Thermal wind equation in stellar convective zones

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According to marginal stability analyses which consider shear, convection and rotation (Rayleigh, 1917; Wasiutynski, 1946; Ledoux, 1958; Chandrasekhar, 1961), stellar convective zones should rotate rigidly, or with constant specific angular momentum l_z. However, the convective zone in the Sun rotates dozens of percent faster at the equator, compared to the mid-latitude regions closer to the poles (Howe, 2009), i.e. neither of these two theoretically predicted configurations.

It has been suggested that the preferred rotation pattern observed in the Sun and other solar-like stars (e.g., Benomar et al., 2018) may instead be governed by a *balance* between gravity, pressure gradients (buoyancy) and inertial forces due to rotation. This balance is expressed most simply in terms of a vorticity balance, with specific application to convectviely unstable regions, which we derive here.

1 Vorticity equation

1.1 Momentum conservation

We start with the momentum conservation equation for a non-magnetic stellar model where viscous dissipation is ignored:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) = -\frac{\nabla P}{\rho} + \nabla \Phi \tag{1}$$

and take the curl Kitchatinov and Ruediger (1995); Thompson et al. (2003)

$$\nabla \times \left(\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) \right) = \nabla \times \left(-\frac{\nabla P}{\rho} + \nabla \Phi \right). \tag{2}$$

The potential term vanishes since $\nabla \times (\nabla f) = 0$. This also means that the term before this only has one non vanishing term $\nabla \times \left(-\frac{\nabla P}{\rho}\right)$ via the product rule for curl. We define the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ and use that

$$\nabla \cdot (\mathbf{v} \otimes \mathbf{v}) = \mathbf{v} (\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla) \mathbf{v} \tag{3}$$

so that we can write our vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\mathbf{v} (\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla) \mathbf{v}) = \frac{\nabla \rho \times \nabla P}{\rho^2}. \tag{4}$$

1.2 Further simplifications

Then we use the identity

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = \nabla v^2/2 - \mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla v^2/2 - \mathbf{v} \times \boldsymbol{\omega} \tag{5}$$

where the first term on the right hand side of Equation (5) vanishes,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\mathbf{v} (\nabla \cdot \mathbf{v}) - \mathbf{v} \times \boldsymbol{\omega}) = \frac{\nabla \rho \times \nabla P}{\rho^2}. \tag{6}$$

Then we use that

$$\nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = \mathbf{v} (\nabla \cdot \boldsymbol{\omega}) + (\boldsymbol{\omega} \cdot \nabla) \mathbf{v} - \boldsymbol{\omega} (\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla) \boldsymbol{\omega}, \tag{7}$$

whose first (and third) term(s) vanish for divergence of a curl (and for mass conservation in the steady state, later). So our vorticity transport equation currently looks like

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\mathbf{v}(\nabla \cdot \mathbf{v})) - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - \boldsymbol{\omega}(\nabla \cdot \mathbf{v}) = \frac{\nabla \rho \times \nabla P}{\rho^2} \tag{8}$$

or in terms of the material derivative

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} = -\nabla \times (\mathbf{v}(\nabla \cdot \mathbf{v})) + (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - \boldsymbol{\omega}(\nabla \cdot \mathbf{v}) + \frac{\nabla \rho \times \nabla P}{\rho^2}. \tag{9}$$

1.3 Imposing a steady state

For a statistically steady state, we have that

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = 0 \tag{10}$$

and the continuity equation gives $\nabla \cdot (\rho \mathbf{v}) = 0$ so that our (vector) equation reads

$$(\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} = \frac{\nabla \rho \times \nabla P}{\rho^2}. \tag{11}$$

We will now make assumptions about the velocity field in the convective zone, and look at this equation component wise.

2 The thermal wind equation

2.1 Assuming a purely azimuthal rotation profile

Assume that the equilibrium state is differential rotation around the z axis Balbus (2009):

$$\mathbf{v} = R\Omega(R, z)\mathbf{e}_\phi. \tag{12}$$

Since the flow is azimuthal, we only need the ϕ component of the vorticity equation

$$((\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}) \cdot \mathbf{e}_\phi = \frac{\nabla \rho \times \nabla P}{\rho^2} \cdot \mathbf{e}_\phi. \tag{13}$$

Note that we will work with both spherical polar (r, ϕ, θ) and cylindrical (R, ϕ, z). For the first term,

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = \frac{1}{R}\frac{\partial}{\partial R}\left(R^2\Omega\right)\mathbf{e}_z - \frac{\partial}{\partial z}(R\Omega)\mathbf{e}_R = \left(R\frac{\partial \Omega}{\partial R} + 2\Omega\right)\mathbf{e}_z - \left(R\frac{\partial \Omega}{\partial z}\right)\mathbf{e}_R \tag{14}$$

which means that this first term vanishes when we dot it with $\mathbf{e}_\phi$. Now we will deal with the second term, where we will use cylindrical grad.

$$\nabla = \frac{\partial}{\partial R} + \frac{1}{R}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial z} \tag{15}$$

divergence operator

$$\nabla \cdot \mathbf{v} = \frac{1}{R}\frac{\partial}{\partial R}(Rv_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(v_\phi) + \frac{\partial}{\partial z}(v_z) \tag{16}$$

$$(\boldsymbol{\omega} \cdot \nabla)\mathbf{v} = ((\nabla \times \mathbf{v}) \cdot \nabla)\mathbf{v} \tag{17}$$

now dot with $\nabla \mathbf{v} \cdot \mathbf{e}_\phi / R = \nabla \Omega$

$$= \left(\left(R\frac{\partial \Omega}{\partial R} + 2\Omega\right)\mathbf{e}_z - \left(R\frac{\partial \Omega}{\partial z}\right)\mathbf{e}_R\right) \cdot \left(\frac{\partial \Omega}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial \Omega}{\partial \phi}\mathbf{e}_\phi + \frac{\partial \Omega}{\partial z}\mathbf{e}_z\right) = 2\Omega\frac{\partial \Omega}{\partial z} = \frac{\partial \Omega^2}{\partial z} \tag{18}$$

We want to write this in terms of spherical polar r and θ, where $R = r\sin\theta$ and $z = r\cos\theta$. Using the chain rule, we can write

$$\frac{\partial}{\partial \theta} = r\cos\theta\frac{\partial}{\partial R} - r\sin\theta\frac{\partial}{\partial z} \tag{19}$$

and

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial z} + \sin\theta\frac{\partial}{\partial R}. \tag{20}$$

Multiplying the first by $\sin\theta$ and the second by $\cos\theta$, we can subtract Equation (19) from Equation (20) to eliminate any R terms, i.e.

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \tag{21}$$

which is the gradient parallel to the rotational axis in spherical polar. Finally, we can write our steady state vorticity equation as

$$\cos\theta\frac{\partial \Omega^2}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \Omega^2}{\partial \theta} = \frac{\nabla\rho\times\nabla P}{R\rho^2}\cdot\mathbf{e}_\phi = \frac{\nabla\rho\times\nabla P}{r\sin\theta\rho^2}\cdot\mathbf{e}_\phi \tag{22}$$

which is known as the *thermal wind equation*.

3 Convective zone step

We can't determine a rotation profile from Equation (22) just yet, but we can expand the right hand side and make some physically motivated simplifying assumptions for convective regions. We will now expand the right hand side Balbus (2009).

3.1 Thermal wind in terms of entropy

We have that

$$(\nabla\rho\times\nabla P)\cdot\mathbf{e}_\phi = \frac{1}{r}\left(\frac{\partial P}{\partial\theta}\frac{\partial\rho}{\partial r} - \frac{\partial P}{\partial r}\frac{\partial\rho}{\partial\theta}\right), \tag{23}$$

but we want to rewrite this in terms of the entropy $S = c_P\sigma$ where c_P is the specific heat capacity $\left(\frac{\partial S}{\partial T}\right)_{P,\mu}$ and $\sigma = \ln P\rho^{-\gamma}$ is the dimensionless entropy variable, and the adiabatic exponent $\gamma = \left(\frac{\partial \ln P}{\partial \ln\rho}\right)_S$. So

$$\frac{\partial S}{\partial\theta} = c_P\frac{\partial}{\partial\theta}\left(\ln\left(P\rho^{-\gamma}\right)\right) = c_P\left[\frac{\frac{\partial P}{\partial\theta}}{P} - \gamma\frac{\frac{\partial\rho}{\partial\theta}}{P}\right] \implies \frac{\partial\rho}{\partial\theta} = \frac{\rho}{c_P\gamma}\left[\frac{1}{P}\frac{\partial P}{\partial\theta} - \frac{\partial S}{\partial\theta}\right]. \tag{24}$$

Similarly for r,

$$\frac{\partial S}{\partial r} = \frac{\rho}{c_P\gamma}\left[\frac{1}{P}\frac{\partial P}{\partial r} - \frac{\partial S}{\partial r}\right]. \tag{25}$$

We can therefore write Equation (23)

$$\left(\frac{\partial P}{\partial\theta}\frac{\partial\rho}{\partial r} - \frac{\partial P}{\partial r}\frac{\partial\rho}{\partial\theta}\right) = \frac{\rho}{c_P\gamma}\left[\left[\frac{1}{P}\frac{\partial P}{\partial\theta} - \frac{\partial S}{\partial\theta}\right]\frac{\partial P}{\partial r} - \left[\frac{1}{P}\frac{\partial P}{\partial r} - \frac{\partial S}{\partial r}\right]\frac{\partial P}{\partial\theta}\right] = \frac{\rho}{c_P\gamma}\left[\frac{\partial S}{\partial r}\frac{\partial P}{\partial\theta} - \frac{\partial S}{\partial\theta}\frac{\partial P}{\partial r}\right] \tag{26}$$

so that the thermal wind equation becomes

$$\cos\theta\frac{\partial \Omega^2}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \Omega^2}{\partial \theta} = \frac{1}{r^2\sin\theta\rho^2}\frac{\rho}{c_P\gamma}\left[\frac{\partial S}{\partial r}\frac{\partial P}{\partial\theta} - \frac{\partial S}{\partial\theta}\frac{\partial P}{\partial r}\right] \tag{27}$$

or

$$\frac{\partial \Omega^2}{\partial r} - \frac{\tan\theta}{r}\frac{\partial \Omega^2}{\partial \theta} = \frac{1}{r^2\sin\theta\cos\theta\rho c_P\gamma}\left[\frac{\partial S}{\partial r}\frac{\partial P}{\partial\theta} - \frac{\partial S}{\partial\theta}\frac{\partial P}{\partial r}\right] \tag{28}$$

3.2 Assumptions about entropy vs pressure gradients

Next, for convective regions we consider whether

$$\frac{\partial S}{\partial r}\frac{\partial p}{\partial\theta} << \frac{\partial S}{\partial\theta}\frac{\partial p}{\partial r} \text{ or } \frac{\partial S}{\partial r}\frac{\partial p}{\partial\theta} >> \frac{\partial S}{\partial\theta}\frac{\partial p}{\partial r}? \tag{29}$$

We first note that $\frac{\partial \ln S}{\partial \ln r} \approx 0$ in convectively unstable regions, but that $\frac{\partial \ln p}{\partial \ln r}$ is large and negative regardless of convective stability. Also, S and p will have small gradients in θ due to rotation's effect on gravity, where hydrostatic equilibrium structure stays more or less the same. The radial pressure gradient dominates above all else (Balbus, 2009), so we keep the term containing that. This will break down in radiative regions “this is justified because in the radiative zone, we can no longer neglect the entropy gradient $\frac{\partial S}{\partial r}$. Note that differential rotation strengthens these arguments to keep the theta gradient. We let gravitational acceleration $g = -\frac{1}{\rho}\frac{\partial P}{\partial r}$ **(ignoring effects of rotation on gravity)** and write this as

$$\frac{\partial \Omega^2}{\partial r} - \frac{\tan\theta}{r}\frac{\partial \Omega^2}{\partial \theta} = \frac{g}{r^2 \sin\theta \cos\theta c_P \gamma}\frac{\partial S}{\partial \theta} = \frac{g}{r^2 \sin\theta \cos\theta \gamma}\frac{\partial \sigma}{\partial \theta}, \tag{30}$$

which is Equation (8) in Balbus 2012. Turbulent wind balance therefore refers to competing (inertial) angular velocity and angular entropy gradients (baroclinic driving). This is an equation for two variables, so if we assume a functional relation between S and Ω then we can solve for the rotation Ω.

4 Balbus functional relation: rotational excess follows entropy deficits

Following Balbus (2009); Balbus et al. (2009), we assume that the entropy is a function of $S = f(\Omega^2)$ (where isentropic and isorotational surfaces coincide), so that $\frac{\partial S}{\partial \theta} = \frac{\mathrm{d}S}{\mathrm{d}\Omega^2}\frac{\partial \Omega^2}{\partial \theta}$. We can therefore write Equation (30)

$$\frac{\partial \Omega^2}{\partial r} = \left(\frac{\tan\theta}{r} + \frac{g\frac{\mathrm{d}S}{\mathrm{d}\Omega^2}}{r^2 \sin\theta \cos\theta c_P \gamma}\right)\frac{\partial \Omega^2}{\partial \theta} \tag{31}$$

To satisfy this equation, Ω^2 must be constant along the characteristics:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = -\left(\frac{\tan\theta}{r} + \frac{g\frac{\mathrm{d}S}{\mathrm{d}\Omega^2}}{r^2 \sin\theta \cos\theta c_P \gamma}\right), \tag{32}$$

and if we make the substitution $y = \sin\theta$, we can re–write this as

$$\frac{\mathrm{d}y^2}{\mathrm{d}r} = 2\sin\theta\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}r} = -\frac{2y^2}{r} - \frac{2g}{r^2 c_P \gamma}\frac{\mathrm{d}S}{\mathrm{d}\Omega^2}. \tag{33}$$

Then after multiplying by r^2 we can write

$$\frac{\mathrm{d}\left(r^2 y^2\right)}{\mathrm{d}r} = -\frac{2g}{c_P \gamma}\frac{\mathrm{d}S}{\mathrm{d}\Omega^2}. \tag{34}$$

Since $\frac{\mathrm{d}S}{\mathrm{d}\Omega^2}$ is a constant along these characteristics we can integrate this

$$y^2 r^2 = r^2 \sin^2\theta = R^2 = A - \frac{\frac{2g}{c_P \gamma}}{r}. \tag{35}$$

So if $S(\Omega^2)$, then surfaces of constant Ω^2 are given by Equation (35). The particular relation that matches helioseismology in Balbus et al. (2009) is

$$f(\Omega^2) = \sigma(r, \theta) - \sigma_r = \sigma' \tag{36}$$

where σ_r is some function of radius alone (e.g. the spherically averaged entropy), σ is the 3D entropy profile, and σ' is called the residual entropy. Since σ' seems to be a solution for the contours in the SCZ, this indicates that pressure gradients and gravity (subject to fictitious rotational forces) balance, leading to the observed rotation profile Ω, which increases with distance from axis R. This dictates the flow in the SCZ, rather than cases of marginal stability, and is shown in Figure 1 and compared to the solar rotation profile (Howe, 2009) (taken from Balbus et al. (2009), Figure 2).

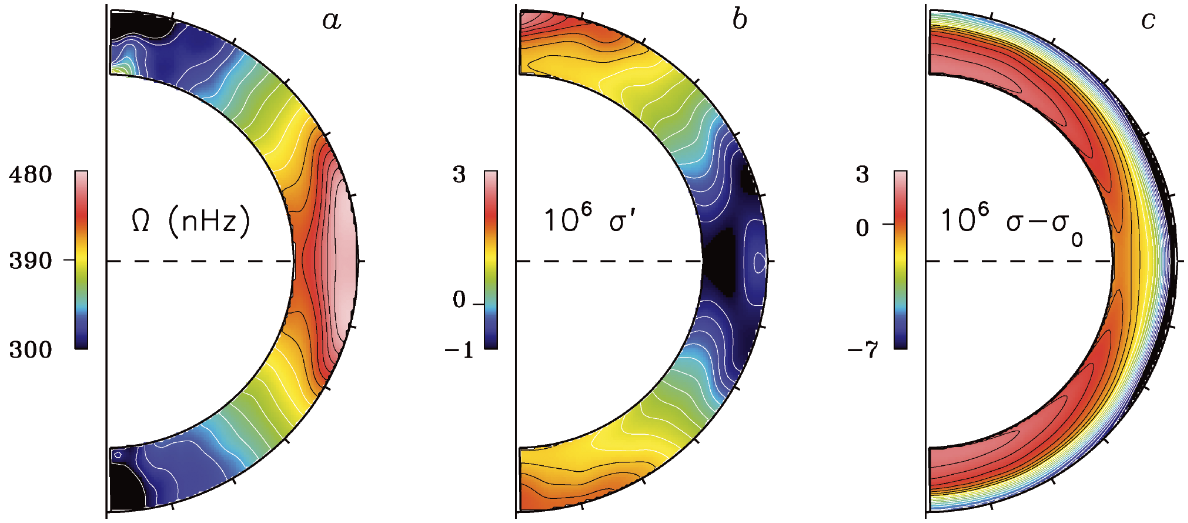

Figure 1: Figure 2. in Balbus et al. (2009), the solar convective zone rotation profile (left) residual entropy (centre) and total entropy (right). The contours of rotation Ω and residual entropy σ' follow each other nicely, but Ω and entropy do not.

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