Resampling Techniques and their Application

-Class 4-

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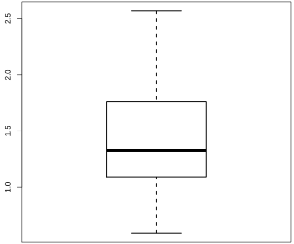
Motivation and Examples-II

The diameter of cork of a Champagne bottle is supposed to be 1.5 cm. If the cork is either too large or too small, it will not fit in the bottle. The manufacturer measures the diameter in a random sample of **n=36** bottles and obtains:

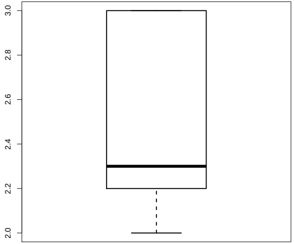



```
x=c(
0.59, 1.23, 1.00, 0.84, 0.88, 1.71,
1.81, 1.84, 2.03, 1.39, 1.30, 1.31,
1.96, 1.33, 2.57, 1.19, 1.01, 2.06,
1.32, 1.55, 1.28, 0.93, 1.63, 1.24,
1.83, 1.81, 0.94, 1.46, 1.25, 1.56,
0.61, 0.83, 1.17, 2.24, 1.68, 1.51)
```

- Data Analysis: Confidence interval and t-test

Motivation and Examples-III

A researcher measures the reaction time of $n = 10$ mice to signal pain when a stitch is applied to their tail.




```
x=c(
2.4, 3.0, 3.0, 2.2, 2.2,
2.2, 2.2, 2.8, 2.0, 3.0)
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Example

The diameter of cork of a Champagne bottle is supposed to be 1.5 cm. If the cork is either too large or too small, it will not fit in the bottle. The manufacturer measures the diameter in a random sample of n=36 bottles. Is there evidence at 1% level that the true mean diameter has moved away from the target?

- Hypothesis: $H_0 : \mu = 1.5$ vs. $H_1 : \mu \neq 1.5$

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mx <- mean(x)
vx <- var(x)
T <- sqrt(36)*(mx-1.5)/sqrt(vx)
p <- 2*min(pt(T,35),1-pt(T,35))
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- Estimator: $\widehat{\mu} = 1.4136$

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- Quality of the estimator? Is the method valid? Is $p = 0.27$ a good estimate?

Example

A researcher measures the reaction time of $n = 10$ mice to signal pain when a stitch is applied to their tail. Compute a 95% confidence interval.

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2.2, 2.2, 2.8, 2.0, 3.0)
mx <- mean(x)
vx <- var(x)
crit <- qt(0.975,9)
lower <- mx-crit/sqrt(10)*sqrt(vx)
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The Resampling Work Flow

Data

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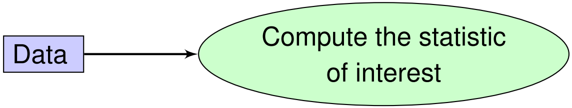

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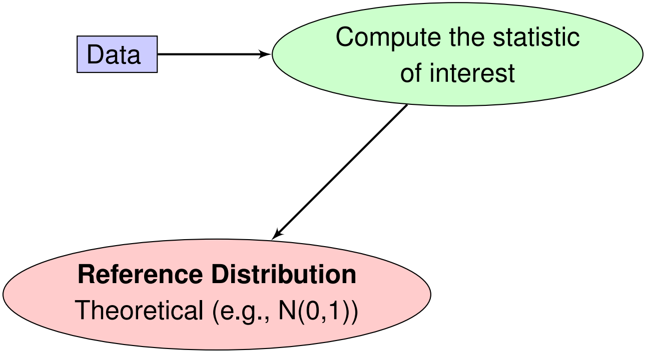

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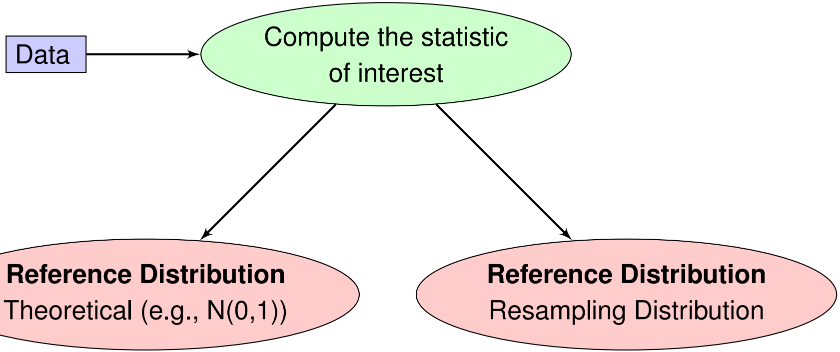

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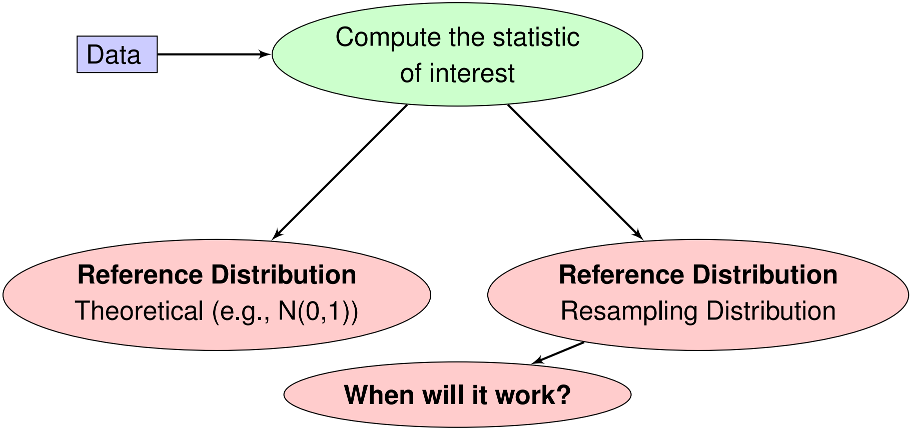

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  - Today, we will study few more

Resampling Distribution

- Randomly sample/generate from data $\mathbf{X} = (X_1, \ldots, X_n)$

$$X_1^*, \ldots, X_n^*$$

such that

$$T^* = \sqrt{n}\frac{\overline{X}_.^* - E(\overline{X}_.^* | \mathbf{X})}{\widehat{\sigma}^*}$$

has a $N(0, 1)$ distribution

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- Compute p-values, critical values etc from the **Resampling Distribution**

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- In R: sample(x,replace=TRUE)

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- Only for more than one group applicable. Why?

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- Note that $E(X_k^*) = 0$, $Var(X_k^*) = \widehat{\sigma}^2$ and $\mu_3(X_k^*) = \widehat{\mu}_3$. (Why?)

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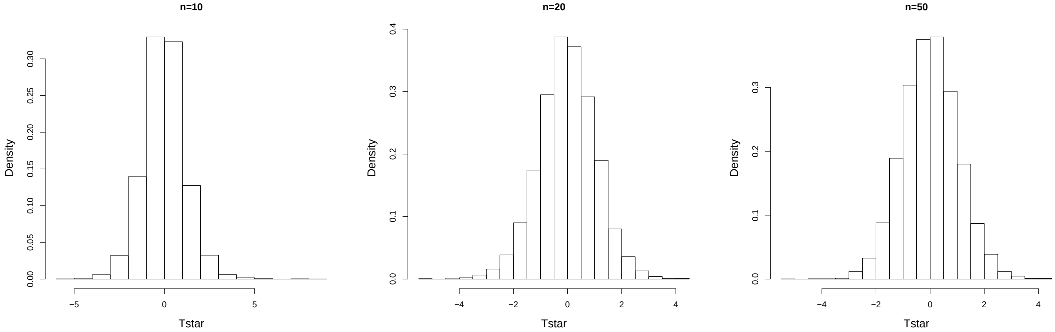
  5. Repeat the steps above a large number of times, e.g. $n_{boot} = 10,000$
  6. Compute the critical value $c^*(1-\alpha/2)$ or the p-value as

$$p = 2\min\left\{\frac{1}{n_{boot}}\sum_{\ell=1}^{n_{boot}} \mathcal{I}(T_\ell \leq T), \frac{1}{n_{boot}}\sum_{\ell=1}^{n_{boot}} \mathcal{I}(T_\ell \geq T)\right\}$$

$\mathcal{I}(x)$ is an indicator

Illustration Resampling Distribution



Compute the $(1-\alpha)$-quantile

What is $E(\overline{X}^*_{.}|\mathbf{X})$?

- The term $E(\overline{X}^*_{.}|\mathbf{X})$ needed to ensure that $E(\overline{X}^*_{.} - E(\overline{X}^*_{.}|\mathbf{X})) = 0$

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  - $E(\widehat{\sigma}^{2,*}|\mathbf{X}) =$
- For all of the different resampling variables

Which method to use?

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  - Generate a matrix of resampling variables (example next slide)

```
skew<-function(x){
n<-length(x)
mx<-mean(x)
sdx<-sd(x)
n/((n-1)*(n-2))*sum(((x-mx)/sdx)^3)}
```

```
myboot <- function(n,Distribution,nboot,nsim){
T=Tboot = Tbootsp =c()
#------------Test Statistic of the Sample------------#
if(Distribution == "Normal"){
x <- matrix(rnorm(n*nsim),ncol=nsim)}
if(Distribution == "Exp"){
x <- matrix(rexp(n*nsim)-1,ncol=nsim)}
mx <- colMeans(x)
vx <- (colSums(x^2)-n*mx^2)/(n-1)
T=sqrt(n)*(mx)/sqrt(vx)
```

```
#------------Simulate the Nonparametric Bootstrap-----#
B <- apply(matrix(1:n,ncol=nboot,nrow=n),2,sample,replace=TRUE)
for(i in 1:nsim){
xstar = matrix(x[,i][B],ncol=nboot,nrow=n)
mxstar = colMeans(xstar)
vxb = (colSums(xstar^2)-n*mxstar^2)/(n-1)
Tstar = sqrt(n)*(mxstar - mx[i])/sqrt(vxb)
p1 = mean(Tstar >= T[i])
p2 = mean(Tstar <= T[i])
Tboot[i] = (2*min(p1,p2)<0.05)
```

```
#-------------Simulate the skew parametric Bootstrap---#
mu3=skew(x[,i])
f=8/mu3^2
xstarsp=matrix(sign(mu3)*sqrt(vx)*((rchisq(n*nboot,f)-f)/sqrt(2*f)),ncol=nboot)
mxbootsp=colMeans(xstarsp)
vxbsp=(colSums(xstarsp^2) - n*mxbootsp^2)/(n-1)
Tstarnp = sqrt(n)*mxbootsp/sqrt(vxbsp)
p1sp = mean(Tstarnp >= T[i])
p2sp = mean(Tstarnp <= T[i])
Tbootsp[i] = (2*min(p1sp,p2sp)<0.05)}
```

```
result = data.frame(n=n,nsim=nsim,nboot=nboot,Dist=Distribution,
T=mean(abs(T)>qt(0.975,n-1)),
Tboot = mean(Tboot), Tbootsp = mean(Tbootsp))
result}
myboot(10,"Exp",1000,1000)
```

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  - Variance of an estimator (e.g. mean)
  - Variance of correlation coefficients
  - Overdispersion parameters, variance of their estimators,...

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- How to estimate $f(\theta)$?

Parameter Estimation - III

Estimation of θ	Properties
Maximum-Likelihood	F must be known
	Algorithm can be difficult
	Algorithm might not converge
	Large sample for distribution
Moment based	F can be unknown
	Computation usually feasible
	Usually exist (no converging issues)
	Small sample approximations
Resampling Methods	

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  - Basically, we simulate data from $\widehat{F}_n$

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  - Estimate the parameter of interest using the values of $\widehat{\theta}_1, \ldots, \widehat{\theta}_{n_{boot}}$

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```
X=c(2.4, 3.0, 3.0, 2.2, 2.2,
2.2, 2.2, 2.8, 2.0, 3.0)
n <- 10
nboot <-10000
B<- apply(matrix(1:n,
ncol=nboot,nrow=n),
2,sample,replace=TRUE)
xstar <- matrix(X[B],
ncol=nboot,nrow=n)
mxstar <- colMeans(xstar)
tauhat2 <- var(mxstar)
```

Project: How Good is the Estimator?

- Use computer simulations to assess the quality of the estimator $\widehat{\tau}^2$
- Compare with $\widehat{\tau}^2_{emp} = \widehat{\sigma}^2 / n$
- Compare the bias and MSE of the estimators
  - Generate n_{sim} random samples from different distributions
  - Compute the estimator upon the sample and safe the value in $\widehat{\theta}_s$
  - Assess the bias and MSE

$$Bias = \frac{1}{n_{sim}} \sum_{s=1}^{n_{sim}} (\widehat{\theta}_s - \theta) \text{ and } MSE = \frac{1}{n_{sim}} \sum_{s=1}^{n_{sim}} (\widehat{\theta}_s - \theta)^2$$

Project

- Task: Estimate the variance of **correlation coefficients** using resampling strategies
- How to get an idea about the true variance?
- Is bootstrap a good way?