Resampling Techniques and their Application

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CHARITÉ
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Organization

- Instructor: Frank Konietschke (Frank.Konietschke@charite.de)
- Assistant: Kerstin Rubarth (Kerstin.Rubarth@charite.de)
- Materials: Available on blackboard (`https://lms.fu-berlin.de`)
- Syllabus: Course outline

Motivation and Examples

An experiment is conducted to study the side effects of a pain reliever on arthritis patients. Out of 480 patients, 60 patients suffered adverse symptom.

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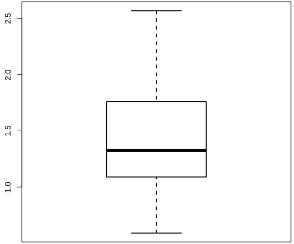
- Quality of the confidence interval?
- We will learn quality criteria and how to investigate them ($\Rightarrow$ **Simulations**)

Motivation and Examples-II

The diameter of cork of a Champagne bottle is supposed to be 1.5 cm. If the cork is either too large or too small, it will not fit in the bottle. The manufacturer measures the diameter in a random sample of **n=36** bottles and obtains:

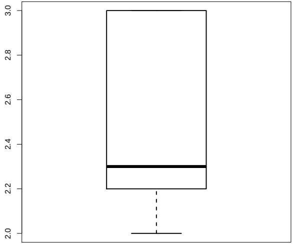



```
x=c(
0.59, 1.23, 1.00, 0.84, 0.88, 1.71,
1.81, 1.84, 2.03, 1.39, 1.30, 1.31,
1.96, 1.33, 2.57, 1.19, 1.01, 2.06,
1.32, 1.55, 1.28, 0.93, 1.63, 1.24,
1.83, 1.81, 0.94, 1.46, 1.25, 1.56,
0.61, 0.83, 1.17, 2.24, 1.68, 1.51)
```

- Study aim: **Estimation** and **Testing**. Quality of the method? Can we do better using resampling?

Motivation and Examples-III

A researcher measures the reaction time of $n = 10$ mice to signal pain when a stitch is applied to their tail.

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x=c(
2.4, 3.0, 3.0, 2.2, 2.2,
2.2, 2.2, 2.8, 2.0, 3.0)
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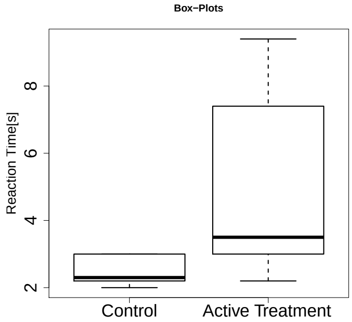

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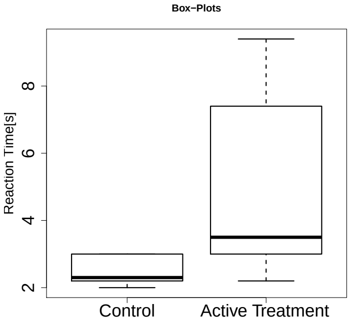

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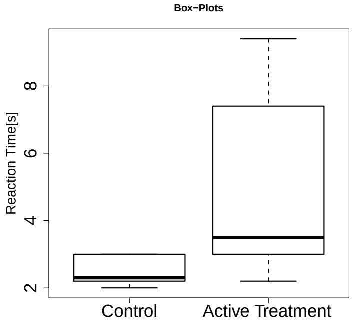

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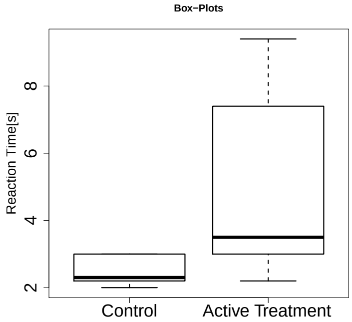

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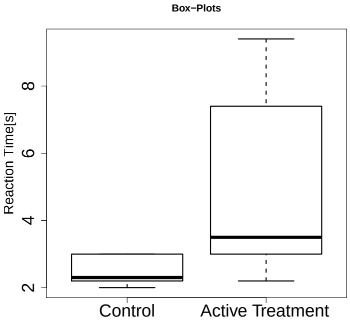

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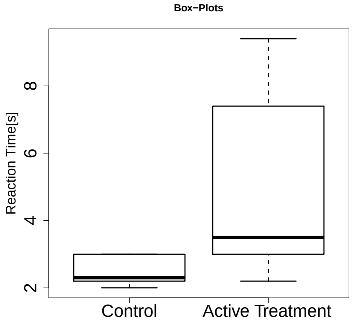

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- Quality of available methods? Can we improve using resampling?

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- We will explore ***Resampling methods*** as modern ways to counter these issues.

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  - Learning simulations will therefore play a major role in this class
  - Understanding of statistical testing theory, p-value etc is fundamental and we therefore refresh today

Review: Basic Idea of Tests of Significance

Example: Every week, two teams ("In-laws" and the "Outlaws") get together and play a football game. They decide who receives the kickoff by a flip of a **fair coin** provided by the In-laws (heads = outlaws receive, tails = in-laws receive).

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    2. For a fair coin, this outcome is so unlikely (extreme) that we can conclude that the coin is not fair.

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  - **Alternative Hypothesis** (H_1):

$$H_1 : p < 50\%$$

    The chance of heads $<$ 50%. (This is the statement we’d like to prove. It says chance variation is not enough to explain the outcome).

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- If such a difference is very unlikely to occur if null hypothesis is true, then we reject the null hypothesis.

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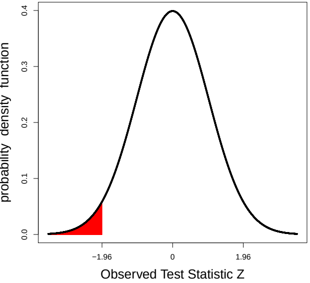
- So, the observed outcome is 3 SD below the expected outcome.
- How likely is the outcome? ($\Rightarrow$ Distribution of Z)

Computation of P-Value

$H_0 : p = p_0$ versus
$H_1 : p < p_0$



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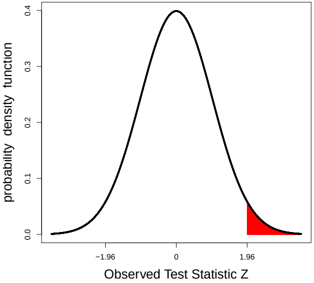
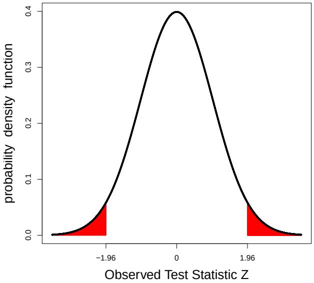


$H_1 : p \neq p_0$



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  - What does this tell us about H_0?
- As our p-value is so small, we conclude that we have enough evidence to reject the null.
- Does this prove that H_0 is false?

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  - Calculate the p-value.
  - Use the p-value and the given significance level to draw the conclusion:
    If p-value $\leq \alpha$, reject H_0. If p-value $> \alpha$, do not reject H_0.

Example

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- Standard error: $SE(\widehat{p}) = \sqrt{\frac{10*90}{480}}$ (under H_0 in %)

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- Standard error: $SE(\widehat{p}) = \sqrt{\frac{10*90}{480}}$ (under H_0 in %)
- Test statistic: $Z = \frac{12.5-10}{\sqrt{\frac{10*90}{480}}} = 1.83$

Example

An experiment is conducted to study the side effects of pain reliever arthritis patients. Out of 480 patients, 60 suffered from adverse events. Is there evidence at 5% level that the proportion of all Bioxx users suffering from adverse symptoms is more than 10%? What if the level was 1%?

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- p-value: : Area to the right of Z under the standard normal curve, Here, $p = 0.034$

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- These errors are defined conditional on the true status (H_0 or H_1).

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- We cannot eliminate the possibility of errors because our decision is based on a sample, and not the whole population.
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- After setting α to be small, the studies are designed in such a way that type II error is minimized.
- What can we do to have both type I and type II error rates acceptably small?

Probabilities of Type I and Type II Errors & P-value - II

- Interpretation of P-value: P-value is **NOT** the probability that H_0 is true. H_0 is either true or not true. It does not vary from sample to sample. P-value tells how likely it is to get the observed sample (or something more extreme) if H_0 is true (Note: H_0 is held fixed). Smaller the P-value, stronger the evidence against H_0.

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- Reject H_0, if $|T| \geq t_{n-1}(1-\alpha/2)$. Or compute the p-value

Example

The diameter of cork of a Champagne bottle is supposed to be 1.5 cm. If the cork is either too large or too small, it will not fit in the bottle. The manufacturer measures the diameter in a random sample of n=36 bottles. Is there evidence at 1% level that the true mean diameter has moved away from the target?

- Hypothesis: $H_0 : \mu = 1.5$ vs. $H_1 : \mu \neq 1.5$

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mx <- mean(x)
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- Is this method of high quality?

Example

A researcher measures the reaction time of $n = 10$ mice to signal pain when a stitch is applied to their tail. Compute a 95% confidence interval.

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2.2, 2.2, 2.8, 2.0, 3.0)
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vx <- var(x)
crit <- qt(0.975,9)
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- Next class: **Simulation of type-1 error**