Week 7&8 Lecture:

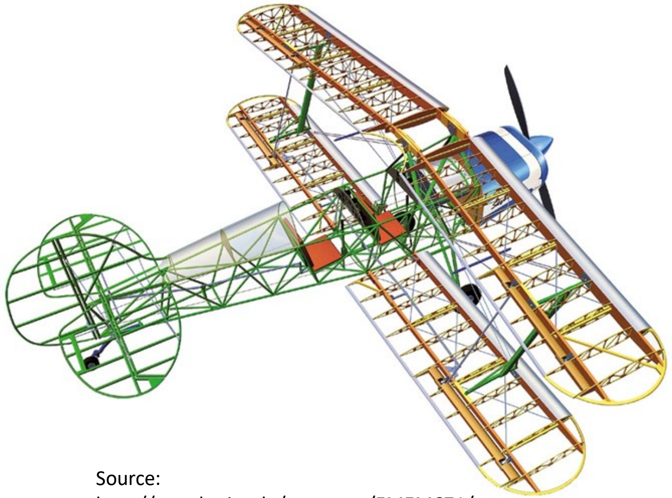
Composite Laminate: Theory and Design

Last update: February 26, 2023

Introduction

- Ancient use of “straw and mud” for house building is probably one of earliest form of composite materials
- Early aircraft structure, consisting of fabric wing skin, wood frame and metal engine, was a mixture of composite *components*
- Modern aircrafts are constructed with a variety of composite materials at increasingly high percentage

Source:http://www.flickr.com/photos/32442418@N02/4199954078/

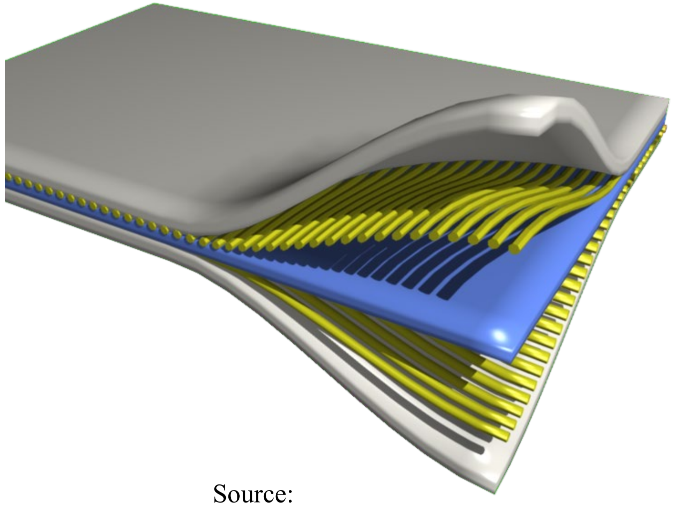


Source: http://people.rit.edu/~pnveme/EMEM671/Images/skybolt_cutaway_white_truss.jpg

Source: CNDE.iastate.edu

Introduction (cont'd)

Composite material or *composite* "is formed by combing two or more different materials, each with its own characteristics, to create a new material/structure whose properties are superior to those of the original components in specific applications"



Source: http://en.wikipedia.org/wiki/Composite_material

Matrices, e.g. resins is the "mud"
Reinforcements, e.g. fibers, is the "straw"

Common composites for modern aircraft structures include

- Reinforced plastics such as fiber-reinforced polymer
- Metal composites with e.g. stitches
- Ceramic composites using composite ceramic and metal matrices

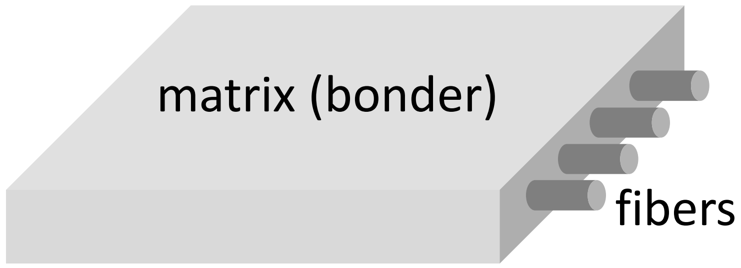
Introduction (cont'd)

- Two most popular fiber-reinforced polymers are carbon fiber reinforced plastic or CFRP and glass fiber-reinforced plastic or GFRP
- A special class is sandwich-structured composite by attaching two thin but stiff skins to a lightweight but thick core such as the so–called honeycombs
- Spray-on foam insulation had been used in space shuttle fleets
- Recent advances include smart composites which uses shape memory resin and microscopic nanocomposites

In this course, we will concentrate on fiber-reinforced composite

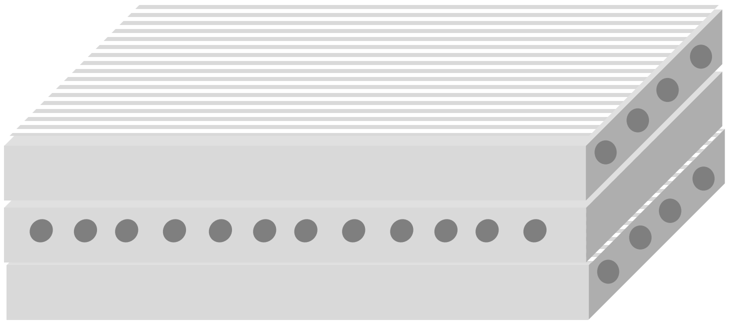
Composite Laminate Structure

Micromechanics:
which fiber & matrix?
what layup?
what mixtures?



A single **ply** contains fibers oriented unidirectional or bidirectional (e.g. a woven fabric). Unidirectional ply is often referred as **lamina**

Laminate theory:
how many plies?
which plies?
what thicknesses?



A stack of laminas (plies) oriented in multi-directions makes up the **laminate**

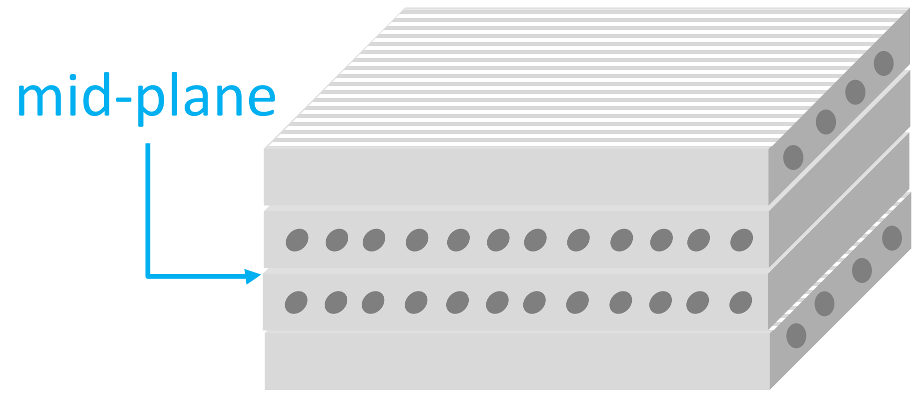
Laminate Notation and Code

4-ply laminate oriented in 0^o, 90^o, 90^o, 0^o sequences can be expressed in several forms:

$$0/90/90/0 = [0,90,90,0] = [0,90]_s$$

where subscript ‘s’ denotes symmetry

As seen on the right, symmetric laminates have plies of identical orientation at same thickness above and below mid-plane



The short-hand form is convenient for repeated pattern:

$$[0,(45,-45)_2]_s=0/45/-45/45/-45/-45/45/-45/45/0$$

Properties of Lamina: Linear Elastic Solid

Each lamina is a linear elastic solid of a certain type:

Material type	Descriptions of property	Number of independent constants
Isotropic	properties are *the same* in all directions	2
Anisotropic	properties are *different* in all directions	21
Orthotropic	three orthogonal planes of symmetry	9
Transversely isotropic	two of three symmetry planes are same	5 -> 4

fiber-reinforced composite

Classical Laminate Theory: Extension from Classical Plate/Beam Theories

- Laminate plate can be treated by extending the classical Kirchhoff-Love plate theory, which in turn is extended from the classical Euler-Bernoulli beam theory

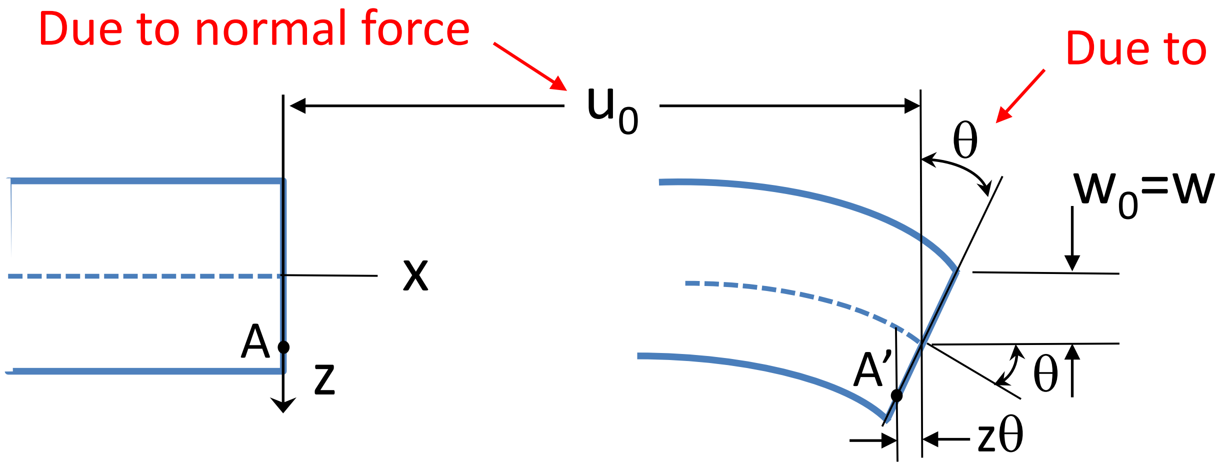
- Kirchhoff assumptions state that, before and after deformation:

  (1) vertical straight lines on the plate remain straight,

  (2) normal to the mid-plane of the plate remain normal, and

  (3) plate thickness remains unchanged

Classical Laminate Theory: Extension from Classical Plate Theory (cont'd)



$$\theta \approx tan\theta = \frac{\Delta w_0}{\Delta x} \approx \frac{\partial w_0}{\partial x}$$

From the geometry and the Kirchhoff assumptions, we have

$$u(x,y) = u_0(x,y) - z\frac{\partial w_0}{\partial x}$$

$$v(x,y) = v_0(x,y) - z\frac{\partial w_0}{\partial y}$$

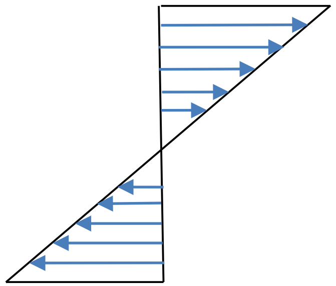
$$w(x,y) = w_0(x,y)$$

u_0, v_0, w_0 : displacements at mid-plane

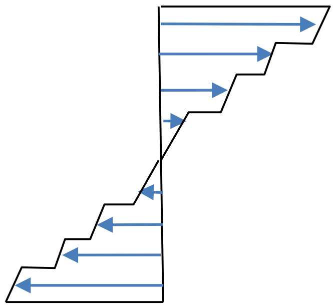
Classical Laminate Theory: Basic Assumptions

- Laminate is thin and wide (length, width >> thickness)
- Perfect in-plane bonding (no slip) between laminas
- Linear strain distribution in the thickness Z direction
- All laminas are linearly elastic (transversely isotropic) and macroscopically homogeneous
- Plane stress/strain – transverse normal and shear strains are negligible

Strain variation



Stress variation

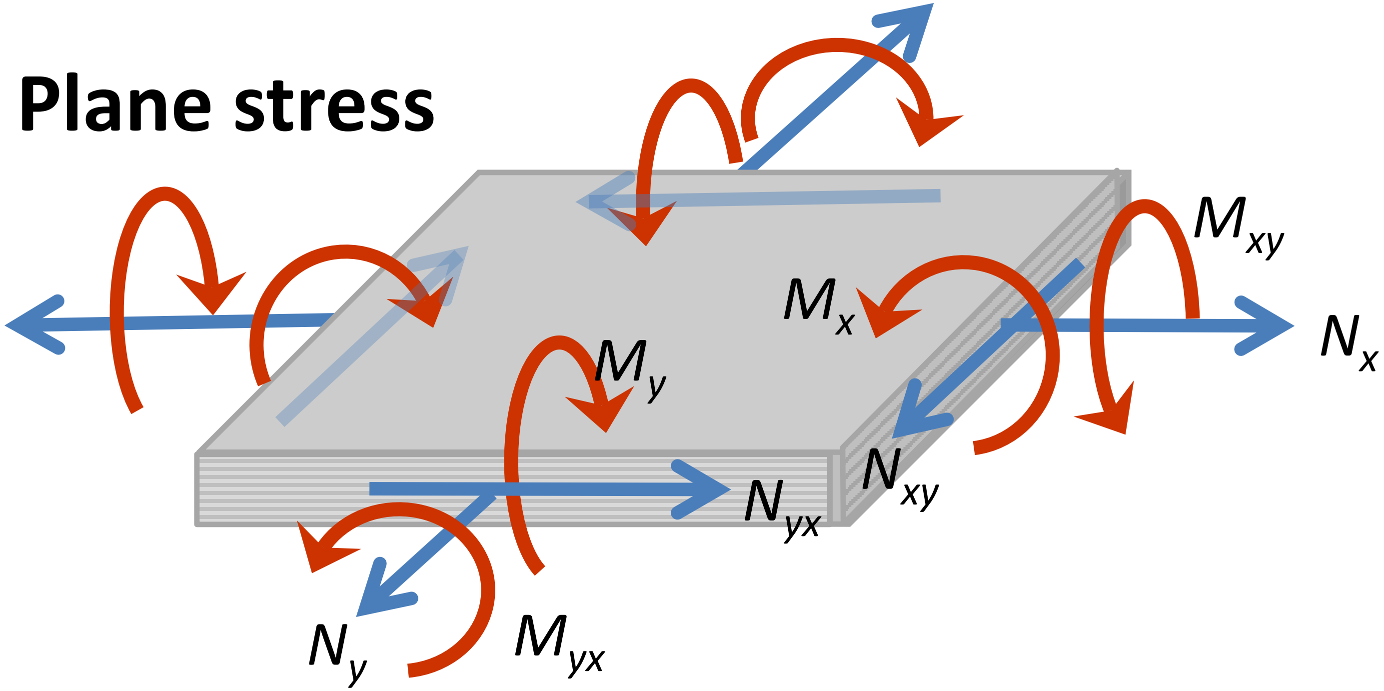


Classical Laminate Theory: Coordinates and Load Orientation

N_x, N_y: resultant normal forces
N_{yx}, N_{xy}: resultant shear forces
M_{yx}, M_{xy}: resultant bending moments

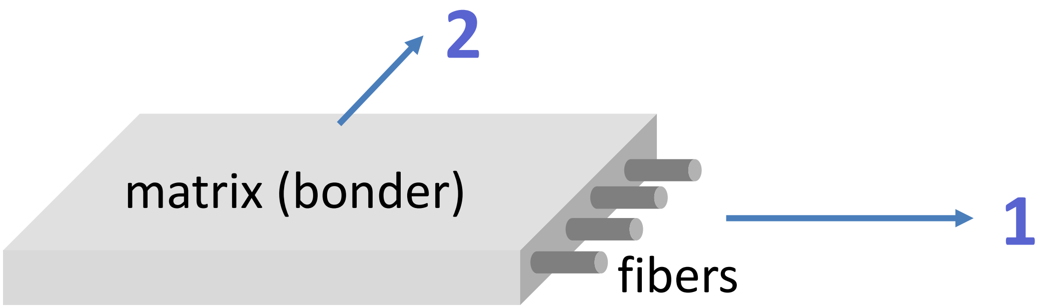
(All per unit length)

Plane stress



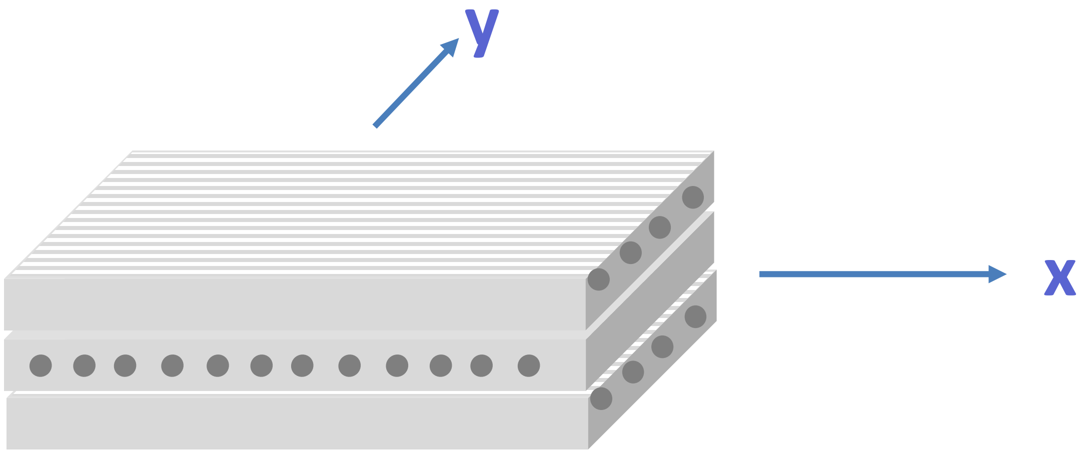
Classical Laminate Theory: Coordinates and Load Orientation (cont'd)

local lamina axes (1,2)



The following pages are to determine the stress-strain in each lamina in the local (1,2) coord. (at different orientation) and find their contributions in the global (x.y) coord.

global laminate axes (x,y)



Classical Laminate Theory: Stress and Strain Relationship

Stresses and strains for the transversely isotropic lamina in the local lamina axes (1,2) can be related by

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

In vector form: $\boldsymbol{\varepsilon} = \mathbf{C}\boldsymbol{\sigma}$ or $\boldsymbol{\sigma} = \mathbf{S}\boldsymbol{\varepsilon}$ where **C** is called compliance matrix and $\mathbf{S}=\mathbf{C}^{-1}$ is the stiffness matrix.

The four independent material constants are longitudinal modulus E_1, transverse modulus E_2, principal Poisson's ratio ν_{12} (or ν_{21}) and shear modulus G_{12}. ν_{12} and ν_{21} are related by $\nu_{12}/E_1 = \nu_{21}/E_2$

Classical Laminate Theory: Rule of Mixtures

Total force in composite is the sum of fiber and matrix components:

$$\sigma_c A_c = \sigma_f A_f + \sigma_m A_m$$

$$\sigma_c = \sigma_f \frac{A_f}{A_c} + \sigma_m \frac{A_m}{A_c} = \sigma_f V_f + \sigma_m V_m$$

(*A*: cross sectional area; V: volume fraction; subscripts: c=composite, f=fibers, m=matrix)

Relating stress to strain leads to

$$\varepsilon_c E_c = \varepsilon_f E_f V_f + \varepsilon_m E_m V_m$$

Pages 14-15 determine the four elastic constants used in page 13

Assuming a perfect bonding between fibers and matrix, so all strains in composite are equal:

$$\varepsilon_c = \varepsilon_f = \varepsilon_m$$

$$\Rightarrow \quad E_L = E_c = E_f V_f + E_m V_m$$ Longitudinal modulus

Classical Laminate Theory: Rule of Mixtures (cont'd)

Transverse modulus

$$\frac{1}{E_T} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

Inplane Poisson's ratio

$$\nu_c = \nu_f V_f + \nu_m V_m$$

Inplane shear modulus

$$\frac{1}{G_c} = \frac{V_f}{G_f} + \frac{V_m}{G_m}$$

Classical Laminate Theory: Coordinate Transformation

Transformation between (1,2)- local lamina axes and (x,y)-global laminate axes is given by

$$\sigma_1 = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

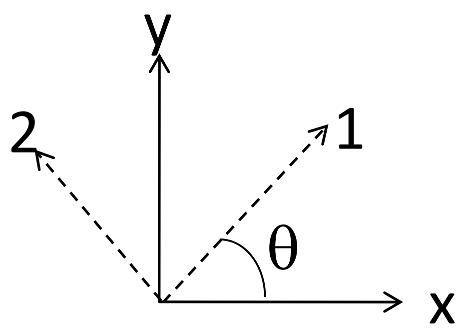
$$\sigma_2 = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{12} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

"project" the stress-strain in local (1,2) coord. in the global (x.y) coord. by coord. rotation

In matrix form with c=cosθ and s=sinθ:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$



In vector form:

σ'=Aσ

Classical Laminate Theory: Coordinate Transformation (cont'd)

Similarly, $\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix}$

By using the Reuter's matrix $\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ we can rewrite

Make use of R to get rid of ½ factor in γ

$$\varepsilon' = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \boldsymbol{RAR^{-1}} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \text{ or } \varepsilon' = \boldsymbol{RAR^{-1}}\varepsilon$$

Inversely $\boldsymbol{\varepsilon} = \boldsymbol{RA^{-1}R^{-1}\varepsilon'}$

Classical Laminate Theory: Coordinate Transformation (cont'd)

Using stress-strain relationship, we can write

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \mathbf{RA}^{-1}\mathbf{R}^{-1} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \mathbf{RA}^{-1}\mathbf{R}^{-1}\mathbf{C} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \mathbf{RA}^{-1}\mathbf{R}^{-1}\mathbf{CA} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

page 13

page 16

Defining $\overline{\mathbf{C}} = \mathbf{RA}^{-1}\mathbf{R}^{-1}\mathbf{CA}$ and $\overline{\mathbf{S}} = \overline{\mathbf{C}}^{-1}$ as the transformed compliance and stiffness matrices respectively, we have

$$\boldsymbol{\sigma} = \overline{\mathbf{S}}\boldsymbol{\varepsilon}$$

Both $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are now related to the global (x,y) coordinates!

Classical Laminate Theory: Laminate Strains

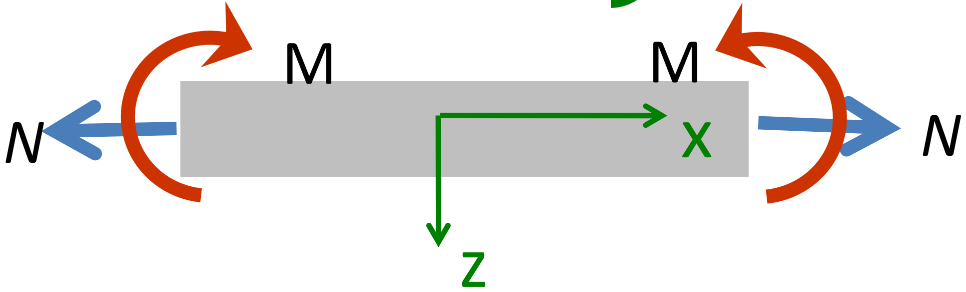
Recall beam theory for combined loading of normal force and bending moment

Strain due to normal force $\varepsilon_x = \dfrac{N}{AE}$

Strain due to bending moment $\varepsilon_x = \dfrac{\sigma_x}{E} = \dfrac{Mz}{EI}$

$$\varepsilon_x = \frac{1}{AE}N + \frac{z}{EI}M$$

eq. (a) referred on page 24



Similar to classical plate theory (pages 8&9), we also assume displacements (u, v) in (x,y) directions are functions of depth Z but not displacement w in z direction:

$$u(x,y) = u_0(x,y) - z\frac{\partial w_0}{\partial x} \qquad v(x,y) = v_0(x,y) - z\frac{\partial w_0}{\partial y}$$

u_0,v_0, w_0: displacements at mid-plane x=y=0

Classical Laminate Theory: Laminate Strains (cont'd)

In terms of linear elasticity (see also Appendix), it can be shown that

basic definition from page 41

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_0}{\partial x} - z\dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial v_0}{\partial y} - z\dfrac{\partial^2 w_0}{\partial y^2} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} - 2z\dfrac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{bmatrix} + z \begin{bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix} \leftarrow K_{xy}$$

plug in same relationships from page 19

K

In vector form $\boldsymbol{\varepsilon} \approx \boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}$

where $\boldsymbol{\varepsilon}^0$ is midplane strain and $\boldsymbol{\kappa}$ is curvature; the third component (from the top) κ_{xy} is called **twisting curvature**

Classical Laminate Theory: Laminate Stresses and Strains

Combing the results from previous slides, we obtain a general transformation for the <u>i-th lamina from local lamina axes to the global laminate axes</u>:

$$\boldsymbol{\sigma} = \overline{\mathbf{S}}\boldsymbol{\varepsilon},\ \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa} \Longrightarrow \boldsymbol{\sigma}_i = \overline{\mathbf{S}}_i\boldsymbol{\varepsilon} = \overline{\mathbf{S}}_i\boldsymbol{\varepsilon}^0 + z\,\overline{\mathbf{S}}_i\boldsymbol{\kappa}$$ eq. (a) referred on page 22

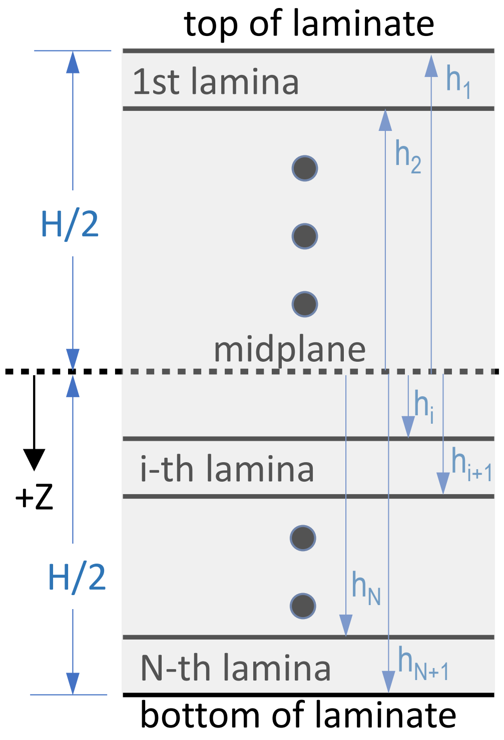
To account for the individual contribution from each of the laminas to the total laminate, we first calculate the net contribution of a lamina by integrating over the thickness of that lamina and then sum up the contributions from all laminas.

For normal force per unit length, *N*, this is done via

$$N = \sum_{i=1}^{K} \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_i\, dz$$ eq. (b) referred on page 22

where $\boldsymbol{\sigma}_i$ is the stress in i-th lamina and z_i and z_{i+1} are measured from laminate midplane to top and bottom of the i-th lamina, respectively. Note that Z axis measures positive from midplane downward

Laminate layout



Classical Laminate Theory: Laminate Stresses and Strains (cont'd)

Separating the mid-plane and curvature components in the stress gives

Plug Eq. (a) into Eq. (b) (shown on page 21)

$$N = \sum_{i=1}^{K}\left(\int_{z_i}^{z_{i+1}} \overline{\mathbf{S}}_i \boldsymbol{\varepsilon}^0 dz + \int_{z_i}^{z_{i+1}} \overline{\mathbf{S}}_i \boldsymbol{\kappa} z dz\right) = \sum_{i=1}^{K}\left(\overline{\mathbf{S}}_i \boldsymbol{\varepsilon}^0 \underbrace{\int_{z_i}^{z_{i+1}} dz} + \overline{\mathbf{S}}_i \boldsymbol{\kappa} \underbrace{\int_{z_i}^{z_{i+1}} zdz}\right)$$

Defining $\boxed{\mathbf{A} = \sum_{i=1}^{K} \overline{\mathbf{S}}_i \left(z_{i+1} - z_i\right)}$ and $\boxed{\mathbf{B} = \frac{1}{2}\sum_{i=1}^{K} \overline{\mathbf{S}}_i \left(z_{i+1}^2 - z_i^2\right)}$ as the "extensional stiffness matrix" and "coupling stiffness matrix" respectively, we have

$$\boxed{N = \mathbf{A}\boldsymbol{\varepsilon}^0 + \mathbf{B}\boldsymbol{\kappa}}$$ Eq. (c) referred on page 23

Eq. (d) referred on page 23

Similarly, we can derive $\boxed{M = \mathbf{B}\boldsymbol{\varepsilon}^0 + \mathbf{D}\boldsymbol{\kappa}}$ if define the "bending stiffness matrix"

$$\boxed{\mathbf{D} = \frac{1}{3}\sum_{i=1}^{K} \overline{\mathbf{S}}_i \left(z_{i+1}^3 - z_i^3\right)}$$

Classical Laminate Theory: Laminate Stresses and Strains (cont'd)

Finally we arrive at a single matrix equation which relates the external forces and moments to the internal laminate stresses and strains (also the laminate structures and properties):

Place Eqs. (c) and (d) shown on page 22 on top and bottom, respectively

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} \qquad \underset{6X1}{\begin{bmatrix} N \\ M \end{bmatrix}} = \underset{6X6}{\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}} \underset{6X1}{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix}} \qquad \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

The **A-B-B-D** matrix is known as the **laminate stiffness matrix** and its inverse is **laminate compliance matrix**.

For symmetric laminates (see page 6), **B**=0 and there is no extension-bending coupling

This can be shown by using the definition of B on page 22 in the layout on page 21

Classical Laminate Theory: Laminate Stresses and Strains (cont'd)

The **A-B-B-D** matrix can be inverted for given *N* and *M*

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix} \Rightarrow \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}$$

similar to eq. (a) on page 19

where

$$\mathbf{a} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D}^*)^{-1}\mathbf{B}\,\mathbf{A}^{-1}$$

$$\mathbf{b} = -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D}^*)^{-1}$$

$$\mathbf{D}^* = \mathbf{D} - \mathbf{B}\,\mathbf{A}^{-1}\,\mathbf{B}$$

$$\mathbf{c} = \mathbf{B}^T$$

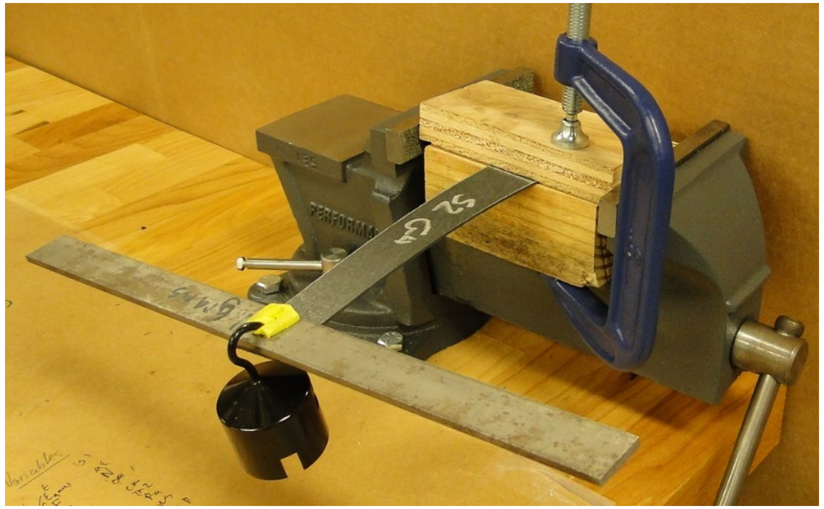
$$\mathbf{d} = (\mathbf{D}^*)^{-1}$$

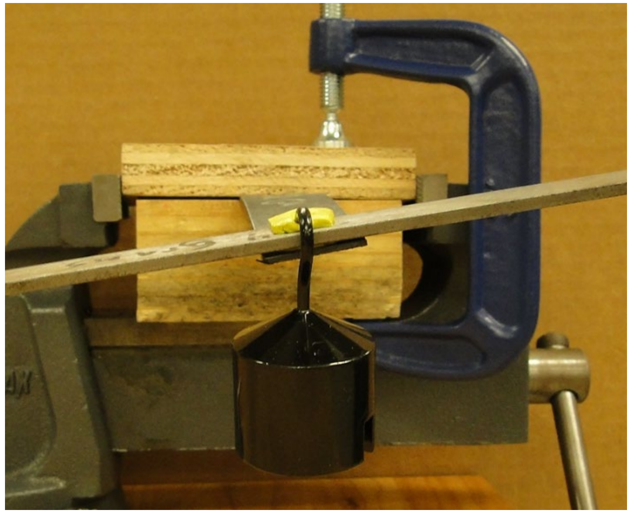
For symmetric laminates (**B**=0), **a**=**A**$^{-1}$and **b**=**c**=0 and **d**=**D**$^{-1}$

Lab 6 – Laminate Layup Design

In our supplemental class web site (https://thermal.cnde.iastate.edu/aere322/), an online laminate "calculator" is available, which implemented the classical laminate theories described above.

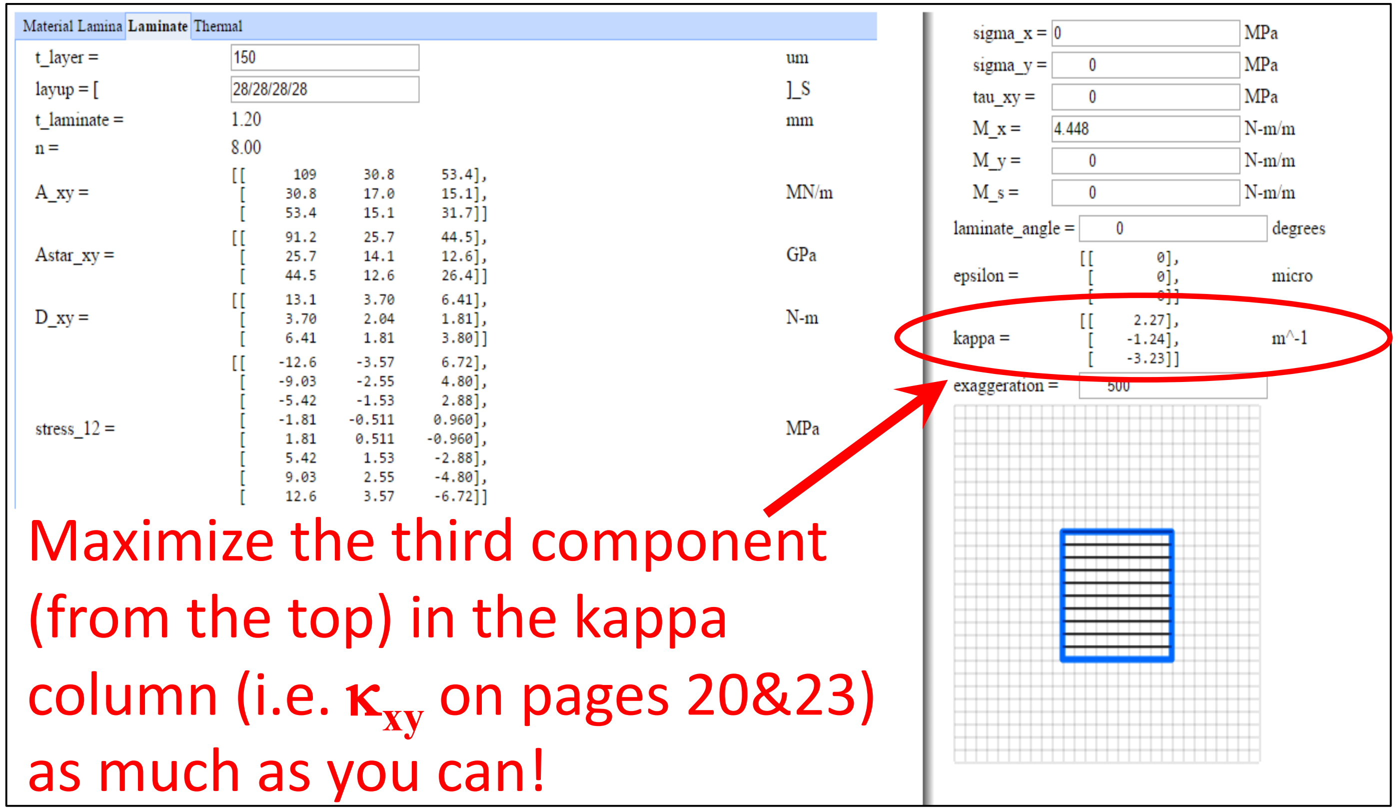
In lab 6, you will use this laminate calculator to design the ply pattern (layup) of an 8-ply symmetric laminate strip such that the end of strip reaches maximum twisting curvature, i.e. twist angle (recall page 20) when subjected to a bending load.





Lab 6 – Laminate Layup Design (cont'd)

Design via online "calculator"



Maximize the third component (from the top) in the kappa column (i.e. κ_{xy} on pages 20&23) as much as you can!

Lab 6 – Laminate Layup Design (cont'd)

show in at least 5 design cycles how to reach your best twist angle at the end of the laminate strip. Included in each cycle:

(1) the design layup you choose,

(2) the reasoning that leads to your choice, and

(3) the value of twisting curvature obtained from the calculator

Main References

1. D. Roylance, "Laminated Composite Plates", http://ocw.mit.edu/courses/materials-science-and-engineering/3-11-mechanics-of-materials-fall-1999/modules/laminates.pdf
2. P. K. Mallick, *Fiber-Reinforced Composites*, CRC Press, 2007. Available online through ISU library: http://www.crcnetbase.com/isbn/9781420005981
3. P. Joyce, http://www.usna.edu/Users/mecheng/pjoyce/composites/Composites_Short_Course.htm

Basic Linear Elasticity

Math Precursor: Taylor's Series Expansion

For a multi-variable function $f = f(x, y, z, \ldots)$, its Taylor's series expansion for change in single variable is

$$f(x + dx, y, z, \ldots) = f(x, y, z, \ldots) + \frac{1}{1!}\frac{\partial f}{\partial x}dx + \frac{1}{2!}\frac{\partial^2 f}{\partial^2 x}dx^2 + \cdots$$

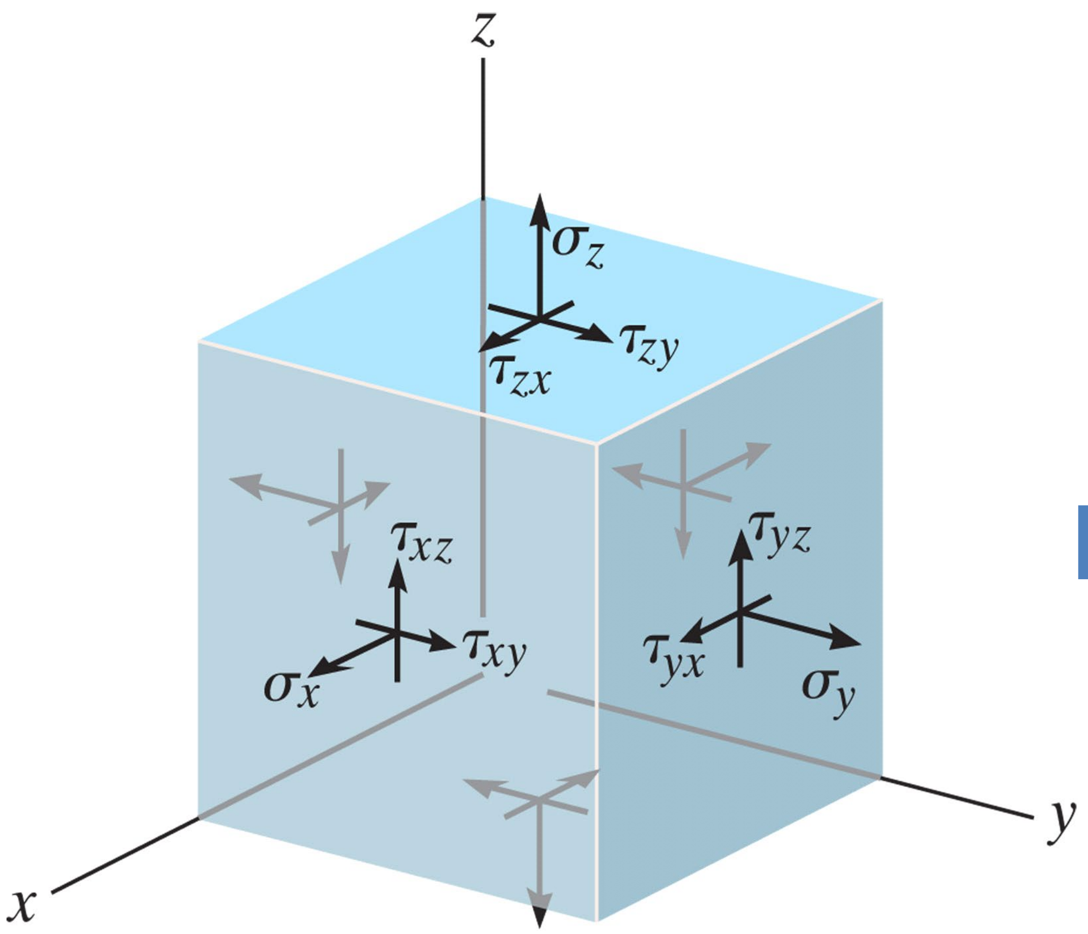
The higher order terms can be neglected for infinitesimal change, such that

$$f(x + dx, y, z, \ldots) \approx f(x, y, z, \ldots) + \frac{\partial f}{\partial x}dx$$

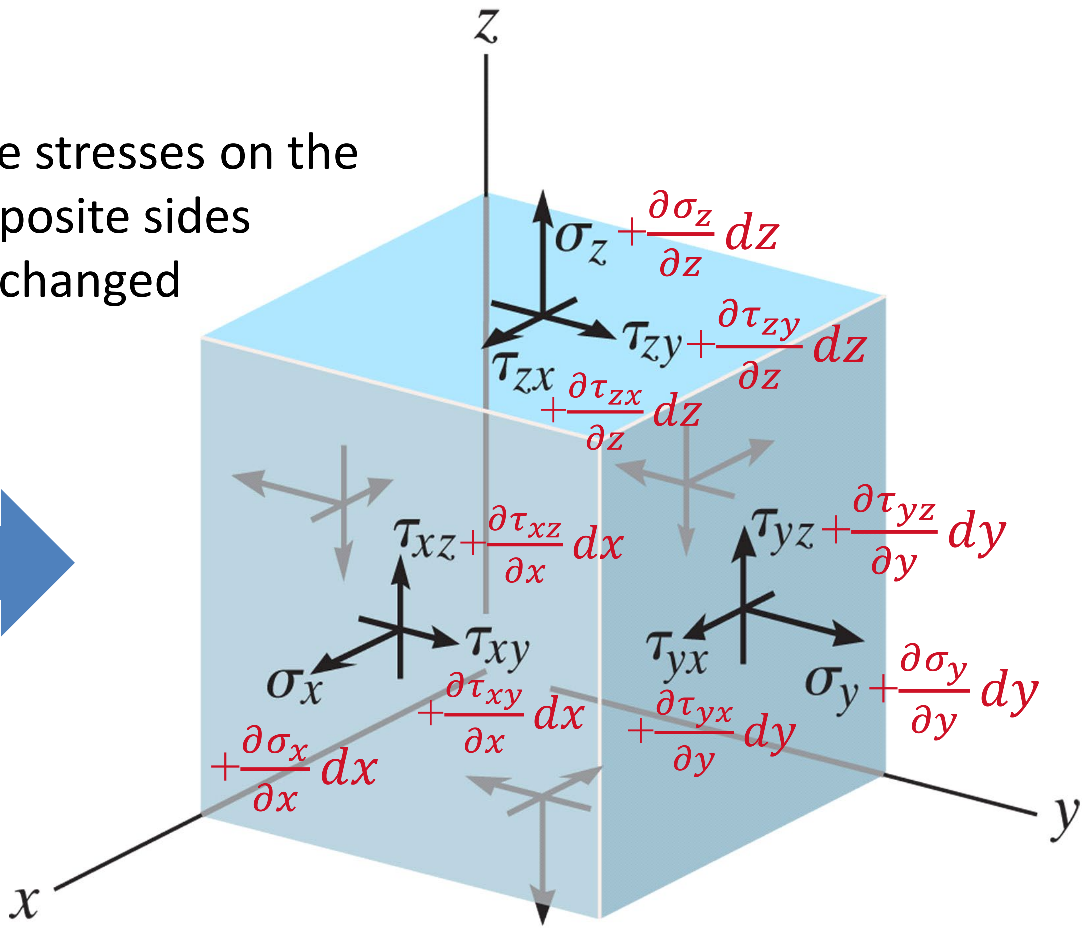
So for $\sigma_x = \sigma_x(x, y, z)$, we have $\sigma_x(x + dx, y, z) \approx \sigma_x(x, y, z) + \frac{\partial \sigma_x}{\partial x}dx$

This expression applies to $\sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}$ and τ_{yz} as well.

Infinitesimal Change of state of stresses



The stresses on the opposite sides unchanged



Equations of Equilibrium

$\Sigma F_x = 0$ with the stresses on the opposite side unchanged,

$$-\sigma_x dydz + \left[\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right] dydz$$

$$-\tau_{yx} dxdz + \left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right] dxdz$$

$$-\tau_{zx} dxdy + \left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right] dxdy$$

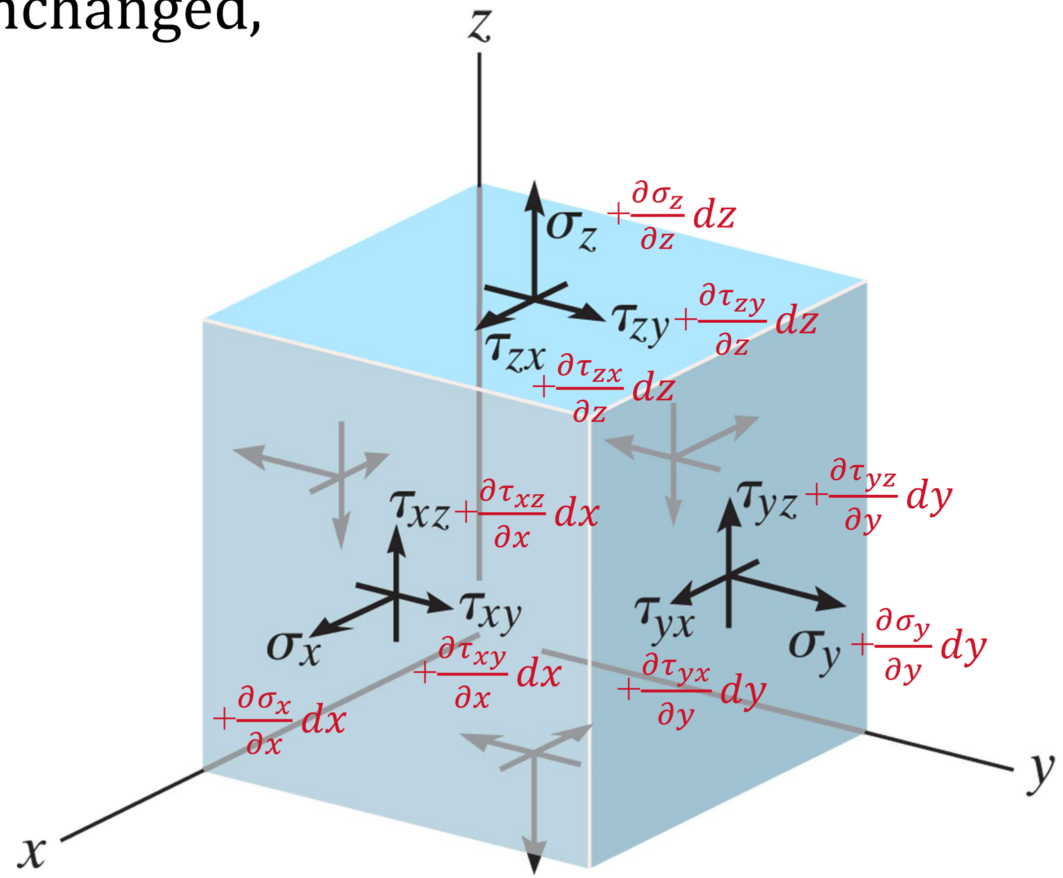
$$+X dxdydz = 0$$

which simplifies to

$$\boxed{\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0}$$

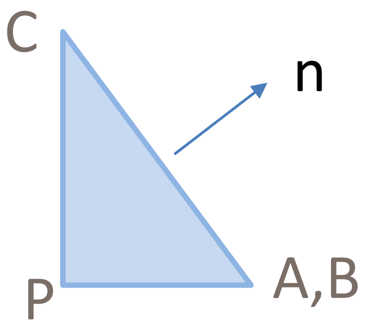
Likewise

$$\boxed{\begin{aligned} &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \\ &\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \end{aligned}}$$



(X, Y, Z: body force per unit volume)

Equations of Equilibrium on Boundary



$\Sigma F_x = 0$:

Dropped; why?

$$\bar{X}dA = \sigma_x dA_x + \tau_{yx} dA_y + \tau_{zx} dA_z - \cancel{\frac{1}{3}\bar{X}dzdA_z}$$

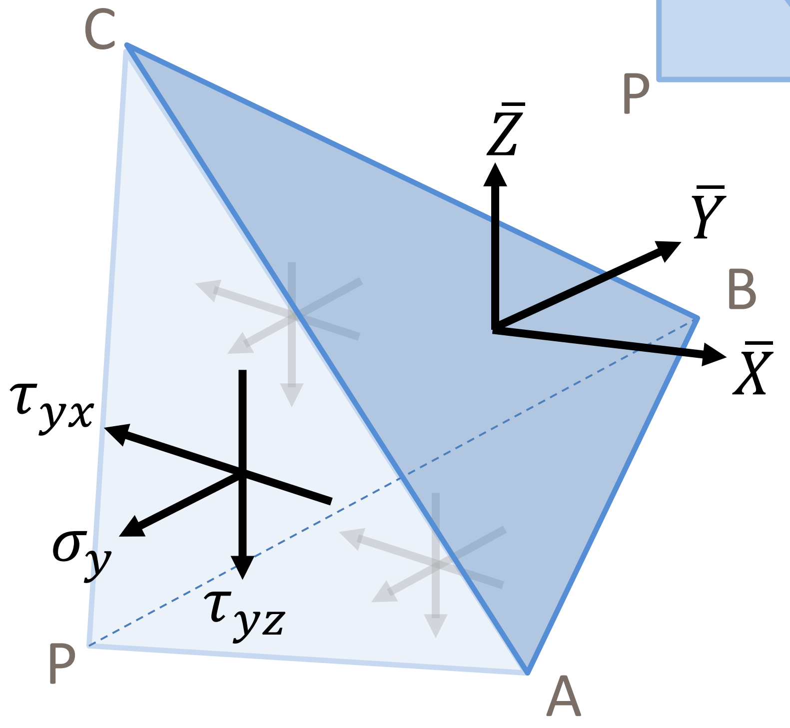
Divided through by *dA* and note that the area ratio equals to the corresponding direction cosine:

$$\bar{X} = \sigma_x \cos(n,x) + \tau_{yx}\cos(n,y) + \tau_{zx}\cos(n,z)$$

$$\bar{Y} = \tau_{yx}\cos(n,x) + \sigma_y \cos(n,y) + \tau_{zy}\cos(n,z)$$

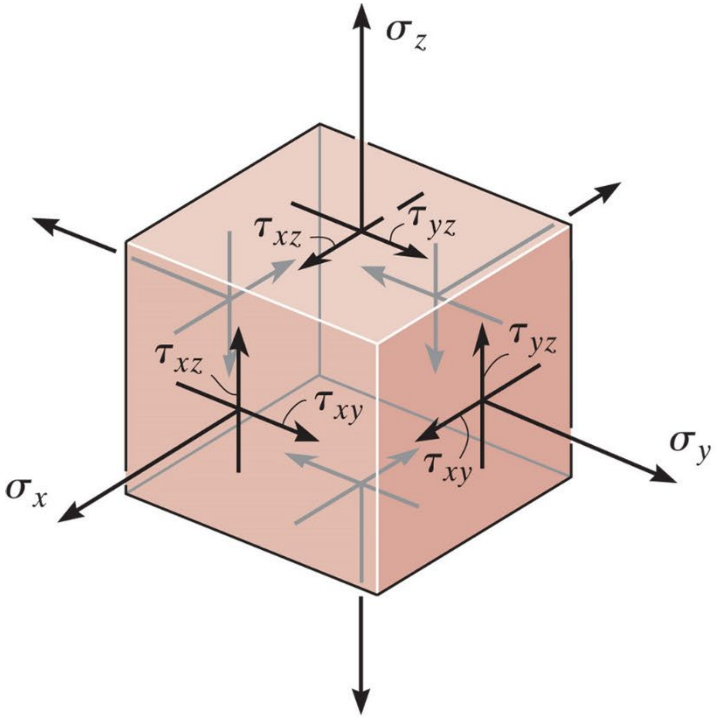
$$\bar{Z} = \tau_{xz}\cos(n,x) + \tau_{zy}\cos(n,y) + \sigma_z \cos(n,z)$$

These are also known as *Cauchy equations*



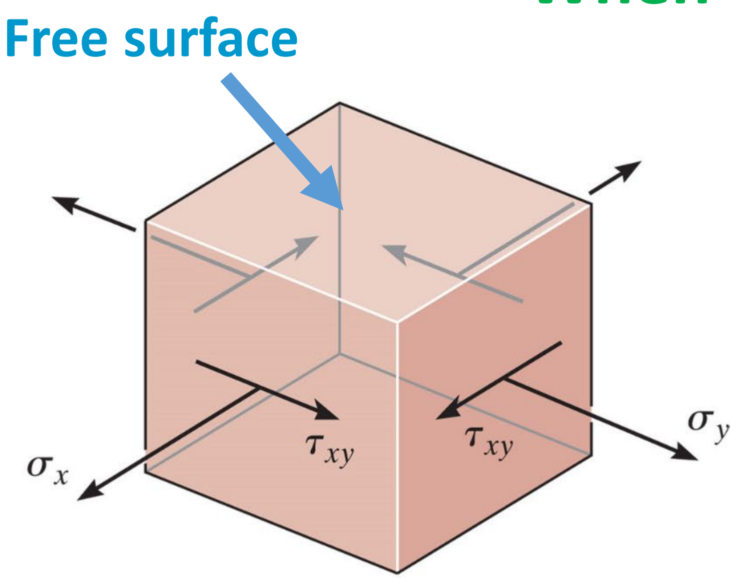
$\bar{X}, \bar{Y}, \bar{Z}$: surface forces per unit area

Plane Stress



General state of stress

9 Stress Components



Plane stress

3 Stress Components

When would this condition happen?

- At a free surface, if Z-direction is perpendicular to the material's local surface
- Through the wall thickness of a "thin" walled pressure vessel
- In a bar or tube under torsion, if 1-direction is the radial direction
- In a bar with an axial load
- In a beam, except in cross section transition zones

Plane Stress (cont'd)

Shown in standardized Simple 2D plane, instead of full 3D cube

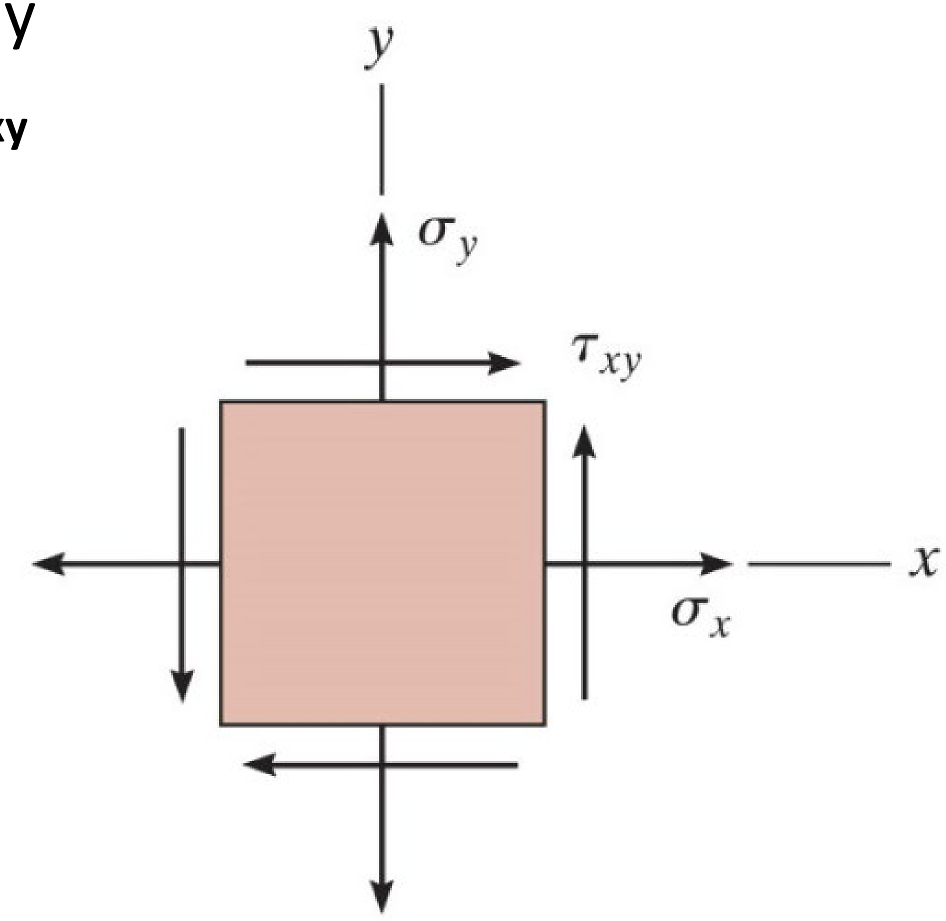
The state of plane stress at a point is uniquely represented by three components, **σ_x, σ_y, τ_{xy}** and acting on an element with a specific orientation

Equations of equilibrium reduce to

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X = 0\,; \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$

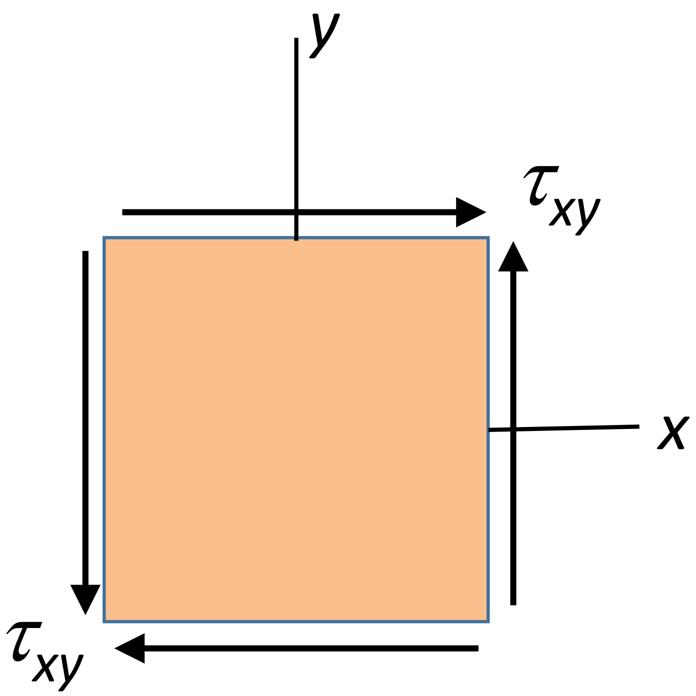
$$\bar{X} = \sigma_x \cos(n, x) + \tau_{yx} \cos(n, y)$$

$$\bar{Y} = \tau_{yx} \cos(n, x) + \sigma_y \cos(n, y)$$



Pure Shear

Plane stress can be further reduced to a "pure shear" state in which only the τ_{xy} component is left



Strain by Elasticity

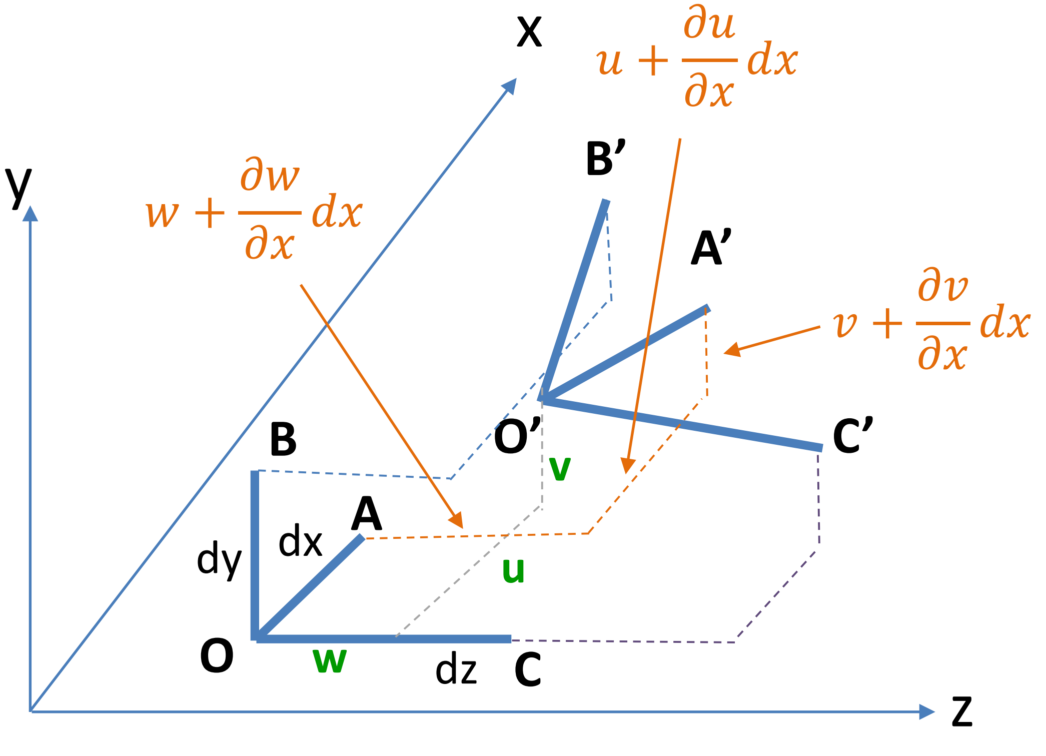
Normal strain $\varepsilon = \lim_{L\to 0}\frac{\Delta L}{L}$ by definition, so

$$\varepsilon_x = \lim_{OA\to 0}\frac{O'A'-OA}{OA}$$

$$(O'A')^2 = \left(dx + u + \frac{\partial u}{\partial x}dx - \mathrm{u}\right)^2$$
$$+\left(v + \frac{\partial v}{\partial x}dx - \mathrm{v}\right)^2$$
$$+\left(w + \frac{\partial w}{\partial x}dx - \mathrm{w}\right)^2$$

$$= dx\sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}$$

Only the three components (in orange) for A-A' displacement shown



u, v, w: displacement

Strain by Elasticity (cont'd)

$$(O'A')^2 = dx\sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2}$$

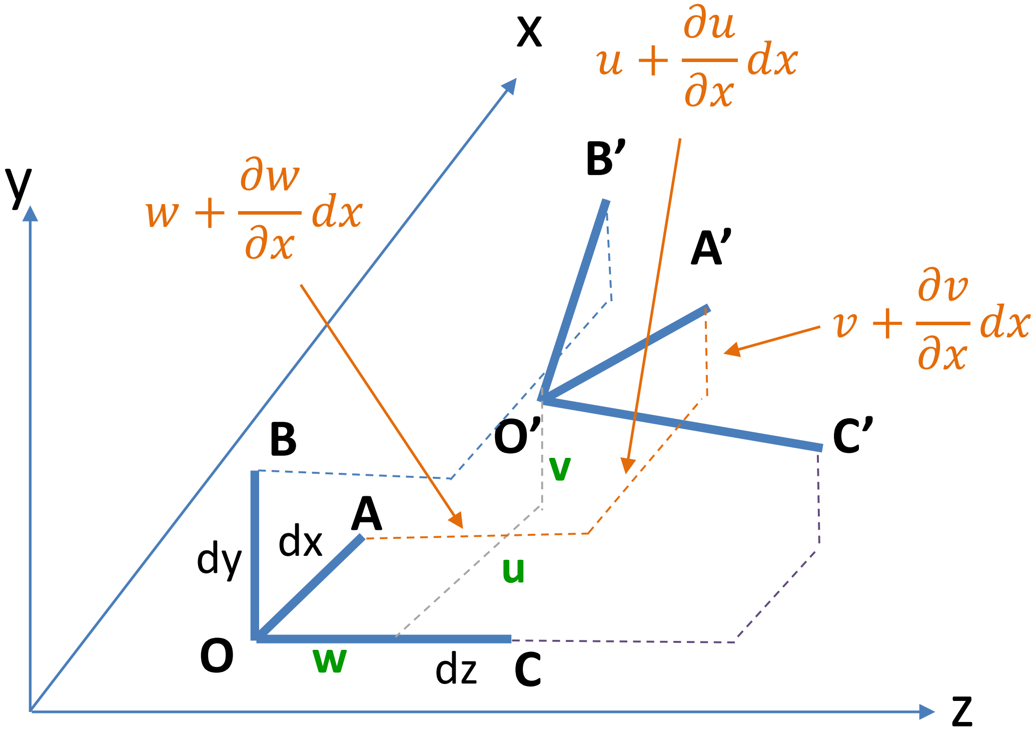
By dropping high order terms and using binomial theorem,

$$(O'A')^2 \approx dx\left(1+2\frac{\partial u}{\partial x}\right)^{\frac{1}{2}} \approx dx\left(1+\frac{\partial u}{\partial x}\right)$$

$$\varepsilon_x = \lim_{OA\to 0}\frac{O'A'-OA}{OA} = \frac{O'A'-dx}{dx} = \frac{\partial u}{\partial x}$$

Likewise, $\varepsilon_y = \dfrac{\partial v}{\partial y}$ and $\varepsilon_z = \dfrac{\partial w}{\partial z}$

Only the three components (in orange) for A-A' displacement shown



u, v, w: displacement

Strain by Elasticity (cont'd)

For small γ_{xz}, $\cos(A'O'C') = \gamma_{xz}$

From trigonometry,

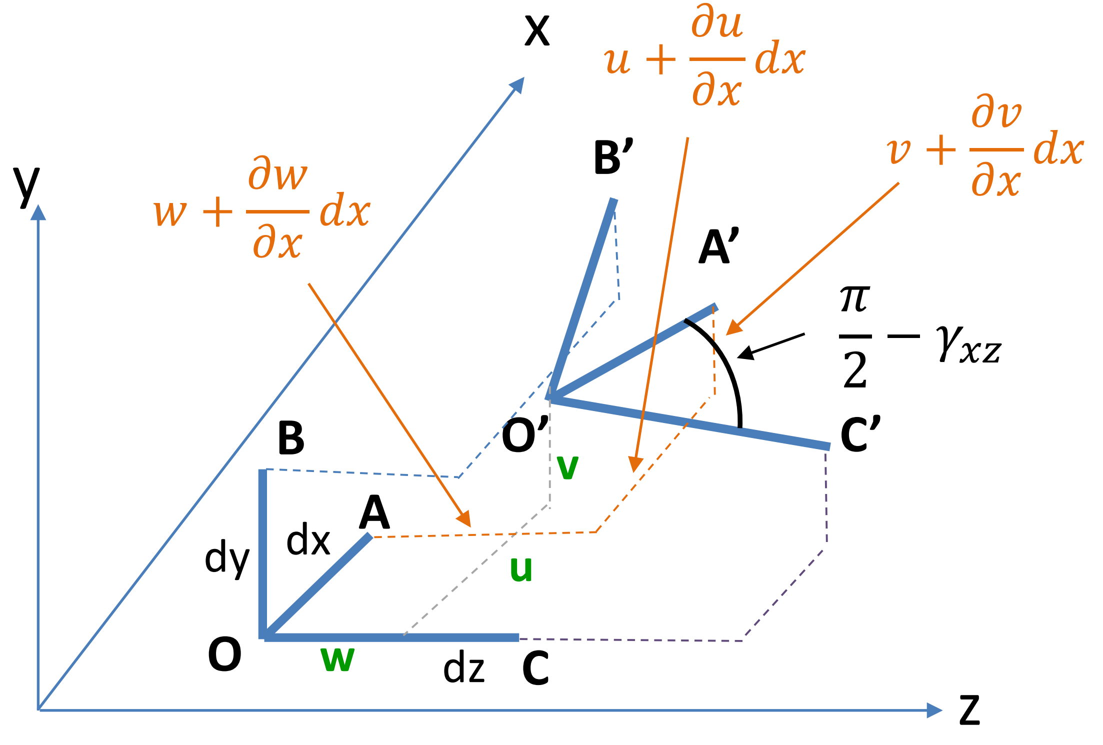
$$\cos(A'O'C') = \frac{(O'A')^2 + (O'C')^2 - (A'C')^2}{2(O'A')(O'C')}$$

$$(O'A') = dx\left(1 + \frac{\partial u}{\partial x}\right) \approx dx,$$

$$(O'C') = dz\left(1 + \frac{\partial w}{\partial z}\right) \approx dz,$$

$$(A'C')^2 = \left(dz - \frac{\partial w}{\partial z}dx\right)^2 + \left(dx - \frac{\partial u}{\partial z}dz\right)^2$$

Only the three components (in orange) for A-A' displacement shown



u, v, w: displacement

Strain by Elasticity (cont'd)

Again by neglecting high order terms,

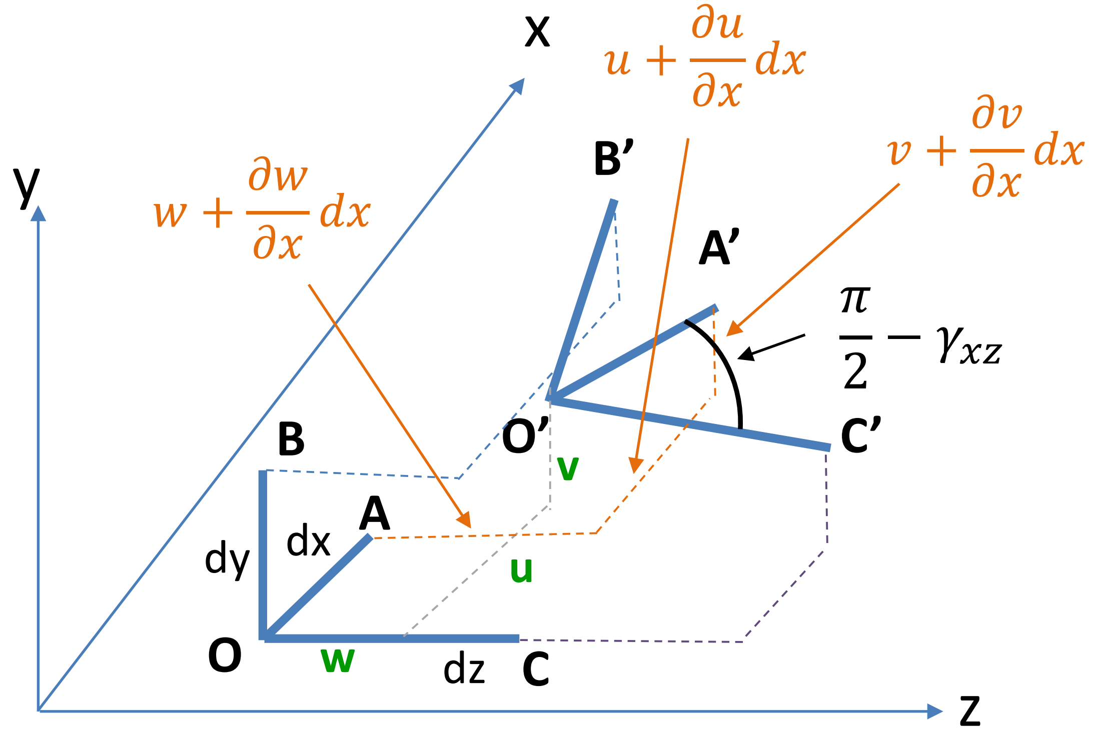
$$\cos(A'O'C') = \frac{2\frac{\partial w}{\partial x}dxdz + 2\frac{\partial u}{\partial z}dxdz}{2dxdz}$$

Finally $\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$

Likewise $\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

Only the three components (in orange) for A-A' displacement shown



u, v, w: displacement

Strain by Elasticity (cont'd)

In summary, for small deformation, linear elasticity has the following forms of strain:

Normal strain

$$\boxed{\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \varepsilon_z &= \frac{\partial w}{\partial z} \end{aligned}}$$

Shear strain

$$\boxed{\begin{aligned} \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{aligned}}$$

Plane strain: only $\varepsilon_x, \varepsilon_y$ and γ_{xy} left