Stellar oscillations and solving for eigenfunctions in polytropes

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Simple prescriptions for stellar oscillations are proving to be extremely useful for interpreting observations of pulsating stars made by e.g. TESS, measuring the spin in the outer layers of the Sun (Howe, 2009) and other solar like stars (Benomar et al., 2018), inferring properties of unseen stellar companions (e.g., Lai, 1997) or planets with e.g. Kepler, and in making predictions about the oscillations of rotating neutron stars observed by LIGO (e.g., Schenk et al., 2002), to name a few.

In these notes we will derive the stellar eigenfunctions for a non-rotating, polytropic stellar model. As well as solving for the oscillation frequencies of modes, which can be measured directly from asteroseismology data, the eigenfunctions can be used to e.g. determine the coupling strength in tidal interactions, which is related to the energy deposited in tides (Press, 1981). When rotation is considered, they are also used to approximate the effect of the Coriolis force onto mode frequencies.

1 Governing equations of stellar structure and oscillations

Before starting the perturbation expansion, we will outline the equations required in the following derivations.

1.1 Equations of hydrodynamics

In fluids, mass conservation is described by the continuity equation, in terms of density ρ and velocity **v**:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{1}$$

In stellar astrophysics, viscosity is generally negligible (very high Reynolds number), and if we ignore magnetic fields and thermal heat conduction we can write the conservation of momentum equation:

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = -\nabla p + \rho \nabla \Phi, \tag{2}$$

where p is the local pressure and Φ is the conservative gravitational potential, i.e.

$$\mathbf{g} = -\nabla \Phi, \tag{3}$$

or in differential form (Poisson's equation)

$$\nabla^2 \Phi = 4\pi G \rho. \tag{4}$$

Conservation of energy per unit volume e is given by

$$\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{j} = \dot{q}, \tag{5}$$

where **j** is the energy flux and $\dot{q}$ is the local heating/cooling rate. Since we consider adiabatic motion we ignore the Gibbs equation which describes entropy conservation.

1.2 Stellar structure equations

We will now focus specifically on stars, which are approximately spherically symmetric so that all properties can be written as functions of r only. This is the form used in the stellar oscillation equations, which we will derive in section 2. For example, Equation (1) for mass conservation becomes

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho. \tag{6}$$

Hydrostatic equilibrium, which describes outward pressure balancing gravity turns Equation (2) into

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho}{r^2}. \tag{7}$$

Energy generation occurs by nuclear burning, so that the stellar luminosity L can be written in terms of the nuclear generation rate ϵ:

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho\epsilon \tag{8}$$

from Equation (5), where $\dot{q} = \rho\epsilon$ is the nuclear energy generation rate. Energy is typically transported by convection or radiation in convectively unstable and stable regions, respectively where the temperature evolves accordingly. These are the known as the *stellar structure equations* and are the key equations for 1D stellar evolution codes.

1.2.1 Poisson and momentum equation

Also, we note that Equation (4) reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = 4\pi G\rho(r) \tag{9}$$

under spherical symmetry, and that Equation (2) reduces to

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \rho\mathbf{g} \tag{10}$$

by setting $\frac{\partial \rho}{\partial t} = 0$ in the continuity equation (Equation (1)). We also set $\epsilon = 0$ since $\tau_{\mathrm{nuc}} >> \tau_{\mathrm{dyn}}$ and ignore structure changes from nuclear burning.

2 Deriving the stellar oscillation equations via linear perturbations

We begin the perturbation expansion (Unno et al., 1989; Christensen-Dalsgaard, 2003) by considering linear perturbations to the (spherically symmetric) stationary background state of the solution to the stellar structure equations, e.g. $\rho = \rho_0 + \rho'$ where ρ_0 is the stationary background and ρ', and we only keep terms where perturbative factors do not exceed first order.

2.1 Continuity equation

We perturb the continuity Equation (1) to get to first order:

$$\frac{\partial \rho'}{\partial t} + \nabla\cdot(\rho_0\mathbf{v}) = 0. \tag{11}$$

2.2 Momentum equation

To derive the equations for oscillations, we first write the Lagrangian displacement perturbation $\delta\mathbf{r}$ (expanded up to first order) in terms of a radial ($\xi_r\mathbf{a}_r$, where $\mathbf{a}_r$ is a unit vector in the radial direction) and horizontal (i.e. polar θ and azimuthal ϕ directions $\boldsymbol{\xi}_h$) component,

$$\delta\mathbf{r} = \xi_r\mathbf{a}_r + \boldsymbol{\xi}_h \tag{12}$$

where the velocity

$$\mathbf{v} = \frac{\partial \boldsymbol{\delta r}}{\partial t} = \frac{\partial \boldsymbol{\xi}}{\partial t}, \tag{13}$$

so that we can write the perturbed momentum equation to first order:

$$\rho_0\frac{\partial \mathbf{v}}{\partial t} = \rho_0\frac{\partial \delta\mathbf{r}^2}{\partial t^2} = -\nabla p' + \rho_0\mathbf{g}' + \rho'\mathbf{g}_0. \tag{14}$$

2.3 Poisson equation

We note that the potential perturbation Φ' is also conservative and will satisfy Poisson's equation,

$$\mathbf{g}' = -\nabla\Phi' \text{ and } \nabla^2\Phi' = 4\pi G\rho', \tag{15}$$

2.4 Momentum equation components

If we put this into the perturbed momentum equation Equation (14) and look at the horizontal component, noting that $\mathbf{g}_0$ is radial only we have

$$\rho_0 \frac{\partial^2}{\partial t^2} (\boldsymbol{\xi}_h) = -\nabla_h p' - \rho_0 \nabla_h \Phi'. \tag{16}$$

Taking the horizontal divergence of this,

$$\rho_0 \frac{\partial^2}{\partial t^2} (\nabla_h \cdot \boldsymbol{\xi}_h) = -\nabla_h \cdot (-\nabla_h p' - \rho_0 \nabla_h \Phi') = -\nabla_h^2 p' - \rho_0 \nabla_h^2 \Phi'. \tag{17}$$

We integrate the perturbed continuity equation (Equation (11)) with respect to time, using Equation (13)

$$\rho' = -\nabla \cdot (\rho_0 \boldsymbol{\delta r}) = -\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \rho_0 \xi_r \mathbf{a}_r \right) - \nabla_h \cdot (\rho_0 \boldsymbol{\xi}_h). \tag{18}$$

Then we use this to replace $\nabla_h \cdot \boldsymbol{\xi}_h$ in Equation (17) so that the horizontal momentum transport equation becomes

$$\rho_0 \frac{\partial^2}{\partial t^2} \left(-\rho' + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \rho_0 \xi_r \mathbf{a}_r \right) \right) = -\nabla_h^2 p' - \rho_0 \nabla_h^2 \Phi'. \tag{19}$$

Next we will look at the radial component of Equation (14) for momentum transport in r,

$$\rho_0 \frac{\partial v_r}{\partial t} = \rho_0 \frac{\partial^2}{\partial t^2} (\xi_r) = -\frac{\partial p'}{\partial r} + \rho_0 g' + \rho' g_0 = -\frac{\partial p'}{\partial r} - \rho_0 \frac{\partial \Phi}{\partial r} + \rho' g_0, \tag{20}$$

where we have used Equation (15). Then we rewrite the perturbed Poisson equation (15) for Φ' in terms of the radial and horizontal divergence

$$\nabla_r^2 \Phi' + \nabla_h^2 \Phi' = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi'}{\partial r} \right) + \nabla_h^2 \Phi' = 4\pi G \rho'. \tag{21}$$

So we have three equations: Poisson (Equation 21) and two momentum equations (Equations 19 and 20). We note that in these three equations, any angular dependence on θ and ϕ appears only in this horizontal gradient operator. The background variables are constant in time since they satisfy the stellar structure equations e.g. $p_0(r,t) = p(r)$. We can write the (time-dependent) perturbations ξ_r, ρ', p' etc. like $p'(r,\theta,\phi,t) = p'(r) f(\theta,\phi) e^{-i\omega t}$, where ω is the oscillation frequency of stable perturbations.

2.5 Solving for the form of the eigenfunctions (details)

We assume that we can separate this angular function $f(\theta,\phi) = y(\theta) z(\phi)$, so it will satisfy the eigenvalue equation for the Laplace operator

$$\left(r^2 \nabla_h + \Lambda \right) f(\theta,\phi) = 0 \tag{22}$$

or

$$\nabla_h^2 f = -\frac{1}{r^2} \Lambda f \tag{23}$$

for eigenvalue Λ. We expand the horizontal divergence so that

$$\frac{1}{\sin^2\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial \phi^2} = -\Lambda f. \tag{24}$$

Now we set $f(\theta,\phi) = y(\theta) z(\phi)$, and after some rearrangement we can write the Equation 23 as

$$\frac{1}{y} \sin\theta \left(\sin\theta \frac{\partial^2 y}{\partial \theta^2} + \cos\theta \frac{\partial y}{\partial \theta} + \Lambda z \right) = \frac{\partial^2 z}{\partial \phi^2} \frac{1}{z} = \text{constant} = \alpha. \tag{25}$$

We start by solving for z, which unlike y only depends on ϕ:

$$\frac{\partial^2 z}{\partial \phi^2} = \alpha z \implies z(\phi) = e^{\pm i \sqrt{\alpha} \phi} = e^{-im\phi} \tag{26}$$

where integer $m = +\sqrt{\alpha}$ (given by periodic boundary condition $z(0) = z(2\pi)$). Then we use this in for the left side of Equation (25), and then make a substitution $x = \cos\theta$, and we are left to solve the (associated) Legendre equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \left[\Lambda - \frac{m^2}{1-x^2}\right] = 0 \tag{27}$$

for some m and Λ. Here we note that we only have regular solutions when ℓ is a positive integer, so that eigenvalues

$$\Lambda = \ell(\ell+1) \implies \nabla_h^2 f = -\frac{\ell(\ell+1)}{r^2} f \tag{28}$$

in Equation (23). Once you see there is an eigenvalue problem in some variable, you can see very soon that SH are a solution. We seek a series solution for x in terms of ℓ and m, which is the associated Legendre polynomial $P_\ell^m(x)$:

$$y = P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell, \tag{29}$$

which is obtained by first suggesting $y(x) = (1-x^2)^{m/2} g(x)$ and then solving for g Whittaker and Watson (1963).

2.6 Solution

The solution for this eigenvalue problem in our angular function f will be spherical harmonics (e.g. Whittaker and Watson, 1963; Abramowitz and Stegun, 1965):

$$f(\theta,\phi) = P_\ell^m(\cos\theta)e^{im\phi} = Y_{\ell m}(\theta,\phi). \tag{30}$$

Now that we have the angular function, we can write the perturbations in terms of spherical harmonics (where we have set $\omega = \omega_{\ell m}$ for particular $Y_{\ell m}$)

$$\xi(r,\theta,\phi,t) = [(\xi_{r,\ell m} + \xi_\mathrm{h}\nabla_{\mathrm{h},\ell\mathrm{m}})Y_l^m(\theta,\phi)]\exp(-\mathrm{i}\omega_{\ell m}t), \tag{31}$$

or just the radial component:

$$\xi_r(r,\theta,\phi,t) = \xi(r)\exp(-\mathrm{i}\omega_{\ell m}t)Y_l^m(\theta,\phi), \tag{32}$$

which are the basic configuration on top of which we can study oscillations (of particular wave number ω_{klm}) and tidal perturbations.

2.7 Coupled system for oscillations

If we write all the perturbations ρ', p', like Equation (32) and use these in Equations (19, 20 and 21), we can divide by the time and angle dependent part, and using Equation (28) we rewrite our Poisson equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) - \frac{\ell(\ell+1)}{r^2}\Phi' = 4\pi G\rho' \tag{33}$$

and the two momentum component equations 19 and 20 (which used continuity) become

$$\omega_{\ell m}^2\rho_0\left(\rho' + \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho_0\xi_r\right)\right) = \frac{\ell(\ell+1)}{r^2}\left(p' + \rho_0\Phi'\right). \tag{34}$$

and

$$-\rho_0\omega_{\ell m}^2\xi_r = -\frac{\mathrm{d}p'}{\mathrm{d}r} - \rho_0\frac{\mathrm{d}\Phi'}{\mathrm{d}r} + \rho' g_0 \tag{35}$$

which is a fourth order system for six variables: $\rho', p', \Phi', \frac{\mathrm{d}\Phi'}{\mathrm{d}t}, \xi_r$ and $\omega_{\ell m}$. If you know how p' is related to ρ' (e.g. you assume a polytropic relation), then the frequencies can be solved for, and modes can be excited/forced by things from tides in to turbulent convection.

3 Solving for the radial eigenfunctions for polytropes

We already discussed the equations for adiabatic oscillations (Equations (33,34 and 35)), in terms of the background density ρ, pressure p and gravitational field Φ, and the linear perturbations in pressure p', density ρ', gravitational field Φ', and also the displacement ξ_r. If we assume a polytropic stellar model we can write $\Gamma = \frac{n+1}{n}$ and $\rho' = \frac{\rho}{\Gamma p} p'$ so that these equations become:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) = \frac{\ell(\ell+1)}{r^2}\Phi' + 4\pi G\frac{\rho}{\Gamma p}p', \tag{36}$$

$$\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho\omega^2\xi_r + \frac{\rho'}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r}, \tag{37}$$

and

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\left(\frac{2}{r} + \frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\xi_r + \frac{1}{\rho}\left(\frac{\ell(\ell+1)}{\omega^2 r^2}\right)p' + p'\Gamma + \frac{\ell(\ell+1)}{\omega^2 r^2}\Phi'. \tag{38}$$

This is a fourth order eigenvalue problem where the discrete eigenvalues, which are the oscillation frequencies ω, for $\frac{\mathrm{d}\Phi'}{\mathrm{d}r}, \Phi', p'$ and ξ_r (when solving, background values will come from the chosen stellar model). We end up with a fourth-order equation for Φ', $\frac{\mathrm{d}\Phi'}{\mathrm{d}r}, p'$ and ξ.

3.1 Aside on the Cowling approximation

In the Cowling approximation the perturbation to the gravitational potential $\Phi' = 0$, so that Equation (33) vanishes and we are left with two equations. These are

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\left(\frac{2}{r} + \frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\xi_r + \frac{1}{\rho}\left(\frac{\ell(\ell+1)}{\omega^2 r^2}\right)p', \tag{39}$$

and

$$\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho\omega^2\xi_r + \frac{\rho'}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}, \tag{40}$$

which is second order solving for ρ' and ξ_r, with ω the eigenvalue. We also note that $\xi_h(r) = \frac{1}{r\omega^2}\left(\frac{p'}{\rho} + \Phi'\right) = \frac{1}{r\omega^2}\frac{p'}{\rho}$ in the Cowling approximation

3.2 Getting density and pressure profiles from stellar model

In either case, we obtain the background density (and pressure) profile by solving the Lane-Emden equation for scaled temperature Θ and polytropic index n

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Theta(r)}{\mathrm{d}r}\right) + \Theta^n = 0, \tag{41}$$

where

$$\rho(r) = \Theta(r)^n, \tag{42}$$

and we scale so that $\rho(0) = 1$. The pressure will be given by

$$P(r) = \kappa\rho(r)^\Gamma = \kappa\Theta(r)^{n+1}, \tag{43}$$

where κ is a constant depending on the polytrope's radius, mass etc.. For an $n = 1.5$ polytrope, $\kappa = 0.4242 G m_1^{1/3} R_1 = 0.4242 Q^{1/3}\xi_0$ where scaled mass $Q = 34.1$ and radius $\xi_0 = 3.65$ (Chandrasekhar, 1939). We will need solutions for eigenfunctions ξ_r, p', ρ' and Φ' in order to solve the tidal coupling problem.

3.3 Boundary conditions for solving for eigenfunctions

Proofs for the following statements can be found on page 228 in Cox (1980). The eigenfunctions must be finite as $r \to 0$, but we see that the differential Equations (36,37,38) are singular there. After performing a Frobenius expansion we have, to leading order:

$$\xi_r \sim r^{\ell-1},\ p' \sim r^\ell \text{ and } \Phi' \sim r^\ell, \tag{44}$$

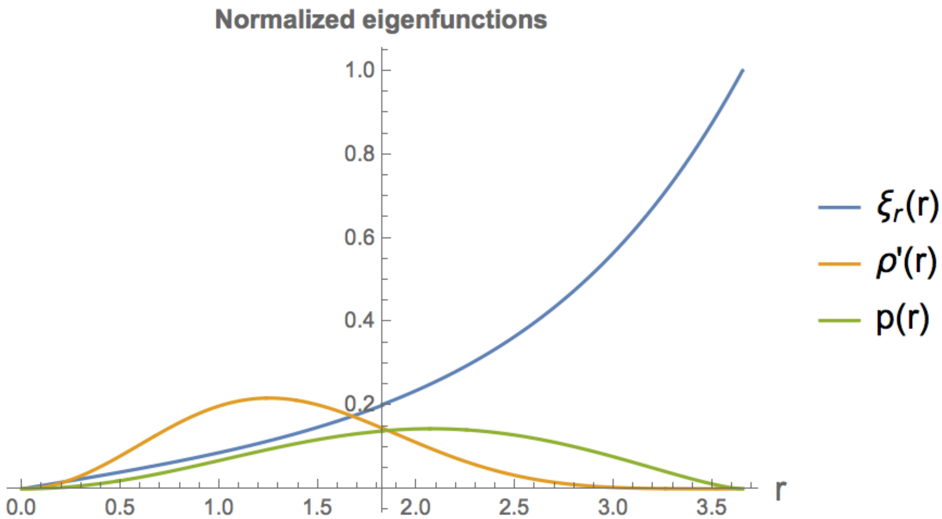

Figure 1: Normalised eigenfunctions ($n = 1$ and $\ell = 2$) for the displacement ξ_r (blue), the perturbed density ρ' (orange) and the perturbed pressure p' (green) solved using the shooting method, were solutions and derivatives match at scaled radius $r = 1.83$, halfway to the scaled polytropic surface $R_1 = 3.65$. (usually, the scaled polytropic radius is ξ. We use r instead to avoid any confusion with the eigenfunctions.)

and the derivative follows

$$\frac{\mathrm{d}\Phi'}{\mathrm{d}r} \sim \ell r^{\ell-1} \tag{45}$$

near this boundary. For continuity of Φ' at the surface $r = R_1$

$$\Phi' \sim r^{-l-1} \tag{46}$$

or equivalently that

$$\frac{\mathrm{d}\Phi'}{\mathrm{d}r} + \frac{\ell+1}{r}\Phi' = 0, \tag{47}$$

and a second boundary condition for $r = R_1$ is that the Lagrangian pressure perturbation vanishes there,

$$\delta p = 0 = p' + \xi_r \frac{\mathrm{d}p}{\mathrm{d}r} \implies p' = -\xi_r \frac{\mathrm{d}p}{\mathrm{d}r}. \tag{48}$$

For the other two boundaries at $r = R_1$, ξ_r and $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$, are free boundary conditions[1].

3.4 Solving numerically

We will implement the shooting method (Numerical Recipes, Press et al., 1986, page 757) in Mathematica. At this stage, the system is still overdetermined, so ω is fixed iteratively, until for a particular choice of ω, it is possible to match the solution starting at the left boundary, with the right boundary. In other words, this procedure turns the overdetermined system into a rank 4 system which we can solve directly. Note that the initial guess will be $O(1)$ since $\omega (= \omega_{k\ell})$ is in units of the dynamical time. At the first iteration in the shooting method, there are four boundary conditions on each side. There are two sets of solutions for each variable; one which is numerically integrated from the left and one from the right. Pairs of solutions are integrated to the midpoint e.g. $R/2$ from either side. If all four functions and their derivatives are equal there, then this is a solution. Otherwise, it iterates until this condition is met. Counting the crossings (nodes) in the solution to determine wave number k. We plot our (normalised) solutions for e.g. p' and ξ_r in Figure 1 for the fundamental mode ($k = 1$) for $n = 1.5$, where $\ell = 2$ (quadrupole). Here, the shooting method solution for the frequency is $\omega_{12} = 1.455$.

[1]Different choices of ξ_r and $\frac{\mathrm{d}\Phi'}{\mathrm{d}r}$ will correspond to a family of scale invariant solutions, which will be normalised so that $\langle \boldsymbol{\xi}_\mathbf{n} | \boldsymbol{\xi}_{\mathbf{n}'} \rangle = \delta_{\mathbf{nn}'}$.

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