5. To inscribe an equilateral triangle in a square so that its angles touch its sides. We make the square *ABGD* and we prolong the line *DG* [fig. 5.25] to *E* to make *DE* equal to *GD*. And on *GE* we construct the semicircle *EAG*. Then with *G* as center and distance *DG* [as radius] we mark *Z* [on the semicircle *EAD*], and with *E* as center and distance *EZ* we mark *H* [on line *DG*]. And we make *AT* equal to *GH*. Then we draw *BT*, *BH*, and *HT*. Then triangle *BTH* is equilateral and it was constructed in the square *ABGD*.

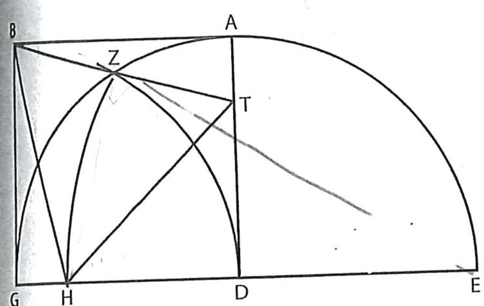

FIGURE 5.25

Proof: For this a lemma is necessary. Let the circle *ABG* be given [fig. 5.26], *AG* its diameter, and *GB* the side of the regular hexagon [inscribed in the circle]. One draws the line *AB*. It is the side of the regular triangle [inscribed in the circle]. One prolongs it to *D*, such that *AD* = *AG*. Since, now, arc *AB* is twice arc *BG*, angle *AGB* is twice angle *BAG*. The two together, however, are a right angle, so angle *AGB* is two-thirds a right angle, and *BAG* is one-third a right angle. Therefore [since the angles of triangle *AGD* are two right angles] there remain for angles *AGD* and *ADG* one and two-thirds of a right angle. However, since they are equally large, each of them is one-half and one-third of a right angle. Since angle *AGB* is two-thirds of a right angle, angle *BGD* is one-sixth of a right angle. Thus, it is proven that if, in a right-angled triangle [*DBG*], the two sides about the right angle are the side of a hexagon [of the circumcircle [of triangle *ABG*]] and the excess of the diameter [of the circumcircle] over the side of the inscribed regular triangle, then the smaller angle of this triangle [*DBG*] is one sixth of a right angle.

Now, in our previous figure [fig. 5.25], in triangle *BHG* the angle *G* is a right angle, formed by *BG* (i.e., the side of the hexagon in circle *EAG*) and *GH* (the

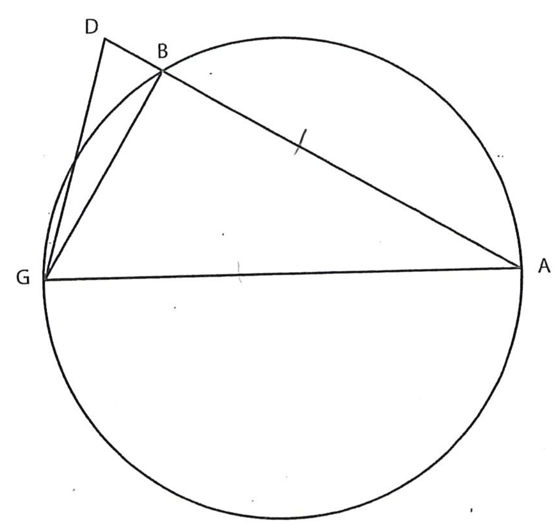

FIGURE 5.26

excess of diameter *EG* over *EZ*, which is the side of the triangle). Thus, by the lemma, angle *GBH* is equal to one-sixth of a right angle. Since the lines *AB* and *AT* are equal to the lines *BG* and *GH*, and the angles at *A* and *G* are each equal to a right angle, angle *ABT* is also equal to one-sixth of a right angle. Thus, angle *TBH* is equal to two-thirds of a right angle. However, the two angles *BTH* and *BHT* are equal to each other, so each of them is equal to two-thirds of a right angle. For that reason, the three sides, *BT*, *BH*, and *TH* are equally large and the triangle *TBH* is equilateral.

Aḥmad ibn Muḥammad ibn ʿAbd al-Jalīl al-Sijzī (late tenth century) was a younger contemporary of al-Kūhī. He was a prolific writer, with at least 50 surviving works to his credit, and an active correspondent. Although he was not of the first rank mathematically, he stressed the need to substitute constructions by conic sections for the ancient verging constructions, such as that used by Archimedes in his construction of the regular heptagon. (He called the latter constructions "moving geometry.") And his treatise on the construction of the astrolabe was an important source for al-Bīrūnī's treatise on the same subject. Al-Sijzī's work on the regular heptagon, which we quote from here, brought him into conflict with his contemporary, Abū al-Jūd, whose erroneous construction he exposed and with whom he exchanged mutual allegations of plagiarism.[85]

Al-Sijzī on the Construction of the Heptagon

In the name of God, the Merciful, the Compassionate. *The book on the Construction of the Heptagon and the Division of the Rectilineal Angle into Three Equal Parts*, by Aḥmad ibn Muḥammad ibn 'Abd al-Jalīl al-Sijzī.

I am astonished that anybody who pursues and occupies himself with the art of geometry, even though he acquires it from the excellent Ancients, thinks that there are weakness and shortcomings in them; and especially when he is a beginner and a student, with so little knowledge of it that he imagines that he can achieve with very little effort things which he believes to be easy to handle and easily understood, although that was far beyond the understanding of those who are trained in this art and skilled in it. I wish I knew of any power, perspicacity, skill and profundity that would favor the opinion that the heptagon can be found by means of the lemmas of somebody who is reading some introductory book—I mean the *Book of Euclid on the Elements*, somebody who has neither skill nor training and finds fault with those who are prominent in this art. What makes it necessary to believe in the weakness of the excellent Archimedes, with his superiority in geometry over the rest of the geometers? He reached such a high level in geometry that the Greeks called him "the geometer Archimedes." None among the Ancients nor any of the later geometers were called by his name because of his excellence in geometry. He took great pains to find out useful things. By his power he completed the tools, the instruments, and the mechanical procedures. He established the lemmas for the heptagon and followed a path leading to success. By his power we have understood the heptagon just as Heron understood the machines by his [Archimedes'] power and his hard work in mathematical matters.

[85]For an account of this episode see [Hogendijk 1984, 242–69].