Extragalactic Astro, HW 4

Mia Morrell

October 16, 2022

---

1 Cosmology, Order of Magnitude Exercise 4

Estimate the particle horizon size at $z \sim 1100$, assuming either the universe is radiation- dominated, or it is matter-dominated, above that redshift.

Solution: From the class notes on cosmology, the particle horizon is the distance a particle could travel between some time $t = 0$ and some later time. Here, we want to find the particle horizon corresponding to some redshift z. Recall that $z = 0$ represents the present time, and $z = \infty$ represents $t = 0$. So we want to integrate the derivative of distance r with respect to redshift z over all redshifts $z \in [z', \infty]$.

To do this integral, we need to derive $\frac{dr}{dz}$. Recall the definition of the scale factor

$$a(t) = \frac{1}{1 + z(t)}. \tag{1}$$

The scale factor scales spatial coordinates to account for universe expansion. Differentiating Eq. 1, we get

$$\frac{dz}{dt} = -\frac{1}{a^2}\frac{da}{dt}. \tag{2}$$

Note the definition of the Hubble parameter, $H(z) = \frac{\dot{a}}{a}$. Plugging this definition into Eq. 2, we get

$$\frac{dz}{dt} = -\frac{1}{a}H(z) = -(1 + z)H(z). \tag{3}$$

Inverting, we get

$$dt = -\frac{dz}{(1 + z)H(z)}. \tag{4}$$

Note that the distance that a photon travels in time Δt must be $a(t)\Delta r = c\Delta t$. Using this to convert dt into dr in Eq. 4, we see that

$$dr = -c\frac{dz}{H(z)} \tag{5}$$

The term $H(z)$ can be found using the Friedmann equation (see https://en.wikipedia.org/wiki/Friedmann_equations)

$$\frac{H^2}{H_0^2} = \Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda. \tag{6}$$

where H_0 is the hubble parameter at the current time. Here we neglect Ω_K and assume a flat universe. Plugging $H(z)$ into Eq. 5 we get

$$dr = -\frac{c}{H_0}dz\left(\Omega_R(1 + z)^4 + \Omega_M(1 + z)^3 + \Omega_\Lambda\right)^{-\frac{1}{2}}. \tag{7}$$

For a matter dominated universe, $\Omega_M \sim 1$ and all other omegas are neglected. Integrating, we get

$$r_H^{(M)}(z) = -\frac{c}{H_0}\int_\infty^{z'} dz(1 + z)^{-\frac{3}{2}} = \frac{c}{H_0}\int_{z'}^{\infty} dz(1 + z)^{-\frac{3}{2}} = -\frac{2c}{H_0}(1 + z)^{-\frac{1}{2}}. \tag{8}$$

For a radiation dominated universe, $\Omega_R \sim 1$ and all other omegas are neglected. Integrating, we get

$$r_H^{(R)}(z) = -\frac{c}{H_0}\int_\infty^{z'} dz(1 + z)^{-2} = -\frac{2c}{H_0}(1 + z)^{-1}. \tag{9}$$

Plugging in $z = 1100$, $H_0 \sim 70$ km/s/Mpc, and $c = 3 \times 10^5$ km/s, we get $r_H^{(M)} \sim 258$ Mpc and $r_H^{(R)} \sim 8$ Mpc.

2 Structure formation, Analytic Exercise 2

Starting from the continuity equation in Equation 6, assuming a flat matter dominated universe ($\Omega_M = 1$), and keeping only first-order terms, derive Equation 11.

Solution: Starting from $\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{v}$ where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla$, we need to show that the peculiar solution $(\delta, \vec{v_p})$ from $\rho = \rho_0(1+\delta)$ and $\vec{v} = \vec{v_0} + \vec{v_p}$ satisfies the continuity equation. That is, we want to show that $\frac{d\delta}{dt} = -\nabla\cdot\vec{v_p}$.

The strategy here is to plug in $\rho = \rho_0(1+\delta)$ and $\vec{v} = \vec{v_0} + \vec{v_p}$ to $\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{v}$ and cancel out the equilibrium solution $\frac{D\rho_0}{Dt} = -\rho_0\nabla\cdot\vec{v_0}$. Following this strategy, we get

$$\frac{D\rho}{Dt} = \left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)[\rho_0(1+\delta)] = -\rho\nabla\cdot\vec{v} = [-\rho_0(1+\delta)\nabla\cdot(\vec{v_0}+\vec{v_p})] \tag{10}$$

$$= \frac{\partial}{\partial t}(\rho_0 + \delta\rho_0) + (\vec{v_0}+\vec{v_p})\cdot\nabla(\rho_0 + \rho_0\delta) = -\rho_0(1+\delta)\nabla\cdot\vec{v_0} - \rho_0(1+\delta)\nabla\cdot\vec{v_p} \tag{11}$$

Canceling out the equilibrium solution, we get

$$\frac{\partial}{\partial t}(\delta\rho_0) + \vec{v_0}\cdot\nabla(\rho_0\delta) + \vec{v_p}\cdot\nabla(\rho_0) + \vec{v_p}\cdot\nabla(\rho_0\delta) = -\rho_0\delta\nabla\cdot\vec{v_0} - \rho_0\nabla\cdot\vec{v_p} - \rho_0\delta\nabla\cdot\vec{v_p} \tag{12}$$

$$= \delta\frac{d}{dt}\rho_0 + \rho_0\frac{\partial}{\partial t}\delta + \delta\vec{v_0}\cdot\nabla\rho_0 + \rho_0\vec{v_0}\cdot\nabla\delta + \vec{v_p}\cdot\nabla\rho_0 + \delta\vec{v_p}\cdot\nabla\rho_0 + \rho_0\vec{v_p}\cdot\nabla\delta \tag{13}$$

Now neglect all terms $\mathcal{O}\left((\delta, \vec{v_p})\right)$:

$$\rho_0\frac{\partial}{\partial t}\delta + \rho_0\vec{v_0}\cdot\nabla\delta = -\rho_0\nabla\cdot\vec{v_p}. \tag{14}$$

Canceling out ρ_0, this is the continuity equation

$$\nabla\cdot\vec{v_p} = -[\frac{\partial}{\partial t}\delta + \vec{v_0}\cdot\nabla\delta] = -\frac{D}{Dt}\delta. \tag{15}$$

Given that $\vec{v_0}$ and the gradient of δ are perpendicular, we derive

$$\nabla\cdot\vec{v_p} = -\frac{\partial}{\partial t}\delta, \tag{16}$$

the desired result.