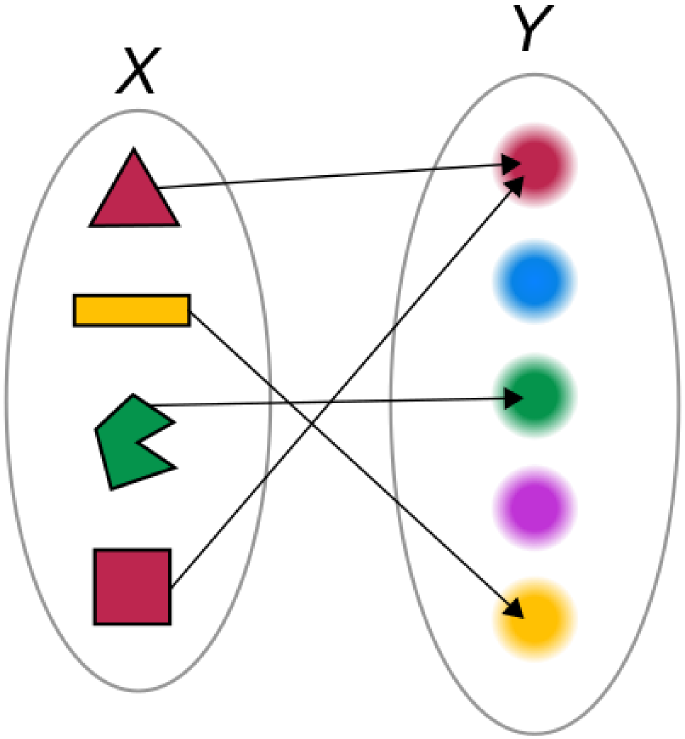
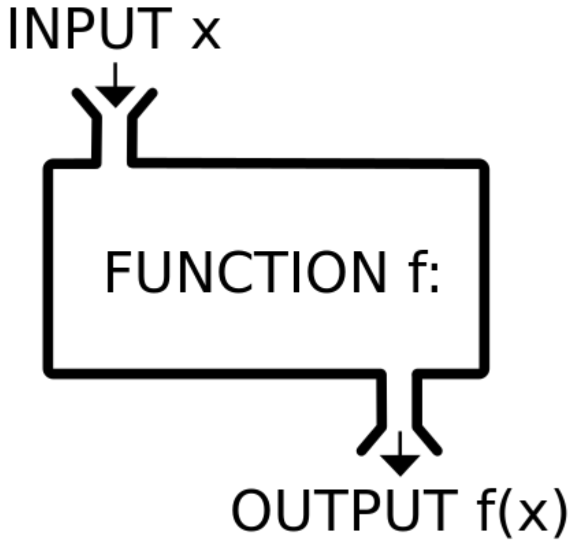
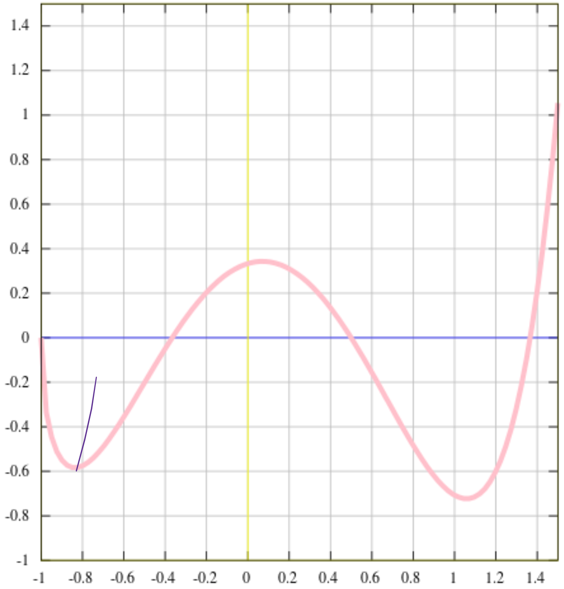
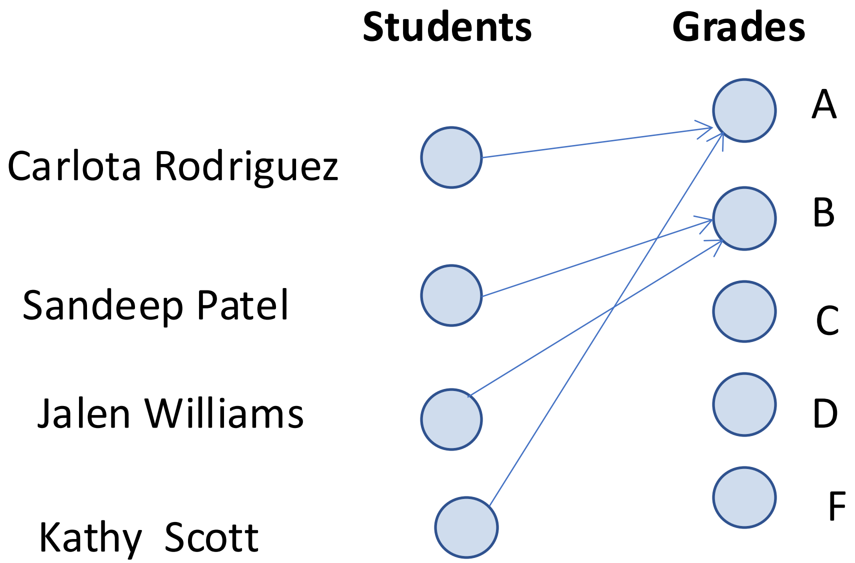
Functions


X
Y

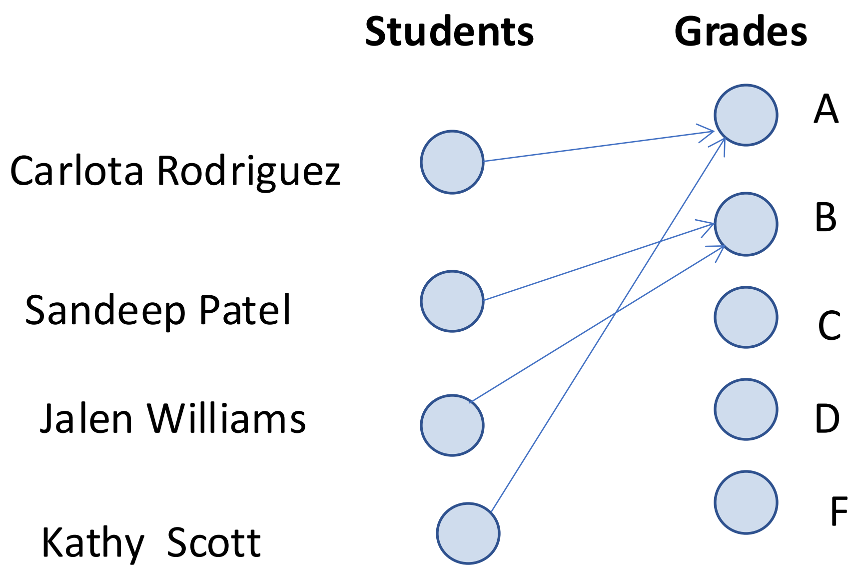

INPUT x
FUNCTION f:
OUTPUT f(x)


1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4


Students
Grades
Carlota Rodriguez
Sandeep Patel
Jalen Williams
Kathy Scott
A
B
C
D
F

What is a function?

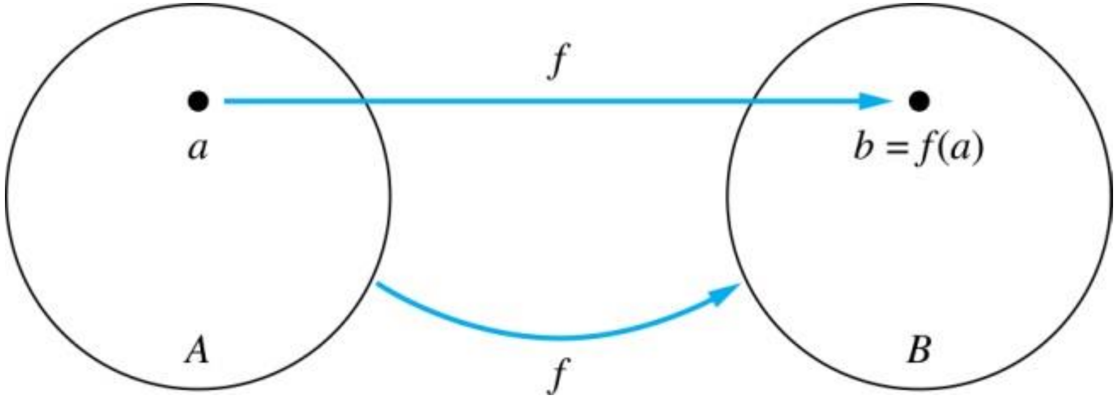
A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element a of A to exactly one element b of B.



$$f(a) = b$$

➢ Functions are also called *mappings* or *transformations*.

Functions - Terminology



Domain

Codomain

Given a function $f\colon A \to B$

- A is called the *domain* of f.
- B is called the *codomain* of f.

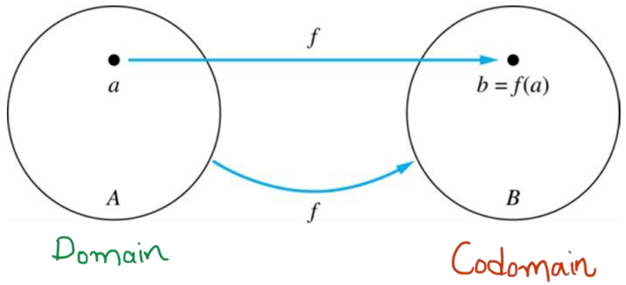
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.

- The *image* (or *range*) of f, denoted by $f(A)$, is the collection of all images of f. That is:

$$f(A) = \{f(a) | a \in A\} = \{b \in B | \exists a \text{ s.t. } f(a) = b\}$$

  - We may not be *mapping* to *every* element in B, so the codomain and range are not necessarily the same set.

Representations of Functions



- An explicit enumeration of the assignment
  - Table, list, etc.
  - Students-grades example

- A formula
  - *E.g.:* $f(x) = x + 1$

- A plot

- A computer program
  - E.g.: A Python program that, given an integer n, produces the n-th Fibonacci Number

Quick Quiz

$f(a) \; = ?$

The image of d is ?

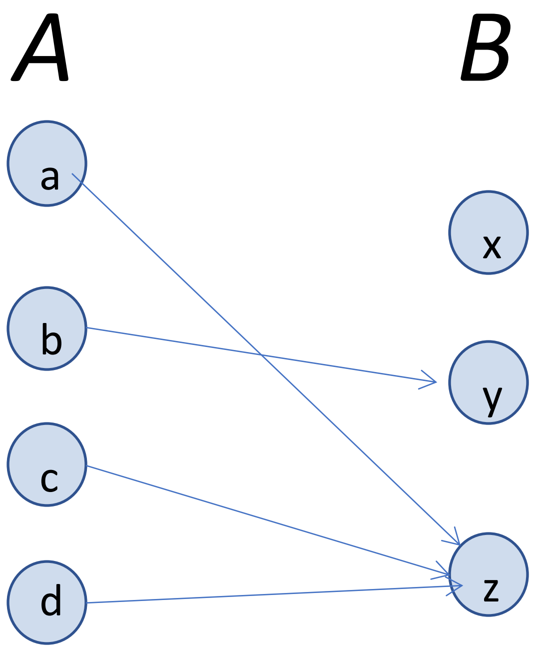
The domain of f is ?

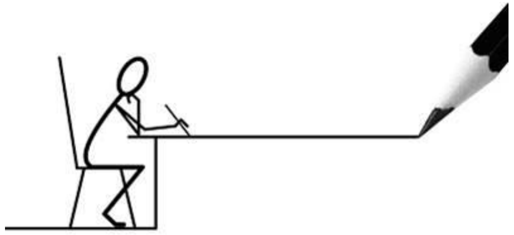
The codomain of f is ?

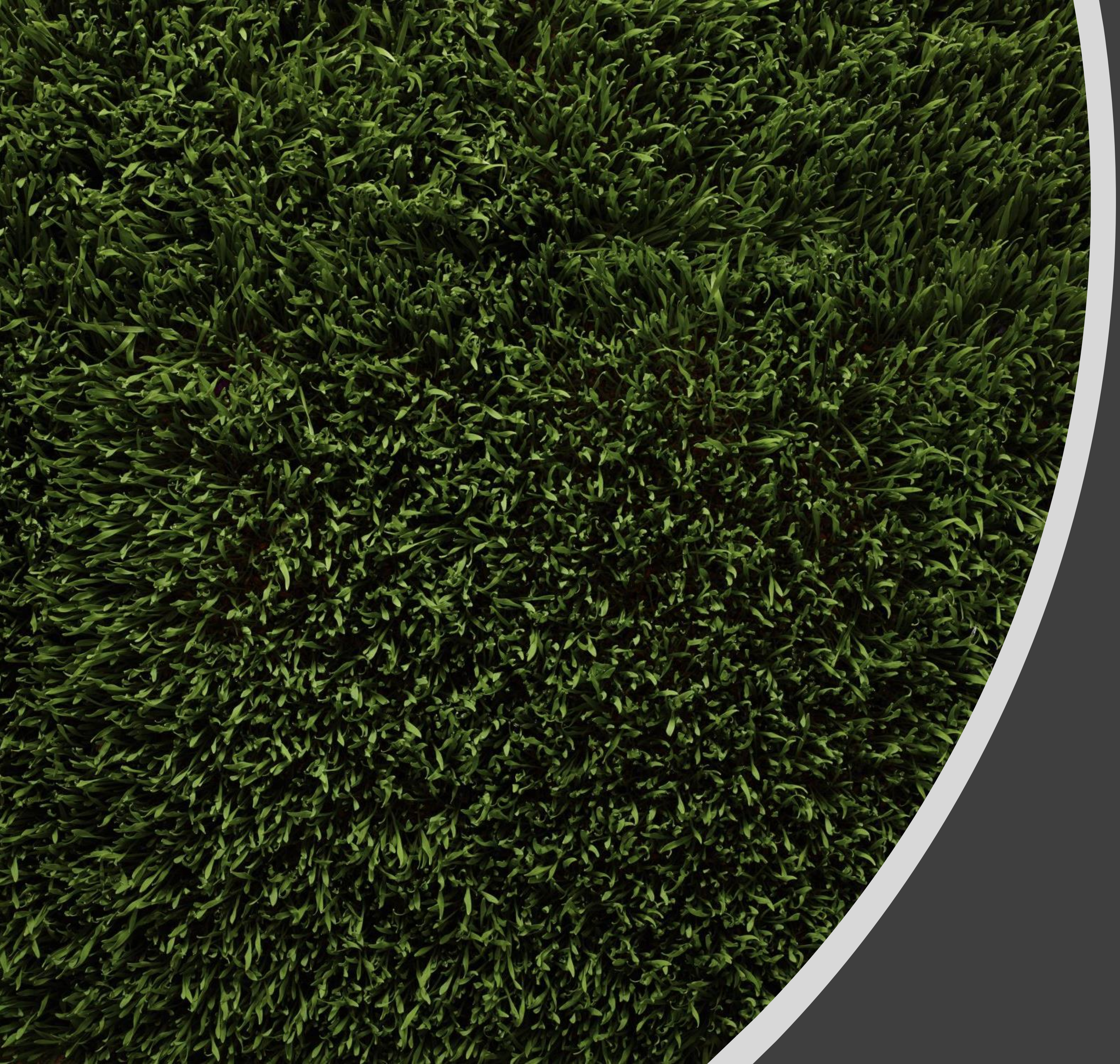
The preimage of y is ?

The image of f: $f(A) \; = ?$

The preimage(s) of z is (are) ?



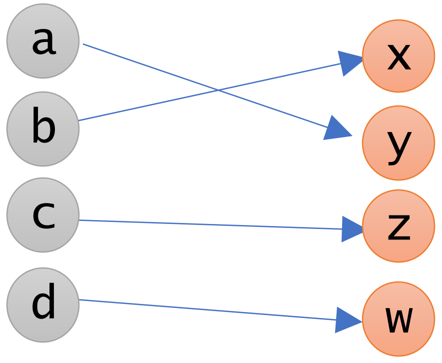




Injective, Bijective, and Surjective Functions

Injective Functions

A function $f: A \rightarrow B$ is called *injective*, or *"one–to–one"*, if it maps different elements of A to different elements of B



That is, f is injective if, for all $x, y \in A$:

$$x \neq y \quad \Longrightarrow \quad f(x) \neq f(y)$$

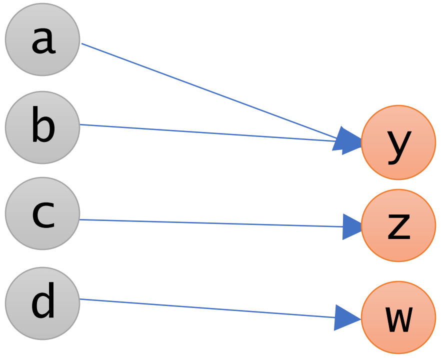
Equivalently: $f(x) = f(y) \quad \Longrightarrow \quad x = y$

Surjective Functions

A function $f: A \to B$ is called *surjective*, or *"onto"*, if it "covers" entirely its codomain. That is, if

$$f(A) = B$$

Equivalently: for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$

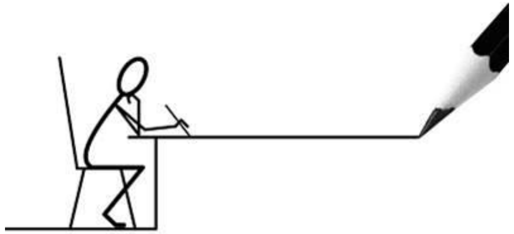


surjective function

Quick Quiz – Injections/Surjections

Let $A = \{a, b, c, d\}$, $B = \{1,2,3\}$, and $f = \{(a, 3), (b, 2), (c, 1), (d, 3)\}$.

Is $f: A \rightarrow B$ injective or surjective?

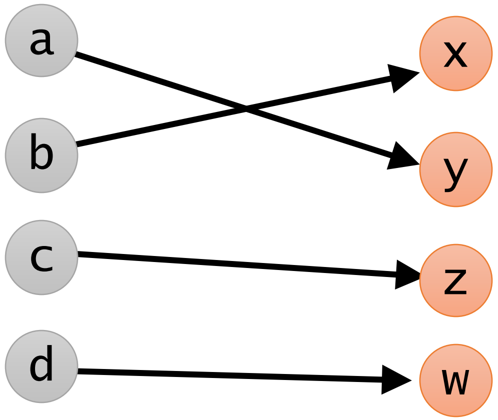


Bijections

A function $f: A \to B$ is called a *bijection*, or *"on-to-one correspondence"*, if it is both an injective *and* a surjective.

surjection injection

Equivalently: every element of the codomain has a <u>unique</u> preimage

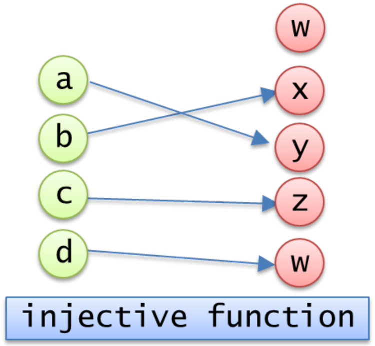


bijective function

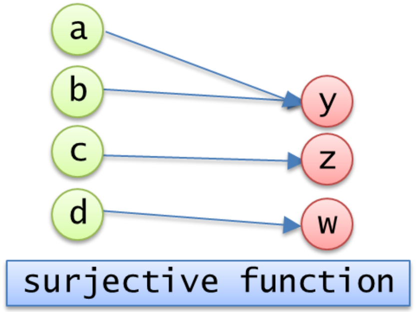
Bijection: Example

Comparing Cardinalities

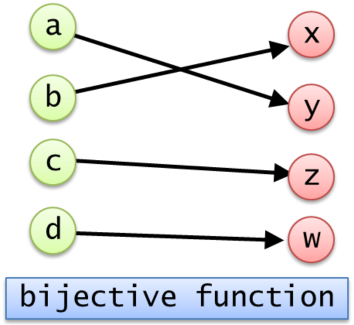
- If $f: A \to B$ is injective, then $|A| \leq |B|$
  - since each element of A has a different image in B



- If $f: A \to B$ is surjective, then $|B| \leq |A|$
  - since each element of B has a preimage in A

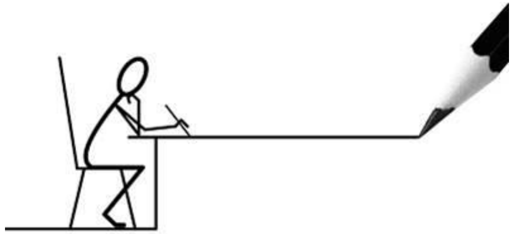


- If $f: A \to B$ is bijection, then $|A| = |B|$
  - i.e. the sets A and B have *exactly* the same size



Hmmm…

- If you can show that there exists a bijection between two sets, you can prove they are **of the same size**
  - (Even if they are both "infinite"!)



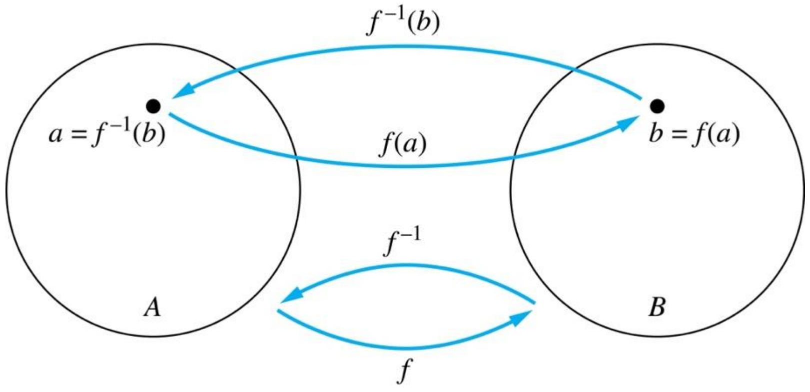
Inverse Functions

Inverse Functions

A function $f: A \rightarrow B$ is *invertible*, if and only if it is a bijection.

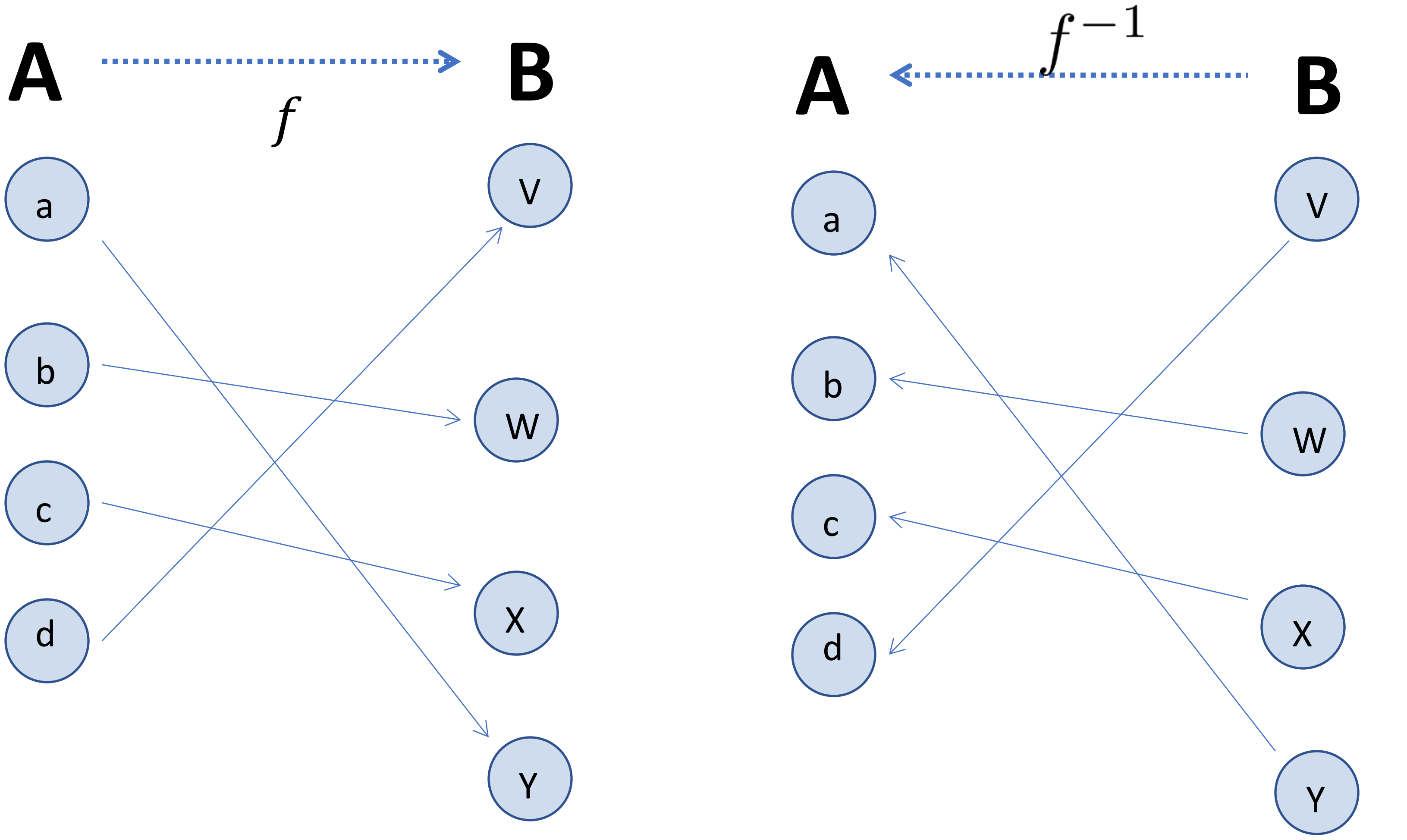
Then, its inverse function, denoted by $f^{-1}: B \rightarrow A$, is the function defined by:

$$f^{-1}(b) = a \quad \Leftrightarrow \quad f(a) = b$$



Notice: $f^{-1}(f(a)) = a$ and $f(f^{-1}(b)) = b$ for all $a \in A$ and $b \in B$

Inverse Functions

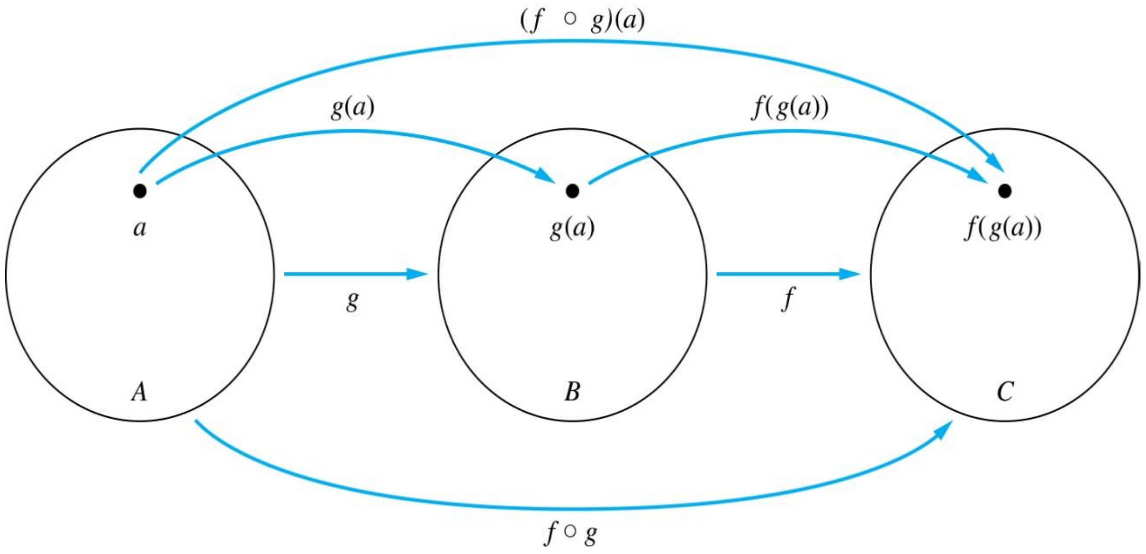


Composition of Functions

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

- **Definition**: Let functions $f\colon B \to C$, $g\colon A \to B$. The *composition* of f with g, is the function $f \circ g\colon A \to C$ defined by

$$(f \circ g)(a) = f(g(a))$$



Composition of Functions

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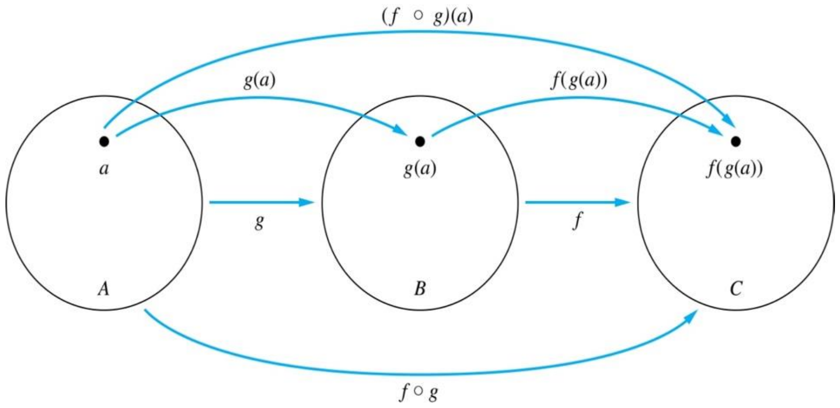
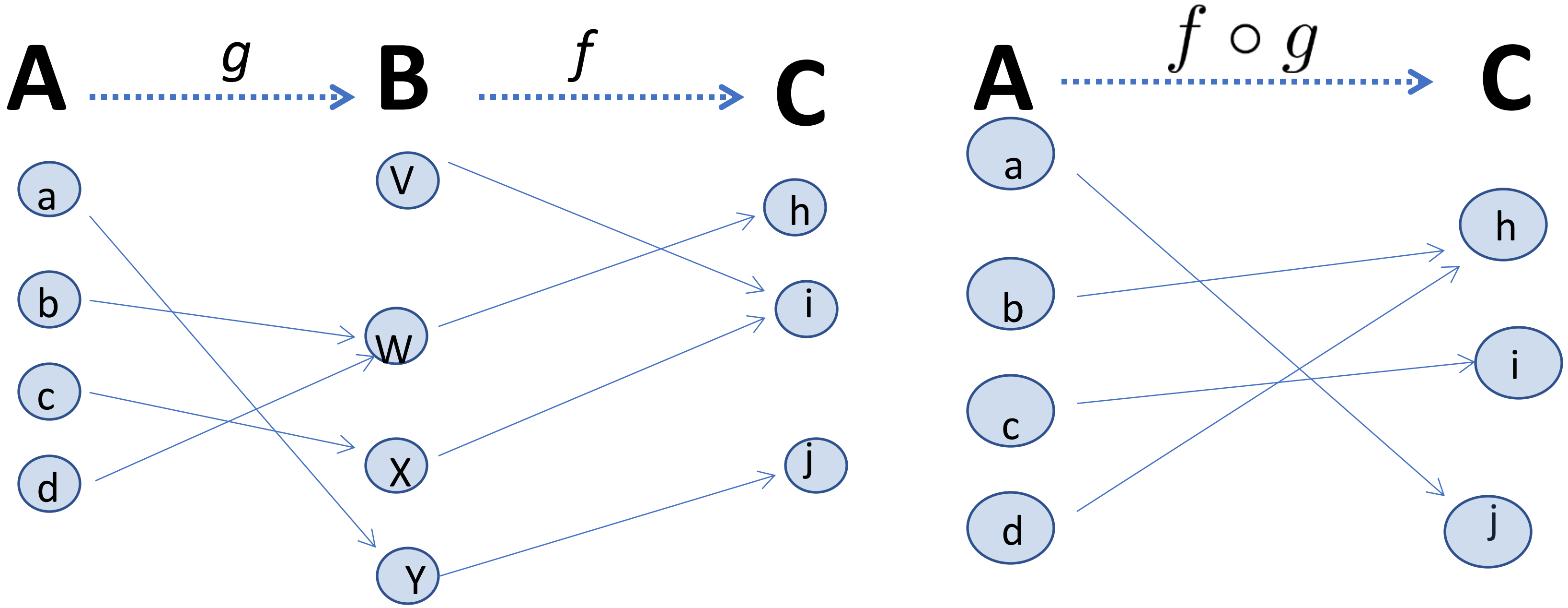


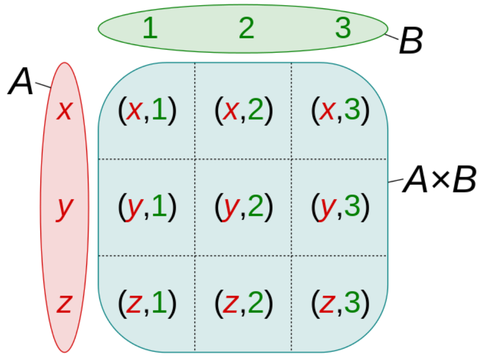
- Example:
  - g(x) maps students to percentage marks
  - f(x) maps percentage marks to letter grades

  - so $(f \circ g)(x)$ maps students to letter grades

Composition



Relations



Binary Relations - Definition

- Let $A,\ B$ be sets. A *binary relation* R is a subset $R \subseteq A \times B$.
- If $(a, b) \in R$ we say that "a is related to b (by R)".
- Notation: aRb

$a \leq b \rightsquigarrow$ "$\leq$" is a binary relation on $\mathbb{N} \times \mathbb{N}$

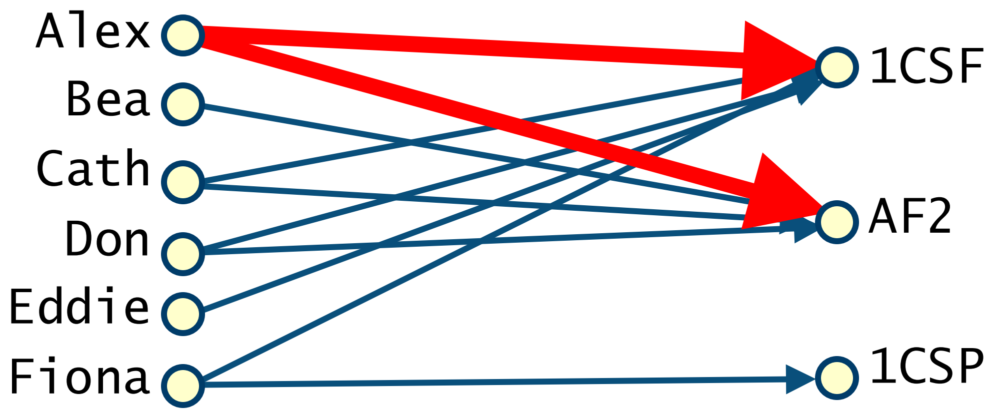
Binary Relations - Example

So: Every function is a relation!
But the opposite not true

- Students: Alex, Bea, Cath, Don, Eddie, Fiona
- Subjects: 1CSF, 1CSP, AF2

- Let R be the relation of students who passed the subjects:

R={(Alex,1CSF),(Alex,AF2),(Bea,AF2),(Cath,AF2),(Cath,1CSF),
(Don,AF2),(Don,1CSF),(Fiona,1CSF),(Eddie,1CSF),(Fiona,1CSP)}



Notice this is **not a function**
e.g. Alex *related* to 1CSF and AF2

Relations are a ***generalization*** functions.

function

$\{(\alpha, f(\alpha))\} \subseteq A \times B$