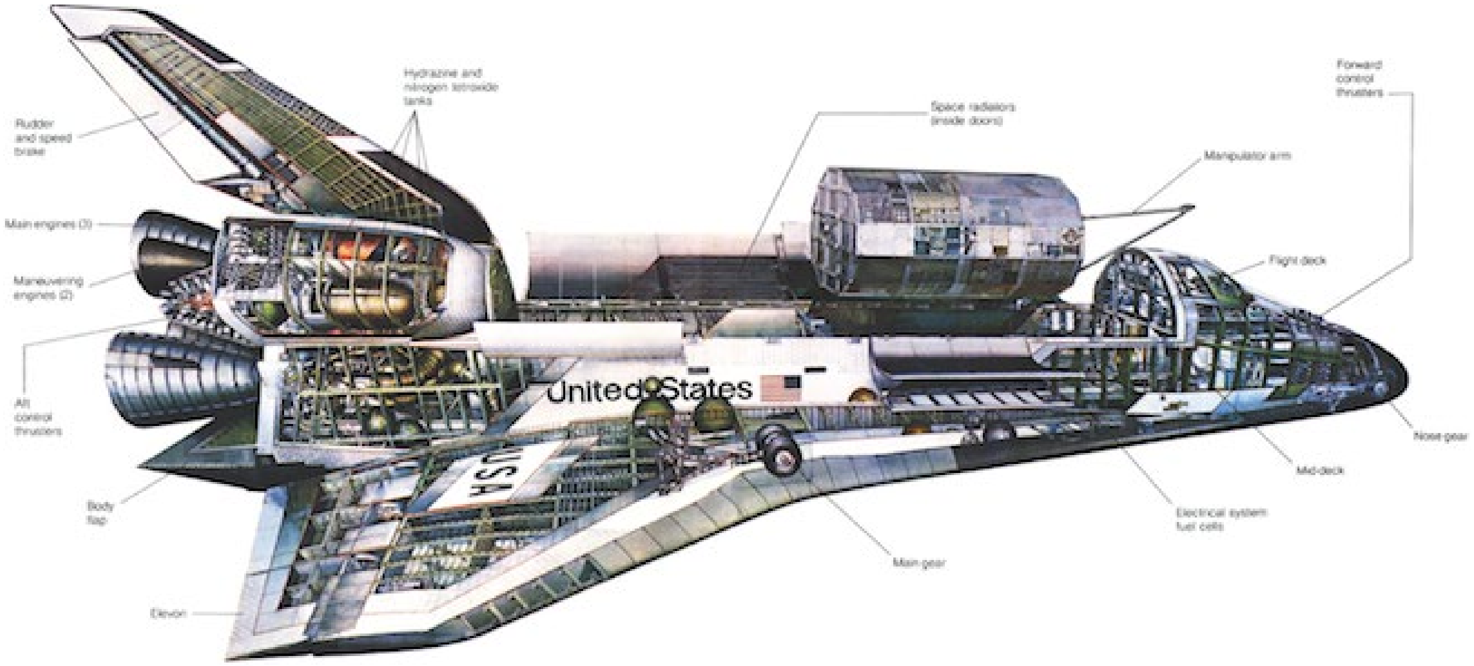
Week 9 Lecture:

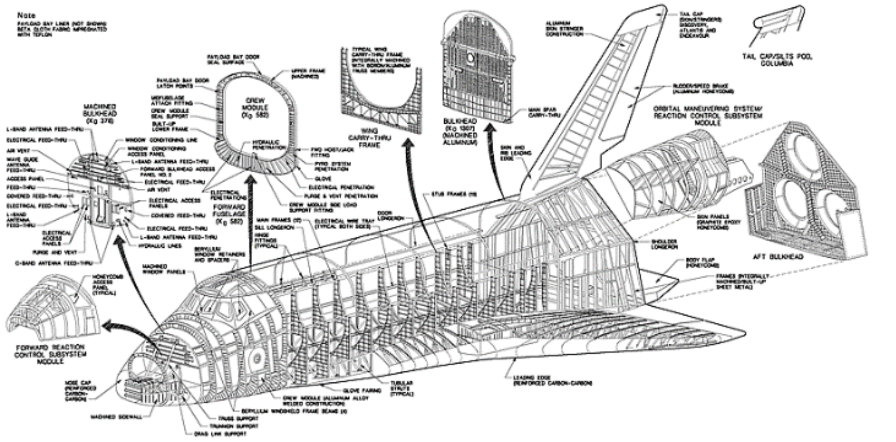
Column Buckling

March 21, 2022

Columns

- Columns are long, straight, prismatic bars subjected to compressive axial loads
- Aeronautic and aerospace vehicles are full of column-related structures





Source: http://www.columbiassacrifice.com/&0_shttlovrvw.htm

Column Buckling

- "Column buckling" – *sudden large* lateral deformation due to a slight increase of an existing load under which the column deformation is slightly perturbed
- In broad scope, "buckling" can happen to entire structure and also under tensile load too!

Source: http://www.civildb.com/buckling-analysis-of-tubular-beam-columns

Source: http://yingloading172.blogspot.com/2011/01/structural-building-defects.html

Structural Failure due to Column Buckling

This off-shore oil platform failure was caused by horizontal hurricane wind force which led to the buckling of its supporting columns

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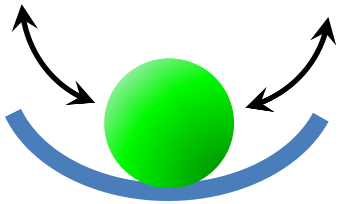
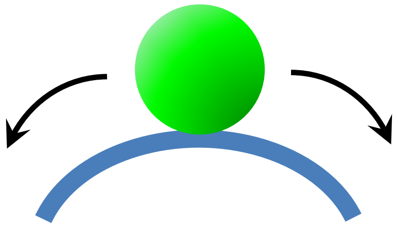
Structural Stability

- Buckling of column (or structure) is not caused by failure of material but by degrading of state of load equilibrium from *stable* to *unstable*
- Initially the loading is in stable equilibrium: column will return to straight configuration if perturbed by small lateral deflection
- When the load reaches a critical value (usually much lower than elastic limit), slightest perturbation will trigger large deflection and column will not return to straight configuration

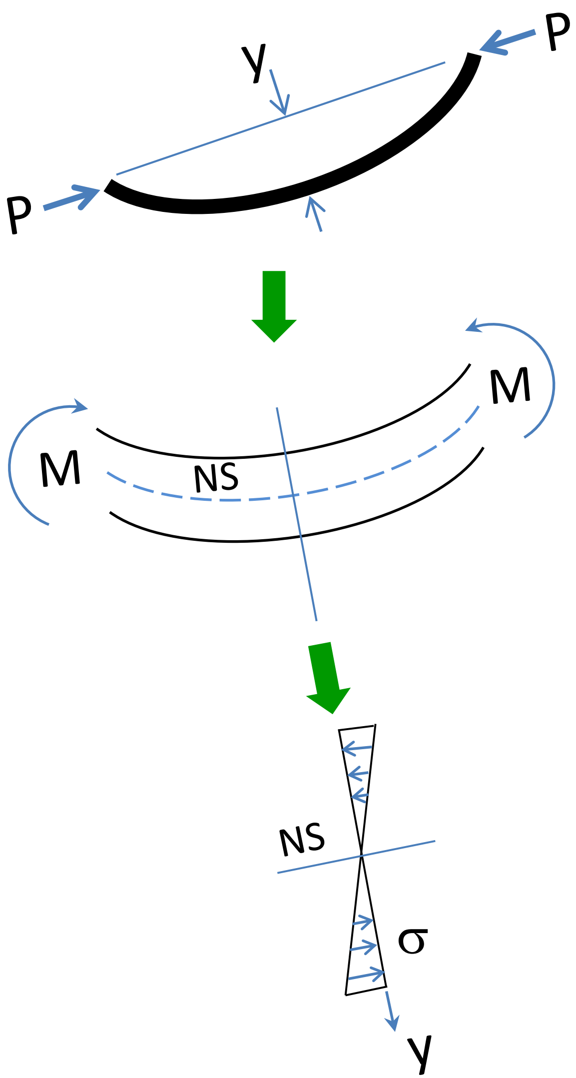


stable



unstable

Analysis of Long Column Buckling



Suppose the load P applied to the column with a small lateral deflection y causes a bending moment M=-Py. The resisting moment M_r in turn gives rise to a normal stress σ acting on the entire cross sectional area A along the axial direction such that

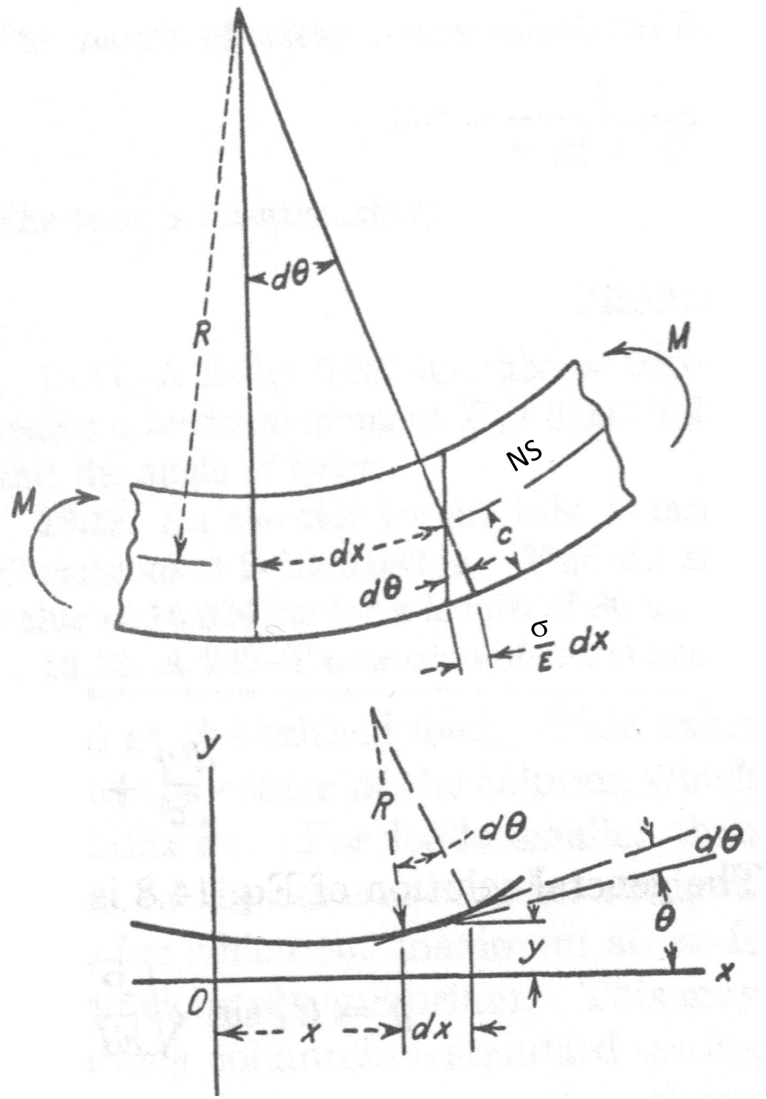
$$M_r = y\int_A \sigma dA$$

where y is measured downwards from the neutral surface (NS). Under elastic limit, $\sigma = E\varepsilon$, in which ε is the axial strain and E is the modulus of elasticity. ε in turn is proportional to y so that

$$M_r = y\int_A const.ydA = const.\int_A y^2 dA = const.I$$

In which I is the moment of inertia of area A about the neutral axis

Analysis of Long Column Buckling (cont'd)



Given M=const. I and σ=const. y, we arrive at the flexure formula

$$\sigma = \frac{My}{I}$$

From the geometry, the outermost elongation in length dx is

$$c\,d\theta = \varepsilon_c dx = \frac{\sigma_c}{E}dx = \frac{Mc}{EI}dx$$

or $$\frac{d\theta}{dx} = \frac{M}{EI}$$

For small lateral deflection y,

$$\theta \approx \tan\theta \approx \frac{dy}{dx}$$

Analysis of Long Column Buckling (cont'd)

Together we have

$$\frac{d\theta}{dx} = \frac{d^2 y}{dx^2} = \frac{M}{EI}$$

Recall M=-Py then

$$\frac{d^2 y}{dx^2} + \frac{Py}{EI} = 0$$

This is a second order homogeneous ordinary differential equation which has solution in the form:

$$y = A\sin\left(\sqrt{\frac{P}{EI}}x\right) + B\cos\left(\sqrt{\frac{P}{EI}}x\right)$$

2nd order homo. O.D.E. solver

$$\frac{d^2 y}{dx^2} + a\frac{dy}{dx} + by = 0$$

Try canonical solution $y = e^{\lambda x}$, and we have the characteristic eq.

$$\lambda^2 + a\lambda + b = 0$$

which has two roots

$$\lambda_{1,2} = \frac{-1}{2}\left(a \pm \sqrt{a^2 - 4b}\right)$$

Thus the O.D.E.'s general solution is

$$y = k_1 e^{\lambda_1 x} + k_2 e^{\lambda_2 x}$$

In complex form, let λ_1=p+iq and λ_2=p-iq

By using Euler formula

$$e^{\pm iz} = \cos z + i\sin z,$$

we finally obtain

$$y = (k_1 + k_2)e^{px}\cos qx + i(k_1 - k_2)e^{px}\sin qx$$

or

$$y = e^{px}(A\cos qx + B\sin qx)$$

Analysis of Long Column Buckling (cont'd)

We use boundary conditions to determine the integration constants A and B: (1) at one end of column, x=0, y=0 leads to B=0, (2) at the other end, x=column length L, y=0 as well, this again gives A=0. For non-trivial solution, we then must have sin()=0 or cos()=0 instead. The former requires:

$$\sqrt{\frac{P}{EI}}L = n\pi, n = 1, 2, \ldots$$

The smallest n=1 gives the so-called Euler buckling load or **critical load** P_{cr}:

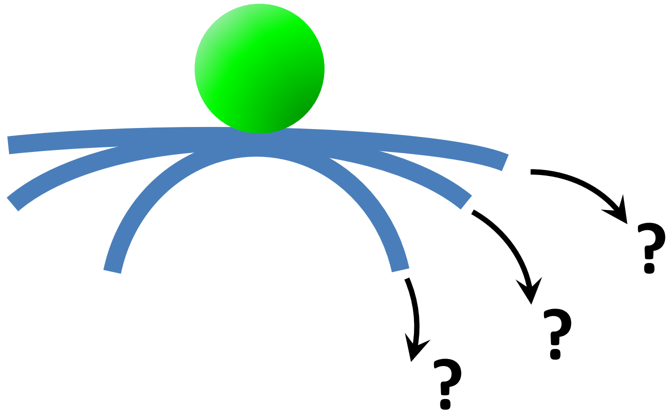
$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Introducing **radius of gyration** $\rho = \sqrt{I/_A}$ (*A* is the cross-sectional area), we obtain the **critical stress** σ_{cr}:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{\left(L/_\rho\right)^2}$$

Slenderness Ratio

The ratio L/ρ is known as the ***slenderness ratio*** which, as we shall see later, determines how stable (or unstable) the column is, given a material property



$$\left.\begin{array}{l}\text{Compression block;}\\ \textit{nominal P/A works}\end{array}\right\} 40 < \frac{L}{\rho} < \left\{\begin{array}{l}140 \text{ for steel}\\ 80 \text{ for timber and aluminum;}\\ \textit{Euler buckling applies}\end{array}\right.$$

Empirical Column Formulas

In the intermediate range, $40 < {}^{L}/_{\rho} < 80$ or 140, empirical formulas have been developed

Gordon-Rankine formula

$$\frac{P}{A} = \frac{\sigma_0}{1 + C_1 \left(\frac{L}{\rho} \right)^2}$$

Straight-line formula

$$\frac{P}{A} = \sigma_0 - C_2 \frac{L}{\rho}$$

parabolic formula

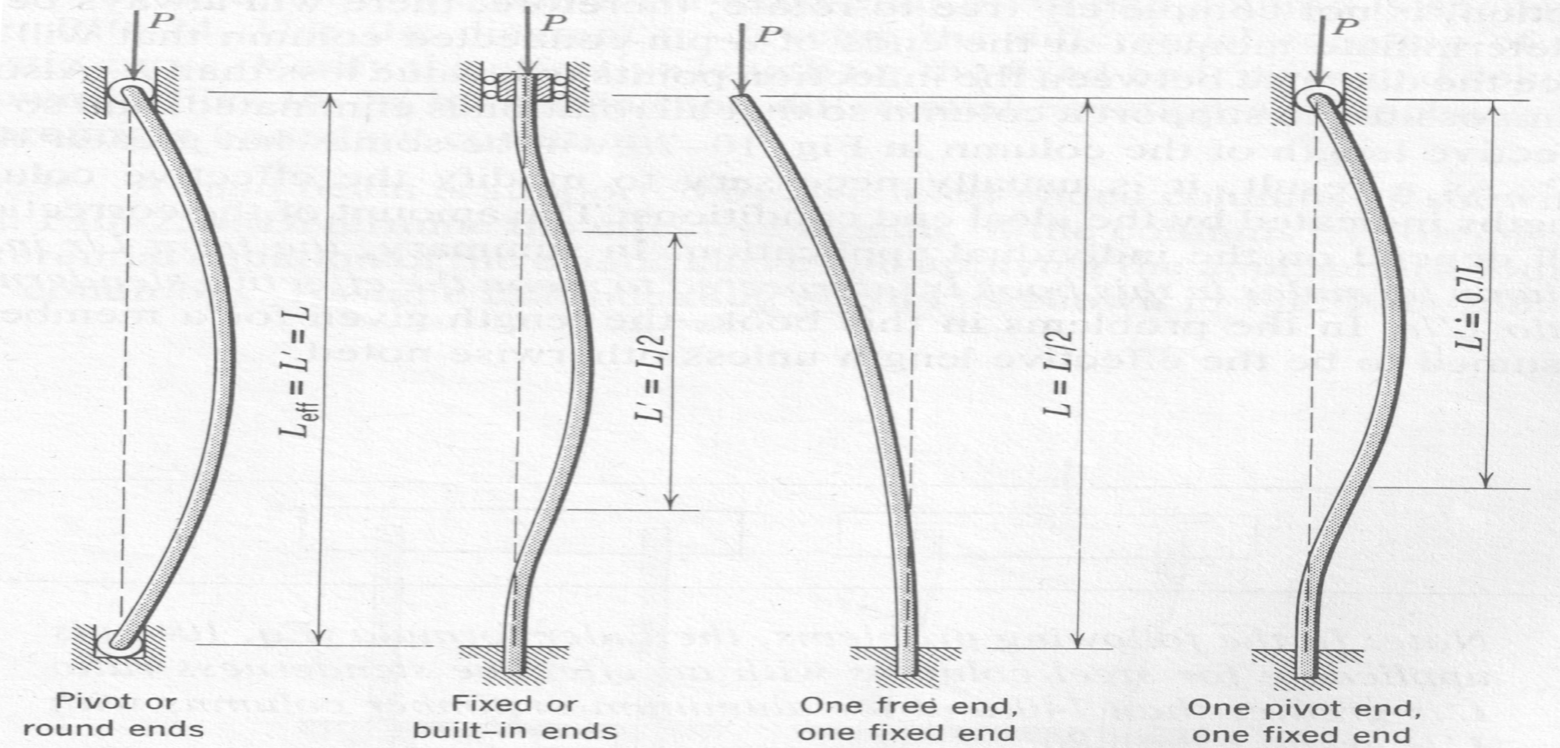
$$\frac{P}{A} = \sigma_0 - C_3 \left(\frac{L}{\rho} \right)^2$$

σ_0 and Cs: empirically determined

Effective Length

For different end conditions, the critical load varies. To account for this, the nominal L is replaced with an *effective length L’*

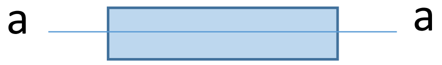
L $\Rightarrow$ **L’** in $P_{cr} = \dfrac{\pi^2 EI}{L^2}$ and $\sigma_{cr} = \dfrac{\pi^2 E}{\left(L/\rho\right)^2}$



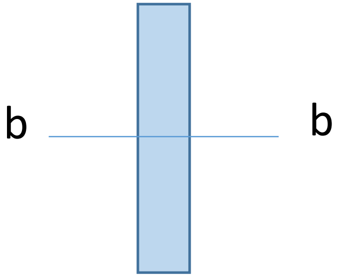
About Which Axis it Buckles?

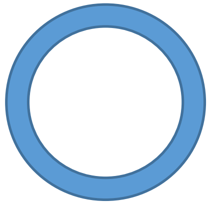
$$P_{cr} = \frac{\pi^2 EI}{L^2} \Rightarrow P_{cr} \propto I \Rightarrow$$

buckle about the “weaker” axis with *least moment of inertia*



vs.





Good choice for column !
(Why?)

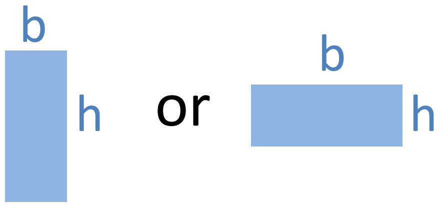
KEY: which shape distributes more around neutral axis?

vs.

A Workout Example

2.5m pivot-ended, 50 x 100-mm rectangular column made of 2014-T6 aluminum alloy with E=73.1GPa and yield strength 414MPa.
Find: slenderness ratio L/ρ, critical load P_{cr} and critical stress σ_{cr}

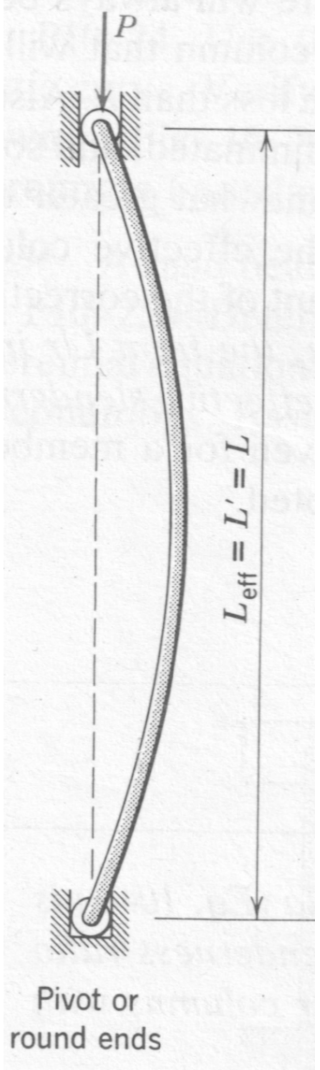
$$\mathrm{I} = \frac{bh^3}{12},\quad \rho = \sqrt{\frac{\mathrm{I}}{A}} = \frac{h}{2\sqrt{3}}$$



? see last slide

Recalling that $\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{\left(L/\rho\right)^2}$, we need minimum ρ to produce minimum σ_{cr}. So choose short side h=50mm:

$$\rho = \frac{50}{2\sqrt{3}} = 14.434mm$$



A Workout Example (cont'd)

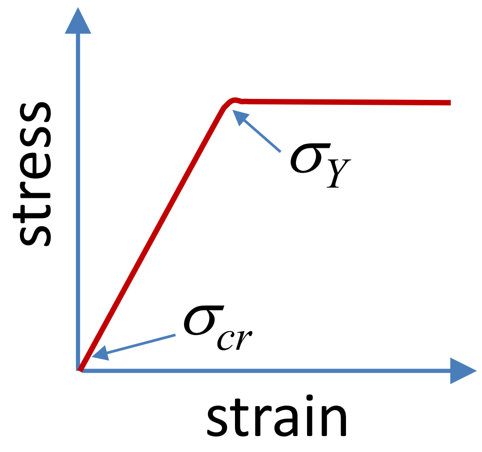
For pivot ends, L'=L. Slenderness ratio L/ρ=2.5x1000/14.434=173.202. Euler long column formula applies (must check this!)

Critical stress $\sigma_{cr} = \frac{\pi^2 E}{\left(L/\rho\right)^2} = \frac{\pi^2 \times 73.1 \times 1000}{173.202^2} = 24.050$ MPa

σ_{cr} < yield strength σ_Y = 414 MPa; valid (must check this also!)

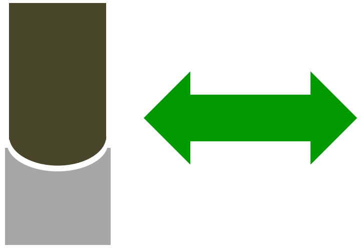
critical load $P_{cr} = \sigma_{cr} A$ = 24.05x(50÷1000)x(100÷1000)=0.120MN

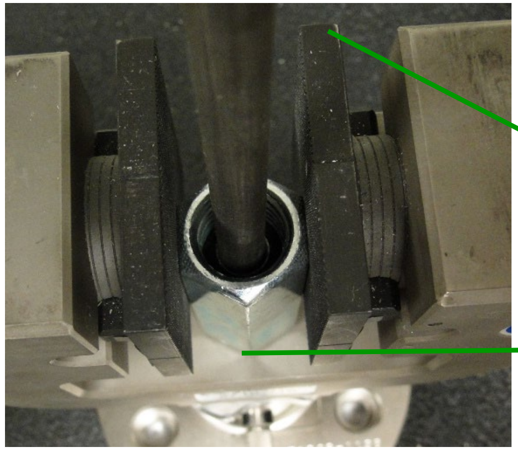
(critical stress σ_{cr}) ÷(yield strength σ_Y)=24.05÷414=5.8% !!!



About the Experiment

- Test buckling of slender steel and aluminum columns of rectangular or circular cross section under different end conditions
- Secure the columns in the Instron load station and apply compression load manually
- Pivot end condition is approximately by fitting round column end into receiving socket

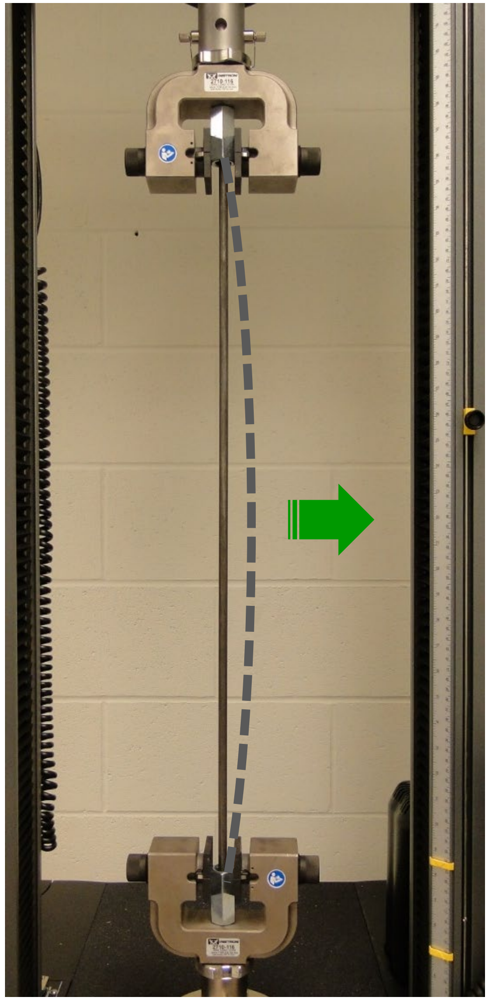




About the Experiment (cont'd)

- Carefully observe the sudden lateral "jolts" while increasing the compression load gradually in fine increments
- Compare the critical loads when "jolts" occur with the prelab predictions for the following cases:

specimen ID	Material	Cross section dimension (inch)	Length (inch)	End condition
I	aluminum	3/8 dia.	30	both pivot (round)
II	aluminum	1/4 x 1	30	both pivot (round)
III	steel	1/4 dia.	30	both pivot (round)
IV	steel	1/4 dia.	24	both pivot (round)
V	steel	1/4 dia.	27.5 (30 original)	one pivot, one fixed



See lab assignment for more details

Main References

1. D. Peery, *Aircraft Structures*, 1950, Ch. 14
2. A. Higdon et al., *Mechanics of Materials*, 4e 1985, Ch. 10