Resampling Techniques and their Application

-Class 2-

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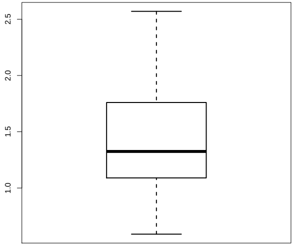
Motivation and Examples-II

The diameter of cork of a Champagne bottle is supposed to be 1.5 cm. If the cork is either too large or too small, it will not fit in the bottle. The manufacturer measures the diameter in a random sample of **n=36** bottles and obtains:

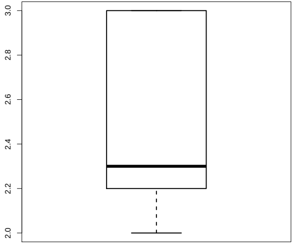



```
x=c(
0.59, 1.23, 1.00, 0.84, 0.88, 1.71,
1.81, 1.84, 2.03, 1.39, 1.30, 1.31,
1.96, 1.33, 2.57, 1.19, 1.01, 2.06,
1.32, 1.55, 1.28, 0.93, 1.63, 1.24,
1.83, 1.81, 0.94, 1.46, 1.25, 1.56,
0.61, 0.83, 1.17, 2.24, 1.68, 1.51)
```

- Data Analysis: Confidence interval and t-test

Motivation and Examples-III

A researcher measures the reaction time of $n = 10$ mice to signal pain when a stitch is applied to their tail.




```
x=c(
2.4, 3.0, 3.0, 2.2, 2.2,
2.2, 2.2, 2.8, 2.0, 3.0)
```

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Example

The diameter of cork of a Champagne bottle is supposed to be 1.5 cm. If the cork is either too large or too small, it will not fit in the bottle. The manufacturer measures the diameter in a random sample of n=36 bottles. Is there evidence at 1% level that the true mean diameter has moved away from the target?

- Hypothesis: $H_0 : \mu = 1.5$ vs. $H_1 : \mu \neq 1.5$

```
mx <- mean(x)
vx <- var(x)
T <- sqrt(36)*(mx-1.5)/sqrt(vx)
p <- 2*min(pt(T,35),1-pt(T,35))
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- Estimator: $\widehat{\mu} = 1.4136$

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- p-value: 2 times the area to the **left** of T under the t curve with 35 df. (Here, p=0.27)
- Quality of the estimator? Is the method valid? Is $p = 0.27$ a good estimate?

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A researcher measures the reaction time of $n = 10$ mice to signal pain when a stitch is applied to their tail. Compute a 95% confidence interval.

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2.2, 2.2, 2.8, 2.0, 3.0)
mx <- mean(x)
vx <- var(x)
crit <- qt(0.975,9)
lower <- mx-crit/sqrt(10)*sqrt(vx)
upper <- mx+crit/sqrt(10)*sqrt(vx)
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- Estimator: $\widehat{\mu} = 2.5$

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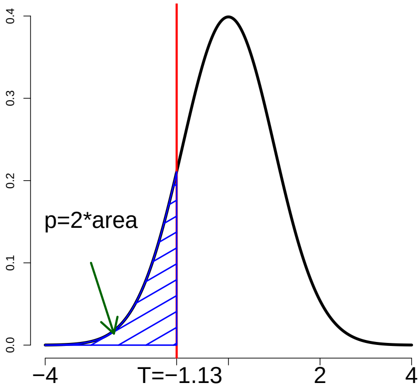

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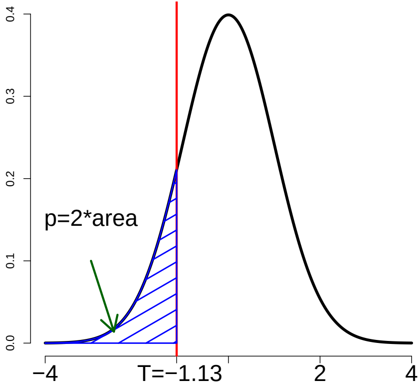

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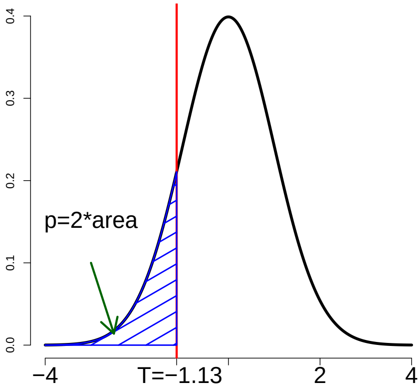

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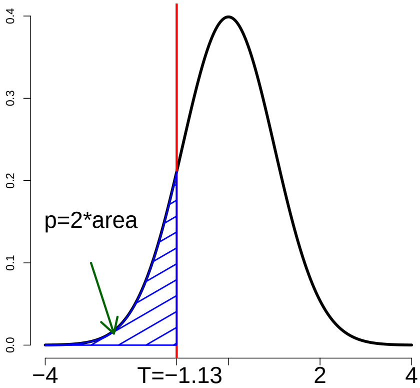

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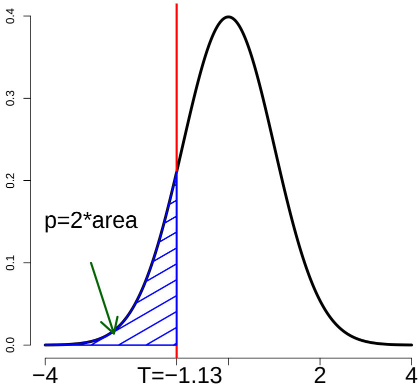
- **What happens if data come from a different distribution?**



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- Can we verify/visualize the distribution from the boxplot?
  ```

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  4. Repeat the above a large number of times
  5. Estimate the type-1 error by averaging the indicators

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- $E(X_k) = 0$ and $Var(X_k) = \sigma^2$
  - $Y_k \sim Exp(\lambda)$: $E(Y_k) = \lambda$ and $Var(Y_k) = \lambda^2$

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  - $Y_k \sim LN(0, 1)$: $E(Y_k) = exp(1/2)$ and $Var(Y_k) = exp(2) - exp(1)$

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  - $Y_k \sim LN(0,1)$: $E(Y_k) = exp(1/2)$ and $Var(Y_k) = exp(2) - exp(1)$
  - ...

```
mysimulation <-function(Needed Parameters){
```

```
Data Generation
```

```
Compute the Test Statistic
Hypothesis Rejected? (0/1)
```

```
Estimate Type-1 error rate
output table}
```

```
mysimulation(...)
```

```
mysimulation <-function(n,s2,nsim){
crit= qt(0.975, n-1) # critical value at 5\% level
ttest <-c(); set.seed(1)
for(i in 1:nsim){ #Begin Simulation Loop

x <- rnorm(n)*sqrt(s2) #generate from normal dist.
#mu=0 and has variance s2

mx <-mean(x); sdx <- sd(x); T<-sqrt(n)*mx/sdx #compute test statistic
ttest[i] <- (abs(T)>=crit) #Reject yes/no
} #End Simulation Loop

result= data.frame(n=n, alpha=0.05, sigma2=s2, tTest=mean(ttest))
result}
mysimulation(10,1,10000)
```

```r
mysimulation <-function(n,s2,nsim){
crit= qt(0.975, n-1) # critical value at 5\% level
ttest <-c(); set.seed(1)
for(i in 1:nsim){ #Begin Simulation Loop
```

```r
x <- (rexp(n)-1)*sqrt(s2) #generate data from exponential
#(hypothesis is TRUE! has variance=s2)
```

```r
mx <-mean(x); sdx <- sd(x); T<-sqrt(n)*mx/sdx #compute test statistic
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result= data.frame(n=n, alpha=0.05, sigma2=s2, tTest=mean(ttest))
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```
mysimulation <-function(n,s2,nsim, Distribution){ #use Distribution as argument
```

```
crit= qt(0.975, n-1) # critical value at 5\% level
ttest <-c()
set.seed(1)
for(i in 1:nsim){ #Begin Simulation Loop
```

```
if(Distribution=="Exp"){#generate data from exponential
x <- (rexp(n)-1)*sqrt(s2) }
if(Distribution=="Normal"){#generate data from normal
x <- rnorm(n)*sqrt(s2) }
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mx <-mean(x); sdx <- sd(x); T<-sqrt(n)*mx/sdx #compute test statistic
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result= data.frame(n=n, alpha=0.05, Dist=Distribution,sigma2=s2, tTest=mean(ttest))
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- Can we take data characteristics into account? ($\Rightarrow$ Resampling)

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    - Variance formula

$$\widehat{\sigma}^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X}_.)^2 = \frac{1}{n-1}\left(\sum_{k=1}^{n} X_k^2 - n\overline{X}_.^2\right)$$

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    - R functions *colMeans(x)*, *colSums(x)*
    - These functions are very fast
    - Variance formula

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X}_.)^2 = \frac{1}{n-1}(\sum_{k=1}^{n} X_k^2 - n\overline{X}_.^2)$$

    - Try to avoid ”for” loops

Efficient Implementation

- Numerical implementation is important
- Use **matrix techniques** to generate the data
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    - R functions *colMeans(x)*, *colSums(x)*
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- Try to avoid ”for” loops
- Generate a matrix of variables (example next slide)

```
mysimulation <-function(n,s2,Distribution,nsim){
crit= qt(0.975, n-1) # critical value at 5\% level
set.seed(1)
```

```
if(Distribution=="Normal"){
x <- matrix(rnorm(n=n*nsim)*sqrt(s2),ncol=nsim)}
if(Distribution=="Exp"){#(Hypothesis is TRUE!)
x <- matrix((rexp(n=n*nsim)-1)*sqrt(s2),ncol=nsim)}
```

```
mx <-colMeans(x)
vx <- (colSums(x^2)-n*mx^2)/(n-1)
T <- sqrt(n)*mx/sqrt(vx)
ttest <- (abs(T)>=crit)
```

```
result= data.frame(n=n,Dist=Distribution,sigma2=s2,ttest=mean(ttest))
result}
mysimulation(10,1,"Normal",10000)
```

Simulation of Point Estimators

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- Which one is better?

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- Which one is better?
- Assess a simulation to compute a) their bias and b) their standard error

Simulation

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  - Generate data (know parameter θ)

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Simulation

- Structure of the Program
  - Generate data (know parameter θ)
  - Begin Simulation Loop
  - Compute estimator and safe the value of $\widehat{\theta}_\ell, \ \ell = 1, \ldots, n_{sim}$
  - End of simulation loop
  - Assess bias and mean square error as

$$bias = \frac{1}{n_{sim}} \sum_{\ell=1}^{n_{nsim}} (\widehat{\theta}_\ell - \theta) \qquad MSE = \frac{1}{n_{sim}} \sum_{\ell=1}^{n_{sim}} (\widehat{\theta}_\ell - \theta)^2$$

```
mysimulation <-function(Needed Parameters){

Data Generation (know true value)

Compute the Point Estimator and safe the value

Compute the bias and MSE
output table}

mysimulation(...)
```

```r
mysimulation <-function(n,s2,Distribution,nsim){

if(Distribution=="Normal"){
x <- matrix(rnorm(n=n*nsim)*sqrt(s2),ncol=nsim)}
if(Distribution=="Exp"){
x <- matrix((rexp(nsim*n)-1)*sqrt(s2),ncol=nsim)} #(Parameter is known)

mx <-colMeans(x)
v1 <- (colSums(x^2)-n*mx^2)/(n-1)
v2<- (colSums(x^2)-n*mx^2)/n

result= data.frame(n=n, sigma2=s2, Distribution= Distribution,
bias.v1=mean(v1-s2), MSE.v1=mean((v1-s2)^2), bias.v2=mean(v2-s2),
MSE.v2=mean((v2-s2)^2))
result}
mysimulation(10,1,"Normal",10000)
```

Project

1. Write a simulation program to assess the quality of 95% confidence interval for mean

2. Let $X_1, \dots, X_n$ have $E(X_k) = \mu$. We want to estimate $\theta = \mu^2$. Which of the estimators

$$\widehat{\theta}_1 = \overline{X}_{\cdot}^2 \qquad \text{or} \qquad \widehat{\theta}_2 = \frac{1}{n(n-1)} \sum_{i \neq j} X_i \cdot X_j$$

   would you recommend?