Iowa State University

Aerospace Engineering

AER E 322 Lab 5
Beam Deflection and Analysis

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2. Objectives

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4. Work Assignments

Refer to Table 1 for the distribution of work during this lab.

Table 1: Work assignments for AER E 322 Lab <lab number>.

Task	**Matthew**	**Peter**	**Natsuki**
Lab Work			
Date Recording	X		X
Exp. Setup		X	X
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Exp. Clean-Up	X	X	X
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5. Materials

The following materials are required for this lab:

- Aluminum beam with cross-sectional dimensions: 12.8 mm × 6.4 mm

- Aluminum beam with cross-sectional dimensions: $12.8\,\mathrm{mm} \times 12.8\,\mathrm{mm}$
- Digital displacement gauge
- Wooden base board
- Wooden roller stand
- Vise clamp
- 100 g, 200 g, 500 g, 1000 g and 1500 g mass
- Ruler
- Wrench

The aluminum beams should be the Aluminum 6061-T6511 alloy, at least 90 cm long, and have a Young's Modulus, E, of 1×10^7 psi or 68.9 GPa.

6. Apparatus

For configurations 1–3, shown in Figure 1, the beam was cantilevered, the load was applied to the free end ($x = 90\,\mathrm{cm}$), and the deflection was measured at $x = \frac{2}{3}L$.

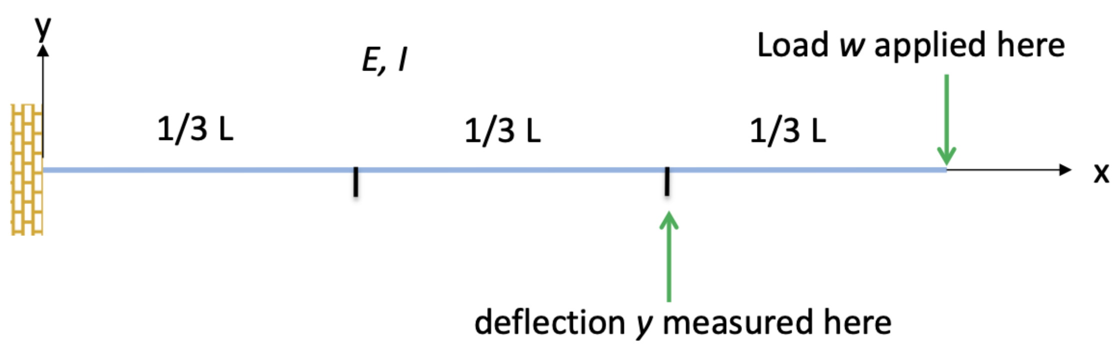

Figure 1: The beam configuration for experiments 1–3.

Table 2 summarizes the differences between configurations 1–3.

For configuration 4, shown in Figure 2, the beam was fixed at one end and—measuring from the fixed end—a roller was positioned at $x = L$, a load was applied at $x = \frac{1}{3}L$, and the

Table 2: Dimensions and loads for configurations 1–3.

Config	Base, b [mm]	Height, h [mm]	Mass [g]	Weight [N]
1	12.8	6.4	100	0.981
2	6.4	12.8	200	1.96
3	12.8	12.8	500	4.905

deflection was measured at $x = \frac{2}{3}L$. The rectangular beam (12.8 mm × 6.4 mm) was used, and the mass was 1000 g, resulting in a 9.81 N load.

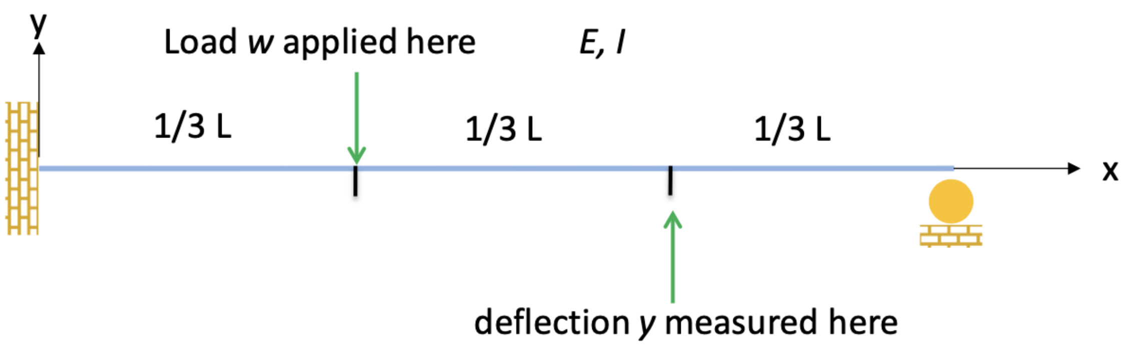

Figure 2: The beam configuration for experiment 4.

For configuration 5, shown in Figure 3, the beam was fixed at one end and—measuring from the fixed end—a roller was positioned at $x = \frac{2}{3}L$, mass m_1 was applied at $x = \frac{1}{3}L$, mass m_2 was applied at $x = L$, and the deflection was measured at $x = L$. The rectangular beam (12.8 mm × 6.4 mm) was used. Mass m_1 was 2500 g, resulting in a 24.5 N load, and mass m_2 was 200 g, resulting in a 1.96 N load.

7. Procedures

8. Data

The displacements in lab were recorded with the digital displacement gauge. The values are tabulated in Table 3.

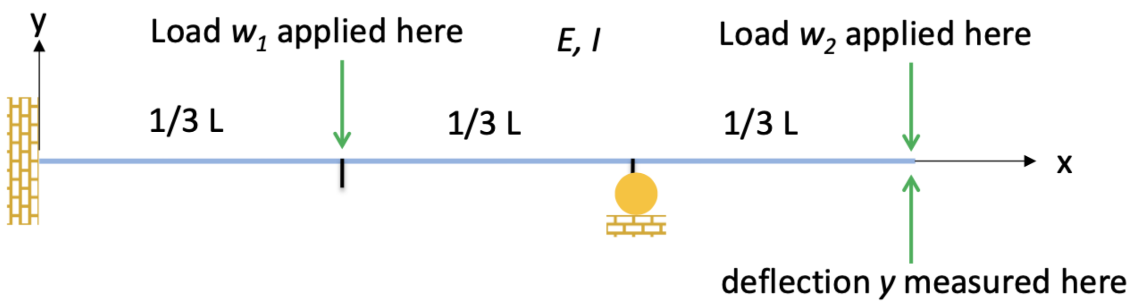

Figure 3: The beam configuration for experiment 5.

Table 3: Tabulated displacements in the five different beam configurations.

Config	Meas. 1 [mm]	Meas. 2 [mm]	Meas. 3 [mm]	Avg [mm]
1	-4.62	-4.59	-4.60	-4.60
2	-3.55	-3.59	-3.57	-3.57
3	-5.01	-5.10	-5.00	-5.04
4	-2.13	-2.11	-2.12	-2.12
5	2.37	2.37	2.22	2.32

9. Analysis

To derive the following deflection expressions, we are given the following equations of superposition for a cantilever beam:

$$0 \leq x \leq L_1 : y = -\frac{wx^2}{6EI}(3L_1 - x) \tag{1}$$

$$L_1 \leq x \leq L : y = -\frac{wL_1^2}{6EI}(3x - L_1) \tag{2}$$

$$L_1 = L : y = -\frac{wx^2}{6EI}(3L - x) \tag{3}$$

where L is the length of the beam, E is the elastic modulus of the beam material, I is the beam moment of inertia, w is the applied load, L_1 is the distance from the fixed end of the beam to the applied load, and x is the point about which deflection is measured. Note that Equation 2 is invalid for $L_1 = L$.

The configuration for test four, shown in Figure 2, consists of a beam fixed at one end with a roller support on the other end. A load is being applied at $\frac{1}{3}$ the length, measured from the fixed end. To calculate the deflection at $\frac{2}{3}$ the length, *i.e.*, $x = \frac{2}{3}L$, we are given the following equation:

$$\nu = -\frac{w}{EI}\frac{23}{4374}L^3 \tag{4}$$

where w is the applied load, E is the elastic modulus, I is the moment of inertia, L is the length of the beam, and ν is the deflection of the beam at $x = \frac{2}{3}L$.

To derive this expression, we first note that the beam configuration is indeterminate. The roller support is redundant and can be replaced by a vertical applied load of magnitude R_y. The beam is now effectively a cantilevered beam with two applied loads. Deflection at a point x can be calculated by summing the deflection at x from each of the applied loads.

The deflection due to the applied load at $x = \frac{1}{3}L$, ν_w, can be derived from Equation 2.

$$\begin{aligned}
\nu_w(x) &= -\frac{wL_1^2}{6EI}(3x - L_1) \\
&= -\frac{w\frac{1}{9}L^2}{6EI}(3x - \frac{1}{3}L) \\
&= -\frac{wL^2}{162EI}(9x - L)
\end{aligned}$$

This expression is true for all $x \in [\frac{1}{3}L, L]$.

The deflection due to the redundant reaction force at $x = L$, ν_{R_y}, can be derived from Equation 3.

$$\begin{aligned}\nu_{R_y}(x) &= -\frac{wx^2}{6EI}(3L - x)\\ &= -\frac{R_y x^2}{6EI}(3L - x)\end{aligned}$$

This expression is true for all $x \in [0, L]$.

To calculate the deflection at $x = \frac{2}{3}L$, we first need to determine the value of R_y. We know from Figure 2, at $x = L$ there is a roller support, and therefore, $\nu(x) = 0$ at $x = L$. We apply this boundary condition below to find R_y.

$$\begin{aligned}\nu(L) = \nu_w(L) + \nu_{R_y}(L) &= 0\\ 0 &= -\frac{wL^2}{162EI}(9L - L) - \frac{R_y L^2}{6EI}(3L - L)\\ 0 &= -\frac{4wL^3}{81EI} - \frac{R_y L^3}{3EI}\\ R_y &= -\frac{4w}{27}\end{aligned}$$

Substituting in the derived expression for R_y into the earlier equation for ν_{R_y}, we find that

$$\nu_{R_y}(x) = \frac{2wx^2}{81EI}(3L - x)$$

Now that we have expressions for both ν_w and ν_{R_y} in terms of w, we can find the derive an equation for the deflection of the beam at $x = \frac{2}{3}L$ in terms of w, E, I, and L.

$$\begin{aligned}
\nu\left(\frac{2}{3}L\right) &= \nu_w\left(\frac{2}{3}L\right) + \nu_{R_y}\left(\frac{2}{3}L\right) \\
&= -\frac{wL^2}{162EI}\left(9\left(\frac{2}{3}L\right) - L\right) + \frac{2w\left(\frac{2}{3}L\right)^2}{81EI}\left(3L - \left(\frac{2}{3}L\right)\right) \\
&= -\frac{5wL^3}{162EI} + \frac{56wL^3}{2187EI} \\
&= \frac{wL^3}{EI}\left(-\frac{5}{162} + \frac{56}{2187}\right) \\
&= -\frac{23wL^3}{4374EI}
\end{aligned}$$

As expected, this equation matches the given expression in Equation 4. Calculating the theoretical deflection is then trivial. We know $w = (1000\,\mathrm{g})(\frac{1\,\mathrm{kg}}{1000\,\mathrm{g}})(9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}) = 9.81\,\mathrm{N}$, $L = 0.90\,\mathrm{m}$, $E = 68.9 \times 10^9\,\mathrm{Pa}$, and $I = \frac{1}{12}(12.8\,\mathrm{mm})(6.4\,\mathrm{mm})^3(\frac{1\,\mathrm{m}}{1000\,\mathrm{mm}})^4 = 2.796 \times 10^{-10}\,\mathrm{m}^4$. Plugging these values into Equation 4, we find that

$$\begin{aligned}
\nu\left(\frac{2}{3}L\right) &= -\frac{23(9.81\,\mathrm{N})(0.90\,\mathrm{m})^3}{4374(68.9 \times 10^9\,\mathrm{Pa})(2.796 \times 10^{-10}\,\mathrm{m}^4)} \\
&= -1.952\,\mathrm{mm} \text{ or } 1.952\,\mathrm{mm} \downarrow
\end{aligned}$$

The configuration for test five, shown in Figure 3, consists of a beam fixed at one end with a roller support at $\frac{2}{3}$ the length. A load is being applied at $\frac{1}{3}$ the length and at the end, measured from the fixed end. To calculate the deflection at the free end of the beam, *i.e.*, $x = L$, we are given the following equation:

$$\nu = -\frac{L^3}{EI}\left(\frac{5w_2}{162} - \frac{w_1}{216}\right) \tag{5}$$

where w_1 is the applied load at $x = \frac{1}{3}L$, w_2 is the applied load at $x = L$, E is the elastic modulus, I is the moment of inertia, L is the length of the beam, and ν is the deflection of the beam at $x = L$.

To derive this expression, we first note that the beam configuration is indeterminate. The roller support is redundant and can be replaced by a vertical applied load of magnitude R_y.

The beam is now effectively a cantilevered beam with three applied loads. Deflection at a point x can be calculated by summing the deflection at x from each of the applied loads.

The deflection due to the applied load at $x = \frac{1}{3}$, ν_{w_1}, can be derived from Equation 2.

$$\begin{aligned}\nu_{w_1}(x) &= -\frac{wL_1^2}{6EI}(3x - L_1) \\ &= -\frac{w_1 \frac{1}{9} L^2}{6EI}(3x - \frac{1}{3}L) \\ &= \frac{w_1 L^2}{162EI}(9x - L)\end{aligned}$$

This expression is true for all $x \in [\frac{1}{3}L, L]$.

The deflection due to the redundant reaction force at $x = \frac{2}{3}L$, ν_{R_y}, can be derived from Equation 2.

$$\begin{aligned}\nu_{R_y}(x) &= -\frac{wL_1^2}{6EI}(3x - L_1) \\ &= -\frac{R_y \frac{4}{9} L^2}{6EI}(3x - \frac{2}{3}L) \\ &= -\frac{2R_y L^2}{81EI}(9x - 2L)\end{aligned}$$

This expression is true for all $x \in [\frac{2}{3}L, L]$.

The deflection due to the applied load at $x = L$, ν_{w_2}, can be derived from Equation 3.

$$\begin{aligned}\nu_{w_2}(x) &= -\frac{wx^2}{6EI}(3L - x) \\ &= -\frac{{w_2}^2}{x} 6EI(3L - x)\end{aligned}$$

This expression is for all $x \in [0, L]$.

To calculate the deflection at $x = L$, we first need to determine the value of R_y. We know from Figure 3), at $x = \frac{2}{3}L$ there is a roller support, and therefore, $\nu(x) = 0$ at $x = \frac{2}{3}L$. We apply this boundary condition below to find R_y.

$$
\begin{aligned}
\nu(\frac{2}{3}L &= \nu_{w_1}(\frac{2}{3}L) + \nu_{R_y}(\frac{2}{3}L) + \nu_{w_2}(\frac{2}{3}L) = 0 \\
0 &= -\frac{w_1 L^2}{162EI}(9(\frac{2}{3}L) - L) - \frac{2R_y L^2}{81EI}(9(\frac{2}{3}L) - 2L) - \frac{w_2 \frac{4}{9}L^2}{6EI}(3L - (\frac{2}{3}L) \\
0 &= -\frac{5w_1}{162} - \frac{8R_y}{81} - \frac{14w_2}{81} \\
R_y &= -\frac{5}{16}(w_1 + \frac{28w_2}{5})
\end{aligned}
$$

Now that we have expressions for the deflection due to all the forces, we can derive an equation for the deflection of the beam at $x = L$ using the method of superposition.

$$
\begin{aligned}
\nu(L) &= \nu_{w_1}(L) + \nu_{R_y}(L) + \nu_{w_2}(L) \\
&= -\frac{w_1 L^2}{162EI}(9L - L) - \frac{2\left(-\frac{5}{16}\left[w_1 + \frac{28w_2}{5}\right]\right)L^2}{81EI}(9L - 2L) - \frac{w_2 L^2}{6EI}(3L - L) \\
&= -\frac{4w_1 L^3}{81EI} + \frac{35\left(w_1 + \frac{28w_2}{5}\right)L^3}{648EI} - \frac{w_2 L^3}{3EI} \\
&= -\frac{L^3}{EI}\left[\frac{4w_1}{81} - \frac{35w_1}{648} - \frac{49w_2}{162} + \frac{w_2}{3}\right] \\
&= -\frac{L^3}{EI}\left(\frac{5w_2}{162} - \frac{w_1}{216}\right)
\end{aligned}
$$

As expected, this equation matches the given expression in Equation 5. Calculating the theoretical deflection is then trivial. We know $w_1 = (2500\,\mathrm{g})(\frac{1\,\mathrm{kg}}{1000\,\mathrm{g}})(9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}) = 24.53\,\mathrm{N}$, $w_2 = (200\,\mathrm{g})(\frac{1\,\mathrm{kg}}{1000\,\mathrm{g}})(9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}) = 1.962\,\mathrm{N}$, $L = 0.90\,\mathrm{m}$, $E = 68.9 \times 10^9\,\mathrm{Pa}$, and $I = \frac{1}{12}(12.8\,\mathrm{mm}) \cdot (6.4\,\mathrm{mm})^3(\frac{1\,\mathrm{m}}{1000\,\mathrm{mm}})^4 = 2.796 \times 10^{-10}\,\mathrm{m}^4$. Plugging these values into Equation 5, we find that

$$
\begin{aligned}
\nu(L) &= -\frac{0.90\,\mathrm{m}}{(68.9 \times 10^9\,\mathrm{Pa})(2.796 \times 10^{-10}\,\mathrm{m}^4)}\left(\frac{5(1.962\,\mathrm{N})}{162} - \frac{24.53\,\mathrm{N}}{216}\right) \\
&= 2.475\,\mathrm{mm} \text{ or } 2.475\,\mathrm{mm} \uparrow
\end{aligned}
$$

Using the theoretical values calculated above and during prelab, Table 4 compares the theoretical deflections to the deflections measured in lab. The deflections measured in lab are an average of three measurements. A negative deflection implies deflection is towards the ground (↓).

Table 4: Theoretical and measured beam deflections for the five configurations in lab.

Configuration	Theoretical [mm]	Measured [mm]	Error
1	−6.416	−4.60	28.3 %
2	−3.208	−3.57	11.3 %
3	−4.010	−5.04	25.6 %
4	−1.952	−2.12	8.61 %
5	2.475	2.32	6.30 %

Two of the configurations, configuration one and configuration three, had particularly high error. For configuration one, this error was likely caused by the smaller surface area exposed to the clamp. In configuration one, the height of the surface the clamp was attached to was 6.4 mm rather than 12.8 mm. This made it very difficult to attach the beam to the board with the clamp, and it is likely that throughout the tests, the beam may have become unstable or unlevel. In configuration three, there was quite a lot of weight on the end of the cantilevered beam with only a clamp fixing the beam to the board. It is very possible the beam was slipping from the clamp under the 500 g mass.

The other tests matched the theoretical values quite well. These amounts of error on such a small scale could be attributed to the accuracy of the device. We have shown in previous labs that the displacement needle is not the most precise or accurate measuring device, and when the scale of displacement is in the single digit millimeters, error of 6 % to 16 % is expected without precise equipment.

It is also possible the weights were not exactly the correct mass, the weights were not placed at exactly the correct location—*i.e.*, the measurements on the beam may have been off by several millimeters or more—or the aluminum beam we used could have had a higher or lower Young's Modulus than the theoretical value. Any of these reasons could have contributed to the displacement being off by several millimeters.

10. Conclusion