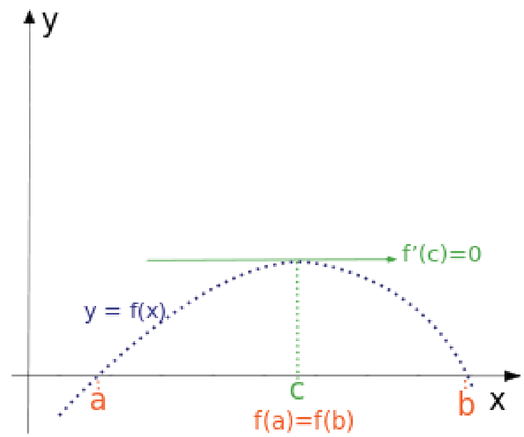
NECESSARY THEOREMS AND L'HOPITAL'S RULE

MTH1035

Semester 1, 2016

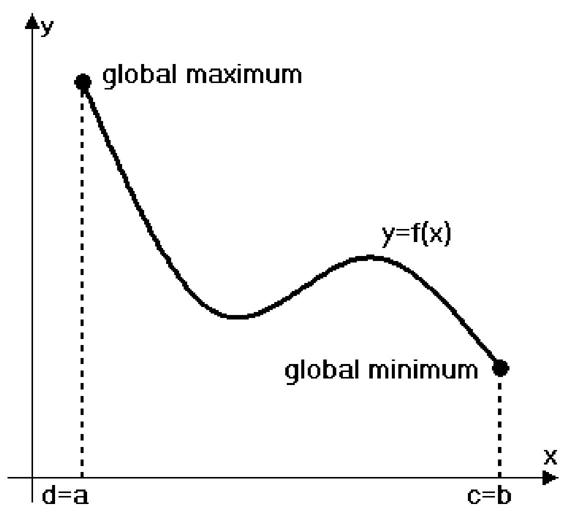
Rolle's Theorem

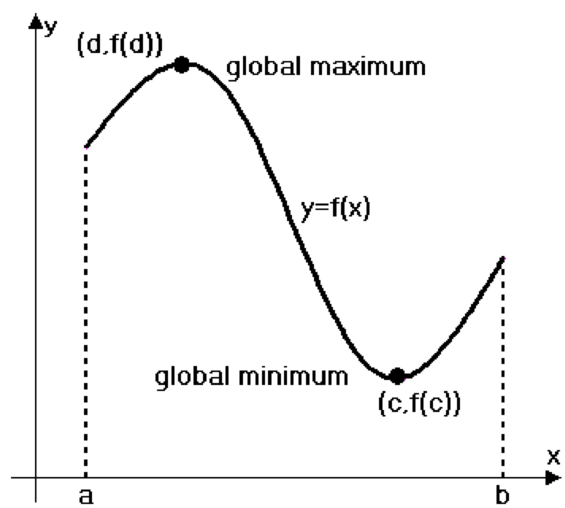
If there are two equal $f(a)$ and $f(b)$ for distinct a and b, there must be at least one c value for which $f'(c) = 0$. In other words, there is at least one stationary point between $f(a) = f(b)$.



Extreme Value theorem

Between some interval $[a, b]$ where $a \neq b$, there is both a minimum and maximum value.





Intermediate Value Theorem

Between a and b every value of $f(a)$ to $f(b)$ is taken (there are infinitely many. Search "countable sets" or "big infinity").

It will be an exercise to prove that for certain polynomials, no zeroes exist using all three of Rolle's, Extreme Value and Intermediate Value theorems.

Mean Value Theorem

If the following conditions are satisfied:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a,b)

Then there exists a c such that

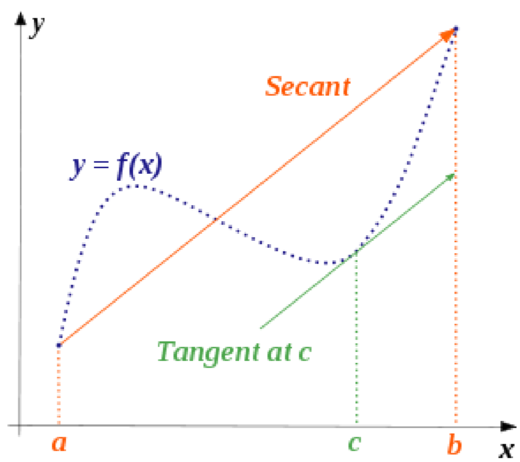
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

In otherwords, there exists a c for which the tangent at $(c, f(c))$ has the same gradient as the secant joining $(a, f(a))$ and $(b, f(b))$.

This is the average gradient.



Cauchy's Mean Value theorem

This is a generalisation of the regular MVT above which is necessary for L'Hopital's Rule. If the following conditions are satisfied:

1. $f(x)$ and $g(x)$ are both continuous on the closed interval $[a, b]$
2. $f(x)$ and $g(x)$ are both differentiable on the open interval
3. $g(a) \neq g(b)$
4. $g'(x) \neq 0$ for all x in (a, b)

Then there exists at least one c such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

L'Hopital's Rule

We are now equipped to rigorously understand L'Hopital's rule.
Say we want to find the following limit:

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

This is often straightforward, but we now consider cases where $\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$.(note: $\frac{0}{0}$ can be thought of as $0 \times \frac{1}{\infty}$. Similarly $\frac{\pm\infty}{\pm\infty}$ can be thought of as $\pm\infty \times \frac{\pm 1}{0}$)
In such cases, $\frac{f(x)}{g(x)}$ is in indeterminate form. This means that we cannot say anything conclusive as to whether the limit exists, let alone what value it may take.

L'Hopital's Rule states that:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

If and only if the limit on the right hand side exists. If this limit is again indeterminate, and ONLY when it is indeterminate, we differentiate again. Otherwise we can say something conclusive about whether the limit exists, and if it does the value it takes as $x \to a$.

Geometric approach: small angle approximation

The small angle approximation is used to simplify laws found in optics, astronomy, mechanics etc. when we are dealing with very small angles (search 'Paraxial approximation', 'Parallax method', 'Motion of a pendulum'.)
For angle θ where $\theta \to 0$,

$$\sin(\theta) \approx \theta$$

Or equivalently, as $\theta \to 0$

$$\frac{\sin(\theta)}{\theta} \approx 1$$

If we were trying to solve

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$$

The result is $\frac{0}{0}$, which is indeterminate.

Applying L'Hopital's rule,

$$\lim_{\theta \to 0} \frac{f(\theta)}{g(\theta)} = \lim_{\theta \to 0} \frac{f'(\theta)}{g'(\theta)} = \lim_{\theta \to 0} \frac{\cos(\theta)}{1} = \frac{1}{1} = \mathbf{1}$$

So we can recover the small angle approximation using L'Hopital's rule.
Also, $\sin(\theta) \approx \theta$ is the Taylor series in sine 'truncated' (higher order polynomicals in θ have been excluded) after one term.

EXERCISES

1. Prove that $2x^3 + x - 2$ has exactly one real root.
   (***Hint:*** *First use the initial value theorem to show that a root exists between 0 and 1. Then use Rolle's Theorem to prove by contradiction that the equation has no other real root.*)

2. Using techniques from L'Hopital's rule, find the small angle approximations for:

   (a) $\tan(\theta)$ as $\theta \to 0$

   (b) $\cos(\theta)$ as $\theta \to 0$

   (c) $\cos(\theta)$ as $\theta \to \frac{\pi}{2}$

REFERENCES

1. Lu, J. (2001) An application of L'Hopital's rule, The College Mathematics Journal, Vol 32, No. 5: 370-372pp.

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3. Stewart, J (2012) Mean Value theorem, L'Hopital's rule. In '*Calculus: Early Transcendentals (7th ed.)*'. Brooks/Cole, Cengage learning, Boston MA. A45-A46

4. Stewart, J (2012) Rolle's Theorem. In '*Calculus: Early Transcendentals (7th ed.)*'. Brooks/Cole, Cengage learning, Boston MA. A45