Firm Investment with Shareholder Inequality*

Michael Carter†

November 10, 2022
Most recent draft available here

Abstract

Does household wealth and income inequality influence the decision-making of corporate firms? I answer this by studying a DSGE model featuring aggregate risk, incomplete markets, households that vary in labor income and wealth, and shareholder-owned firms. With household heterogeneity and incomplete markets, shareholder value is not well defined. I resolve this classic problem with a mutual fund that holds the production firms and a private equity sector that forces producers to maximize their net market value (or cum-dividend share price). These intermediaries jointly pin down the equilibrium market stochastic discount factor, which depends on the distribution of household wealth. I use this model to study the interaction between firm behavior and different types of household inequality. I find the increase in household earnings risk observed from 1970 to 2010 causes firms to accumulate more capital, lowering the volatility of consumption, and explaining 60% of the observed decline in dividend yields. I then examine the role of wealth inequality through unanticipated redistribution shocks. More inequality leads to higher investment, wages, and output, though at a significant welfare cost to poor households.

JEL Codes: D52, E13, G12, L21

Keywords: Incomplete markets, Asset prices, Firm objectives

*I gratefully acknowledge the valuable comments and suggestions from Aubhik Khan, Julia Thomas, Kyle Dempsey, and participants at the OSU macro colloquium and the Midwest Macro conference. A finite horizon version of this paper was presented under the title, "How Does Household Earnings Risk Influence Aggregate Investment?". All errors are my own.

†Department of Economics, Ohio State University. Email: carter.1537@osu.edu

1 Introduction

When markets are incomplete and households face both individual and aggregate risk, households engage in precautionary savings.[1] In models where households directly hold capital and rent it to firms, this precautionary savings motive takes the form of increased capital stock. But in the data, household direct investment is typically less than a third the size of corporate investment.[2] To understand investment over the business cycle, we need to know how corporate firms make dynamic investment decisions on behalf of their shareholders.

If household risk and firm dynamics are both important to understanding macroeconomic fluctuations, why are models with these features not more common? Economists typically model firms as entities that maximize value for their shareholders. But with incomplete markets, the valuation of payoffs in future uncertain states depends on the household's individual state. An investment choice that maximizes value for one shareholder may not maximize value for another. To resolve this issue, I consider a model where aprivate equity firm threatens to take over production firms that do not maximize net market value (or cum-dividend share price). In equilibrium, firms will value future payoffs using a share-weighted average of their shareholders' expected marginal rates of substitution across states. I use this model to show that an increase in household earnings risk increases investment by firms, lowers the price-earnings ratio, and lowers aggregate consumption volatility over the business cycle.

The primary contribution of this paper is a mechanism that disciplines dynamic firm choices when shareholders have different valuations of future states. I utilize this mechanism to study two topics that previously could not be addressed. First, I examine how increasing household wage risk changes firm behavior. I then test a set of unanticipated wealth redistribution shocks to see how the wealth distribution itself changes firm dynamics. I also demonstrate that alternate discounting schemes proposed in the literature are inconsistent with observed shareholder and intermediary behavior. Firms that discount using discounting methods common in the literature have a lower value with the same capital stock than firms that discount with the market discount factor I derive in this paper.

Under incomplete markets, payoffs in each state are not uniquely priced. This uncer-

[1] Aiyagari (1994) documents that households accumulate more capital in response to uninsurable individual risk. Krusell and Smith (1998) find that this precautionary savings further increases in a setting with aggregate uninsurable risk.

[2] In 2019, aggregate investment was composed of corporate investment (64%), household investment (20%), sole proprietorships and partnerships (10%), and nonprofits or other tax exempt institutions (6%) according to the BEA Fixed Asset Investment data.

tainty about the value of payoffs causes two problems. First, it becomes more difficult to price assets. An asset can generally be priced as $P = \mathbb{E}[M'X]$ where X is a vector of payoffs and M is the stochastic discount factor (SDF). With complete markets, this SDF vector is the vector of state-contingent claim prices. When markets are incomplete, however, there are infinitely many potential discount factors M that satisfy $P = \mathbb{E}[M'X]$ if there is more than one aggregate state.[3]

The second, related issue is that firms no longer have a well-defined objective in incomplete markets. Firms generally want to maximize shareholder value. When markets are complete, maximizing shareholder value means maximizing payoffs in future states valued by the SDF. Under incomplete markets, each shareholder might have slightly different valuations of payoffs in each future state and would therefore disagree about the optimal investment choice.

To resolve the joint problems of asset pricing and firm discounting, I introduce a pair of financial agents that find the stochastic discount factor (or pricing kernel) and discipline the production sector's investment and dividend choices. A mutual fund finds the market price for payoffs across states, while a private equity firm pins down the discounting regime for production firms.

In my model, production firms own capital, produce each period using a decreasing returns[4] production technology, invest for the future, and pay dividends. Households save in equity through an equity mutual fund, which bundles shares of all production firms into a single investment instrument. This mutual fund is similar to capital mutual funds in Carlstrom and Fuerst (1997), though the mutual fund in this paper holds shares of production firms instead of capital. The mutual fund's bundling of production shares plays the dual role of simplifying the household's problem to a single continuous choice variable and preventing production firms from becoming financial innovators.[5] The primary benefit of prohibiting financial innovation is that it keeps the model tractable.

The private equity firm looks for opportunities to take over production firms and make

[3] Following the textbook explanation from Campbell (2018), suppose there are N assets and S aggregate states, with N<S. Then the matrix P is 1×N, the vector of payoffs X is S×N, and the discount factor M' is 1×S. The vector of payoffs has a maximum rank of N, so there are infinitely many M' that satisfy the pricing equation. In my setting, there is one asset (N=1) and seven potential future aggregate states (S=7) in each date and state.

[4] The assumption of decreasing returns means that firms earn real economic profits which the firm can either pay out as dividends or reinvest into the business. Carceles-Poveda and Coen-Pirani (2010) show that the firm's value is exactly equal to its capital stock with constant returns to scale, even with heterogeneous shareholders.

[5] Financial innovation can happen when a firm creates a new set of payoffs that were not spanned by the previous set of possible investment choices. If a firm promises a tiny deviation in one future aggregate state, households could trade this firm purely as a financial asset, even if it does not meaningfully change output.

them more valuable, similar to the outside manager proposed by Grossman and Hart (1979). If the private equity firm can find an alternate investment plan that is supported by the mutual fund, it will take over the production firm for a single period. If the private equity firm cannot find an alternate capital investment choice that is strictly preferred by the mutual fund, it does not act in that period. In equilibrium, the private equity firm will not be able to improve on the choices of production firms. The key difference between my approach and Grossman and Hart (1979) is that their manager compensates shareholders based on each household's *perceived* price perceptions. In my model, the private equity firm compensates the mutual fund using the equilibrium, *market* price perceptions. I demonstrate that firms who discount according to the method proposed by Grossman and Hart (1979) (and used by Carceles-Poveda and Coen-Pirani (2010)) have a lower market value at the same capital level than firms who discount using the discount factor in my model.

In equilibrium, each household buys or sells shares of equity at a market-clearing price. Each household's shareholding choice depends on the aggregate price, future payoffs, and their current idiosyncratic state. The mutual fund measures a pricing kernel as the *post-trade*[6] share-weighted marginal rates of substitution of all marginal shareholders. This pricing kernel determines the price the mutual fund is willing to pay for a production firm given its future returns (which depend on its current capital level and investment decision). To prevent the private equity firm from finding a deviation, the production firms will value future payoffs with the pricing kernel found by the mutual fund. Discounting future payoffs with the same pricing kernel as the mutual fund results in firms maximizing shareholder value[7]by maximizing their net market value (or cum-dividend share price).

I test the relationship between household risk and macroeconomic aggregates by modeling the observed increase in earnings variance from 1970 to 2010 in the United States.[8] I find this higher level of risk results in a lower expected rate of return on capital, higher

[6]Weighing marginal rates by post-trade share weights is a key difference from the literature. Grossman and Hart (1979) and Carceles-Poveda and Coen-Pirani (2009) use pre-trade weights, which I will show result in firms with lower value at the same capital level. Constantinides and Duffie (1996) and Constantinides and Ghosh (2017) use post-trade weights, but their models feature zero trade, so measuring the discount factor post-trade is unimportant in their setting.

[7]While firms maximize their value, the equilibrium of this model will generally not be constrained efficient. Production firms are atomistic, so they do not consider their investment's impact on wages and other prices. A social planner will be able to find an improvement by changing aggregate capital and shifting wages.

[8]Heathcote, Storesletten and Violante (2010) document the increase in persistent earnings risk over this time period and find that it leads to an increase in consumption. However, they focus on labor market results and set interest rate outside the model, while I focus on capital investment and have an endogenous rate of return on savings.

aggregate investment, less volatile output, and less volatile aggregate consumption. As earnings risk increases, aggregate demand for savings increases because households want to hold more insurance against negative labor productivity shocks. Firms see household demand for savings, so they increase their investment. This leads to a higher capital stock, which smooths out productivity shocks during recessions. However, the increased capital stock lowers the marginal product of capital, which translates to a lower dividend yield. The model can explain 60% of the observed fall in dividend yield and 55% of the rise in the price-earnings ratio for S&P 500 stocks.[9]

I also show that the wealth distribution directly influences firm behavior. The most extreme case I consider transfers all wealth to 5% of households as an unanticipated shock. Firms in this setting are entirely owned by rich households who have a low marginal propensity to consume and therefore a low valuation of current dividends. Knowing that their owners place relatively little value on current dividends, firms increase their investment. At the date of the wealth transfer, investment increases by over 30% which comes at the cost of aggregate consumption falling by nearly 10%. As firms increase their investment, the capital stock increases, which increases aggregate output, equity price, and wages. Aggregate consumption remains below the baseline for nearly 20 years before the increased level of output finally offsets the increase in investment. However, these headline results mask much lower happiness among low-wealth households. Despite increased wages, low wealth households can't save out of poverty because the price of savings is high and the return to savings is low.[10] They remain stuck near the borrowing constraint until they eventually get high productivity shocks and start slowly saving away from the borrowing constraint. To my knowledge, this is the first paper that examines the effect of wealth redistribution shocks in a setting with dynamic firms.

The discounting approach I find in this paper has a few useful properties. First, it nests the representative household case. This allows me to see how representative household economies differ from economies with idiosyncratic risk. Second, it can nest a setting where trade is exogenously prohibited, as is the case in Krusell, Mukoyama and Smith (2011). I use this nesting to show that no-trade models result in excessive capital savings relative to models with trade in incomplete markets. The discounting approach in my paper also nests settings where there is endogenously zero trade, as is the case in

[9] A model where households directly hold capital (like Krusell and Smith (1998)) would not be able to explain as much of this variation, though that difference is primarily driven by the assumption of decreasing returns in my setting. In setting with constant returns, my model exactly replicates the results in Krusell and Smith (1998).

[10] Greenwald et al. (2021) find that low interest rates increase inequality by making low-wealth households worse off and benefit high-wealth households better off. My results are consistent with their finding while also suggesting that wealth inequality can be an underlying cause of lower rates of return.

Constantinides and Duffie (1996) or Constantinides and Ghosh (2017).

Related Literature An extensive literature studies the role of household risk in incomplete markets where households own capital. Aiyagari (1994) develops this in a setting without aggregate risk, which Krusell and Smith (1998) extends with aggregate risk. Challe and Ragot (2016) documents the role of household precautionary savings with unemployment risk over the business cycle. I contribute to this literature by including a production sector that operates a decreasing returns to scale production technology. When this is the case, shareholder-owned firms earn real economic profits and must decide how much of these profits to pay out as dividends or reinvest in the business through investment.

The closest link between household risk and firm behavior generally comes from the entrepreneurship literature. Cagetti and De Nardi (2006) describe a setting where household wealth generates a distribution of entrepreneur firms. In their setting, however, small businesses take their discounting directly from their owners. My approach focuses on larger corporate firms who are responsible for 60% of total investment, while small businesses only account for about 10% of investment.

My paper is most closely related to the firm discounting and price perception literature. Early work by Drèze (1974) describes the problem of uncertainty in the firm's valuation of future payoffs. Grossman and Hart (1979) proposed aggregating discount factors weighed by current shareholding to try to discipline the production sector's problem. Carceles-Poveda and Coen-Pirani (2009) used this aggregation method to study how firm behavior under the proposed discount valuation. These models share two common shortcomings. First, the compensations to shareholders rely on off-equilibrium perceptions of price changes. Each household has a different belief about how prices will change after a deviation in investment. Second, the discount factor used by firms cannot be used to find the value of shares of the firm, except in the special case of constant returns.[11] In contrast, I construct a model where price perceptions are consistent with equilibrium. Additionally, the stochastic discount factor in my model both disciplines the firm's choices and prices assets. Firms that weigh the future by the methods proposed by Grossman and Hart (1979) and Carceles-Poveda and Coen-Pirani (2009) will be worth less than firms that weigh the future using my methodology because my approach takes into account *post-trade* optimal conditions.

Instead of using price perceptions, Krusell, Mukoyama and Smith (2011) exogenously

[11] Carceles-Poveda and Coen-Pirani (2010) prove that the firm's value is equal to its capital stock with constant returns to scale and shareholder heterogeneity.

imposes zero trade, which results in a setting where a single household pins down the aggregate discount factor. My model nests their approach and demonstrates that it results in excessive investment. Empirically, Gormsen and Huber (2022) documents that firms have higher discount rates than what is implied by the cost of equity. My model replicates this as payoffs are more valued in low states than high states, which results in firms who look like they are risk averse.

Bejan (2020) finds that financial innovation can break the standard link between firm value and discounted returns. Her model focuses on the case of a group of stakeholders called the *control group*. The production firm operates to maximize the preferences of this subgroup. This setting would be particularly useful when studying the problem of a firm with a block of a few, distinct shareholders. My setting is more general and assumes that control of a firm is pinned down by the external threat of a shareholder challenge. Moreover, the pricing kernel described in this paper can be used to value any asset, including those governed by a control group.

The asset pricing literature also relates closely to my work. Constantinides and Duffie (1996), Braun and Nakajima (2012), and Constantinides and Ghosh (2017) combine household marginal rates of substitution to create an aggregate stochastic discount factor. The discount factors calculated in their settings differs from the standard result from representative household models, which partly explains the equity premium. However, these papers construct an income process that results in zero trade while my model allows for shareholders to change over time. Marcet and Singleton (1999) finds asset prices are higher with higher income risk, but they only focus on the price and not the discount factor required to find that price. Krueger and Lustig (2010) documents that a lack of insurance for idiosyncratic risk only shifts the price of aggregate risk if household risk is uncorrelated with aggregate risk. Household wage income risk is correlated with aggregate risk in my setting, so their result is consistent with my findings. Paron (2021) documents a similar phenomenon in continuous time models.

The paper proceeds as follows. Section 2 describes the model environment in detail, with particular focus given to the problem of the private equity firm. Section 3 describes the conditions required for equilibrium and shows how I derive the aggregate stochastic discount factor. Section 4 discusses the algorithm I use to solve the model. Section 5 discusses business cycles moments and impulse responses with varying levels of realistic idiosyncratic risk. Section 8 concludes.

2 Model Environment

In this model economy, production firms own capital and make meaningful intertemporal decisions on behalf of their shareholders. Households face idiosyncratic labor productivity risk and can only save in equity. Households save in equity through a mutual fund which bundles shares of production firms. Finally, a private equity firm tries to take over a production firm with the support of the mutual fund.

I begin the description of this economy with details about the maximization problem facing each household, the production firms, and the mutual fund. Once the dynamic agents are introduced, I describe the off-equilibrium private equity firm.

2.1 Households

There is a single production good which is used for both consumption and investment. This good is the model's numeraire. There are a unit measure of households in this economy, identified by their start of period assets a and idiosyncratic labor productivity η. Each household has identical, time-separable, concave, strictly increasing preferences over consumption. Each supplies labor inelastically and saves in equity a. I assume η is a Markov chain; $\eta \in \mathbf{N} \equiv \{\eta_1, \ldots, \eta_{N_\eta}\}$, where $\Pr(\eta' = \eta_j | \eta = \eta_i) = \pi_{ij} \geq 0$ and $\sum_{j=1}^{N_\eta} \pi_{ij} = 1$ for each $i = 1, \ldots, N_\eta$. For simplicity and without loss of generality, I assume higher indexed values of η denote higher productivity levels: $\eta_1 < \eta_2 < \cdots < \eta_{N_\eta}$.

The household's asset holding in the mutual fund is given by $a \in \mathbf{A} \subset \mathbf{R}$. The set $\mathbf{A}$ is bounded above by $\bar{A}$ and below by $\underline{a}$. The upper bound $\bar{A}$ is set outside the model and is chosen at a high enough level such that no households choose it in equilibrium. The lower bound $\underline{a}$ is a parameter in the model. The lower bound must fall in the range $[\underline{A}, 1]$, where $\underline{A}$ is the natural borrowing limit. I define this limit as the smallest level of debt a household could service conditional on entering the period with that debt and holding the lowest productivity draw. I derive an expression for the natural borrowing limit in Appendix C. If $\underline{a} = 0$, this constraint would prohibit short sales. If $\underline{a} = 1$, the economy is in exogenously-imposed autarky, similar to the no-trade scenario described in Krusell, Mukoyama and Smith (2011). I discuss similarities between my method and theirs further in Section 3.4.1.

I summarize the distribution of households over (a, η) using the probability measure μ_H defined on the Borel algebra $\mathcal{S}$ generated by the open subsets of the product space, $\mathbf{S}_H = \mathbf{A} \times \mathbf{N}$.

I require two more components to fully define the aggregate state. The first is aggregate exogenous TFP z. I assume z is a Markov chain; $z \in \mathbf{Z} \equiv \{z_1, \ldots, z_{N_z}\}$, where $\Pr(z' = z_n | z = z_m) = \pi_{mn} \geq 0$ and $\sum_{n=1}^{N_z} \pi_{mn} = 1$ for each $m = 1, \ldots, N_z$. As with labor productivity, I assume higher indexed levels of z are more productive: $z_1 < z_2 < \cdots < z_{N_z}$.

The final component of the aggregate state is the distribution of firms over their start of period capital, $k \in \mathcal{K} \subset \mathbf{R}_{++}$. Similar to households, I summarize the distribution of firms over k using the probability measure μ_F. The aggregate state of the economy is then $\mathbb{Z} \equiv (z, \mu_H, \mu_F)$.

The per-productivity-unit wage $w(\mathbb{Z})$ is taken as given by the household. In each state, the price of equity is expressed as $P(\mathbb{Z})$ which pays dividends $D(\mathbb{Z})$. These are equilibrium prices which the household takes as given when making its decisions.

I now describe the recursive problem of each household in the economy. Let $V(a, \eta_i; \mathbb{Z})$ be the start of period value of a household with assets a, productivity η_i, and the aggregate state given by $\mathbb{Z} \equiv (z_m, \mu_H, \mu_F)$. The dynamic problem of each household is given by:

$$V(a, \eta_i; \mathbb{Z}) = \max_{c, a'} u(c) + \beta \sum_{j=1}^{N_\eta} \pi_{ij} \sum_{n=1}^{N_z} \pi_{mn} V(a', \eta_j; z_n, \mu'_H, \mu'_F) \tag{1}$$

$$\text{s.t. } c + P(\mathbb{Z}) a' \leq \left(P(\mathbb{Z}) + D(\mathbb{Z})\right) a + w(\mathbb{Z}) \eta_i \tag{2}$$

$$\underline{a} \leq a' \tag{3}$$

$$\mu'_F = \Gamma_F(\mathbb{Z}), \qquad \mu'_H = \Gamma_H(\mathbb{Z})$$

where β is the common subjective discount factor. Equation 3 is the debt limit if $\underline{a} < 0$, a ban on short sales if $\underline{a} = 0$, or a minimum savings rule if $\underline{a} > 0$.

The distribution of households over productivity and shareholding evolves over time according to a mapping Γ_H which depends on the current aggregate state. That is, $\mu'_H = \Gamma_H(z, \mu_H, \mu_F)$. This evolution depends on the asset choices of households in the previous period and the realization of idiosyncratic shocks. The distribution of firms over capital is similar, with $\mu'_F = \Gamma_F(z, \mu_H, \mu_F)$. The household takes both of these laws of motion as given when making its shareholding choice.

Let $c(a, \eta; \mathbb{Z})$ and $a(a, \eta; \mathbb{Z})$ be the decision rules for consumption and future shareholding of a household with current state (a, η) and aggregate state $\mathbb{Z}$.

2.2 Equity Mutual Fund

A risk-neutral mutual fund bundles shares of the production firms and sells the bundle to households as equity. Each period, the intermediary collects dividends from production firms, chooses how many shares of each production firm it wants to hold for the next period, and pays out aggregate dividends to households. Aggregate dividends are the dividends collected from production firms plus the net revenue from changing its shareholding of production firms.

The mutual fund chooses aggregate dividends D (with decision rule $D(\mathbb{Z})$) and its portfolio of future shareholding in production firms $\{s'_k\}$ to maximize its net market value. It buys shares $\{s'_k\}$ in each firm indexed by their capital level k at price $p(k;\mathbb{Z})$ and collects dividends $d(k;\mathbb{Z})$. The goal of the intermediary is to maximize net market value, with payoffs in future states valued by the price vector χ, which the intermediary takes as given. [12]

The mutual fund's recursive problem is written as:

$$J(\{s_k\};\mathbb{Z}) = \max_{\{s'_k\}} D + \sum_{n=1}^{N_z} \chi(z_n|\mathbb{Z}) J(\{s'_k\}; z_n, \mu'_H, \mu'_F) \tag{4}$$

$$\text{s.t.} \quad D \leq \int_{\mathcal{K}} \left((p_k + d_k)s_k - p_k s'_k \right) \mu(dk) \tag{5}$$

$$\mu'_F = \Gamma_F(\mathbb{Z}), \qquad \mu'_H = \Gamma_H(\mathbb{Z})$$

where $\chi(z_n|\mathbb{Z})$ is the pricing kernel or stochastic discount factor, which values payments in future state $\{z_n, \mu'_H, \mu'_F\}$ conditional on the current aggregate state $\mathbb{Z}$. This discount factor is taken as given by the mutual fund. In Section 3.3, I describe the equilibrium properties of this discount factor.

2.3 Production Firms

A unit measure of production firms produce a homogeneous output using labor n and start of period capital stock k. They produce using a strictly increasing and concave production function $y = zF(k, n)$. The variable z is the common exogenous stochastic TFP level which was described in the household section.

A firm enters each period with its predetermined stock of capital, $k \in \mathbf{K} \subset \mathbf{R}_{++}$. The goal of each production firm is to maximize dividends plus discounted future value, with payoffs in future states valued by the price vector $\tilde{\chi}$. Each firm chooses labor to maximize period profits, then selects future capital and current dividends. A portion of the firm's capital stock δ depreciates each period. The firm pays a convex adjustment cost $I(k', k)$ that depends on both its current and future capital levels.

The mutual fund values future payoffs at χ while the production firms value future

[12] As in Makowski (1983), maximizing net market value expands the budget constraint of all households who held positive shares at the start of the period.

payoffs at $\tilde{\chi}$.[13] Each production firm takes the price vector $\tilde{\chi}$ as given. One important distinction here is that the discount factor used by each production firm $\tilde{\chi}$ is not assumed to be the same as the financial intermediary's discount factor χ.

As before, I use a shorthand for the aggregate state $\mathbb{Z} \equiv (z_m, \mu_H, \mu_F)$. Each production firm's problem can be written recursively as:

$$G(k; z_m, \mu_H, \mu_F) = \max_{n,k'} d + \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) G(k'; z_n, \mu'_H, \mu'_F) \tag{6}$$

$$\text{s.t.} \quad d + k' + I(k', k) \leq z_m F(k, n) - w(\mathbb{Z})n + (1-\delta)k \tag{7}$$

$$\mu'_F = \Gamma_F(\mathbb{Z}), \qquad \mu'_H = \Gamma_H(\mathbb{Z})$$

where $\tilde{\chi}(z_n|\mathbb{Z})$ is the aggregate valuation for dividends paid in future state $\{z_n, \mu'_H, \mu'_F\}$ conditional on the current aggregate state. This discount factor is taken as given by each production firm and is described in detail in section 3.3. The value of future payoffs is always positive, so the firm's budget constraint will always hold with equality.

Let $k(k; \mathbb{Z})$ and $d(k; \mathbb{Z})$ be the decision rules for future capital and dividends of a firm with current capital k and aggregate state $\mathbb{Z}$.

2.4 Private Equity

Finally, I introduce a private equity firm to discipline the choices of the production sector. The private equity firm is inspired by Grossman and Hart (1979) with some meaningful modifications. The private equity firm has access to a technology that allows it to implement an alternate production plan if and only if it can secure support from the firm's owners, the mutual fund.[14] If the private equity firm cannot find a strictly profitable deviation for any production firm, the private equity firm does not act in the period.

To find a profitable deviation, the private equity firm first proposes some alternate investment plan $\hat{k}'$. As the private equity firm is only able to alter the plans of a single firm, it will not shift aggregate prices. The private equity fund then chooses a side payment $\xi(\hat{k})$ to the mutual fund to compensate it for changing the value of its portfolio. If the portfolio becomes more valuable after the change, the mutual fund would pay the private equity firm to implement the change ($\xi(\hat{k}) < 0$). If the portfolio becomes less valuable, the private equity firm would have to compensate the mutual fund ($\xi(\hat{k}') > 0$).

[13] These will be the same in equilibrium, but not because it was imposed as a modeling assumption.

[14] I consider a case where the private equity firm instead directly compensates marginal shareholders in Appendix A

The private equity firm's problem is written as:

$$\max_{\xi(\hat{k}'),\hat{k}'} -\xi(\hat{k}') \tag{8}$$

$$\text{s.t.} \quad J(\{s_k\};\mathbb{Z}) < \hat{J}(\{s_k\};\mathbb{Z}|\hat{k}') \tag{9}$$

where hat variable $(\hat{J})$ denotes the change in the mutual fund's value conditional on one of the production firms changing its capital investment to $\hat{k}'$ and after receiving the side payment $\xi(\hat{k}')$. Equation 9 is the mutual fund's participation constraint. It will vote unanimously in favor of the new plan only if it is made better off than it was before the deviation.

When the private equity firm chooses an alternate scheme $\hat{k}'$, it changes share price and dividend payments both today and in the future. To make this problem tractable, I first assume that the firm takes over a tiny, identical mass of firms with size ε. I then take $\varepsilon \to 0$ to find the atomistic limit. I write the new level of dividends as $\hat{D} \equiv (1-\varepsilon)D + \varepsilon\hat{d}$ and the new price level as $\hat{P} \equiv (1-\varepsilon)P + \varepsilon\hat{p}$. There is no need to include changes in shareholding because the mutual fund holds all shares and, in equilibrium, it will continue to hold shares at the new price level.

Below is the mutual fund's problem when the private equity firm successfully (off-equilibrium) takes over a production firm:

$$\hat{J}(\{s_k\};\mathbb{Z}) = \max_{\{s_k'\}} D + \sum_{n=1}^{N_z} \chi(z_n|\mathbb{Z}) J(\{s_k'\}; z_n, \mu_H', \hat{\mu}_F') \tag{10}$$

$$\begin{aligned} \text{s.t.} \quad & D \leq (1-\varepsilon) \int_{\mathcal{K}} \left((p_k + d_k)s_k - p_k s_k'\right) \mu(dk) \\ & \quad + \varepsilon(\hat{p} + \hat{d}) \\ & \hat{\mu}_F' = \Gamma_F(\mathbb{Z}), \qquad \mu_H' = \Gamma_H(\mathbb{Z}) \end{aligned} \tag{11}$$

The key difference from Equations 4-5 is that the price and dividends have changed for a small ε mass of firms and that the distribution of firms is slightly different $\hat{\mu}_F'$.

2.5 Discussion of Assumptions

Before continuing the analysis, I would like to pause briefly to discuss the rationality of the modeling choices presented above. Specifically, I want to discuss the mutual fund and the private equity firm.

2.5.1 Benefits of a Mutual Fund

Is it reasonable to impose a mutual fund as the sole savings instrument for household savings? Retirement fund managers typically suggest shifting a portfolio toward bonds while approaching retirement age, so there is some empirical evidence for portfolio management beyond simple market savings. However, common financial advice suggests that saving in equity will generally outperform stock picking. Warren Buffett famously (Perry, 2017) wagered in 2008 that a low-cost S&P 500 index fund would outperform a portfolio of actively managed hedge funds. Further, the SCF documents that safe assets only make up about 10% of wealth for the top 90% of households.

Is it reasonable to expect that a mutual fund will support net market value maximization by the firms it owns? As discussed in the introduction, shareholder challenges tend to be more successful when they target low market value firms. Additionally, mutual funds are legally required to act in their shareholders' best interest. Pursuing net market value maximization expands the budget constraint of households with long positions in equity.

The mutual fund plays three key roles in this economy. It prevents production firms from becoming financial innovators, it simplifies the problem of price discovery, and it makes this problem tractable.

First and most importantly, it prevents atomistic production firms from becoming financial innovators. DeAngelo (1981), Makowski (1983), and Krouse (1985) argue that shareholders will be unanimous in supporting the firm's decision to maximize net market value if firms are sufficiently small. That is, shareholders have to believe that a firm's deviation will not change the set of available prices or future outcomes. By imposing a financial intermediary between the household and the production sector, a production firm will not be able to change the available choice set for households when it produces differently than its peers. Preventing financial innovation keepts the problem more tractable.

Second, it simplifies the potential problem of price discovery. It could be difficult for every production firm to ask its shareholders how they value payoffs across time, aggregate those answers, and predict future shareholding. A financial intermediary sector could much more realistically study markets and make prices available to production firms.[15] In a full information model, this isn't particularly necessary, but it is a helpful feature for future work.

Finally, the mutual fund reduces the size of the problem to something tractable. As

[15] An example of this behavior comes from Investor Relations departments at large firms. These groups frequently interact with institutional investors, who might provide feedback about how a proposed capital investment plan will change share prices.

written, the model now features a distribution of households over shareholding and productivity and a distribution of firms over capital. If shareholders were allowed to own individual production firms,[16] shareholding would become a portfolio choice and the distribution of households would increase by the number of firms in the economy. Firms would also need to know who specifically holds their shares and how their decisions impacts those specific households, including the portfolio balancing effects of a change in capital.

The tractability result also has important implications for equilibrium. Imagine a setting where households directly invested in firms. Two production firms might start the period identically, but for some reason attract different types of shareholders. If one firm attracts poor shareholders, it will likely invest less than the firm that attracts rich shareholders. If they are ex-ante identical, which firm will attract which type of shareholders? The answer isn't immediately clear. Rather than spending effort to track shareholder-to-firm combinations, it is much more straightforward to impose a financial intermediary.

While it is a strong assumption that a mutual fund has primary voting power for all shares, it is consistent with recent trends in the structure of equity markets. Fichtner, Heemskerk and Garcia-Bernardo (2017) and Edelman, Thomas and Thompson (2014) document that large financial intermediaries control a majority of voting shares and that they are required to vote in their shareholders' best interests.

2.5.2 Private Equity Firms

The problem of the private equity firm is modeled from observed shareholder proxy battles. Fos (2017) documents that a majority of shareholder challenges state their goal as increasing market value. Further, shareholder challenges that target market value tend to be more likely to succeed.

This modeling choice is the most important one in this paper because it ultimately pins down the objective of the production firm. Other authors approach this question differently. Grossman and Hart (1979) and Carceles-Poveda and Coen-Pirani (2009) assume that the production firm tries to maximize start-of-period shareholder value and that those shareholders have control over the firm. These authors also consider cases where the date-zero shareholders control the firm. These assumptions each lead to slightly different capital choices by production firms (as documented in Carceles-Poveda and Coen-

[16] Clearly, households do differentiate their portfolios for a variety of reasons. There is also evidence from Brockman et al. (2022) that firm risk earns a return premium over the market rate. This would not be the case with a single mutual fund holding all production firms. Private or closely-held corporate firms may also be inconsistent with the mutual fund presented here. These cases leave room for future research.

Pirani (2009)). A micro-founded private equity firm eliminates the need for assumptions about firm control.

One weakness of this modeling choice is its assumption that all production firms care about a shareholder challenge. Closely-held private firms, entrepreneur firms, partnerships, sole proprietorships, nonprofits, and corporate firms with large blocks of insider shareholding[17] are all types of firms who might reasonably ignore a shareholder challenge. The research on the distribution of wealth (De Nardi and Fella, 2017) documents that entrepreneurs can also generate higher rates of return than the market. Because corporate firms are responsible for nearly two thirds of aggregate investment, adding these types of firms to the model is best left as an exercise for future work.

3 Equilibrium Definition and Properties of the Discount Factor

In this section, I describe the conditions for a recursive competitive stock market equilibrium. Then, I discuss properties of the equilibrium stochastic discount factor.

3.1 Recursive Competitive Stock Market Equilibrium

A Recursive Competitive Stock Market Equilibrium is a set of functions,

$$\{w, G, \chi, \tilde{\chi}, d, n, k, p, d, J, P, D, V, s, c\}$$

that jointly solve the household, firm, and mutual fund's problems, and clear the markets for goods, labor, production firm shares, and mutual fund shares, as described by the following:

i. V solves Eq. 1 with policy functions $\{c, a\}$

ii. J solves Eq. 4 with policy function D

iii. G solves Eq. 6 with policy functions $\{d, n, k\}$

iv. The market for equity clears in each date and state:

$$1 = \int_S a(a, \eta, \mathbb{Z})\mu(d[a \times \eta])$$

[17]Berkshire Hathaway is an example of such a firm. 43.3% of Berkshire Hathaway stock is held by directors and executive officers of the company as of March 2, 2022. A firm like this is likely immune to all but the most united shareholder challenge. `https://www.berkshirehathaway.com/meet01/2022proxy.pdf`

v. The intermediary holds all shares of the production firms $1 = s'_k \; \forall \; k$ with positive mass

vi. The labor market clears:

$$\int_{\mathcal{K}} n(k;\mathbb{Z})\mu(dk) = \int_{\mathcal{S}} \eta\mu(d[a \times \eta])$$

vii. The goods market clears:

$$\int_{\mathcal{K}} (k(k;\mathbb{Z}) - (1-\delta)k)\,\mu(dk) + \int_{\mathcal{S}} c(a,\eta;\mathbb{Z})\mu(d[a \times \eta]) = \int_{\mathcal{K}} zF(k,n(k;\mathbb{Z}))\mu(dk)$$

viii. Γ_H is defined by:

$$\mu'(A,\eta_j) = \int_{\{a,\eta_i|(a(a,\eta_i;\mathbb{Z}))\in A\}} \pi_{ij}\mu(d[a \times \eta_i]) \quad \forall \; (A,\eta_j) \in \mathcal{S}$$

ix. Γ_H is defined by:

$$\mu'(k) = \int_{\{k|(k(k;\mathbb{Z}))\in K\}} \mu(dk) \quad \forall \; k \in \mathcal{K}$$

x. The private equity firm cannot find a profitable deviation for any production firm(the maximum of Equation 8 is zero).

While not explicitly listed, χ and $\tilde{\chi}$ are determined by the conditions above. The financial intermediary's discount factor χ is determined by market clearing for equity. The production firm's discount factor $\tilde{\chi}$ is determined by the last equilibrium condition and will be derived in Section 3.3.[18]

3.2 Optimal Choices

I begin by describing the conditions that pin down optimal choices for each type of agent. With those optimal choices in mind, I will construct a pair of discount factors that are consistent with both clearing the equity market (χ) and surviving a proxy battle ($\tilde{\chi}$).

Each household's optimal choice of a' satisfies:

$$Pu'(c) = \beta\mathbb{E}_{\eta',z'}(P' + D')u'(c') + \lambda_a \tag{12}$$

[18]There are many values of $\tilde{\chi}$ that can survive a shareholder challenge when $N_z \geq 2$. However, the discount factor I derive in Section 3.3 pins down a family of discount factors that jointly lead to the same prices and allocations.

where $u'(c)$ is the marginal utility of consumption in the current period and $\mathbb{E}$ reflects the expectation of transitioning over both idiosyncratic state η' and aggregate productivity state z'. Future outcomes $P',\ D',\ c'$ are each optimal choices of each type of agent in each realized future state. The λ_a term reflects the fact that some households may want to save less than is allowed by the minimum savings constraint described in Equation 3. This term will be equal to zero for households who choose $a' > \underline{\mathrm{a}}$.

The financial intermediary chooses shareholding of each production firm, which has optimality conditions:

$$p(k;\mathbb{Z}) = \sum_{n=1}^{N_z} \chi(z_n|\mathbb{Z})[p'(k(k;\mathbb{Z}); z_n, \mu'_H, \mu'_F) + d'(k(k;\mathbb{Z}); z_n, \mu'_H, \mu'_F)] \quad \forall\, k \in \mathcal{K} \tag{13}$$

The price the mutual fund is willing to pay for a firm with capital level k and a vector of future payoffs $p',\ d'$ depends on the future price and dividend it can expect to receive, weighed by some discounting χ for each future state. Because the financial intermediary is not bound by a short sales constraint, the *law of one price* will hold.[19] The mutual fund will price future payoffs at the same rate for all of the production firms that it owns.

Each production firm's optimal choices are given by:

$$1 + \frac{\partial I(k',k)}{\partial k'} = \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) D_1 G(k'; z_n, \mu'_H, \mu'_F) \tag{14}$$

Because each firm is atomistic, it takes as given future prices and its shareholders' valuation of payoffs in future states. Because the firm is owned by the mutual fund, the firm cannot become a financial innovator. This is similar to the case in Makowski (1983) but with a different mechanism. The firm only weighs the lower dividends against the change in future valued payments. In equilibrium, the distribution of firms will be degenerate with all of the mass at a single capital level.

Finally, I describe the choices of the private equity firm, which will discipline the production firms' discount factor $\tilde{\chi}$. When deriving the private equity firm's optimal choices, I assume it takes over a mass of identical production firms of size ε. This is assumed to be small enough that the private equity firm does not have market power to change other firms' behavior in the current or future date. Then I take the limit as $\varepsilon \to 0$ to find the specific results at the atomistic firm limit.

The private equity firm chooses a capital deviation $\hat{k}'$ for a mass ε of production firms with current capital level k_i. A capital deviation changes the firm's value to the mutual

[19]The law of one price is common in the finance literature. See Chapter 4 of Campbell (2018).

fund across three channels. It changes dividends d, equity price p, and the vector of future returns $\{p'_n + d'_n\}$.

These production firm changes pass through directly to the mutual fund's balance sheet. The mutual fund will not change its shareholding level. Rather, the price the mutual fund is willing to pay changes as described in Equation 13, depending on the vector of future returns $\hat{p}'$ & $\hat{d}'$.

The firm chooses $\hat{k}'$ and side payment $\xi(\hat{k}')$ satisfying:

$$
\begin{aligned}
[\xi(\hat{k}')] : \qquad & 1 = \Omega \\
[\hat{k}'] : \qquad & \varepsilon\Omega\frac{\partial p}{\partial \hat{k}'} = \varepsilon\Omega\left(\frac{\partial(p+d)}{\partial \hat{k}'} + \sum_{n=1}^{N_z}\chi(z_n|\mathbb{Z})\frac{\partial(p'_n + d'_n)}{\partial \hat{k}'} + \underbrace{o(\varepsilon)}_{\to 0}\right)
\end{aligned}
\tag{15}
$$

where Ω is the multiplier on the participation constraint described in Equation 9 and ε are the mass of firms controlled. The term $o(\varepsilon)$ accounts for changes to the firm's value through channels other than direct price and dividend, like the change on other firms' value in the next period if some mass ε of competitors behaved differently than expected. This term goes to zero in the atomistic limit. Rearranging yields:

$$
-\frac{\partial d}{\partial \hat{k}'} = \sum_{n=1}^{N_z}\chi(z_n|\mathbb{Z})\frac{\partial(p' + d')}{\partial \hat{k}'} \tag{16}
$$

The capital choice of the private equity firm $\hat{k}'$ determined by Equation 16 is the same as the capital choice of the production firm k' determined by Equation 14 if the production firm's value is equal to its net market value: $G(k;\mathbb{Z}) = p(k;\mathbb{Z}) + d(k;\mathbb{Z})$.

Lemma 1. *If the production firm discounts future states with the mutual fund's discount factor χ, its choice of k' is the same as the private equity firm's optimal deviation $\hat{k}'$ and will therefore survive a proxy challenge.*

The proof is by construction. Suppose $G(k;\mathbb{Z}) = p(k;\mathbb{Z}) + d(k;\mathbb{Z})$. I rewrite the Benveniste-Scheinkman condition and the production firm's change in dividends with respect to capital as:

$$
D_1 G(k;\mathbb{Z}) = \frac{\partial(p+d)}{\partial k} \tag{17}
$$

$$
\frac{\partial d}{\partial k'} = -(1 + \frac{\partial I(k',k)}{\partial k'}) \tag{18}
$$

I substitute these expressions back into the production firm's optimal choices and combine with the private equity firm's optimality condition. Finally, I evaluate this at $k' = \hat{k}'$, which will be the case in equilibrium. Together, these substitutions yield:

$$1 + \frac{\partial I(k',k)}{\partial k'} = \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) D_1 G(k'; z_n, \mu'_H, \mu'_F) \qquad \text{(14 repeated)}$$

$$-\frac{\partial d}{\partial k'} = \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) \frac{\partial(p'+d')}{\partial k'} \qquad \text{(substituting 17 \& 18)}$$

$$\sum_{n=1}^{N_z} \chi(z_n|\mathbb{Z}) \frac{\partial(p'+d')}{\partial \hat{k}'} = \sum_{n=1}^{N_z} \tilde{\chi}(z_n|\mathbb{Z}) \frac{\partial(p'+d')}{\partial k'} \qquad (19)$$

Lemma 2. *The production firm survives a proxy challenge if its discount factor $\tilde{\chi}$ is the same as the financial intermediary's discount factor χ.*

This expression can also be derived by directly compensating marginal shareholders. I discuss that alternate setting (with identical results) in Appendix A.

3.3 Solving the Intermediary's Discount Factor

I construct an aggregate discount factor from each household's optimality conditions as described in Equation 12. To simplify notation, I suppress the current aggregate state $\mathbb{Z} \equiv \{z_m, \mu_H, \mu_F\}$ and the transition of future distributions. I first rewrite each household's Euler equation for shareholding in terms of the equity price. I then multiply through by future shareholding choice a' (Equation 20) and aggregate over all households (21):

$$Pu'(c) = \beta \sum_{n=1}^{N_z} \pi_{mn}(P'_n + D'_n) \sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn}) + \lambda_a \qquad \text{(12 rewritten)}$$

$$P = \frac{\beta \sum_{n=1}^{N_z} \pi_{mn}(P'_n + D'_n) \sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} + \frac{\lambda_a}{u'(c)}$$

$$Pa' = \left(\frac{\beta \sum_{n=1}^{N_z} \pi_{mn}(P'_n + D'_n) \sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} \right) a' + \frac{\lambda_a}{u'(c)} a' \qquad (20)$$

$$\int_{\mathcal{S}} Pa' \mu(d[a \times \eta]) = \int_{\mathcal{S}} \left(\left(\frac{\beta \sum_{n=1}^{N_z} \pi_{mn}(P'_n + D'_n) \sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} \right) a' + \frac{\lambda_a}{u'(c)} a' \right) \mu(d[a \times \eta]) \qquad (21)$$

where c_{jn} is a shorthand for the household's consumption rule when it transitions to idiosyncratic productivity level $\eta' = \eta_j$, aggregate TFP transitions to level $z' = z_n$, and the shareholding in the next period is the solution to the household's maximization problem: $c_{jn} \equiv c(a(a, \eta; \mathbb{Z}), \eta_j; z'_n, \mu'_H, \mu'_F)$.

When the stock market clears ($1 = \int_{\mathcal{S}} a(a, \eta, \mathbb{Z})\mu(d[a \times \eta])$), the left hand side of Equation 21 is equal to the equity price P. I can express equity price as a function of weighted marginal rates of substitution and future payoffs:

$$P = \sum_{n=1}^{N_z} (P'_n + D'_n) \left(\beta\pi_{mn} \int_{\mathcal{S}} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} a' \mu(d[a \times \eta]) \right) + \int_{\mathcal{S}} \frac{\lambda_a}{u'(c)} a' \mu(d[a \times \eta]) \quad (22)$$

A stochastic discount factor directly values an asset conditional on a vector of future returns without an additive wedge. Therefore, I want to eliminate Lagrange multiplier λ_a in Equation 22. I start by rearranging the household's optimal choice for a' to be in terms of $\lambda_a / u'(c)$, which will allow me to simplify equation 22.

$$\frac{\lambda_a}{u'(c)} = P - \frac{\beta \sum_{n=1}^{N_z} \pi_{mn}(P'_n + D'_n) \sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} \quad \text{(12 rearranged)}$$

For households who choose $a' > \underline{a}$, the expression above is equal to zero. I introduce an indicator function that will let me separate out households who are at the savings limit against those who save more than the minimum.

$$\mathbb{I} = \begin{cases} 1 & a' > \underline{a} \\ 0 & a' = \underline{a} \end{cases}$$

With this indicator I separate out Equation 22 into households who are and aren't bound by the savings condition.

$$\begin{aligned} P = &\sum_{n=1}^{N_z} (P'_n + D'_n) \left(\beta\pi_{mn} \int_{\mathcal{S}} \mathbb{I} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} a' \mu(d[a \times \eta]) \right) + \\ &\sum_{n=1}^{N_z} (P'_n + D'_n) \left(\beta\pi_{mn} \int_{\mathcal{S}} (1 - \mathbb{I}) \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} \underline{a} \mu(d[a \times \eta]) \right) + \\ &\int_{\mathcal{S}} (1 - \mathbb{I}) \left(P - \frac{\beta \sum_{n=1}^{N_z} \pi_{mn}(P'_n + D'_n) \sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} \right) \underline{a} \mu(d[a \times \eta]) \end{aligned}$$

With some additional algebra, this leads to the equation:

$$P = \sum_{n=1}^{N_z} (P'_n + D'_n) \frac{\left(\beta \pi_{mn} \int_{\mathcal{S}} \mathbb{I} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} a' \mu(d[a \times \eta]) \right)}{\int_{\mathcal{S}} \mathbb{I} a' \mu(d[a \times \eta])} \tag{23}$$

This equation says that the value of equity is determined by discounted future payoffs. The discounting comes from shareholders' expected marginal rates of substitution across aggregate states, weighted by their end of period shareholding. I define this discount factor as:

$$\chi(z_n|\mathbb{Z}) \equiv \beta \pi_{mn} \frac{\int_{\mathcal{S}} \mathbb{I} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} a' \mu(d[a \times \eta])}{\int_{\mathcal{S}} \mathbb{I} a' \mu(d[a \times \eta])} \tag{24}$$

Before moving on, I would like to discuss a special case of the discount factor if $\underline{a} = 0$. In that case, the discount factor simplifies to:

$$\chi(z_n|\mathbb{Z}) = \pi_{mn} \beta \int_{\mathcal{S}} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} a' \mu(d[a \times \eta]) \tag{25}$$

This form of the discount factor is identical to the one proposed in Drèze (1974). However, the discount factor in that paper is assumed rather than being derived as an equilibrium expression. Additionally, the discount factor proposed by Drèze (1974) cannot handle cases where the minimum savings constraint is binding while the discount factor described in Equation 24 can.

3.4 Analysis

As discussed in Lemma 1, the production firm will survive a shareholder challenge if its discount factor is the same as the mutual fund's. Therefore, I will simplify notation in the remainder of this paper and use χ to describe the aggregate stochastic discount factor.

In the sections below, I discuss the equilibrium discount factor. I describe properties of the discount factor, describe how it compares to discount factors proposed in the literature, and briefly discuss uniqueness and unanimity with the discount factor above.

3.4.1 Properties of the Discount Factor

The discount factor described in Equation 24 features a number of useful properties. I discuss below how it nests a number of standard models, including the representative household case, the exogenous no-trade case, constant returns to production environments, and the Makowski (1983) Criterion. Additionally, this discount factor results in net market value maximization, which is consistent with observed firm behavior.

First, it neatly nests the representative household discount factor. With a representative household, the future shareholding choice is always $a' = 1$. And because there is no idiosyncratic risk, the distribution is degenerate. That discount factor can be written as:

$$\chi_{\text{rep}}(z_n|\mathbb{Z}) = \pi_{mn}\beta\frac{u'(c'_n)}{u'(c)}$$

which is a special case of the discount factor I derive in Equation 24. This discount factor is standard in the literature featuring dynamic firms, such as Khan and Thomas (2013).

Another useful feature is that this discount factor nests the exogenous no-trade approach proposed by Krusell, Mukoyama and Smith (2011). In their model, the minimum savings rule for equity is $\underline{\text{a}} = 1$, which requires all households to save the median number of shares. In that setting, only a single shareholder (or type of shareholders) would not want to choose $a' < 1$, meaning $\mathbb{I} = 0$ for all shareholders except one. Their discount factor is then:

$$\chi_{\text{KMS}}(z_n|\mathbb{Z}) = \pi_{mn}\beta\frac{\sum_{j=1}^{N_\eta}\pi_{ij}u'(c'_{jn})}{u'(c)} \qquad \text{for } \eta_i = \max \mathbf{N} \tag{26}$$

If there is only a single household (or single type of household) who would not want to choose $a' < \underline{\text{a}}$, then my model exactly replicates this result. There is only one household for whom the indicator function $\mathbb{I}$ in Equation 24 is nonzero, so that household's marginal rate of substitution is the stochastic discount factor.

In a constant returns environment without adjustment costs, the price of a production firm p is equal to its next-period capital stock k'. This is a standard result in the literature, as in Carceles-Poveda and Coen-Pirani (2009).

The proposed discount factor also meets a criterion as set out by Makowski (1983). That criterion requires the discount factor used by the firm satisfy $P = \max[\sum_n SDF_{a,\eta_i}(P'_n + D'_n)]$, where SDF_{a,η_i} is the stocahastic discount factor across aggregate states for the household indexed by $\{a, \eta_i\}$. However, every household that chooses $a' > \underline{\text{a}}$ will satisfy this condition as shown by the optimality conditions in Equation 12.[20]

One downside of the Murkowski criterion is that it does not identify a unique discount factor. Every household that chooses to hold more than the minimum level of assets will satisfy the criterion, and each of these discount factors may lead to the firm making a different choice in capital. Because my discount factor uses information from all shareholders in equilibrium, it is distinct.

Another useful property of this discount factor is that it implicitly maximizes value weighed by current shareholding. As in DeAngelo (1981), maximizing the firm's net value expands each household's budget constraint by the size of their current shareholdings. This wealth effect allows households to choose more consumption or savings. The price effect and the change to future payoffs can be ignored because the shareholder can freely adjust their future shareholding choice.

3.4.2 My Approach Relative to Alternatives

The two most common alternate discount factors stem from Grossman and Hart (1979) and Carceles-Poveda and Coen-Pirani (2009). For each, they set the discount factor to:

$$\chi_{\text{alt}}(z_n|\mathbb{Z}) = \pi_{mn}\beta \int_{\mathcal{S}} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} \mathbf{W}_{a,\eta} \mu(d[a \times \eta]) \tag{27}$$

where $\mathbf{W}_{a,\eta}$ is the model-specific weighting for shareholder indexed by $\{a, \eta\}$. Grossman and Hart (1979) consider weights equal to either date zero holdings ($\mathbf{W}_{a,\eta} = a_{0_{,a,\eta}}$) or beginning of period holdings ($\mathbf{W}_{a,\eta} = a_{a,\eta}$). In comparison, my model uses $\mathbf{W}_{a,\eta} = a'_{a,\eta}$ in the special case $\underline{a} = 0$, as shown in Equation 25.

How does my model differ from these expressions? First, I explicitly allow for cases where $\underline{a} \neq 0$ by only considering households who choose to be shareholders. Second, the discount factor in my model is built up explicitly from equilibrium conditions. The alternate methods simply assume that the firm weighs future payoffs by these weighting factors. The most important difference is that the discount factor used by firms in my work is also the pricing kernel.

From the mutual fund's problem, *any* asset is priced as $p = \sum_{n=1}^{N_z} \chi(z_n|\mathbb{Z})(p'_n + d'_n)$. However, this relationship does not necessarily hold with these alternate weights.[21] This is the case because the alternate discount factors do not utilize the asset market clearing condition in their calculation.

[20]Households who choose $a' = \underline{a}$ will have $\lambda_a > 0$, which means only their SDF will not satisfy the Murkowski criterion.

[21]Except in the special case of constant returns to production or if households don't trade shares.

How inaccurate would the guess of equity price be if one were to incorrectly guess that aggregate price is given by $P = \sum_{n=1}^{N_z} \chi_{\text{alt},n}(P'_n + D'_n)$? For this analysis, I assume $\underline{a} = 0$ so the numerator of χ from Equation 24 goes to 1. With that assumption in place, the price error is given by:

$$P_{err} = \sum_{n=1}^{N_z} (P'_n + D'_n) \left(\beta \pi_{mn} \int_S \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c'_{jn})}{u'(c)} (a - a') \mu(d[a \times \eta]) \right) \tag{28}$$

In a complete markets setting where all households have the same marginal rate of substitution across states, the error will be zero. Similarly, the error would be zero in cases where all households endogenously choose not to trade. However, the error would be large in cases where households make large trades across periods. For example, a setting with stochastic, idiosyncratic changes to the discount factor β (as in Krusell and Smith (1998)) would likely generate large buying and selling of shares.

I discuss the numerical difference between my model and the discounting approach proposed by Grossman and Hart (1979) in Section 5.4. A firm that discounts using the Grossman discounting will not survive a proxy challenge in my model. However, production firms that follow the initial-shareholding discounting regime will not maximize net market value, so they would be taken over by the private equity firm.

3.4.3 Unanimity

A common question in this literature is around the concept of unanimity. In a setting that requires unanimity, the goal is for all shareholders to want the firm to pursue exactly the same capital investment plan. Carceles-Poveda and Coen-Pirani (2009) shows that this is the case in models with constant returns and no short selling. However, DeAngelo (1981) and Makowski (1983) instead propose that ex-ante shareholders unanimously prefer for the firm to engage in net value (or cum-dividend price) maximization. The best a firm can do for its shareholders is to maximally expand their budget sets, which is done by maximizing the firm's net value.

In my model, households are unanimous in their preference for firm value maximization, as is the case in DeAngelo (1981). They are not unanimous, however, in the firm's exact choice of future capital. Some households would be better off if the firm increased its capital choice and vice-versa. However, a lack of unanimity about specific capital plans is not a shortcoming of the model. In observed equity markets, shareholder votes about capital choices (or firm management in general) are relatively rare. Fos (2017) documents that the most frequent cause for shareholder challenges are poor stock performance, while proxy challenges that target capital structure tend to be less successful. The empirical data support ignoring unanimity in capital choice as long as firms are maximizing their net market value.

4 Algorithm

The solution algorithm utilizes a modified version of the backward induction developed by Reiter (2009). Rather than solving the value function with a fixed law of motion and separately simulating to find the true law of motion (as in Krusell and Smith (1998)), this method uses a *proxy distribution* of households at each aggregate state and finds a law of motion simultaneously with solving the value function. A simulation is run to ensure that the proxy distribution accurately anticipates the distribution of households across aggregate states.

I begin by discretizing the aggregate state z into 7 states using the Tauchen algorithm. I similarly discretize idisoyncratic productivity levels η into 7 states. I then need to choose a proxy aggregate state. I use the total level of log capital, which serves as a good measure of aggregate total wealth. I linearly space log capital M into 9 grid points. Finally, I discretize the choice values for a' and k' on a grid with 99 points for shareholding and 299 points for capital.

I use a naive guess of the distribution of households by assuming that all households start with shareholding $a = 1$ and productivity is distributed at the stationary level. I start with a guess for the aggregate law of motion, equity prices, and dividends at each state. I guess that $K' = K$, which also pins down aggregate dividends. The guess of share price is trivial, but I start by assuming $P = \beta D/(1-\beta)$. This guess of the price is consistent with an asset priced in a riskless Lucas economy. Finally, I also guess starting levels for the firm's value G, the household's value V, and the household's period marginal utility of consumption MUC. For a starting guess, I set MUC^0 as the marginal utility of consuming the wage endowment.

In each iteration o, the algorithm proceeds as follows:

1 **Outer Loop:** In each aggregate state indexed by (z, M):

  (a) **Solve LOM:** Guess a future aggregate state M'_g, which implies dividends D_g

    i. **Clear the Equity Market via bisection:** Guess a price for equity P.

    ii. Solve each household's optimal choice of a' given the previous V^o, law of motion M'_g, dividends D_g, and future prices and dividends P^o & D^o.

iii. Measure total shareholding $A(P) = \int_{\mathcal{S}} a' \mu(d[a \times \eta])$.

   A. If $A(P) - 1 >$precision, there is too much demand for shares, so the price needs to rise. Return to step 1(a)i

   B. If $1 - A(P) >$precision, there is insufficient demand for shares, so the price is too high and needs to fall. Return to step 1(a)i.

iv. Once the share price has cleared the equity market, measure the implied stochastic discount factor (Equation 24) using the MUC^o array.

v. **Consistency with aggregates:** Solve the firm's problem given the discount factor and firm values G^o. This yields k'.

   A. If $k' - K' >$precision, the guess of K' was too low. Guess a higher K' and return to step 1a

   B. If $K' - k' >$precision, the guess of K' was too high. Guess a lower K' and return to step 1a

(b) Once capital choice is consistent, update the guess of $P^{o+1}, D^{o+1}, K'^{o+1}$.

2 With a consistent guess of aggregates, solve the household's and the firm's problem conditional on the updated guesses of $P^{o+1}, D^{o+1}, K'^{o+1}$

(a) $V^{o+1} = u(c) + \beta V^o$

(b) $MUC^{o+1} = u'(c)$

(c) $G^{o+1} = d + \beta G^o$

3 If the norm is sufficiently small $|V^{o+1} - V^o| <$precision, proceed to the simulation. Otherwise, return to step 1.

In the algorithm above, I search for price via bisection. Conditional on a guess of M', demand for shares is weakly decreasing in price, so there is a single price that clears the equity market. I solve the household's problem and with the endogenous grid method and I solve the firm's problem via golden section search.

Once I've solved the value functions and aggregate laws of motion with the proxy distribution, I simulate the economy to calculate a new reference distribution. In the simulation, I draw a TFP shock on the grid, then solve it at the previously determined level of aggregate capital using the process described in step 1 above. However, instead of using the proxy distribution, I use the distribution of households from the previous simulation step. I run this simulation for 750 periods to "pre-heat" before tracking the distribution of households in each date and with each realization of shocks on the z grid. I simulate 750 periods, then update the reference distribution as described in Reiter (2009). After updating the reference distribution, I solve the household's value function again. I typically run this process four times in total, though results generally don't change after the second update of the reference distribution.

5 Business Cycle Moments and Impulse Responses

I now apply my method to a simple business cycle example to see how aggregate behavior varies with idiosyncratic risk. To begin, I specify explicit forms for the utility and production functions. Households value consumption with CRRA utility of the form:

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

where σ is the relative risk aversion parameter. I assume a Cobb-Douglas production function: $F(k, n) = zk^{\alpha}n^{\gamma}$ with $\alpha + \gamma \in (0, 1]$. I further assume a quadratic adjustment cost function: $I(k', k) = \frac{\psi}{2k}(k' - k)^2$. If $\psi = 0$, this nests a model without adjustment costs.

5.1 Model Parameters

Table 1 summarizes the model parameters that are constant across iterations. All results are from a parameterized model which captures rough trends.

σ	δ	β	α	γ	σ_z	ρ_z	$\underline{a}$	ψ
3.0	0.070	0.96	0.30	0.55	0.0120	0.7898	0	4.0

Table 1: Summary of economic parameters. σ is the coefficient of relative risk aversion. δ is depreciation. β is the household's subjective discount factor. α is capital's share of production and γ is labor's share. σ_z is standard deviation of TFP and ρ_z is the persistence of the TFP shock. $\underline{a}$ is the borrowing constraint, with 0 indicating no short sales are allowed. ψ is the scale of the quadratic adjustment cost.

The period of the model is annual. The coefficient of relative risk aversion σ is set to 3.0, which is on the high end of levels used in the real business cycle literature. Depreciation is set to 7%, which approximately matches the investment to capital ratio in the United States in the post-war period. The subjective discount factor β is set to 0.96, which would imply a real interest rate of 4% in a riskless representative agent economy. The adjustment parameter ψ is in the range of values in the literature, which typically vary from 0.15-150.

Labor's share of production γ is estimated from NIPA tables as total payments to labor divided by GDP, which is approximately 55%. I choose α at the low end of the literature's range at 0.3, which implies a total return to scale of 0.85. For TFP, I calculate Solow residuals for 1956-2019. I estimate TFP's AR(1) process with a standard deviation of 1.204% and a persistence of 78.98%.

Idiosyncratic Risk

		Variance of Log Earnings	
Setting	σ_η	Data	Model
Rep HH	0.0	0.0	-
1970	0.1154	0.25	0.25
2010	0.1548	0.45	0.45

Table 2: Comparison of parameter values across settings. σ_η is the standard deviation of shocks to the persistent earnings process. The persistence ρ_η is 0.973 from Heathcote, Storesletten and Violante (2010).

For idiosyncratic risk, I consider three cases. I start with a baseline model with zero idiosyncratic risk, which will represent the representative agent case. For the cases with idiosyncratic risk, I consider risk levels matching 1970 and 2010. For both, I use estimates from Heathcote, Storesletten and Violante (2010) for the size and persistence of the idiosyncratic income shock. I target the 1970's and 2010's with shock sizes of 0.1154 and 0.1548, respectively.[22] For each, the persistence of the shock is set to 0.973. These shock sizes target the variance of log earnings documented in Heathcote, Storesletten and Violante (2010). I do not include the transitory shocks[23]featured in their models and instead only focus on the persistent shocks.

5.2 Business Cycle Moments

Table 1 describes statistics about business cycles in this economy with three different types of productivity risk. Aggregate risk and aggregate labor supply are held constant across settings.

An increase in income risk reduces the standard deviation of output, consumption, investment, and share price. Production firms hold higher capital stocks, which delivers more market value to shareholders. Households use this higher average market value to smooth consumption across riskier idiosyncratic states.

As idiosyncratic risk increases, the average return on equity falls from 2.6% to 1.9%. This reflects two driving factors. First, the demand for shares rises as idiosyncratic risk increases. Households want to ensure themselves against a low idiosyncratic productivity shock, so they save in equity, which drives up the price of the asset and lowers

[22]While Heathcote, Storesletten and Violante (2010) document a number of causes for the increase in variance, I model it as a change to a technology parameter. If the variance in earnings is driven by endogenous choices, my model will overstate results.

[23]Including transitory shocks will increase the precautionary savings motive. This would amplify any results I find when moving away from the representative agent case.

Business Cycle Moments

	Representative Household Economy					
	Y	C	I	D	P	r^e
Average	0.649	0.525	0.124	0.168	4.061	4.1%
σ/μ	2.58%	1.92%	5.46%	0.75%	4.35%	52.8%
SDX/SDY	*1.000*	0.744	2.121	0.290	1.690	20.5
CORR(X,Y)	*1.000*	0.996	0.990	0.653	0.998	0.206
AutoCorr	0.862	0.884	0.832	0.987	0.875	-0.129

	1970's wage risk ($\sigma_\eta = 0.10$)					
	Y	C	I	D	P	r^e
Average	0.693	0.539	0.154	0.158	6.032	2.6%
σ/μ	2.56%	1.80%	5.31%	0.65%	4.24%	81.4%
SDX/SDY	*1.000*	0.702	2.072	0.254	1.656	31.8
CORR(X,Y)	*1.000*	0.994	0.992	-0.123	0.998	0.227
AutoCorr	0.860	0.884	0.834	0.860	0.875	-0.129

	2010's wage risk ($\sigma_\eta = 0.1414$)					
	Y	C	I	D	P	r^e
Average	0.717	0.544	0.172	0.150	7.684	1.9%
σ/μ	2.55%	1.76%	5.11%	0.89%	4.25%	107.2%
SDX/SDY	*1.000*	0.692	2.009	0.348	1.671	42.1
CORR(X,Y)	*1.000*	0.992	0.990	-0.398	0.996	0.244
AutoCorr	0.859	0.887	0.829	0.806	0.879	-0.128

Table 3: Columns are output, consumption, investment, dividends, equity price, and realized return on equity, respectively. σ/μ is the standard deviation divided by the average. SDX/SDY is the relative standard deviation of the variable divided by the relative standard deviation of output. CORR(X,Y) describes the variable X's correlation with output. AutoCorr is the variable's correlation with itself over time.

average returns. Second, idiosyncratic risk increases the household's desire for dividend payments in low productivity states. This means firms will invest more, which lowers the rate of return on capital. This result is consistent with Huggett (1993) and Aiyagari (1994), which both find that rate of return falls with individual risk.

When households directly own capital, higher investment would be considered precautionary savings. However, the firm directly owns capital, so the concept of precautionary savings can't exactly be considered the same way as it is used in the literature. The firm has no utility function, is not risk averse, and doesn't face any change in productivity risk across these scenarios, so why would a firm engage in precautionary savings? This entirely driven by the household's risk aversion and insurance against earnings risk. Wage is perfectly correlated with output, so every household has a higher marginal utility of consumption in low productivity states. With higher idiosyncratic income risk, each household has a higher expected marginal utility of consumption in each aggregate state (due to Jensen's Inequality). As risk increases, households value payoffs in low productivity states more than they did in a riskless environment. Firms see that households value payoffs more in low states, so they save more in good states to ensure a higher stream of dividend payments.

5.2.1 Model Fit of Financial Moments

With a simple shift in earnings risk, my model is able to replicate additional secular trends in equity markets. The first is a trend toward lower dividend yields. In the 1970's, dividends were roughly 3.5% of equity price, while they are now closer to 2%. My model features dividend yields of 2.6% on average with a 1970's level of risk, which falls to 2.0% with the 2010's level of idiosyncratic risk. Dividend yields fell by 45% in the data and 25% in my model, so I explain 55% of the fall in dividend yields with only a shift in idiosyncratic risk. As idiosyncratic risk increases, households want more savings for low productivity states. This leads to an increase in average investment, which increases capital stock and asset prices. With decreasing returns to scale, dividends increase but at a lower rate than the capital stock increases.

Pricing Ratios Over Time

	Data		Model	
Year	Dividend Yield	PE Ratio	Dividend Yield	PE Ratio
1970	3.5%	15.1	2.6%	19.3
2010	1.9%	21.2	2.0%	23.8
Change	-45.7%	40.3%	-25.3%	23.1%

Table 4: Dividend yields and price earnings ratios evaluated at 1970 and 2010. Dividend yield and PE ratio are from Shiller (2016). Model dividend yield is evaluated as average dividend divided by average price and PE ratio is price divided by the sum of investment and dividends. The only change in the model is from idiosyncratic earnings risk which rises from 0.11 in 1970 to 0.15 in 2010, as described in Table 2

My model also replicates the rise in the price earnings ratio. Shiller (2016) documents the PE ratio as rising 40% between 1970 and 2010. My model generates an increase in PE ratio of 22%. An increase in household income risk explains 55% of the rise in the price earnings ratio.

5.3 Impulse Responses

How does idiosyncratic risk change recovery from recessions? I start by simulating the economy for 900 periods with TFP at the median level. This produces a steady state distribution of households and a constant level of capital. I then hit the economy with a negative TFP shock that is 1 standard deviation in size (2.14% below average). I simulate for 100 periods with the TFP shock decaying naturally at the rate of $\rho_z = 0.7337$. In each simulated date, I solve the equity price and law of motion using the process I described in Section 4.

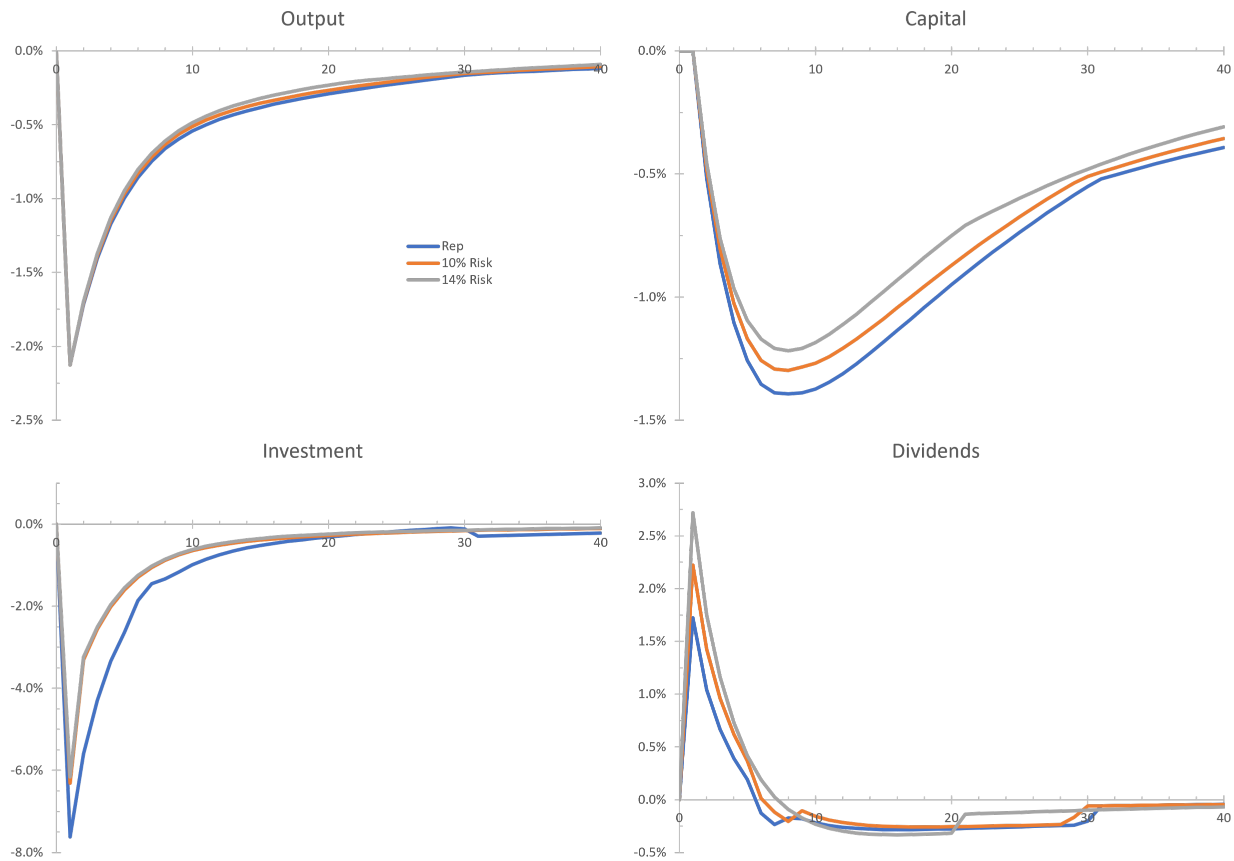

Figure 1: Response of dividends to a 2.14% negative TFP shock. The baseline representative household case is shown in blue. The 10% income risk representing 1970 is shown in gold and the 14% income risk representing 2010 is shown in grey. Note that the lumps in the dividend impulse response are approximately 1.4E-4 in size, which is only 0.02% of steady state output.

In Figure 1, I show that the economy without idiosyncratic risk goes into a deeper recession than the economies with some level of idiosyncratic risk. During a recession, dividend payments increase to help smooth consumption. In a representative household economy, dividends are higher than the steady state for 6 years. The economies with 1970's (2010's) household earnings risk feature dividends higher than the steady state for 7 (8) years. While not pictured, the price of equity falls by less than output but stays low longer due to the firm's lower capital level over the recession.

The increased smoothing in output and dividends comes from a higher initial capital stock which can be safely drawn down during a recession. Firms can then pay more in dividends without cutting into the capital stock as severely.

Dividends are procyclical in the data and in the representative household business cycle moments. However, dividends appear strongly countercyclical at the beginning of the TFP recession in Figure 1 before becoming correlated with output after 8 years. What causes this discrepancy? At the date of the shock, output falls significantly. To smooth consumption, firms increase their dividend payment, but households can also sell shares to finance consumption. Households who have low wealth are very exposed to fluctuations in the wage, so they will sell their shares while high wealth households can accumulate shares. As the recession continues, equity winds up more in the hands of high productivity households who can afford to save.

As the capital stock falls, so does the share price. Despite dividends below the steady state level after 8 years, the lower price of savings means that dividend yields actually rise as the recession continues.

5.4 Household Distributions

I now briefly discuss the distribution of wealth in my model as I change levels of idiosyncratic risk. In the representative economy, there is clearly a degenerate distribution of wealth. Table 5 shows how the wealth distribution changes with increasing levels of household earnings risk.

Wealth Distribution

	Percentage of wealth held by top X of households				
Model	1%	5%	10%	20%	30%
10% earnings risk	4.3%	16.9%	29.3%	48.4%	63.1%
14% earnings risk	4.6%	17.8%	30.4%	49.5%	63.6%
Data[24]	30%	51%	64%	79%	88%

Table 5: Distribution of household wealth by top percentiles of households. Evaluated at the average TFP level and the average aggregate capital level.

As the variance of income risk increases, the wealthiest households accumulate more wealth. This finding is standard in the literature, so this result primarily demonstrates that this model generates expected results.

The wealth distribution is not remotely close to the data, but that is a common feature in the household heterogeneity literature without certain features. There are a few common approaches to rectify this issue. They include stochastic preferences (β heterogeneity), rates of return that increase with wealth, and alternate income schemes. Any of these extensions would fit well into my model and could help better target the wealth distribution. However, I leave that as an exercise for later work.

Additionally, allowing negative wealth or short sales would both concentrate more wealth at the top of the distribution as it creates some households with negative wealth. This paper does not explicitly target the distribution of wealth, so I do not pursue these modeling changes at this time.

5.5 Comparison to Other Discounting Regimes

I now compare some key business cycle results from my model to results generated by alternate discounting regimes. I set the persistence of earnings to $\rho_\eta = 0.8$ and the standard deviation of the earnings process to $\sigma_\eta = 0.2$.[25] I consider three alternatives - discounting using current shareholders' discount factors, discounting at the implied safe rate, and discounting with a limited subset of interior shareholders. Discounting using current shareholders' valuation is the method proposed by Grossman and Hart (1979) and evaluated by Carceles-Poveda and Coen-Pirani (2009). The second approach, discounting each future state at one rate dependent on the current state, is more common in the New Keynesian literature. The final approach, discounting with a limited subset of shareholders, is described in Krusell, Mukoyama and Smith (2011) as a method to get around the question of aggregation altogether.

When constructing these alternate discount factors, I make a few small changes to the model environment. When discounting using current shareholder valuations, I calculate an alternate discount factor $\tilde{\chi}(z_n|\mathbb{Z})$ using the same formula as Equation 24, but I replace future shareholding weights a' with current shareholding a.

When I evaluate safe rate discounting, I use the standard expression to find the stochastic discount factor described in Equation 24, but I assume the firm discounts the future using the sum of these weights: $\bar{\chi}(\mathbb{Z}) \equiv \sum \chi(z_n|\mathbb{Z})$.

To find the business cycle moments in the environment with limited participation, I set $\underline{a} = 0.975$. This minimum savings rule is close to the zero trade setting proposed by Krusell, Mukoyama and Smith (2011). I plan on evaluating $\underline{a} \to 1$, but my algorithm needs to be refined to handle truly zero-trade cases.

To compare settings, I run the algorithm described in Section 4 with each variant discount factor. I then run a simulation with the same TFP shocks and measure the result

[25] These parameter values are chosen for run time speed. They result in less income risk than the examples discussed in the business cycle setting. This means well-calibrated estimates of earnings risk will increase the size of these differences.

from the last 750 periods.[26] Table 6 describes results.

Discounting Comparison - Average Percentage Difference

Discounting Method	Y	C	I	$\frac{P}{K}$	$\frac{P}{E}$
Current Shareholders	0.002	0.001	0.007	-0.008	-0.003
Safe Rate	0.081	0.033	0.268	-0.307	-0.119
Limited Participation	7.969	2.591	29.11	20.23	43.79

Table 6: Comparison of three alternate discounting proposals. The table above compares averages for each model relative to the baseline across a 750 period simulation. Columns are output, consumption, investment, price to capital (or price to book) ratio, and price to earnings ratio. For example, discounting the future using current shareholders' marginal rates of substitution results in an economy with 0.002% higher average output relative to the baseline scenario.

The first two alternate models generate similar behavior. They both result in higher average output, consumption, and investment. This happens because firms generally undervalue payouts in low states when using alternate discount factors. Because firms undervalue payouts in low states, households know that they will have less insurance in low states, so demand for shares will rise. As demand for shares increases, the interest rate falls (or the discount factor rises), which increases firm investment. The lower rate of return results in lower price to book and price to earnings ratios than are generated by my model. Firms that discount using either of these alternate discount rates would not survive a proxy challenge in the benchmark model.

While these gaps are small, the key problem with discounting using the current shareholder methodology is that the SDF that prices assets is different than the discount factor that governs firm behavior. With a representative firm, this difference isn't critical because the equity price is the same as firm prices. But in a model with firm heterogeneity, an analysis of firm price would not be feasible. With the SDF described in Equation 24, any asset can be priced given a set of future payments.

The limited participation case results in much higher average output, consumption, and investment. Unlike the other alternates considered, price to book and price to earnings ratios are significantly higher under the limited participation setting. With limited participation, only the highest productivity shareholders price the equity asset. They have a lot to lose if the economy shifts to a low aggregate state and their productivity falls to the low level. And because they cannot dissave when they receive a low labor productivity shock, their marginal valuation of payoffs in future states is much higher than it is in the standard model. This results in much higher price to book and price to earnings ratios in the limited participation setting.

[26] All markets still clear in these alternate settings. However, the private equity will generally be able to find a profitable deviation under these scenarios.

6 Wealth Shocks

I next use unanticipated wealth redistribution shocks to examine the role of wealth inequality in shaping outcomes. I will show that, in general, increasing wealth inequality leads to higher investment and output. Additionally, these redistributive wealth shocks are much more persistent than TFP shocks.

In this experiment, I first simulate the economy for 900 periods without aggregate productivity shocks to find a steady state level for the economy. The end of this 900 year simulation is date 0. In date 1, I shock the economy with an unanticipated wealth redistribution shock. At the beginning of period of the shock, I redistribute assets to match a target distribution. I then simulate the economy as described in Section 4. I run a total of four experiments. One shocks the economy back to equality, the next puts all wealth in the hands of 5% of households, and two experiments that replicate the wealth distributions in 1970 and 2010. I also include comparisons to a single standard deviation negative TFP shock to put the results from Section 5.3 in perspective. When performing these redistribution shocks, I distribute wealth evenly across productivity levels. I do not change the distribution of individual household labor productivity and I do not assume that high or low productivity households are any more likely to hold shares.

Wealth Shocks and Distributions

Setting	Bottom 90%	10%-5%	5%-1%	Top 1%
Baseline	69.6%	12.6%	13.2%	4.6%
Equality	90.0%	5.0%	4.0%	1.0%
5% have all	0.0%	0.0%	80.0%	20.0%
1970	30.0%	15.3%	27.1%	27.6%
2010	24.3%	13.1%	23.1%	39.5%

Table 7: Distribution of wealth in each redistribution shock setting. The baseline case is from the 2010 wealth case and is consistent with with the second line in Table 5. The "equality" case distributes all wealth evenly among all households, while the "5% have all" case distributes all wealth equally among the top 5% of households. Finally, the 1970 and 2010 cases are from Table B1 in the online appendix of Saez and Zucman (2016).

Table 7 describes the distribution of wealth in each shock scenario. The cases "equality" and "5% have all" represent simple cases where wealth inequality is either shut down or very concentrated. The latter case will resemble an economy with exogenous market segmentation. I next consider redistributing wealth to the levels seen in 1970 and 2010. I use the distributions from Table B1 of the online appendix of Saez and Zucman (2016).[27]

[27] While their data uses multiple sources of wealth (housing, bonds, borrowing, etc), it still serves as a useful benchmark for the model. Wealth in this paper is only accumulated through a single channel, so comparing to total wealth is the closest natural proxy.

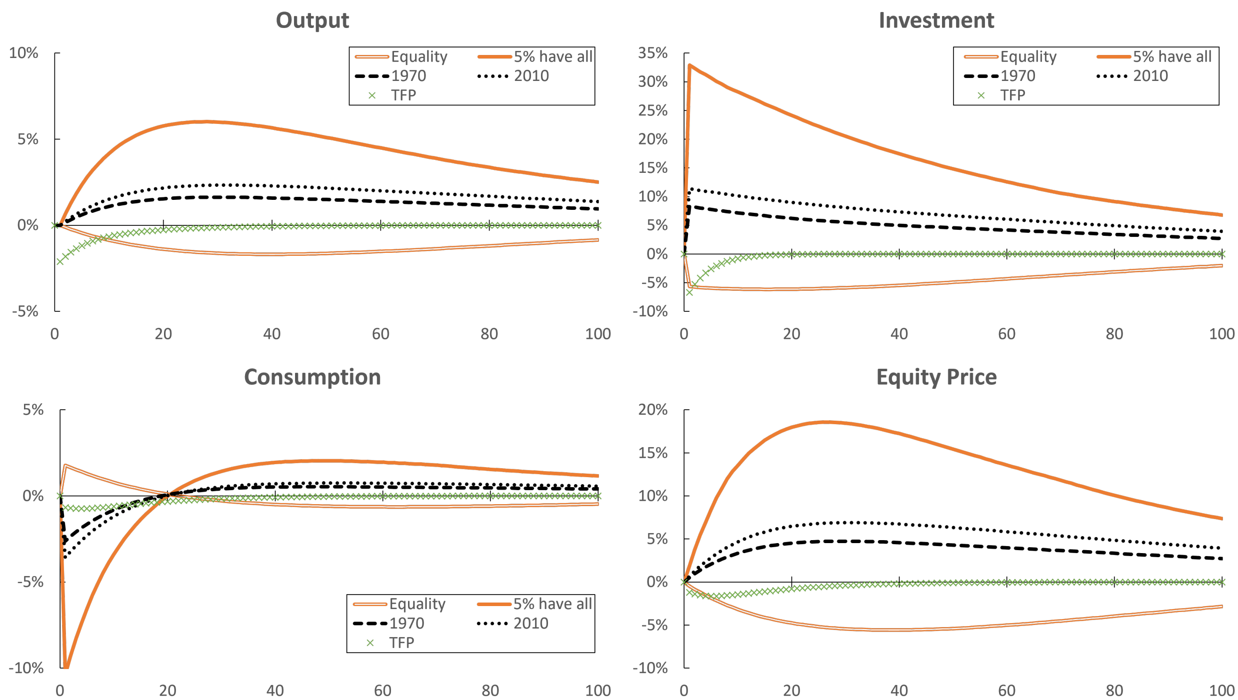

Figure 2: Impulse responses to unanticipated wealth redistribution shocks. Each of these starts from a steady state with zero TFP innovations for 900 years, then are shocked in date 1. Wealth shocks reallocate wealth to the specified levels. "Equality" reallocates all wealth equally. "5% have all" gives 5% of households all wealth. 1970 and 2010 distribute wealth to the levels documented in Table B1 of the online appendix of Saez and Zucman (2016). The "TFP" line does not exogenously redistribute wealth but instead shows the effect of a single TFP negative shock for comparison.

Figure 2 documents the results of these experiments. In cases where inequality increased (all except the TFP shock and the equality case), investment immediately increases. Rich households are relatively more patient than other households, so firms increase their investment. Higher investment leads to a growth in equity price and output. Aggregate consumption falls while the economy focuses on building a large capital stock, though it eventually becomes higher than the baseline 20 years after the initial shock.

In the equality case, no households are near their borrowing limit, but no houses are very rich. Households don't need much precautionary savings because they have large buffer stocks already, so investment falls by about 5% every year for nearly 60 years. Lower investment leads to lower output and lower equity prices. However, aggregate consumption increases in the first 20 years as the economy dissaves.

Relative to a TFP shock, wealth redistribution shocks are incredibly persistent. Output returns to within 0.5% of the steady state within 13 years of a productivity shock. In comparison, it takes between 124 and 200 years for output to return to its steady state after a wealth distribution shock.

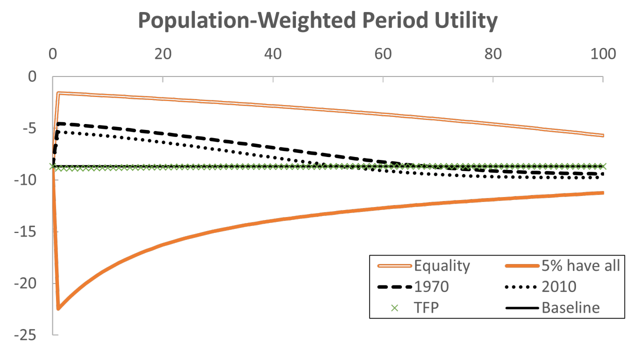

Figure 3: Population-weighted period utility after four wealth distribution shocks and a TFP shock. The units, weighted utility, do not have a natural scale . Each of these starts from a steady state with zero TFP innovations for 900 years, then are shocked in date 1. Wealth shocks reallocate wealth to the specified levels. "Equality" reallocates all wealth equally. "5% have all" gives 5% of households all wealth. 1970 and 2010 distribute wealth to the levels documented in Table B1 of the online appendix of Saez and Zucman (2016). The "TFP" line does not exogenously redistribute wealth but instead shows the effect of a single TFP negative shock for comparison.

Why are recoveries from wealth shocks so protracted? In a high inequality setting, rich households hold a higher level of equity than in the baseline case. They face little risk, so they can afford to be patient in waiting for returns. They prefer high investment, which leads to high equity price, low dividend payments, and lower rates of return on equity. Low wealth households then have little incentive to hold shares because the rate of return is so low. Low wealth households have low consumption, so their marginal utility is very high. Coupled with higher than average equity prices in an inequality setting, the cost of investing is high and they save very slowly.

Consumption and output both eventually increase after a shock toward more inequality, but this doesn't say anything about welfare. I examine the welfare implications of a redistribution shock by plotting changes to population-weighted utility[28] relative to the steady state.

Figure 3 shows the population-weighted utility under each experiment. Shifting toward equality increases utility in every period despite the fact that it lowered aggregate consumption after the first 20 periods. Fewer households are near the borrowing constraint, so few households have very low utility after the shock. In contrast, the experiment that eliminates the wealth of 95% of households leads to large utility losses. Many households can only afford to consume their wages and have very high marginal utility. Utility rises quickly over the first 20 years after the shock as capital, output, and wages rise. But this rise in wage does not make up for the fact that the low rate of return on equity makes it

[28]The population-weighted utility metric doesn't fully capture the distribution effects of the policy, it can still be a useful starting point. In the future, I plan to update this to compensating variation.

unappealing for low-wealth households to increase their savings. As is the case in Greenwald et al. (2021), low rates of return hurt low-wealth households.

Utility changes in the 1970 and 2010 wealth shock cases are likely driven by the choice of how to redistribute wealth. When I shock the economy, the average household in the bottom 90% of wealth loses wealth. However, they are still far enough away from the borrowing constraint that they can use their wealth to insure away from negative productivity shocks. But with increases in capital, the rate of return falls and fewer low-wealth households choose to hold equity. The interest rate channel primarily drives the fall in utility below the baseline case seen starting around period 60. Wealth shocks have a long lasting effect on economic outcomes.

7 Robustness

In this section, I first demonstrate that changes to the model environment behave as expected when I utilize the discount factor described in this paper. I then perform a numerical robustness test to show that the discount factor accurately predicts equilibrium changes to price.

7.1 Results with Alternate Settings

When returns to scale are constant, my model replicates the standard finding that a firm's value is equal to its future capital choice. In this special case, my model and the approach suggested by Grossman and Hart (1979) or Carceles-Poveda and Coen-Pirani (2009) are identical.

Higher adjustment costs ψ will typically reduce the volatility of investment. This parameter can be used to ensure that the dividend process becomes procyclical as is the case in the data. And as adjustment costs increases, the average price of equity does not change significantly, but it becomes much more volatile. In the baseline case $\psi = 1$, price is 1.3 times as volatile as output. If adjustment costs become very high ($\psi = 150$), the standard deviation of equity price becomes 3.4 times as large as the standard deviation of output.

With log utility ($\sigma = 1$), an increase in earnings risk causes similar changes as the baseline case, though with lower magnitude. This result is intuitive. As risk aversion falls, an increase in risk has less of an impact on outcomes.

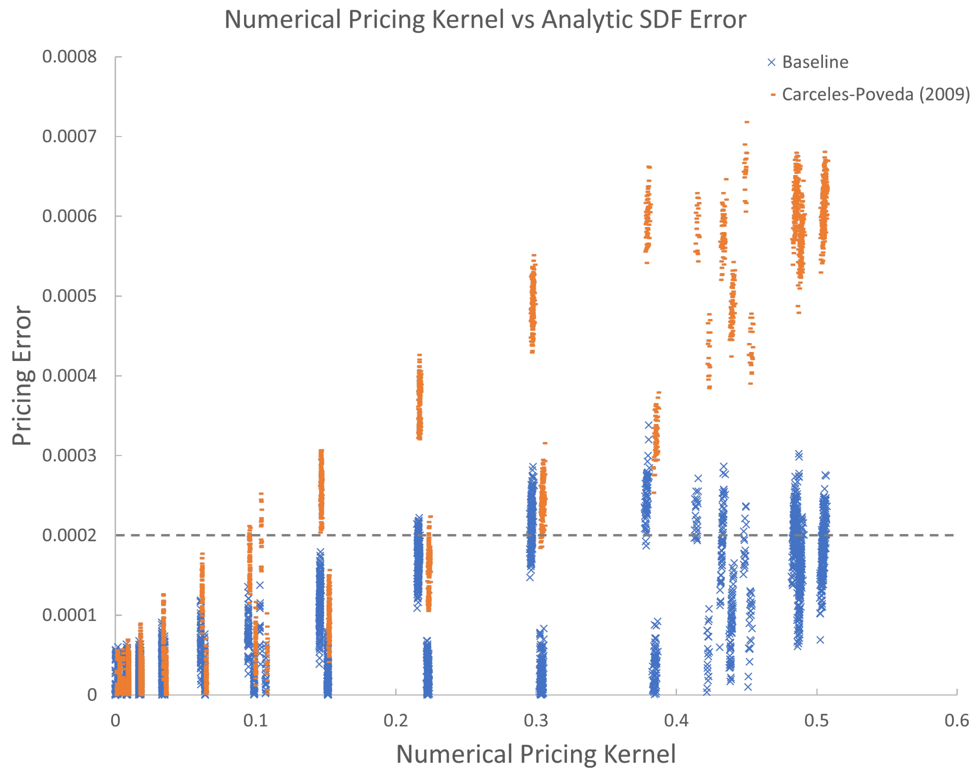

Figure 4: Error between numerical pricing kernel and the analytic SDF. The X axis shows the numerical change in price divided by the size of the change to future dividends. The Y axis shows the absolute value of the error between the stochastic discount factor and the numerical pricing kernel, as described in Equation 30. These comparisons are run in the 1970 labor risk setting with $\underline{a}' = 0.25$ in order to force more households to be near their borrowing constraint. Results are from the last 750 periods of a 1,500 period simulation. The size of future deviations is 1E-4 and the numerical precision for price is 1E-8, so any pricing errors below 0.0002 (the gray line) are within numerical tolerance.

7.2 Numerical Comparison

I verify the accuracy of my analytic stochastic discount factor by computing a numerical change in price during the simulation steps of the algorithm. The stochastic discount factor accurately prices changes to future returns and has a lower error rate than the method proposed by Carceles-Poveda and Coen-Pirani (2009).

I test numerical accuracy during the simulation phase of the algorithm described in Section 4.[29] In each date, I first find the market clearing price and the aggregate law of motion that satisfy equilibrium conditions given the distribution of households and firms. Then, I cycle through each future state z_n' and set future dividends as $\hat{D}_n' = D_n' + 0.0001$. I then find the market clearing price $\hat{P}$ given the change to future dividends. If the discount

[29] I re-calibrate the model setting $\underline{a}' = 0.25$. This choice doesn't shift aggregate outcomes significantly, but it does put more households close to the borrowing limit.

factor proposed in this paper is accurate, we should expect:

$$\chi(z_n'|\mathbb{Z}) = \lim_{\hat{D}_n' \to D_n'} \frac{\hat{P} - P}{\hat{D}_n' - D_n'} \tag{29}$$

The error of my discount factor is the absolute value of the difference between the numerical change in price and the discount factor's anticipated change in price. That is:

$$\text{Error} = \left| \chi(z_n'|\mathbb{Z}) - \frac{\hat{P} - P}{\hat{D}_n' - D_n'} \right| \tag{30}$$

Figure 4 shows the relationship between observed change in price on the X axis and error size on the Y axis for both the benchmark model and the model from Carceles-Poveda and Coen-Pirani (2009). On average, the stochastic discount factor differs from the numerical calculation by 6.2E-5 while the method using start of period shareholder marginal rates of substitution misses by 1.7E-4. Given the numerical test is for a proposed deviation of 1E-4, the analytic discount factor's error in guessing price is only 6.2E-9 (or 1.7E-8 for the current shareholder method). The computational solution method only clears price to an accuracy of 1E-8, so any "pricing errors" in the figure smaller than 0.0002 are within numerical precision.

In Figure 4, the numerical pricing kernel is primarily driven by transition probabilities. With adjustment costs, firms generally don't change their capital stock significantly. TFP is then the largest driver of rates of return across periods.

The error increases with the size of observed price change. The correlation between error and price change is driven by endogenous changes to shareholding. With larger expected changes, households start to change their shareholding choices, and price begins to deviate from the SDF.

As shown in Figure 4, the discounting proposed in this paper has a third lower error than the next most common approach. The difference in error between my model and the method from Carceles-Poveda and Coen-Pirani (2009) is relatively small. As described in Equation 28, the price error is a weighted sum of marginal rates of substitution multiplied by the size of shareholding changes. Households only generally make small changes to shareholding from one period to the next, so the error should be relatively small.

8 Conclusion

In this paper, I derive an equilibrium discounting mechanism that firms can use to maximize their value when they are owned by heterogeneous shareholders. I show that an increase in idiosyncratic household productivity risk increases capital investment levels, lowers equity rate of return, and results in less volatile aggregate consumption sequences. Additionally, a rough calibration of the increase in idiosyncratic risk explains half of the fall in dividend yields observed between the 1970's and the 2010's.

The model presented here has numerous avenues for refinement. First, I plan on better matching the wealth distribution by implementing a β heterogeneity plan as is the case in Krusell and Smith (1998). I expect the increase in wealth inequality will drive additional precautionary savings and will increase share price. Second, I plan on modeling the income process more seriously. Skewness of earnings is procyclical in the data with a long right tail. I anticipate this would also lead to more precautionary savings and higher equity price. Temporary, one period earnings shocks are also a feature of the data that would likely shift firm behavior in my model.

There are a number of possible extensions of this framework, but I focus here on two. In future work, I plan on extending my model to examine the interaction between firm and household heterogeneity over the last 40 years. My discounting method can be immediately extended to a setting with firm heterogeneity with only a small change to the solution algorithm. There is evidence that household earnings are becoming more dispersed over time (Hubmer, Krusell and Smith., 2021) alongside evidence that firm dynamism is falling (Akcigit and Ates, 2022). These phenomena have not been studied in tandem in part because there is not a consensus method for linking heterogeneous shareholders to firm choices.

The next natural extension is in the realm of common ownership. The current frontier of common ownership literature (Azar and Vives, 2021) does not feature endogenous valuation of future states. My approach would extend that literature by providing it with a consistent method for valuing future payoffs when owned by heterogeneous shareholders.

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Appendices

A Alternate Problem of the Private Equity Firm

In this section, I derive the private equity firm's problem if it compensates marginal equity shareholders directly. This alternate formulation results in an identical stochastic discount factor. I include it because it more closely matches the original approach of Grossman and Hart (1979) with one additional assumption (Assumption 1 below).

Assumption 1. *Shareholders only have voting rights if they hold more than the minimum number of shares.*

Assumption 2. *A private equity firm needs 100% of voting shares to approve a capital deviation for it to be approved.*

Assumption 3. *A successful deviation will be small enough that households do not change their shareholding choice.*

Assumptions 1 and 2 require that the private equity firm make side payments to all marginal[30] investors. Households who are made better off by the alternate investment plan $\hat{k}'$ will vote for the plan and pay the private equity firm to implement it. Households who are made worse off by $\hat{k}'$ will be compensated for their votes so that they are at least indifferent to the alternate plan. In this case, the side payment is the price of their vote.

I define the set of shareholder who choose $a' > \underline{a}$ as $\hat{\mathcal{S}} \subset \mathcal{S}$. The private equity firm's problem is written as:

$$\max_{\xi(a,\eta|\hat{k}'),\hat{k}'} \int_{\hat{\mathcal{S}}} -\xi(a,\eta|\hat{k}')\mu(d[a \times \eta]) \tag{31}$$

$$\text{s.t.} \quad u(c) + \beta\mathbb{E}V(a'; z', \mu'_H, \mu'_F) \leq u(\hat{c}) + \beta\mathbb{E}V(a'; z', \mu'_H, \hat{\mu}'_F) \quad \forall (a,\eta) \in \hat{\mathcal{S}} \tag{32}$$

$$\hat{c} \equiv w\eta + \xi + (\hat{P} + \hat{D})a - \hat{P}a'$$

where hat variables $(\hat{c}, \hat{P}, \hat{D}, \hat{\mu}'_F)$ denote the components that shift when $\hat{k}'$ changes. When the private equity firm chooses an alternate scheme $\hat{k}'$, it changes prices and dividend payments today and in the future.

Just as in the baseline case, the private equity firm targets a mass $\varepsilon_{\to 0}$ of identical production firms. Unlike the baseline case, the firm needs to consider how its choice of alternate capital $\hat{k}'$ changes current prices.

[30] Marginal investors, as described earlier, are households who choose to hold more than the minimum number of shares.

The private equity firm's optimal choices are given as:

$$[\xi(\hat{k}')]: \qquad \mu_i = \Omega_i u'(c)$$

$$[\hat{k}']: \quad \varepsilon\Omega_i \frac{\partial p}{\partial \hat{k}'} a' u'(c) = \varepsilon\Omega_i \left(\frac{\partial p + \partial d}{\partial \hat{k}'} a u'(c) + \sum_{n=1}^{N_z} \beta\pi_{,m} \frac{\partial p' + \partial d'}{\partial \hat{k}'} a' \sum_{j=1}^{N_\eta} \pi_{ij} u'(c') + \underbrace{o(\varepsilon)}_{\to 0} \right) \tag{33}$$

where Ω_i is the Lagrange multiplier on each household's participation constraint. If capital rises, households lose out by having to pay a higher price today for future shareholding and they lose out on some amount of dividends. They then benefit from higher price and dividend payments in the future.

If the production firm was already maximizing its net market value (or cum-dividend share price), then $\frac{\partial p' + \partial d'}{\partial k'} = 0$. In equilibrium, $k' = \hat{k}'$. Integrating over households and substituting in the equilibrium result that production firms are net value maximizers yields:

$$-\frac{\partial d}{\partial k'} = \sum_{n=1}^{N_z} \frac{\partial p_n' + \partial d_n'}{\partial k'} \frac{\left(\beta\pi_{mn} \int_{\mathcal{S}} \mathbb{I} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c_{jn}')}{u'(c)} a' \mu(d[a \times \eta]) \right)}{\int_{\mathcal{S}} \mathbb{I} a' \mu(d[a \times \eta])} \tag{34}$$

Finally, I substitute in the production firm's optiamlity conditions:

$$\sum_{n=1}^{N_z} \frac{\partial p_n' + \partial d_n'}{\partial k'} \tilde{\chi}(z_n) = \sum_{n=1}^{N_z} \frac{\partial p_n' + \partial d_n'}{\partial k'} \frac{\left(\beta\pi_{mn} \int_{\mathcal{S}} \mathbb{I} \frac{\sum_{j=1}^{N_\eta} \pi_{ij} u'(c_{jn}')}{u'(c)} a' \mu(d[a \times \eta]) \right)}{\int_{\mathcal{S}} \mathbb{I} a' \mu(d[a \times \eta])} \tag{35}$$

With the assumptions laid out at the beginning of this section, the firm's discounting regime $\tilde{\chi}$ is consistent with the the mutual fund's pricing kernel χ.

B Micro-Founded Private Equity Firm

Here, I model an alternate setting where the private equity firm looks more like one seen in the data. Equilibrium results and discount factors are identical. The only change is the narrative description and one of the equilibrium conditions.

Each period, a deep-pocketed risk-neutral private equity firm attempts to find an arbitrage opportunity by taking over a production firm. To do so, it purchases 100% of shares of a single production firm from the mutual fund *before* dividends are paid. The private equity firm then implements a new investment plan, keeps the dividends, and sells remaining shares back to the mutual fund. The private equity firm only acts if it can earn a strictly positive profit from taking over the production firm.

The mutual fund willingly sells shares to the private equity firm at the price $p + d$, which leaves its budget constraint entirely unchanged. It is also willing to buy back the shares at the new price $\hat{p}$ as long as that price satisfies the mutual fund's first order condition in Equation 13.

Under this alternate setting, the private equity firm's problem is now written as:

$$\max_{\hat{k}'} p(k;\mathbb{Z};\hat{k}') + d(k;\mathbb{Z};\hat{k}') - \Big(p(k;\mathbb{Z};k') + d(k;\mathbb{Z};\hat{k}')\Big) \tag{36}$$

and the final equilibrium condition described in Section 3 is replaced with, "The private equity firm cannot find a profitable deviation, which is satisfied when the maximum of Equation 36 is zero."

The private equity firm's optimal choices are the same with this specification as in the main body of the text. Nothing has changed about the mutual fund's problem, so the discount factor in Equation 24 carries through to this setting. All results are identical.

C Endogenous Borrowing Limit

The natural borrowing limit is set such that the household will always be able to service its debt in aggregate state $\mathbb{Z}$. Additionally, it must be able to service this debt if it starts with the lowest level of savings $a = \underline{\mathrm{A}}$ and with the lowest productivity draw $\eta = \eta_1$. That is:

$$\underbrace{0}_{c} + P(\mathbb{Z})\underline{\mathrm{A}} = (P(\mathbb{Z}) + D(\mathbb{Z}))\underline{\mathrm{A}} + w(\mathbb{Z})\eta_1 \tag{37}$$

$$\Rightarrow \underline{\mathrm{A}} = \max \frac{-w(\mathbb{Z})\eta_1}{D(\mathbb{Z})} \tag{38}$$

If the utility function satisfies the standard Inada condition,[31] then households will never choose $a' = \underline{\mathrm{A}}$.

[31] $\lim_{c\to 0} u'(c) = +\infty$