HYPERBOLIC FUNCTIONS

MTH1035

Semester 1, 2016

Definitions of Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Oddness/Evenness

One of the many parallels with trigonometric functions is the oddness and evenness of hyperbolic sine and cosine, namely:

$$\sinh(-x) = -\sinh(x)$$

$$\cosh(-x) = \cosh(x)$$

Power Series comparison

In MTH1030 you have learned how to construct the first few terms of the power series of some function. Recall that the power series expansions of sine and cosine are as follows:

$$sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

The power series expansions of hyperbolic sine and cosine are as follows:

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

Unit circle and Hyperbola

Trigonometric functions are easily understood by relating point $P(x,y) = P(\cos(t), \sin(t))$ on the unit circle $x^2+y^2 = 1$, where t is the radial angle from the positive x axis.
The analogy for hyperbolic functions is relating a point $Q(x,y) = Q(\cosh(t), \sinh(t))$ to the unit hyperbola $x^2-y^2 = 1$, where t is $2 \times A$, where A is the area between the line from the origin to Q, and the unit hyperbola.

Definitions of Inverse Hyperbolic functions

Inverse functions are found by reflecting the hyperbolic function (restricted domain for cosh to make it one to one) in the line $y = x$. The results are as follows:

$$\text{arcsinh}(x) = \ln(x + \sqrt{x^2+1})$$

$$\text{arccosinh}(x) = \ln(x + \sqrt{x^2-1})$$

$$\text{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Derivatives of Hyperbolic Functions

Using the exponential definitions of sinh, cosh, tanh you can prove to yourself using the chain rule that:

$$\frac{d}{dx}(\sinh(g(x))) = g'(x)\cosh(g(x))$$

$$\frac{d}{dx}(\cosh(g(x))) = g'(x)\sinh(g(x))$$

$$\begin{aligned}\frac{d}{dx}(\tanh(g(x))) &= \frac{g'(x)}{\cosh(g(x))^2} \\ &= g'(x)\text{sech}(g(x))^2\end{aligned}$$

Taking note of the similarities with derivatives of trigonometric functions, with some minor sign changes.

Derivatives of Inverse Hyperbolic Functions

Using the logarithmnic definitions of arcsinh, arccosh, arctanh you can also prove using the chain rule that:

$$\frac{d}{dx}(\text{arcsinh}(g(x)) = \frac{g'(x)}{\sqrt{g(x)^2+1}}$$

$$\frac{d}{dx}(\text{arccosh}(g(x)) = \frac{g'(x)}{\sqrt{g(x)^2-1}}$$

$$\frac{d}{dx}(\text{arctanh}(g(x)) = \frac{g'(x)}{1-g(x)^2}$$

Which are again similar to derivatives of inverse trigonometric functions, except for some sign changes.

More identities

1. **Basic Identities**

$$\cosh(x)^2 - \sinh(x)^2 = 1$$

$$\cosh(x)^2 + \sinh(x)^2 = \cosh(2x)$$

$$sech(x)^2 + \tanh(x)^2 = 1$$

$$\coth(x)^2 - \mathrm{cosech}(x)^2 = 1$$

2. **Angle addition**

$$\sinh(\alpha+\beta) = \sinh(\alpha)\cosh(\beta) + \sinh(\beta)\cosh(\alpha)$$

$$\sinh(\alpha-\beta) = \sinh(\alpha)\cosh(\beta) - \sinh(\beta)\cosh(\alpha)$$

$$\cosh(\alpha+\beta) = \cosh(\alpha)\cosh(\beta) + \sinh(\beta)\sinh(\alpha)$$

$$\cosh(\alpha-\beta) = \cosh(\alpha)\cosh(\beta) - \sinh(\beta)\sinh(\alpha)$$

(Using oddness/evenness)

3. **Reciprocal Hyperbolic functions and their derivatives**

$$\mathrm{cosech}(x) = \frac{2}{e^x - e^{-x}}$$

$$\frac{d}{dx}(\mathrm{cosech}(g(x))) = -g'(x)\mathrm{cosech}(g(x))\coth(g(x))$$

$$\mathrm{sech}(x) = \frac{2}{e^x + e^{-x}}$$

$$\frac{d}{dx}(\mathrm{sech}(g(x))) = -g'(x)\tanh(g(x))sech(g(x))$$

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\begin{aligned}\frac{d}{dx}(\coth(g(x))) &= -g'(x)\mathrm{cosech}(g(x))^2 \\ &= g'(x)(1 - \coth(g(x))^2\end{aligned}$$

EXERCISES

1. Show that $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$
2. Show that $\frac{d}{dx}(\cosh^{-1}(x)) = \frac{1}{\sqrt{x^2-1}}$
3. Calculate $\sinh(x) + \cosh(x)$
   (a) In terms of exponentials
   (b) as a series expression

   Are your answers to (a) and (b) consistent with what you know about the power series of e^x?
4. **APPLICATION: Special Relativity**
   In special relativity, inertial reference frames move at a constant velocity relative to each other. Say that frame 2 moves faster than frame 1, which moves faster than frame 0. The velocity of reference frame 2 relative to reference frame 0 is:
   $$v_2 = \frac{v_1 + v_{12}}{1 + v_1 v_{12}} \tag{1}$$
   where v_1 is the velocity of frame 1 relative to frame 0, and v_{12} is the velocity of frame 2 relative to frame 1, all as factors of c (speed of light).
   Contrast to if we were to classically add velocities and naively say that $v_2 = v_1 + v_{12}$, the formula prevents v_2 from ever being greater than 1, i.e greater than c.
   (a) By introducing rapidity θ, where
   $$v_1 = \tanh(\theta_1)$$
   $$v_{12} = \tanh(\theta_{12})$$
   $$v_2 = \tanh(\theta_2)$$
   Show that $\theta_{12} = \theta_1 + \theta_2$ by substituting these into (1).
   (b) What happens when v_1 and v_2 are $<<$ 1 (i.e. very small compared to the speed of light c) ? Is this consistent with the classical Newtonian velocity addition described above?
   (c) Sketch $v_2 = \tanh(\theta_2)$. Given that $\theta_{12} = \theta_1 + \theta_2$, what happens to v_2 as $\theta \to \pm\infty$? Can v_2 exceed the speed of light under this transformation?

   The velocity parameter of rapidity adds linearly, unlike velocity which are coupled combinations of the other velocities. This transformation is a nice way to express that even if classically the two velocities v_{12} AND v_1 add to greater than c, the velocity of reference frame 0 as observed in inertial reference frame 2 (v_2) can NEVER exceed the speed of light.
5. **CHALLENGE**
   Using Euler's formula,
   $$e^{iz} = \cos(z) + i\sin(z)$$
   and that
   $$\cosh(iz) = \cos(z), \sinh(iz) = i\sin(z)$$
   $$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$
   Prove the following:
   (a) $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$
   (b) $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

   (Hint: let $z = x + y$)

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