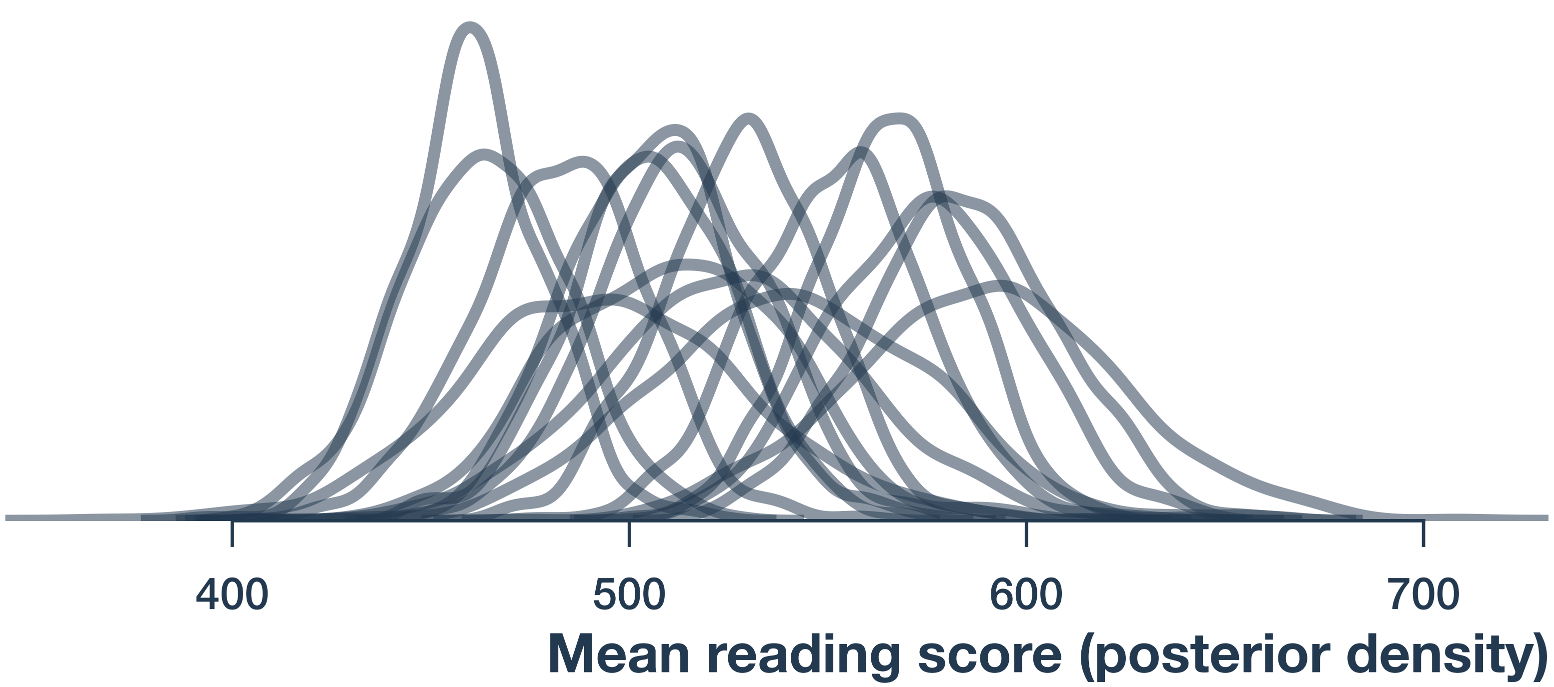
$$\mu_{ik} = \alpha_k$$

Each classroom k has its own average score α_k

$$\alpha_k \sim \mathrm{Norm}(500, 100)$$

$$\sigma \sim \mathrm{Unif}(0, 100)$$

Variability in α_k accounts for some inter-student variation



Partial pooling

Strategy 3 (partial pooling)

Include a separate parameter for each classroom's average score, but model those averages as random draws from a normal distribution.

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \alpha_k$$

Each classroom k still has its own average score α_k

$$\alpha_k \sim \text{Norm}(\gamma, \eta)$$

The prior distribution for classroom averages is estimated from the data

$$\sigma \sim \text{Unif}(0, 100)$$

$$\gamma \sim \text{Norm}(500, 100)$$

$$\eta \sim \text{Unif}(0, 100)$$

No pooling versus partial pooling

No pooling

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \alpha_k$$

$$\alpha_k \sim \text{Norm}(500, 100)$$

$$\sigma \sim \text{Unif}(0, 100)$$

Partial pooling

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \alpha_k$$

$$\alpha_k \sim \text{Norm}(\gamma, \eta)$$

Parameters γ and η describe 'typical' classrooms, allowing information to be shared among all of the α_k estimates.

$$\sigma \sim \text{Unif}(0, 100)$$

σ measures variability within each classroom

$$\gamma \sim \text{Norm}(500, 100)$$

$$\eta \sim \text{Unif}(0, 100)$$

η measures variability between all classrooms

Partial pooling

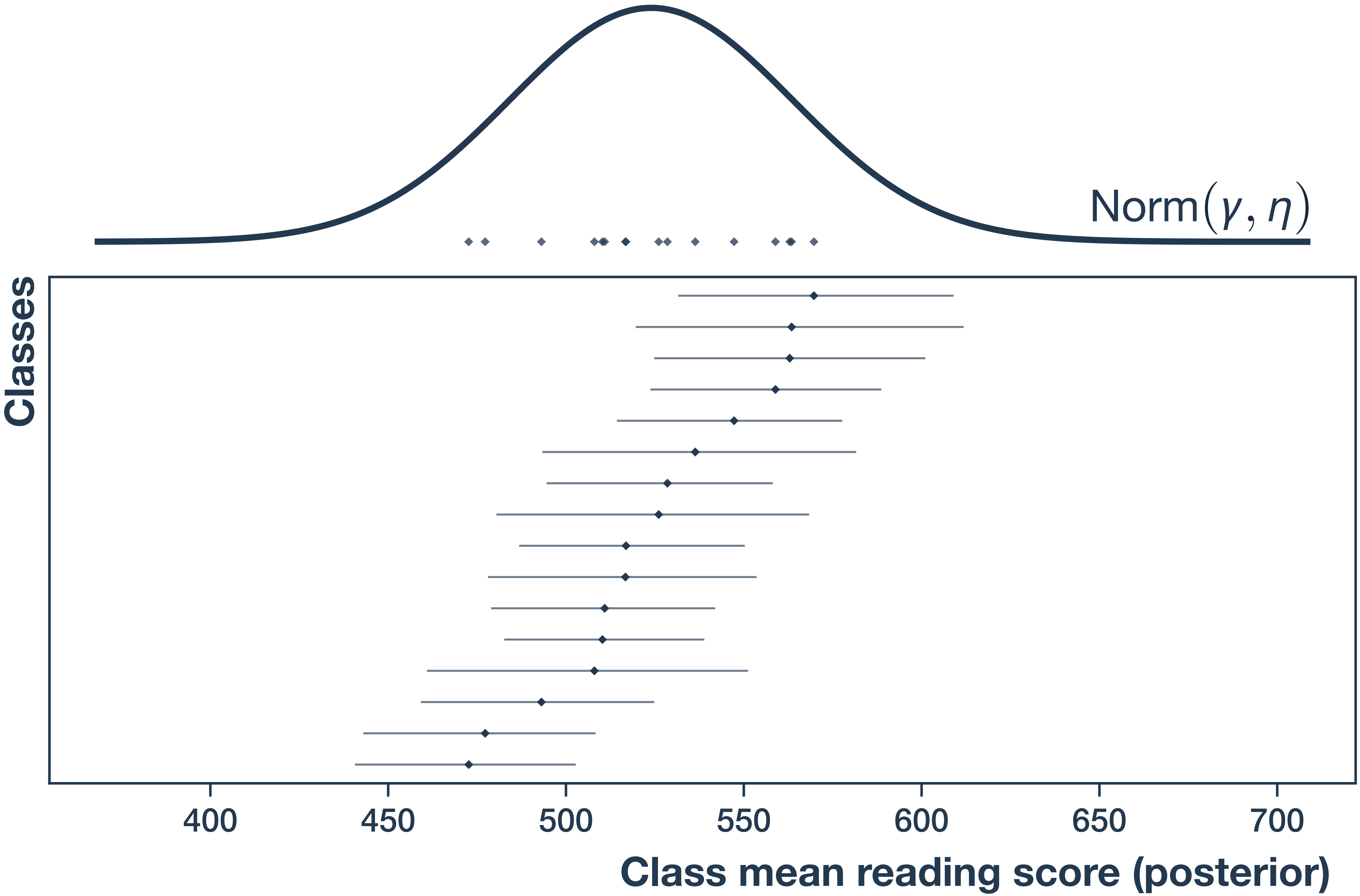
$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$
$$\mu_{ik} = \alpha_k$$

$$\alpha_k \sim \text{Norm}(\gamma, \eta)$$

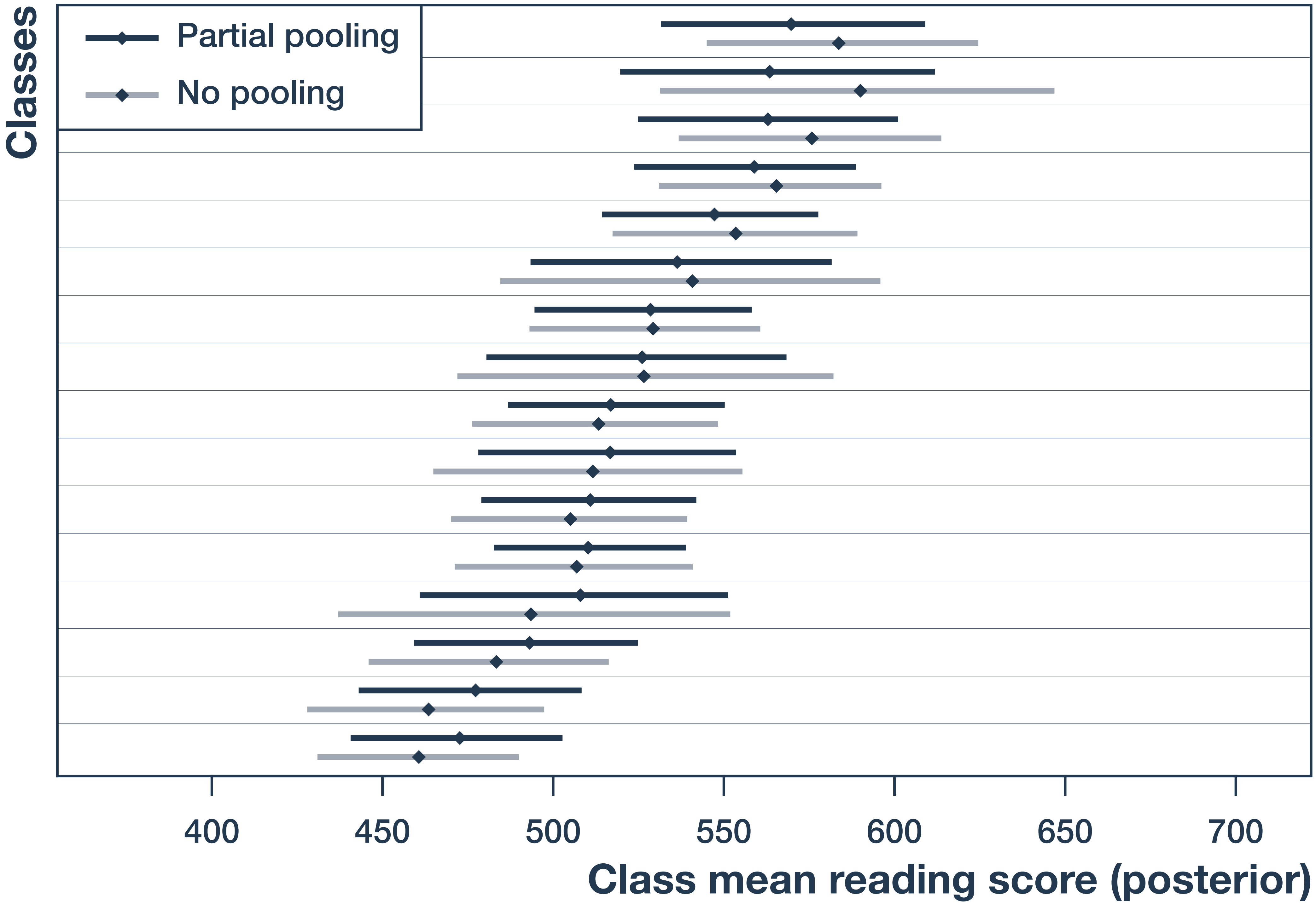
$$\sigma \sim \text{Unif}(0, 100)$$
$$\gamma \sim \text{Norm}(500, 100)$$
$$\eta \sim \text{Unif}(0, 100)$$

	Mean		***90% credible interval***
$\boldsymbol{\sigma}$	50.80	42.73	58.21
$\boldsymbol{\gamma}$	523.95	503.52	545.22
$\boldsymbol{\eta}$	39.78	22.90	59.29
$\boldsymbol{\alpha_1}$	492.93	461.80	526.99
$\boldsymbol{\alpha_2}$	472.53	441.40	503.23
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{\alpha_{16}}$	525.66	481.70	569.47

Partial pooling



Partial pooling



Summary

Complete pooling

- Disregards nested levels
- Pools all data into same group
- Precise estimate of mean
- **Underfit:** Errs systematically in prediction

WAIC: 767.0

Eff. Param: 1.7

Partial pooling (random effects)

- Group-level estimates in population context
- Mutual distribution allows information sharing
- Small groups take cues from the rest of the groups
- **"Just right" fit:** Optimal balance of information pooling

WAIC: 751.0

Eff. Param: 12.5

No pooling (fixed effects)

- Independent estimate for each group
- No information shared between groups
- Imprecise estimates for smaller groups
- **Overfit:** Does poorly in out-of-sample prediction

WAIC: 754.4

Eff. Param: 15.6