Wednesday 7 February 2024
14.00– 16.00 BST
Duration: 2 hours
Timed exam – fixed start time

Level 2 - DEGREE of BSc Software Engineering (Graduate Apprenticeship)

Practical Algorithms
COMPSCI1021

Answer All Questions

This examination paper is an open book, online assessment, and is worth a total of 80 marks.

Question 1 – Algorithms, Searching & Sorting, and Complexity (20 marks in total)

1. (a) Consider the following algorithm **some_algo** described in pseudocode. It takes as input an array **array** of integers and it (creates and) returns a list **output**. It internally also uses a Boolean variable **test** and two integer variables **i** and **j**.

```
define some_algo (array)
    output = []
    for i from 0 to length(array)-1
        j = 0
        test = TRUE
        while (test == TRUE) AND (j < length(output))
            if output[j] == array[i]
                test = FALSE
            j = j + 1
        if (test == TRUE)
            output.append(array[i])
  return output
```

Answer the following questions regarding the **some_algo** algorithm above:

(i) What is the output of the algorithm when run on input `[1, 2, 1, 3]`?

[3 marks]

(ii) In one sentence, describe what the algorithm does. That is, given a certain input, how does this algorithm transform it? (Note that you are *not* asked to describe what the code does line-by-line.)

[3 marks]

(iii) Which of the following characterize the (worst-case, asymptotic) running time of the algorithm? Explain your answer and, additionally, give an example of a worst-case input that forces the algorithm to execute the maximum number of steps.

(A) Linear (B) Quasi-linear (C) Quadratic

[5 marks]

(b) Characterize the following statements regarding Big-O notation as "True" or "False". Note that no justification is required.

(i) $3n^2 = O(n^2)$
(ii) $n - 7 = O(n)$
(iii) $2n^3 + n^2\log(n) = O(n^3)$
(iv) $100 = O(1)$
(v) $\log(n) = O(n)$

[5 marks]

(c) Let A be an be an array of integers, whose elements are already sorted in ascending order. The length of A is n. Assume that, for any integer i such that $1 \leq i \leq n$, we can query the value of the i-th element of array A, namely $A[i]$, and this operation costs a single computational step.

You are now given an integer value k. Explain why it is possible to check whether an element of A has value k by performing at most $O(\log(n))$ steps.

[4 marks]

Question 2 – Data Collections and Graphs (20 marks in total)

2. **(a)** Recall that $\mathcal{P}(S)$ denotes the powerset of a set S. You are given sets $X = \{1, 3, 5\}$, $Y = \{3, 7\}$, and $Z = \{2, 6, 8\}$. Determine the following (no justification needed):

(i) $X \cup Y$

[1 mark]

(ii) $|X \times Y|$

[1 mark]

(iii) $|Y - X|$

[1 mark]

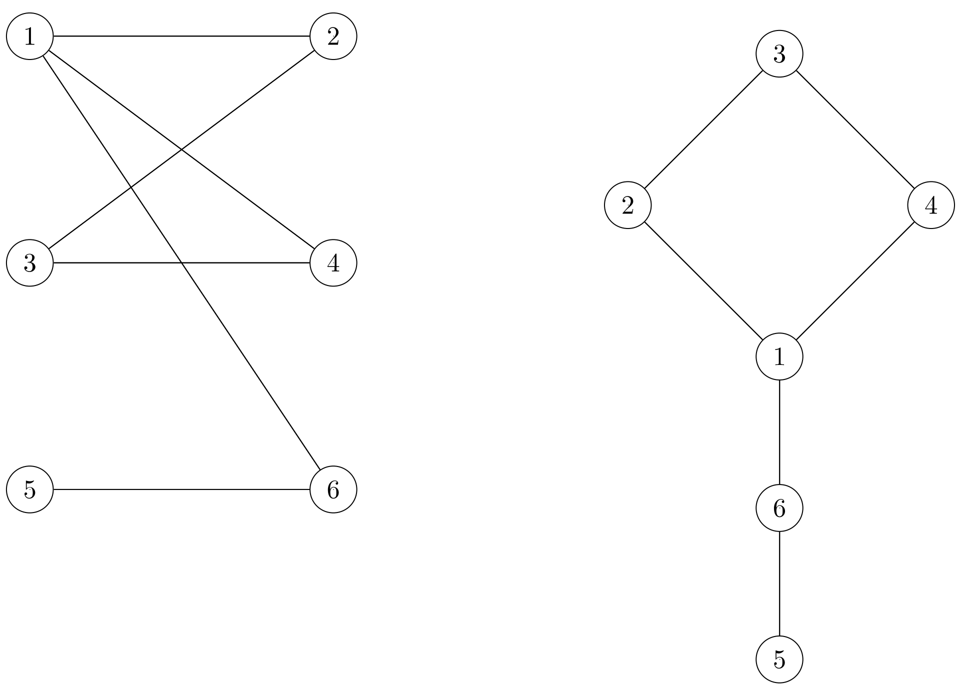
(iv) $\mathcal{P}(Y) \cup Y$

[1 mark]

(v) $|\mathcal{P}(X \cup Z)|$

[1 mark]

(b) Consider the following graphs G_1 and G_2:



G_1 G_2

Choose the right answer below. The following matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

is the adjacency matrix of:

I. only graph G_1

II. only graph G_2

III. both graphs G_1 and G_2

IV. none of the above.

There is no need to justify your answer, but wrong answers get a *negative* mark of -1.

[3 marks]

(c) Let function $f: \mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = 2^n$, where $\mathbb{N} = \{0,1,2,3,\dots\}$ are the natural numbers.

(i) What it the *preimage* of 64? What is the preimage of $f(4176)$?

[2 marks]

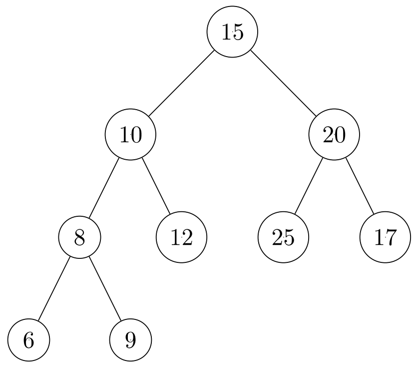
(ii) Is f one-to-one? Is f onto? Is f bijective? Is f invertible? Justify each of your answers.

[4 marks]

(iii) Let $g: \mathbb{N} \longrightarrow [0, \infty)$ be the function defined by $g(n) = \log_2 n$. Determine the composite functions $f \circ g$ and $g \circ f$.

[2 marks]

(d) You are given the following graph:



(i) Explain why this graph is a tree.

[1 mark]

(ii) Is this a binary tree? Justify your answer.

[1 mark]

(iii) Is the tree (when rooted at node 15) a binary search tree (BST)? Justify your answer.

[1 mark]

(iv) Is the tree (when rooted at node 15) balanced? Justify your answer.

[1 mark]

Question 3 – Sequences, Induction, Recursion, and Counting (20 marks in total)

3. **(a)** Is it true that the following inequality holds for all positive integers n?

$$2^{n+1} \leq (n+1)^2$$

Justify your answer.

[4 marks]

(b) Let $\{a_n\}$ be the sequence defined by the recurrence relation $a_n = 3a_{n-1}$, and the initial condition $a_0 = \frac{1}{81}$.

Find a closed-form formula for the sequence $\{a_n\}$, using the iterative method.

[5 marks]

(c) Using mathematical induction, prove that the following inequality holds for all positive integers n:

$$\frac{3}{1} + \frac{3}{3} + \frac{3}{5} + \cdots + \frac{3}{2n-1} \leq n + 2.$$

[6 marks]

(d) In the 4th year at the University of Glasgow students must take exactly 6 courses over two semesters. They have the choice of taking 3 or 4 courses from a set of 8 courses in semester 1, and 2 or 3 courses from a different set of 9 courses in semester 2.

In how many ways can a student satisfy these requirements? Explain your answer, including the rules used and showing some calculations (final numeric result is not required).

[5 marks]

Question 4 – Probability and Formal Reasoning (20 marks in total)

4.

(a) What is the conditional probability that a randomly generated bit string of length 8 contains at least 6 consecutive 0's, given that the last (or rightmost) bit is 1? Explain your answer.

[5 marks]

(b) Using either a truth table or the laws of logical equivalence, show that the following equivalence is valid: $(\neg p \lor (\neg q \land p)) \equiv p \rightarrow \neg q$. If using the laws of logical equivalence, state the law you have used at each step.

[4 marks]

(c) Suppose the domains of discourse are S for students and F for faculty members (e.g., lecturers and advisers of studies) and we have the following predicates:

G(x): x is a student in the Graduate Apprenticeship in Software Engineering (GA) programme.

L(y): y is a lecturer in the GA programme.

A(y,x): y is the advisor of studies for x.

H(x): x is an Honours student.

(i) Express the following English statement in logical formulae using the predicates above: *Some GA students are Honours students.*

[2 marks]

(ii) Express the following English statement in logical formulae using the predicates above: *All advisors of studies for GA students are lecturers in the GA programme.*

[3 marks]

(iii) Express in concise (good) English without variables the following logical formula:

$\exists y \in F.\ \exists x \in S.\ L(y) \land A(y,x) \land G(x) \land H(x)$

[2 marks]

(d) Using the definition of odd and even integers, prove the following statement is true for all positive integers $n \in Z^+$: "if n is odd then $6 \cdot n + 3$ is odd". State what method of proof you use.

[4 marks]