Resampling Techniques and their Application

-Class 9-

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  - Less costs (potentially)
- Measurements from the same subject **are not necessarily independent**

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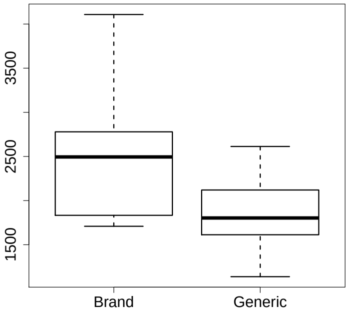

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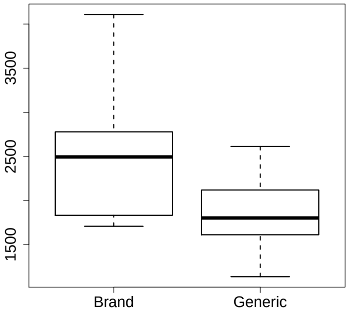

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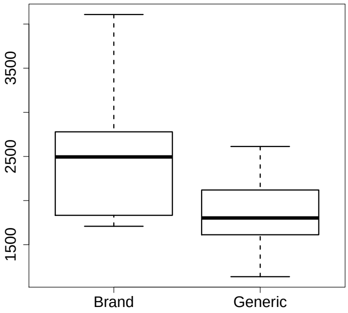

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- Aim: $H_0 : \mu_1 = \mu_2$ and confidence interval

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    - Measures the degree of the (linear) relationship between X_k and Y_k
    - On average, $(\underbrace{(X_k - \mu_1)}_{\lesseqgtr 0}\underbrace{(Y_k - \mu_2)}_{\lesseqgtr 0}) \lesseqgtr 0$

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  - Reject H_0, if $|T| \geq t_{1-\alpha/2}(n-1)$

Example Evaluation

```
brand=c(4108,2526,2779,3852,1833, 2463,2059,1709,1829,2594)
generic=c(1755,1138,1613,2254,1310,2120,1851,1878,1682,2613)
plot(brand,generic,pch=19,cex=1.3)
n=length(brand)
x=cbind(brand,generic)
var(x)

diff=brand-generic
mD=mean(diff)
vd=var(diff)

T=sqrt(n)*mD/sqrt(vd)
pvalue=2*min(pt(T,n-1), 1-pt(T,n-1))

t.test(brand,generic,paired=TRUE)
```

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- But how? Differences?

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    - Resampling from all data and thus ignoring dependencies

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- Note that centering is not necessary (why?)

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- This method is equivalent to....

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Resampling Using Original Data

- **Data $\mathbf{X} = (X_{11}, \ldots, X_{2n})$ (fixed values)**
- **Permutation** randomly permute the $2n$ observations X^*_{ik} in $\mathbf{X}$
- Example $\mathbf{X} = (1, 2, 3, 4, 5) \Rightarrow$
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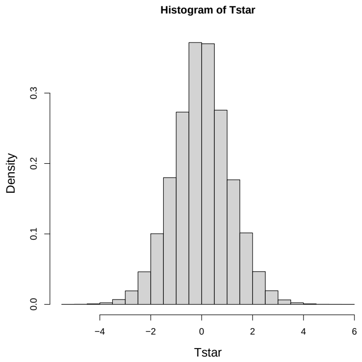
- Reference: Konietschke and Pauly (2015)

Illustration

```
x=brand
y=generic
plot(x,y,pch=19,cex=1.3)
n=10
d=x-y
T=sqrt(n)*mean(d)/sd(d)
pvalue=2*min(pt(T,n-1),1-pt(T,n-1))
pvalue
Tstar=c()
xy=c(x,y)
for(i in 1:100000){
xstar=sample(xy) #permutation overall
dstar=xstar[1:n]-xstar[(n+1):(2*n)]
Tstar[i]= sqrt(n)*mean(dstar)/sd(dstar)
}
pstar= 2*min(mean(Tstar<=T),mean(Tstar>=T))
pstar
```



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\mathbf{X}_k &= (F_1^{-1}(\Phi(Z_{1k})), F_2^{-1}(\Phi(Z_{2k}))), \; \mathbf{Z}_k \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \text{ (quantile method)}, \; \text{or} \\
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- $\Phi(x)$: CDF of $N(0, 1)$

Project

- In a paired data setting, permuting data overall and thus ignoring the dependency is somewhat counter intuitive. Verify the validity of the method for the paired t-test in a simulation study at 5% level of significance. Use $n_{sim} = 10,000$ and $n_{perm} = 10,000$ permutation runs. Generate bivariate normal data with variance $\sigma_i^2 = 1$ and different covariances $\sigma \in \{-0.95, -0.5, 0, 0.5, 0.95\}$ and sample sizes $n \in 10, 20$.