Solving Systems of Differential Equations

for harmonic oscillator problems

Solving first order ODE's

Separable first order ODE's

This method can be used when we have a first order ODE of the form:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

We then rearrange so that everything involving x is on the left hand side, while everything involving y is on the right hand side.

$$g(y)dy = f(x)dx$$

Integrating both sides, we can find a solution for y in terms of x (provided that f and g are integrable functions.

Integrating factor

Motivation

Say we have a linear ODE of the following form:

$$\frac{dy}{dx} + p(x)y = r(x)$$

Now, it would be ideal if we could integrate both sides like we did with first order separable equations. However after inspection we can see that we cannot simultaneously decouple the $\frac{dy}{dx}$ and $p(x)y$ to do that.
BUT there is a neat trick we can use so that we can integrate both sides to find a solution for y in terms of x.
We could integrate both sides if the equation was of the form

$$\frac{d}{dx}(y \times a(x)) = b(x)$$

In other words, we want to utilize the product rule so that the LHS of the first equation can be expressed as the derivative of a product. If this is the case then we can integrate both sides with respect to x:

$$\int \frac{d}{dx}(y \times a(x))dx = \int b(x)dx$$

$$y \times a(x) = \int b(x)dx$$

So...

$$y = \frac{1}{a(x)} \int b(x)dx$$

In other words, if the LHS of (1) is of the form

$$\frac{dy}{dx}a(x) + a'(x)y$$

Then the equation is exact and we can integrate both sides directly.

Integrating factor step

In order for our equation to be exact, we multiply the whole equation by the integrating factor. This is defined as follows:

$$I(x) = e^{\int p(x)dx}$$

Now (1) can be written in exact form,

$$I(x)\frac{dy}{dx} + I(x)p(x)y = I(x)r(x)$$

$$e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)y = e^{\int p(x)dx}r(x)$$

Finally...

$$\frac{d}{dx}(y(x) \times e^{\int p(x)dx}) = e^{\int p(x)dx}r(x)$$

$$\int \frac{d}{dx}(y(x) \times e^{\int p(x)dx})dx = \int e^{\int p(x)dx}r(x)dx$$

$$y(x) \times e^{\int p(x)dx} = \int e^{\int p(x)dx}r(x)dx$$

So,

$$\boxed{y(x) = \frac{1}{e^{\int p(x)dx}} \int e^{\int p(x)dx}r(x)dx}$$

Which may involve integration by parts or substitution methods.

Solving second order ODE's

Method of undetermined co-efficients

We will first look at solving linear, homogeneous, constant co-efficient equations of the form

$$y'' + by' + cy = 0$$

Where A, B are real constants.
We suggest a solution of the form

$$\boxed{y = e^{\lambda x}}$$

Where $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$. Substituting this into the our ODE,

$$\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$

Taking out $e^{\lambda x}$ as a common factor,

$$e^{\lambda x}(\lambda^2 + b\lambda + c) = 0$$

Using the null factor law,

$$e^{\lambda x} = 0 \; or \; \lambda^2 + b\lambda + c = 0$$

But $e^{\lambda x} > 0$ for all x and λ. So we solve $\lambda^2 + b\lambda + c = 0$ where

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

For homogeneous, constant co-efficient, linear ODE's, λ single handedly determines the form of y(x), so four distinct scenarios will be gone through.

Real, distinct eigenvalues

For $b^2 - 4c > 0$, there will be two real and distinct eigenvalues, call them λ_1, λ_2. So our full solution will be:

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

Noting that we have 2 arbitrary constants A, $B \in \mathbb{R}$ from the second order nature of the equation.

Real, repeated eigenvalues

For $b^2 - 4c = 0$, there will be only one distinct eigenvalue, call it λ. However because of the second order nature, we seek TWO linearly independent solutions. To do this, set the constant in front of $e^{\lambda x}$ as $Ax + B$. So the full solution is

$$y(x) = Axe^{\lambda x} + Be^{\lambda x}$$

So although we had only one distinct eigenvalue, we can still get two linearly independent solutions.

Complex eigenvalues (Real AND imaginary part)

For $b^2 - 4c < 0$ and $b \neq 0$, λ will have two distinct solutions with both real and imaginary components. In fact the solutions are complex conjugates of the form $a \pm ib$. So $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$. This suggests a solution of the form

$$y(x) = Ae^{(a+ib)x} + Be^{(a-ib)x}$$

This is equivalent to

$$y(x) = Ae^{ax}e^{ibx} + Be^{ax}e^{-ibx}$$

$$= e^{ax}(Ae^{ibx} + Be^{-ibx})$$

Now, recall Eulers formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

So

$$y(x) = e^{ax}(A(\cos(bx) + i\sin(bx)) + B(\cos(-bx) + i\sin(-bx)))$$

$$= e^{ax}(A(\cos(bx) + i\sin(bx)) + B(\cos(bx) - i\sin(bx)))$$

$$= e^{ax}((A + B)\cos(bx) + i(A - B)\sin(bx))$$

Now, A and B are just constants, so we re-write the solution as

$$y(x) = e^{ax}(F\cos(bx) + G\sin(bx))$$

Which is a growing or decaying oscillatory function with F, $G \in \mathbb{C}$.

Complex eigenvalues (Only imaginary part)

When $b^2 - 4c < 0$ and $b = 0$, we have a special case of complex eigenvalues where $\lambda = 0 \pm ib$, in other words

$$y(x) = e^{0x}(F\cos(bx) + G\sin(bx))$$

$$= F\cos(bx) + G\sin(bx)$$

Which is a purely oscillatory sinusoidal function whose amplitude does not grow or decay. An example of this solution is simple harmonic motion, which is represented by the second order differential equation

$$y'' + y = 0$$

Our characteristic equation is $\lambda^2 + 1 = 0$. So $\lambda = \pm i$ and the full solution to the ODE is

$$y(x) = F\cos(x) + G\sin(x)$$

In later calculus courses you will find that F and G can usually be determined by two ‘boundary’ or ‘initial’ conditions. But for now we only care about the general solution.

Method of inspection for r(x)

We now move onto constant co-efficient *inhomogeneous* linear ODE's of the form

$$y'' + by' + cy = r(x)$$

This is equivalent to

$$y'' + by' + cy = r(x) + 0$$

The neat thing about dealing with *linear* ODE's is that solutions of each term on the RHS can be added together. So the full solution

$$y(x) = y_h + y_p$$

Where y_h is the homogeneous solution, whose exponential solutions are determined by the characteristic equation in the method of undetermined co-efficients outlined above. In this section we will determine y_p from $r(x)$ and write the full solution $y(x)$.
In general, to determine $r(x)$ we make a guess based on the form of the RHS. A problem arises when this guess is the same as one of the solutions to the homogeneous equation, but this is easily dealt with.

Polynomials

Say we want to find the solution to the following differential equation:

$$y'' + 3y' + 2y = 2x^2 + 1$$

For the homogeneous solution to $y'' + 3y' + 2y = 0$, we get $y_h = Ae^{2x} + Be^x$. To find y_p we look at the right hand side. It is a polynomial of maximum order 2, so we suggest a solution which is a general polynomial of order 2:

$$y_p = c_1x^2 + c_2x + c_3$$

. We are looking for a *particular* solution, so we must solve for c_1, c_2 and c_3. To do this, we differentiate the general form of y twice, then substitute into the left hand side.

$$y_p' = 2c_1x + c_2, y_p'' = 2c_1$$

Substituting these into the RHS, the DE becomes

$$2c_1 + 3(2c_1x + c_2) + 2(c_1x^2 + c_2x + c_3) = 2x^2 + 1$$

In order to solve for the co-efficients, we group terms with the same powers of x and equate with the same terms on the RHS. Expanding, then factorising again we get

$$2c_1x^2 + (6c_1 + 2c_2)x^1 + (2c_1 + 3c_2 + 2c_3)x^0 = 2x^2 + 0x^1 + 1x^0$$

Equating each co-efficient in front of different powers of x we get three equations:

$$2c_1 = 2$$

$$6c_1 + 2c_2 = 0$$

$$2c_1 + 3c_2 + 2c_3 = 1$$

So $c_1 = 1$, $c_2 = -3$, $c_3 = 4$.

$$\therefore y_p = x^2 - 3x + 4$$

And the full solution including the homogeneous solution is

$$y(x) = x^2 - 3x + 4 + Ae^{2x} + Be^x$$

Noting that we have two arbitrary constants in the full solution as expected.

cos and sin

We want to find the solution to the following differential equation:

$$y'' - y = 2\cos(x)$$

For the homogeneous solution to $y'' - y = 0$ we have $y_h = Ae^x + Be^{-x}$ For the particular solution, we suggest a solution of the form

$$y_p = c_1\cos(x) + c_2\sin(x)$$

We then follow the same procedure as for the polynomial solution.

$$y_p' = -c_1\sin(x) + c_2\cos(x)$$

$$y_p'' = -c_1\cos(x) - c_2\sin(x) = -y_p$$

Substituting this into the DE,

$$-y_p - y_p = 2\cos(x)$$

$$-2y_p = 2\cos(x)$$

$$\implies y_p = -\cos(x)$$

So the full solution is

$$y(x) = Ae^x + Be^{-x} - \cos(x)$$

We now move onto when our guess is a solution to the homogeneous equation. We want to solve

$$y'' + y = 2\cos(x)$$

Solving the homogeneous equation $y'' + y = 0$,

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

This suggests a solution $y_h = A\cos(x) + B\sin(x)$. If we look at the RHS, we would suggest a solution of the form

$$y_p = c_1\cos(x) + c_2\sin(x)$$

However, we can see that regardless of what c_1 and c_2 are, this 'particular' solution is included in the homogeneous solution. So we suggest a solution of the form

$$y_p = (c_1 + c_2x)\cos(x) + (c_3 + c_4x)\sin(x)$$

From this, we can solve for c_1, c_2, c_3, and c_4 then combine with the homogeneous solution for the full solution. Again we expect only two arbitrary constants.

Exponentials

We want to find the solution to the following differential equation:

$$y'' - y = e^{2x}$$

For the homogeneous solution to $y'' - y = 0$ we have $y_h = Ae^x + Be^{-x}$ For the particular solution, we suggest a solution of the form

$$y_p = c_1e^{2x}$$

We then follow the same procedure as for the polynomial solution.

$$y_p' = 2c_1 e^{2x}$$

$$y_p'' = 4c_1 e^{2x} = 4y_p$$

Substituting this into the DE,

$$4y_p - y_p = e^{2x}$$

$$\implies y_p = \frac{1}{3}e^{2x}$$

So the full solution is

$$y(x) = Ae^x + Be^{-x} + \frac{1}{3}e^{2x}$$

We now move onto when our guess is a solution to the homogeneous equation. We want to solve

$$y'' - y = e^x$$

Solving the homogeneous equation $y'' - y = 0$,

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

This suggests a solution $y_h = Ae^x + Be^{-x}$. If we look at the RHS, we would suggest a solution of the form

$$y_p = c_1 e^x + c_2 e^{-x}$$

However, we can see that regardless of what c_1 and c_2 are, this 'particular' solution is included in the homogeneous solution. So we suggest a solution of the form

$$y_p = (c_1 + c_2 x)e^x$$

From this, we can solve for c_1, c_2 then combine with the homogeneous solution for the full solution. Again we expect only two arbitrary constants.

$$y_p = (c_1 + c_2 x)e^x = c_1 e^x + c_2 x e^x$$

$$y_p' = c_1 e^x + c_2 x e^x + c_2 e^x$$

$$y_p'' = c_1 e^x + c_2 x e^x + c_2 e^x + c_2 e^x = c_1 e^x + c_2 x e^x + 2c_2 e^x$$

Substituting this into the original ODE,

$$\cancel{c_1 e^x} + \cancel{c_2 x e^x} + 2c_2 e^x - (\cancel{c_1 e^x} + \cancel{c_2 x e^x}) = e^x$$

So,

$$y_p'' - y_p = 2c_2 e^x = (1 \times e^x) + (0 \times x e^x)$$

Equating co-efficients of e^x,

$$c_1 = 0,\ 2c_2 = 1 \implies c_2 = \frac{1}{2}$$

$$y_p = \frac{1}{2}x e^x$$

Finally we write the full solution as

$$y = (A + \frac{1}{2}x)e^x + Be^{-x}$$

Solving systems of first order coupled DE's

We are given a pair of coupled first order ODE's

$$\frac{du}{dx} = 6u + 16v$$

$$\frac{dv}{dx} = -u - 4v$$

By inspection, there is no obvious way to isolate u and v using what we know about first order ODE's. We will look at two independent ways to solve such problems, step-by-step. The first involves solving ONE second order ODE, and the second involves eigenvalues and eigenvectors.

Decoupling by differentiation

1. **Differentiate both equations with respect to x to arrive at two second order, coupled ODE's.**

$$\frac{d^2u}{dx^2} = 6\frac{du}{dx} + 16\frac{dv}{dx} = 6(6u + 16v) + 16(-u - 4v)$$

$$\frac{d^2v}{dx^2} = -\frac{du}{dx} - 4\frac{dv}{dx} = -(6u + 16v) - 4(-u - 4v)$$

   Given the definitions of $\frac{du}{dx}$ and $\frac{dv}{dx}$.
   Simplifying,

$$\frac{d^2u}{dx^2} = 20u + 32v$$

$$\frac{d^2v}{dx^2} = -2u$$

   We want our second order ODE's to be in terms of one variable but these are still coupled.

2. **Get expressions of u and v in terms of v and u and their derivatives to arrive at two second order, uncoupled ODE's.**
   From (1) and (2) we have that

$$v = \frac{1}{16}(\frac{du}{dx} - 6u)$$

$$u = -\frac{dv}{dx} - 4v$$

   Substituting these into (5) and (6),

$$\frac{d^2u}{dx^2} = 20u + \frac{32}{16}(\frac{du}{dx} - 6u)$$

$$\frac{d^2v}{dx^2} = -2(\frac{dv}{dx} + 4v)$$

   Simplifying,

$$\frac{d^2u}{dx^2} = 20u + 2\frac{du}{dx} - 12u$$

$$\implies \frac{d^2u}{dx^2} - 2\frac{du}{dx} - 8u = 0$$

$$\frac{d^2v}{dx^2} = -2(-\frac{dv}{dx} - 8v)$$

$$\implies \frac{d^2v}{dx^2} - 2\frac{dv}{dx} - 8v = 0$$

3. **Solve the characteristic equations**

$$\lambda_u^2 - 2\lambda_u - 8 = 0$$

$$(\lambda_u - 4)(\lambda_u + 2) = 0 \implies \lambda_u = 4 \; and \; \lambda_u = -2$$

$$\lambda_v^2 - 2\lambda_v - 8 = 0$$

$$(\lambda_v - 4)(\lambda_v + 2) = 0 \implies \lambda_v = 4 \; and \; \lambda_v = -2$$

4. **Construct full solutions for $u(x)$ and $v(x)$** Given the results for λ_u and λ_v we suggest that

$$u(x) = c_1e^{4x} + c_2e^{-2x}$$

$$v(x) = c_3e^{4x} + c_4e^{-2x}$$

But we had two first order equations, so should expect only two arbitrary constants to come up in our solutions. We can substitute these into one of (1) or (2) to end up with only two constants instead of four. Using (1),

$$u(x) = c_1e^{4x} + c_2e^{-2x}$$

$$\frac{du}{dx} = 4c_1e^{4x} - 2c_2e^{-2x}$$

$$4c_1e^{4x} - 2c_2e^{-2x} = 6(c_1e^{4x} + c_2e^{-2x}) + 16(c_3e^{4x} + c_4e^{-2x})$$

$$= (6c_1 + 16c_3)e^{4x} + (6c_2 + 16c_4)e^{-2x}$$

Equating co-efficients,

$$4c_1 = 6c_1 + 16c_3 \implies c_1 = -8c_3$$

$$-2c_2 = 6c_2 + 16c_4 \implies c_2 = -2c_4$$

So the full solutions to the two first order, linear, coupled ODE's is

$$u(x) = -8c_3e^{4x} - 2c_4e^{-2x}$$

$$v(x) = c_3e^{4x} + c_4e^{-2x}$$

Decoupling using eigenvalues and eigenvectors

We will use techniques from linear algebra to obtain the same solution.

1. **Rewrite the system of equations as a matrix system**

$$\frac{d}{dx}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

In other words, $\underline{\mathbf{v}}' = A\underline{\mathbf{v}}$ for $\underline{\mathbf{v}} = \begin{bmatrix} u \\ v \end{bmatrix}$ and $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$

2. **Find the eigenvalues of** A

$$det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 6 & -16 \\ 1 & \lambda + 4 \end{vmatrix} = 0 \implies (\lambda - 6)(\lambda + 4) + 16 = 0$$

$$\lambda^2 - 2\lambda - 24 + 16 = 0$$

$$\lambda^2 - 2\lambda - 8 = 0$$

$$\implies \lambda_1 = 4, \ \lambda_2 = -2$$

3. **Find the eigenvectors of A**
λ ***= 4 case***:

$$\begin{bmatrix} -2 & -16 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \underline{\mathbf{0}}$$

$$\implies \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \end{bmatrix} = \underline{\mathbf{v}}_1$$

λ ***= -2 case***:

$$\begin{bmatrix} -8 & -16 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \underline{\mathbf{0}}$$

$$\implies \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \underline{\mathbf{v}}_2$$

4. **Construct the full solution by multiplying each eigenvector by its repective eigenvalue solution:**

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = c_3 e^{4x}\begin{bmatrix} -8 \\ 1 \end{bmatrix} + c_4 e^{-2x}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Compare this to the result when decoupling by differentiation. They are equivalent! While we might have thought that eigenvalues were only useful in the realm of linear algebra, we see that they have powerful applications to calculus.

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