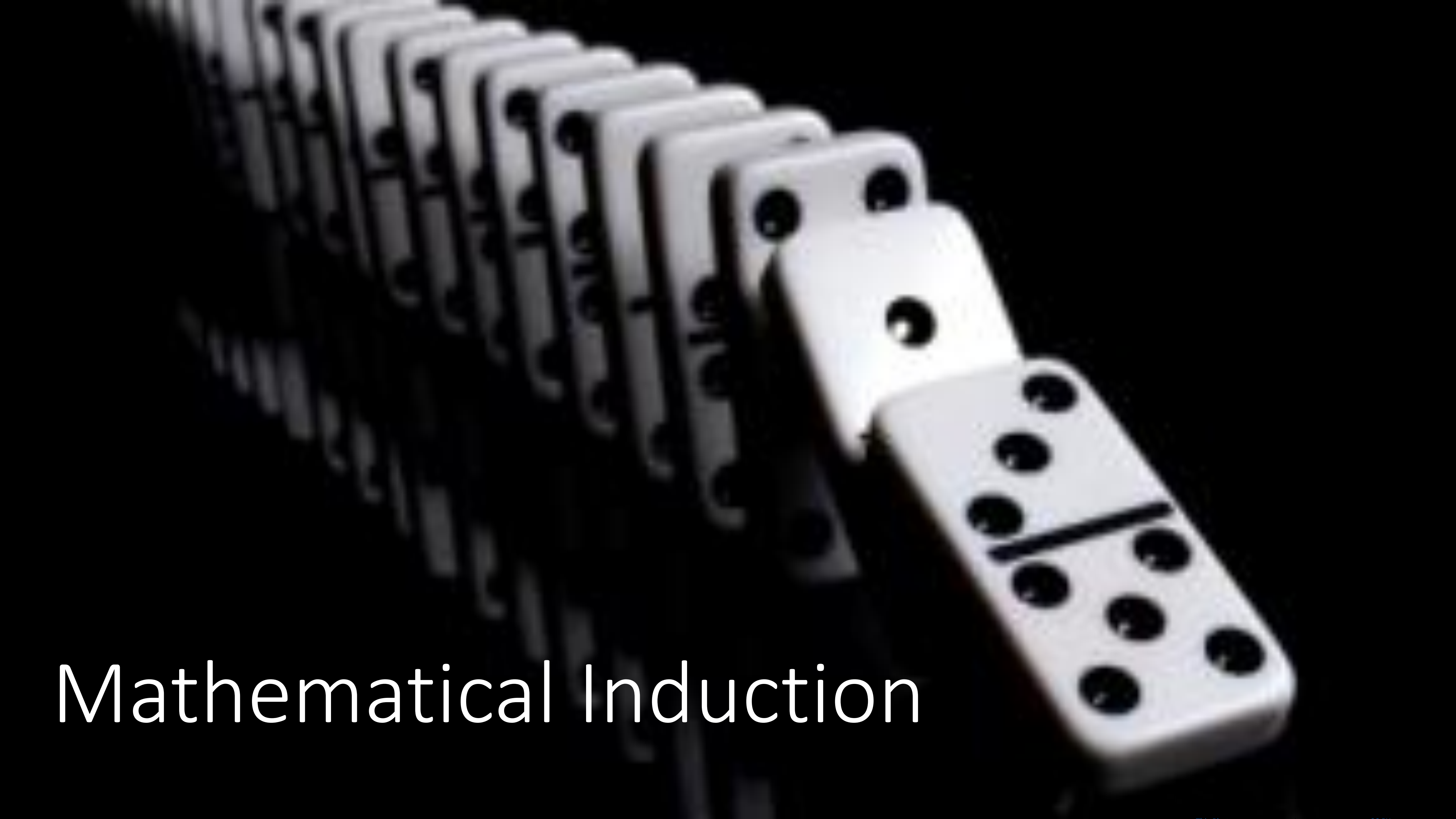
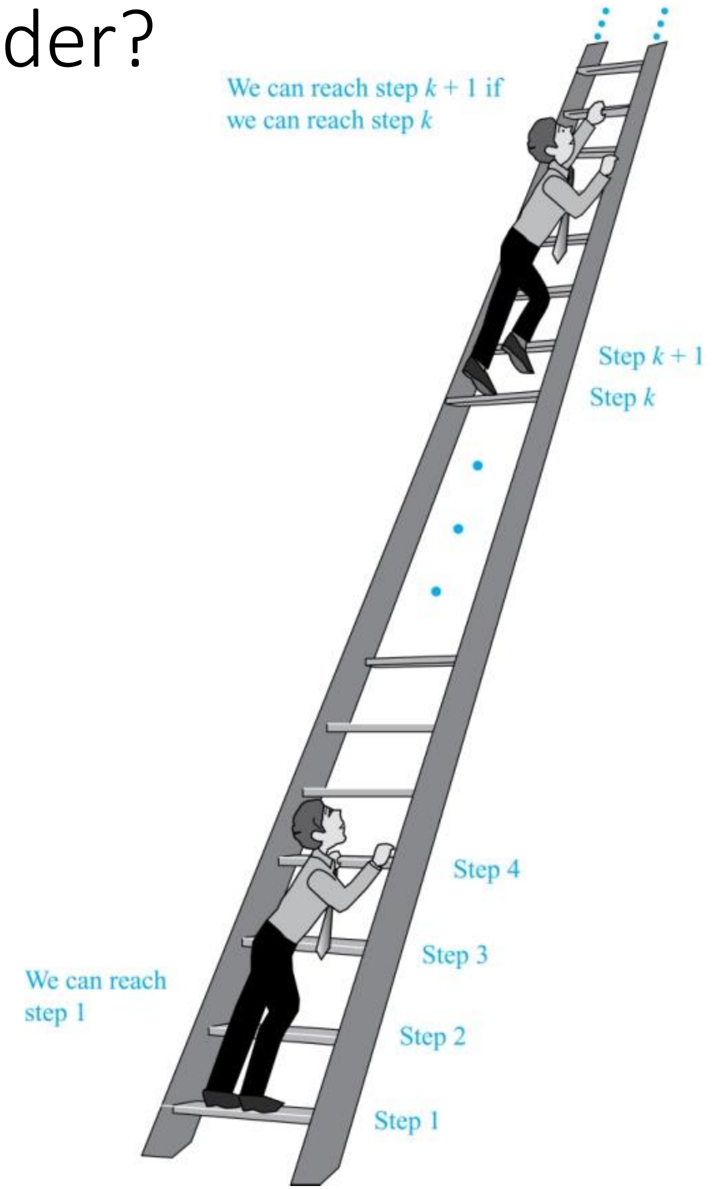


Mathematical Induction

Climbing an Infinite Ladder

Can we reach ANY step in this *infinite* ladder?

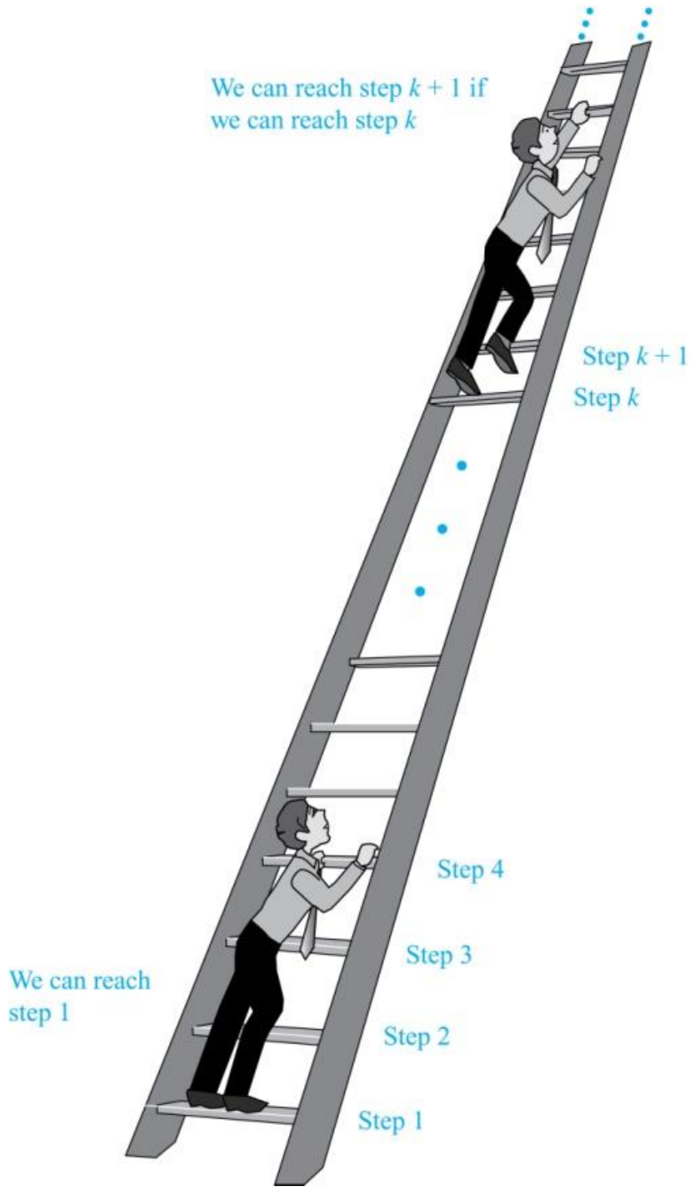


Climbing an Infinite Ladder
Can we reach ANY step in this *infinite* ladder?

Suppose we have an infinite ladder, and we can show:

1. We can reach the first rung of the ladder.
2. If we can reach a *particular* rung of the ladder, then we can reach the next rung.

** or, equivalently, any <u>arbitrary</u> rung of the ladder*



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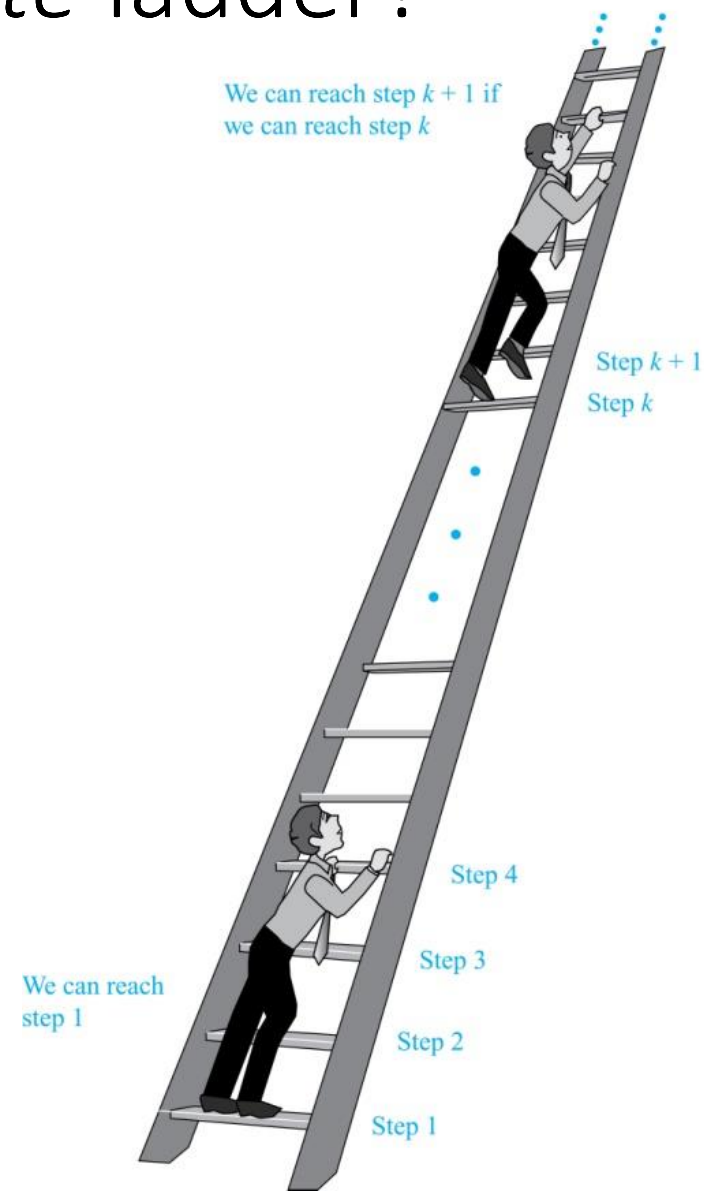
From (1), we can reach the first rung.

Then by applying (2), we can reach the second rung.

Applying (2) again, the third rung. And so on.

We can apply (2) any number of times to reach any particular rung, no matter how high up.

This example motivates **proof by mathematical induction**.



Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, we complete these steps:

- *Basis Step*: Show that $P(1)$ is true.
- *Inductive Step*:
  - First assume: $P(k)$ is true for an arbitrary k (***inductive hypothesis***)
  - Then show: If $P(k)$ is true, then $P(k + 1)$ is true for all positive integers k.
    - That is, show: $P(k) \rightarrow P(k + 1) \; \forall \; k$.
    - (Using the if-then, or *implication* symbol "→", and *for all* operator "∀")

Where "P" is a what is called a proposition.

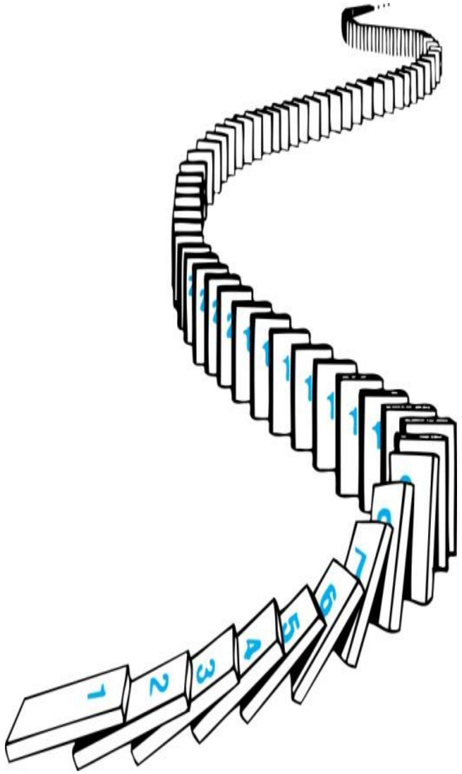
A proposition is a statement that must have a truth value; that is, it must be either true or false.

P(n) is that proposition with a variable that has been set to the value "n".

Another view of how Mathematical Induction Works

Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

Let $P(n)$ be the proposition that the nth domino is knocked over.



We know that the first domino is knocked down, i.e., $P(1)$ is true .

We also know that if whenever the kth domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Hence, all dominos are knocked over.

$P(n)$ is true for all positive integers n.

Proving a Summation Formula by Mathematical Induction

Example: Show that: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$1+2+3+\cdots+n$

Note: We have to have a conjecture **first.** Mathematical induction can be used to prove it correct. It is not a tool for *discovering* such conjectures.

Proving a Summation Formula by Mathematical Induction

Example: Show that: $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Solution:

Basis Step: Show that $P(1)$ is true.
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Proving a Summation Formula by Mathematical Induction

Example: Show that: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Solution:

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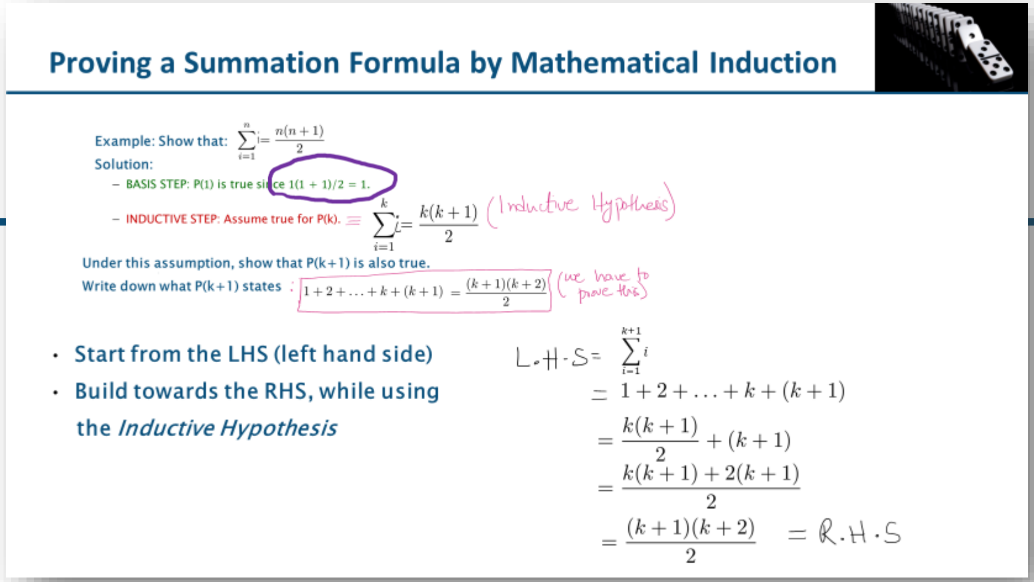

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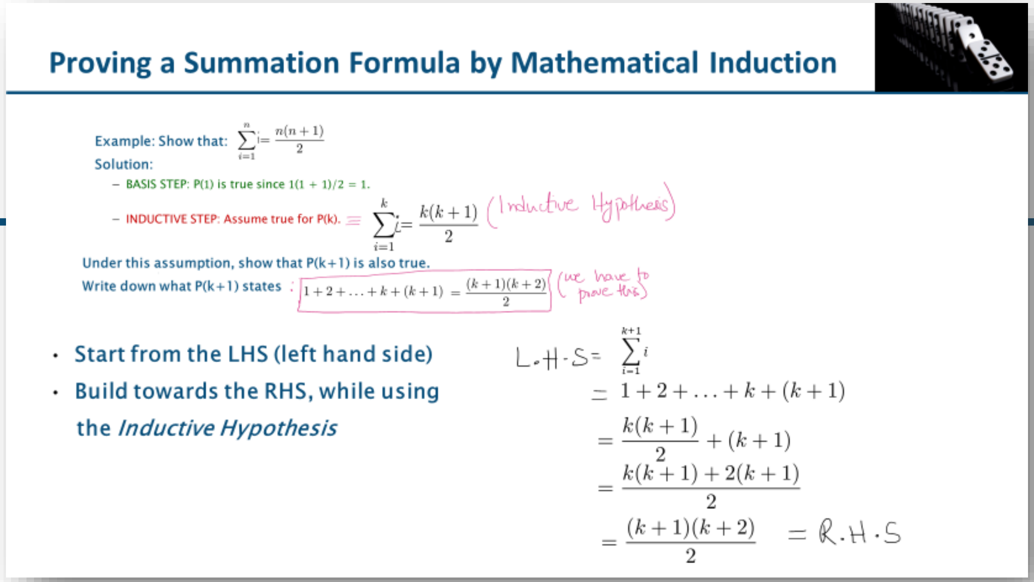

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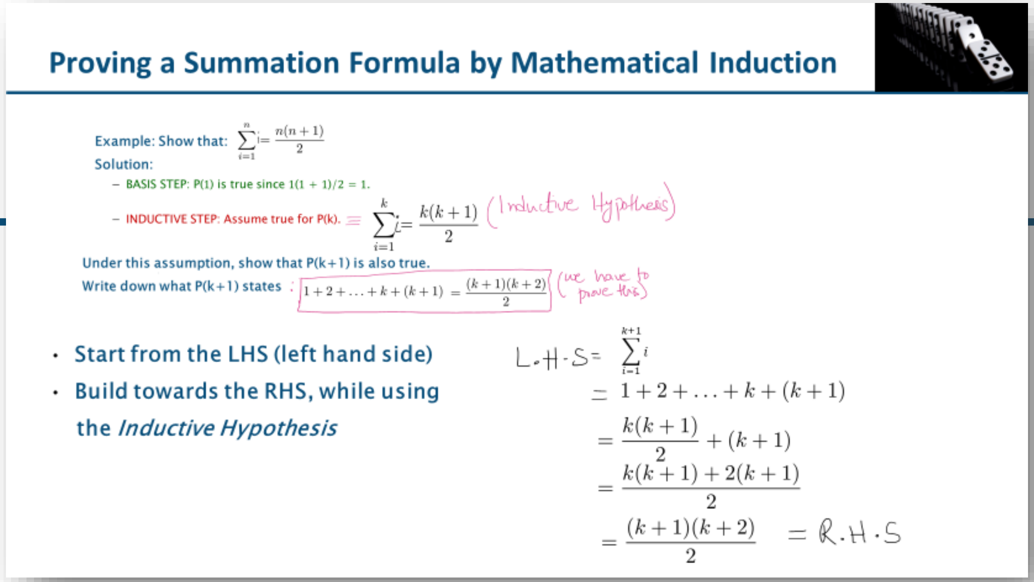

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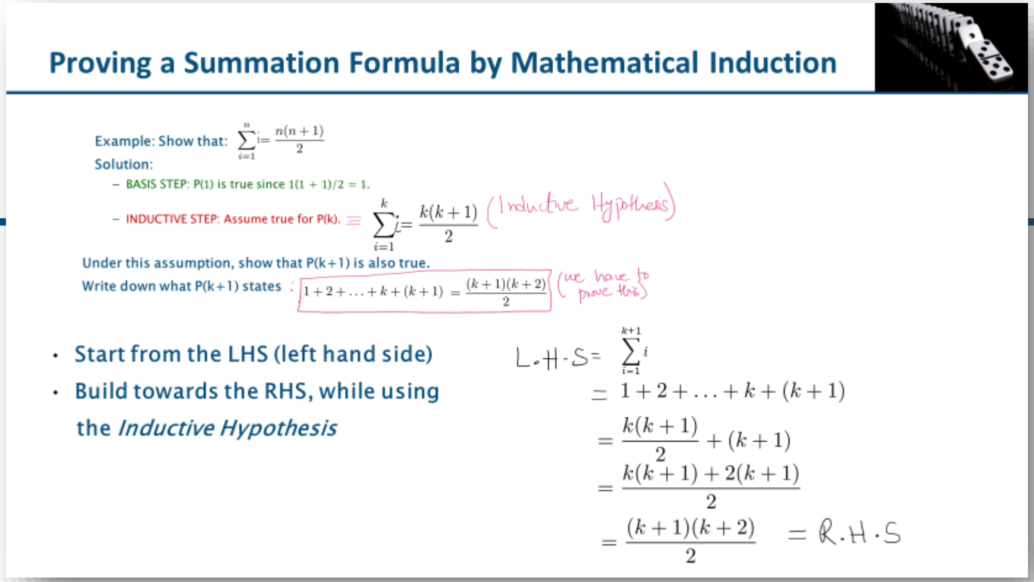

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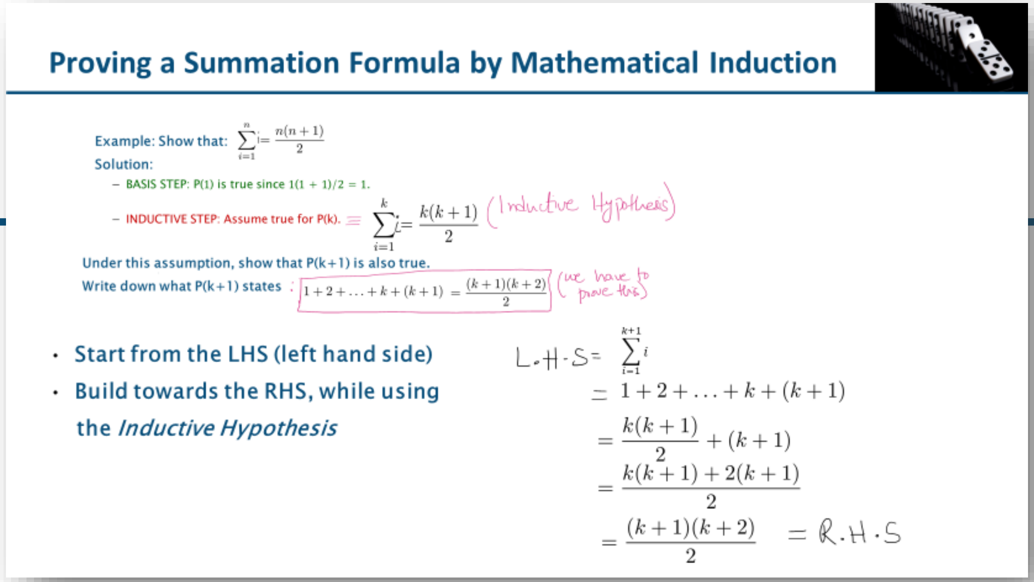

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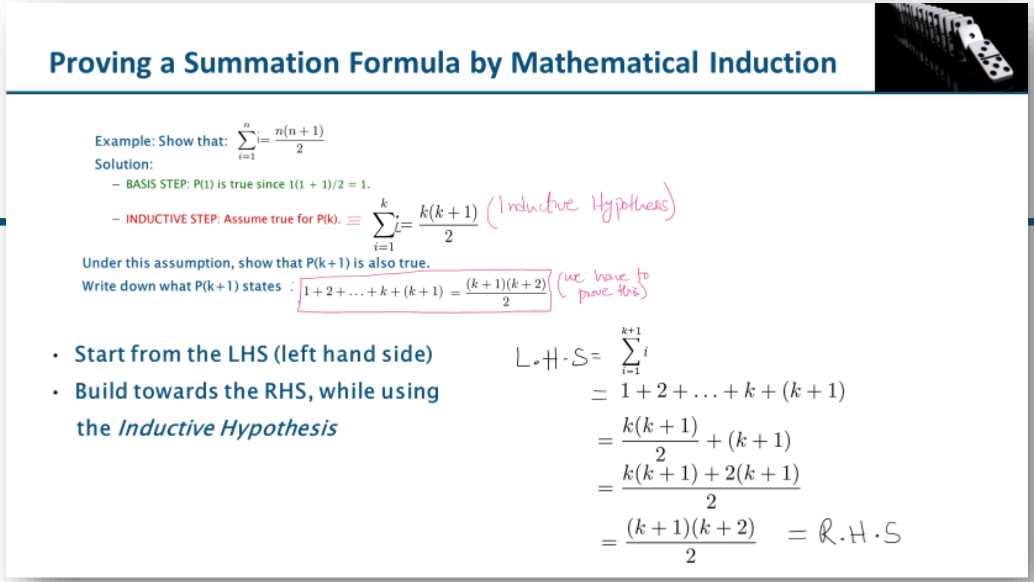

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– then show P(k+1) is true *using the inductive hypothesis, i.e. P(k)*

– in many cases this involves show that the lhs = rhs of some equation

– in such cases start manipulating the lhs

keeping an eye on rhs as this is where you want to end up

look out for where you can *use* the inductive hypothesis so you can use it

Practice ☺