1 Wave Equation

This **PDE** (Partial Differential Equation) is called the **Wave Equation**:

$$u_{tt} = c^2 u_{xx}$$

where c is a constant called "the speed of the wave".

The **unknown** function $u(x,t)$ can be used to describe the **height** of a wave relative to the equilibrium $u = 0$, over a region with x being the position variable. The function $u(x,t)$ depends on t, which indicates that the height of the wave would change with time.

We aim to solve $u(x,t)$ in the **finite domain**:

$$0 < x < l$$

where l is a constant being the length of the domain.

Suppose we are given the following **initial conditions**:

$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x)$$

where $\phi(x)$ describes the **shape** of the wave at time $t = 0$, and $\psi(x)$ describes the **verticle velocity** of the wave at time $t = 0$.

Because we are solving the equation on a finite domain, we must also set the **boundary conditions**. Let's first use the easiest type of boundary conditions:

$$u(0,t) = 0 \qquad u(l,t) = 0$$

These boundary conditions indicates that the wave is **fixed** at both ends of an interval (with $u = 0$). Imagine a guitar string with length l which is fixed at both ends, but are free to vibrate (and have a non-zero **height**) in the region between the 2 endpoints, after the string is struck at $t = 0$.

This fully sets up the problem that we are interested in. We will present a technique to solve this problem, called **separation of variables**. The same technique will be used in other problems, where we may change the **PDE** in question, or we may change the **boundary conditions**.

Separation of Variables
We will **guess** that the solution $u(x,t)$ will be the following form:

$$u(x,t) = X(x) \cdot T(t)$$

where $X(x)$ is a function **only** dependent on x,
and $T(t)$ is a function **only** dependent on t,
and the solution $u(x,t)$ will be an expression involving x **multiplied** to an expression involving t. i.e. The variables x and t are **separated**.

For notational convenience, we may supress the dependence, and write:

$$u(x,t) = X \cdot T$$

as the letter X indicates a function depending on x only, similarly for T.
Next, we plug in our guess into the PDE:

$$u_{tt} = c^2 u_{xx}$$

For example, let's calculate u_t. Our guess is $u = X(x) \cdot T(t)$, since $X = X(x)$ only involves x, it is regarded as a constant when we take the derivative against t:

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\Big(X(x) \cdot T(t)\Big) = X(x)\frac{\partial}{\partial t}T(t) = X(x) \cdot T'(t) = X \cdot T'$$

where T' implicitly means derivative against t since T is only a function of t.
Similarly:

$$u_{tt} = X \cdot T''$$
$$u_x = X' \cdot T$$
$$u_{xx} = X'' \cdot T$$

where X' implicitly means derivative against x since X is only a function of x.

Thus the PDE becomes:

$$X \cdot T'' = c^2 X'' \cdot T$$

Next, we would like to **separate** the variables onto 2 sides of the equation. By **convention**, we choose to put all terms involving t on the **left** side of the equation, and all terms involving x on the **right** side of the equation. We also put any constant on the **left** side as well.

Thus, after dividing the term c^2XT on both sides, the equation becomes:

$$\frac{T''}{c^2T} = \frac{X''}{X} \qquad \Big(= F \Big)$$

Let's call the resulting expression F. By separating the variables, we can show that the quantity F must be a **constant**, not dependent on x nor t:
If we take $\dfrac{\partial F}{\partial x}$, we can use the expression on the left:

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\Big(\frac{T''}{c^2T}\Big) = 0$$

since this expression does not involve x, so it is regarded as a constant.
Similarly, if we take $\dfrac{\partial F}{\partial t}$, we can use the expression on the right:

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial t}\Big(\frac{X''}{X}\Big) = 0$$

since this expression does not involve t, so it is regarded as a constant.

Since both derivative are 0, F must not depend on x nor t, so it must be a constant, called the **separation constant**.
By **convention**, we set this constant to be $-\lambda$, where $\lambda > 0$:

$$\frac{T''}{c^2T} = \frac{X''}{X} = -\lambda$$

We will later show why the separation constant **must** be negative, hence we choose to write it as $-\lambda$ with $\lambda > 0$. Let's assume this is the case for now.

We may rewrite the above equation into 2 equations:

$$\begin{aligned} \frac{T''}{c^2T} &= -\lambda & \qquad \frac{X''}{X} &= -\lambda \\ T'' &= -\lambda c^2 T & \qquad X'' &= -\lambda X \\ T'' + \lambda c^2 T &= 0 & \qquad X'' + \lambda X &= 0 \end{aligned}$$

These are **second order linear equations**, which can be easily solved.

Since we have chosen $\lambda > 0$, we see the solutions must be sin and cos:

$$T(t) = A\cos(\sqrt{\lambda}\,ct) + B\sin(\sqrt{\lambda}\,ct) \qquad X(x) = C\cos(\sqrt{\lambda}\,x) + D\sin(\sqrt{\lambda}\,x)$$

Now, we plug in the **boundary conditions**.
Recall that our guess was $u(x,t) = X(x)\cdot T(t)$.

$$u(0,t) = 0 \qquad \Rightarrow \qquad X(0)T(t) = 0 \qquad \Rightarrow \qquad X(0) = 0$$

as we solved $T(t) \neq 0$. So

$$X(0) = C\cos(\sqrt{\lambda}\,0) + D\sin(\sqrt{\lambda}\,0) = C$$

Thus, we conclude $C = 0$, from the boundary condition at $x = 0$.

$$u(l,t) = 0 \qquad \Rightarrow \qquad X(l)T(t) = 0 \qquad \Rightarrow \qquad X(l) = 0$$

$$X(l) = D\sin(\sqrt{\lambda}\,l) = 0$$

Thus, either $D = 0$ or $\sin(\sqrt{\lambda}\,l) = 0$.

If $D = 0$, then $X(x) = 0$ for all x, which means $u(x,t) = 0$ for all (x,t).
This is in fact a solution, called the **trivial solution** of the PDE $u_{tt} = c^2 u_{xx}$.
We are never interested in the trivial solution.

Thus $\sin(\sqrt{\lambda}\,l) = 0$, which only happens if the angle is a integer multiple of π:

$$\sqrt{\lambda}\cdot l = n\pi$$
$$\sqrt{\lambda} = \frac{n\pi}{l}$$

where we may choose n to be any positive integer.
Thus for each n, we have a solution, which we label as $u = u_n$:

$$u_n(x,t) = X(x)T(t) = D_n \sin(\frac{n\pi x}{l})\Big(A_n\cos(\frac{n\pi ct}{l}) + B_n\sin(\frac{n\pi ct}{l})\Big)$$

We can ignore $n = 0$ as $\sin 0 = 0$ would give the trivial solution again.
We can also ignore D_n, as the D_nA_n and D_nB_n can be regarded as 2 new constants.

Since each value of n gives a solution to the PDE, the **general solution** must be a **linear combination** of the solutions u_n that we guessed:

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{l})\Big(A_n\cos(\frac{n\pi ct}{l}) + B_n\sin(\frac{n\pi ct}{l})\Big)$$

Now we apply the **initial conditions**.

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{l})\Big(A_n \cos(\frac{n\pi ct}{l}) + B_n \sin(\frac{n\pi ct}{l})\Big)$$

Let's first focus on $u(x,0) = \phi(x)$. Setting $t = 0$ above , we have

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})$$

This is a very interesting and important equation.
The series on the right is called the **Fourier (sine) Series** of $\phi(x)$.

Recall that for **Taylor Series**, we approximate a function $f(x)$ with an infinite polynomial:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{where} \qquad c_n = \frac{f^{(n)}(0)}{n!}$$

For **Fourier Series**, we approximate a function $f(x)$ with an infinite sum, but not with polynomials x^n, but with **sine** (and/or cosines), with different frequencies which increases with n:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}) \qquad \text{where } 0 < x < l$$

Similar to Taylor Series, for most functions we encounter that aren't too badly behaved, the approximation would get better and better, as we use more and more terms in the series.

We can actually compute the coefficients very easily, using the following:

$$\int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} = \begin{cases} 0 & \text{if } m \neq n \\ \frac{l}{2} & \text{if } m = n \end{cases}$$

For completeness, let's also state the corresponding fact involving cosine:

$$\int_0^l \cos\frac{n\pi x}{l} \cos\frac{m\pi x}{l} = \begin{cases} 0 & \text{if } m \neq n \\ \frac{l}{2} & \text{if } m = n\,(\neq 0) \end{cases} \qquad \int_{-l}^l \sin\frac{n\pi x}{l} \cos\frac{m\pi x}{l} = 0 \qquad \text{for any n, m}$$

These are called the **Orthogonal Relations for Fourier Series**.

To compute the coefficients in the Fourier Series, start with the series:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}) \qquad \text{where } 0 < x < l$$

Let me randomly choose an integer m.
(Think of m as a fixed number such as 3.)
I will compute the coefficient A_m, by **multiplying** $\sin(\frac{m\pi x}{l})$ on both sides of the equation above, and **integrate** over 0 to l.

$$\int_0^l f(x) \sin(\frac{m\pi x}{l}) = \sum_{n=1}^{\infty} A_n \int_0^l \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l})$$

The right side has an infinite sum of integrals, but fortunately, every single one of the integral would be equal to zero, **except one**, when the summation variable n is equal to m. For that particular term when $n = m$, the integral would be equal to $\frac{l}{2}$. Thus

$$\int_0^l f(x) \sin(\frac{m\pi x}{l}) = 0 + 0 + ... + A_m \frac{l}{2} + 0 + 0 + ...$$

$$A_m = \frac{2}{l} \int_0^l f(x) \sin(\frac{m\pi x}{l})$$

Thus, given $f(x)$, we can solve for each coefficient in the Fourier Series, A_m, by computing the above integral.

Now, coming back to our original PDE problem, the solution was

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{l}) \Big(A_n \cos(\frac{n\pi ct}{l}) + B_n \sin(\frac{n\pi ct}{l}) \Big)$$

which involves infinitely many unknown constants A_n and B_n.

From the initial condition $u(x,0) = \phi(x)$, we had,

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})$$

Using the formula above, we have

$$A_n = \frac{2}{l} \int_0^l \phi(x) \sin(\frac{n\pi x}{l})$$

To apply the other initial condition $u_t(x,0) = \psi(x)$, we need $u_t(x,t)$ first:

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{l})\Big(- A_n \sin(\frac{n\pi ct}{l}) + B_n \cos(\frac{n\pi ct}{l})\Big)\Big(\frac{n\pi c}{l}\Big)$$

where we have gotten an extra constant term from chain rule.
Setting $t = 0$, using $u_t(x,0) = \psi(x)$,

$$\psi(x) = \sum_{n=1}^{\infty} B_n \Big(\frac{n\pi c}{l}\Big) \sin(\frac{n\pi x}{l})$$

Once again, to solve B_n, we multiply $\sin(\frac{m\pi x}{l})$ and integrate:

$$\int_0^l \psi(x) \sin(\frac{m\pi x}{l}) = \sum_{n=1}^{\infty} B_n \Big(\frac{n\pi c}{l}\Big) \int_0^l \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l})$$

$$\int_0^l \psi(x) \sin(\frac{m\pi x}{l}) = B_m \Big(\frac{m\pi c}{l}\Big) \frac{l}{2}$$

(We may now switch the letter m back to n, so it looks nicer.)

$$B_n = \frac{2}{n\pi c} \int_0^l \psi(x) \sin(\frac{n\pi x}{l})$$

This concludes our solution to the wave equation on the finite domain, with the given initial condition and boundary condition. We have found the solution $u(x,t)$ as an infinite sum, which involves constants A_n and B_n given by the formula above.

An Important Example

Consider the initial conditions with $\psi(x) = 0$. This corresponds to the fact that the string was held at a shape $\phi(x)$, and then released with no initial velocity given. This is the typical situation when a guitar string is plucked by a musician. The solution $u(x,t)$ would model the shape of the string for future times. In this scenario, $B_n = 0$ for all n. If we furthermore choose $\phi(x) = \sin(\frac{m\pi x}{l})$ for some integer m, then $A_n = 0$ for all n, except $A_m = 1$. We get only **one** term in the sum of $u(x,t)$, and this is called the **m-th harmonic**, where the string vibrates at a frequency of $\frac{m\pi c}{l}$.

$$u(x,t) = \sin(\frac{m\pi x}{l}) \cos(\frac{m\pi ct}{l})$$

Separation Constant $(-\lambda)$
Why must the separation constant be negative (i.e. $\lambda > 0$)?
Recall the 2 differential equations are given by:

$$T'' + \lambda c^2 T = 0 \qquad X'' + \lambda X = 0$$

Let's first consider the easy case: $\lambda = 0$
This gives

$$T'' = 0 \qquad X'' = 0$$

These are actually a "special" form of the second order linear equations.
The solutions are:

$$T(t) = At + B \qquad X(x) = Cx + D$$

Now we apply the **boundary conditions**.

$$u(0,t) = 0 \quad \Rightarrow \quad X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

$$X(0) = D$$

Thus, we conclude $D = 0$, from the boundary condition at $x = 0$.

$$u(l,t) = 0 \quad \Rightarrow \quad X(l)T(t) = 0 \quad \Rightarrow \quad X(l) = 0$$

$$X(l) = Cl = 0$$

Since $l \neq 0$ as the length of the interval, we conclude $C = 0$.
But this would mean $X(x) = 0$, so $u(x,t) = X(x)T(t) = 0$ for all (x,t), which is the trivial solution that we do not need.
Thus there are no (non-trivial) solutions when $\lambda = 0$.

Now let's consider the case: $\lambda < 0$
For notational convenience, set $\lambda = -\mu$ where $\mu > 0$.
We have

$$T'' - \mu c^2 T = 0 \qquad X'' - \mu X = 0$$

These are also important forms of the second order linear equations.
The solutions are:

$$T(t) = Ae^{\sqrt{\mu}\,ct} + Be^{-\sqrt{\mu}\,ct} \qquad X(x) = Ce^{\sqrt{\mu}\,x} + De^{-\sqrt{\mu}\,x}$$

Now we apply the **boundary conditions**.

$$u(0,t) = 0 \quad \Rightarrow \quad X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

$$X(0) = C + D = 0$$

This gives $D = -C$, from the boundary condition at $x = 0$.

$$u(l,t) = 0 \quad \Rightarrow \quad X(l)T(t) = 0 \quad \Rightarrow \quad X(l) = 0$$

$$X(l) = Ce^{\sqrt{\mu}\,l} + De^{-\sqrt{\mu}\,l} = 0$$

Using the previous condition of $D = -C$, we get

$$Ce^{\sqrt{\mu}\,l} - Ce^{-\sqrt{\mu}\,l} = 0$$

$$C(e^{\sqrt{\mu}\,l} - e^{-\sqrt{\mu}\,l}) = 0$$

Thus, either $C = 0$ or $e^{\sqrt{\mu}\,l} - e^{-\sqrt{\mu}\,l} = 0$.
If $C = 0$, then $D = 0$ also. So $X(x) = 0$ for all x, which means $u(x,t) = 0$ for all (x,t), which is again the trivial solution that we do not need.

On the other hand, if

$$e^{\sqrt{\mu}\,l} - e^{-\sqrt{\mu}\,l} = 0$$

$$e^{\sqrt{\mu}\,l} = e^{-\sqrt{\mu}\,l}$$

$$\sqrt{\mu}\,l = -\sqrt{\mu}\,l$$

$$\sqrt{\mu} + \sqrt{\mu} = 0$$

$$2\sqrt{\mu} = 0$$

Thus $\mu = 0$, which is a contradiction as we assumed $\mu > 0$.

Thus, there are no (non-trivial) solutions when $\lambda < 0$.

This confirms that the separation constant is **negative**, which is why we always label the constant as $-\lambda$ with $\lambda > 0$.

Question 1
Find the solution to the wave equation on $0 < x < l$,
with $u(0,t) = 0$, $u(l,t) = 0$, and:
1. $\phi(x) = \sin\left(\dfrac{2\pi x}{l}\right)$, $\psi(x) = \sin\left(\dfrac{3\pi x}{l}\right)$
2. $\phi(x) = 1$, $\psi(x) = 0$
3. $\phi(x) = x$, $\psi(x) = 0$

2 Heat Equation

This **PDE** is called the **Heat Equation** (also called **Diffusion Equation**):

$$u_t = ku_{xx}$$

where k is a constant called "the rate of diffusion" (of heat perhaps).

Notice that the heat equation is extremely similar to the wave equation, yet in fact it describes a completely different phenomenon: diffusion of heat through a 1D medium (such as a metal rod). The **unknown** function $u(x,t)$ would be the temperature at the point x at time t. Alternatively, the heat equation can also be used to describe the diffusion of ink in a rod of water, where $u(x,t)$ would be the concentration of ink at the point x at time t.

Again, we aim to solve $u(x,t)$ in the **finite domain**:

$$0 < x < l$$

where l is a constant being the length of the domain.

Suppose we are given the following **initial condition**:

$$u(x,0) = \phi(x)$$

where $\phi(x)$ describes the **initial heat distribution** at time $t = 0$.
Since the heat equation only involves 1 derivative in t, we actually only need 1 initial condition to solve the problem.

We must also set the **boundary conditions**. This time, let's showcase another type of boundary conditions involving derivatives:

$$u_x(0,t) = 0 \qquad u_x(l,t) = 0$$

These boundary conditions indicates that the **derivative against** x is **always zero** both ends of an interval, which can be interpreted as no heat is entering or leaving the domain. In other words, the 2 ends of the rod are **insulated** (but the temperature $u(0,t)$ and $u(l,t)$ at both ends are free to increase or decrease as time goes on).

This fully sets up the problem that we are interested in. We will of course proceed with **Separation of Variables**.

Separation of Variables
We will **guess** that the solution $u(x, t)$ will be the following form:

$$u(x, t) = X(x) \cdot T(t)$$

Thus the PDE $u_t = ku_{xx}$ becomes:

$$X \cdot T' = kX'' \cdot T$$

After dividing the term kXT on both sides, the equation becomes:

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

where by **convention**, we have again set the separation constant to be $-\lambda$, where $\lambda > 0$. (Let's assume again that the separation constant must be negative as before.)

We may rewrite the above equation into 2 equations:

$$\begin{aligned} \frac{T'}{kT} &= -\lambda & \qquad \frac{X''}{X} &= -\lambda \\ T' &= -\lambda kT & X'' &= -\lambda X \\ T' + \lambda kT &= 0 & X'' + \lambda X &= 0 \end{aligned}$$

The first equation is a **first order linear equation**.
The second equation is a **second order linear equation**.
The solutions are given by:

$$T(t) = Ae^{-\lambda kt} \qquad X(x) = C\cos(\sqrt{\lambda}\,x) + D\sin(\sqrt{\lambda}\,x)$$

(Since we have chosen $\lambda > 0$, the solution to the second order equation must be sin and cos.)

Now, we plug in the **boundary conditions**.
To use the boundary conditions $u_x(0, t) = 0$ and $u_x(l, t) = 0$, we need to calculate $u_x(x, t)$. Recall that our guess was

$$u(x, t) = X(x) \cdot T(t)$$

$$u_x(x, t) = X'(x) \cdot T(t)$$

Note that

$$X'(x) = -C\sin(\sqrt{\lambda}\,x)(\sqrt{\lambda}) + D\cos(\sqrt{\lambda}\,x)(\sqrt{\lambda})$$

So

$$u_x(0,t) = 0 \quad\Rightarrow\quad X'(0)T(t) = 0 \quad\Rightarrow\quad X'(0) = 0$$
$$X'(0) = -C\sin(\sqrt{\lambda}\,0)(\sqrt{\lambda}) + D\cos(\sqrt{\lambda}\,0)(\sqrt{\lambda}) = D\sqrt{\lambda}$$

Since we assumed $\lambda > 0$, we conclude $D = 0$, from the boundary condition at $x = 0$.

$$u_x(l,t) = 0 \quad\Rightarrow\quad X'(l)T(t) = 0 \quad\Rightarrow\quad X'(l) = 0$$
$$X'(l) = -C\sin(\sqrt{\lambda}\,l)(\sqrt{\lambda}) = 0$$

Thus, either $C = 0$ or $\sin(\sqrt{\lambda}\,l) = 0$.

If $C = 0$, then $X(x) = 0$ for all x, which means $u(x,t) = 0$ for all (x,t). This gives the trivial solution which we do not need.

Thus $\sin(\sqrt{\lambda}\,l) = 0$, which only happens if the angle is a integer multiple of π:

$$\sqrt{\lambda}\cdot l = n\pi$$
$$\sqrt{\lambda} = \frac{n\pi}{l}$$

where we may choose n to be any positive integer.

Recall that we assumed $u(x,t) = X(x)T(t)$, with

$$T(t) = Ae^{-\lambda kt} \qquad X(x) = C\cos(\sqrt{\lambda}\,x) + D\sin(\sqrt{\lambda}\,x)$$

Now that $D = 0$, we have a solution for each n, which we label as $u = u_n$:

$$u_n(x,t) = X(x)T(t) = A_n\cos(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2kt}$$

In this case, we can **not** ignore $n = 0$ as $\cos 0 = 1$, which does not give the trivial solution.

Since each value of n gives a solution to the PDE, the **general solution** must be a **linear combination** of the solutions u_n that we guessed:

$$u(x,t) = \sum_{n=0}^{\infty} A_n\cos(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2kt}$$

Notice that the sum now starts at $n = 0$.

Now we apply the **initial condition** $u(x,0)=\phi(x)$.

$$u(x,t)=\sum_{n=0}^{\infty}A_n\cos(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2kt}$$

$$u(x,0)=\phi(x)=\sum_{n=0}^{\infty}A_n\cos(\frac{n\pi x}{l})$$

The series on the right is called the **Fourier (cosine) Series** of $\phi(x)$.
Note that the reason why we got the cosine series instead of sine, is because we chose the **boundary conditions** to involve the **derivative** of u instead.

Recall the **Orthogonal Relations for Fourier Series**:

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}=\begin{cases}0 & \text{if } m\neq n\\ \frac{l}{2} & \text{if } m=n\,(\neq 0)\\ l & \text{if } m=n=0\end{cases}$$

Typically, the next step would be choosing an integer m.
However, since we are working with a cosine series, we must distinguish $m=0$ and $m\neq 0$.

$$\phi(x)=\sum_{n=0}^{\infty}A_n\cos(\frac{n\pi x}{l})$$

First, let's consider $m=0$.
Let me multiply $\cos(\frac{m\pi x}{l})=1$ and integrate over 0 to l:

$$\int_0^l\phi(x)\cos(\frac{m\pi x}{l})=\sum_{n=0}^{\infty}A_n\int_0^l\cos(\frac{n\pi x}{l})\cos(\frac{m\pi x}{l})$$

All terms on the right side would be zero, **except one**, when the summation variable n is equal to $m=0$.

$$\int_0^l\phi(x)\cos(\frac{m\pi x}{l})=A_0\int_0^l\cos(\frac{0\pi x}{l})\cos(\frac{0\pi x}{l})+0+0+...$$

$$\int_0^l\phi(x)\cos(\frac{m\pi x}{l})=A_0\int_0^l 1=A_0l$$

$$A_0=\frac{1}{l}\int_0^l\phi(x)\qquad\qquad(m=0)$$

For $m \neq 0$, every term would be zero, **except one**, when the summation variable n is equal to m, where the integral would be equal to $\frac{l}{2}$:

$$\int_0^l \phi(x)\cos(\frac{m\pi x}{l}) = \sum_{n=0}^{\infty} A_n \int_0^l \cos(\frac{n\pi x}{l})\cos(\frac{m\pi x}{l})$$

$$\int_0^l \phi(x)\cos(\frac{m\pi x}{l}) = 0 + 0 + ... + A_m\frac{l}{2} + 0 + 0 + ...$$

$$A_m = \frac{2}{l}\int_0^l \phi(x)\cos(\frac{m\pi x}{l}) \qquad m \neq 0$$

Thus, given $\phi(x)$, we can solve for each coefficient in the Fourier Series, A_m, by computing the above integrals. Unfortunately, the formulas for A_m are different for $m \neq 0$ and $m = 0$, so one needs to do the calculation for the $m = 0$ term separately.

This concludes our solution to the heat equation on the finite domain, with the given initial condition and boundary condition. We have found the solution $u(x,t)$ as an infinite sum, which involves constants A_n given by the formulas above.

Separation Constant $(-\lambda)$
We have found that $n = 0$ ($\lambda = 0$) for the cosine series does yield a non-trivial term in the sum

$$u_0(x,t) = A_0$$

which is a constant function.
Indeed, if we try the separation constant $\lambda = 0$, the 2 differential equations become:

$$T' = 0 \qquad X'' = 0$$

$$T(t) = A \qquad X(x) = Cx + D$$

Now we apply the **boundary conditions**.

$$u_x(0,t) = 0 \quad \Rightarrow \quad X'(0)T(t) = 0 \quad \Rightarrow \quad X'(0) = 0$$

$$X'(0) = C$$

Thus, we conclude $C = 0$, from the boundary condition at $x = 0$.

$$u_x(l,t) = 0 \qquad \Rightarrow \qquad X'(l)T(t) = 0 \qquad \Rightarrow \qquad X'(l) = 0$$
$$X'(l) = C = 0$$

This time, the second boundary condition yielded the same result as the first. In other words, no restrictions are set on the values A nor D.
Thus our guess is $u(x,t) = X(x)T(t) = D \cdot A$, which is exactly the constant function solution for the term $n = 0$ in the sum, where we have instead combined the 2 constants and wrote

$$u_0(x,t) = A_0$$

Now let's consider the case: $\lambda < 0$
For notational convenience, set $\lambda = -\mu$ where $\mu > 0$.
We have

$$T' - \mu kT = 0 \qquad\qquad X'' - \mu X = 0$$

The solutions are:

$$T(t) = Ae^{\mu kt} \qquad\qquad X(x) = Ce^{\sqrt{\mu}\,x} + De^{-\sqrt{\mu}\,x}$$

Now we apply the **boundary conditions**.
We first need $u_x(x,t) = X'(x)T(t)$.

$$X'(x) = Ce^{\sqrt{\mu}\,x}\sqrt{\mu} - De^{-\sqrt{\mu}\,x}\sqrt{\mu}$$

$$u_x(0,t) = 0 \qquad \Rightarrow \qquad X'(0)T(t) = 0 \qquad \Rightarrow \qquad X'(0) = 0$$
$$X(0) = C\sqrt{\mu} - D\sqrt{\mu} = 0$$
$$C - D = 0 \qquad\qquad (\text{since } \mu \neq 0)$$

This gives $C = D$, from the boundary condition at $x = 0$.

$$u_x(l,t) = 0 \qquad \Rightarrow \qquad X'(l)T(t) = 0 \qquad \Rightarrow \qquad X'(l) = 0$$
$$X(l) = Ce^{\sqrt{\mu}\,l}\sqrt{\mu} - De^{-\sqrt{\mu}\,l}\sqrt{\mu} = 0$$

Using the previous condition of $C = D$, we get

$$Ce^{\sqrt{\mu}\,l}\sqrt{\mu} - Ce^{-\sqrt{\mu}\,l}\sqrt{\mu} = 0$$

$$C\sqrt{\mu}(e^{\sqrt{\mu}\,l} - e^{-\sqrt{\mu}\,l}) = 0$$

Thus, either $C = 0$ or $e^{\sqrt{\mu}\,l} - e^{-\sqrt{\mu}\,l} = 0$.
If $C = 0$, then $D = 0$ also. So $X(x) = 0$ for all x, which means $u(x,t) = 0$ for all (x,t), which is again the trivial solution that we do not need.

On the other hand, if

$$e^{\sqrt{\mu}\,l} - e^{-\sqrt{\mu}\,l} = 0$$

$$\sqrt{\mu}\,l = -\sqrt{\mu}\,l$$

Thus $\mu = 0$, which is a contradiction as we assumed $\mu > 0$.

Thus, there are no (non-trivial) solutions when $\lambda < 0$.

This confirms that the separation constant is **negative** ($\lambda > 0$), but also may be **zero** ($\lambda = 0$). Thus for **boundary conditions** that involves derivatives, we need to take into account the $n = 0$ term.

Question 2
Find the solution to the heat equation on $0 < x < l$,
with $u(0,t) = 0$, $u(l,t) = 0$, and $u(x,0) = \phi(x)$.

Note that the boundary conditions now do not involve derivatives. That is to say, the temperature is **fixed** at both ends of the rod. This can be interpreted as the rod is attached to icebaths on both ends.

Question 3
Find the solution to the heat equation on $0 < x < l$,
with $u(0,t) = 0$, $u_x(l,t) = 0$, and $u(x,0) = \phi(x)$.

This is sometimes called a "mixed" boundary condition.

Question 4
Find the solution to the heat equation on $-l < x < l$,
with $u(-l,t) = u(l,t)$, $u_x(-l,t) = u_x(l,t)$, and $u(x,0) = \phi(x)$.

This is called **periodic boundary condition**, where the point $x = l$ and $x = -l$ are viewed as the "same" point. $u(x,t)$ can be interpreted as the temperature of a **circular** rod with length (circumference) $2l$. Be careful when using the **Orthogonal Relations** as the domain is now $-l < x < l$.

3 Schrödinger Equation

This **PDE** is called the (time dependent) **Schrödinger Equation**

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + V\,\Psi$$

where:

1. $\Psi(x,t)$ is the **unknown complex** function, which somewhat describes the position of a particle in Quantum Mechanics.
2. i is the complex number, with $i^2 = -1$.
3. $\hbar$ called "h bar", is a physical constant, approximately 10^{-34}.
4. m is a constant, being the mass of the particle.
5. $V = V(x)$ is the **potential function**, which **only depends on** x.

Given a complicated $V(x)$, this equation is mostly impossible to solve. Thus, we will take $V(x)$ to be of the following simple form:

$$V(x) = \begin{cases} 0 & 0 < x < l \\ \infty & \text{else} \end{cases}$$

That is to say, the particle will be free to roam between $0 < x < l$, but would **not** be allowed outside of $0 < x < l$. (In some sense, the force is so overwhelmingly large at both endpoints, that the particle is confined between the interval. This setup is sometimes called "particle in a box".)

With this choice of $V(x)$, we simplify the equation immensely:

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx}$$

We aim to solve $\Psi(x,t)$ in the **finite domain**:

$$0 < x < l$$

We still need some initial condition and some boundary condition. Since the Schrödinger Equation is used specifically in Quantum Mechanics, we must first learn some basics.

Wave Function

$\Psi(x,t)$ is called the **wave function**. In general, it is a function that would involve complex numbers. The function describes the position of the particle, as it is the **probability amplitude** for the position of the particle. In other words, $|\Psi(x,t)|^2$ is the **probablity density** for the position of the particle:

$$\left\{\begin{array}{c}\text{The probability of finding the}\\ \text{particle between } a \text{ and } b\text{, at time } t\end{array}\right\} = \int_a^b |\Psi(x,t)|^2\,dx$$

where $|\Psi|^2 = \overline{\Psi}\cdot\Psi$, involving the **complex conjugate**: $\overline{z} = a - ib$.

In our case, since the particle can not be outside the interval, so we must have $\Psi(x,t) = 0$ outside the interval, only then we would have the probability of being outside the interval to be zero:

$$\int_{-\infty}^{0} |0|^2 = 0 \qquad \int_{l}^{\infty} |0|^2 = 0$$

On the other hand, $\Psi(x,t)$ must be differentiable, so it must be continuous. Thus the **boundary conditions** must be chosen to be:

$$\Psi(0,t) = 0 \qquad \Psi(l,t) = 0$$

Suppose we are also given the following **initial condition**:

$$\Psi(x,0) = \phi(x)$$

where $\phi(x)$ describes the **initial state** of the particle at time $t = 0$.

This fully sets up the problem that we are interested in. We will of course proceed with **Separation of Variables**. The method will be identical to the previous 2 equations. However, since $\Psi(x,t)$ is complex, all values that we encounter during our calculation may also be complex.

Recall polar coordinates for complex numbers, given $z = a + ib$:

$$z = a + ib = Re^{i\theta}$$

Lastly, since $|\Psi(x,t)|^2$ is a probability density, it must be **normalized**:

$$\int_0^l |\Psi(x,t)|^2\,dx = 1 \qquad \underset{t=0}{\Longrightarrow} \qquad \int_0^l |\phi(x)|^2\,dx = 1$$

Separation of Variables
We will **guess** that the solution $\Psi(x,t)$ will be the following form:

$$\Psi(x,t) = X(x) \cdot T(t)$$

Thus the PDE, $i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx}$ becomes:

$$i\hbar X \cdot T' = -\frac{\hbar^2}{2m}X'' \cdot T$$

After dividing the term XT on both sides, the equation becomes:

$$i\hbar\frac{T'}{T} = -\frac{\hbar^2}{2m}\frac{X''}{X} = E$$

where by **convention**, we have set the separation constant to be E, where $E > 0$. It turns out E would be the **energy** of this particular solution.

We may rewrite the above equation into 2 equations:

$$\begin{aligned} i\hbar\frac{T'}{T} &= E & \qquad -\frac{\hbar^2}{2m}\frac{X''}{X} &= E \\ T' &= -\frac{iE}{\hbar}T & \qquad X'' &= -\frac{2mE}{\hbar^2}X \\ T' + \frac{iE}{\hbar}T &= 0 & \qquad X'' + \frac{2mE}{\hbar^2}X &= 0 \end{aligned}$$

The first equation is a **first order linear equation**.
The second equation is a **second order linear equation**.
The solutions are given by:

$$T(t) = Ae^{-\frac{iEt}{\hbar}} \qquad X(x) = C\cos\left(\sqrt{\frac{2mE}{\hbar^2}}\,x\right) + D\sin\left(\sqrt{\frac{2mE}{\hbar^2}}\,x\right)$$

(Since we have chosen $E > 0$, the solution to the second order equation must be sin and cos.)

Note the solution for $T(t)$. This is sometimes called the **time evolution** term. It is a **complex exponential**, which by **Euler's Formula**:
$e^{i\theta} = \cos\theta + i\sin\theta$, means that $T(t)$ is in fact sinusoidal, so $\Psi(x,t)$ somehow **oscillates** as time goes on, hence the name **wave function**.

Now, we plug in the **boundary conditions**.
Recall that our guess was $\Psi(x,t) = X(x) \cdot T(t)$.

$$\Psi(0,t) = 0 \qquad \Rightarrow \qquad X(0)T(t) = 0 \qquad \Rightarrow \qquad X(0) = 0$$

as we solved $T(t) \neq 0$. So

$$X(0) = C\cos(\sqrt{\frac{2mE}{\hbar^2}}\,0) + D\sin(\sqrt{\frac{2mE}{\hbar^2}}\,0) = C$$

Thus, we conclude $C = 0$, from the boundary condition at $x = 0$.

$$\Psi(l,t) = 0 \qquad \Rightarrow \qquad X(l)T(t) = 0 \qquad \Rightarrow \qquad X(l) = 0$$

$$X(l) = D\sin(\sqrt{\frac{2mE}{\hbar^2}}\,l) = 0$$

Thus, either $D = 0$ or $\sin(\sqrt{\frac{2mE}{\hbar^2}}\,l) = 0$.

If $D = 0$, then $X(x) = 0$ for all x, which means $\Psi(x,t) = 0$ for all (x,t).
The trivial solution can not satisfy $\int_0^l |\psi(x,t)|^2\,dx = 1$, so it is not acceptable.

Thus $\sin(\sqrt{\frac{2mE}{\hbar^2}}l) = 0$, which only happens if the angle is a integer multiple of π:

$$\sqrt{\frac{2mE}{\hbar^2}} \cdot l = n\pi \qquad\qquad \Rightarrow \qquad\qquad \sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{l}$$

$$E = E_n = \frac{n^2\pi^2\hbar^2}{2ml^2}$$

for each positive interger n, E_n is the energy of the **n-th stationary state**.
For each n, the wave function of the **n-th stationary state**, $\Psi = \Psi_n$:

$$\Psi_n(x,t) = X(x)T(t) = A_n\sin(\frac{n\pi x}{l})e^{-\frac{iE_nt}{\hbar}}$$

The **general solution** must be a **linear combination** of the solutions Ψ_n that we guessed:

$$\Psi(x,t) = \sum_{n=1}^{\infty} A_n\sin(\frac{n\pi x}{l})e^{-\frac{iE_nt}{\hbar}}$$

Now we apply the **initial conditions**.

$$\Psi(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}) e^{-\frac{iE_n t}{\hbar}}$$

We have the initial state as $\Psi(x,0) = \phi(x)$.

$$\Psi(x,0) = \phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})$$

We once again arrive at the **Fourier (sine) Series** of $\phi(x)$.
Using the **orthogonality relations**, we have

$$A_n = \frac{2}{l}\int_0^l \phi(x)\sin(\frac{n\pi x}{l})$$

Note that since $\phi(x)$ may be complex, A_n may also be complex.

Normalization
Recall that since $|\Psi(x,t)|^2$ is the probability density, we need to make sure it would be normalized for all time:

$$\int_0^l |\Psi(x,t)|^2\, dx = 1$$

We know that $|\phi(x)|^2$ must be normalized, as it is given initially. It turns out that this would guarantee $|\Psi(x,t)|^2$ will be normalized for all future times:

Suppose

$$\int_0^l |\phi(x)|^2\, dx = 1$$

We write $\phi(x)$ as a Fourier Series as above, recall that $|\phi(x)|^2 = \overline{\phi}\cdot\phi$,

$$|\phi(x)|^2 = \overline{\Big(\sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})\Big)}\Big(\sum_{m=1}^{\infty} A_m \sin(\frac{m\pi x}{l})\Big)$$

where we must use 2 different summation index, as we are multiplying two series together. i.e. We may get the term $n=1$ multiplied to $m=1$, and also $n=1$ with $m=2$ etc.

Using the properties of complex conjugation, and $\sin(\frac{n\pi x}{l})$ is a real number,

$$|\phi(x)|^2 = \Big(\sum_{n=1}^{\infty} \overline{A_n} \sin(\frac{n\pi x}{l})\Big)\Big(\sum_{m=1}^{\infty} A_m \sin(\frac{m\pi x}{l})\Big)$$

Using distribution property of multiplying 2 sums,

$$|\phi(x)|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \overline{A_n} A_m \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l})$$

Now, putting on the integral, recall that we know $|\phi(x)|^2$ is normalized,

$$1 = \int_0^l |\phi(x)|^2\, dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \overline{A_n} A_m \int_0^l \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l})\, dx$$

Using the **orthogonality relations**, the integral would be zero if $m \neq n$. In other words, the integral picks out only the terms where $m = n$: Every term in the second summation over m, would be equal to 0 if $m \neq n$, and the only non-zero term is when $m = n$.

$$1 = \sum_{n=1}^{\infty} \overline{A_n} A_n \int_0^l \sin(\frac{n\pi x}{l}) \sin(\frac{n\pi x}{l})\, dx$$

$$1 = \sum_{n=1}^{\infty} |A_n|^2 \frac{l}{2}$$

must be true if $|\phi(x)|^2$ was normalized initially. (Note: $|A_n|^2 = \overline{A_n}\, A_n$)

Now, we can compute whether $|\Psi(x,t)|^2$ would be normalized for future times.

$$\int_0^l |\Psi(x,t)|^2\, dx = \int_0^l \overline{\Big(\sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}) e^{-\frac{iE_n t}{\hbar}}\Big)} \Big(\sum_{m=1}^{\infty} A_m \sin(\frac{m\pi x}{l}) e^{-\frac{iE_m t}{\hbar}}\Big)\, dx$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \overline{A_n} A_m e^{+\frac{iE_n t}{\hbar}} e^{-\frac{iE_m t}{\hbar}} \int_0^l \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l})\, dx$$

Notice that due to complex conjugation, the time evolution term for the index n sum has now become $+i$.

By the **orthogonality relations**, the integral would pick out the term $m = n$,

$$\int_0^l |\Psi(x,t)|^2 \, dx = \sum_{n=1}^{\infty} \overline{A_n} A_n e^{+\frac{iE_n t}{\hbar}} e^{-\frac{iE_n t}{\hbar}} \int_0^l \sin(\frac{n\pi x}{l}) \sin(\frac{n\pi x}{l}) \, dx$$

Now notice that after $m = n$, the time evolution terms cancels out to be 1 exactly, as $E_n \neq E_m$ in general, but definitely $E_n = E_n$.

$$\int_0^l |\Psi(x,t)|^2 \, dx = \sum_{n=1}^{\infty} |A_n|^2 \frac{l}{2}$$

which must equal 1, by assumption of $\displaystyle\int_0^l |\phi(x)|^2 \, dx = 1$.

Thus $|\Psi(x,t)|^2$ does in fact stay normalized for all future time, which means it indeed can be a probability density.

Question 5

Define the **expected value** (average position of the particle) at time t to be

$$< x >= \int_0^l \overline{\Psi(x,t)} \, x \, \Psi(x,t) \, dx = \int_0^l x \, |\Psi(x,t)|^2 \, dx$$

Given $\phi(x)$, find the (real) normalization constant N. Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. Find $< x >$ for all time. Does $< x >$ oscillate in time? If so, at what frequency?

1. $\phi(x) = N \sin(\frac{n\pi x}{l})$, the n-th stationary state.

2. $\phi(x)$ is an **even mix** of the first two stationary states:

$$\phi(x) = N \Big(\sin(\frac{\pi x}{l}) + \sin(\frac{2\pi x}{l}) \Big)$$

Recall:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

3. $\phi(x)$ is a mix of the first two stationary states, with a **relative phase**:

$$\phi(x) = N \Big(\sin(\frac{\pi x}{l}) + e^{i\alpha} \sin(\frac{2\pi x}{l}) \Big) \qquad \alpha \text{ real constant}$$

In particular, study the special cases of $\alpha = \frac{\pi}{2}$ and $\alpha = \pi$.

4 Fourier Series

Full Fourier Series
For a differentiable function $f(x)$ defined on $(-l, l)$, the **Full Fourier Series** is given by:

$$f(x) = C_0 + \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi x}{l} + D_n \sin \frac{n\pi x}{l} \right)$$

$$C_0 = \frac{1}{2l} \int_{-l}^{l} f(x) \qquad C_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \qquad D_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}$$

The Fourier Sine and Cosine Series on $(0, l)$ are both special cases of the Full Fourier Series on $(-l, l)$.

Parseval's Equality
For notational purposes, let's write a Fourier Series using a new index k:

$$f(x) = \sum_{k=0}^{\infty} a_k X_k(x) \qquad a < x < b$$

For the Fourier Sine Series, it is just a change of letter: $X_k(x) = \sin \frac{k\pi x}{l}$.
For the Fourier Cosine Series: $X_k(x) = \cos \frac{k\pi x}{l}$ (including $k = 0$)

For a Full Fourier Series in the new index k, the first few functions are:
$X_0(x) = 1$, $X_1(x) = \cos \frac{\pi x}{l}$, $X_2(x) = \sin \frac{\pi x}{l}$, $X_3(x) = \cos \frac{2\pi x}{l}$, $X_4(x) = \sin \frac{2\pi x}{l}$
and the constants a_k are the constants before the functions $X_k(x)$:
$a_0 = C_0$, $a_1 = C_1$, $a_2 = D_1$, $a_3 = C_2$, $a_4 = D_2$

In all 3 cases, we are writing $f(x)$ over an interval $(0, l)$ or $(-l, l)$, as a linear combination of some functions $X_k(x)$, with some coefficients a_k in front.

If $\int_a^b |f(x)|^2$ is finite, then:

$$\int_a^b |f(x)|^2 = \sum_{k=0}^{\infty} |a_k|^2 \int_a^b |X_k(x)|^2$$

where, in general, we use the complex absolute value: $|z|^2 = \overline{z} \cdot z$.

Integration of Fourier Series
We can always integrate the Full Fourier Series term by term.

More precisely, for a differentiable function $f(x)$ defined on $(-l, l)$, we can consider its **anti-derivative**, $F(x)$ where $F'(x) = f(x)$ and $F(0) = 0$. The integral of the Fourier Series would be exactly equal to $F(x)$.

$$f(x) = C_0 + \sum_{n=1}^{\infty} \Big(C_n \cos \frac{n\pi x}{l} + D_n \sin \frac{n\pi x}{l} \Big)$$

$$F(x) = C + C_0 x + \sum_{n=1}^{\infty} \Big(\frac{C_n l}{n\pi} \sin \frac{n\pi x}{l} - \frac{D_n l}{n\pi} \cos \frac{n\pi x}{l} \Big)$$

where C is the constant of integration, which can be found by using the orthogonality relations: $C = \dfrac{1}{2l} \displaystyle\int_{-l}^{l} F(x)$ (for the Full Fourier Series)

Similarly, we can integrate the Fourier Sine/Cosine Series term by term.

Differentiation of Fourier Series
In general, we can **not** differentiate a Fourier Series.

Consider the example of the Fourier Sine Series of $f(x) = 1$ on $(0, \pi)$:

$$1 = \sum_{\text{n odd}} \frac{4}{n\pi} \sin nx$$

If we attempt to take the derivative on the right (term by term), we get

$$\sum_{\text{n odd}} \frac{4}{\pi} \cos nx$$

Notice that with the cancellation of n, the series **no longer converges**, as $\dfrac{4}{\pi} \cos nx$ does not get smaller (does not decrease to 0) as n increases.

The problem gets even worse if we take the derivative again,

$$\sum_{\text{n odd}} \frac{-4n}{\pi} \sin nx$$

where $\dfrac{-4n}{\pi} \sin nx$ gets larger and larger as n increases.

This shows that if we blindly take the derivative of a Fourier Series, we may get a Series that doesn't even exist in the first place, let alone having the derivative of the Fourier Series equaling to the derivative of $f(x)$. Fortunately, if we assume an extra condition, we can take derivative as usual.

For a differentiable function $f(x)$ defined on $(-l, l)$ with $f(-l) = f(l)$, we can consider its **derivative**, $f'(x)$. The derivative of the Fourier Series would be exactly equal to the derivative, $f'(x)$.

$$f(x) = C_0 + \sum_{n=1}^{\infty} \Big(C_n \cos \frac{n\pi x}{l} + D_n \sin \frac{n\pi x}{l} \Big)$$

$$f'(x) = \sum_{n=1}^{\infty} \Big(-\frac{C_n n\pi}{l} \sin \frac{n\pi x}{l} + \frac{D_n n\pi}{l} \cos \frac{n\pi x}{l} \Big)$$

For a function $f(x)$ defined on $(0, l)$, having a Fourier Sine or Cosine Series, we must extend $f(x)$ onto $(-l, l)$, before using the result above.

Question 6
1. Find the Fourier Sine Series of $f(x) = 1$ on $(0, \pi)$.
2. Plug in $x = \frac{\pi}{2}$. What series do you get? What about $x = 1$?
(Compare with the **Taylor Series** of $arctan(x)$ at $x = 1$).

Question 7
1. Find the Fourier Sine Series of $f(x) = x$ on $(0, l)$.
2. Plug in $x = \frac{l}{2}$. What series do you get?
3. Integrate and find the Fourier Cosine Series of $f(x) = x^2$ on $(0, l)$.
Find the constant of integration using the orthogonality relations.
4. Plug in $x = 0$ and $x = l$. These are **p-series** with $p = 2$.
5. Integrate and find the Fourier Sine Series of $f(x) = x^3$ on $(0, l)$.
Note: the function x has its corresponding Fourier Sine (and Cosine) Series.
6. Find the value of the **p-series** with $p = 4$.

Question 8
1. Find the Full Fourier Series of $f(x) = x^2$ on $(-l, l)$. Note: $f(x)$ is even.
Can we differentiate the Series to get the Full Fourier Series of $f(x) = x$?
2. Using Parseval's Equality, find the value of the **p-series** with $p = 2$ and $p = 4$.

5 Vector Function

So far, we have considered **multi-variable functions**.
The function takes in 2 (or 3) inputs, and gives out 1 output:

$$f(x, y) : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$$

We can also have **vector functions**, which gives **multiple outputs**.
In general, we can have

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

$$f \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

It is called a vector function, since f have many outputs, so the outputs as a whole can be regarded as a vector.

$$f(\vec{x}) = \vec{y}$$

Each output, may depend on **all of the inputs**, and so each output is a **coordinate function** that depends on all of the input variables.
In other words, a vector function is made up of many multi-variable functions.

$$f \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, ..., x_m) \\ f_2(x_1, x_2, ..., x_m) \\ \vdots \\ f_n(x_1, x_2, ..., x_m) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

This makes extending the definitions from multi-variable functions to vector functions very easy:

A vector function f is integrable/continuous/differentiable,
when all of its coordinate functions are integrable/continuous/differentiable.

If we focus only within 3 dimensions, there are 2 new types of functions.

Parametrization
When f goes from a lower dimensional space, to a higher dimensional space, we sometimes call f a **parametrization**.

1D Parametrization

$$f(t) : \mathbb{R} \to \mathbb{R}^2 \qquad \textbf{OR} \qquad f(t) : \mathbb{R} \to \mathbb{R}^3$$

where we have used the variable t as the input of f.
Thus for each t, $f(t) = (x, y)$ is a point in 2D, or $f(t) = (x, y, z)$ in 3D.

If we view t as time, then we can view $f(t)$ as the **position** of an object. Thus $f(t)$ describes the position of an object, as time goes on, which will traverse out a **curve**.
Since a curve is a 1D object, this is called a **1D parametrization**.

In fact, this is one of the most important ways to describe a curve in higher dimensions, and finding a parametrization for a given curve is an important skill in Calculus.

Example
Given a circle of radius R at the origin, we can choose the parametrization to be:

$$f(t) = (R\cos t, R\sin t) = (x, y) \qquad t \in [0, 2\pi]$$

where $x = R\cos t$, $y = R\sin t$ in polar coordinates.
The parametrization starts at $t = 0$, which is the point $(R, 0)$. Then as t increases, $f(t)$ traverses the circle **counter-clockwise**, as $t = \theta$ is the angle, and back to $(R, 0)$ when $t = 2\pi$.

However, parametrization is not unique. The same circle can be parametrized by:

$$f(t) = (R\sin t, R\cos t) = (x, y) \qquad t \in [0, 2\pi]$$

where $x = R\cos t$, $y = R\sin t$ is **not** the standard polar coordinates.
The parametrization starts at $t = 0$, which is the point $(0, R)$. Then as t increases, $f(t)$ traverses the circle **clockwise**, since t is now the angle between the point (x, y) and the positive y-axis.
We still traverse the whole circle, but this is the less natural parametrization.

When we need a parametrization of a curce, we typically choose the easiest one.

2D Parametrization

$$f(u,v) : \mathbb{R}^2 \to \mathbb{R}^3$$

where we have used the variables u, v as the inputs of f.
Thus for each (u,v) in $2D$, $f(u,v) = (x,y,z)$ is a point in 3D.

Given a 2D region in the domain $\mathbb{R}^2$ (such as a square), $f(u,v)$ will create a **surface** in 3D.
Since a surface is a 2D object, this is called a **2D parametrization**.

Going from left to right:
We may think that f **lifts up** a 2D region in $\mathbb{R}^2$ into 3D, and maybe stretches the region somewhat, depending on the definition of f.

Going from right to left:
Given a surface S in 3D, we can try to find a parametrization for this surface $f(u,v) = (x,y,z)$. f **pulls back** this (curved) surface S in 3D with (x,y,z) as variables, into a **flat** region in 2D with (u,v) as variables.

We know that an equality involving x, y, z gives a surface in 3D: $g(x,y,z) = 0$
This gives an alternative way of describing a surface in 3D. Finding a parametrization for a given surface is also an important skill in Calculus.

Example
If we consider the plane $z = x + y + 10$ that is above the unit square on the xy-plane, we may choose the parametrization to be:

$$f(u,v) = (u, v, u+v+10) = (x,y,z) \qquad u \in [0,1] \quad v \in [0,1]$$

This is called the **natural parametrization**, as we have set $u = x$ and $v = y$.
For any point $(x,y) = (u,v)$ on the xy-plane, f takes in this point, and gives out the same point with a new **3rd coordinate**, or **height**, with the 3rd component defined by f: $z = u + v + 10$.
This **lifts up** the unit square from the xy-plane, onto the plane $z = x+y+10$.
However, notice that the square is now slanted, and also stretched.
In this case, the plane $z = x + y + 10$ is flat, but in general, f lifts up a flat region in 2D, into a curved surface in 3D, with some stretching.

In reverse, we can view f pulls back a (possibly complicted) surface in 3D, (in this case $z = x + y + 10$), into a flat region in 2D (the unit square).

Coordinate Transformation
When f goes between spaces of the same dimension, we sometimes call f a **coordinate transformation**.

$$f(u,v) : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \textbf{OR} \qquad f(u,v,w) : \mathbb{R}^3 \to \mathbb{R}^3$$

where $f(u,v) = (x,y)$, and $f(u,v,w) = (x,y,z)$.

Going from left to right:
Given a 2D region in the domain $\mathbb{R}^2$ (such as a square), $f(u,v) = (x,y)$ will produce another 2D region in $\mathbb{R}^2$, with some stretching as defined by f.

Going from right to left:
Given a (possibly complicated) region in 2D with variables (x,y), we can try to find a coordinate transformation f for this region, where $f(u,v)$ **transforms a simple region** in the uv-plane, onto the original region in the xy-plane.
We have found a new set of coordinates (u,v), to describe the more complicated region in the xy-plane.

Similarly, we can also do this for regions in 3D.

Example
Given a **solid** circle of radius R at the origin, $x^2 + y^2 \leq R^2$, it is a region in 2D. We know that representing this region is complicated in (x,y), where we get expressions such as $y = \sqrt{R^2 - x^2}$.

However, we may consider **polar coordinates**:

$$f(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y) \qquad r \in [0,R] \quad \theta \in [0,2\pi]$$

where for each point (r,θ) in the "$r-\theta$" plane, $f(r,\theta) = (x,y)$ is a point in the solid circle in the xy-plane.
In other words, f **transforms a solid rectangle** in the "$r-\theta$" plane, into the solid circle in the xy-plane. Instead of working with a circle, we now can work with a rectangle instead, as we have found **better coordinates** to describe the original region in the xy-plane.

For f to turn a rectangle into a circle, it must stretch the rectangle at multiple places, at different rates, which is based on the definition of f.

6 Basic Types of 1D Parametrization

1. Natural Parametrization
For a curve in $\mathbb{R}^2$ given by $y = g(x)$, where $x \in (a, b)$,
we set t to be x:

$$\alpha(t) = (t, g(t)) \qquad t \in (a, b)$$

The parametrization traverses the curve from a to b.

Similarly, if a curve is given by $x = h(y)$, where $y \in (c, d)$,

$$\alpha(t) = (h(t), t) \qquad t \in (c, d)$$

2. Line Segment
For a line segment from $\vec{a}$ to $\vec{b}$,
we utilize the formula $\vec{r} = t\vec{v} + \vec{r_0}$.
Set $\vec{v} = \vec{b} - \vec{a}$, $\vec{r_0} = \vec{a}$

$$\alpha(t) = t(\vec{b} - \vec{a}) + \vec{a} \qquad t \in (0, 1)$$

3. Part of Circle
For parts of a circle with radius R

$$\alpha(t) = (Rcos(t), Rsin(t))$$

t represent the angle, the range of t depends on the range of the angle of the particular curve, from the smaller angle to the larger angle.
The parametrization traverses the circle counterclockwise.

4. Part of Ellipse
After getting the standard equation of the ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$,
we set the quantities inside the bracket to be $cos(t)$ and $sin(t)$

$$\alpha(t) = (a \cdot cos(t), b \cdot sin(t))$$

The value t does **not** exactly represent the angle anymore, except when it matches the axis, namely $0, \pi/2, \pi, 3\pi/2, 2\pi$. The range of t still depends upon the particular curve. The parametrization also traverses counterclockwise.

7 Basic Types of 2D parametrization

1. Natural Parametrization
For a surface S that can be described by $z = g(x, y)$, we take the projection of S onto the xy-plane. Let the **projection** be the set A in the plane.

$$\alpha(u, v) = (u, v, g(u, v)) \qquad (u, v) \in A$$

Similar formula can be obtained for projection onto different planes.
For $y = h(x, z)$, we can project onto the xz-plane, with A in xz-plane.

$$\alpha(u, v) = (u, h(u, v), v) \qquad (u, v) \in A$$

For $x = k(y, z)$, we can project onto the yz-plane, with A in yz-plane.

$$\alpha(u, v) = (k(u, v), u, v) \qquad (u, v) \in A$$

2. Cylindrical Coordinates
Recall $x^2 + y^2 = r^2$, where r is the radius to the z-axis.
If the projection of S onto the xy-plane is circular, then for $z = g(x, y)$

$$\alpha(r, \theta) = (rcos\theta, rsin\theta, g(rcos\theta, rsin\theta))$$

where r and θ corresponds to the radius and angle of the **projection** on the xy-plane.
We take the function $g(x, y)$ and replace all x, y with r and θ.

For the cylindrical shell of radius R, and the height $z \in (c, d)$:

$$\alpha(\theta, z) = (Rcos\theta, Rsin\theta, z)$$

The range of θ depends on the angle of the shell, and $z \in (c, d)$ depends on the height. Notice that R is now a constant in the parametrization as the radius do **not** change for the cylindrical shell.

For a disk with radius R at height $z = H$:

$$\alpha(r, \theta) = (rcos\theta, rsin\theta, H)$$

where r and θ corresponds to the radius and angle, and z is a fixed value.

3. Rotation of graph using Cylindrical Coordinates
For $x^2 + y^2 = r^2$, the surface can be plotted in 2D in r-z plane, and rotated the graph around z-axis to attain the surface in 3D, where the r-axis becomes both x and y axis. A full rotation corresponds to $\theta \in (0, 2\pi)$.
For $z = z(r)$:

$$\alpha(r, \theta) = (rcos\theta, rsin\theta, z(r))$$

where the domain of r depends on the endpoint values of the 2D plot.

For $r = r(z)$:

$$\alpha(\theta, z) = (r(z)cos\theta, r(z)sin\theta, z)$$

where the domain of z depends on the endpoint values of the 2D plot.

4. Ellipse Projection
It is very similar to cylindrical coordinates.
If the projection onto the xy-plane is an ellipse, then write the equation as:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Then for $z = g(x, y)$

$$\alpha(r, \theta) = (a \cdot rcos\theta, b \cdot rsin\theta, g(a \cdot rcos\theta, b \cdot rsin\theta)) \qquad r \in (0, 1)$$

Again, we need to replace x, y in $g(x, y)$ with r and θ.
θ somewhat corresponds to angle same as the 2D ellipse case.

5. Spherical coordinates
For the surface of the sphere of radius R:

$$\alpha(\phi, \theta) = (R \sin\phi \cos\theta, R \sin\phi \sin\theta, R \cos\phi)$$

Note that the radius R is a constant. In this case, the normal vector is:

$$|\frac{\partial\alpha}{\partial\phi} \times \frac{\partial\alpha}{\partial\theta}| = |(R^2 sin^2\phi cos\theta, R^2 sin^2\phi sin\theta, R^2 sin\phi cos\phi)| = R^2 sin\phi$$

6. Plane
Recall the equation of plane $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$.
Rewrite the equation in terms of $z = g(x, y)$, then take the projection onto xy-plane, and use Natural Parametrization.

Question 9

Parametrize the following sets.

1. C is the curve $y = 9 - x^2$ from $x = -1$ to $x = 2$.

2. C is the left half of the circle with radius 6 in the **counterclockwise** direction.

3. C is the ellipse $x^2/25 + y^2/9 = 1$ from the negative x-axis to the positive y-axis in the **counterclockwise** direction.

4. The surface given by $z = 2 - 3y + x^2$ over the triangle on the xy-plane with vertices (0, 0), (2, 0), (2, -4).

5. The surface given by $z = 3x^2 + 3y^2$, bounded below $z = 6$.

6. The surface is the upper hemisphere of radius 2.

7. The surface of the intersection of the solid cylinder $x^2 + y^2 \leq 9$ with the plane $x + z = 5$.

8 Vector Field

A function from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$\vec{F}(x, y) = (F_1(x, y), F_2(x, y))$$

A function from $\mathbb{R}^3$ to $\mathbb{R}^3$:

$$\vec{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$$

are called **vector fields**.
This is because for each point (x, y), we plug into $\vec{F}$ and we attain another pair of points. However, this new pair of points can be interpreted as a vector or **arrow**.
We can draw that arrow at the point (x, y) in 2D plane.

To draw a vector field, first pick a point (x, y) in 2D plane.
Attain the point $\vec{F}(x, y) = (u, v)$, where (u, v) is 2 numbers.
Find (u, v) on the 2D plane, and connect the arrow from 0 to (u, v).
Now move the arrow to the point (x, y) that we started with.

Repeating the process gives many arrows on the 2D plane, called the vector field.

Define the ∇ operator as a 'vector':

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

Define the **divergence** of $\vec{F}$ as the dot product:

$$div\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3$$

Define the **curl** of $\vec{F}$ as the cross product:

$$curl\vec{F} = \nabla \times \vec{F} = det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

For a scalar function $f(x, y, z)$, we still have the **gradient**:

$$\nabla f = (\frac{\partial}{\partial x}f, \frac{\partial}{\partial y}f, \frac{\partial}{\partial z}f)$$

Note that:
∇f is a **vector**.
$\nabla \cdot \vec{F}$ is a **scalar**.
$\nabla \times \vec{F}$ is a **vector**.

We can construct 2nd derivatives with the ∇ operator.

For **scalar** function $f(x, y, z)$:

The divergence of the gradient is the **Laplacian** operator which is a **scalar**:

$$\nabla \cdot \nabla f = \nabla^2 f = \frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f + \frac{\partial^2}{\partial z^2}f$$

The curl of gradient is always the **zero vector**:

$$\nabla \times (\nabla f) = \vec{0}$$

For **vector** function $\vec{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$:

The gradient of the divergence is **NOT** the Laplacian, and it's a **vector**:

$$\nabla(\nabla \cdot \vec{F}) = \nabla(\frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3) \neq 0$$

The divergence of curl is always the **zero scalar**:

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

The curl of curl is not important, but it is a **vector**:

$$\nabla \times (\nabla \times \vec{F}) \neq 0$$

Note that the bracket is very important and must be calculated first.

Question 10
Sketch the following vector fields, by picking some easy points.

a) $\vec{F}(x, y) = (2x, -2)$

b) $\vec{F}(x, y) = (y - 1, x + y)$

Question 11
Compute the divergence and curl of the following vector fields.

a) $\vec{F}(x, y, z) = (2xy^2, 3z, 4y)$

b) $\vec{F}(x, y, z) = (x + z^2, x^2y^3, -z + 3x)$

9 k-Dimensional Volume

In $\mathbb{R}^n$, sets have n-dimensional volume: the integral of $f = 1$ over the set. Sometimes we are interested in the k-dimensional volume of a set, where $k < n$.

For example, think of a string. That seems to be a 1 dimensional object, so it has a 1 dimensional volume, length. However, I can put it on the surface of a desk, and now it is a part of a 2 dimensional space. Its 2 dimensional volume, area, is zero, since we say strings, or lines in 2D, have no area.

However, just because we put the string on the desk, its 1 dimensional volume still must exist, shouldn't it?

Define a set $S \in \mathbb{R}^n$ to be **k-dimensional** if there is a **parametrization**:

$$\alpha : \mathbb{R}^k \to \mathbb{R}^n$$

for the set $A \subset \mathbb{R}^k$, the image of A, $\alpha(A) = S$
α takes the set A in a lower dimension, into a higher dimensional space.

For example, Let A to be a piece of paper on the desk of 2D space.
You pick up the piece of paper and hold it in mid air.
Then S is the set of points that the paper occupies in 3D space.

The act of you picking up the paper, can be regarded as the function α.

$$\alpha : \mathbb{R}^2 \to \mathbb{R}^3$$

where $\alpha(A) = S$, with $A \subset \mathbb{R}^2$ and $S \subset \mathbb{R}^3$.

There are many different kinds of parametrization, which takes the set A from a lower dimensional space, onto the set S in a higher dimensional space.

To attain the k-dimensional volume, we need to return (pull-back) this set S in the higher dimensional space, to A in the the lower dimensional space. We can calculate the k-dimensional volume of A using the integral of $f = 1$, since it is in a k-dimensional space. To attain the k-dimensional volume of S, we must take into account of the fact that the parametrizaiton α may have stretched the set A onto set S (possibly in an uneven manner).

10 Line Integral

We focus on domain that are of 1 dimension in nature.

Let C represent the curve. Then the **arclength**, the 1D volume, is

$$\text{Arclength} = \int_C 1 = \int_a^b |\alpha'(t)|\, dt$$

where $[a, b]$ is the domain of the parametrization.

Since C is 1D, the parametrization for C in $\mathbb{R}^2$ is:

$$\alpha(t) = (x(t), y(t)) \qquad t \in (a, b)$$

Similarly in $\mathbb{R}^3$:

$$\alpha(t) = (x(t), y(t), z(t)) \qquad t \in (a, b)$$

The quantity which determines the stretching factor of the parametrization is

$$|\alpha'(t)|$$

$\alpha'(t) = (x'(t), y'(t), z'(t))$ is interpreted as a vector, and the absolute value of a vector will introduce the square root.

Using another notation:

$$\text{Arclength} = \int_C 1 = \int_C 1\ ds = \int_a^b |\alpha'(t)|dt$$

The (scalar) line integral is usually written with the differential ds, this indicates that the extra factor to multiply is $|\alpha'(t)|$.

We can integrate any other function instead of the function 1, so we generalize the line integral to any function $f(x, y)$ for $\mathbb{R}^2$ and $f(x, y, z)$ for $\mathbb{R}^3$

$$\int_C f \cdot ds = \int_a^b f \circ \alpha\ |\alpha'(t)|dt$$

The term $f \circ \alpha$ means replace all variables x, y, z with the variable t according to $\alpha(t)$.

We also define the line integral with respect to x and y by the notation

$$\int_C f\,dx = \int_a^b f\circ\alpha\cdot x'(t)dt$$

$$\int_C f\,dy = \int_a^b f\circ\alpha\cdot y'(t)dt$$

Notice that the differential dx indicates we only multiply by the derivative $x'(t)$ and not the square root. It does **not** mean integrating with respect to x.

Consider the **vector function** $\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$.
We can also integrate $\vec{F}$ over the curve:

$$\int_C \vec{F} = \int_C \vec{F}\cdot d\vec{r} = \int_a^b (\vec{F}\circ\alpha)\cdot\alpha'(t)dt$$

Notice the similarity in the formula. The only notable change is the multiply is a **dot product** between vectors.
If we write $\vec{F} = (P,Q)$, and $\alpha'(t) = (x'(t), y'(t))$:

$$\int_a^b (\vec{F}\circ\alpha)\cdot\alpha'(t)dt = \int_a^b (P\circ\alpha)\cdot x'(t) + (Q\circ\alpha)\cdot y'(t)dt = \int_C P\,dx + Q\,dy$$

In other words, the vector line integral can be interpreted as the sum of 2 scalar integrals with respect to x and y. This can be regarded as only a **difference in notation**, as they express the same quantity. Nonetheless, this parallel will be important later on for other techniques introduced.

Similar definition is used for curves in $\mathbb{R}^3$ as well.
Properties of integral is similar to previous properties of regular integrals, the most notable being:
If $C = C_1 \cup C_2$ as 2 curves joined together, then

$$\int_C f = \int_{C_1} f + \int_{C_2} f$$

Always integrate from a small value of t to a bigger value of t.
Sometimes, C comes with a direction, and α has a direction as well since t goes from a to b. When the 2 directions are opposite, we need to multiply a **minus sign** to our answer.

The **exception** is the scalar integral with ds which will not change sign.
i.e. The curve C for a scalar integral would not have a direction.

11 Summary of Line Integral

To compute a line integral by definition:

Step 1:
Attain a Parametrization $\alpha(t)$ with $t \in (a, b)$ and its direction.

$$\alpha(t) = (x(t), y(t)) \qquad \textit{or} \qquad \alpha(t) = (x(t), y(t), z(t))$$

Compute the derivative $\alpha'(t)$

Step 2:
Recognize the integral as either

Scalar Integral	vs.	**Vector Integral**
Given $f(x, y)$ or $f(x, y, z)$		Given $\vec{F}(x, y) = (F_1, F_2)$ or $\vec{F}(x, y, z) = (F_1, F_2, F_3)$
$\int_C f\, ds = \int_a^b (f \circ \alpha) \cdot \lvert\alpha'(t)\rvert dt$		$\int_C \vec{F} \cdot d\vec{r} = \int_a^b (\vec{F} \circ \alpha) \cdot \alpha'(t) dt$

$f \circ \alpha$ and $\vec{F} \circ \alpha$ means replacing all x, y, z in the function with the variable t using α. Since $\alpha'(t)$ is a vector, we need to take the absolute value to multiply to f, but we can multiply to $\vec{F}$ directly using the **dot product**.

The integral may be written in different forms:

$$\int_C f\, dx = \int_a^b f \circ \alpha \cdot x'(t) dt \qquad\qquad \vec{F}(x, y) = (P, Q)$$

$$\int_C f\, dy = \int_a^b f \circ \alpha \cdot y'(t) dt \qquad\qquad \int_C \vec{F} \cdot d\vec{r} = \int_C P\, dx + Q\, dy$$

Step 3:
Check whether the direction of α matches the direction of the curve. If they don't match, multiply by a minus sign onto the final answer.

However the scalar integral with ds does not change sign.

(The **arclength** is defined as the scalar line integral of the function $f = 1$)

Question 12

Compute the line integral.

a) $\int_C 2x\,ds$ where C is the curve $y = 9 - x^2$ from $x = -1$ to $x = 2$.

b) $\int_C y^2 - 10xy\,ds$ where C is the left half of the circle with radius 6.

c) $\int_C 2x\,ds$ where C is the line segment from (1, 0) to (0, 1), then followed by the circle of radius 1 from (0, 1) to (1, 0) counterclockwise.

d) $\int_C x\,dy - xy\,dx$ where C is the circle of radius 1 from (0, 1) to (0, -1) in the **clockwise** direction.

e) Let $\vec{F}(x, y) = (y^2, x - 2y)$

Compute $\int_C \vec{F} \cdot d\vec{r}$ where C is the line segment from (1, 3) to (4, 5).

f) Let $\vec{F}(x, y) = (2x^2, y^2 - 1)$

Compute $\int_C \vec{F} \cdot d\vec{r}$ where C is the ellipse $x^2/25 + y^2/9 = 1$ from the positive y-axis to the negative x-axis in the **clockwise** direction.

12 Gradient Theorem

Recall Fundamental Theorem of Calculus: For $g(x) = f'(x)$

$$\int_a^b g(x) = \int_a^b f'(x) = f(b) - f(a)$$

The value of an integral only depends upon the endpoints of the interval.
We generalize this idea to higher dimension, with the derivative being ∇f.
Since ∇f is a vector, we have a vector integral.

Gradient Theorem:
For a curve C in $\mathbb{R}^n$ with endpoints being $\vec{a}$ and $\vec{b}$,
if $\vec{F} = \nabla f$ where f is a scalar function:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{r} = \int_{\vec{a}}^{\vec{b}} \nabla f \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$$

To calculate the value of the integral, **simply plug in** $\vec{b}$ and $\vec{a}$ into f.

In $\mathbb{R}^2$, recall for $\vec{F} = (P, Q)$, the equivalent notation:

$$\int_C P\,dx + Q\,dy = \int_C \vec{F} \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$$

Thus, one technique to calculate a vector line integral $\int_C \vec{F}$ is to ask whether if there exist a function f such that $\vec{F} = \nabla f$, and if so, find f.
In $\mathbb{R}^2$, for $\vec{F} = (P, Q)$ differentiable over **simply connected** domain, if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Then there exist f such that $\vec{F} = \nabla f$, and $\vec{F}$ is called **conservative**.
In $\mathbb{R}^3$, the condition for existence of f is:

$$\nabla \times \vec{F} = \vec{0}$$

Note: The 2 derivative conditions above give $\vec{F}$ is a **closed vector field**, but **simply connected** (no holes in the domain), gives conservative.

In cases where $\vec{F} = \nabla f$, the integral from $\vec{a}$ to $\vec{b}$ will be the same answer regardless of the path C taken from $\vec{a}$ to $\vec{b}$.
Furthermore, if the path C is **closed**, meaning it forms a perfect loop, then the starting point and the endpoin is the same, so the integral is zero.

To solve for f, we can preceed as follows:
Set

$$\vec{F} = (P, Q) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial x} = P \qquad\qquad \frac{\partial f}{\partial y} = Q$$

We integrate both sides with respect to the given variables, but we need to add an extra factor that depends on the variables that are **not** integrated to.

$$\int \frac{\partial f}{\partial x}\, dx = f = \int P\, dx + g(y) \qquad\qquad \int \frac{\partial f}{\partial y}\, dy = f = \int Q\, dy + h(x)$$

We match both expressions of f to solve for $g(y)$ and $h(x)$. Note that $g(y)$ can only have variable y inside, and similarly to $h(x)$.

Then

$$\int_C P\, dx + Q\, dy = \int_C \vec{F} \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$$

where $\vec{a}$ and $\vec{b}$ are the endpoints of C.

Example

$\vec{F}(x, y) = (3x^2 - 2xy + 2, 6y^2 - x^2 + 3)$

$$\frac{\partial f}{\partial x} = 3x^2 - 2xy + 2 \qquad\qquad \frac{\partial f}{\partial y} = 6y^2 - x^2 + 3$$

$$\int \frac{\partial f}{\partial x}\, dx = \int 3x^2 - 2xy + 2\, dx + g(y) \qquad \int \frac{\partial f}{\partial y}\, dy = \int 6y^2 - x^2 + 3\, dy + h(x)$$

$$f = x^3 - x^2 y + 2x + g(y) \qquad\qquad f = 2y^3 - x^2 y + 3y + h(x)$$

Matching terms, we see that x^3 and $2x$ do not appear on the right side, so they must be in $h(x)$. Similarly, $2y^3 + 3y$ do not appear on the left side, so they must be in $g(y)$. $-x^2 y$ appear on both sides, so it just stays put.
Thus

$$h(x) = x^3 + 2x \qquad\qquad g(y) = 2y^3 + 3y$$

Notice that $h(x)$ only has x inside, and $g(y)$ only have y.

$$f(x, y) = x^3 - x^2 y + 2x + 2y^3 + 3y$$

Question 13
a) Let $f(x,y) = x^3(3-y^2)+4y$.
Compute $\int_C \nabla f \cdot d\vec{r}$ where C is given by $\alpha(t) = (3-t^2, 5-t)$ with $t \in (-1,2)$

b) Let $f(x,y) = yx^2$.
Compute $\int_C \nabla f \cdot d\vec{r}$ where C is the right part of the ellipse $x^2+2y^2 = 4$ in the **counter-clockwise** direction.

Question 14
Determine whether the vector field is conservative.
If so, find f such that $\vec{F} = \nabla f$.
a) $\vec{F}(x,y) = (2x+4y, 2x-2y)$
In $\mathbb{R}^2$, remember to first check $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

b) $\vec{F}(x,y) = (3x^2-2xy+2, 6y^2-x^2+3)$

c) $\vec{F}(x,y) = (2xy^2+2y, 2x^2y+2x)$

d) $\vec{F}(x,y) = (2xy^3+e^x\cos(y), e^x\sin(y)-3x^2y^2)$

e) $\vec{F}(x,y,z) = (2z^4-2y-y^3, z-2x-3xy^2, 6+y+8xz^3)$
Strategy: check that $\nabla \times \vec{F} = \vec{0}$.
When you integrate with respect to x, remember to add a function that depend on y **and** z, maybe call it $g(y,z)$. Similarly, we would get extra functions $h(x,z)$ and $k(x,y)$ as well.

Question 15
Compute the line integral.
a) $\vec{F}(x,y) = (4x^3+3y+2y^3/x^3, 3x-3y^2-3y^2/x^2)$
$\int_C \vec{F} \cdot d\vec{r}$ where C is the line segment from (2, 2) to (1, 0).

b) $\vec{F}(x,y) = (3x^2e^{2y}+4ye^{4x}, -2+2x^3e^{2y}+e^{4x})$
$\int_C \vec{F} \cdot d\vec{r}$ where C is given by $\alpha(t) = (3t, t^2)$ with $t \in (1,2)$.

c) Let $\vec{F}(x,y) = (y^2-4y+5, 2xy-4x-9)$
$\int_C \vec{F} \cdot d\vec{r}$ where C is the upper half of the ellipse $x^2/9+y^2/4 = 4$ with the **counter-clockwise** direction.

13 Green's Theorem

Here is another way to interpret Fundamental Theorem of Calculus:

To calculate an integral of a function over some domain, it suffices to calculate the values of the anti-derivative at the boundary of the domain.

Let us write the boundary of domain D as ∂D.
Then the 'generalized Stoke's theorem' is:

$$\int_D d\omega = \int_{\partial D} \omega$$

ω is a 'function-like' object, the letter d stands for 'some kind' of derivative.

A simple observation is:
The derivative symbol d can be transfered onto the integral next to the domain D, becoming ∂D

This translates to the main idea of the theorem:
To calculate an integral of the derivative of a function over a doamin, it suffices to calculate the value of function at the boundary of the domain.

We will be seeing many reincarnations of this idea as we progress to the end of the course.

We can already say, the Fundamental Theorem of Calculus is one special case of the theorem.
To calculate the integra of the derivative of f in 1D, we only need to evaluate f at the boundary.

$$\int_a^b f'(x) = f(b) - f(a)$$

Similarly, Gradient theorem is also a special case of the theorem.

$$\int_C \nabla f = \int_{\vec{a}}^{\vec{b}} \nabla f = f(\vec{b}) - f(\vec{a})$$

A curve in $\mathbb{R}^n$ has 2 points as boundary, and we evaluate f at these points.

Now we introduce another reincarnation of this idea in 2D.

Green's Theorem in 2D
For $\vec{F}(x, y) = (P, Q)$ differentiable over an area D with boundary being curve C:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

The last integral is a regular 2D integral over the domain D.

The last integral is 'some kind' of derivative of $\vec{F}$ over the domain D.
To calculate this integral of a derivative, we can evaluate the function $\vec{F}$ at the boundary of the domain instead. Because there are infinitely many points in the boundary of the domain, we need to perform an integral of $\vec{F}$ over the boundary of D, which is a 1D curve.

If there are several curves C that are boundary of D, we need to integrate over all the curves of the boundary.

The direction of the curves must be given such that:
When traversing the curve in this direction, the domain D is kept on the **left** side of the curve.

For a simple blob of area D, the curve C is on the outside, and must be traversed **counter-clockwise**.

If the blob D has a hole inside, then the hole would produce an additional piece of boundary of D. The inner curve surrounding the hole must be traversed **clockwise**.

Thus, when calculating a line integral, if the curve C is closed, we can turn the integral into a 2D regular integral. If the direction of the curve do not match the requirement of the theorem, we simply multiply by **minus sign** at the end.

In very rare cases, a line integral with C not being closed, can still be evaluated by using Green's Theorem. We need to create another (easy) curve that when joined with C, creates a closed curve, and thus bounding a domain D.

Question 16

Compute the line integral.

a) $\int_C yx^2\,dx - x^2\,dy$ where C is given by the left half of the circle with radius 5 **counter-clockwise**, followed by the line segment from (0, -5) to (0, 5).

b) $\int_C (y^4 - 2y)\,dx + (-6x + 4xy^3)\,dy$ where C is the boundary of the rectangle with $x \in (0, 3)$, $y \in (0, 2)$, traversed **clockwise**.

c) Let $\vec{F}(x, y) = (y^3 - xy^2, 2 - x^3)$.
$\int_C \vec{F} \cdot d\vec{r}$ where C is the boundary of the **solid** circle in the first quadrant with radius 4 (centered at the origin), traversed **counter-clockwise**.

d) Let $\vec{F}(x, y) = (2y - x^2, 7x + y^2)$.
$\int_C \vec{F} \cdot d\vec{r}$ where C is the union of 2 circles. The first circle is of radius 1, **clockwise**. The second circle is of radius 3, **counter-clockwise**.

e) Let $\vec{F}(x, y) = (y^2x + x^2, x^2y + x - e^{sin(y)})$.
$\int_C \vec{F} \cdot d\vec{r}$ where C is the top half of the circle of radius 1 **counter-clockwise**.
Note: The curve C is **not** a closed loop. We can add a curve to create a loop.

Question 17

Consider the **Magnetic Field** $\vec{F}(x, y) = (\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2})$.

Compute $\int_C \vec{F} \cdot d\vec{r}$. Note: $\vec{F}$ is **not** differentiable at the origin.

1. Let C be a circle centered at the origin, radius R counterclockwise.

2. Let C be an **arbitrary** loop that does **not** enclose the origin.

3. Let C be an **arbitrary** loop, encircling the origin once counterclockwise.
Strategy: We can not use Green's Theorem directly as $\vec{F}$ is **not** differentiable at the origin. We can add an extra small circle of radius $R = \epsilon$ around the origin, and apply the theorem on the region in between.

4. Let C be an **arbitrary** curve starting from a point on the positive x-axis, rotating counterclockwise, ending at a point on the negative x-axis, with the curve always above the x-axis.
Strategy: We can try to enclose an area by adding the horizontal line on the x-axis, but we must avoid the origin where $\vec{F}$ is not differentiable. So we introduce a half circle of radius $R = \epsilon$ around the origin.

14 Surface Integral

We focus on domain that are of 2 dimension in nature.

Let S represent the surface. Then the **surface area**, the 2D volume, is

$$\text{Surface Area} = \int_S 1 = \int_A |\frac{\partial\alpha}{\partial u} \times \frac{\partial\alpha}{\partial v}|\, du\, dv$$

where A is the **2D domain** of the parametrization α, so the integral on the right is a **2D regular integral** (where we may need to use **2D change of variables** in order to evaluate the integral).

Since S is 2D, the parametrization for S in $\mathbb{R}^3$ is:

$$\alpha(u, v) = (x(u, v), y(u, v), z(u, v)) \qquad (u, v) \in A$$

The quantity which determines the stretching factor of the parametrization is

$$|\frac{\partial\alpha}{\partial u} \times \frac{\partial\alpha}{\partial v}|$$

Both $\dfrac{\partial\alpha}{\partial u}$ and $\dfrac{\partial\alpha}{\partial v}$ are interpreted as vectors. The cross product of them is also a vector, anad the absolute value of a vector will introduce the square root.

Using another notation:

$$\text{Surface Area} = \int_S 1 = \int_S 1\ dS = \int_A |\frac{\partial\alpha}{\partial u} \times \frac{\partial\alpha}{\partial v}|\, du\, dv$$

The (scalar) surface integral is usually written with the differential dS, this indicates that the extra factor to multiply is $|\dfrac{\partial\alpha}{\partial u} \times \dfrac{\partial\alpha}{\partial v}|$.

We can integrate any other function instead of the function 1, so we generalize the surface integral to any function $f(x, y, z)$ for $\mathbb{R}^3$

$$\int_S f \cdot dS = \int_A f \circ \alpha\ |\frac{\partial\alpha}{\partial u} \times \frac{\partial\alpha}{\partial v}|\, du\, dv$$

The term $f \circ \alpha$ means replace all variables x, y, z with the variable (u, v) according to $\alpha(u, v)$.

Consider the **vector function** $\vec{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$. We can also integrate $\vec{F}$ over the surface:

$$\int_S \vec{F} = \int_C \vec{F} \cdot d\vec{S} = \int_a^b (\vec{F} \circ \alpha) \cdot \Big(\frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v}\Big)\, du\, dv$$

Notice the similarity in the formula. The only notable change is the multiply is a **dot product** between vectors.

The Normal Vector

The **normal vector** of a surface S with parametrization α is:

$$\vec{n} = \frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v}$$

As S is typically a curved surface, $\vec{n}$ would depend on (x, y, z). As the name suggests, at each point, the vector $\vec{n}$ is **normal** (perpendicular) to the surface.

We know that if a vector $\vec{v}$ is normal to a surface, then so is **any scalar multiple** of $\vec{v}$. However, this definition of $\vec{n}$ is the particular normal vector that is of interest.

Given a surface S with parametrization α, the normal vector can have 2 **directions** to point toward. The direction of the normal vector $\vec{n}$ is called the **normal direction**, which is determined by α. The normal direction of a 2D surface is analogous to the direction of a 1D curve.

On the other hand, the **length** of $\vec{n}$ being the cross product of the 2 vectors, determines the stretching factor of the parametrization α.

For the scalar integral, we multiply by $|\vec{n}|$, to take into account of the stretching. For the vector integral, we multiply (dot product) the normal vector $\vec{n}$, to take into account of both the direction and the stretching.

Similar to the line integral, given a surface, we find the most common parametrization α, which comes with a normal direction $\vec{n}$. The surface S also comes with a direction. When the 2 directions are opposite, we need to multiply a **minus sign** to our answer.

Again, the **exception** is the scalar integral with dS which will not change sign.

15 Summary of Surface Integral

To integrate over a surface S by definition, it is similar to the line integral.
Step 1:
Find a parametrization of the surface:

$$\alpha(u,v) = (x(u,v), y(u,v), z(u,v))$$

with A being the domain of parametrization.
The **normal vector** on the surface parametrized by α is given by

$$\vec{n} = \frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v}$$

The partial derivatives are intepreted as **vectors** so they have a cross product. This quantity acts as the derivative of α, that we multiply to the integral.

Step 2:
Recognize the integral as either

Scalar Integral	vs.	**Vector Integral**
Given $f(x,y,z)$		Given $\vec{F}(x,y,z) = (F_1, F_2, F_3)$
$\int_S f\, dS = \int_A (f \circ \alpha)\ \lvert \frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v} \rvert$		$\int_S \vec{F} \cdot d\vec{S} = \int_A (\vec{F} \circ \alpha) \cdot \left(\frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v} \right)$

$f \circ \alpha$ and $\vec{F} \circ \alpha$ means replacing all x, y, z in the function with the variable u, v using α. Since $\frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v}$ is a vector, we need to take the absolute value to multiply to f, but we can multiply to $\vec{F}$ directly using **dot product**.

Notice both integral are done over the **domain of parametrization** A as a 2D regular integral, so we may need to use **change of variables** to evaluate.

Step 3:
The surface S come with a **normal direction** as well. For vector integrals, if the normal vector of α does not match the normal direction of S, we multiply by a **minus sign**.

The **surface area** of S is of course given by

$$\text{Surface Area} = \int_S 1 = \int_A \lvert \frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v} \rvert$$

Question 18
Compute the Surface Area.

a) The surface area of a sphere of radius R.

b) The surface area of $z = 2 - 3y + x^2$, that is over the triangle on the xy-plane with vertices (0, 0), (2, 0), (2, -4).

c) The surface area of $z = 3x^2 + 3y^2$, that is below $z = 6$.

d) $f(x, y) = z = y^2 - x^2 + c$ where c is constant.
Find the surface area of the graph between 2 cylinders centred at the origin with radius 1 and 2.

e) Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$, that is inside the cylinder $x^2 - ax + y^2 = 0$.

Question 19
Let S be the portion of the sphere at the origin with radius 2, with $z \geq 1$.
Compute $\int_S x^2 + y^2 \, dS$

Question 20
Compute the **flux** of the vector field through the surface: $\int_S \vec{F} \cdot d\vec{S}$.

a) Let $\vec{F}(x, y, z) = (2x, 2y, 2z)$.
$S = \{x^2 + y^2 = 9, 0 \leq z \leq 5\}$ be the **capless** cylinder, with outward normal.

b) Let $\vec{F}(x, y, z) = (0, 0, x^2 + y^2)$.
Let S be the disk of radius 3 at $z = 3$, with upward normal.

c) Let $\vec{F}(x, y, z) = (3x^2, 2y, 8)$.
Let S be the portion of the plane $-2x + y + z = 0$ for $(x, y) \in [0, 2] \times [0, 2]$, with normal in the -z direction.

d) Let $\vec{F}(x, y, z) = (z, 0, 0)$.
Let S be the triangle given by $(1, 0, 0), (0, 2, 0), (0, 1, 1)$, with upward normal.
Compute $\int_S \vec{F} \cdot d\vec{S}$.

16 Divergence and Stokes Theorem

Now we introduce 2 more reincarnations of the idea that:
To calculate an integral of the derivative of a function over a domain, it suffices to calculate the value of function at the boundary of the domain.

Divergence Theorem
For a domain D, consider the boundary S as a surface being with normal vector pointing **outward**:

$$\int_D \nabla \cdot \vec{F}\, dV = \int_S \vec{F} \cdot d\vec{S}$$

The integral on the left is a **regular 3D integral**.
The integral on the right is a **vector surface integral**.

If there is a 3D hole in D, then the surface around the hole inside D would have normal vector pointing **into the hole**.

Stokes Theorem
For a surface S, consider the boundary C as a curve:

$$\int_S (\nabla \times \vec{F})\, d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

The curve must be oriented so that as we travel along the curve C with our head pointing in the normal direction of the surface S, the surface S is kept on our **left** side.

The integral on the left is a **vector surface integral**.
The integral on the right is a **vector line integral**.

If there is a 2D hole in S, then the curve around the hole would have **opposite** direction compared to the curve on the outside, due to the **left** requirement.

These theorem allows us to change one type of into another type, and choose the easiest integral to compute. When the orientation do not match, we simply multiply by a **minus sign**.
In very rare cases, when the surface is **not** closed, we can still use these Theorems by adding an (easy) surface to create a closed surface.

Question 21
Let $\vec{F} = (3xy^2, xe^z, z^3)$.
Let S be the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the plane $z = -1$ and $z = 2$, oriented with the normal direction outward.
Compute $\int_S \vec{F} \cdot d\vec{S}$.

Question 22
Let $\vec{F} = (2ycosz, e^x sinz, xe^y)$.
Let S is the upper hemisphere of radius 2, with the normal direction being upward.
Compute $\int_S (\nabla \times \vec{F}) \cdot d\vec{S}$.

Strategy:
Use Stokes Theorem on the boundary (curve) of the surface.

Question 23
Let $\vec{F} = (yz, 2x, 3x)$.
Let C be the curve of intersection by plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise when viewed from above.
Compute $\int_C \vec{F} \cdot d\vec{r}$ using Stokes Theorem.

Question 24
Let $\vec{F} = (ze^{siny}, 3x^2, xyz)$.
Let S be the surface of the ellipsoid given by $x^2 + \dfrac{y^2}{4} + z^2 = 1$, with $y > 0$.
Let $\alpha(u, v) = (u, 2\sqrt{1 - u^2 - v^2}, v)$, be the natural parametrization of S, which determines the normal direction of S. What is the domain of α?
Compute $\int_C \vec{F} \cdot d\vec{r}$, where C is the boundary of S, with the direction of C determined by Stokes' Theorem.

Question 25
Let $\vec{F} = (e^{sin(y)} yz^2, arctan(xz), 3 + y)$.
Let S be the surface given by $z = 1 - x^2 - y^2$ above the xy-plane, oriented with the normal direction outward.
Compute $\int_S \vec{F} \cdot d\vec{S}$.

Note: the surface is **not** closed.

17 Technique Summary of Line Integral

Step 0:
Recognize whether it is a scalar integral or a vector integral.
Scalar integral is characterized by the quantity ds, dx, dy.
However the integral of the form:

$$\int_C P\,dx + Q\,dy = \int_C \vec{F} \cdot d\vec{r}$$

is in fact a vector integral in different notation.

If it is a **scalar integral**, then the **only** method is compute by definition.
Parametrize the curve and multiply by $|\alpha'(t)|$.

When it is a **vector integral**:
Step 1: Test for conservative vector field.
In $\mathbb{R}^2$, check

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

In $\mathbb{R}^3$, check $\nabla \times \vec{F} = 0$.
If it holds, then solve the equation $\vec{F} = \nabla f$ for the function f, then

$$\int_C P\,dx + Q\,dy = \int_C \vec{F} \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$$

Step 2: Check for closed loop
In $\mathbb{R}^2$, the curve C may be closed and bounds an area, try to use Green's Theorem to turn the line integral into a 2D regular integral.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

In $\mathbb{R}^3$, the curve C may be closed and bounds a surface, try to use Stokes Theorem to turn the line integral into a surface integral.

$$\int_C \vec{F} \cdot d\vec{r} = \int_S (\nabla \times \vec{F})\, d\vec{S}$$

Step 3: Use definition to evaluate integral
Parametrize the curve and multiply by $\alpha'(t)$ as a dot product.

Finally, when an orientation is off, multiply by **minus sign**.

18 Technique Summary of Surface Integral

Step 0:
Recognize whether it is a scalar integral or a vector integral.
Scalar integral is characterized by the quantity dS.

If it is a **scalar integral**, then the **only** method is compute by definition.
Parametrize the surface and multiply by

$$|\frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v}|$$

When it is a **vector integral**:
Step 1: Check for closed surface
In $\mathbb{R}^3$, the surface S may be closed and bounds a volume, try to use Divergence Theorem to turn the surface integral into a 3D regular integral.

$$\int_S \vec{F} \cdot d\vec{S} = \int_D \nabla \cdot \vec{F}\ dV$$

Step 2: Check Stokes Theorem
If the integrand is $\nabla \times \vec{F}$, then we should try to use Stokes Theorem:

$$\int_S (\nabla \times \vec{F})\ d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

Compute the line integral instead using the previous techniques.

Step 3: Use definition to evaluate integral
Parametrize the surface and multiply by $\dfrac{\partial \alpha}{\partial u} \times \dfrac{\partial \alpha}{\partial v}$ as a dot product.

Finally, when an orientation is off, multiply by **minus sign**.

Question 26

Consider the **Electric Field** $\vec{F}(x, y, z) = \dfrac{1}{(x^2 + y^2 + z^2)^{3/2}}(x, y, z)$.

Compute $\int_S \vec{F} \cdot d\vec{S}$. Note: $\vec{F}$ is **not** differentiable at the origin.

1. Let S be a sphere centered at the origin, radius R with outward normal.
2. Let S be an **arbitrary** closed surface that does **not** enclose the origin.
3. Let S be an **arbitrary** surface, enclosing the origin with outward normal. **Strategy:** We can not use Divergence Theorem directly as $\vec{F}$ is **not** differentiable at the origin. We can add an extra small sphere of radius $R = \epsilon$ around the origin, and apply the theorem on the region in between.
4. Let S be an **arbitrary** dome with outward normal, that is sitting on the xy-plane, with $z \geq 0$, having a boundary C, which is a closed curve on the xy-plane.
   Strategy: We can try to enclose a volume by adding the horizontal plane on the xy-plane enclosed by C, but we must avoid the origin where $\vec{F}$ is not differentiable. So we introduce a half sphere of radius $R = \epsilon$ around the origin.

Question 27

Let $\vec{F}(x, y, z) = (y \ln(y^2 + 1), \arctan(x^2 z), y^2)$.
Let S be given by $z = 4 - x^2 - y^2$ which is above the xy-plane, with upward normal.
Compute $\int_S \vec{F} \cdot d\vec{S}$.

Question 28

Recall **integration by parts**:

$$\int_a^b u\, dv = u\, v\Big|_a^b - \int_a^b du\, v$$

This can be interpreted as we are "moving" one derivative "d" from v onto u, and in doing so, we get an extra boundary term "without" the derivative "d", along with a minus sign. Show the analogous of integration by parts in higher dimensions, called **Green's Identities**:

$$\int_D \nabla v \cdot \nabla u\, dV = \int_S (v \nabla u) \cdot d\vec{S} - \int_D v \Delta u\, dV$$

$$\int_D (u \Delta v - v \Delta u)\, dV = \int_S (u \nabla v - v \nabla u) \cdot d\vec{S}$$