IOWA STATE UNIVERSITY

Aer E 322: Aerospace Structures Laboratory

Week 13 Lecture:

Vibration 101 and FFT Primer

April 18, 2022

Vibration 101

Introduction to Vibration Analysis

- Vibration is the study of oscillatory behaviors of physical systems
- Discrete vs. continuous
- Linear vs. nonlinear
- Deterministic vs. random
- Resonance

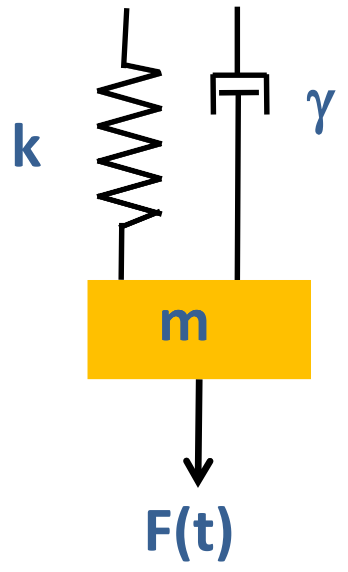
In this course, we emphasize on basic 1-DOF discrete system

General Single Degree-of-Freedom Discrete System

- Prototype or **building block** of more complicate systems

$$my''(t) + \gamma y'(t) + ky(t) = F(t) \qquad y'' = \frac{d^2y}{dt^2}, y' = \frac{dy}{dt}, etc.$$

Initial conditions: $y(0)=y_0$ (measured at equilibrium), $y'(0)=v_0$



k: spring stiffness;
γ: damping coefficient;
v_0: initial velocity

Vibration of Single DOF Discrete System

$$my''(t) + \gamma y'(t) + ky(t) = F(t)$$

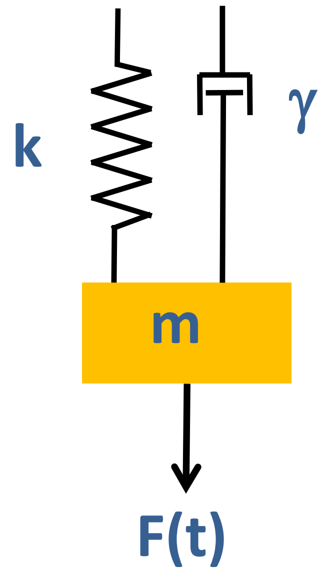
Undamped $\gamma=0$

Free $F=0$

Damped $\gamma\neq0$

Forced $F \neq 0$

vibration



1. Undamped Free Vibration

No external exciting force, no damping: $F(t)=0$, $\gamma=0$

$$\longrightarrow \quad my''(t) + ky(t) = 0$$

The general solution is (recall the 2nd order ODE solver in earlier lecture)

$$y(t) = \frac{v_0}{\omega_0}\sin\omega_0 t + y_0\cos\omega_0 t, \quad \omega_0 = \sqrt{k/m}$$

or

$$y(t) = R\cos(\omega_0 t - \theta), \quad R = \sqrt{y_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \quad \theta = \arctan\frac{v_0}{\omega_0 y_0}$$

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

ω_0: natural frequency, $=2\pi f_0$, R: amplitude, θ: phase, T: period

2. Damped Free Vibration

No external exciting force: F(t)=0 ➡ $my''(t) + \gamma y'(t) + ky(t) = 0$

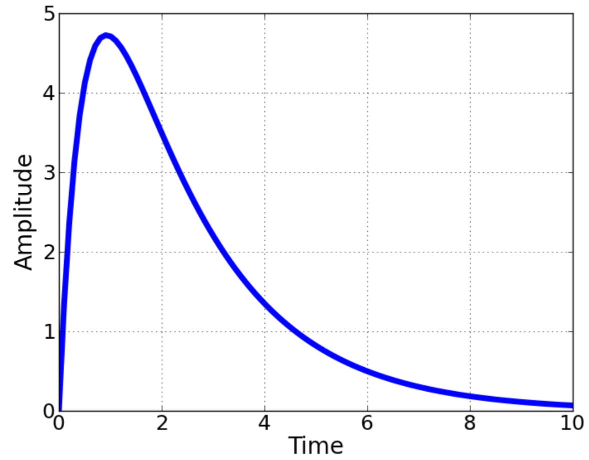
The characteristic equation is

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

which leads to three cases:

(1) $\gamma^2 - 4mk > 0:$ $\quad y(t) = \frac{r_2 y_0 - v_0}{r_2 - r_1} e^{r_1 t} + \frac{r_1 y_0 - v_0}{r_1 - r_2} e^{r_1 2t}, \quad r_1, r_2 < 0$

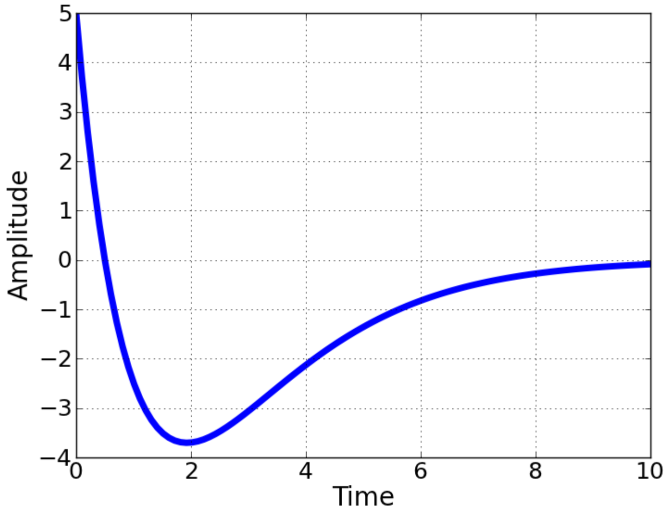
This case is called **overdamped** that oscillation quickly dies out exponentially; no period



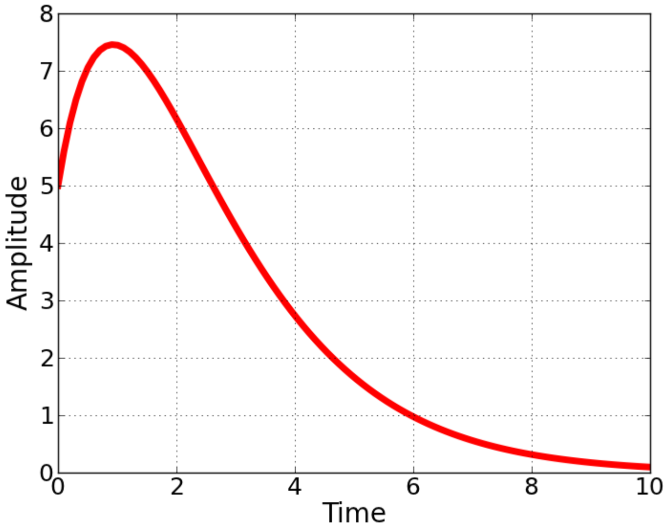
2. Damped Free Vibration (cont'd)

(2) $\gamma^2 - 4mk = 0: \quad y(t) = \left[y_0 + \left(\frac{\gamma y_0}{2m} + v_0 \right) \right] \exp\left(\frac{-\gamma t}{2m} \right), \quad \frac{\gamma}{2m} > 0$

Characteristic equation has only one repeated root. The second root can be found by, e.g. method of reduction of order. This case is known as **critically damped** that oscillation also dies out quickly; no period



Initial conditions determine oscillation shapes



2. Damped Free Vibration (cont'd)

(3) $\gamma^2 - 4mk < 0:$ $\quad y(t) = \left[y_0 \cos \mu t + \left(\frac{\gamma y_0}{2\mu m} + \frac{v_0}{\mu} \right) \sin \mu t \right] \exp\left(\frac{-\gamma t}{2m} \right)$

or $y(t) = R\cos(\mu t - \theta)\exp\left(\frac{-\gamma t}{2m} \right)$

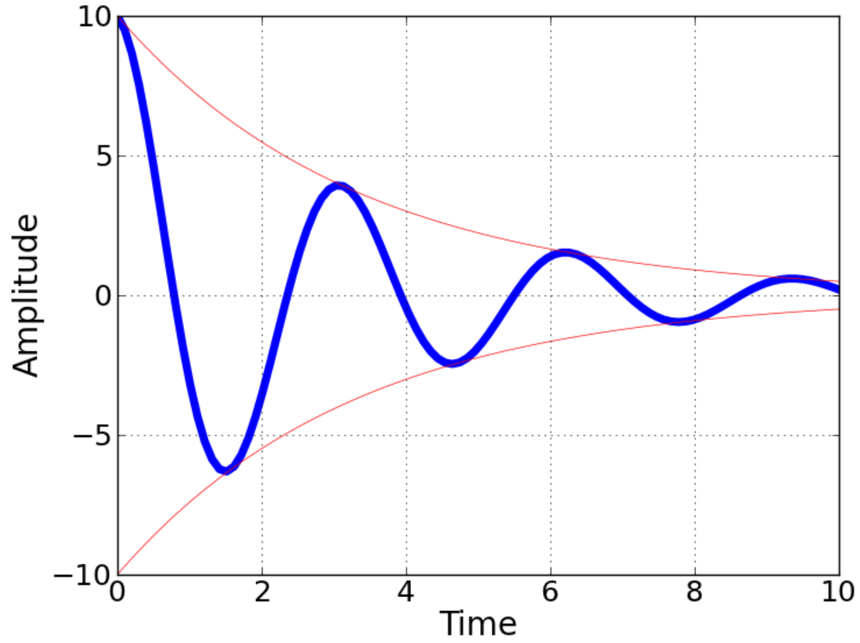
where $\mu = \frac{\sqrt{4mk - \gamma^2}}{2m}$ is called **quasi frequency**

This case is **underdamped** that oscillation dies out slower with **quasi period** $T=2\pi/\mu$

Note that in all cases the damper absorbs energy, so as expected

$$t \to \infty, \quad y(t) \to 0$$

Application: shock absorber of cars



3. Damped Forced Vibration

Add an external sinusoidal force F(t)=$F_0\cos\omega t$ back

$$\Longrightarrow \quad my''(t)+\gamma y'(t)+ky(t)=F_0\cos\omega t$$

This is an inhomogeneous ODE whose solution has two parts. One is called *complementary solution, $y_c(t)$,* which is the solution to the homogeneous equation as before

(1) $\gamma^2-4mk>0: \quad y(t)=\dfrac{r_2y_0-v_0}{r_2-r_1}e^{r_1t}+\dfrac{r_1y_0-v_0}{r_1-r_2}e^{r_12t}, \quad r_1,r_2<0$

(2) $\gamma^2-4mk=0: \quad y(t)=\left[y_0+\left(\dfrac{\gamma y_0}{2m}+v_0\right)\right]\exp\left(\dfrac{-\gamma t}{2m}\right), \quad \dfrac{\gamma}{2m}>0$

(3) $\gamma^2-4mk<0: \quad y(t)=\left[y_0\cos\mu t+\left(\dfrac{\gamma y_0}{2\mu m}+\dfrac{v_0}{\mu}\right)\sin\mu t\right]\exp\left(\dfrac{-\gamma t}{2m}\right)$

$y_c(t)$ responds to initial conditions and is **transient** since

$$t\to\infty, \quad y_c(t)\to 0$$

3. Damped Forced Vibration (cont'd)

The other is a particular solution, $y_p(t)$, which can be found by, e.g. method of undetermined coefficients:

$$y_p(t) = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \cos(\omega t - \theta)$$

where $\theta = \arcsin\left(\frac{\gamma\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}\right)$

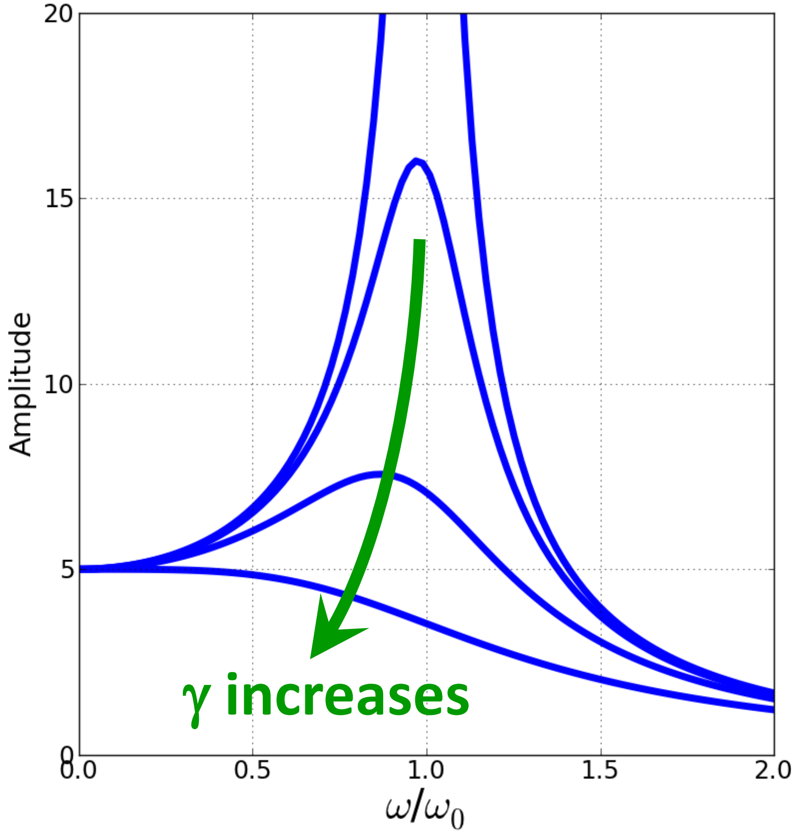
$Y_p(t)$ is called **forced response** (to the external sinusoidal force) or steady-state solution. Its oscillation has the same frequency ω and persists as long as the external force is present

3. Damped Forced Vibration - Resonance

$$y_p(t) = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}\cos(\omega t - \theta) = R\cos(\omega t - \theta)$$

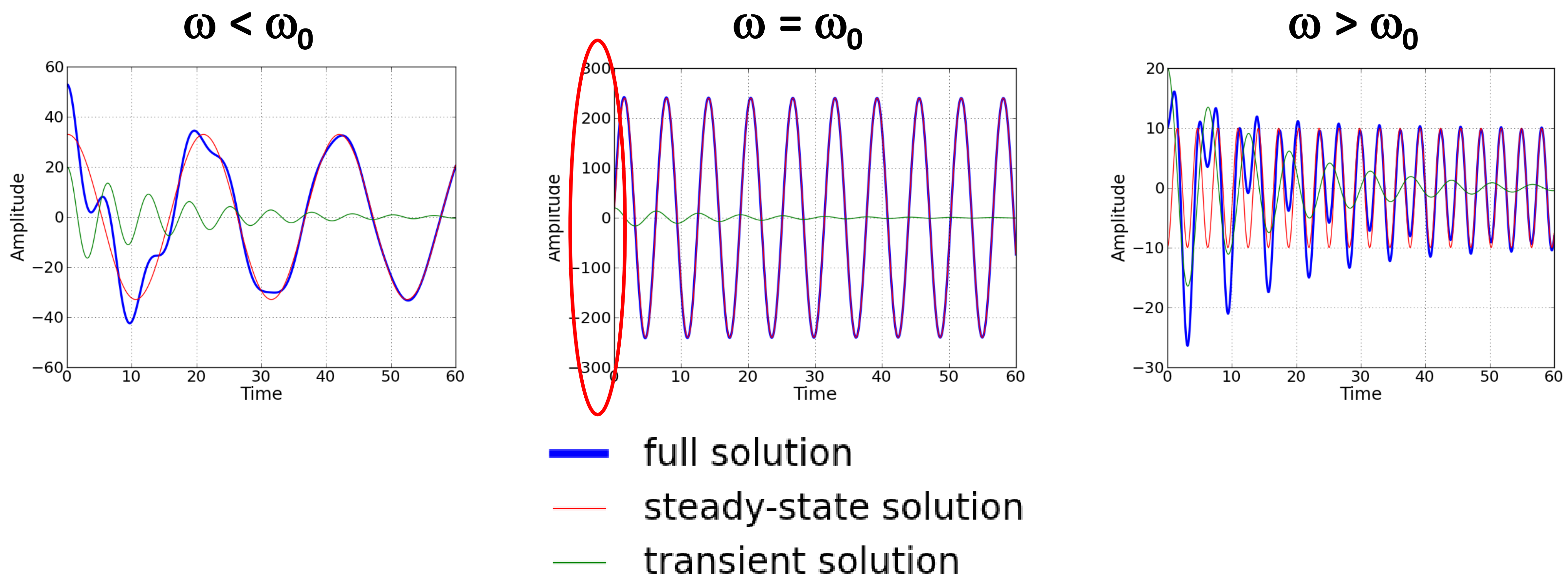
Note that R becomes large as damping γ decreases and ω is in the vicinity of ω_0. In the limit, R $\rightarrow \infty$ as $\gamma \rightarrow 0$ given $F_0 \neq 0$ and $\omega = \omega_0 \neq 0$!

This phenomenon, known as **resonance**, is of critical importance to any physical structure



3. Damped Forced Vibration (cont'd)

Underdamped cases



4. Undamped Forced Vibration

No damping but an external sinusoidal force: $\gamma=0$, $F(t)=F_0\cos\omega t$

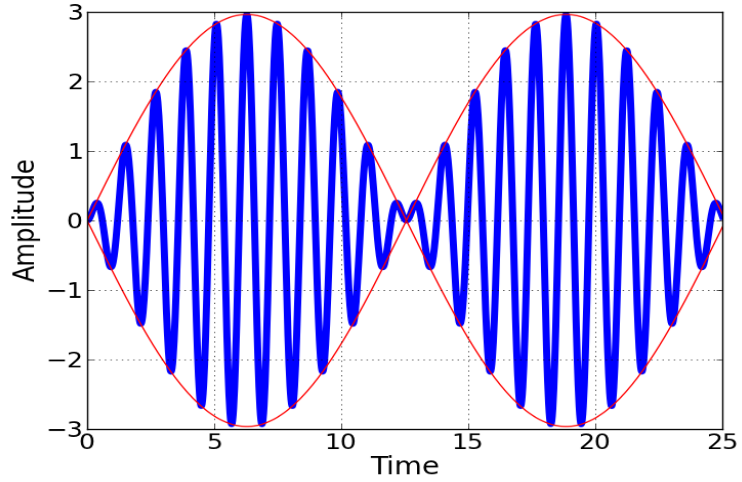
$$\longrightarrow \quad my''(t) + ky(t) = F_0 \cos\omega t$$

(1) $\omega \neq \omega_0$:

$$y(t) = \left[y_0 - \frac{F_0}{m(\omega_0^2 - \omega^2)} \right] \cos\omega_0 t + \frac{v_0}{\omega_0} \sin\omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos\omega t$$

If the system is at rest initially, then $y_0=v_0=0$ *and* $y(t)$ can be re-arranged to

$$y(t) = \underbrace{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2}}_{\text{envelope}} \sin\frac{(\omega_0 + \omega)t}{2}$$



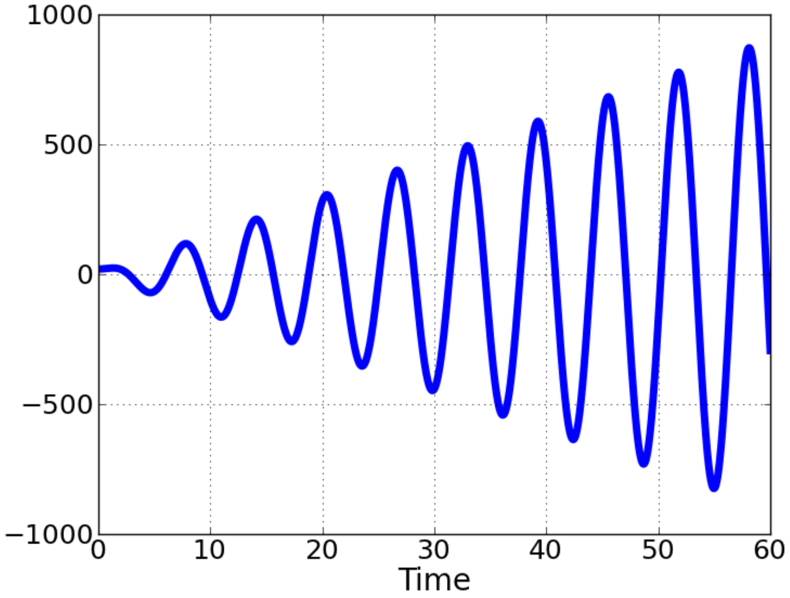
This phenomenon, known as **beat**, occurs with e.g. two tuning forks of nearly identical frequencies

4. Undamped Forced Vibration (cont'd)

(2) $\omega = \omega_0$:

$$y(t) = y_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t + \frac{F_0}{2m\omega_0} t \sin\omega_0 t$$

The oscillation becomes unbounded as time goes on!



Vibration of Single DOF Discrete System: Recap

$$my''(t) + \gamma y'(t) + ky(t) = F(t)$$

amplitude A
frequency ω
natural frequency ω_0

Undamped
$\gamma=0$

Free
$F=0$

continuous vibration with $\omega=\omega_0$ and constant A

transient vibration with decayed A: (1) overdamped, (2) critically damped and (3) underdamped with quasi ω

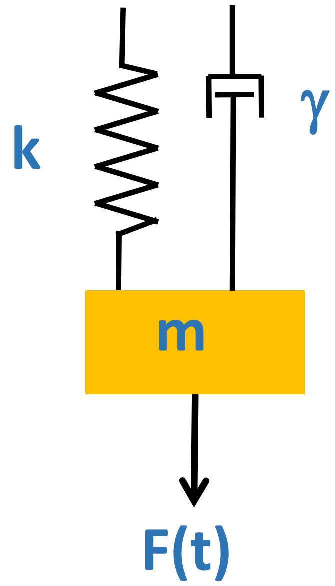
(1) $\omega \neq \omega_0$: continuous vibration with variable A (beat phenomenon), (2) $\omega = \omega_0$: A unphysically unbounded

continuous vibration with variable A initially: (1) overdamped, (2) critically damped and (3) underdamped

Damped
$\gamma \neq 0$

Forced
$F \neq 0$

$\omega = \omega_0$ **resonance**!



Structural Failures Due to Aerodynamic Resonance

Resonance coupled with aerodynamic effects can result in catastrophic structure failures. The historic Tocoma Bridage collapse is one well-known example in civic structures. In aircraft structure, similar phenomena known as **flutter** also occur. This area of study is called **aeroelasticity**

Tacoma bridge collapse

https://www.youtube.com/watch?v=KqqyAZDpV6c

Aircraft flutter crash

https://www.youtube.com/watch?v=egDWh7jnNic

FFT Primer

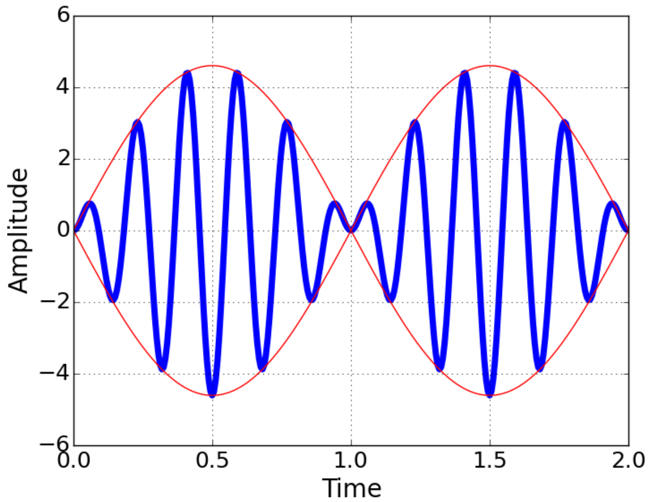
Fourier Transform

In studying vibration, **Fourier transform** (FT) is the standard tool which relates signal between time and frequency domains

$$f(t) = \int_{-\infty}^{\infty} F(f) e^{i2\pi f t}\, dt \;\Leftrightarrow\; F(f) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi f t}\, df \qquad \omega = 2\pi f$$

Fast Fourier transform (FFT) is an *efficient* numerical implementation of the above continuous FT and is almost always done via computers

$$y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2} \sin\frac{(\omega_0 + \omega)t}{2}$$

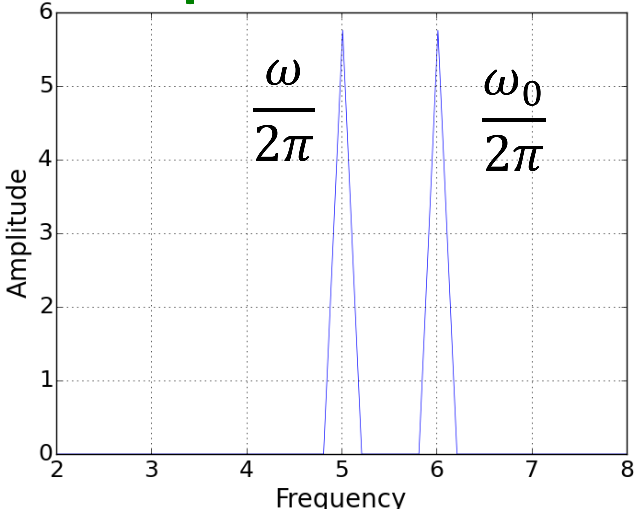


FFT

IFFT

I: inverse

spectrum



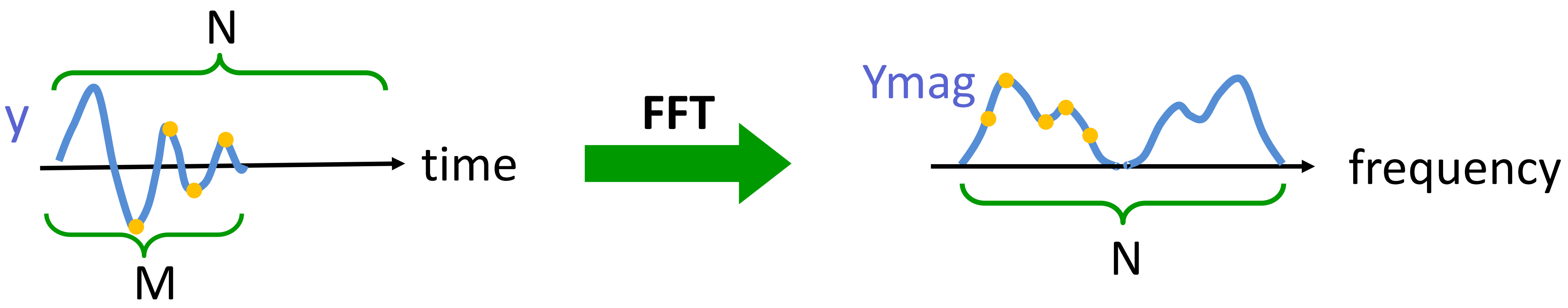
Spectral Analysis by FFT: Basic Usage

Nowadays most software systems make FFT quite easy to use. Taking MATLAB for example, we can perform FFT via a simple call like this:

Y = fft(y, N)

where fft() is the MATLAB built-in FFT function, y is the input array storing the real time domain signal data points of length M, and Y is the output array of length N containing the complex frequency domain spectra. Note that $N \geq M$. Then by taking the absolute value of Y we can obtain its real spectral magnitude:

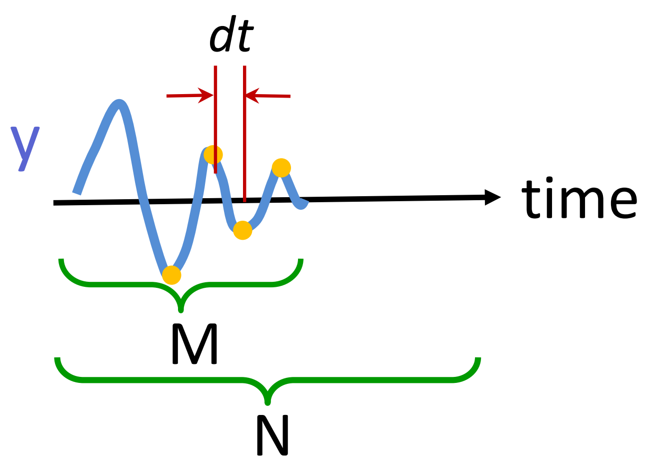
Ymag = abs(Y)



See how all these are done in the MATLAB demo code "fftsin.m" available in class web site under Misc Documents and in Canvas

Spectral Analysis by FFT: Basic Usage (cont'd)

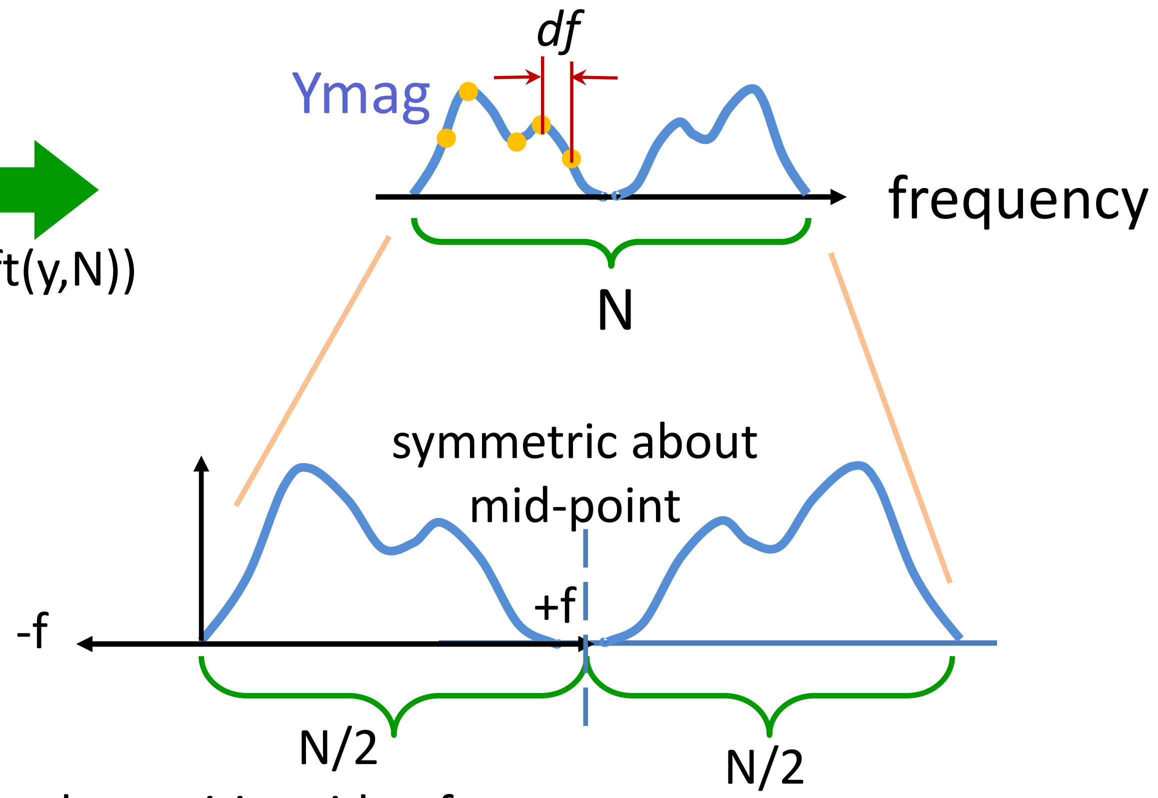
time sampling interval:



FFT

Ymag=abs(fft(y,N))

frequency sampling interval:



$$df = \frac{1}{Ndt}$$

represents the <u>positive</u> side of frequency of the continuous FT

take this half by convention

represents the <u>negative</u> side of frequency of the continuous FT

Spectral Analysis by FFT: Issues and Pitfalls

FFT is probably the most popular tool for spectral analysis, but is also frequently misused and misinterpreted. Under the hood FFT actually implements the discrete version of FT and is expected to take on periodic data. You therefore need to learn how to use FFT and interpret its results properly in conforming with the continuous FT. Below we point out some common issues and pitfalls.

Signal Sampling

Spectral analysis is all about finding significant characteristics such as peaks in the spectra. As you can see, the ability of detecting fine spectral peaks relies on high frequency resolution, i.e. by maintaining a sufficiently small frequency sampling interval df. From $df=1/(Ndt)$, we see equivalently this requires a large value of the product Ndt.

Spectral Analysis by FFT: Issues and Pitfalls

Signal Sampling (cont'd)

So the first rule-of-thumb is to sample the time signal as long as possibly can, i.e. to obtain a lengthy *N*. In the meantime, however, we also need to keep the time sampling interval *dt* sufficiently small to satisfy the so-called **Nyquist sampling criterion**:

$$dt < \frac{1}{2f_{max}}$$

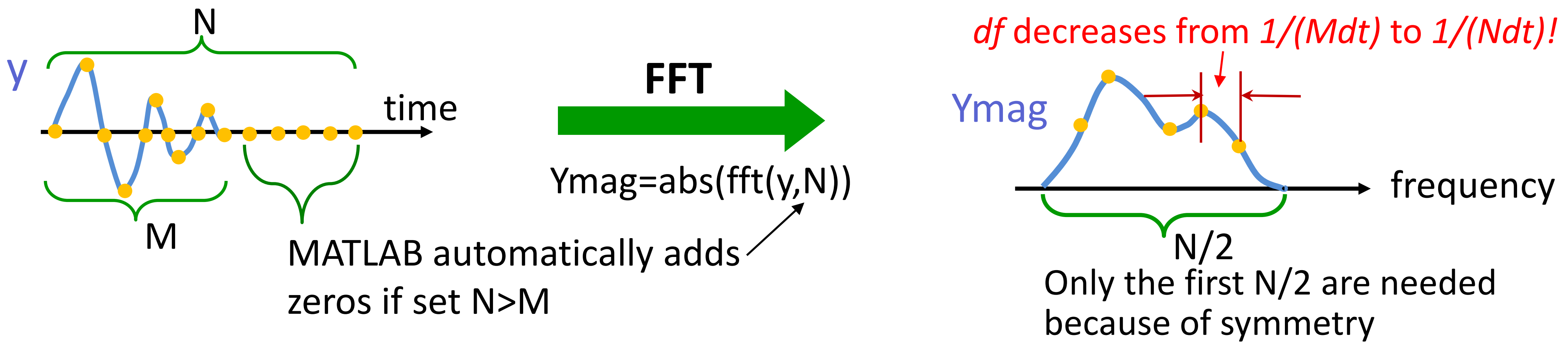
where f_{max} is signal's highest frequency which can be estimated by iterating FFT trials if not known in advance. In practice, smaller *dt* than $1/(2f_{max})$ is routinely used

If the Nyquist criterion is not satisfied, a rippling distortion of the spectra known as **Aliasing** will happen. Similar ripple effects or side lobes can occur if the time signal is truncated too soon, i.e. not sampled long enough

Spectral Analysis by FFT: Issues and Pitfalls

Signal Sampling (cont'd)

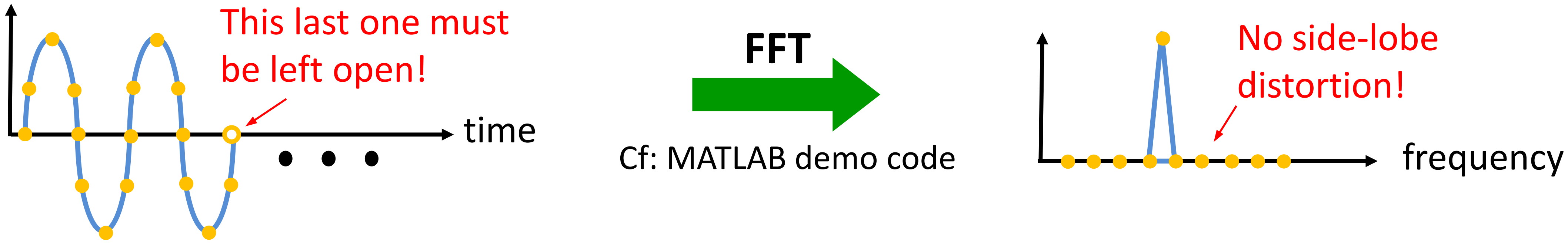
One popular technique for improving frequency resolution *(but not the contents!)* is known as **zero-padding** by simply padding the time signal with extra zeros. While holding *dt* the same in *df=1/(Ndt)*, these zeros increase *N* and decrease *df*, i.e. improve the frequency resolution. The underlying improvement comes from interpolating the spectra by the *sinc* function (=*sin(f)/f*). But zero-padding only works well for signals of finite duration. If the time signal is expected to be periodic, zero-padding actually breaks the periodicity, and the *sinc* interpolation makes frequency resolution worse by introducing extra side lobes in the spectra.



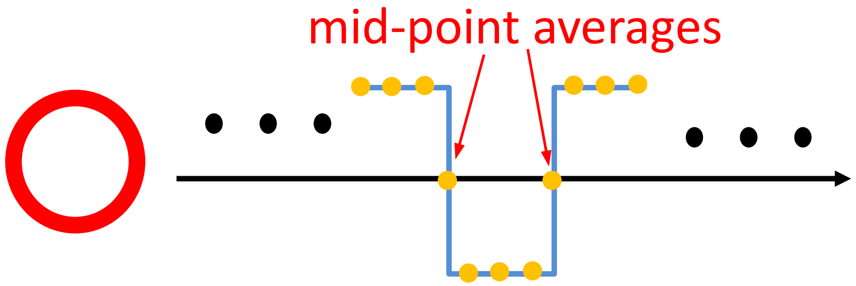
Spectral Analysis by FFT: Issues and Pitfalls

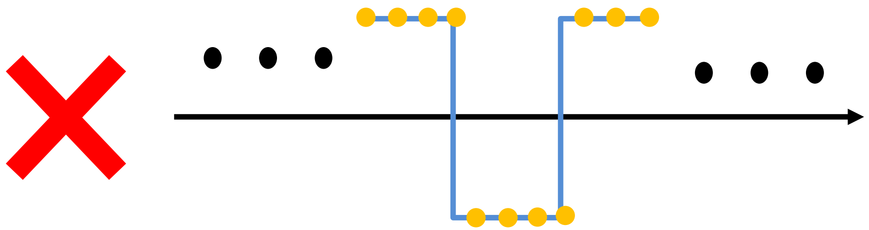
Signal Sampling (cont'd)

If you know your signal is periodic (e.g. a sine wave), try to sample exactly multiple periods of the time signal (as shown) to avoid aliasing-like distortion.



Often the time signal has discontinuities. Be sure to set "average" at these discontinuities (again to reduce side lobes):





Spectral Analysis by FFT: Issues and Pitfalls

Amplitude Scaling

When analyzing signals in frequency domain, what usually matter are the peak locations, not their amplitudes. But if you do need to quantify the spectral amplitudes, extra care must be taken in handling amplitude scaling. This is largely due to the unfortunate fact that there is no universal expression defined for the continuous FT, and variations exist among the different definitions in specifying the amplitude multiplier to the Fourier integrals (as seen on page 19). Depending on which FT definition your FFT code follows, different amplitude scaling factors will have to be used.

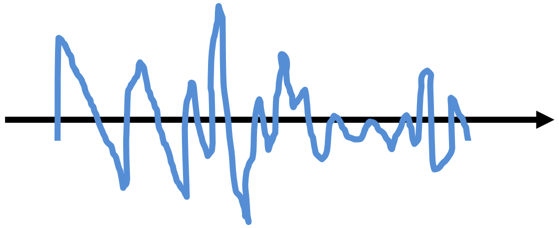
Spectral Analysis by FFT: Issues and Pitfalls

Amplitude Scaling (cont'd)

A proper amplitude scaling is often determined by observing the signal trend and making the best educational guess. Below we show two amplitude scaling factors applied to different signal types. These amplitude scaling factors are needed to keep the signal amplitude of the MATLAB FFT output (shown on page 20) consistent with that of the FT pair (defined on page 19)

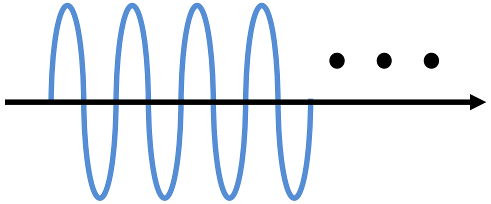
General signal

Y = dt*fft(y, N)



Signal likely to be periodic

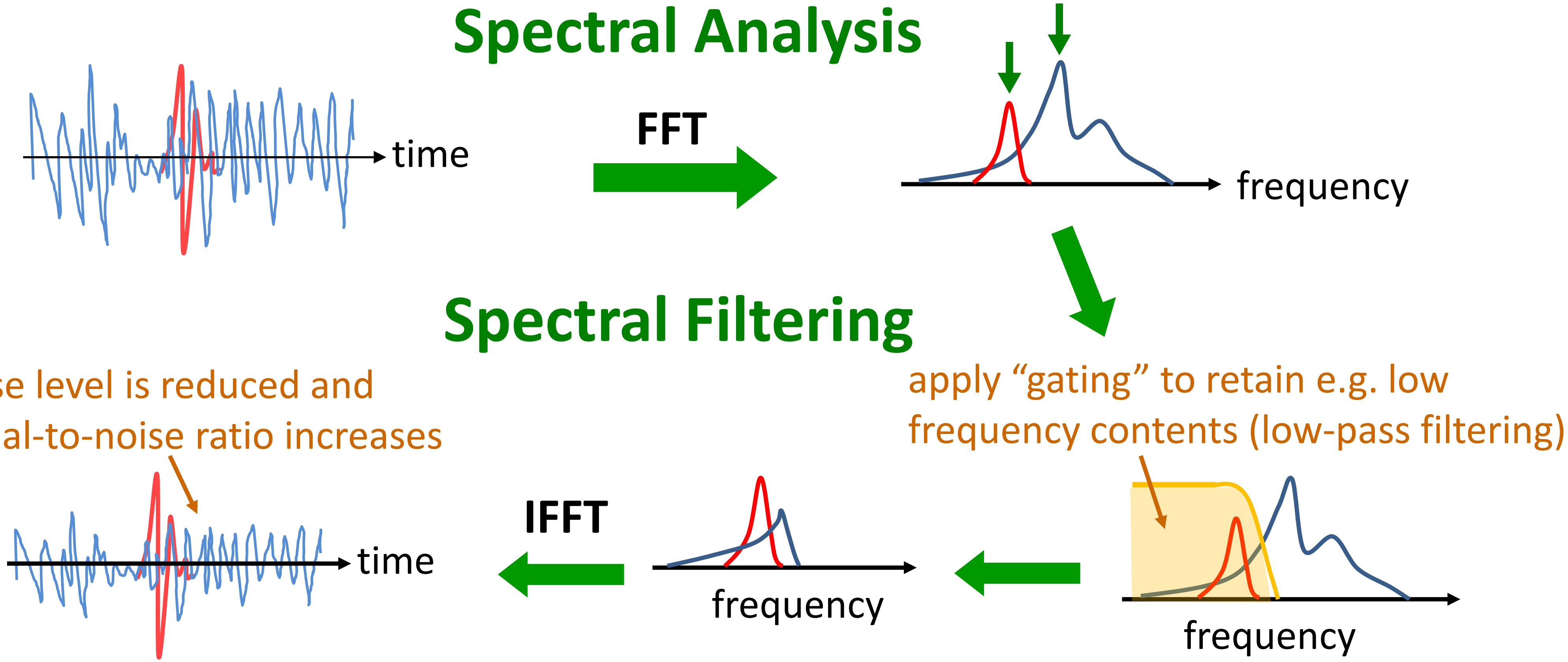
Y = (1/N)*fft(y, N)



Spectral Analysis and Filtering

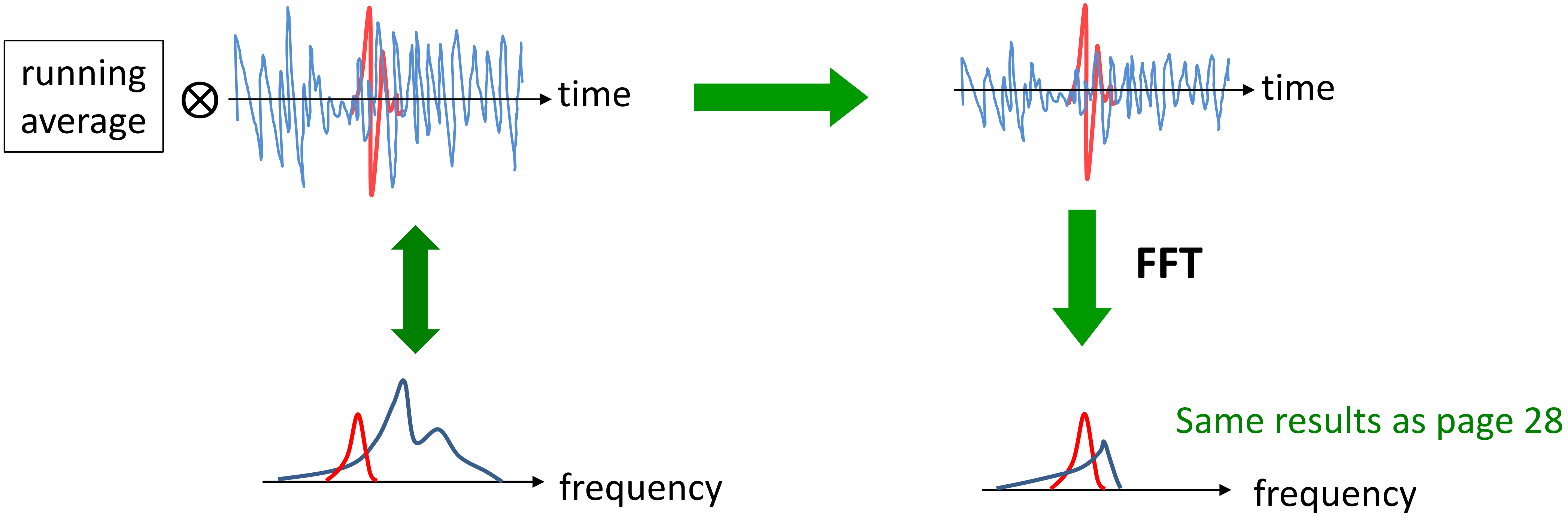
Two important applications of FFT are

- Spectral analysis (e.g. identifying key frequency and strength)
- Spectral filtering (i.e. removing unwanted frequency contents)



Spectral Filtering (cont'd)

Spectral filtering can also be done directly in time domain. The ***running average*** technique you learned earlier this semester is actually an effective low-pass filter. You can get the same result by first applying it in time domain and followed by an FFT



References

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E Oran Brigham “The Fast Fourier Transform and Its Applications” Prentice Hall, 1988