Resampling Techniques and their Application

-Class 13-

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Example: Yeast Study

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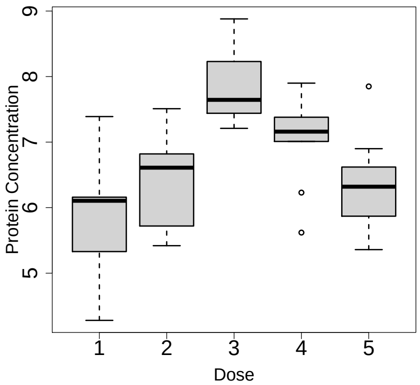

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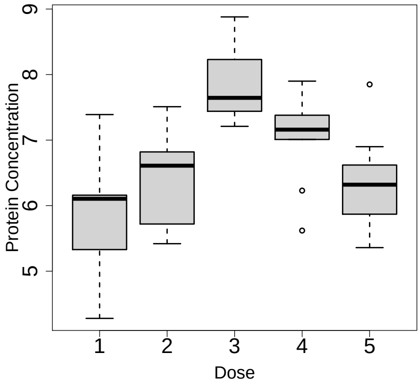

ID	M1	M2	M3	M4	M5
1	6.16	6.04	7.21	7.23	6.22
2	4.28	5.42	7.44	6.23	6.03
3	5.26	5.72	7.40	7.02	5.87
4	6.11	6.65	7.44	7.09	6.62
5	6.15	6.67	7.79	5.62	5.80
6	5.33	7.50	8.23	7.38	6.42
7	5.47	6.82	7.94	7.01	6.57
8	6.10	6.57	8.73	7.90	6.90
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- $\boldsymbol{C}$: Contrast matrix

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```
x=matrix(c(6.16, 6.04, 7.21, 7.23, 6.22,
4.28, 5.42, 7.44, 6.23, 6.03,
5.26, 5.72, 7.40, 7.02, 5.87,
6.11, 6.65, 7.44, 7.09, 6.62,
6.15, 6.67, 7.79, 5.62, 5.80,
5.33, 7.50, 8.23, 7.38, 6.42,
5.47, 6.82, 7.94, 7.01, 6.57,
6.10, 6.57, 8.73, 7.90, 6.90,
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Xbar=colMeans(x)

Vhat=var(x)
```

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- So, contrast of interest is the *GrandMean* contrast

$$\mathbf{C} = \begin{pmatrix} \frac{d-1}{d} & -\frac{1}{d} & \cdots & -\frac{1}{d} \\ -\frac{1}{d} & \frac{d-1}{d} & \cdots & -\frac{1}{d} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{d} & -\frac{1}{d} & \cdots & \frac{d-1}{d} \end{pmatrix} = \mathbf{I} - \frac{1}{d}\mathbf{J}$$

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- $\mathbf{I} = diag(d)$ and $\mathbf{J}$: $d \times d$ matrix of 1's
- Note that $\mathbf{c}_\ell' \boldsymbol{\mu} = \mu_\ell - \frac{1}{d} \sum_{j=1}^{d} \mu_j$
- $\boldsymbol{C}$ is also known as centering matrix

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```
library(multcomp)
library(MASS)
C=contrMat(n=rep(10,5),"GrandMean")
CX=C%*%Xbar
CVhat = C%*%Vhat%*%t(C)
W=n*t(CX)%*%ginv(CVhat)%*%CX
pvalue= 1-pchisq(W, d-1)
```

Wald-Type Test: Properties

Advantages

-
-
-
-

Disadvantages

-
-
-
-

ANOVA-Type Statistic

- $\widehat{\boldsymbol{V}}$ causes issues in the WTS, let us remove it

$$
\begin{aligned}
A_1 &= n(\boldsymbol{C}\overline{\mathbf{X}}.)' \left[\boldsymbol{C}\boldsymbol{C}'\right]^+ \boldsymbol{C}\overline{\mathbf{X}}. \\
&= n\overline{\mathbf{X}}'_. \boldsymbol{C}' \left[\boldsymbol{C}\boldsymbol{C}'\right]^+ \boldsymbol{C}\overline{\mathbf{X}}. \\
&= n\overline{\mathbf{X}}'_. \boldsymbol{T}\overline{\mathbf{X}}.
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- The final test is given by

$$
\begin{aligned}
A &= n\overline{\mathbf{X}}_\cdot' \boldsymbol{T}\overline{\mathbf{X}}_\cdot / Trace(\boldsymbol{T}\widehat{\boldsymbol{V}}), \\
f &= \frac{Trace(\boldsymbol{T}\widehat{\boldsymbol{V}})^2}{Trace(\boldsymbol{T}\widehat{\boldsymbol{V}}\boldsymbol{T}\widehat{\boldsymbol{V}})}
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$$

- Under H_0, ATS has a $F(f, \infty)$ distribution

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```
TT <- t(C)%*%ginv(C%*%t(C))%*%C
TrTV <-sum(c(diag(TT%*%Vhat)))
A <- n*t(Xbar)%*%TT%*%Xbar/TrTV

TVTV<-TT%*%Vhat%*%TT%*%Vhat
TrTVTV <-sum(c(diag(TVTV)))
f <- TrTV^2/TrTVTV

pvalue <- 1-pf(A,f,10^10)

#10^10=infty, arbitrary high nr#
```

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R Package MANOVA.RM

- R-package *MANOVA.RM*

```
data=data.frame(x=c(x),ID=rep(1:10,5),
dose=sort(rep(1:5,10)))
library(MANOVA.RM)

fit<-RM(x~dose, subject="ID", data=data)

summary(fit)
```

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  - Overall result and multiple comparisons may be incompatible

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  - Idea: For each partial hypothesis one t-Test
  - Compare the t-values with one critical value (Gabriel, 1969)

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  - $\widehat{\boldsymbol{V}}$: empirical covariance matrix
- Variance of a contrast

$$\sigma_\ell^2 = Var(\boldsymbol{c}_\ell' \overline{\mathbf{X}}_{\cdot}) = \boldsymbol{c}_\ell' \boldsymbol{V} \boldsymbol{c}_\ell, \quad \widehat{\sigma}_\ell^2 = \boldsymbol{c}_\ell' \widehat{\boldsymbol{V}} \boldsymbol{c}_\ell$$

Multiple Contrast Tests

- Estimators
  - $\overline{\mathbf{X}}_{\cdot} = (\overline{X}_{1\cdot}, \ldots, \overline{X}_{a\cdot})'$ (vector of means)
  - $\widehat{\boldsymbol{V}}$: empirical covariance matrix
- Variance of a contrast

$$\sigma_\ell^2 = Var(\boldsymbol{c}_\ell' \overline{\mathbf{X}}_{\cdot}) = \boldsymbol{c}_\ell' \boldsymbol{V} \boldsymbol{c}_\ell, \quad \widehat{\sigma}_\ell^2 = \boldsymbol{c}_\ell' \widehat{\boldsymbol{V}} \boldsymbol{c}_\ell$$

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Parametric MCTP

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MCTP: Properties

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Disadvantages

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