Week 14 Lecture:

Structural Dynamics 101

April 25, 2022

Introduction to Structural Dynamics Analysis

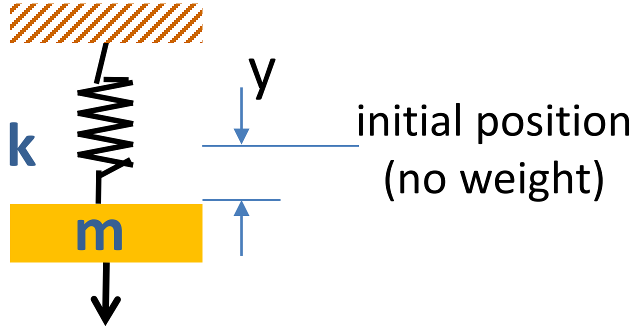
- In Week 13 lecture you have learned the fundamentals of mechanical vibration. These include free and forced vibration with or without damping, and resonance.
- This lecture continues on in-depth analysis of structural dynamics, particularly for determining natural frequencies and mode shapes
- We will emphasize on the uses of equivalent 1-DOF mass-spring system and 2-DOF dynamic stiffness method

Determining Natural Frequency Using Equivalent 1-DOF Mass-Spring System

In Week 13 lecture we learned that undamped free vibration has the natural frequency

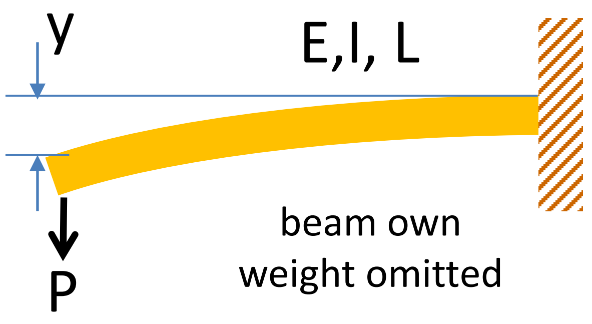
$$\omega = \sqrt{\frac{k}{m}}$$

$$y = \frac{mg}{k}$$



determined by the stiffness *k* and mass m. The same mass-spring principle can be easily applied to other simple structures, such as this cantilever beam of length L with end load P:

$$y = \frac{P}{k} = \frac{PL^3}{3EI} \Longrightarrow k = \frac{3EI}{L^3}$$

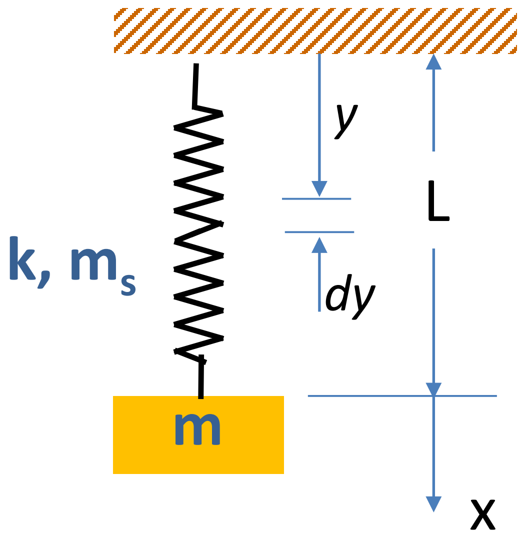


Thus $\omega = \sqrt{\frac{3EI}{mL^3}}$

Determining Natural Frequency Using Equivalent 1-DOF Mass-Spring System (cont'd)

The motion of 1-DOF system can often be handled by *Rayleigh method* using energy method and the **equivalent lumped mass** or *effective mass* referenced to a specified point.

Consider the mass-spring system again but with spring's own weight added. Let *x'* be the velocity of the lumped mass *m*. Assuming that the velocity of spring element *dy* located at a distance *y* from the top fixed end vary linearly, then the kinetic energy of element *dy* is



$$\frac{1}{2}\left(m_s \frac{dy}{L}\right)\left(\frac{y}{L}x'\right)^2$$

Determining Natural Frequency Using Equivalent 1-DOF Mass-Spring System (cont'd)

Then the total added kinetic energy of spring's own weight is

$$\frac{1}{2}\int_0^L \left(\frac{m_s}{L}\right)\left(\frac{y}{L}x'\right)^2 dy = \frac{1}{2}\frac{m_s}{3}(x')^2$$

So the equivalent lumped mass or effective mass is found to be one third of the spring mass. Hence the natural frequency of the mass-spring system with spring's own weight included needs to be revised to

$$\omega = \sqrt{\frac{k}{m + \frac{m_s}{3}}}$$

Model Free Normal Mode Vibration by 2-DOF Mass-Spring System

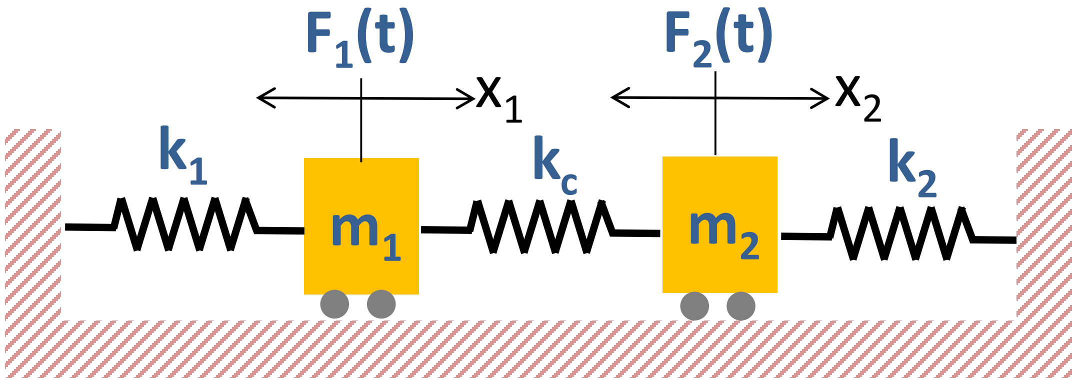
From FBDs

$$m_1 x_1''(t) + (k_1 + k_c)x_1(t) - k_c x_2(t) = F_1(t)$$

$$m_2 x_2''(t) - k_c x_1(t) + (k_2 + k_c)x_2(t) = F_2(t)$$

In matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} + \begin{bmatrix} (k_1 + k_c) & -k_c \\ -k_c & (k_2 + k_c) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$



Model Free Normal Mode Vibration by 2-DOF Mass-Spring System (cont'd)

To determine the natural frequencies and mode shapes of the system, consider the homogeneous equations

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} + \begin{bmatrix} (k_1 + k_c) & -k_c \\ -k_c & (k_2 + k_c) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We also assume simple harmonic motion such that

$$x_i(t) = A_i \sin \omega t, \quad x_i''(t) = -\omega^2 A_i \sin \omega t$$

Then

$$\begin{bmatrix} (k_1 + k_c) & -k_c \\ -k_c & (k_2 + k_c) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Model Free Normal Mode Vibration by 2-DOF Mass-Spring System (cont'd)

Or

$$\begin{bmatrix} k_1 + k_c - \omega^2 m_1 & -k_c \\ -k_c & k_2 + k_c - \omega^2 m_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has no-trivial solutions only if its determinant equals to zero, i.e.

$$\begin{vmatrix} k_1 + k_c - \omega^2 m_1 & -k_c \\ -k_c & k_2 + k_c - \omega^2 m_2 \end{vmatrix} = 0$$

Expanding the determinant yields the characteristic equation for ω^2:

$$\omega^4 - \left(\frac{k_1 + k_c}{m_1} + \frac{k_2 + k_c}{m_2} \right) \omega^2 + \frac{k_1 k_2 + (k_1 + k_2) k_c}{m_1 m_2} = 0$$

Square roots of the two solutions to the characteristic equation, ω_1 and ω_2, are the two natural frequencies of the mass-spring system.

Model Free Normal Mode Vibration by 2-DOF Mass-Spring System (cont'd)

Two corresponding amplitude ratios can be obtained from the matrix equations as

$$\left(\frac{A_1^{(1)}}{A_2^{(1)}}\right)=\left(\frac{A_1}{A_2}\right)_{\omega=\omega_1}=\frac{k_c}{k_1+k_c-\omega_1^2 m_1}=\frac{k_2+k_c-\omega_1^2 m_2}{k_c}=r_1$$

$$\left(\frac{A_1^{(2)}}{A_2^{(2)}}\right)=\left(\frac{A_1}{A_2}\right)_{\omega=\omega_2}=\frac{k_c}{k_1+k_c-\omega_2^2 m_1}=\frac{k_2+k_c-\omega_2^2 m_2}{k_c}=r_2$$

Following this, two corresponding vectors, called modal vectors, can be selected as

$$\begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \end{pmatrix}=\begin{pmatrix} r_1 A_2^{(1)} \\ A_2^{(1)} \end{pmatrix},\quad \begin{pmatrix} A_1^{(2)} \\ A_2^{(2)} \end{pmatrix}=\begin{pmatrix} r_2 A_2^{(2)} \\ A_2^{(2)} \end{pmatrix}$$

These vectors can be normalized by arbitrarily assigning, say 1, to $A_2^{(1)}$ and $A_2^{(2)}$

Mathematically, these natural frequencies and modal vectors are known as the eigenvalues and eigenvectors

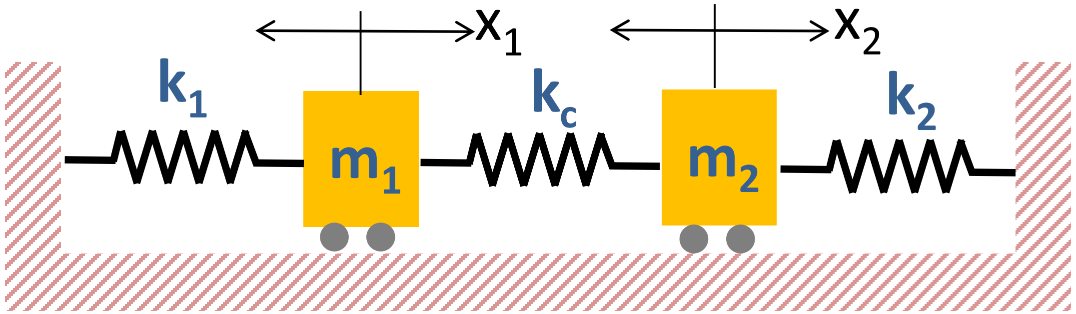
Free Vibration Example of 2-DOF Mass-Spring System

Let $m_1=m_2=m$ and $k_1=k_2=k_c=k$ in our 2-DOF mass-spring system, then the characteristics equation reduces to

$$\omega^4 - \frac{4k}{m}\omega^2 + \frac{3k^2}{m^2} = 0$$

The two natural frequencies, as the solutions to the characteristics equations, are

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}}$$



Free Vibration Example of 2-DOF Mass-Spring System (cont'd)

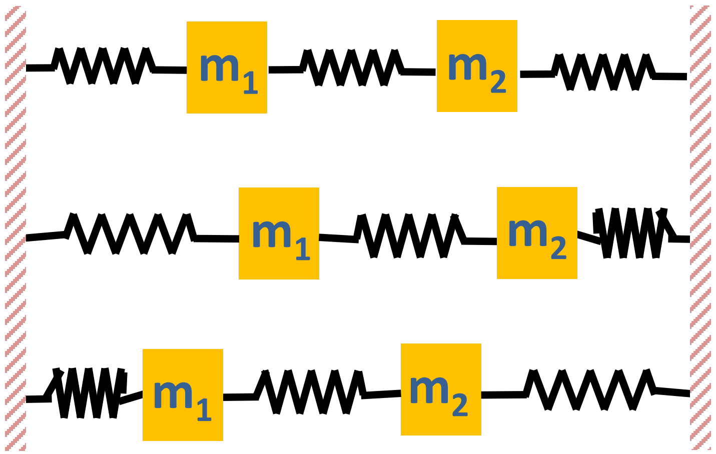
$$\begin{pmatrix} \dfrac{A_1^{(1)}}{A_2^{(1)}} \end{pmatrix} = \begin{pmatrix} \dfrac{A_1}{A_2} \end{pmatrix}_{\omega=\omega_1} = \frac{k+k-(k/m)m}{k} = 1,$$

$$\begin{pmatrix} \dfrac{A_1^{(2)}}{A_2^{(2)}} \end{pmatrix} = \begin{pmatrix} \dfrac{A_1}{A_2} \end{pmatrix}_{\omega=\omega_2} = \frac{k+k-(3k/m)m}{k} = -1$$

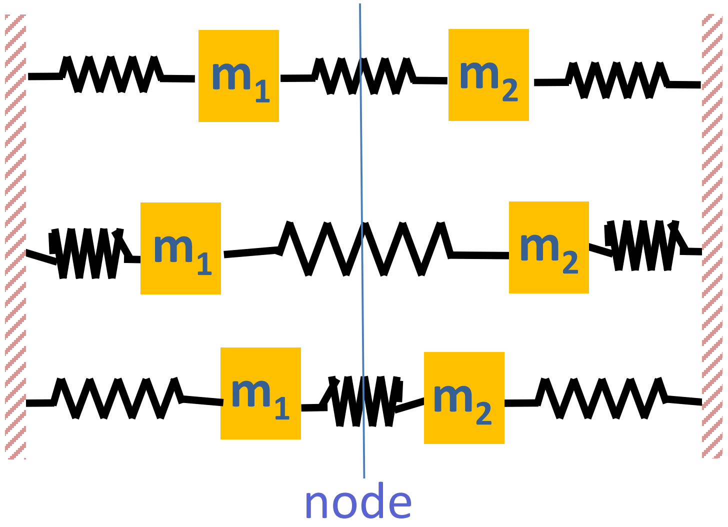
Modal vectors

$$\begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} A_1^{(2)} \\ A_2^{(2)} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Mode #1: $\omega=\omega_1$ **"in sync"**

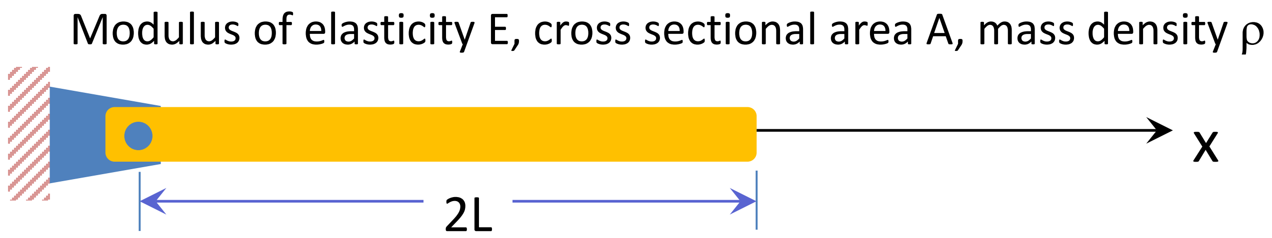


Mode #2: $\omega=\omega_2$ **"opposite"**



Dynamic Stiffness Method for Bar Vibration

The normal mode concept applied to 2-DOF mass-spring system can be further extended to stiffness method *dynamically* for determining natural frequencies and modal shapes of complex structures. This is demonstrated in the following 1-D bar example.



It is understood that a bar element, commonly used in truss structure with pin joints, can only take axial force and has one DOF at each end

Following what have learned earlier in stiffness method on beam and frame from AerE 321, we can derive the local stiffness matrix for a bar element of length *L*, cross sectional *A* and modulus of elasticity *E*:

$$k = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Dynamic Stiffness Method for Bar Vibration (cont'd)

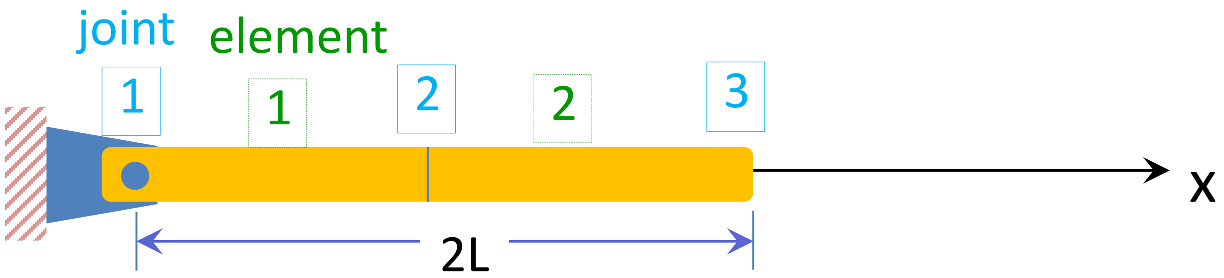
We discretize the 2L bar into two elements of length L each:

node number

$$k_{element\,\#1} = \frac{AE}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \begin{array}{cc} 2 & 3 \end{array} \end{array} \frac{AE}{L} = k_{element\,\#2}$$

$$k_{global} = \frac{AE}{L}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$



Modulus of elasticity E, cross sectional area A, mass density ρ

Dynamic Stiffness Method for Bar Vibration (cont'd)

Following the lumped mass approach, we divide the element mass evenly between the two end joints:

$$m_{local,element} = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$m_{global,bar} = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the matrix equation for the entire bar can be assembled likewise

$$\left(\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Dynamic Stiffness Method for Bar Vibration (cont'd)

Since joint 1 is fixed, we can use the boundary condition there to reduce the matrix equation down to

$$\left(\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho AL}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Like before, we solve the corresponding characteristics equation for the natural frequencies ω

$$\omega_1 = \sqrt{(2-\sqrt{2})\mu}, \;\; \omega_2 = \sqrt{(2+\sqrt{2})\mu}, \; \mu = \sqrt{\frac{E}{\rho L^2}}$$

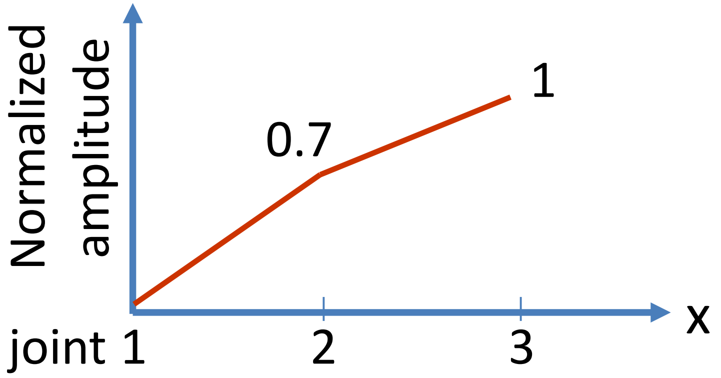
The modal vectors that determine the mode shapes can be obtained similarly:

$$\begin{pmatrix} A_2^{(1)} \\ A_3^{(1)} \end{pmatrix} = \begin{pmatrix} 0.7 \\ 1 \end{pmatrix}, \;\; \begin{pmatrix} A_2^{(2)} \\ A_3^{(2)} \end{pmatrix} = \begin{pmatrix} -0.7 \\ 1 \end{pmatrix}, \;\; A_1^{(1)} = A_1^{(2)} = 0$$

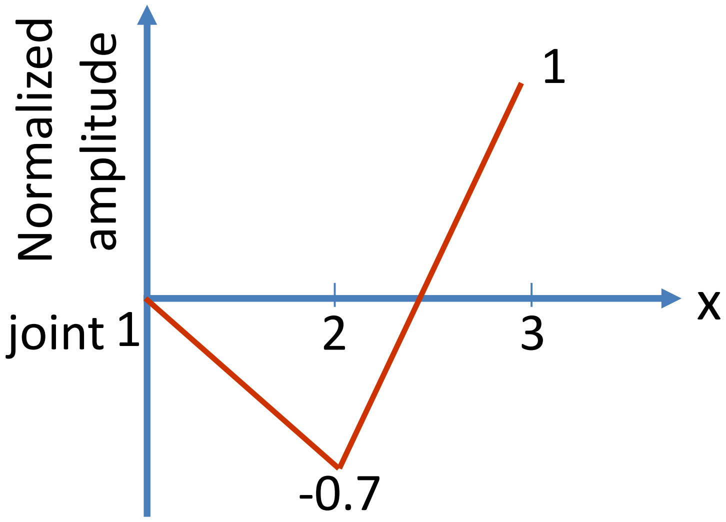
Dynamic Stiffness Method for Bar Vibration (cont'd)

The resulting mode shapes are similar to those of the 2-DOF mass-spring system. Depending on the excitation direction, the bar can be entirely in tension or in compression at first natural frequency ω_1, and in second mode ω_2, the bar is partially tension and compression or compression and tension.

Mode #1: $\omega = \omega_1$ "in sync"



Mode #2: $\omega = \omega_2$ "opposite"



Stiffness Method for Structural Dynamics: Application to Lab 10 Model Structure Building

The bar example illustrated above can be readily extended to beam, frame and other structures using large number of DOFs to determine higher natural frequencies and modal shapes.

In practice, however, the computation for solving high order eigenvalue problem quickly becomes much more involved and time consuming. In reality, the higher modes die down faster than the lower ones, and thus in general only the first few modes are of interest.

Nevertheless, for Lab 10 3D model structure building (skipped in F20 and S21 due to pandemic), what you learn here should provide you a good starting point for tackling dynamic analysis of some simple structures such as your masterpiece. *Give it a try and Good luck!*

How about this?!

References

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H.A. Buchholdt and S.E. Moossavi Nejad, *Structural Dynamics for Engineers*, second edition, ICE Publishing, 2012

Daryl L. Logan, *A First Course in the Finite Element Method*, fourth edition, Thomson Learning, 2007