PROOFS

MTH1035

Semester 1, 2016

Introduction to logic

Proofs are used to convince others that a statement is true. Statements are either true of false, so cannot be both. When proving mathematical statements, words such as 'if' or 'or' have precise meanings.
In the truth tables given, P, Q and R denote statements,

1 Direct Proof

Here we deal with **If . . . then . . .** statements.

Table 1: Truth table: Direct proofs

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

In other words, **If P then Q** is true when P and Q are both true, or P is false. The statement is false when P is true and Q is false.

e.g. Show that for every integer x, if x is even then x^2 is even.

Proof

Let $x = 2n$, $n \in \mathbb{R}$
So $x^2 = (2n)^2 = 4n^2$
But $4n^2$ is even for all $n \in \mathbb{R}$.
$\therefore$ if x is even then x^2 is even.

2 Proof by cases

Here, **P or Q** ($P \vee Q$) is defined as true when P is true or both P and Q are true, and is false when *both* P and Q are false.
Also, **P and Q** ($P \wedge Q$) is defined as true when both P and Q are true, false when P is false or Q is false or *both* P and Q are false. Here P and Q are assumptions (statements), and R is the implication (statements).

e.g. Show that for $x \in \mathbb{Z}$, $x^2 + 3x + 5$ is an odd integer.
The cases to consider here are odd x and even x.

Table 2: Truth table: Proof by Cases

P	Q	$P \vee Q$	R	$P \vee Q \Rightarrow R$
T	T	T	T	T
T	T	T	F	F
T	F	T	T	T
T	F	T	F	F
F	T	T	T	T
F	T	T	F	F
F	F	F	T	T
F	F	F	F	T

Proof

Case 1: x is even
Then $x = 2n$ for some $n \in \mathbb{Z}$ So

$$x^2 + 3x + 5 = (2n)^2 + 3(2n) + 5$$

$$= 4n^2 + 6n + 5$$

$$= 4n^2 + 6n + 4 + 1$$

$$2(2n^2 + 3n + 2) + 1$$

Since $2n^2 + 3n + 2 \in \mathbb{Z}$; it follows that $x^2 + 3x + 5$ is odd.

Case 2: x is odd
Then $x = 2k + 1$ for some $k \in \mathbb{Z}$
So

$$x^2 + 3x + 5 = (2k + 1)^2 + 3(2k + 1) + 5$$

$$= 4k^2 + 4k + 1 + 6k + 3 + 5$$

$$= 4k^2 + 10k + 9$$

$$2(2k^2 + 5k + 4) + 1$$

Since $2k^2 + 5k + 4 \in \mathbb{Z}$, it follows that $x^2 + 3x + 5$ is odd.

3 Proof by contrapositive

Table 3: Truth table: Proof by Contrapositive

P	Q	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

NOT P ($\neg P$) is defined as true when P is false, and false when P is true.

Let $x \in \mathbb{Z}$. Show that if $5x + 7$ is even then x is odd.

Proof:

Contrapositive method: Assume that x is even Then $x = 2n$ for some $n \in \mathbb{Z}$. So,

$$5x + 7 = 5(2n) + 7$$

$$= 10n - 7$$

$$= 10n - 8 + 1$$

$$= 2(5a - 4) + 1$$

Since $5a - 4$, $n \in \mathbb{Z}$ it follows that $5x + 7$ is odd.

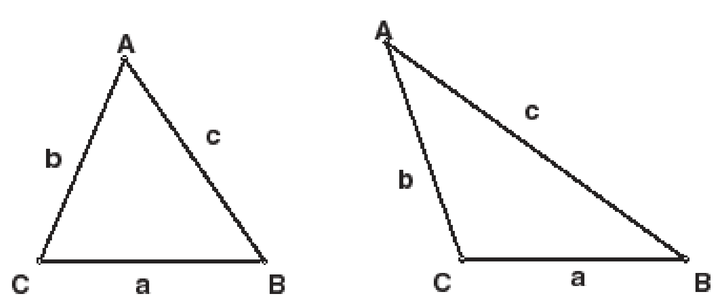
4 Proof by contradiction

You can prove a statement by assuming that it is false and deducing a contradiction. We want to prove that P is true.

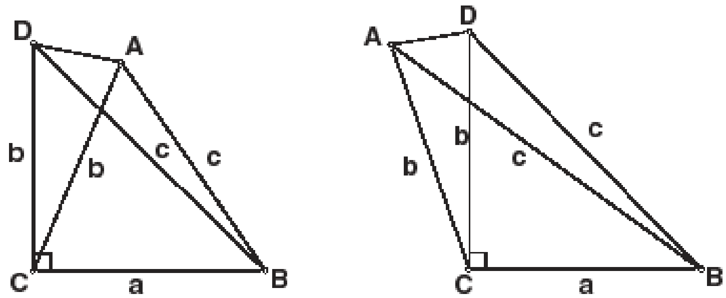
1. Assume P is false
2. Try and deduce Q, which we know to be false
3. If Q is deduced to be true, our assumption about P being false must be wrong

$\therefore$ P is true.

e.g. Pythagoras' theorem states that if a triangle is right angled, its sides satisfy $a^2 + b^2 = c^2$ (right angle $\rightarrow a^2 + b^2 = c^2$). However the converse $((a^2 + b^2 = c^2 \rightarrow$ right angled triangle).
The latter can be shown using a proof by contradiction.
Suppose we do *not* have a right angled triangle. We consider two cases: acute and obtuse triangles.



- Erect a perpendicular line segment CD



- By Pythagoras' theorem (the former), BD = c
- So ACB and ABD form isosceles triangles

It can also be seen that

- CDA = CAD
- BDA = DAB - (1)

However from the definitions of the triangles,

- BDA < CDA = CAD < DAB for acute case
- DAB < CAD = CDA < BDA for obtuse case -(2)

(1) and (2) contradict eachother (only one can be true).
So by assuming that we do not have right angled triangles, we arrive at a contradiction.
$\therefore a^2 + b^2 = c^2 \to$ right angled triangle

5 Biconditional "if and only if" proofs

Table 4: Truth table: Direct proofs

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

Here, $\neg Q$ is the assumption, $\neg P$ the implication.

e.g. Let $x \in \mathbb{Z}$: Then x^2 is even *iff* (if and only if) x is even
Proof:

Direct proof

Assume x is even. Then $x = 2a$ for some $a \in Z$:

$$\therefore x^2 = (2a)^2 = 4a^2 = 2(2a^2)$$

Because $2a^2 \in \mathbb{Z}$; it follows that x^2 is even.

By the contrapositive

Assume that x is odd. Then $x = 2b + 1$ for some $b \in \mathbb{Z}$:

$$\therefore x^2 = (2b+1)^2 = 4b^2 + 4b + 1$$

$$= 2(2b^2 + 2b) + 1$$

Because $2b^2 + 2b \in \mathbb{Z}$; it follows that x^2 is odd.

6 Proof by mathematical induction

Well-Ordering principle

The well-ordering principle is necessary for the proof of mathematical induction. It is a result of set theory. You should just have a qualitative understanding of this result.

Result

A non-empty set of real numbers is said to be well ordered if every non-empty subset of S has a least element. e.g. If S = $\{1, 2, 3\}$, the non-empty subsets are $\{1, 2, 3\}$, $\{2, 3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1\}$, $\{2\}$ and $\{3\}$.
Each subset has a least element and hence S = $\{1, 2, 3\}$ is well-ordered. In other words, any non-empty set of positive integers has a least element.

Well-ordering principle

The set of natural numbers, $\mathbf{N}$ is well-ordered.
Let $S \subseteq \mathbf{N}$, where

- $1 \in S$
- when $k \in S$, then $k + 1 \in S$

Then $S \in \mathbf{N}$ This is an axiom which defines the natural numbers.

Principle of mathematical induction

Proof by induction involves the following:

- Base case
- Inductive step

e.g. Prove by induction that $n^2 + n$ is divisible by 2:

Proof

If $n = 1$ (Base case): We have $(1)^2 + (1) = 2$ which is divisible by 2
Assume that the proposition is true for $n = k$, i.e. $k^2 + k$ is divisible by 2:

Inductive step

If $n = k + 1$:

$$(k+1)^2 + (k+1) = (k+1)((k+1)+1)$$

$$= (k+1)(k+2)$$

which is the product of two consecutive numbers. One of these must be a multiple of 2 and hence is divisible by 2.

EXERCISES

Direct proof

1. Prove that if x is an odd integer then $9x + 5$ is an even integer.
2. Let $n \in Z$: Prove that if $1 - n^2 > 0$, then $3n + 2$ is an even integer.

Proof by Cases (Exhaustion)

Definition:
A natural number is called composite if it is the product of other natural numbers, all of which are greater than 1 ,e.g. $187 = 17 \times 11$

3. Prove the following:
   If a natural number is of the form $10^{3n} + 1$, then it is a composite number.
   Note: $10^{3n} + 1$ is a number of the form $1000 \ldots 1$ (has $3n - 1$ zeros).
   (*Hint*: Use the identity $a^3 + b^3 = (a + b)(a^2 + ab + b^2)$)

Proof by Contrapositive

Definition: Two integers are said to have the same parity if they are both odd or both even.

4. Prove the following:
   Theorem: If x and y are two integers for which $x + y$ is even then x and y have the same parity.
5. Let $x \in \mathbb{Z}$: Prove that if $7x - 8$ is an even integer then x is an even integer.
6. Prove the following:
   Theorem: If a and b are real numbers such that their product ab is irrational, then either a or b must be an irrational number.

Proof by Contradiction

7. Prove that there are no rational number solutions to the equation $x^3 + x + 1 = 0$
8. Show that if a is a rational number and b is an irrational number, then $a + b$ is an irrational number.
9. Prove that $\sqrt{2}$ is an irrational number.

Biconditional Proofs ("If and only if")

10. Prove that a number is divisible by 4 if and only if its last two digits form a number that is divisible by 4.
11. Prove that the lines

$$ax + by + e = 0$$

$$cx + dy + f = 0$$

are parallel if and only if $ad - bc = 0$

Proof by Induction

12. Prove that for all positive integers n,

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$$

13. Use a direct proof to show that if x is odd, then $x + 1$ is even. Extend this to an iff proof.