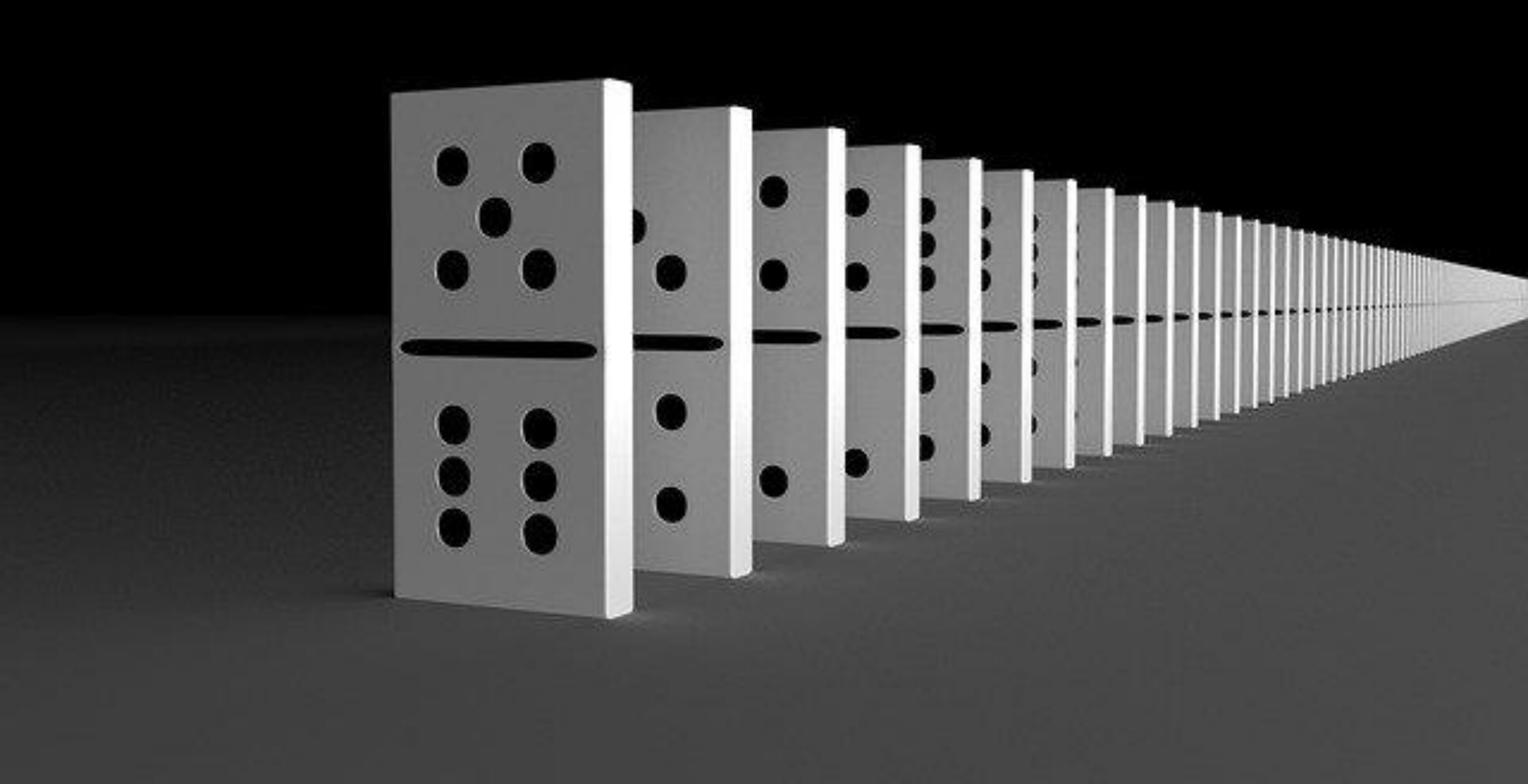
Sequences and Recurrence Relations

SEQUENCES



Sequences and Summations: Motivation

- Sequences are an ORDERED LIST of elements
  - Can be finite, or infinite
- Used in Discrete Maths and Computer Science in many ways, and in many other disciplines, ranging from botany to music
- Provide solutions to certain “counting problems”
  - (e.g. counting number of steps in an algorithms → Algorithmic Complexity)
- They are an important DATA STRUCTURE
  - In this context sequences are generally referred to as **“lists”.** E.g. Lists in Python
- We often need to work with SUMS OF ELEMENTS IN AN ORDERED LIST
  - *Many trigonometric functions, transcendental function, important mathematical constants, are defined in terms of summations over sequence of terms; not our concern in this topic though...*

Sequences: Definition

- Sequences are ***ordered lists of elements***.
  - `1, 2, 3, 5, 8`
  - `1, 3, 9, 27, 81, ...`
  - order matters (`1,2,3` is not the same as `1,3,2`)
  - repetitions are allowed (and are "meaningful", so `1,2,2,3` is not the same sequence as `1,2,3`)
  - *sometimes simply called "lists"*

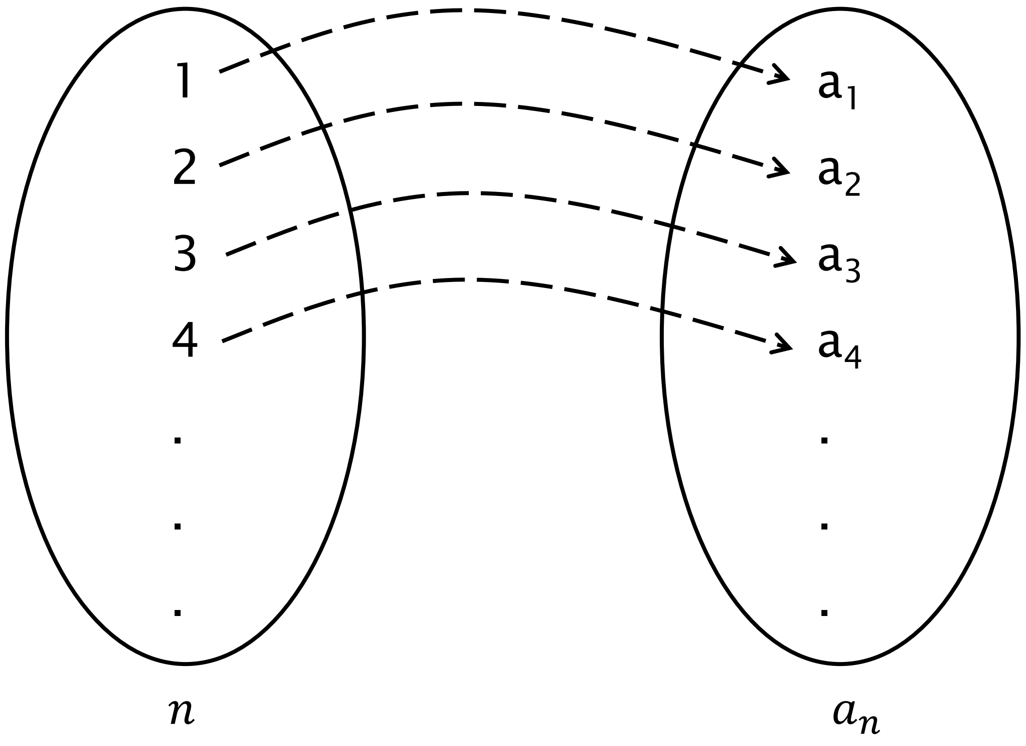
- *You can refer to the concept of "Lists" and "Array" data structures, which are very similar to the concept of "Sequences" we are discussing here*

Sequences: Definition; More Formal

- Formally, a sequence $(a_n)_{n=0}^{\infty}$ is a *function* over the natural numbers:

$$a: \mathbb{N} \longrightarrow \mathbb{R}$$

- Examples:
  - If a_n is the sequence of all odd natural numbers, then $(a_n) = (1, 3, 5, \dots)$.
    - That is, $a_n = 2n + 1$
  - The powers of 2: $(1, 2, 4, 8, 16, \dots)$. That is, $a_n = 2^n$.



Two Special Sequences

- Arithmetic Progression
- Geometric Progression

Arithmetic Progression: Motivating Example

- **Arrays in memory start at a particular address**
  - E.g. let's say an Array A starts at address (in decimal): 1000
- **Each element in the array is of a particular size**
  - Say A is an array of integers, so each element's size is 4 bytes (occupies 4 addresses)
- **In this scenario, the starting addresses of the elements in the array will form an arithmetic progression**
  - Starting addresses of array:
    - 1000
    - 1004
    - $1008,$
    - $1012, \dots$
  - Or, equivalently:
    - $1000 + \mathbf{0} \times 4$
    - $1000 + \mathbf{1} \times 4,$
    - $1000 + \mathbf{2} \times 4$
    - $1000 + \mathbf{3} \times 4, \dots$
  - Direct formula for address of n^{th} element (where $n = 0,1,2, \dots$): $1000 + \boldsymbol{n} \times 4$

Arithmetic Progression

- An *arithmetic progression* is a sequence with a common, *fixed* difference between *any* two consecutive terms.
- Therefore, it has the form

$$a_0, a_0 + d, a_0 + 2d, a_0 + 3d, \dots$$

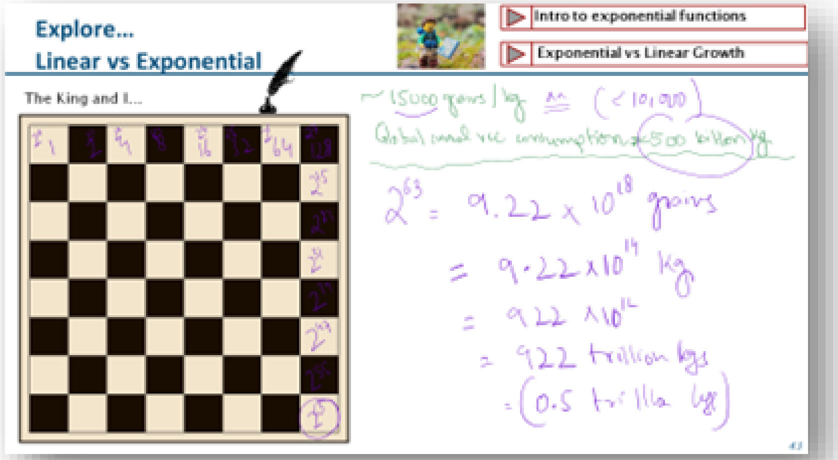
where a_0 is the *initial term* and d is the *difference*.

- That is, $\boldsymbol{a_n = a_0 + nd}$ for all $\boldsymbol{n} \in \mathbb{N}$.
- Examples
  - $a_0 = 1$ and $d = 1$: $(1,\ 2,\ 3,\ 4, \dots)$
  - $a_0 = -1$ and $d = -1$: $(-1,\ -2,\ -3,\ -4,\ \dots)$
  - $a_0 = 1/2$ and $d = 1/2$: $(1/2,\ 1,\ 3/2,\ 2,\ \dots)$
  - $a_0 = 1000$ and $d = 4$: $(1000, 1004, 1008, 1012,\ \dots)$

Geometric Progression: Motivating Example

- Any quantity going by a particular percentage at regular intervals
- E.g., say population of a place is initially 100,000, and grows by 2% every year
- So year-wise population
  - $100{,}000$
  - $100{,}000 \times 1.02$
  - $100{,}000 \times 1.02 \times 1.02 = 100{,}000 \times 1.02^2$
  - $100{,}000 \times 1.02 \times 1.02 \times 1.02 = 100{,}000 \times 1.02^3$
- Population at year = n (where for first year, n = 0)
  - $100{,}000 \times 1.02^n$

Geometric Progression: Another Motivating Example



$a = 1$

$r = 2$

Geometric Progression

- An *geometric progression* is a sequence with a common, *fixed* ratio between *any* two consecutive terms.
- Therefore, it has the form

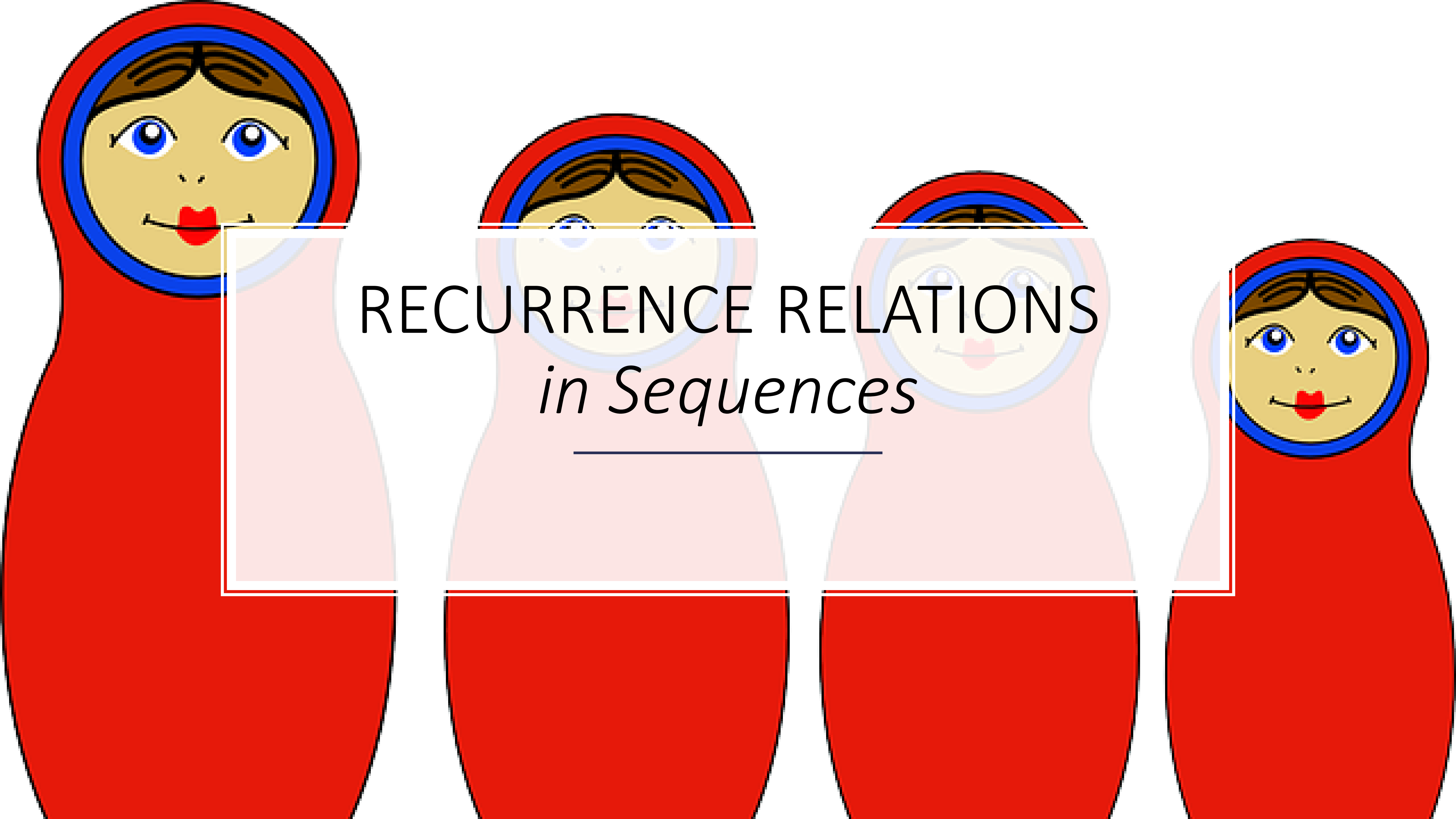
$$a_0, \; a_0 r, a_0 r^2, a_0 r^3, \ldots$$

where a_0 is the *initial term* and r is the *ratio*.

- That is, $\boldsymbol{a_n = a_0 r^n}$ for all $\boldsymbol{n} \in \mathbb{N}$.
- Examples
  - $a_0 = 1$ and r= 2: $(1, \; 2, \; 4, \; 8, \ldots)$
  - $a_0 = 1$ and r= -1: $(1, \; -1, \; 1, \; -1, \; \ldots)$
  - $a_0 = 1$ and r= $1/2$: $(1, \; 1/2, \; 1/4, \; 1/8, \; \ldots)$
  - $a_0 = 100{,}000$ and r= 1.02: $(100000, 102000, 104040, 106121, \ldots)$

What to these progressions look like when plotted?

- Download this spreadsheet, and experiment with
  - different values of a and d (for arithmetic progression)
  - different values of a and r (for geometric progression)



RECURRENCE RELATIONS
in Sequences

Representing Sequences

Sequences can sometimes be arbitrary with no "pattern":

E.g. $A = \{4, 7, 56, 12312, 3, 1, 0, 1, 2, 2\}$

We are more interested in Sequences that follow a certain PATTERN, where we can specify the terms by:

a) defining terms *in terms of previous terms(s)* (recurrence relation)

e.g. Arithmetic and Geometric Progressions, and/or

b) giving a certain *FORMULA* for a term at any position we wish (i.e the n^{th} term) – next subsection

Representing Sequences: RULES / RECURRENCE RELATION

We can define a Sequences by specifying a RULE to find successive elements.

That is, we *specify the first (or first few elements)*, and then define a *recurrence* relation on how to calculate subsequent terms.

E.g. for the specifying the sequence of even numbers:

Representing Sequences: RULES / RECURRENCE RELATION

We can define a Sequences by specifying a <u>RULE to find successive elements.</u>

That is, we *specify the first (or first few elements)*, and then define a <u>*recurrence* relation</u> on how to calculate subsequent terms.

E.g. for the specifying the sequence of even numbers:

$$a_n = a_{n-1} + 2$$ (Recurrence Relation)

$$a_0 = 0$$ (Initial Condition)

$$\{a_n\} = 0, 2, 4, 6, \ldots$$ (Resulting Sequence, i.e. "Solution")

Recurrence Relations: Example 1

Let (a_n) be a sequence defined by the recurrence relation

$$a_0 = 2$$
$$a_n = a_{n-1} + 3$$

What are the values of the terms a_1, a_2 and a_3 ?

Recurrence Relations: Example 1

Let (a_n) be a sequence defined by the recurrence relation

$$a_0 = 2$$
$$a_n = a_{n-1} + 3$$

What are the values of the terms a_1, a_2 and a_3 ?

- **Solution**: We see from the recurrence relation that:

$$\begin{aligned} a_1 &= a_0 + 3 = 2 + 3 = 5 \\ a_2 &= a_1 + 3 = 5 + 3 = 8 \\ a_3 &= a_2 + 3 = 8 + 3 = 11 \end{aligned}$$

Recurrence Relations: Example 2

Let (a_n) be a sequence defined by the recurrence relation

$$a_0 = 3$$
$$a_1 = 5$$
$$a_n = a_{n-1} - a_{n-2} \qquad \text{(for } n = 2, 3, \ldots\text{)}$$

What are the values of a_2 and a_3 ?

- Solution:

Recurrence Relations: Example 2

Let (a_n) be a sequence defined by the recurrence relation

$$a_0 = 3$$
$$a_1 = 5$$
$$a_n = a_{n-1} - a_{n-2} \qquad \text{(for } n = 2, 3, \ldots\text{)}$$

What are the values of a_2 and a_3 ?

- **Solution**: We see from the recurrence relation that:

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

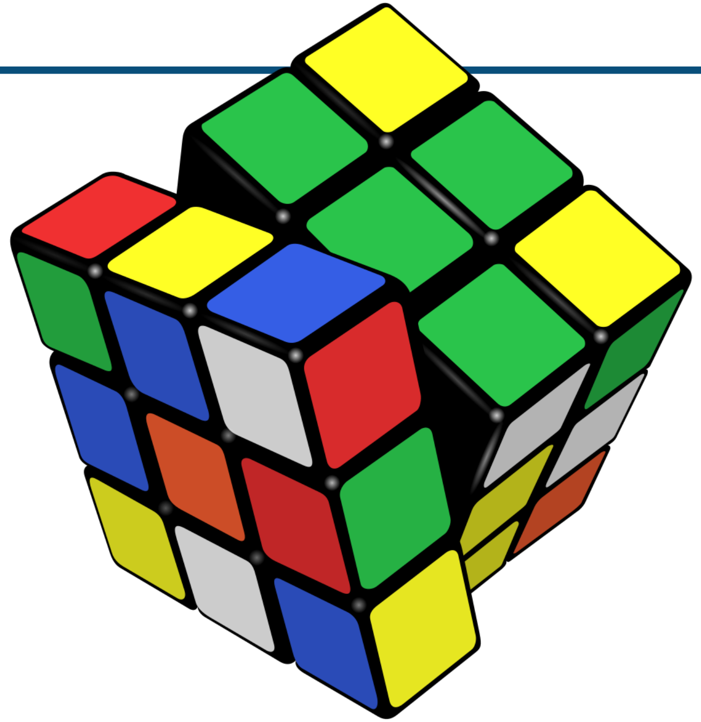
$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Recurrence Relation Example: Fibonacci Sequence

The *Fibonacci sequence* $(f_0, f_1, f_2, \dots)$ is defined by:

- Initial conditions: $f_0 = 0, \quad f_1 = 1$
- Recurrence relation: $f_n = f_{n-1} + f_{n-2}$

$$f_2 = f_1 + f_0 = 1 + 0 = 1$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8$$



“SOLVING” RECURRENCE RELATIONS

$$a_n = a_{n-1} + 2$$ (Recurrence Relation)

$$a_0 = 0$$ (Initial Condition)

$$\{a_n\} = 0,2,4,6,\ldots$$ (Resulting Sequence, i.e. "Solution")

Problem?

$$a_n = a_{n-1} + 2$$ (Recurrence Relation)
$$a_0 = 0$$ (Initial Condition)
$$\{a_n\} = 0,2,4,6,\ldots$$ (Resulting Sequence, i.e. "Solution")

- Say that, given a recurrence relation we want to find the nth element of a sequence (e.g. `n = 1 billion`)
  - You can consider the sequence of even numbers for illustration

- We *can* find it through the use of recurrence relations, but we will have to compute the previous `999,999,999` elements first before we can compute the `1 billionth` element

- In other words, this is an `O(n)` computation

- Can you tell me, *directly*, what the `1 billionth` even number is?

Simple Examples to Show we can do it!

- The sequence is: natural numbers
- The sequence is: even numbers
- The sequence is: odd numbers
- Yes, we can do it!

Cunning plan?

Representing Sequences: FORMULAS

We can specify the terms by giving a certain FORMULA for a DIRECTLY COMPUTING term at any position we wish (i.e the n^{th} term)

Representing Sequences: FORMULAS

We can specify the terms by giving a certain FORMULA for a term at <u>any position we wish</u> (i.e the n^{th} term) – <u>O(1)!!</u>

Example: Consider the sequence $\{a_n\} = 0, 2, 4, 6, \ldots$

- the n^{th} term in this sequence can be given *<u>directly</u>* by this *formula*

$$a_n = 2n$$

Now compute the 1 billionth term of this sequence: how many *steps* did you take?

"Solving" Recurrence Relations

Representing Sequences: RULES / RECURRENCE RELATION

We can also define a Sequences by specifying a RULE to find successive elements.

That is, we *specify the first (or first few elements)*, and then define a *recurrence* relation on how to calculate subsequent terms.

E.g. for the same sequence as before (even numbers):

$$a_n = a_{n-1} + 2$$ (Recurrence Relation)

$$a_0 = 0$$ (Initial Condition)

$$\{a_n\} = 0,2,4,6,\ldots$$ (Resulting Sequence, i.e. "Solution")

Representing Sequences: FORMULAS

Sequences can sometimes be arbitrary with no "formula" that connects their value to their "index": E.g. A = {4, 7, 56, 12312, 3, 1, 0, 1, 2, 2}

We are more interested in Sequences that follow a certain PATTERN, where we can specify the terms by giving a certain FORMULA for a term at any position we wish (i.e the n^{th} term)

Example: Consider the sequence $\{a_n\}$ where

Notation for representing a sequence

Not to be confused with notation for a set. (Note the context)

$$a_n = 2n \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

The n^{th} term in the sequence is given by this "formula"

$$\{a_n\} = 0,2,4,6,\ldots$$

Solving Recurrence Relation

Going from a Recurrence Relation TO a Formula for the n^{th} term

Solving Recurrence Relations

- Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.

- Such a formula is called a *closed formula*.

- Various methods for solving recurrence relations.

- Here we illustrate by example the method of iteration.

Solving Recurrence Relations

- Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.

- Such a formula is called a *closed formula*.

- Various methods for solving recurrence relations.

- Here we illustrate, by example, the iterative method.

Iterative Solution Method

- “Brute force" method of solving a recurrence relation
- Also known as forward substitution:

1. Start with initial condition
2. Work upwards/forward until you reach a_n in terms of a_0 (initial condition) and constants *only*
3. Try to identify the pattern and derive the formula

Iterative Solution Method : Example

Working upward, forward substitution

1. Start with initial condition,
2. Work upward until you reach a_n in terms of a_0 (initial condition) and constants only
3. "Deduce" formula

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$

for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$

Iterative Solution Method 1: Example

Working upward, forward substitution

1. Start with initial condition,
2. Work upward until you reach a_n
3. "Deduce" formula

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$

$$a_1 = a_0 + 3 = 2 + 3$$

$$a_2 = a_1 + 3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_3 = a_2 + 3 = (2 + 3 \cdot 2) + 3 = 2 + 3 \cdot 3$$

$$a_4 = a_3 + 3 = (2 + 3 \cdot 3) + 3 = 2 + 3 \cdot 4$$

.

.

$$a_n = a_{n-1} + 3 = 2 + 3(n-1) + 3 = 2 + 3n$$

Iterative Solution Method : Example

Working upward, forward substitution

1. Start with initial condition,
2. Work upward until you reach a_n
3. “Deduce” formula

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$

$$a_1 = a_0 + 3 = 2 + 3$$

$$a_2 = a_1 + 3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_3 = a_2 + 3 = (2 + 3 \cdot 2) + 3 = 2 + 3 \cdot 3$$

$$a_4 = a_3 + 3 = (2 + 3 \cdot 3) + 3 = 2 + 3 \cdot 3$$

spot the pattern

.

.

apply here

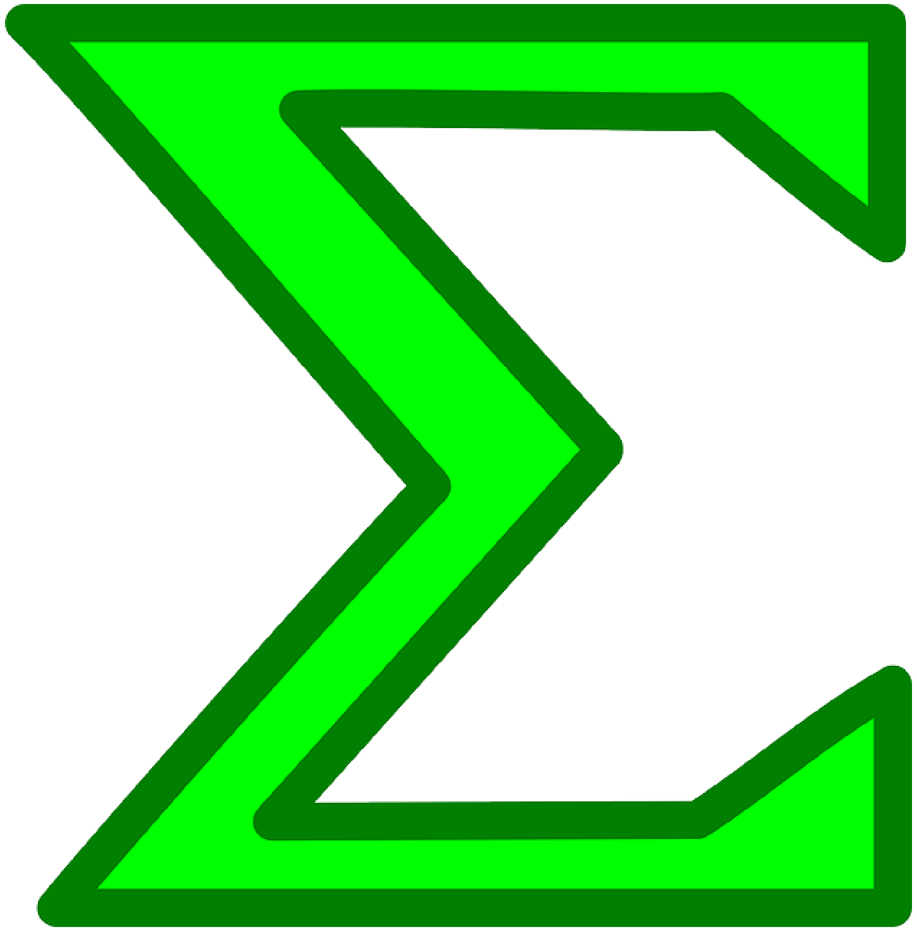
$$a_n = a_{n-1} + 3 = 2 + 3(n-1) + 3 = 2 + 3n$$

Some Useful Sequences

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

SUMMATIONS



Summations and product – Examples

Suppose we have a sequence $\mathbf{a_1, a_2, a_3, \ldots}$

$$\boxed{\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n}$$

$$\boxed{\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_{n-1} \cdot a_n}$$

“Sigma” for sum and “Pi” for product

Summations – Notation

Suppose we have a sequence $a_1, a_2, a_3, \ldots$

lower limit

upper limit

$$\sum_{i=m}^{n} a_i$$

index of summation

Summations – Examples

The sum of the first hundred positive integers

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + \cdots + 99 + 100$$

What is the answer?

Example: Summing up first *n* natural numbers

Example 1

- Input: integer n
- Output: the sum of the first n numbers

```
SUMS1(n)
  i := 0
  sum := 0
  while i < n
    increment i
    sum := sum + i
  return sum
```

Operations
O(1)
O(1)
O(n)
O(n)
O(n)
O(1)

- T(n) = O(1)+O(1)+O(n)+O(n)+O(n)+O(1) = O(n)
- Can we do better?

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vs.

Example 1: Improved

- Input: integer n
- Output: the sum of the first n numbers

Summation rule

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

```
SUMS2(n)
  sum := n * (n+1)/2
  return sum
```

Operations
O(1)
O(1)

- T(n) = O(1) + O(1) = O(1)
  - No loops!

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Clever Carl
https://nrich.maths.org/2478

Summations – Examples

So the sum of the first hundred positive integers

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + \cdots + 99 + 100$$

is:

$$\sum_{i=1}^{n} i = n \cdot (n+1)/2$$

Summations and Sets

- More generally, we can specify the indices to be used for summation by referring to a set *S*:

$$\sum_{j \in S} a_j$$

- Examples:

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$
  from the sequence $\{a_n\}$

- The notation:

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \prod_{m \le j \le n} a_j$$

  represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Geometric Series

Sum of n natural numbers