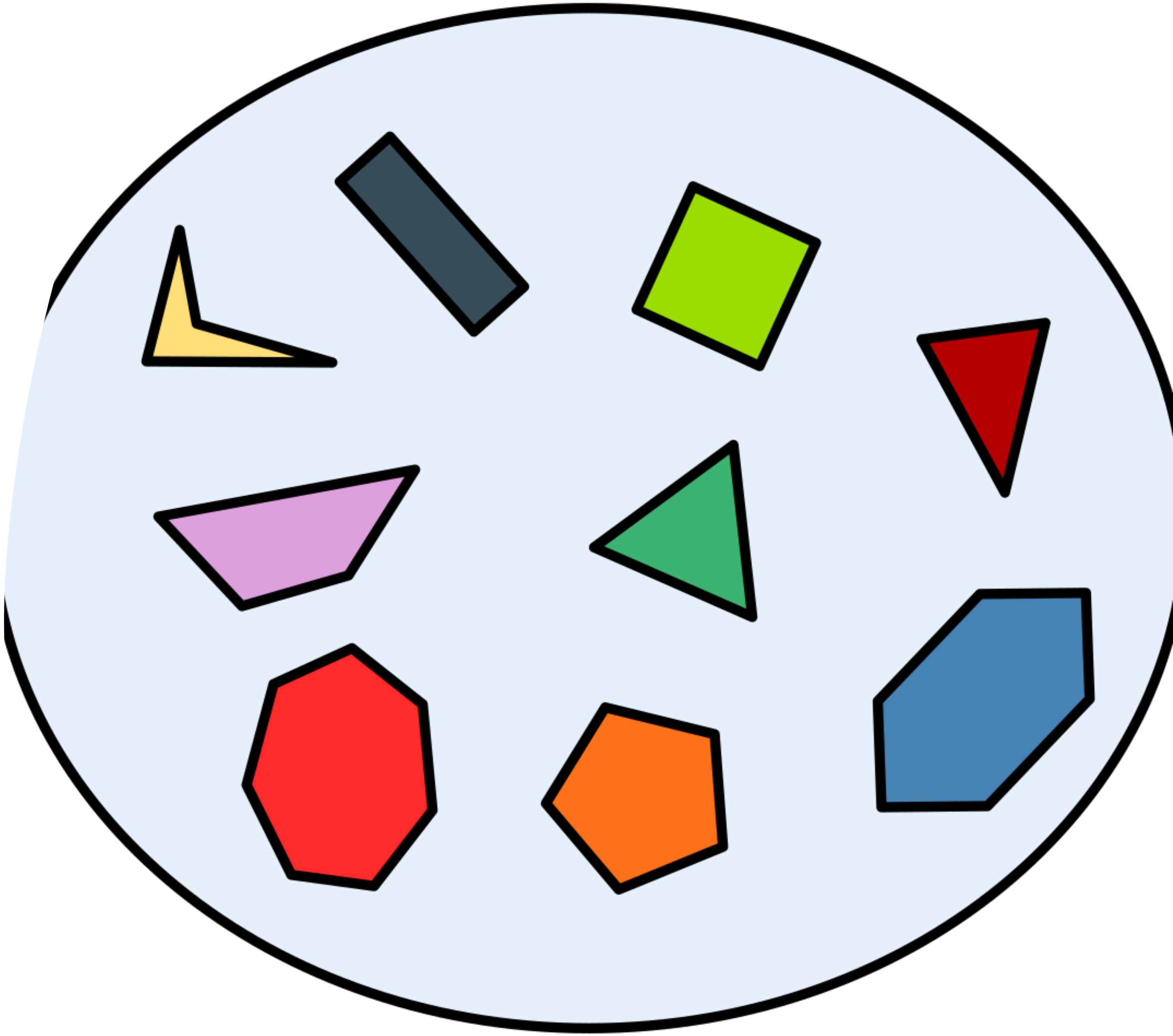
Sets

The “Mathsy” Perspective



SETS - MOTIVATION

Study *collections* in an *organized fashion*

Sets – Introduction

Set theory is a very **important** area of **mathematics.**

The **set** is considered *the most fundamental notion* in **all of mathematics!**

The **notion of functions** is built **on top of** sets:

A function assigns to each element of one set, exactly one element of another set

Set: Definition

- A **set** is a collection of objects (called its *elements*/members) with a key property: **membership**
  - That is: *"is this element a member of this set or not"?*

- Hence a *set* is an unordered collection of (unique) objects.
  - so, no concept of first element, n^{th} element, neighbouring element, etc.
  - repetition does not matter
    - has no impact on whether or not an element is a member of a set

- A set is said to *contain* its elements.

- The notation $a \in A$ denotes that a is an element of the set A.
  - If a is *not* a member of A, we write: $a \notin A$

Notations for Describing a Set

- Roster notation

- Set-builder notation

Roster Notation

- $S = \{a, b, c, d\}$

- Order has no effect:

$$S = \{a, b, c, d\} = \{b, c, a, d\}$$

- Multiplicity is irrelevant:

$$S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$$

- Ellipses (…) may be used to describe a set without listing all its members, when the pattern can be clearly derived from the context:

$$S = \{a, b, c, d, \dots, z\}$$

$$S = \{1, 2, 3, \dots\}$$

Roster Notation — Limitations

- Using the roster method, try to describe:
  - All pair of natural numbers (x,y) that satisfy the equation $x + y = 1000$
  - All positive rational number

➢ Either not possible, or very inconvenient…

Set-Builder Notation

- Specify the property/properties that ***characterize*** its membership:

$$S = \{x \mid x \text{ is a positive integer less than } 100\}$$
$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$
$$P = \{x \in \mathbb{Z}_+ \mid x \text{ is odd and } x < 10\}$$

- A predicate may be used for succinctness

(where a predicate is basically a logical statement that can be True or False, depending on some variable(s).)

$$S = \{x \mid P(x)\}$$

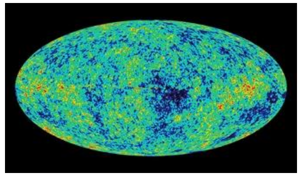
$|$ can be read as "such that"
$\in$ can be read as "belongs to"
$\boldsymbol{P(x)}$ here means P is a *predicate* that is True for x

- *Example:* $S = \{x \mid Prime(x)\}$
- Example: Set of positive rational numbers*:*

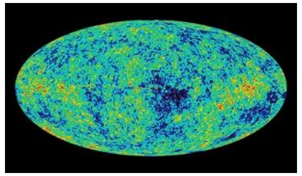
$$\mathbb{Q}_+ = \{x \in \mathbb{R} \mid x = p/q, \text{for some positive integers } p, q\}$$

Some Common Sets

- Natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
- Integers $\mathbb{Z} = \{\dots -3,\ -2, -1, 0, 1, 2, 3, \dots\}$
- Positive integers $\mathbb{Z}_{+} = \{1, 2, 3, 4, 5, \dots\}$
- Rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} \middle| \, p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$
- Real numbers $\mathbb{R}$



Universal Set and Empty Set

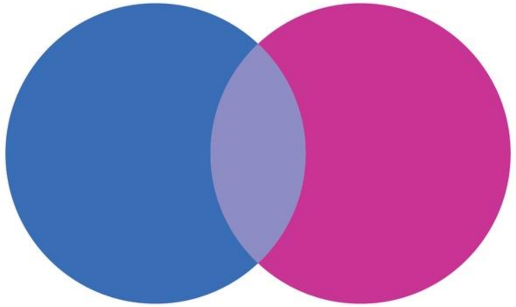


Universal Set and Empty Set

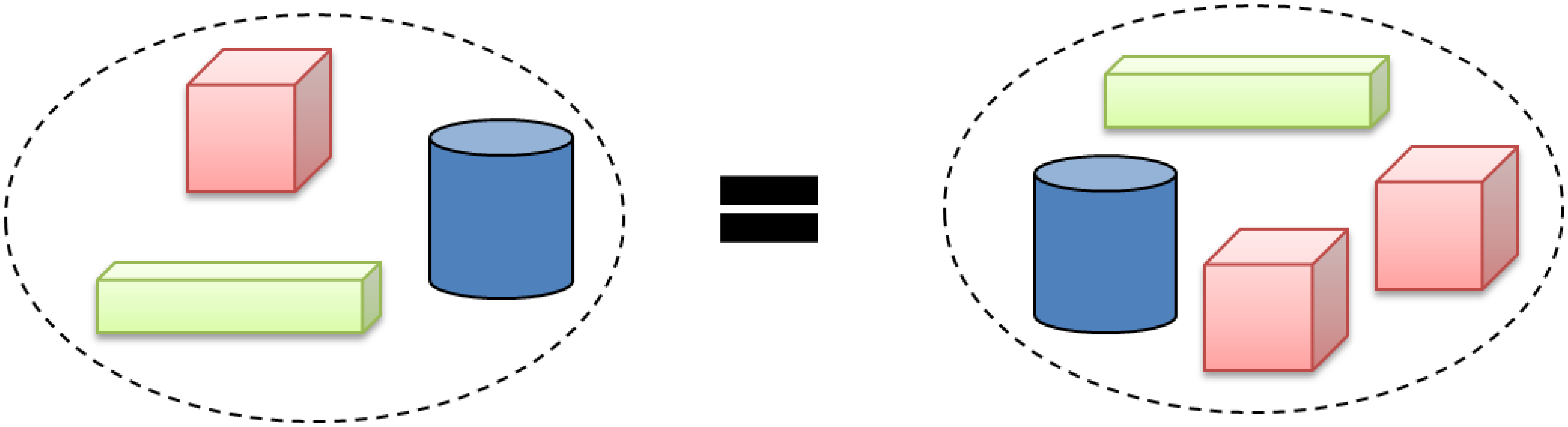
- The *universal set* U is the set containing *everything* currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Depends on **context**.

"Venn Diagram" is useful for illustrating relationships between sets

- The empty set $\emptyset = \{\}$ is the set with no elements.



Set Equality



Some pre-requisite symbols

$\forall$ *For all*

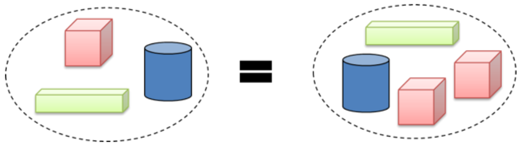
A $\rightarrow$ B ***If*** *A is true,* ***then*** *B is true*

$A \leftrightarrow B$ *A is true* ***if and only if*** *B is true*

$\wedge$ *Logical* ***AND***

$\vee$ *Logical* ***OR***

Set Equality



Set Equality

Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if

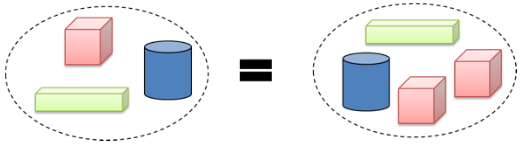
$$\forall x(x \in A \leftrightarrow x \in B)$$

- We write $A = B$ if A and B are equal sets.
- Examples:

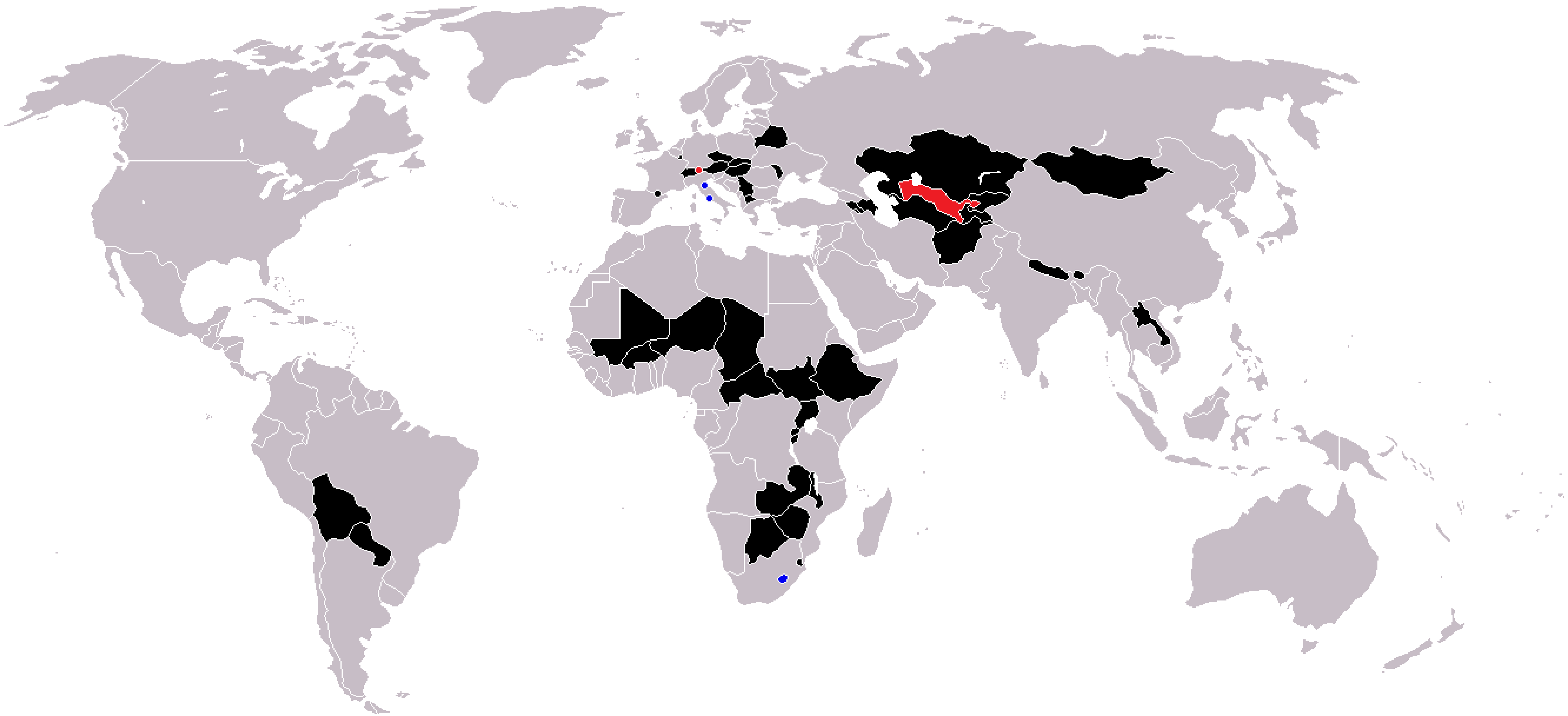
$$\{1, 3, 5\} = \{3, 5, 1\}$$

$$\{1, 5, 5, 5, 3, 3, 1\} = \{1, 3, 5\}$$

$$\{1, 3, 7\} \neq \{1, 3\}$$



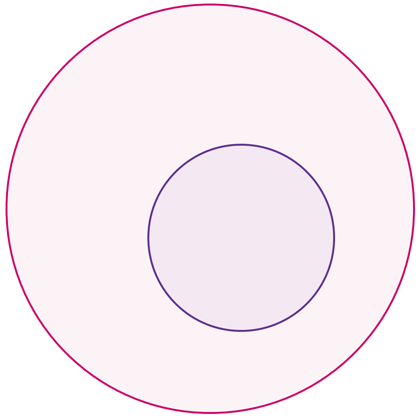
Subsets



https://upload.wikimedia.org/wikipedia/commons/4/48/Landlocked_countries.png

Subsets — Definition

- The set A is a *subset* of B, denoted by $A \subseteq B$, if and only if: every element of A is also an element of B.
- Formally: $A \subseteq B$ if and only if $\forall x(x \in A \rightarrow x \in B)$
- Note that:
  - $\emptyset \subseteq S$ for every set S.
  - $S \subseteq S$ for every set S.

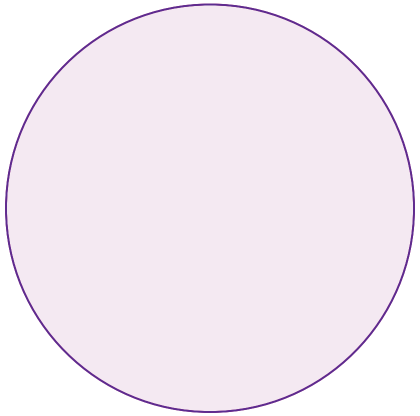


Venn diagram

Subsets and Equality

- Two set are **equal** if and only if they are **subsets of each other**

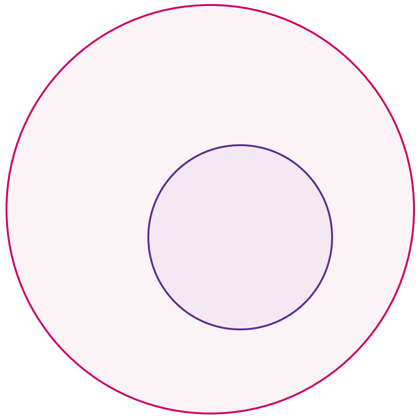
$$\mathrm{A} = \mathrm{B} \quad \text{if and only if} \quad A \subseteq B \wedge B \subseteq A$$



“Proper” Subset

- We say that A is a *proper* subset of B, if and only if:
  - A is a subset of B, but A is *not* equal to B:

$$A \subseteq B \ \wedge \ A \neq B$$



- In other words:
  - Every element of A is also an element of B
  - **But** there are some elements of B that are *not* in A

Set Cardinality

Set Cardinality

The *cardinality* of a finite set A, denoted by $|A|$,
is the number of *(distinct)* elements of A.

Examples:

- $|\emptyset| = 0$
- Let S be the set of all letters of the English alphabet. Then $|S| = 26$
- $|\{1, 2, 3\}| = 3$
- $|\{\emptyset\}| = 1$
- Set of integers?

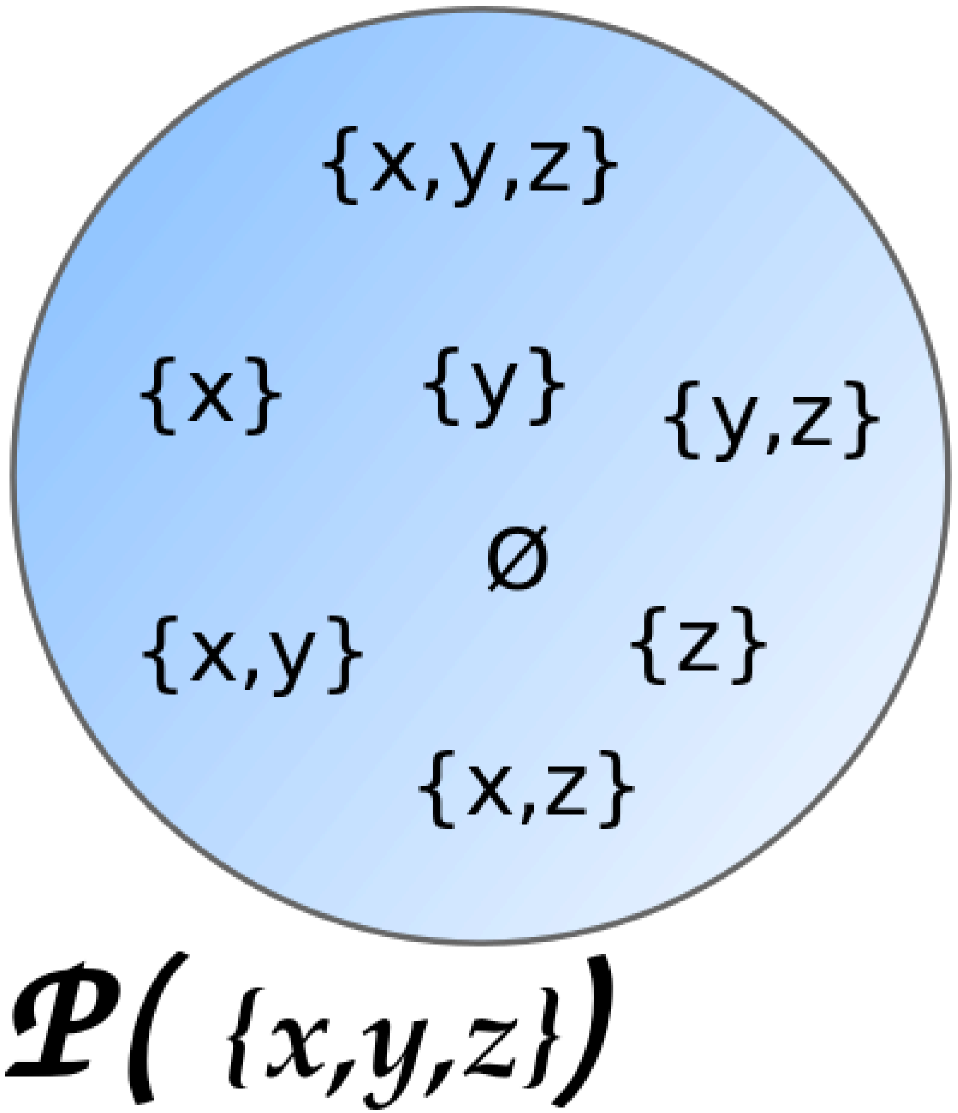
Set Cardinality

The *cardinality* of a finite set *A*, denoted by $|A|$,
is the number of *(distinct)* elements of *A*.

Examples:

- $|\emptyset| = 0$
- Let S be the set of all letters of the English alphabet. Then $|S| = 26$
- $|\{1, 2, 3\}| = 3$
- $|\{\emptyset\}| = 1$
- The set of integers is infinite! (∞)
    - The "cardinality" of infinite sets is a somewhat trickier topic.
    - Not all "infinities" are equal!

Power Sets



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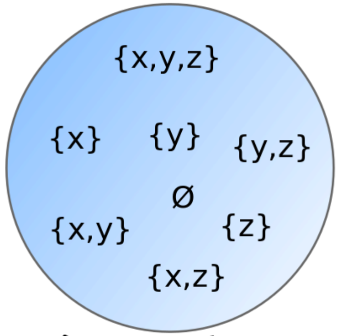
Power Sets

The *power set* of a given set A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

- Example: If $S = \{a, b, c\}$ then

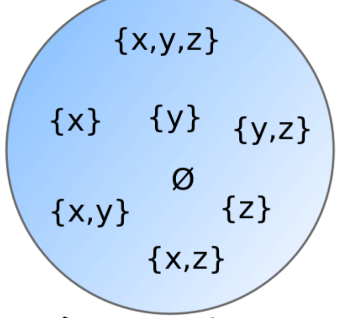
$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

- If a set has n elements, then the cardinality (i.e. size) of the power set is 2^n. That is,
  - $|A| = n \rightarrow |\mathcal{P}(A)| = 2^n$



$\mathcal{P}(\{x,y,z\})$

Cardinality of Power Sets



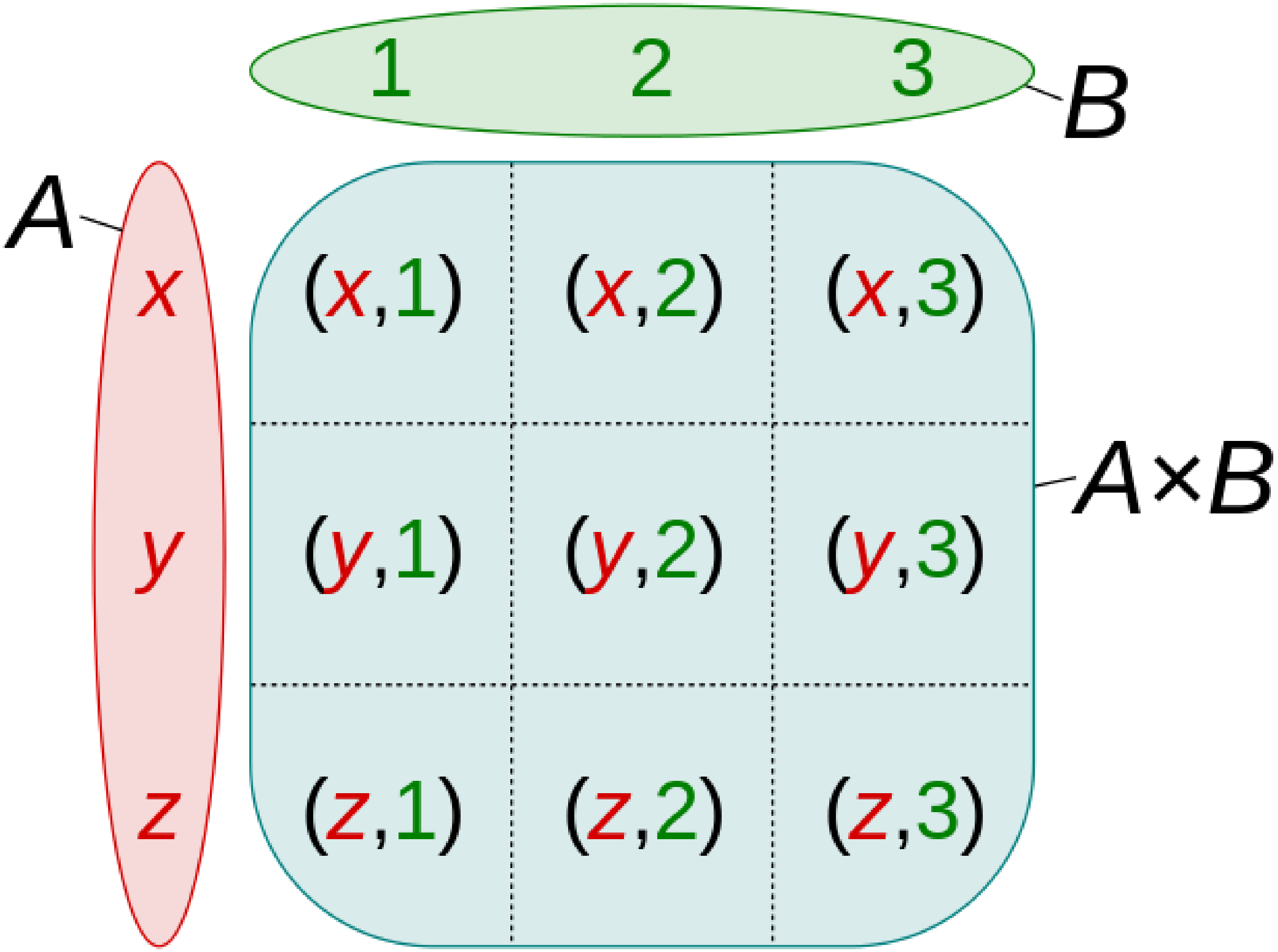
$\mathcal{P}(\{x,y,z\})$

- Let $S = \{a, b, c\}$. Each *subset* of S can be represented as a **bit string** of length $|S| = 3$, where each component is:
  - 1 if corresponding element is a member of S
  - 0 if not a member

a	b	c	
0	0	0	∅
0	0	1	{c}
0	1	0	{b}
0	1	1	{b,c}
1	0	0	{a}
1	0	1	{a,c}
1	1	0	{a,b}
1	1	1	{a,b,c}

- Therefore, the size of $\mathcal{P}(S)$ is 2^n:
  - the number of bit strings of length n
    (we will see at a later unit exactly why…)

Cartesian Product



René Descartes
(1596-1650)

Cartesian Product: Two Sets

The ***Cartesian product*** of two sets A and B, denoted by $A \times B$, is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example:

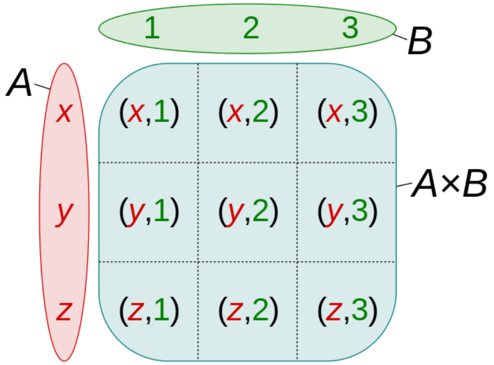
$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Note: $|A \times B| = |A| \cdot |B|$

$6 = 2 \cdot 3$

Can be generalized to *any* number of sets…



Cartesian Product Example: Cluedo!

If

S = set of all suspects
W = set of all weapons
L = set of all locations

Then the cartesian product

$$S \times W \times L$$

will describe exactly all possible "murder scenarios", i.e., all possible "configurations" of the game.

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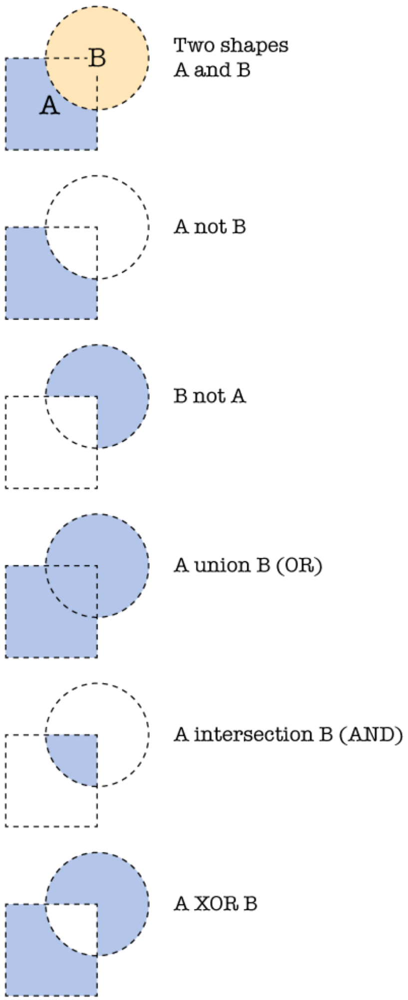
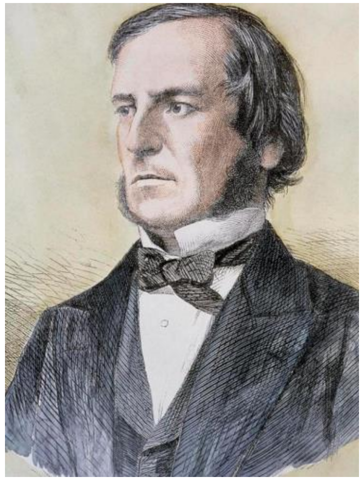
Set Operations

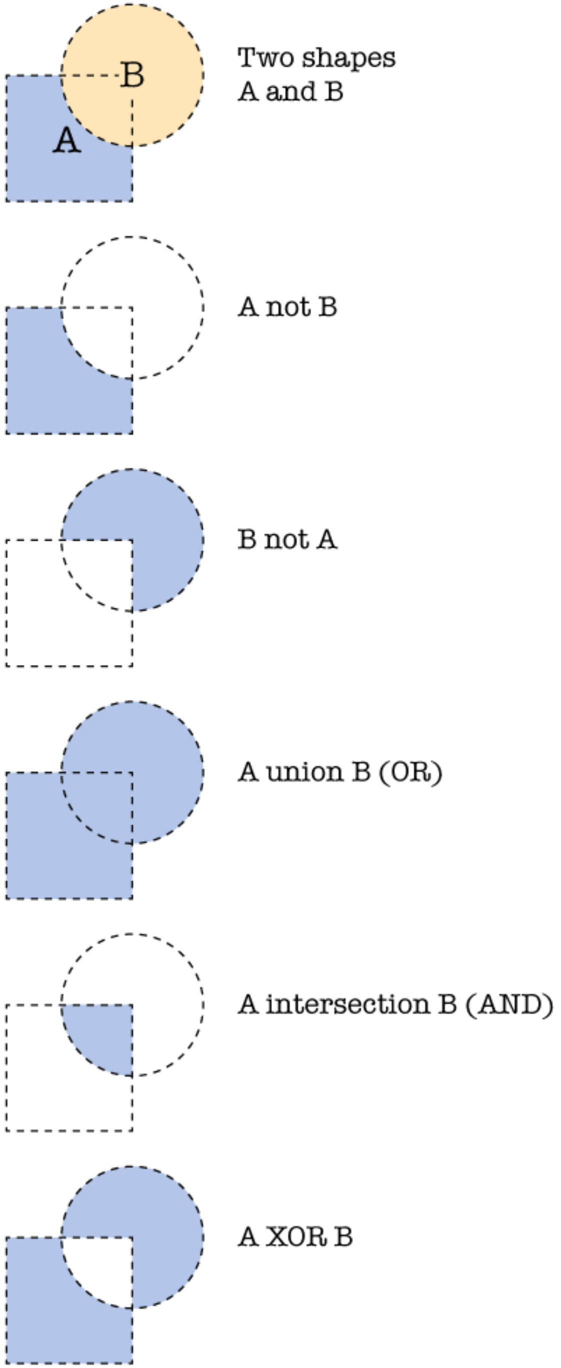


Boolean Algebra!

- Set theory can be interpreted as an instance of an algebraic system called a *Boolean Algebra*.
  - We will look at another Boolean algebra later called “propositional calculus”
- The operators in set theory are analogous to the corresponding Boolean operators we have already seen
- Recall the definition of universal set *U*
  - All sets are assumed to be subsets of *U*



George Boole (1815–1864)



Union

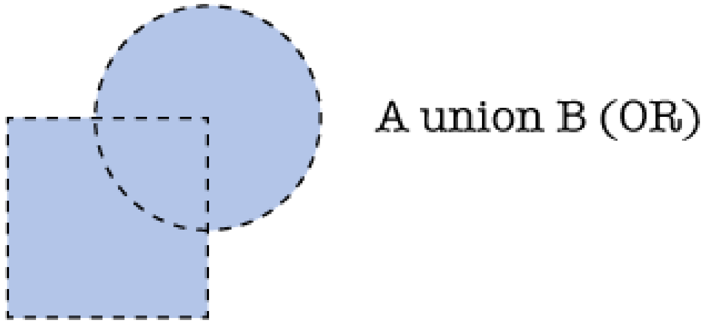
- **Definition**: The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x | x \in A \vee x \in B\}$$

Analogous to the logical OR

- **Example**: What is $\{1, 2, 3\} \cup \{3, 4, 5\}$?

Answer: $\{1, 2, 3, 4, 5\}$



$A \cup B$

Intersection

- **Definition**: The *intersection* of sets *A* and *B*, denoted by *A* ∩ *B*, is

$$\{x | x \in A \land x \in B\}$$

Analogous to the logical AND

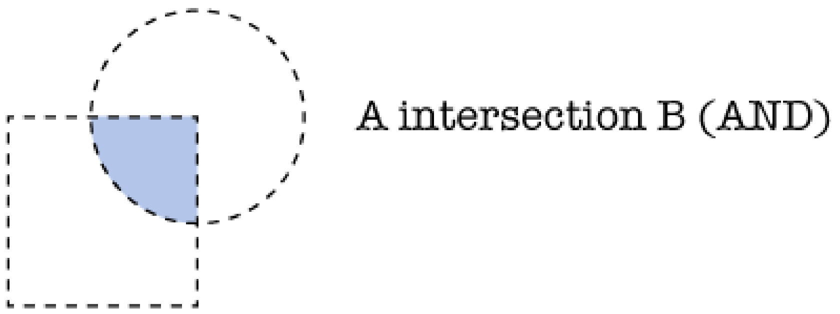
  - Note: if the intersection $A \cap B$ is *empty*, then sets A and B are said to be *disjoint*.

- **Example**: What is $\{1, 2, 3\} \cap \{3, 4, 5\}$?

  Answer: $\{3\}$

- **Example:** What is $\{1, 2, 3\} \cap \{4\,, 5, 6\}$?

  Answer: ∅



A* ∩ *B

Multiple sets

Notation for unions and intersections over sets $A_1, A_2, \ldots, A_n$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Complement

Definition: the *complement* of A (with respect to a given universal set U), denoted by $\bar{A}$ (or A^c) is the set:

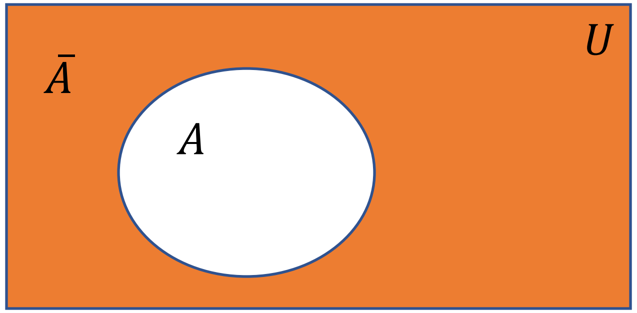
$$\bar{A} = \{x \in U \mid x \notin A\}$$

Analogous to the logical NOT

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$?

negation: $x \leq 70$

Answer: $\{1, 2, \ldots, 70\}$



Venn Diagram for Complement

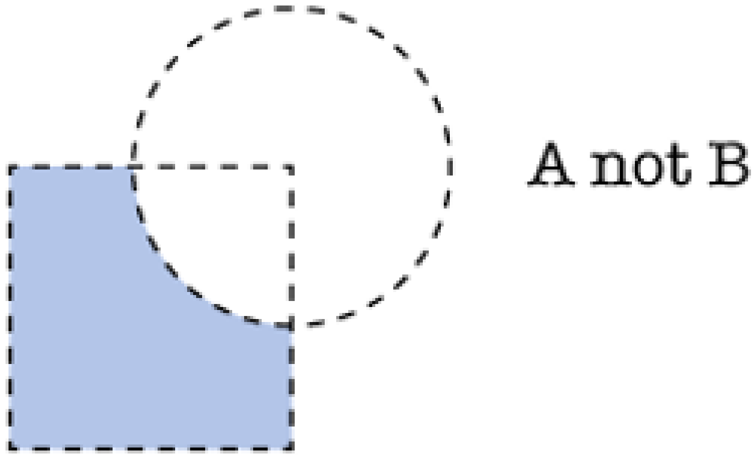
$\bar{A}$

Difference

Definition: The *difference* of A from B, denoted by $A - B$ (or $A \setminus B$), is the set containing the all elements of A that are *not* in B.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \bar{B}$$

- $A - B$ is also called the complement of B with respect to A



$A - B$

Set Identities & Equivalences

- Identity laws $A \cup \emptyset = A$ $A \cap U = A$
- Domination laws $A \cup U = U$ $A \cap \emptyset = \emptyset$
- Idempotent laws $A \cup A = A$ $A \cap A = A$
- Double Complement law $\overline{(\overline{A})} = A$

Set Identities & Equivalences

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$\overbrace{A \cup (B \cup C)}^{A \cup B \cup C} = \overbrace{(A \cup B) \cup C}^{A \cup B \cup C}$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$x \cdot (y + z) \qquad x \cdot y + x \cdot z$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities & Equivalences

- De Morgan's laws $\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Absorption laws $A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$

  ↳ more generally:
  if $A \subseteq B$ then: $A \cup B = B$ $A \cap B = A$
- Complement laws $A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$
- Complement conversion to difference $\boxed{A \cap \bar{B} = A - B.}$