# Extremely low mass Helium white dwarfs in close, detached double white dwarf binaries

In this notebook, we detail the

1) finite temperature mass radius relation, and
2) basic tidal heating timescale

for helium composition “extremely low mass white dwarf” (ELM WD) in detached double white dwarf binaries (DWDBs)
These are Figure 1 and Figure 3 in McNeill and Hirai 2025 (submitted).

## physical constants

In[498]:=

$$\text{afun} = \frac{\text{GG}^{1/3}\ ((\text{m1} + \text{m2}))^{1/3}}{(\text{f})^{2/3}\ \pi^{2/3}};$$

In[499]:=

$$\text{Rsol} = 6.995 \times 10^{10};$$
$$\text{Msol} = 2 \times 10^{33};$$
$$\text{Mchirpf[m11\_, m22\_]} = \frac{(\text{m11 m22})^{3/5}}{(\text{m11} + \text{m22})^{1/5}};$$
$$\text{G} = 6.67 \times 10^{-8};$$
$$\text{c} = 3 \times 10^{10};$$
$$\text{c} = 3 \times 10^{10};$$
$$\text{Msol} = 2 \times 10^{33};$$
$$\text{mHz} = 0.001;$$
$$\text{kK4} = 10^{4};$$
$$\sigma = 5.67 \times 10^{-5};$$

## Figure 1 : Finite temperature mass radius relation

Here we present a contour plot for finite temperature white dwarfs. This is a fit for Panei et al 2000’s Figure 3.

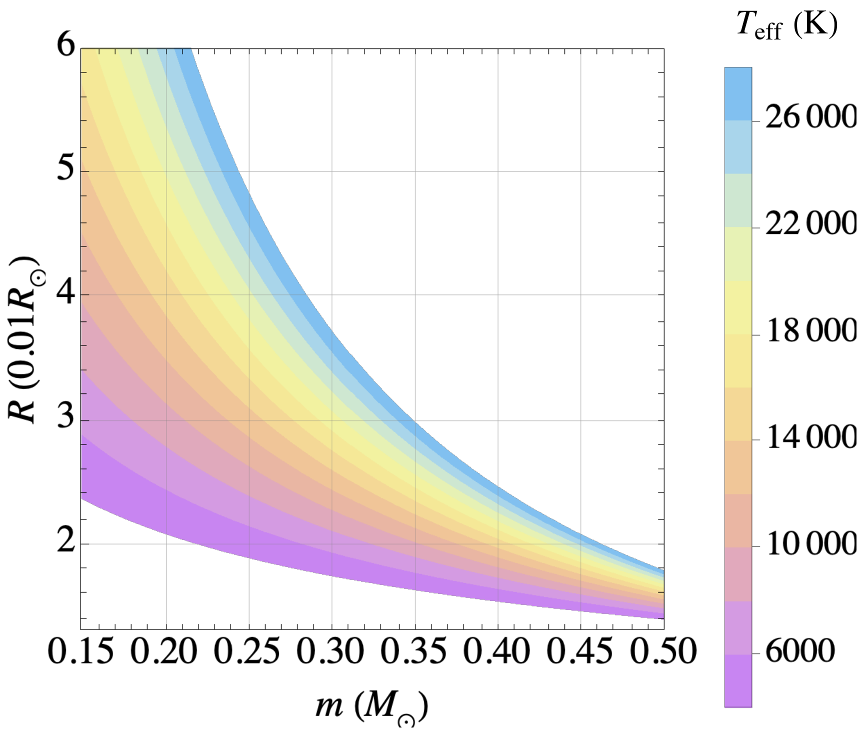
In[509]:=

```
labels = Directive[FontSize → 18, FontFamily → "Times", Black];
```

In[510]:=

```
contourprim = ContourPlot[
  (1.1798232975286564`*^47 Log[mmm]^2 - 1.4023785637137418`*^48 Log[mmm]
      Log[0.74269870382108113407122043463241581874`15.954589770191005 rr] +
     4.167288525500679`*^48
      Log[0.74269870382108113407122043463241581874`15.954589770191005 rr]^2) /
   (7.70388920852663`*^43 + 3.4482434674930335`*^44 Log[mmm] +
     3.858565034251842`*^44 Log[mmm]^2) , {mmm, 0.15, 0.5},
  {rr, 1.3, 6}, Contours → {1, 4000, 6000, 8000, 10 000, 12 000, 14 000,
    16 000, 18 000, 20 000, 22 000, 24 000, 26 000, 28 000},
  ImageSize → Medium, ColorFunction → "Pastel", Axes → True,
  FrameLabel → {Style["m (M⊙)", 20, Black], Style["R (0.01R⊙)", 20, Black]},
  FrameTicksStyle → Directive[FontSize → 20, Black],
  ContourStyle → None, ScalingFunctions → {None, None},
  BaseStyle → {FontSize → 20},
  PlotLegends → Placed[BarLegend[Automatic,
     LegendLabel → Style["Teff (K)", Black], LabelStyle → labels], {After, Top}],
  PlotRange → {{0.15, 0.5}, {1.3, 6}, {4000, 28 000}},
  LabelStyle → (FontFamily → "Times"), GridLines → Automatic
 ]
```

Out[510]=



In[511]:=

```
cplot1 = ContourPlot[
   (1.1798232975286564`*^47 Log[mmm]^2 - 1.4023785637137418`*^48 Log[mmm]
        Log[0.74269870382108113407122043463241581874`15.954589770191005 rr] +
      4.167288525500679`*^48
        Log[0.74269870382108113407122043463241581874`15.954589770191005 rr]^2) /
    (7.70388920852663`*^43 + 3.4482434674930335`*^44 Log[mmm] +
      3.858565034251842`*^44 Log[mmm]^2) , {mmm, 0.15, 0.5},
   {rr, 1.3, 6}, Contours → {1, 4000, 6000, 8000, 10 000, 12 000, 14 000,
     16 000, 18 000, 20 000, 22 000, 24 000, 26 000, 28 000},
   ImageSize → Medium, ColorFunction → "Pastel", Axes → True,
   FrameLabel → {Style["m (M⊙)", Bold, 20], Style["R(0.01R⊙)", Bold, 20]},
   FrameTicksStyle → Directive[FontSize → 20],
   ContourStyle → None, ScalingFunctions → {None, None},
   LabelStyle → (FontFamily → "Times"), BaseStyle → {FontSize → 20},
   PlotLegends → Placed[BarLegend[Automatic,
      LegendLabel → Style["Teff (K)", 18], LabelStyle → labels], {After, Top}],
   PlotRange → {{0.15, 0.5}, {1.3, 3.25}, {4000, 20 000}},
   BaseStyle → Directive[Opacity[1]]];
```

Now we will make a scatter plot of the measured R, m and T. Taking data compiled from ZTF (Brown et al. 2011; Burdge et al. 2019a,b, 2020a,b)

In[512]:=

```
mtest = {0.32, 0.45, 0.167, 0.32, 0.3, 0.33, 0.38,
   0.28, 0.4, 0.36, 0.36, 0.323, 0.335, 0.26, 1, 1, 0.27, 0.19};
rtest = {2.319, 2.069, 5.70, 2.90, 2.80, 2.49,
   2.24, 2.5, 2.2, 2.9, 2.2, 2.98, 2.75, 3.53, 1, 1, 2.794, 5.1};
Ttest = 1000 {12.8, 26.45, 20, 18.25, 15.3, 16.8,
    19.9, 12, 20.4, 26, 16.5, 26, 19, 16.4, 28, 4, 13.4, 16.4};
df2 = Transpose[{mtest, rtest, Ttest}];
pts2 = df2;
Graphics[{AbsoluteThickness[3], Point[pts2⟦All, {1, 2}⟧,
    VertexColors → ColorData["Pastel"] /@ Rescale[pts2⟦All, 3⟧]]},
  AspectRatio → 1, Frame → True];
stylesTemp = ColorData["Pastel"] /@ Rescale[pts2⟦All, 3⟧];
Pltfun[ii_] := ListPlot[{pts2⟦All, {1, 2}⟧⟦ii⟧},
   PlotRange → {{0.1, 1}, {1, 6}}, AspectRatio → 1, PlotMarkers → {"★", 18},
   PlotStyle → {{stylesTemp⟦ii⟧}}, LabelStyle → (FontFamily → "Times"),
   PlotLegends → {Style["R(m,Teff) of detached WD", 16]}];
```

In[520]:=

```
outline = ListPlot[pts2⟦All, {1, 2}⟧, PlotRange → {{0.1, 0.5}, {1, 6}},
   AspectRatio → 1, PlotMarkers → {"★", 24}, PlotStyle → {{Black}}, PlotLegends →
    {Style["from Table 1", 18]}, LabelStyle → (FontFamily → "Times")];
```

In[521]:=

```
outline = ListPlot[pts2[[All, {1, 2}]], PlotRange → {{0.1, 0.5}, {1, 6}},
   AspectRatio → 1, PlotMarkers → {"★", 24}, PlotStyle → {{Black}},
   PlotLegends → {Style["from Table 1", 18, Bold]},
   LabelStyle → (FontFamily → "Times")];
```

In[522]:=

In[523]:=

```
Show[ListPlot[pts2[[All, {1, 2}]], PlotRange → {{0.9, 1.2}, {0.5, 6}},
   AspectRatio → 1, PlotMarkers → {"★", 24}, PlotStyle → {{Black}}, PlotLegends →
    {Style["from Table 1", 18]}, LabelStyle → (FontFamily → "Times")],
  Pltfun[1], Pltfun[2], Pltfun[3], Pltfun[4], Pltfun[5], Pltfun[6],
  Pltfun[7], Pltfun[8], Pltfun[9], Pltfun[10], Pltfun[11], Pltfun[12],
  Pltfun[13], Pltfun[14], Pltfun[15], Pltfun[16], Pltfun[17], Pltfun[18]];
```

We also define the "cold" white dwarf mass radius relation for completely degenerate white dwarfs from Verbunt & Rappaport 1998

In[524]:=

$$\text{Regg[m\_]} := 0.0114 \left((m/1.44)^{-2/3} - (m/1.44)^{2/3}\right)^{1/2} \left(1 + 3.5 \left(m/\left(5.7\times10^{-4}\right)\right)^{-2/3} + \left(\left(5.7\times10^{-4}\right)/m\right)\right)^{-2/3}$$

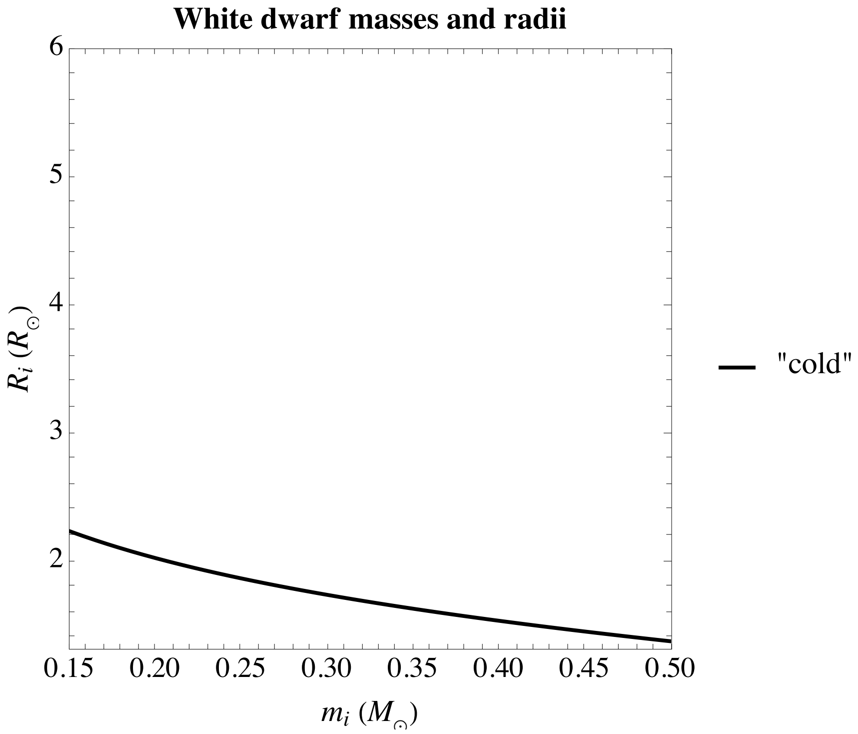
In[525]:=

```
plotegg = Plot[100 Regg[m], {m, 0.15, 0.5}, AspectRatio → 1,
   AxesLabel → {Style["mass (solar)", 16], Style["radius (solar)", 16]},
   BaseStyle → {FontSize → 15}, LabelStyle → (FontFamily → "Times"),
   PlotStyle → {Black, Thick}, PlotRange → {{0.15, 0.5}, {100 × 0.013, 100 × 0.06}},
   PlotLegends → {Style["R(m)", 16, Italic]},
   FrameLabel → {Style["mi (M⊙)", 16], Style["Ri (R⊙)", 16],
     Style["White dwarf masses and radii", Bold, 16], Style["Ri (10^8cm)", 16]},
   BaseStyle → {FontSize → 15}, LabelStyle → (FontFamily → "Times"),
   Frame → True, FrameTicks → {{{.1, .2, .3, .4, .5}, {Automatic},
      Charting`ScaledTicks[{#/(Rsol/10^8) &, Rsol/10^8 # &}]}}];
```

In[526]:=

```
Plot[100 Regg[m], {m, 0.15, 0.5}, AspectRatio → 1,
 AxesLabel → {Style["mass (solar)", 16], Style["radius (solar)", 16]},
 BaseStyle → {FontSize → 15}, LabelStyle → (FontFamily → "Times"),
 PlotStyle → {Black, Thick}, PlotRange → {{0.15, 0.5}, {100 × 0.013, 100 × 0.06}},
 PlotLegends → {Style[" \"cold\" ", 16]}, FrameLabel → {Style["mi (M⊙)", 16],
   Style["Ri (R⊙)", 16], Style["White dwarf masses and radii", Bold, 16]},
 BaseStyle → {FontSize → 15}, LabelStyle → (FontFamily → "Times"),
 Frame → True, FrameTicks → {{{.1, .2, .3, .4, .5}, {Automatic},
    ChartingScaledTicks[{#/(Rsol/10^8) &, Rsol/10^8 # &}]}}]
```

Out[526]=



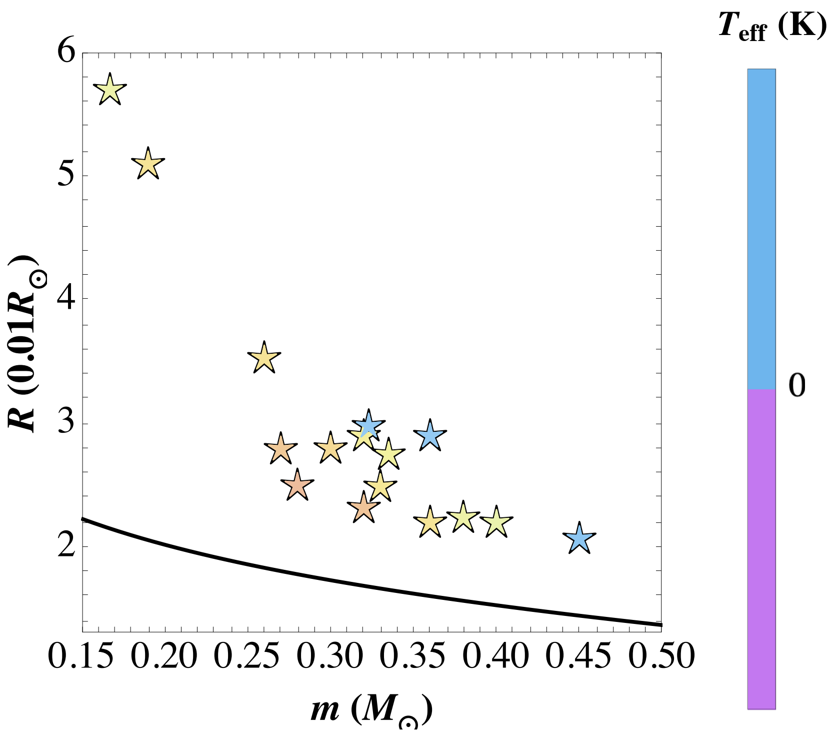
In[527]:=

```
contourprimempty = ContourPlot[
  0 (1.1798232975286564`*^47 Log[mmm]^2 - 1.4023785637137418`*^48 Log[mmm]
       Log[0.742698703821081134071220434632415818740`15.954589770191005 rr] +
      4.167288525500679`*^48
       Log[0.742698703821081134071220434632415818740`15.954589770191005 rr]^2) /
    (7.70388920852663`*^43 + 3.4482434674930335`*^44 Log[mmm] +
      3.858565034251842`*^44 Log[mmm]^2) , {mmm, 0.15, 0.5},
  {rr, 1.3, 6}, Contours → {1, 4000, 6000, 8000, 10 000, 12 000, 14 000,
    16 000, 18 000, 20 000, 22 000, 24 000, 26 000, 28 000},
  ImageSize → Medium, ColorFunction → "Pastel", Axes → True,
  FrameLabel → {Style["m (M⊙)", Bold, 20], Style["R (0.01R⊙)", Bold, 20]},
  FrameTicksStyle → Directive[FontSize → 20],
  ContourStyle → None, ScalingFunctions → {None, None},
  BaseStyle → {FontSize → 20},
  PlotLegends → Placed[BarLegend[Automatic,
     LegendLabel → Style["Teff (K)", Bold], LabelStyle → labels], {After, Top}],
  PlotRange → {{0.15, 0.5}, {1.3, 6}, {4000, 28 000}},
  LabelStyle → (FontFamily → "Times")
 ];
```

In[528]:=

```
Show[contourprimempty, plotegg, outline, Pltfun[1], Pltfun[2],
 Pltfun[3], Pltfun[4], Pltfun[5], Pltfun[6], Pltfun[7], Pltfun[8],
 Pltfun[9], Pltfun[10], Pltfun[11], Pltfun[12], Pltfun[13],
 Pltfun[14], Pltfun[15], Pltfun[16], Pltfun[17], Pltfun[18]]
```

Out[528]=

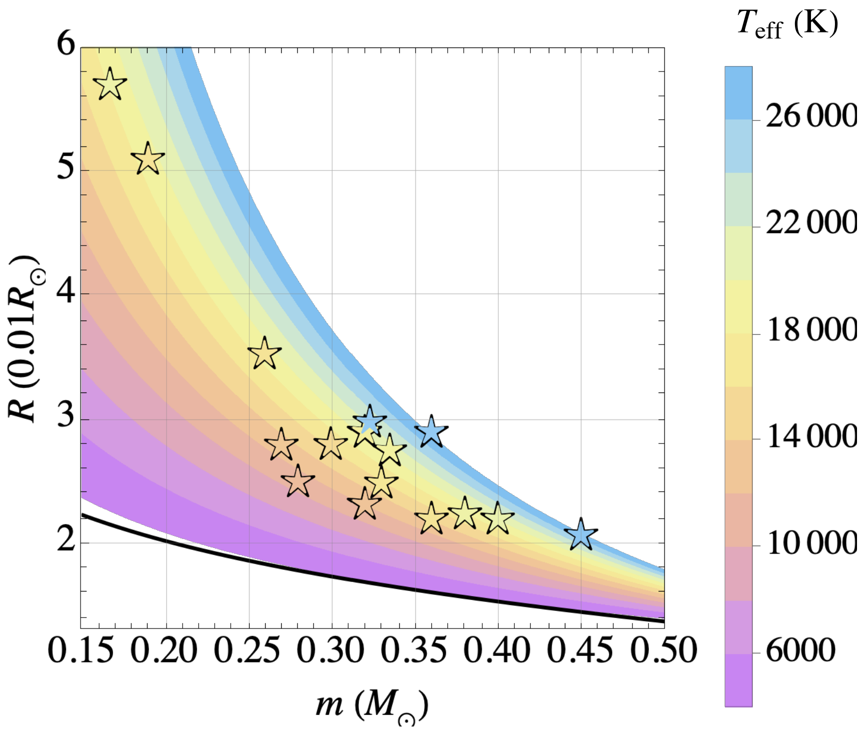


Putting these all together becomes Figure 1:

In[529]:=

```
Show[contourprim, plotegg, outline, Pltfun[1], Pltfun[2],
 Pltfun[3], Pltfun[4], Pltfun[5], Pltfun[6], Pltfun[7], Pltfun[8],
 Pltfun[9], Pltfun[10], Pltfun[11], Pltfun[12], Pltfun[13],
 Pltfun[14], Pltfun[15], Pltfun[16], Pltfun[17], Pltfun[18]]
```

Out[529]=




# Figure 2 : tidal heating vs cooling regime

Using the tidal friction timescale defined in Hut 1981, the gravitational wave frequency *f* increase due to tidal heating is

In[530]:=

$$\text{fdotTD1[m1\_, m2\_, fGW\_, R1\_]} = \frac{18\ \text{fGW}^{13/3}\ \text{m2}\ \pi^{13/3}\ \text{R1}^5 \left(\frac{\text{fGW}}{2}\right)}{G^{5/3}\ \text{m1}\ (\text{m1}+\text{m2})^{5/3}}\ \text{konQ1}$$

Out[530]=

$$\frac{1.17044\times 10^{15}\ \text{fGW}^{16/3}\ \text{konQ1 m2 R1}^5}{\text{m1}\ (\text{m1}+\text{m2})^{5/3}}$$

From gravitational waves the *f* increase is given by Peters 1964

In[531]:=

$$\text{fdotGW[m1\_, m2\_, fGW\_]} = \frac{96\ \pi^{8/3}\ G^{5/3}\ (\text{fGW})^{11/3}\ (\text{Mchirpf[m1, m2]})^{5/3}}{5\ c^5}\ \text{// FullSimplify}$$

Out[531]=

$$1.83501\times 10^{-62}\ \text{fGW}^{11/3} \left(\frac{(\text{m1 m2})^{3/5}}{(\text{m1}+\text{m2})^{1/5}}\right)^{5/3}$$

Here we solve for an appropriate k/Q value . Calibrated to Piro 2011.

In[532]:=

```
ratiosol = konQ1 /. Solve[fdotTD1[0.26 Msol, 0.5 Msol, 0.0026, 0.0371 Rsol] ==
     0.06 fdotGW[0.26 Msol, 0.5 Msol, 0.0026] , konQ1][[1]]
```

Out[532]=

$$7.682 \times 10^{-12}$$

For J1539

In[533]:=

```
ratiosol = konQ1 /. Solve[fdotTD1[0.21 Msol, 0.61 Msol, 0.0048, 0.0314 Rsol] ==
     0.1 fdotGW[0.21 Msol, 0.61 Msol, 0.0048] , konQ1][[1]]
```

Out[533]=

$$7.6605 \times 10^{-12}$$

We choose k/Q value:

In[534]:=

$$\texttt{kQratio} = 8 \times 10^{-12};$$

The fits from Panei et al. 2000 for temperature dependent mass radius relation, defined in McNeill and Hirai 2025 are:

In[535]:=

$$\texttt{RPanei[m\_, T\_]} := 0.0132 \times 10^{-0.00177\, T^{1/2}}\, m^{0.148-0.00941\, T^{1/2}}$$

In[536]:=

$$\texttt{Rscale[m1a\_, T1a\_]} := 10^{-0.02792426461145596` + 0.7641778013995925` \sqrt{T1a}}\, m1a^{0.14797691065884058` - 0.9408955042478873` \sqrt{T1a}}$$

In[537]:=

```
WDid = {J0538, J0533, J2029, J0722, J1749, J1901, J2243, J0651, J1539};
m1prims = {0.32, 0.167, 0.32, 0.33, 0.28, 0.36, 0.323, 0.26, 0.21};
T1prims = {12.8, 20, 18.25, 16.8, 12, 26, 26.3, 16.53, 10} 1000;
m2secs = {0.45, 0.652, 0.3, 0.38, 0.4, 0.36, 0.335, 0.5, 0.61};
Porb = {866.6, 1233.97, 1252.06, 1422.55, 1586.03, 2436.11, 528, 765, 414.8};
fGWs = 2 / Porb;
```

The Roche lobe frequency is given by Paczyński 1971

In[543]:=

$$\texttt{fGWRL} = \frac{2^{3/2}}{9\,\pi} \left( \frac{G\ \texttt{m1prims}\ \texttt{Msol}}{(\texttt{Rscale[m1prims 10, T1prims / 10000] Rsol / 100})^{3}} \right)^{1/2}$$

Out[543]=

{0.00971922, 0.0016704, 0.00780715, 0.00879815,
 0.00789618, 0.00812451, 0.00610308, 0.00539583, 0.005439}

White dwarf cooling is given by Mestel 1952, Hurley and Shara 2003:

In[544]:=

```
testZ = 0.02;
Atest = 4;
Lsol = 3.826 × 10^33;
Tcold = 4000;
```

In[548]:=

$$L2a[m1\_, t\_, Z\_] := \frac{300\ m1\ Z^{0.4}}{(Atest\ (t+0.1))^{1.18}};$$

In[549]:=

$$L2b[m1\_, t\_, Z\_] := \frac{300\ (9000\ Atest)^{5.3}\ m1\ Z^{0.4}}{(Atest\ (t+0.1))^{6.48}};$$

Here we numerically solve the cooling timescale, which is the time from formation to Tcold which we defined as 4000 K .

In[550]:=

$$\tau cools2 = \left(t\ /.\ Table\left[NSolve\left[7.1\times10^{-4}\ (Rsol\ RPanei[m1prims[[i]], Tcold])^{2}\ Tcold^{4} == Piecewise[\{\{L2a[m1prims[[i]], t, testZ], t<9000\}, \{L2b[m1prims[[i]], t, testZ], t>9000\}\}]\ Lsol, t\right], \{i, 1, 9\}\right]\right)\ //\ Flatten$$

NSolve: NSolve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

NSolve: NSolve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

NSolve: NSolve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

General: Further output of NSolve::ratnz will be suppressed during this calculation.

Out[550]=

{9325.86, 3851.99, 9325.86, 9410.13, 8830.64, 9652.56, 9351.34, 7840.16, 5564.45}

Then we calculate the tidal friction timescale defined in Hut 1981

In[551]:=

$$\tau mergeTDfixMyr2 = Table\left[\left(\frac{2}{3}\times\frac{2}{18}\ \left(\left(G^{5/3}\ m1prims[[i]]\ (m1prims[[i]]\ Msol + m2secs[[i]]\ Msol)^{5/3}\ kQratio^{-1}\right)\Big/ \left(fGWs[[i]]^{13/3}\ m2secs[[i]]\ \pi^{13/3}\ (Rsol\ /\ 100\ Rscale[m1prims[[i]]\ 10, T1prims[[i]]\ /\ 10\,000])^{5}\right)\right)\right)\Big/\ \left(3.15\times10^{7}\times10^{6}\right), \{i, 1, 9\}\right]$$

Out[551]=

{711.981, 10.9677, 1766.42, 4434.85, 4887.26, 35 675.7, 18.1169, 58.9895, 4.58024}

Merger timescale from Peters 1963

The time until Roche contact from present day until Roche lobe frequency (using Paczyński 1971’s)

In[552]:=

$$\tau RL = Table\left[Integrate\left[\left(\frac{96\ \pi^{8/3}\ G^{5/3}}{5\ c^{5}}\ (Mchirpf[m1prims[[i]]\ Msol, m2secs[[i]]\ Msol])^{5/3}\ f^{11/3}\right)^{-1}, \{f, fGWs[[i]], fGWRL[[i]]\}\right], \{i, 1, 9\}\right]\Big/\ \left(3.146\times10^{7}\times10^{3}\right)$$

Out[552]=

{1369.82, 374.761, 5136.56, 5823.65, 8590.42, 23 891.3, 339.511, 946.937, 61.9168}

In[553]:=

```
df1 = Transpose[{-Log10[τmergeTDfixMyr2 / τcools2], T1prims, -Log[τRL]}]
```

Out[553]=

```
{{1.11722, 12 800., -7.22243},
 {2.54557, 20 000, -5.92629}, {0.722596, 18 250., -8.54414},
 {0.326716, 16 800., -8.66968}, {0.256927, 12 000, -9.0584},
 {-0.56773, 26 000, -10.0813}, {2.71279, 26 300., -5.82751},
 {2.12355, 16 530., -6.85323}, {3.08453, 10 000, -4.12579}}
```

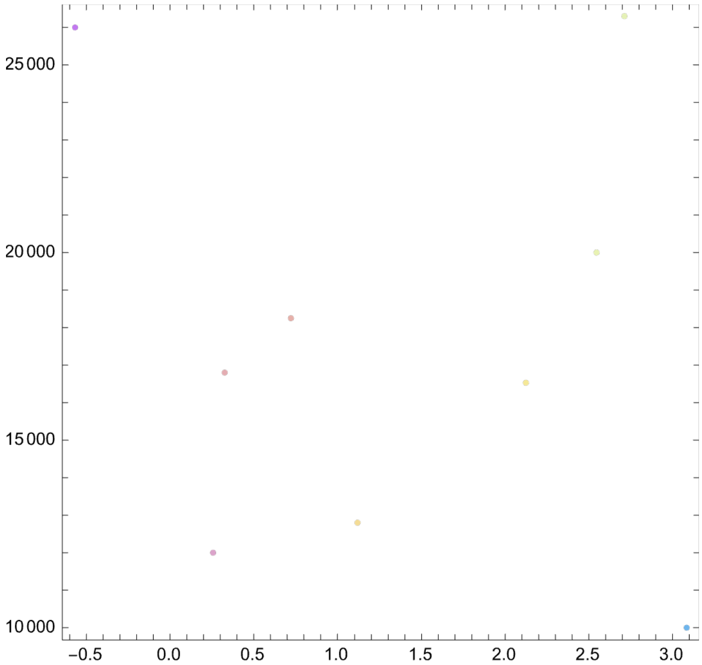
In[554]:=

```
pts2 = df1
Graphics[{AbsoluteThickness[3], Point[pts2〚All, {1, 2}〛,
   VertexColors → ColorData["Pastel"] /@ Rescale[pts2〚All, 3〛]]},
 AspectRatio → 1, Frame → True]
```

Out[554]=

```
{{1.11722, 12 800., -7.22243},
 {2.54557, 20 000, -5.92629}, {0.722596, 18 250., -8.54414},
 {0.326716, 16 800., -8.66968}, {0.256927, 12 000, -9.0584},
 {-0.56773, 26 000, -10.0813}, {2.71279, 26 300., -5.82751},
 {2.12355, 16 530., -6.85323}, {3.08453, 10 000, -4.12579}}
```

Out[555]=




In[556]:=

```
stylesTemp = ColorData["AvocadoColors"] /@ Rescale[pts2〚All, 3〛]
```

Out[556]=

In[557]:=

```
Pltfun[ii_] :=
 ListPlot[{pts2〚All, {1, 2}〛〚ii〛}, PlotRange → {{-1.5, 4.2}, {6000, 32 000}},
  AspectRatio → 1, PlotMarkers → {"★", 18},
  PlotStyle → {{stylesTemp〚ii〛}}, LabelStyle → (FontFamily → "Times")]
```

In[558]:=

```
br2 = ListPlot[Transpose[{-Log10[τmergeTDfixMyr2 / τcools2], T1prims}],
   AspectRatio → 1, PlotMarkers → {"★", 25}, PlotStyle → {{Black}, "*"},
   Frame → True, LabelStyle → (FontFamily → "Times"),
   FrameLabel → {Style["Log10[τcool/τTD]"], Style["Teff (K)", 16]},
   BaseStyle → {FontSize → 16}, PlotRange → {{-1.5, 4.2}, {6000, 32 000}},
   PlotLegends → {Style["Q1/k1= 3.5 × 10^9", 16]}, GridLines → Automatic];
```

In[559]:=

```
labels = Directive[FontSize → 18, FontFamily → "Times"];
```

In[560]:=

```
cptrack =
  ContourPlot[ tscale, {ratio, 0, 4}, {tscale, Min[Log10[τRL]], Max[Log10[τRL]]},
   Contours → Table[(i + 1) 0.1, {i, 0, 43}], ImageSize → Medium,
   ColorFunction → (ColorData["AvocadoColors"]),
   Axes → True, FrameTicksStyle → Directive[FontSize → 18],
   ContourStyle → None, ScalingFunctions → {None, None, None},
   PlotLegends → Placed[BarLegend[Automatic, LegendLabel →
       Style["Log10(τRL in kyr)", 18], LabelStyle → labels], {After, Top}],
   PlotRange → {{0, 4}, {6000, 32 000}, {2, 4.5}}, Frame → True,
   LabelStyle → (FontFamily → "Times"),
   FrameLabel → {Style["Log10[τcool/τTD]", 18], Style["Teff (K)", 18]},
   GridLines → Automatic, FrameStyle → Automatic];
```

In[561]:=

```
cptrack = ContourPlot[ tscale, {ratio, 0, 4},
   {tscale, 2, 5}, Contours → Table[(i + 1) 0.1, {i, 0, 43}],
   ImageSize → Medium, ColorFunction → (ColorData["AvocadoColors"]),
   Axes → True, FrameTicksStyle → Directive[FontSize → 18],
   ContourStyle → None, ScalingFunctions → {None, None, None},
   PlotLegends → Placed[BarLegend[Automatic, LegendLabel →
       Style["Log10(τRL in kyr)", 18], LabelStyle → labels], {After, Top}],
   PlotRange → {{0, 4}, {6000, 32 000}, {2.5, 4.5}}, Frame → True,
   LabelStyle → (FontFamily → "Times"),
   FrameLabel → {Style["Log10[τcool/τTD]", 18], Style["Teff (K)", 18]},
   GridLines → Automatic, FrameStyle → Automatic];
```

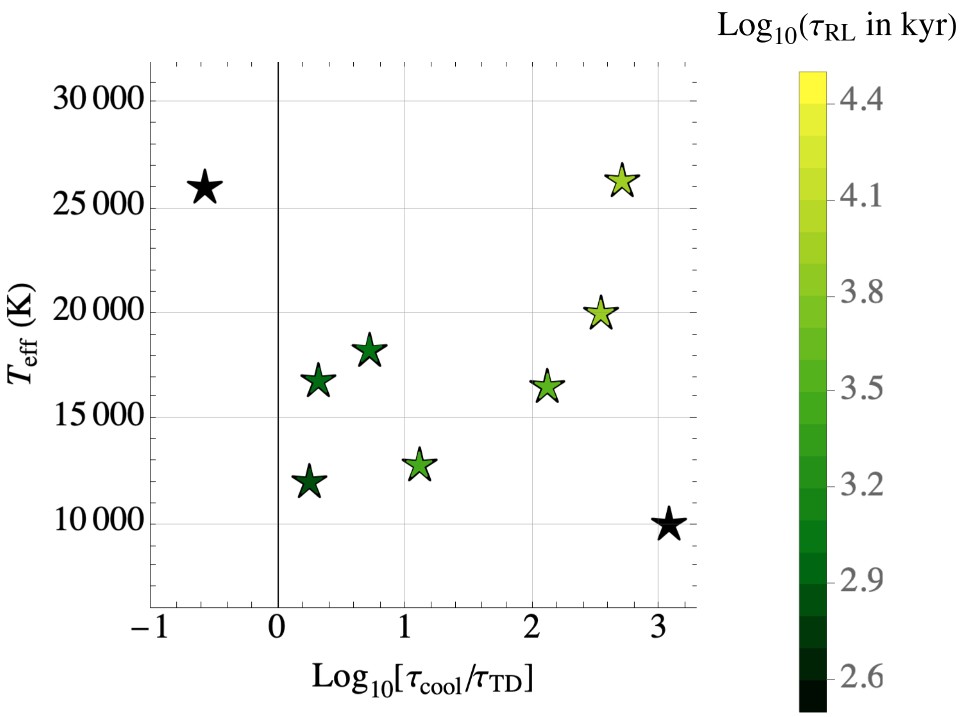
In[562]:=

```
cptrack = ContourPlot[ tscale, {ratio, 0, 4},
   {tscale, 2.5, 4.5}, Contours → Table[(i + 1) 0.1, {i, 0, 43}],
   ImageSize → Medium, ColorFunction → (ColorData["AvocadoColors"]),
   Axes → True, FrameTicksStyle → Directive[FontSize → 18, Black],
   ContourStyle → None, ScalingFunctions → {None, None, None},
   PlotLegends → Placed[BarLegend[Automatic, LegendLabel →
       Style["Log10(τRL in kyr)", 18], LabelStyle → labels], {After, Top}],
   PlotRange → {{0 - 1, 3.3}, {6000, 32 000}, {2.5, 4.5}}, Frame → True,
   LabelStyle → (FontFamily → "Times"), FrameLabel →
    {Style["Log10[τcool/τTD]", 18, Black], Style["Teff (K)", 18, Black]},
   GridLines → Automatic, FrameStyle → Automatic];
```

In[563]:=

```
Show[cptrack, br2, Pltfun[1], Pltfun[2], Pltfun[3],
 Pltfun[4], Pltfun[5], Pltfun[6], Pltfun[7], Pltfun[8]]
```

Out[563]=




In[564]:=

```
τfricTD =
 Table[(1/9 ((G^(5/3) m1prims⟦i⟧ (m1prims⟦i⟧ Msol + m2secs⟦i⟧ Msol)^(5/3) kQratio^-1) /
      (fGWs⟦i⟧^(13/3) m2secs⟦i⟧ π^(13/3) (Rsol / 100 Rscale[m1prims⟦i⟧ 10,
            T1prims⟦i⟧ / 10 000])^5))) / (3.15 × 10^7 × 10^6), {i, 1, 9}]
```

Out[564]=

```
{1067.97, 16.4516, 2649.62, 6652.28, 7330.89, 53 513.5, 27.1754, 88.4843, 6.87036}
```

In[565]:=

```
τTempTD =
 Table[(1/9 ((G^(5/3) m1prims⟦i⟧ (m1prims⟦i⟧ Msol + m2secs⟦i⟧ Msol)^(5/3) kQratio^-1) /
      (fGWs⟦i⟧^(13/3) m2secs⟦i⟧ π^(13/3) (Rsol / 100 Rscale[m1prims⟦i⟧ 10,
            T1prims⟦i⟧ / 10 000])^5))) / (3.15 × 10^7 × 10^6), {i, 1, 9}]
```

Out[565]=

```
{1067.97, 16.4516, 2649.62, 6652.28, 7330.89, 53 513.5, 27.1754, 88.4843, 6.87036}
```

In[566]:=

```
Log10[τcools2 / τfricTD]
```

Out[566]=

```
{0.941129, 2.36948, 0.546505, 0.150625,
 0.0808355, -0.743821, 2.5367, 1.94746, 2.90844}
```

In[567]:=
```
br4 = ListPlot[Transpose[{-Log10[τfricTD / τcools2], T1prims}],
   AspectRatio → 1, PlotMarkers → {"★", 25}, PlotStyle → {{Black}, "*"},
   Frame → True, LabelStyle → (FontFamily → "Times"),
   FrameLabel → {Style["Log10[τcool/τTD]"], Style["Teff (K)", 16]},
   BaseStyle → {FontSize → 16}, PlotRange → {{-1.5, 4.2}, {6000, 32 000}},
   PlotLegends → {Style["Q1/k1= 3.5 × 10^9", 16]}, GridLines → Automatic];
```

In[568]:=
```
τcools2
```

Out[568]=
```
{9325.86, 3851.99, 9325.86, 9410.13, 8830.64, 9652.56, 9351.34, 7840.16, 5564.45}
```

In[569]:=
```
list1grey = Transpose[{{-2, 0}, {0, 0}}];
list2grey = Transpose[{{-2, 0}, {400 000, 400 000}}];
```

In[571]:=
```
plotgrey =
  ListPlot[{list2grey, list1grey}, Mesh → All, PlotMarkers → None, Joined → True,
   Filling → {1 → {2}}, FillingStyle → {Blend[{Gray, Gray, Black}], Opacity[.3]},
   PlotStyle → {{Blend[{Gray, Gray, Black}], Opacity[0.5]}},
   PlotRange → {{-2, 4}, {0, 222 000}}, AspectRatio → 1, Frame → True,
   LabelStyle → (FontFamily → "Times"), GridLines → Automatic,
   PlotLegends → {Style["J0651 primary (Ω0=0)", 16]}];
```

In[572]:=
```
Correlation[-Log10[τmergeTDfixMyr2 / τcools2], T1prims]
```

Out[572]=
```
-0.130405
```

In[573]:=
```
-Log10[τmergeTDfixMyr2 / τcools2]
```

Out[573]=
```
{1.11722, 2.54557, 0.722596, 0.326716,
 0.256927, -0.56773, 2.71279, 2.12355, 3.08453}
```

Here we calculate the cooling and tidal heating timescales for J1539 in particular .

In[574]:=
```
J1539τ =
```

$$\left(\frac{2}{3} \times \frac{2}{18} \left(\left(G^{5/3}\ 0.21\ (0.21\ \text{Msol} + 0.61\ \text{Msol})^{5/3}\ \text{kQratio}^{-1}\right) \Big/ \left(0.0048^{13/3} \times 0.61\ \pi^{13/3}\ (\text{Rsol} / 100\ \text{Rscale}[0.21 \times 10, 10\,000 / 10\,000])^{5}\right)\right)\right) \Big/ \left(3.15 \times 10^{7} \times 10^{6}\right)$$

Out[574]=
```
4.67023
```

In[575]:=

```
J1539τcool = (t /. NSolve[7.1 × 10^-4 (Rsol RPanei[0.21, Tcold])^2 Tcold^4 ==
      Piecewise[{{L2a[0.21, t, testZ], t < 9000}, {L2b[0.21, t, testZ], t > 9000}}]
       Lsol, t]) // Flatten
```

NSolve: NSolve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[575]=

```
{5564.45}
```

In[576]:=

```
J1539τratio = -Log10[J1539τ / J1539τcool ][[1]]
```

Out[576]=

```
3.07608
```

In[577]:=

```
J1539temp = 10 000;
```

In[578]:=

```
(J1539τ / J1539τcool)^-1
```

Out[578]=

```
{1191.47}
```

In[579]:=

```
mtps = ResourceFunction["PolygonMarker"]["Triangle", {Offset[10], π},
   {EdgeForm[Blend[{Cyan, Blue, Cyan}]], FaceForm[stylesTemp[[9]]]}];
```

In[580]:=

```
stylesTemp[[8]];
```

In[581]:=

```
J1539plot = ListPlot[{Transpose[{J1539τratio, 0.98 J1539temp}]},
   PlotMarkers → {mtps}, Joined → False, PlotStyle → {{stylesTemp[[9]]}},
   PlotRange → {{-1, 4}, {6000, 22 000}}, AspectRatio → 1,
   Frame → True, LabelStyle → (FontFamily → "Times"),
   FrameLabel → {Style["f_GW (Hz)"], Style["T_eff (K)", 16]},
   BaseStyle → {FontSize → 16}, GridLines → Automatic,
   PlotLegends → {Style["J1539 limit", 16]}];
```

In[582]:=

```
J1539τratio
```

Out[582]=

```
3.07608
```

In[583]:=

In[584]:=

```
J1539plot = ListPlot[{Transpose[{J1539τratio, 0.96 J1539temp}]},
   PlotMarkers → {mtps}, Joined → False, PlotStyle → {{stylesTemp[[9]]}},
   PlotRange → {{-1, 3.3}, {6000, 32 000}}, AspectRatio → 1,
   Frame → True, LabelStyle → (FontFamily → "Times"),
   FrameLabel → {Style["f_GW (Hz)"], Style["T_eff (K)", 16]},
   BaseStyle → {FontSize → 16}, GridLines → Automatic,
   PlotLegends → {Style["J1539 limit", 16]}];
```

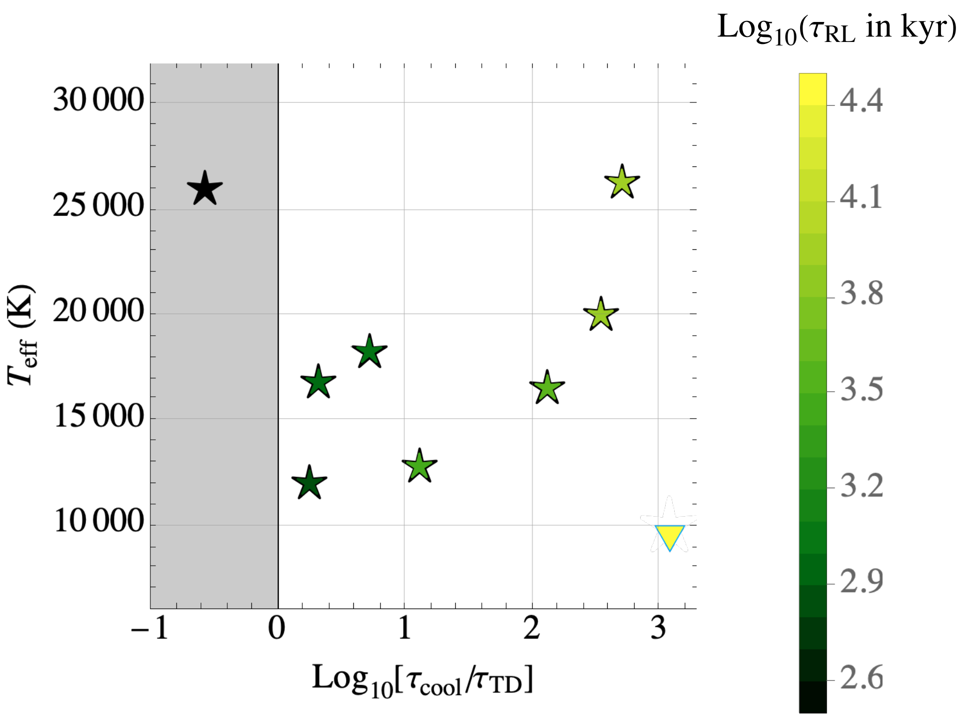
In[585]:=

```
Pltfunwhite[ii_] :=
 ListPlot[{pts2⟦All, {1, 2}⟧⟦ii⟧}, PlotRange → {{-1.5, 4.2}, {6000, 32 000}},
  AspectRatio → 1, PlotMarkers → {"★", 40},
  PlotStyle → {White}, LabelStyle → (FontFamily → "Times")]
```

In[586]:=

```
Show[cptrack, plotgrey, br2, J1539plot, Pltfun[1], Pltfun[2], Pltfun[3],
 Pltfun[4], Pltfun[5], Pltfun[6], Pltfun[7], Pltfun[8], Pltfunwhite[9], J1539plot]
```

Out[586]=

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