Firm Investment with Shareholder Heterogeneity*

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Abstract

Does rising household income risk change aggregate investment and output by corporate firms? To answer this question, I study a stochastic, general equilibrium model featuring aggregate risk, incomplete markets, households who are heterogeneous in productivity and wealth, and shareholder-owned firms that own capital. The firm's objective is typically not well defined in models like this, so I introduce a pair of financial intermediaries that jointly pin the firm's behavior. In equilibrium, firms maximize their net market value (or cum-dividend share price), which is determined by the end of period shareholder weighted marginal rate of substitution. This method nests both the standard representative household case and a variety of no-trade models which are popular in the literature. I then apply this model to the data to see how a change in earnings variance between 1970 and 2010 changes firm dynamics. Higher levels of household idiosyncratic risk causes firms to accumulate more capital, which results in lower volatility of consumption and output over the business cycle. I also find that the observed change in the idiosyncratic productivity process for US households can explain 60% of the decline in dividend yields between 1970 and 2010 as well as 50% of the rise in price earnings ratio.. Additionally, my model generates the pro-cyclical price-earnings ratio observed in the data.

JEL Codes: D52, E13, G12, L21

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1 Introduction

Does shareholder wage risk change firm investment behavior and, if so, does an increase in household risk meaningfully shape business cycle dynamics? Wage risk in incomplete markets drives precautionary savings when households directly hold capital. However, households are only responsible for 20% of aggregate investment, while corporate firms perform 60-65% of investment.[1] Further, the literature documents numerous cases where firm choices play an important role in understanding macroeconomic dynamics.

If household risk and firm dynamics are both important to understanding macroeconomic fluctuations, why aren't models with these features more common? The goal of the firm is generally not well defined when it is owned by heterogeneous shareholders. Each shareholder's valuation of payoffs over different aggregate states will depend on their current idiosyncratic income and wealth levels. If this is the case, how does the firm value future payoffs? To answer this question, I construct a model with uninsurable aggregate and household risk featuring a set of financial intermediaries who discipline a consistent valuation of payoffs across future states. In equilibrium, firms will value future payoffs using a share-weighted average of their shareholders' expected marginal rates of substitution across states. I find an increase in household earnings risk increases investment by firms, lowers the price-earnings ratio, and lowers aggregate consumption volatility over the business cycle.

Households save in aggregate equity through a mutual fund[2] which bundles shares of all production firms into a single investment instrument. This plays the dual role of simplifying the household's problem to a single continuous choice variable and preventing production firms from becoming financial innovators.[3] I also include a private equity firm that disciplines the choices of the production firm, similar to the one proposed by Grossman and Hart (1979). The private equity firm takes over the production firm if it can find an alternate investment scheme supported by the mutual fund, which is replicates behavior observed in shareholder proxy challenges.

This paper contributes an approach for aggregating heterogeneous shareholders' preferences that is consistent with observed firm and proxy behavior. The discount factor I

[1]Households perform approximately 20% of aggregate investment, while partnerships, sole proprietor businesses, and nonprofits make up the remainder.

[2]Similar to a capital mutual fund described in Carlstrom and Fuerst (1997), though the mutual fund in my model holds equity rather than capital.

[3]Financial innovation can happen when a firm creates a new set of payoffs that were not spanned by the previous set of possible investment choices. If a firm promises a tiny deviation in one future state of nature, households could trade this firm purely as a financial asset, even if it doesn't meaningfully change output.

describe nests the standard representative household case and a setting with exogenously imposed zero trade, as in Krusell, Mukoyama and Smith (2011). I also demonstrate that alternate discounting schemes proposed in the literature are inconsistent with observed shareholder and intermediary behavior. This theory contribution enables researchers to build models featuring rich heterogeneity among both firms and households.

This paper also contributes a link between increasing household risk and key macroeconomic aggregates. From 1970 to 2010, the variance in earnings has increased in the US. I find this higher level of risk results in a lower expected rate of return on capital, higher aggregate investment, less volatile output, and less volatile aggregate consumption. Additionally, the model can explain 30% of the observed fall in dividend yield and 20% of the rise in the price to earnings ratio for S&P 500 stocks.

Related Literature A large literature studies the role of household risk in incomplete markets where households own capital. Aiyagari (1994) develops this in a setting without aggregate risk while Krusell and Smith (1998) includes aggregate risk. Challe and Ragot (2016) document the role of household precautionary savings with unemployment risk over the business cycle. I contribute to this literature by including a production sector that makes a meaningful choice between profit and rental payments. Busch et al. (2022) document skewness in earnings growth is procyclical. While this skewness is not a feature in my model, my model results would be stronger if this skewness were included.

The closest link between household risk and firm behavior generally comes from the entrepreneurship literature. Cagetti and De Nardi (2006) describe a setting where household wealth generates a distribution of entrepreneur firms. In their setting, however, small businesses take their discounting directly from their owners. My approach focuses on larger corporate firms who are responsible for 60% of total investment, while small businesses only account for about 20%.

My paper is most closely related to the literature on firm discounting and price perception. Early work by Drèze (1974*a*) describes the problem of uncertainty in the firm's valuation of future payoffs. To remedy this, Grossman and Hart (1979) proposed aggregating discount factors weighed by current shareholding. Carceles-Poveda (2009) used this aggregation method to study firm behavior under the proposed discount valuation. However, those models depend on compensating households for their off-equilibrium perception of price changes. In contrast, I construct a model where price perceptions are consistent with equilibrium. I also find that methods based on aggregating current shareholder preferences do not result in net market value maximization. Firms that weigh the future by the methods proposed by Grossman and Hart (1979) and Carceles-Poveda

(2009) will be worth less than firms that weigh the future using my methodology because my approach takes into account post-trade optimal conditions. However, the results between my discounting and theirs are close, which is consistent with the horse race performed in Carceles-Poveda (2009). Instead of using price perceptions, Krusell, Mukoyama and Smith (2011) exogenously impose zero trade, which results in a setting where a single household pins down the aggregate discount factor. My model nests their approach. Empirically, Gormsen and Huber (2022) document firms have higher discount rates than what is implied by the cost of equity. My model replicates this as payoffs are more valued in low states than high states, which results in firms who look like they are risk averse.

The asset pricing literature also relates closely to my work. Constantinides and Duffie (1996), Braun and Nakajima (2012), and Constantinides and Ghosh (2017) combine household marginal rates of substitution to create an aggregate stochastic discount factor. However, these papers construct an income process that results in zero trade while my model allows for shareholders to change over time. Marcet and Singleton (1999) find asset prices are higher with higher income risk, but they only focus on the price and not the discount factor required to find that price. Krueger and Lustig (2010) document that a lack of insurance for idiosyncratic risk only shifts the price of aggregate risk if household risk is uncorrelated with aggregate risk. Household wage income risk is correlated with aggregate risk, so their result reinforces my findings. Paron (2021) documents a similar phenomenon in continuous time models.

To tie my model's assumptions back to observed behavior, I rely on a number of works examining the interaction between shareholders, intermediaries, and firms. Carlstrom and Fuerst (1997) sets up an environment where risky firms rent capital from a single financial intermediary. I follow the literature by extending their approach to an equity market. The private equity challenge is based on theory from Grossman and Hart (1979) but supported in data by Fos (2017). Fos finds that shareholder challenges are most common and are more likely to succeed when they attempt to increase firm value, which is consistent with the challenge by the proposed private equity firm in my model. Fichtner, Heemskerk and Garcia-Bernardo (2017) and Edelman, Thomas and Thompson (2014) document that large financial intermediaries control a majority of voting shares and that they are required to vote in their shareholders' best interests. This matches my model's assumption that the private equity firm directly interfaces with the financial intermediary instead of individual shareholders.

The paper proceeds as follows. Section 2 describes the model environment in detail, with particular focus given to the problem of the private equity firm. Section 3 describes the conditions required for equilibrium and shows how I derive the aggregate stochastic discount factor. Section 4 discusses the algorithm I use to solve the model. Section 5 discusses business cycles moments and impulse responses with varying levels of realistic idiosyncratic risk. Section 6 concludes.

2 Model Environment

In this model economy, production firms own capital and make meaningful intertemporal decisions on behalf of their shareholders. Households face idiosyncratic labor productivity risk and can only save in aggregate equity. I begin the description of this economy with details about the maximization problem facing each household, the production firms, and the mutual fund (financial aggregator). Once the dynamic agents are introduced, I describe the off-equilibrium private equity firm.

2.1 Households

There is a single production good which is used for both consumption and investment. This good is the model's numeraire. There are a unit measure of households in this economy, identified by their start of period assets a and idiosyncratic labor productivity η. Each household has identical, time-separable, concave, strictly increasing preferences over consumption. Each supplies labor inelastically and saves in aggregate equity a. I assume η is a Markov chain; $\eta \in \mathbf{N} \equiv \{\eta_1, \ldots, \eta_{N_\eta}\}$, where $\Pr(\eta' = \eta_j | \eta = \eta_i) = \pi_{ij} \geq 0$ and $\sum_{j=1}^{N_\eta} \pi_{ij} = 1$ for each $i = 1, \ldots, N_\eta$. For simplicity and without loss of generality, I assume higher indexed values of η denote higher productivity levels: $\eta_1 < \eta_2 < \cdots < \eta_{N_\eta}$.

The household's asset holding in the mutual fund is given by $a \in \mathbf{A} \subset \mathbf{R}$. The set $\mathbf{A}$ is bounded above by 1 and below by $\underline{\mathrm{a}}$. This lower bound $\underline{\mathrm{a}}$ is a parameter in the model. The lower bound must fall in the range $[\underline{\mathrm{A}}, 1]$, where $\underline{\mathrm{A}}$ is the natural borrowing limit. I define this limit as the smallest level of debt a household could service conditional on entering the period with that debt and holding the lowest productivity draw. I derive an expression for the natural borrowing limit in Appendix C. If $\underline{\mathrm{a}} = 0$, this constraint would prohibit short sales. If $\underline{\mathrm{a}} = 1$, the economy is in exogenously-imposed autarky, similar to the no-trade scenario described in Krusell, Mukoyama and Smith (2011). I discuss similarities between my method and theirs further in Section 3.4.1.

I summarize the distribution of households over (a, η) using the probability measure μ_H defined on the Borel algebra $\mathcal{S}$ generated by the open subsets of the product space, $\mathbf{S}_H = \mathbf{A} \times \mathbf{N}$.

I require two more components to fully define the aggregate state. The first is aggregate exogenous TFP z. I assume z is a Markov chain; $z \in \mathbf{Z} \equiv \{z_1, \ldots, z_{N_z}\}$, where $\Pr(z' = z_n | z = z_m) = \pi_{mn} \geq 0$ and $\sum_{n=1}^{N_z} \pi_{mn} = 1$ for each $m = 1, \ldots, N_z$. As with labor productivity, I assume higher indexed levels of z are more productive: $z_1 < z_2 < \cdots < z_{N_z}$.

The final component of the aggregate state is the distribution of firms over their start of period capital, $k \in \mathcal{K} \subset \mathbf{R}_{++}$. Similar to households, I summarize the distribution of firms over k using the probability measure μ_F. The aggregate state of the economy is then (z, μ_H, μ_F).

The per-productivity-unit wage $w(z, \mu_H, \mu_F)$ is taken as given by the household. In each state, the price of equity (or shares in the mutual fund) is expressed as $P(z, \mu_H, \mu_F)$ which pays dividends $D(z, \mu_H, \mu_F)$. These are equilibrium prices which the household takes as given when making its decisions.

I now describe the recursive problem of each household in the economy. Let $V(a, \eta_i; z_m, \mu_H, \mu_F)$ be the start of period value of a household with assets a, productivity η_i, and the aggregate state given by (z_m, μ_H, μ_F). For convenience of notation, I define a shorthand for the aggregate state as $\mathbb{Z} \equiv (z_m, \mu_H, \mu_F)$. The dynamic problem of each household is given by:

$$V(a, \eta_i; z_m, \mu_H, \mu_F) = \max_{c, a'} u(c) + \beta \sum_{j=1}^{N_\eta} \pi_{ij} \sum_{n=1}^{N_z} \pi_{mn} V(a', \eta_j; z_n, \mu'_H, \mu'_F) \tag{1}$$

$$\text{s.t. } c + P(\mathbb{Z})a' \leq (P(\mathbb{Z}) + D(\mathbb{Z}))\, a + w(\mathbb{Z})\eta_i \tag{2}$$

$$\underline{\mathrm{a}} \leq a' \tag{3}$$

$$\mu'_F = \Gamma_F(\mathbb{Z}), \qquad \mu'_H = \Gamma_H(\mathbb{Z})$$

where β is the commmon subjective discount factor. Equation 3 describes the minimum savings policy, where $\underline{\mathrm{a}} \in [\underline{\mathrm{A}}, 1]$ and $\underline{\mathrm{A}}$ is the natural borrowing limit as described above and in Appendix C.

The distribution of households over productivity and shareholding evolves over time according to a mapping Γ_H which depends on the current aggregate state. That is, $\mu'_H = \Gamma_H(z, \mu_H, \mu_F)$. This evolution depends on the asset choices of households in the previous period and the realization of idiosyncratic shocks. The distribution of firms over capital is similar, with $\mu'_F = \Gamma_F(z, \mu_H, \mu_F)$. The household takes both of these laws of motion as given when making its shareholding choice.

Let $c(a, \eta; \mathbb{Z})$ and $a(a, \eta; \mathbb{Z})$ be the decision rules for consumption and future shareholding of a household with current state (a, η) and aggregate state $\mathbb{Z}$.

2.2 Equity Mutual Fund

A risk-neutral mutual fund (or equity financial aggregator) bundles shares of the production firms and sells the bundle to households as aggregate equity. Each period, the intermediary collects dividends from production firms, chooses how many shares of each production firm it wants to hold for the next period, and pays out aggregate dividends to households. Aggregate dividends are the dividends collected from production firms plus the net revenue from changing its shareholding of production firms.

The financial aggregator chooses aggregate dividends $D(z, \mu_H, \mu_F)$ and its portfolio of future shareholding in production firms $\{s'_k\}$ to maximize its net market value. It buys shares $\{s'_k\}$ in each firm indexed by their capital level k at price $p(k; \mathbb{Z})$ and collects dividends $d(k; \mathbb{Z})$. The goal of the intermediary is to maximize net market value, with payoffs in future states valued by the price vector χ, which the intermediary takes as given. As in Makowski (1983), maximizing net market value expands the budget constraint of all households who held shares at the start of the period and is therefore unanimously supported.[4]

The financial aggregator's recursive problem is written as:

$$J(\{s_k\}; \mathbb{Z}) = \max_{\{s'_k\}} D + \sum_{n=1}^{N_z} \chi(z_n|\mathbb{Z}) J(\{s'\}_{\mathcal{K}}; z_n, \mu_H, \mu_F) \tag{4}$$

$$\text{s.t.} \quad D \leq \int_{\mathcal{K}} \left((p_k + d_k) s_k - p_k s'_k \right) dk \tag{5}$$

$$\mu'_F = \Gamma_F(\mathbb{Z}), \qquad \mu'_H = \Gamma_H(\mathbb{Z})$$

where $\chi(z_n|\mathbb{Z})$ is some price for value paid in future state $\{z_n, \mu'_H, \mu'_F\}$ conditional on the current aggregate state. This discount factor is taken as given by the financial aggregator. In Section 3.3, I describe the equilibrium properties of this discount factor.

2.3 Production Firms

A unit measure of production firms produce a homogeneous output using labor n and start of period capital stock k. They produce using a strictly increasing and concave production function $y = zF(k, n)$. The variable z is the common exogenous stochastic TFP level which was described in the household section.

A firm enters each period with its predetermined stock of capital, $k \in \mathbf{K} \subset \mathbf{R}_{++}$. The goal of each production firm is to maximize dividends plus discounted future value, with

[4] I need to flesh this out a bit.

payoffs in future states valued by the price vector $\tilde{\chi}$. The firm chooses labor to maximize period profits, then selects future capital and current dividends. A portion of the firm's capital stock δ depreciates each period. The firm pays a convex adjustment cost $I(k', k)$ that depends on both its current and future capital levels.

The firm takes the price vector $\tilde{\chi}$ as given. One important distinction here is that the discount factor used by the production firm $\tilde{\chi}$ is not assumed to be the same as the financial intermediary's discount factor χ. The mutual fund values future payoffs at χ while the production firms value future payoffs at $\tilde{\chi}$. These will be the same in equilibrium, but not because it was imposed as a modeling assumption.

As before, I use a shorthand for the aggregate state $\mathbb{Z} \equiv (z_m, \mu_H, \mu_F)$. The production firm's problem can be written recursively as:

$$G(k; z_m, \mu_H, \mu_F) = \max_{n,k'} d + \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) G(k'; z_n, \mu'_H, \mu'_F) \tag{6}$$

$$\text{s.t.} \quad d + k' + I(k', k) \leq z_m F(k, n) - w(\mathbb{Z})n + (1-\delta)k \tag{7}$$

$$\mu'_F = \Gamma_F(\mathbb{Z}), \qquad \mu'_H = \Gamma_H(\mathbb{Z})$$

where $\tilde{\chi}(z_n|\mathbb{Z})$ is the aggregate valuation for dividends paid in future state $\{z_n, \mu'_H, \mu'_F\}$ conditional on the current aggregate state. This discount factor is taken as given by the production firm and is described in detail in section 3.3.

Let $k(k;\mathbb{Z})$ and $d(k;\mathbb{Z})$ be the decision rules for future capital and dividends of a firm with current capital k and aggregate state $\mathbb{Z}$.

2.4 Private Equity

Finally, I introduce a private equity firm (or management firm) to discipline the choices of the production sector. The management firm is inspired by Grossman and Hart (1979) with some meaningful modifications. The management firm has access to a technology that allows it to implement an alternate production plan if and only if it can secure support from the firm's owners, the mutual fund.[5] If the private equity firm cannot find a strictly profitable deviation for any production firm, the management firm does not act in the period.

To find a profitable deviation, the manager first proposes some alternate investment plan $\hat{k}'$. The manager is only able to alter the plans of a single firm, so it will not shift aggregate prices. The private equity fund then chooses a side payment $\xi(\hat{k})$ to the mutual

[5] I consider a case where the private equity firm instead directly compensates marginal shareholders in Appendix B

fund to compensate it for changing the value of its portfolio. If the portfolio becomes more valuable after the change ($\xi(\hat{k}) < 0$), the mutual fund would pay the private equity firm to implement the change. If the portfolio becomes less valuable ($\xi(\hat{k}') > 0$), the private equity firm would have to compensate the mutual fund. If the private equity firm cannot find a strictly profitable deviation for any production firm, the private equity firm does not act in the period.

The manager's problem is written as:

$$\max_{\xi(\hat{k}'),\hat{k}'} \; -\xi(\hat{k}') \tag{8}$$

$$\text{s.t.} \quad J(\{s_k\};\mathbb{Z}) < \hat{J}(\{s_k\};\mathbb{Z}|\hat{k}') \tag{9}$$

where hat variable ($\hat{J}$) denotes the change in the mutual fund's value conditional on one of the production firms changing its capital investment to $\hat{k}'$ and after receiving the side payment $\xi(\hat{k}')$. Equation 9 is the mutual fund's participation constraint. It will vote unanimously in favor of the new plan only if it is made better off than it was before the deviation.

When the management firm chooses an alternate scheme $\hat{k}'$, it changes prices and dividend payments today and in the future. To make this problem tractable, I first assume that the firm takes over a tiny, identical mass of firms with size ε. I then take $\varepsilon \to 0$ to find the atomistic limit. I write the new level of dividends as $\hat{D} \equiv (1-\varepsilon)D + \varepsilon\hat{d}$ and the new price level as $\hat{P} \equiv (1-\varepsilon)P + \varepsilon\hat{p}$. There is no need to include changes in shareholding because the mutual fund holds all shares and, in equilibrium, it will continue to hold shares at the new price level.

2.5 Discussion of Assumptions

Before continuing the analysis, I would like to pause briefly to discuss the rationality of the modeling choices presented above. Specifically, I want to discuss the mutual fund and the private equity firm.

2.5.1 Benefits of a Mutual Fund

Is it reasonable to impose a financial intermediary? On the one hand, retirement fund managers typically suggest shifting a portfolio toward bonds while approaching retirement age. On the other hand, Warren Buffett famously (Perry, 2017) wagered in 2008 that a low-cost S&P500 index fund would outperform a portfolio of actively managed hedge funds. With infinitely lived agents, it might be reasonable to ignore the popular bond portfolio advice in favor of letting shareholders only care about wealth maximization. However, limiting the number of investment channels reduces each household's opportunity to insure against exogenous shocks.

Is it reasonable to expect that a private equity firm will support net market value maximization by the firms it owns? As discussed in the introduction, shareholder challenges tend to be more successful when they target low market value firms.

The mutual fund plays three key roles in this economy. It prevents production firms from becoming financial innovators, it simplifies the problem of price discovery, and it makes this problem tractable.

First and most importantly, it prevents atomistic production firms from becoming financial innovators. DeAngelo (1981), Makowski (1983), and Krouse (1985) argue that shareholders will be unanimous in supporting the firm's decision to maximize net market value if firms are sufficiently small. That is, shareholders have to believe that a firm's deviation will not change the set of available prices or future outcomes. By imposing a financial intermediary between the household and the production sector, a production firm will not be able to change the available choice set for households when it produces differently than its peers.

Second, it simplifies the potential problem of price discovery. It could be difficult for every production firm to ask its shareholders how they value payoffs across time, aggregate those answers, and predict future shareholding. A financial intermediary sector could much more realistically study markets and make prices available to production firms.[6] In a full information model, this isn't particularly necessary, but it is a helpful feature for future work.

Finally, the financial aggregator reduces the size of the problem to something tractable. As written, the model now features a distribution of households over shareholding and productivity and a distribution of firms over capital. If shareholders were allowed to own individual production firms, shareholding would become a portfolio choice and the distribution of households would increase by the number of firms in the economy. Firms would also need to know who specifically holds their shares and how their decisions impacts those specific households, including the portfolio balancing effects of a change in capital.

This last portion also has important implications for equilibrium. Imagine a setting where households directly invested in firms. Two production firms might start the period

[6] An example of this behavior comes from Investor Relations departments at large firms. These groups frequently interact with institutional investors, who might provide feedback about how a proposed capital investment plan will change share prices.

identically, but for some reason attract different types of shareholders. If one firm attracts poor shareholders, it will likely invest less than the firm that attracts rich shareholders. If they are ex-ante identical, which firm will attract which type of shareholders? The answer isn't immediately clear. Rather than spending effort to track shareholder-to-firm combinations, it is much more straightforward to impose a financial intermediary.

This is not to say the mutual fund is an assumption without flaws. Clearly, households do differentiate their portfolios for a variety of reasons. There is also evidence from Brockman et al. (2022) that firm risk earns a return premium over the market rate. This would not be the case with a single mutual fund holding all production firms. Private or closely-held corporate firms may also be inconsistent with the mutual fund presented here. However, these downsides leave room for future research and are not enough to fully discredit the value of an equity intermediary.

2.5.2 Private Equity Firms

The problem of the private equity firm is modeled from observed shareholder proxy battles. Fos (2017) documents that a majority of shareholder challenges state their goal as increasing market value. Further, shareholder challenges that target market value tend to be more likely to succeed.

This modeling choice is the most important one in this paper because it ultimately pins down the objective of the production firm. Other authors approach this question differently. Grossman and Hart (1979) and Carceles-Poveda (2009) assume that the production firm tries to maximize start-of-period shareholder value and that those shareholders have control over the firm. These authors also consider cases where the date-zero shareholders control the firm. These assumptions each lead to slightly different capital choices by production firms (as documented in Carceles-Poveda (2009)). A micro-founded private equity firm eliminates the need for assumptions about firm control.

One weakness of this modeling choice is its assumption that all production firms are worried about a shareholder challenge. Closely-held private firms, entrepreneur firms, partnerships, sole proprietorships, nonprofits, and corporate firms with large blocks of insider shareholding[7] are all types of firms who might reasonably ignore a shareholder challenge. The research on the distribution of wealth (De Nardi and Fella, 2017) documents that entrepreneurs can generate higher rates of return than the market. Because corporate firms are responsible for nearly two thirds of aggregate investment, adding

[7] Berkshire Hathaway is an example of such a firm. 43.3% of Berkshire Hathaway stock is held by directors and executive officers of the company as of March 2, 2022. https://www.berkshirehathaway.com/meet01/2022proxy.pdf

these types of firms to the model is best left as an exercise for future work.

3 Equilibrium Definition and Properties of the Discount Factor

In this section, I describe the conditions for a recursive competitive stock market equilibrium. Then, I discuss properties of the equilibrium stochastic discount factor.

3.1 Recursive Competitive Stock Market Equilibrium

A Recursive Competitive Stock Market Equilibrium is a set of functions,

$$\{w, G, \chi, \tilde{\chi}, d, n, k, p, d, J, P, D, V, s, c\}$$

that jointly solve the household, firm, and financial aggregator's problems, and clear the markets for goods, labor, production firm shares, and aggregate equity, as described by the following:

i. V solves Eq. 1 with policy functions $\{c, a\}$

ii. J solves Eq. 4 with policy function D

iii. G solves Eq. 6 with policy functions $\{d, n, k\}$

iv. The market for aggregate equity clears in each date and state:

$$1 = \int_{\mathcal{S}} a(a, \eta, \mathbb{Z}) \mu(d[a \times \eta])$$

v. The intermediary holds all shares of the production firms $1 = s'_k \ \forall \ k \in \mathcal{K}$

vi. The labor market clears:

$$\int_{\mathcal{K}} n(k; \mathbb{Z}) \mu(dk) = \int_{\mathcal{S}} \eta \mu(d[a \times \eta])$$

vii. The goods market clears:

$$\int_{\mathcal{K}} \left(k(k; \mathbb{Z}) - (1-\delta)k\right) \mu(dk) + \int_{\mathcal{S}} c(a, \eta; \mathbb{Z}) \mu(d[a \times \eta]) = \int_{\mathcal{K}} zF(k, n(k; \mathbb{Z})) \mu(dk)$$

viii. Γ_H is defined by:

$$\mu'(A, \eta_j) = \int_{\{a,\eta_i|(a(a,\eta_i;\mathbb{Z}))\in A\}} \pi_{ij}\mu(d[a \times \eta_i]) \quad \forall\, (A, \eta_j) \in \mathcal{S}$$

ix. Γ_H is defined by:

$$\mu'(k) = \int_{\{k|(k(k;\mathbb{Z}))\in K\}} \mu(dk) \quad \forall\, k \in \mathcal{K}$$

x. The management firm cannot find a profitable deviation for any production firm, which is satisfied when the maximum of Equation 8 is zero.

While not explicitly listed, χ and $\tilde{\chi}$ are determined by the conditions above. The production firm's discount factor $\tilde{\chi}$ is determined by the last equilibrium condition and will be derived in further detail in Section 3.3. The financial intermediary's discount factor χ is determined by market clearing for aggregate equity.

It is worth noting here that χ and $\tilde{\chi}$ are not unique given all other prices and allocations if $N_z \geq 3$. However, the discount factors I derive in Section 3.3 pin down a family of discount factors that jointly lead to the same prices and allocations. I describe the affine transformations of the discount factors in Appendix A.

3.2 Optimal Choices

I begin by describing the conditions that pin down optimal choices for each type of agent. With those optimal choices in mind, I will construct a pair of discount factors that are consistent with both clearing the aggregate equity market (χ) and surviving a proxy battle ($\tilde{\chi}$).

Each household's optimal choice of a' satisfies:

$$Pu'(c) = \beta\mathbb{E}_{\eta',z'}(P' + D')u'(c') + \lambda_a \tag{10}$$

where $u'(c)$ is the marginal utility of consumption in the current period and $\mathbb{E}$ reflects the expectation of transitioning over both idiosyncratic state η' and aggregate productivity state z'. Future outcomes P', D', c' are each optimal choices of each type of agent in each realized future state. The λ_a term reflects the fact that some households may want to save less than is allowed by the minimum savings constraint described in Equation 3. This term will be equal to zero for households who choose $a' > \underline{a}$.

The financial intermediary chooses shareholding of each production firm, which has optimality conditions:

$$p(k;\mathbb{Z}) = \sum_{n=1}^{N_z} \chi(z_n|\mathbb{Z})[p'(k(k;\mathbb{Z});z_n,\mu'_H,\mu'_F) + d'(k(k;\mathbb{Z});z_n,\mu'_H,\mu'_F)] \quad \forall\, k \in \mathcal{K} \tag{11}$$

The price the aggregator is willing to pay for a firm with capital level k and a vector of future payoffs $p',\ d'$ depends on the future price and dividend it can expect to receive, weighed by some discounting χ for each future state. Because the financial intermediary is not bound by a short sales constraint, the *law of one price* will hold.[8] The aggregator will price future payoffs at the same rate for all of the production firms that it owns.

Each production firm's optimal choices are given by:

$$1 + \frac{\partial I(k',k)}{\partial k'} = \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) D_1 G(k';z_n,\mu'_H,\mu'_F) \tag{12}$$

Because each firm is atomistic, it takes as given future prices and its shareholders' valuation of payoffs in future states. Because the firm is owned by the mutual fund, the firm cannot become a financial innovator. This is similar to the case in Makowski (1983) but with a different mechanism. The firm only weighs the lower dividends against the change in future valued payments. In equilibrium, the distribution of firms will be degenerate with all of the mass at a single capital level.

Finally, I describe the choices of the management firm, which will discipline the production firms' discount factor $\tilde{\chi}$. When deriving the manager's optimal choices, I assume it takes over a mass of identical production firms of size ε. This is assumed to be small enough that the manager does not have market power to change other firms' behavior in the current or future date. Then I take the limit as $\varepsilon \to 0$ to find the specific results at the atomistic firm limit.

The manager chooses a capital deviation $\hat{k}'$ for a mass ε of production firms with current capital level k_i. A capital deviation changes the firm's value to the mutual fund across three channels. It changes dividends d, equity price p, and the vector of future returns $\{p' + d'\}$.

These production firm changes pass through directly to the mutual fund's balance sheet. The mutual fund will not change its shareholding level. Rather, the price the mutual fund is willing to pay changes as described in Equation 11, depending on the vector of future returns $\hat{p}'$ & $\hat{d}'$.

[8] The law of one price is common in the finance literature. See Chapter 4 of Campbell (2018).

The firm chooses $\hat{k}'$ and side payment $\xi(\hat{k}')$ satisfying:

$$[\xi(\hat{k}')]: \qquad 1 = \Omega$$

$$[\hat{k}']: \qquad \varepsilon\Omega\frac{\partial p}{\partial \hat{k}'} = \varepsilon\Omega\left(\frac{\partial p + \partial d}{\partial \hat{k}'} + \sum_{n=1}^{N_z}\chi(z_n|\mathbb{Z})\frac{\partial p' + \partial d'}{\partial \hat{k}'} + \underbrace{o(\varepsilon)}_{\to 0}\right) \tag{13}$$

where Ω is the multiplier on the participation constraint described in Equation 9 and ε are the mass of firms controlled. The term $o(\varepsilon)$ accounts for changes to the firm's value through channels other than direct price and dividend, like the change on other firms' value in the next period if some mass ε of competitors behaved differently than expected. This term goes to zero in the atomistic limit. Rearranging yields:

$$-\frac{\partial d}{\partial \hat{k}'} = \sum_{n=1}^{N_z}\chi(z_n|\mathbb{Z})\frac{\partial p' + \partial d'}{\partial \hat{k}'} \tag{14}$$

The capital choice of the private equity firm $\hat{k}'$ determined by Equation 14 is the same as the capital choice of the production firm k' determined by Equation 12 if the production firm's value is equal to its net market value: $G(k;\mathbb{Z}) = p(k;\mathbb{Z}) + d(k;\mathbb{Z})$.

Lemma 1. *If the production firm discounts future states with the mutual fund's discount factor χ, its choice of k' is the same as the management firm's optimal deviation $\hat{k}'$ and will therefore survive a proxy challenge.*

The proof is by construction. Suppose $G(k;\mathbb{Z}) = p(k;\mathbb{Z}) + d(k;\mathbb{Z})$. I rewrite the Benveniste-Scheinkman condition and the production firm's change in dividends with respect to capital as:

$$D_1G(k;\mathbb{Z}) = \frac{\partial p + \partial d}{\partial k} \tag{15}$$

$$\frac{\partial d}{\partial k'} = -(1 + \frac{\partial I(k',k)}{\partial k'}) \tag{16}$$

I substitute these expressions back into the production firm's optimal choices and combine with the management firm's choices, which yields:

$$1 + \frac{\partial I(k', k)}{\partial k'} = \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) D_1 G(k'; z_n, \mu_H', \mu_F') \qquad \text{(12 repeated)}$$

$$-\frac{\partial d}{\partial k'} = \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) \frac{\partial p' + \partial d'}{\partial k'} \qquad \text{(substituting 15 \& 16)}$$

$$\sum_{n=1}^{N_z} \chi(z_n|\mathbb{Z}) \frac{\partial p' + \partial d'}{\partial \hat{k}'} = \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) \frac{\partial p' + \partial d'}{\partial k'} \qquad \text{(substituting 14)}$$

The production firm survives a proxy challenge if its discount factor $\tilde{\chi}$ is the same as the financial intermediary's discount factor χ. This expression can also be derived by directly compensating marginal shareholders. I discuss that alternate setting (with identical results) in Appendix B.

3.3 Analytic Form of the Intermediary's Discount Factor

I construct an aggregate discount factor from each household's optimality conditions as described in Equation 10. To simplify notation, I suppress the current aggregate state $\mathbb{Z} \equiv \{z_m, \mu_H, \mu_F\}$ and the transition of future distributions. I first rewrite each household's Euler equation for shareholding in terms of the aggregate equity price. I then multiply through by future shareholding choice a' (Equation 17) and aggregate over all households (18):

$$Pu'(c) = \beta \sum_{n=1}^{N_z} \pi_{mn}(P_n' + D_n') \sum_{j=1}^{N_\eta} \pi_{ij} u'(c_{jn}') + \lambda_a \qquad \text{(10 rewritten)}$$

$$P = \frac{\beta \sum_{n=1}^{N_z} \pi_{mn}(P_n' + D_n') \sum_{j=1}^{N_\eta} \pi_{ij} u'(c_{jn}')}{u'(c)} + \frac{\lambda_a}{u'(c)}$$

$$Pa' = \left(\frac{\beta \sum_{n=1}^{N_z} \pi_{mn}(P_n' + D_n') \sum_{j=1}^{N_\eta} \pi_{ij} u'(c_{jn}')}{u'(c)} \right) a' + \frac{\lambda_a}{u'(c)} a' \qquad (17)$$

$$\int_{\mathcal{S}} Pa' \mu(d[a \times \eta]) = \int_{\mathcal{S}} \left(\left(\frac{\beta \sum_{n=1}^{N_z} \pi_{mn}(P_n' + D_n') \sum_{j=1}^{N_\eta} \pi_{ij} u'(c_{jn}')}{u'(c)} \right) a' + \frac{\lambda_a}{u'(c)} a' \right) \mu(d[a \times \eta]) \qquad (18)$$

where c_{jn} is a shorthand for the household's consumption rule when it transitions to idiosyncratic productivity level $\eta' = \eta_j$, aggregate TFP transitions to level $z' = z_n$, and the shareholding in the next period is the solution to the household's maximization problem: $c_{jn} \equiv c(a(a, \eta; \mathbb{Z}), \eta_j; z'_n, \mu'_H, \mu'_F)$.

When the stock market clears in equilibrium ($1 = \int_{\mathcal{S}} a(a, \eta, \mathbb{Z})\mu(d[a \times \eta])$), the left hand side of Equation 18 is equal to the aggregate equity price P. I can express aggregate equity price as a function of weighted marginal rates of substitution and future payoffs:

$$P = \sum_{n=1}^{N_z} (P'_n + D'_n) \left(\beta \pi_{mn} \int_{\mathcal{S}} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} a' \mu(d[a \times \eta]) \right) + \int_{\mathcal{S}} \frac{\lambda_a}{u'(c)} a' \mu(d[a \times \eta]) \quad (19)$$

A stochastic discount factor directly values an asset conditional on a vector of future returns without an additive wedge. Therefore, I want to eliminate Lagrange multiplier λ_a in Equation 19. I start by rearranging the household's optimal choice for a' to be in terms of $\lambda_a / u'(c)$, which will allow me to simplify equation 19.

$$\frac{\lambda_a}{u'(c)} = P - \frac{\beta \sum_{n=1}^{N_z} \pi_{mn} (P'_n + D'_n) \sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} \quad \text{(10 rearranged)}$$

For households who choose $a' > \underline{a}$, the expression above is equal to zero. I introduce an indicator function that will let me separate out households who are at the savings limit against those who save more than the minimum.

$$\mathbb{I} = \begin{cases} 1 & a' > \underline{a} \\ 0 & a' = \underline{a} \end{cases}$$

With this indicator I separate out Equation 19 into households who are and aren't bound by the savings condition.

$$\begin{aligned} P = &\sum_{n=1}^{N_z} (P'_n + D'_n) \left(\beta \pi_{mn} \int_{\mathcal{S}} \mathbb{I} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} a' \mu(d[a \times \eta]) \right) + \\ &\sum_{n=1}^{N_z} (P'_n + D'_n) \left(\beta \pi_{mn} \int_{\mathcal{S}} (1 - \mathbb{I}) \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} \underline{a} \mu(d[a \times \eta]) \right) + \\ &\int_{\mathcal{S}} (1 - \mathbb{I}) \left(P - \frac{\beta \sum_{n=1}^{N_z} \pi_{mn} (P'_n + D'_n) \sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} \right) \underline{a} \mu(d[a \times \eta]) \end{aligned}$$

With some additional algebra, this leads to the equation:

$$P = \sum_{n=1}^{N_z} (P'_n + D'_n) \frac{\left(\beta \pi_{mn} \int_{\mathcal{S}} \mathbb{I} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} a' \mu(d[a \times \eta]) \right)}{\int_{\mathcal{S}} \mathbb{I} a' \mu(d[a \times \eta])} \tag{20}$$

This equation says that the value of aggregate equity is determined by discounted future payoffs. The discounting comes from shareholders' expected marginal rates of substitution across aggregate states, weighted by their end of period shareholding. I define this discount factor as:

$$\chi(z_n | \mathbb{Z}) \equiv \beta \pi_{mn} \frac{\int_{\mathcal{S}} \mathbb{I} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} a' \mu(d[a \times \eta])}{\int_{\mathcal{S}} \mathbb{I} a' \mu(d[a \times \eta])} \tag{21}$$

Before moving on, I would like to discuss a special case of the discount factor if $\underline{a} = 0$. In that case, the discount factor simplifies to:

$$\chi(z_n | \mathbb{Z}) = \pi_{mn} \beta \int_{\mathcal{S}} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} a' \mu(d[a \times \eta]) \tag{22}$$

This form of the discount factor is identical to the one proposed in Drèze (1974*a*). However, the discount factor in that paper is assumed rather than being derived as an equilibrium expression. Additionally, the discount factor proposed by Drèze (1974*a*) cannot handle cases where the minimum savings constraint is binding while the discount factor described in Equation 21 can.

3.4 Analysis

As discussed in Lemma 1, the production firm will survive a shareholder challenge if its discount factor is the same as the mutual fund's. Therefore, I will simplify notation in the remainder of this paper and use χ to describe the aggregate stochastic discount factor.

In the sections below, I discuss the equilibrium discount factor. I describe properties of the discount factor, describe how it compares to discount factors proposed in the literature, and briefly discuss uniqueness and unanimity with the discount factor above.

3.4.1 Properties of the Discount Factor

The discount factor described in Equation 21 features a number of useful properties. I discuss below how it nests a number of standard models, including the representative household case, the exogenous no-trade case, constant returns to production environments, and the Makowski (1983) Criterion. Additionally, this discount factor results in net market value maximization, which is consistent with observed firm behavior.

First, it neatly nests the representative household discount factor. With a representative household, the future shareholding choice is always $a' = 1$. And because there is no idiosyncratic risk, the distribution is degenerate. That discount factor can be written as:

$$\chi_{\text{rep}}(z_n|\mathbb{Z}) = \pi_{mn}\beta\frac{u'(c'_n)}{u'(c)}$$

which is a special case of the discount factor I derive in Equation 21. This discount factor is standard in the literature featuring dynamic firms, such as Khan and Thomas (2013).

Another useful feature is that this discount factor nests the exogenous no-trade approach proposed by Krusell, Mukoyama and Smith (2011). In their model, the minimum savings rule for equity is $\underline{a} = 1$, which requires all households to save the median number of shares. In that setting, only a single shareholder (or type of shareholders) would not want to choose $a' < 1$, meaning $\mathbb{I} = 0$ for all shareholders except one. Their discount factor is then:

$$\chi_{\text{KMS}}(z_n|\mathbb{Z}) = \pi_{mn}\beta\frac{\sum_{j=1}^{N_\eta}\pi_{ij}u'(c'_{jn})}{u'(c)} \qquad \text{for } \eta_i = \max \mathbf{N} \tag{23}$$

If there is only a single household (or single type of household) who would not want to choose $a' < \underline{a}$, then my model exactly replicates this result. There is only one household for whom the indicator function $\mathbb{I}$ in Equation 21 is nonzero, so that household's marginal rate of substitution is the stochastic discount factor.

In a constant returns environment without adjustment costs, the price of a production firm p is equal to its next-period capital stock k'. This is a standard result in the literature, as in Carceles-Poveda (2009).

The proposed discount factor also meets a criterion as set out by Makowski (1983). That criterion requires the discount factor used by the firm satisfy $P = \max[\sum_n SDF_{a,\eta_i}(P'_n + D'_n)]$, where SDF_{a,η_i} is the stocahastic discount factor across aggregate states for the household indexed by $\{a, \eta_i\}$. However, every household that chooses $a' > \underline{a}$ will satisfy this condition as shown by the optimality conditions in Equation 10.[9]

One downside of the Murkowski criterion is that it does not identify a unique discount factor. Every household that chooses to hold more than the minimum level of assets will satisfy the criterion, and each of these discount factors may lead to the firm making a slightly different choice in capital. Because my discount factor uses information from all shareholders in equilibrium, it is distinct.

Another useful property of this discount factor is that it implicitly maximizes value weighed by current shareholding. As in DeAngelo (1981), maximizing the firm's net value expands each household's budget constraint by the size of their current shareholdings. This wealth effect allows households to choose more consumption or savings. The price effect and the change to future payoffs can be ignored because the shareholder can freely adjust their future shareholding choice.

3.4.2 My Approach Relative to Alternatives

The two most common alternate discount factors stem from Grossman and Hart (1979) and Drèze (1974*b*) (or Carceles-Poveda (2009)). For each, they set the discount factor to:

$$\chi(z_n|\mathbb{Z}) = \pi_{mn}\beta \int_{\mathcal{S}} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} \mathbf{W}_{a,\eta} \mu(d[a \times \eta]) \tag{24}$$

where $\mathbf{W}_{a,\eta}$ is the model-specific weighting for shareholder indexed by $\{a, \eta\}$. Grossman and Hart (1979) assume $\mathbf{W}_{a,\eta} = a_{0a,\eta}$, which corresponds with the shareholding at the start of time. Carceles-Poveda (2009) instead use $\mathbf{W}_{a,\eta} = a_{a,\eta}$, which corresponds with shareholding at the start of each period. In comparison, my model uses $\mathbf{W}_{a,\eta} = a'_{a,\eta}$ in the special case $\underline{a} = 0$, as shown in Equation 22.

How does my model differ from these expressions? First, I explicitly allow for cases where $\underline{a} \neq 0$ by only considering households who choose to be shareholders. Second, the discount factor in my model is built up explicitly from equilibrium conditions. The alternate methods simply assume that the firm weighs future payoffs by these weighting factors.

I discuss the numerical difference between my model and the one proposed by Grossman and Hart (1979) in Section 5.4. However, a firm that discounts using the Grossman discounting will not survive a proxy challenge in my model. Their production firms do not maximize net market value, so they would be taken over by the private equity firm.

[9]Households who choose $a' = \underline{a}$ will have $\lambda_a > 0$, which means only their SDF will not satisfy the Murkowski criterion.

3.4.3 Uniqueness of the Discount Factor

While my approach gives a specific form for the discount factor, there are trivially many transformations of the discount factor that are consistent with the defined equilibrium. The discount factor only needs to satisfy two conditions in equilibrium. First, the equity market needs to clear, which happens when $p(k;\mathbb{Z}) = \sum_{n=1}^{N_z} \chi(z_n|\mathbb{Z})[p'(k(k;\mathbb{Z}); z_n, \mu'_H, \mu'_F) + d'(k(k;\mathbb{Z}); z_n, \mu'_H, \mu'_F)]$. Second, the firm's choice of capital k' must solve the firm's problem: $G(k; z_m, \mu_H, \mu_F) = \max_{n,k'} d + \sum_{n=1}^{N_z} \chi(z_n|\mathbb{Z}) G(k'; z_n, \mu'_H, \mu'_F)$. I've already shown that $\chi(z_n|\mathbb{Z})$ satisfies these conditions, but there are trivially many transformations that also satisfy these conditions as long as there are more than two aggregate states ($N_z > 2$).

Lemma 2. *Let $\bar{k}'$ be an allocation that solves the firm's problem G given the aggregate state $\mathbb{Z}$ and the vector of discount factors $\chi(\mathbb{Z})$ defined in Equation 21. Any vector $\bar{\chi}$ can also a valid discount factor if $\bar{k}'$ is still a solution to the firm's problem and the firm still sells for a price of p.*

The proof is provided in more details in Appendix A.

3.5 Unanimity

A common question in this literature is around the concept of unanimity. In a setting that requires unanimity, the goal is for all shareholders to want the firm to pursue exactly the same capital investment plan. Carceles-Poveda (2009) shows that this is the case in models with constant returns and no short selling. However, DeAngelo (1981) and Makowski (1983) instead propose that ex-ante shareholders unanimously prefer for the firm to engage in net value (or cum-dividend price) maximization. The best a firm can do for its shareholders is to maximally expand their budget sets, which is done by maximizing the firm's net value.

In my model, households are unanimous in their preference for firm value maximization, as is the case in DeAngelo (1981). They are not unanimous, however, in the firm's exact choice of future capital. Some households would be better off if the firm increased its capital choice and vice-versa. However, a lack of unanimity about specific capital plans is not a shortcoming of the model. In observed equity markets, shareholder votes about capital choices (or firm management in general) are relatively rare. Fos (2017) documents that the most frequent cause for shareholder challenges are poor stock performance, while proxy challenges that target capital structure tend to be less successful. The empirical data support ignoring unanimity in capital choice as long as firms are maximizing their net market value.

4 Algorithm

The solution algorithm utilizes a modified version of the backward induction developed by Reiter (2009). While I still use a proxy distribution to find consistent behavior, I also require the representative household to behave consistently with perceived aggregates. Details are described below.

I begin by discretizing the aggregate state z into 7 states using the Tauchen algorithm. I similarly discretize idisoyncratic productivity levels η into 7 states. I then need to choose a proxy aggregate state. I use the total level of log capital, which serves as a good measure of aggregate total wealth. I linearly space log capital M into 9 grid points. Finally, I discretize the choice values for a' and k' on a grid with 99 points for shareholding and 299 points for capital.

I begin with a guess for the proxy distribution of households over shareholding and productivity at each point on the aggregate grid (z, M). A naive guess of the distribution could be to assume that all households start with shareholding $a = 1$ and productivity is distributed at the steady state level.

I also start with a guess for the aggregate law of motion, equity prices, and dividends at each state. I guess that $K' = K$, which also pins down aggregate dividends. The guess of share price is trivial, but I start by assuming $P = \beta D/(1-\beta)$. This guess of the price is consistent with an asset priced in a riskless Lucas economy. Finally, I also guess starting levels for the firm's value G, the household's value V, and the household's period marginal utility of consumption MUC.

In each iteration o, the algorithm proceeds as follows:

1 **Outer Loop:** In each aggregate state indexed by (z, M):

   (a) **Solve LOM:** Guess a future aggregate state M'_g, which implies dividends D_g

      i. **Clear the Equity Market:** Guess a price for aggregate equity P.

      ii. Solve each household's optimal choice of a' given the previous V^o, law of motion M'_g, dividends D_g, and future prices and dividends P^o & D^o.

      iii. Measure total shareholding $A(P) = \int_S a' \mu(d[a \times \eta])$.

         A. If $A(P) - 1$ >precision, there is too much demand for shares, so the price needs to rise. Return to step 1(a)i

         B. If $1 - A(P)$ >precision, there is insufficient demand for shares, so the price is too high and needs to fall. Return to step 1(a)i.

      iv. Once the share price has cleared the equity market, measure the implied stochastic discount factor (Equation 21) using the MUC^o array.

v. **Consistency with aggregates:** Solve the firm's problem given the discount factor and firm values G^o. This yields k'.

A. If $k' - K'$ >precision, the guess of K' was too low. Guess a higher K' and return to step 1a

B. If $K' - k'$ >precision, the guess of K' was too high. Guess a lower K' and return to step 1a

(b) Once capital choice is consistent, update the guess of $P^{o+1}, D^{o+1}, K'^{o+1}$.

2 With a consistent guess of aggregates, solve the household's and the firm's problem conditional on the updated guesses of $P^{o+1}, D^{o+1}, K'^{o+1}$

(a) $V^{o+1} = u(c) + \beta V^o$

(b) $MUC^{o+1} = u'(c)$

(c) $G^{o+1} = d + \beta G^o$

3 If the norm is sufficiently small $|V^{o+1} - V^o|$ <precision, proceed to the simulation. Otherwise, return to step 1.

In the algorithm above, I search for price via bisection. Conditional on a guess of M', demand for shares is weakly decreasing in price, so there is a single price that clears the equity market. I solve the household's problem and with the endogenous grid method and I solve the firm's problem via golden section search.

Once I've solved the value functions and aggregate laws of motion with the proxy distribution, I simulate the economy to calculate a new reference distribution. In the simulation, I draw a TFP shock on the grid, then solve it at the previously determined level of aggregate capital using the process described in step 1 above. However, instead of using the proxy distribution, I use the distribution of households from the previous simulation step. I run this simulation for 750 periods to "pre-heat" before tracking the distribution of households in each date and with each realization of shocks on the z grid. I simulate 750 periods, then update the reference distribution as described in Reiter (2009). After updating the reference distribution, I solve the household's value function again. I typically run this process four times in total, though results generally don't change after the second update of the reference distribution.

5 Business Cycle Moments and Impulse Responses

I now apply my method to a simple business cycle example to see how aggregate behavior varies with idiosyncratic risk. To begin, I specify explicit forms for the utility and production functions. Households value consumption with CRRA utility of the form:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$$

where σ is the relative risk aversion parameter. I assume a Cobb-Douglas production function: $F(k,n) = zk^{\alpha}n^{\gamma}$ with $\alpha + \gamma \in (0,1]$. I further assume a quadratic adjustment cost function: $I(k',k) = \frac{\psi}{2k}(k'-k)^2$. If $\psi = 0$, this nests a model without adjustment costs.

5.1 Model Parameters

Table 1 summarizes the model parameters that are constant across iterations. All results are from a parameterized model which captures rough trends.

σ	δ	β	α	γ	σ_z	ρ_z	$\underline{a}$	ψ
3.0	0.070	0.96	0.30	0.55	0.0143	0.7337	0	1.0

Table 1: Summary of economic parameters. σ is the coefficient of relative risk aversion. δ is depreciation. β is the household's subjective discount factor. α is capital's share of production and γ is labor's share. σ_z is standard deviation of TFP and ρ_z is the persistence of the TFP shock. $\underline{a}$ is the borrowing constraint, with 0 indicating no short sales are allowed. ψ is the scale of the quadratic adjustment cost.

The period of the model is annual. The coefficient of relative risk aversion σ is set to 3.0, which is on the high end of levels used in the real business cycle literature. Depreciation is set to 7%, which approximately matches the investment to capital ratio in the United States in the post-war period. The subjective discount factor β is set to 0.96, which would imply a real interest rate of 4% in a riskless representative agent economy. The adjustment parameter ψ is in the range of values in the literature, which typically vary from 0.15-150.

Labor's share of production γ is estimated from NIPA tables as total payments to labor divided by GDP, which is approximately 55%. For capital share and the TFP process, I use the estimates from Fernald (2012). Capital's share of output is in the range of 30-35%. I choose α at the low end of this range at 0.3, which implies a total return to scale of 0.85. I estimate TFP's AR(1) process with a standard deviation of 1.43% and a persistence of 73.37%. This level of persistence is a bit lower than the literature, but it leads to a standard deviation of TFP of 2.1%, which is consistent with the literature.

For idiosyncratic risk, I consider three cases. I start with a baseline model with zero idiosyncratic risk, which will represent the representative agent case. For the cases with

Idiosyncratic Risk

Setting	σ_η	Variance of Log Earnings	Observed Variance of Log Earnings
Rep HH	0.0	0.0	-
1970	0.1	0.19	0.25
2010	0.1414	0.37	0.45

Table 2: Comparison of parameter values across settings. σ_η is the standard deviation of shocks to the persistent earnings process. The persistence ρ_η is 0.973 from Heathcote, Storesletten and Violante (2010).

idiosyncratic risk, I consider two special cases. For both, I use estimates from Heathcote, Storesletten and Violante (2010) for the size and persistence of the productivity shock. I target the 1970's and 2010's with shock sizes of 0.10 and 0.20, respectively. For each, the persistence of the shock is set to 0.90. These shock sizes target the variance of log earnings documented in Heathcote, Storesletten and Violante (2010). I do not include the transitory shocks[10]featured in their models and instead only focus on the persistent shocks.

5.2 Business Cycle Moments

Table 1 describes statistics about business cycles in this economy with three different types of productivity risk. Aggregate risk and aggregate labor supply are held constant across settings.

An increase in income risk reduces the standard deviation of output, consumption, investment, and share price. Production firms hold higher capital stocks, which delivers more market value to shareholders. Households use this higher average market value to smooth consumption across riskier idiosyncratic states.

As idiosyncratic risk increases, the average return on equity falls from 3.9% to 3.5%. This reflects two driving factors. First, the demand for shares rises as idiosyncratic risk increases. Households want to ensure themselves against a low idiosyncratic productivity shock, so they save in equity, which drives up the price of the asset and lowers average returns. Second, idiosyncratic risk increases the household's desire for dividend payments in low productivity states. This means firms will invest more, which lowers the rate of return on capital.

When households directly own capital, higher investment would be considered precautionary savings. However, the firm directly owns capital, so the concept of precautionary savings can't exactly be considered the same way as it is used in the literature. The firm has no utility function, is not risk averse, and doesn't face any change in productivity risk

[10]Including transitory shocks will increase the precautionary savings motive. This would amplify any results I find when moving away from the representative agent case.

Business Cycle Moments

	Representative Household Economy					
	Y	C	I	D	P	r^e
Average	0.6498	0.5255	0.1243	0.1681	4.0605	4.06%
σ/μ	2.78%	1.75%	7.83%	1.82%	3.84%	4.23%
SDX/SDY	*1.000*	0.630	2.813	0.652	1.381	1.520
CORR(X,Y)	*1.000*	0.952	0.957	-0.386	0.942	-0.995
AutoCorr	0.839	0.929	0.752	0.721	0.937	0.810
	1970's wage risk ($\sigma_\eta = 0.10$)					
	Y	C	I	D	P	r^e
Average	0.6856	0.5369	0.1487	0.1599	5.5834	2.83%
σ/μ	2.74%	1.63%	7.33%	2.24%	3.61%	4.61%
SDX/SDY	*1.000*	0.597	2.679	0.817	1.317	1.683
CORR(X,Y)	*1.000*	0.943	0.963	-0.575	0.936	-0.992
AutoCorr	0.833	0.932	0.753	0.690	0.937	0.796
	2010's wage risk ($\sigma_\eta = 0.1414$)					
	Y	C	I	D	P	r^e
Average	0.7128	0.5436	0.1692	0.1515	7.1339	2.11%
σ/μ	2.69%	1.55%	6.94%	2.71%	3.52%	4.99%
SDX/SDY	*1.000*	0.577	2.579	1.007	1.307	1.856
CORR(X,Y)	*1.000*	0.930	0.965	-0.657	0.923	-0.990
AutoCorr	0.827	0.936	0.748	0.679	0.941	0.784

Table 3: Columns are output, consumption, investment, dividends, equity price, and realized return on equity, respectively. σ/μ is the standard deviation divided by the average. SDX/SDY is the relative standard deviation of the variable divided by the relative standard deviation of output. CORR(X,Y) describes the variable X's correlation with output. AutoCorr is the variable's correlation with itself over time.

across these scenarios, so why would a firm engage in precautionary savings? This entirely driven by the household's risk aversion and insurance against earnings risk. Wage is perfectly correlated with output, so every household has a higher marginal utility of consumption in low productivity states. With higher productivity risk, each household has a higher expected marginal utility of consumption in each aggregate state (due to Jensen's Inequality). As risk increases, households value payoffs in low productivity states more than they did in a riskless environment. Firms see that households value payoffs more in low states, so they save more in good states to ensure a higher stream of dividend payments.

5.2.1 Model Fit of Financial Moments

With a simple shift in earnings risk, my model is able to replicate additional secular trends in equity markets. The first is a trend toward lower dividend yields. In the 1970's, dividends were roughly 3.5% of equity price, while they are now closer to 2%. My model features dividend yields of 3% on average with a 1970's level of risk which falls to 2.1% with the 2010's level of idiosyncratic risk. Dividend yields fell by 45% in the data and 27% in my model, so I explain over 60% of the fall in dividend yields with only a shift in idiosyncratic risk.

Pricing Ratios Over Time

	Data		Model	
Year	Dividend Yield	PE Ratio	Dividend Yield	PE Ratio
1970	3.5	15.1	2.9%	18.1
2010	1.9	21.2	2.1%	22.2
Change	-45.7%	40.3%	-27.6%	22.6%

Table 4: Dividend yields and price earnings ratios evaluated at 1970 and 2010. Dividend yield and PE ratio are from Shiller (2016). Model dividend yield is evaluated as average dividend divided by average price and PE ratio is price divided by the sum of investment and dividends. The only change in the model is from idiosyncratic earnings risk which rises from 0.10 in 1970 to 0.1414 in 2010.

My model also replicates the rise in the price earnings ratio. Shiller (2016) documents the PE ratio as rising 40% between 1970 and 2010. My model generates an increase in PE ratio of 22%. An increase in household income risk explains 55% of the rise in the price earnings ratio.

5.3 Impulse Responses

How does idiosyncratic risk change recovery from recessions? I start by simulating the economy for 900 periods with TFP at the median level. This produces a steady state distribution of households and a constant level of capital. I then hit the economy with a negative TFP shock that is 1 standard deviation in size (2.14% below average). I simulate for 100 periods with the TFP shock decaying naturally at the rate of $\rho_z = 0.7337$. In each simulated date, I solve the equity price and law of motion using the process I described in Section 4.

In Figure 1, I show that the economy without aggregate risk goes into a deeper recession than the economies with some level of idiosyncratic risk. During a recession, dividend payments increase to help smooth consumption. In a representative household economy, dividends are higher than the steady state for 6 years. The economies with 1970's (2010's) household earnings risk feature dividends higher than the steady state for 7 (8) years.

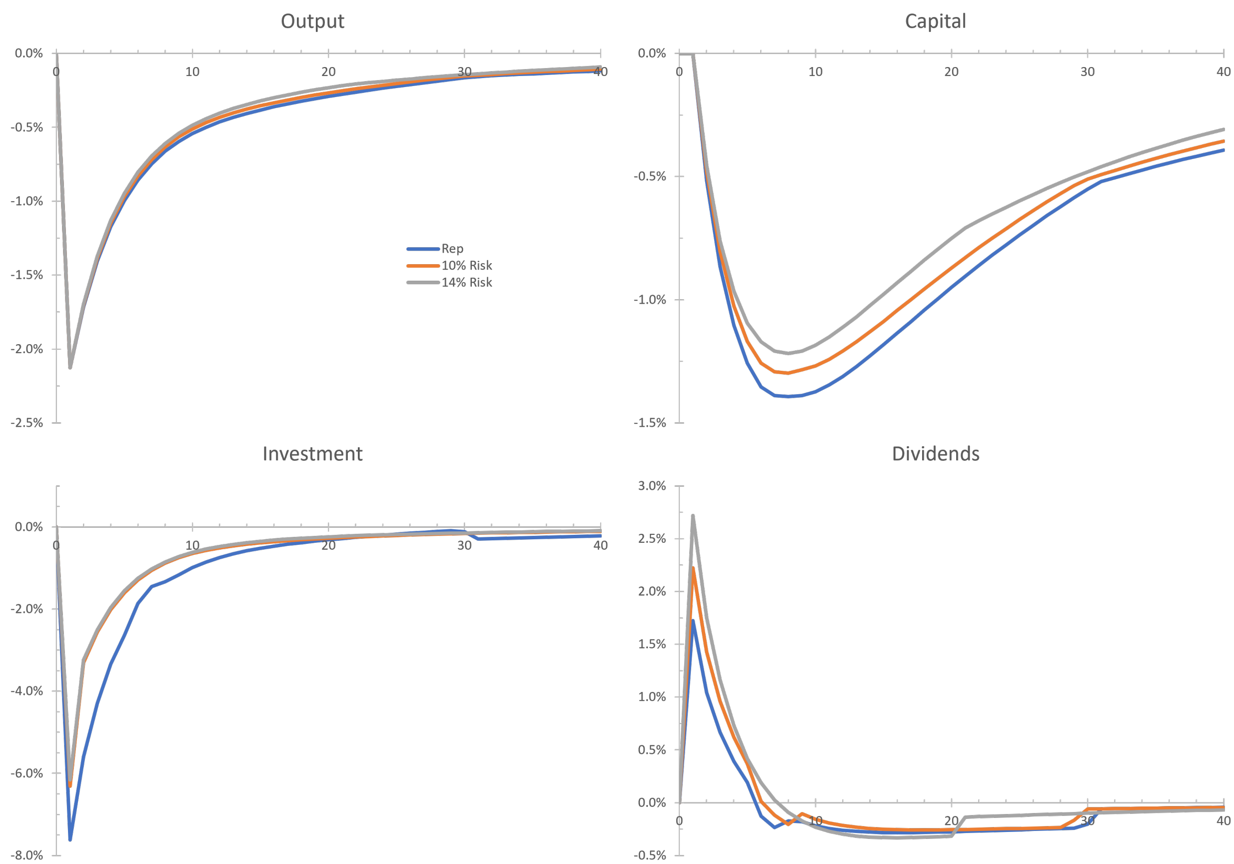

Figure 1: Response of dividends to a 2.14% negative TFP shock. The baseline representative household case is shown in blue. The 10% income risk representing 1970 is shown in gold and the 14% income risk representing 2010 is shown in grey. Note that the lumps in the dividend impulse response are approximately 1.4E-4 in size, which is only 0.02% of steady state output.

While not pictured, the price of equity falls by less than output but stays low longer due to the firm's lower capital level over the recession.

The increased smoothing in output and dividends comes from a higher initial capital stock which can be safely drawn down during a recession. Firms can then pay more in dividends without cutting into the capital stock as severely.

5.4 Household Distributions

I now briefly discuss the distribution of wealth in my model as I change levels of idiosyncratic risk. In the representative economy, there is clearly a degenerate distribution of wealth. Table 5 shows how the wealth distribution changes with increasing levels of household earnings risk.

As the variance of income risk increases, the wealthiest households accumulate more wealth. This finding is standard in the literature, so this result primarily demonstrates

Wealth Distribution

Model	Percentage of wealth held by top X of households 1%	5%	10%	20%	30%
10% earnings risk	4.3%	16.9%	29.3%	48.4%	63.1%
14% earnings risk	4.6%	17.8%	30.4%	49.5%	63.6%
Data[11]	30%	51%	64%	79%	88%

Table 5: Distribution of household wealth by top percentiles of households. Evaluated at the average TFP level and the average aggregate capital level.

that this model generates expected results.

The wealth distribution is not remotely close to the data, but that is a common feature in the household heterogeneity literature without certain features. There are a few common approaches to rectify this issue. They include stochastic preferences (β heterogeneity), rates of return that increase with wealth, and alternate income schemes. Any of these extensions would fit well into my model and could help better target the wealth distribution. However, I leave that as an exercise for later work.

Additionally, allowing negative wealth or short sales would both concentrate more wealth at the top of the distribution as it creates some households with negative wealth. This paper does not explicitly target the distribution of wealth, so I do not pursue these modeling changes at this time.

5.5 Comparison to Other Discounting Regimes

I now compare some key business cycle results from my model to results generated by alternate discounting regimes. I set the persistence of earnings to $\rho_\eta = 0.8$ and the standard deviation of the earnings process to $\sigma_\eta = 0.2$.[12] I consider three alternatives - discounting using current shareholders' discount factors, discounting at the implied safe rate, and discounting with a limited subset of interior shareholders. Discounting using current shareholders' valuation is the method proposed by Grossman and Hart (1979) and evaluated by Carceles-Poveda (2009). The second approach, discounting each future state at one rate dependent on the current state, is more common in the New Keynesian literature. The final approach, discounting with a limited subset of shareholders, is described in Krusell, Mukoyama and Smith (2011) as a method to get around the question of aggregation altogether.

[12]These parameter values are chosen for run time speed. They result in less income risk than the examples discussed in the business cycle setting. This means well-calibrated estimates of earnings risk will increase the size of these differences.

When constructing these alternate discount factors, I make a few small changes to the model environment. When discounting using current shareholder valuations, I calculate an alternate discount factor $\tilde{\chi}(z_n|\mathbb{Z})$ using the same formula as Equation 21, but I replace future shareholding weights a' with current shareholding a.

When I evaluate safe rate discounting, I use the standard expression to find the stochastic discount factor described in Equation 21, but I assume the firm discounts the future using the sum of these weights: $\bar{\chi}(\mathbb{Z}) \equiv \sum \chi(z_n|\mathbb{Z})$.

To find the business cycle moments in the environment with limited participation, I set $\underline{a} = 0.975$. This minimum savings rule is close to the zero trade setting proposed by Krusell, Mukoyama and Smith (2011). I plan on evaluating $\underline{a} \to 1$, but my algorithm needs to be refined to handle truly zero-trade cases.

To compare settings, I run the algorithm described in Section 4 with each variant discount factor. I then run a simulation with the same TFP shocks and measure the result from the last 750 periods. Table 6 describes results.

Discounting Comparison - Average Percentage Difference

Discounting Method	Y	C	I	$\frac{P}{K}$	$\frac{P}{E}$
Current Shareholders	0.002	0.001	0.007	-0.008	-0.003
Safe Rate	0.081	0.033	0.268	-0.307	-0.119
Limited Participation	7.969	2.591	29.11	20.23	43.79

Table 6: Comparison of three alternate discounting proposals. The table above compares averages for each model relative to the baseline across a 750 period simulation. Columns are output, consumption, investment, price to capital (or price to book) ratio, and price to earnings ratio. For example, discounting the future using current shareholders' marginal rates of substitution results in an economy with 0.002% higher average output relative to the baseline scenario.

The first two alternate models generate similar behavior. They both result in higher average output, consumption, and investment. This happens because firms generally undervalue payouts in low states when using alternate discount factors. Because firms undervalue payouts in low states, households know that they will have less insurance in low states, so demand for shares will rise. As demand for shares increases, the interest rate falls (or the discount factor rises), which increases firm investment. The lower rate of return results in lower price to book and price to earnings ratios than are generated by my model. Firms that discount using either of these alternate discount rates would not survive a proxy challenge in the benchmark model.

The limited participation case results in much higher average output, consumption, and investment. Unlike the other alternates considered, price to book and price to earnings ratios are significantly higher under the limited participation setting. With limited participation, only the highest productivity shareholders price the aggregate equity asset. They have a lot to lose if the economy shifts to a low aggregate state and their productivity falls to the low level. And because they cannot dissave when they receive a low labor productivity shock, their marginal valuation of payoffs in future states is much higher than it is in the standard model. This results in much higher price to book and price to earnings ratios in the limited participation setting.

5.6 Robustness

When returns to scale are constant, my model replicates the standard finding that a firm's value is equal to its future capital choice. In this special case, my model and the approach suggested by Grossman and Hart (1979) or Carceles-Poveda (2009) are identical.

Higher adjustment costs ψ will typically reduce the volatility of investment. This parameter can be used to ensure that the dividend process becomes procyclical as is the case in the data. And as adjustment costs increases, the average price of equity does not change significantly, but it becomes much more volatile. In the baseline case $\psi = 1$, price is 1.3 times as volatile as output. If adjustment costs become very high ($\psi = 150$), the standard deviation of equity price becomes 3.4 times as large as the standard deviation of output.

6 Conclusion

In this paper, I derive an equilibrium discounting mechanism that firms can use to maximize their value when they are owned by heterogeneous shareholders. I show that an increase in idiosyncratic household productivity risk increases capital investment levels, lowers equity rate of return, and results in less volatile aggregate consumption sequences. Additionally, a rough calibration of the increase in idiosyncratic risk explains 20% of the fall in dividend returns observed between the 1970's and the 2010's.

With decreasing returns, shareholders are not unanimous about their preference for the firm's capital choice. They are, however, unanimous in their desire for the firm to maximize the firm value each period. This result is standard in the literature, but my method is unique in its ability to nest alternate models.

There are two natural extensions to this literature. First, I plan on extending my model to examine the interaction between firm and household heterogeneity over the last 40 years. My discounting method can be immediately extended to a setting with firm heterogeneity with only a small change to the solution algorithm. There is evidence that household earnings are becoming more dispersed over time (Hubmer, Krusell and Smith.,

2021) alongside evidence that firm dynamism is falling (Akcigit and Ates, 2022). These phenomena have not been studied in tandem in part because there is not a consensus method for linking heterogeneous shareholders to firm choices.

The next natural extension is in the realm of common ownership. The current frontier of common ownership literature (Azar and Vives, 2021) does not feature endogenous valuation of future states. My approach would extend that literature by providing it with a consistent method for valuing future payoffs when owned by heterogeneous shareholders.

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Appendices

A Affine Transformations of the Discount Factor

There are trivially many discount factors that jointly satisfy:

$$k' = \arg\max_{k'} d + \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) J(k'; z_n, \mu'_H, \mu'_F) \tag{25}$$

$$p = \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) J(k'; z_n, \mu'_H, \mu'_F) \tag{26}$$

Suppose χ_0 is the discount factor vector calculated by equation 21, which is size $[N_z\ 1]$. From the firm's first order condition, the choice of k' solves

$$1 = \sum_{n=1}^{N_z} \chi_n \frac{\partial \pi'_z}{\partial k'}$$

which is a simplified version Equation 12 when adjustment cost is zero. Then, define Π as the value of $\frac{\partial \pi'_z}{\partial k'}$ evaluated at $k' = k^*$. Π is also a vector of size $[N_z\ 1]$. Finally, define $J \equiv J(k^*)/p$ as a final vector of size $[N_z\ 1]$.

A discount factor that induces the optimal savings level and leads to a firm value that matches the firm value is given by any χ that solves:

$$[1\ \ 1] = \chi' \begin{bmatrix} J \\ \Pi \end{bmatrix} \tag{27}$$

As long as there are more than two exogenous TFP states that occur with positive probability, χ has trivially many possible values that satisfy equilibrium.

B Alternate Problem of the Manager

In this section, I derive the private equity firm's problem if it compensates marginal equity shareholders directly.

Assumption 1. *Shareholders only have voting rights if they hold more than the minimum number of shares.*

Assumption 2. *A private equity firm needs 100% of voting shares to approve a capital deviation for it to be approved.*

Assumptions 1 and 2 require that the manager make side payments to all marginal[13] investors. Households who are made better off by the alternate investment plan $\hat{k}'$ will vote for the plan and pay the private equity firm to implement it. Households who are made worse off by $\hat{k}'$ will be compensated for their votes so that they are at least indifferent to the alternate plan.

I define the set of shareholder who choose $a' > \underline{a}$ as $\hat{\mathcal{S}} \subset \mathcal{S}$. The manager's problem is written as:

$$\max_{\xi(a,\eta|\hat{k}'),\hat{k}'} \int_{\hat{\mathcal{S}}} -\xi(a,\eta|\hat{k}')\mu(d[a \times \eta]) \tag{28}$$

$$\text{s.t.} \quad u(c) + \beta \mathbb{E} V(a'; z', \mu'_H, \mu'_F) \leq u(\hat{c}) + \beta \mathbb{E} V(a'; z', \mu'_H, \hat{\mu}'_F) \quad \forall (a,\eta) \in \hat{\mathcal{S}} \tag{29}$$
$$\hat{c} \equiv w\eta + \xi + (\hat{P} + \hat{D})s - \hat{P}s'$$

where hat variables $(\hat{c}, \hat{P}, \hat{D}, \hat{\mu}'_F)$ denote the components that shift when $\hat{k}'$ changes. When the management firm chooses an alternate scheme $\hat{k}'$, it changes prices and dividend payments today and in the future.

Equation 29 reflects the participation constraint for each household. In Appendix B.1, I show that satisfying this constraint results in lower aggregate demand for shares.

B.1 Participation Constraint Leads to Lower Equity Demand

I begin with both the household's and the management firm's optimality conditions. For the household, I suppress the λ_a term because the manager knows they will not change their savings choice a' with a minor change to prices. In the manager's problem, I also rearrange the shareholding choice to only influence the first term on the right hand side.

$$Pu'(c) = \beta \mathbb{E}_{\eta',z'}(P' + D')u'(c') \tag{10 repeated}$$
$$\frac{\partial P}{\partial \hat{k}'}u'(\hat{c}) = \frac{\partial P + D}{\partial \hat{k}'}\frac{a}{a'}u'(\hat{c}) + \beta \mathbb{E}_{\eta',z'}\frac{\partial P' + D'}{\partial \hat{k}'}u'(\hat{c}') \tag{13 repeated}$$

If the production firm fails the proxy challenge, it was choosing a k' that did not maximize its net market value. Therefore, the first term on the right hand side of Equation 13 is strictly positive for households whose shareholding choice satisfies $\frac{a}{a'} > 0$. I can

[13] Marginal investors, as described earlier, are households who choose to hold more than the minimum number of shares.

therefore rewrite Equation 13 as:

$$\frac{\partial P}{\partial \hat{k}'} u'(\hat{c}) > \beta \mathbb{E}_{\eta',z'} \frac{\partial P' + D'}{\partial \hat{k}'} u'(\hat{c}') \tag{13.2}$$

The cost of holding shares increased by more than the benefit of holding shares. Therefore, when the household reoptimizes and chooses $\hat{a}'|\hat{k}'$, it will choose to hold fewer shares.

What about the case for shareholders switch to or from short selling - that is, shareholders for whom $\frac{a}{a'} < 0$? For these households, the sign on Equation 13.2 is reversed. The benefit of investing is now lower than the cost of investing, so these households will choose to invest more. However, these shareholders will not be a large enough population to result in a net increase in demand for shares. To see this, I return to the aggregate form of the manager's optimizaiton problem:

$$\frac{\partial P}{\partial \hat{k}'} \int_{\hat{\mathcal{S}}} s_i' \mu(d[a \times \eta]) = \frac{\partial P + D}{\partial \hat{k}'} \int_{\hat{\mathcal{S}}} s_i \mu(d[a \times \eta]) + \beta \mathbb{E}_{z'} \frac{\partial P' + D'}{\partial \hat{k}'} \int_{\hat{\mathcal{S}}} s' \mathbb{E}_{\eta'} \frac{u'(\hat{c}')}{u'(\hat{c})} \tag{14 repeated}$$

As before, the production firm will fail the proxy challenge if it chooses k' that does not maximize its net market value. The first term on the right hand side of Equation 14 is positive as long as the shareholders who choose $a' > \underline{a}$ had positive initial shareholding in aggregate at the start of the period.[14] Therefore, the aggregate cost of holding shares increases by more than the aggregate benefit of holding shares, so aggregate demand for shares will fall.

B.2 Non-marginal shareholders as non-adjustors

Shareholders who choose $a' = \underline{a}$ would not want to adjust their shareholding level given some alternate investment plan $\hat{k}'$. PROOF

C Endogenous Borrowing Limit

The natural borrowing limit is set such that the household will always be able to service its debt in aggregate state $\mathbb{Z}$. Additionally, it must be able to service this debt if it starts with the lowest level of savings $a = \underline{A}$ and with the lowest productivity draw $\eta = \eta_1$.

[14] This should almost always be the case. This could only be violated if short selling households are less likely to be constrained by a minimum savings constraint than households with positive shares.

That is:

$$\underbrace{0}_{c} + P(\mathbb{Z})\underline{\text{A}} = (P(\mathbb{Z}) + D(\mathbb{Z}))\underline{\text{A}} + w(\mathbb{Z})\eta_1 \tag{30}$$

$$\Rightarrow \underline{\text{A}} = \max \frac{-w(\mathbb{Z})\eta_1}{D(\mathbb{Z})} \tag{31}$$

If the utility function satisfies the standard Inada condition[15], then households will never choose $a' = \underline{\text{A}}$.

[15] $\lim_{c \to 0} u'(c) = +\infty$