LIMITS AND ASYMPTOTIC EXPANSIONS

MTH1035

Semester 1, 2016

This lesson will go through some applications of integration techniques you've learnt. Recall that when using integration by parts,

$$\int fg'dx = fg - \int f'gdx$$

We choose f and g carefully such that $\int f'gdx$ can be evaluated. However for some integrals, there is no choice of f and g that will give us an exact answer, and we are left to keep integrating by parts in a recursive fashion. Integrating by parts recursively is used to evaluate asymptotic expansions of special functions. An asymptotic series expansion is a series expansion of a function in variable x. For large x, the asymptotic expansion is an arbitrarily good approximation to the function, where the function can be convergent or divergent.

Stirling's approximation

Stirling's approximation states the following:

$$\ln(N!) = N\ln(N) - N \tag{1}$$

This is particularly useful for large N, necessary for thermodynamical situations in statistical mechanics. The approximation will be found using Riemann sums.

Proof

We start by finding the integral of $\ln(x)$.
Using the product rule,

$$\frac{d}{dx}(x\ln(x)) = \ln(x) + 1$$

But this is of little use. Try:

$$\frac{d}{dx}(x\ln(x) - x) = \ln(x) + 1 - 1 = \ln(x)$$

Integrating both sides,

$$\int \frac{d}{dx}(x\ln(x) - x)dx = \int \ln(x)dx$$

$$\therefore \int \ln(x)dx = x\ln(x) - x$$

Now we have an expression for the RHS of (1). In thermodynamical systems we deal with very large n, so let's take the integral from 1 to n (since there is a singularity at n = 0).

$$\int_1^n \ln(x)dx = [x\ln(x) - x]_{x=1}^{x=n}$$

$$= n\ln(n) - n + 1$$

But for large n,

$$\int_1^n \ln(x)dx \approx n\ln(n) - n \tag{2}$$

We will now evaluate the RHS of (2) with geometric arguments, using a Riemann sum. The area under $f(x) = \ln(x)$ can be approximated by

$$A = width \times (\ln(1) + \ln(2) + \ln(3) \ldots \ln(n))$$

$$= 1 \times (\ln(1 \cdot 2 \cdot 3 \ldots n)$$

$$= \ln(n!)$$

So if n is large,

$$\int_1^n \ln(x)dx = [x\ln(x) - x]_{x=1}^{x=n} = \ln(n!)$$

$$\therefore N\ln(N) - N \approx \ln(N!)$$

General Method

1. Assign f and g to perform integration by parts.
2. The result will contain an integral. Integrate this by parts, keeping the assignments for f and g consistent.
3. The result of this will also contain an integral. Keep integrating by parts until you have enough terms to notice a pattern in the terms which have been integrated.
4. Express the result as the first few terms + the last term + remainder term (integral)

Error function erf(x)

The error function is defined as follows:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{3}$$

To calculate the asymptotic expansion, we express this as

$$\text{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \tag{4}$$

where

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt + \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1$$

As with the gamma function, the asymptotic expansion will be obtained by integration by parts.
So we let $u = t^2 \implies t = u^{1/2}$. Therefore when evaulating integrals, $x \to x^2$.

$$\frac{dt}{du} = \frac{1}{2} u^{-1/2} \implies dt = \frac{1}{2} u^{-1/2} du$$

Continuing from (4),

$$\text{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \frac{1}{2} \int_{x^2}^\infty e^{-u} u^{-1/2} du = 1 - \frac{1}{\sqrt{\pi}} \int_{x^2}^\infty e^{-u} u^{-1/2} du \tag{5}$$

Integration by parts

For $I_0(x) = \int_x^\infty e^{-u}u^{-1/2}du$, let $f = u^{1/2}$ and $g' = e^{-u}$

$$\therefore \int_{x^2}^\infty e^{-u}u^{-1/2}du = [-e^{-u}u^{-1/2}]_{x^2}^\infty - \int_{x^2}^\infty -e^{-u}\frac{1}{2}u^{-3/2}du$$

So at this stage,

$$\operatorname{erf}(x) = 1 - \frac{1}{\sqrt{\pi}}([-e^{-u}u^{1/2}]_{x^2}^\infty + \int_x^\infty e^{-u}\frac{1}{2}u^{-3/2}du) = 1 - \frac{1}{\sqrt{\pi}}([e^{-x^2}x^{-1}] + \int_{x^2}^\infty e^{-u}\frac{1}{2}u^{-3/2}du)$$

$$\operatorname{erf}(x) = 1 - \frac{1}{\sqrt{\pi}}[\frac{e^{-x^2}}{x}] + \frac{1}{2}\int_{x^2}^\infty e^{-u}u^{-3/2}du) \tag{6}$$

Now define $I_1(x) = \int_x^\infty e^{-u}u^{-3/2}du$, let $f = u^{3/2}$ and $g' = e^{-u}$

$$\therefore \int_{x^2}^\infty e^{-u}u^{-3/2}du = [-e^{-u}u^{-3/2}]_{x^2}^\infty - \int_{x^2}^\infty -e^{-u}\frac{3}{2}u^{-5/2}du$$

So,

$$\operatorname{erf}(x) = 1 - \frac{1}{\sqrt{\pi}}[\frac{e^{-x^2}}{x}] + \frac{1}{2}[[-e^{-u}u^{-3/2}]_{x^2}^\infty - \frac{3}{2}\int_{x^2}^\infty -e^{-u}u^{-5/2}du])$$

$$= 1 - \frac{1}{\sqrt{\pi}}[\frac{e^{-x^2}}{x}] + \frac{1}{2}[[e^{-x^2}x^{-3}] - \frac{3}{2}\int_{x^2}^\infty -e^{-u}u^{-5/2}du])$$

$$\therefore \boxed{\operatorname{erf}(x) = 1 - \frac{1}{\sqrt{\pi}}([e^{-x^2}](\frac{1}{x} + \frac{1}{2x^3}) - \frac{3}{2}\int_{x^2}^\infty -e^{-u}u^{-5/2}du)}$$

So

$$I_2(x) = \int_{x^2}^\infty e^{-u}u^{-5/2}du = [[e^{-x^2}x^{-5}] - \frac{5}{2}\int_{x^2}^\infty -e^{-u}u^{-7/2}du)$$

$$I_3(x) = \int_{x^2}^\infty e^{-u}u^{-7/2}du = [[e^{-x^2}x^{-7}] - \frac{7}{2}\int_{x^2}^\infty -e^{-u}u^{-9/2}du)$$

$$\ldots$$

$$I_n(x) = \int_{x^2}^\infty -e^{-u}u^{-n-1/2}du = [[e^{-x^2}x^{-(2n+1)}] - (-n-1/2)\int_{x^2}^\infty -e^{-u}u^{-n-3/2}du])$$

So given these results,

$$\operatorname{erf}(x) = 1 - \frac{1}{\sqrt{\pi}}(e^{-x^2}(\frac{1}{x} + \frac{1}{2x^3} - \frac{3}{4x^5} + \frac{7}{8x^7} + \ldots + (-1)^n\frac{1\cdot3\cdot5\ldots(2n-1)}{2^nx^{2n+1}}) + (-1)^{n+1}\frac{1\cdot3\cdot5\cdot\ldots(2n+1)}{2^{n+1}x^{2n+3}}) \tag{7}$$

So in general,

$$\operatorname{erf}(x) = 1 - (\frac{e^{-x^2}}{\sqrt{\pi}}\sum_{n=0}^N(-1)^n\frac{1\cdot3\cdot5\ldots(2n-1)}{2^nx^{2n+1}}) + R_{n+1}(x) \tag{8}$$

Where

$$R_{n+1}(x) = (-1)^{n+1}\frac{1}{\sqrt{\pi}}\frac{1\cdot3\cdot5\cdot\ldots(2n+1)}{2^{n+1}x^{2n+3}}I_{n+1}(x)$$

Recall that we defined I_x:

$$I_n = \int_{x^2}^{\infty} t^{-n-1/2}e^{-t}dt = \frac{e^{-x^2}}{x^{2n+1}} - (n+\frac{1}{2})I_{n+1}(x)$$

Now,

$$|I_n(x)| = |\int_{x^2}^{\infty} t^{-n-1/2}e^{-t}dt| = \frac{e^{-x^2}}{x^{2n+1}} - (n+\frac{1}{2})I_{n+1}(x)$$

$$\leq \frac{1}{x^{2n+1}}\int_{x^2}^{\infty} e^{-t}dt$$

$$|I_n(x)| \leq \frac{e^{-x^2}}{x^{2n+1}}$$

So,

$$R_{n+1}(x) \leq \frac{1\cdot 3\cdot 5\cdot \ldots (2n+1)}{2^{n+1}\sqrt{\pi}}\frac{e^{-x^2}}{x^{2n+3}}$$

So what we have done is define an integral as a function of $x \to \infty$ and a remainder term $R_{n+1}(x)$, which is an integral.

Gamma function

The Gamma function is defined as follows:

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt \tag{9}$$

We will integrate this by parts. Assign t^{x-1} as f and e^{-t} as g'. Integrating once by parts:

$$\int_0^{\infty} t^{x-1}e^{-t}dt = [-e^{-t}t^{x-1}]_{t=0}^{t=\infty} - \int_0^{\infty}(x-1)t^{x-2}\cdot -e^{-t}dt$$

$$= [-e^{-t}t^{x-1}]_{t=0}^{t=\infty} + (x-1)\int_0^{\infty} t^{x-2}e^{-t}dt$$

Integrating by parts again,

$$\int_0^{\infty} t^{x-2}e^{-t}dt = [-e^{-t}t^{x-2}]_{t=0}^{t=\infty} - \int_0^{\infty}(x-2)t^{x-3}\cdot -e^{-t}dt$$

So,

$$\Gamma(x) = [-e^{-t}t^{x-1}]_{t=0}^{t=\infty} - (x-1)([-e^{-t}t^{x-2}]_{t=0}^{t=\infty} - \int_0^{\infty}(x-2)t^{x-3}\cdot -e^{-t}dt)$$

$$\ldots$$

$$\Gamma(x) = [-e^{-t}t^{x-1}]_{t=0}^{t=\infty} + (x-1)([-e^{-t}t^{x-2}]_{t=0}^{t=\infty} + (x-2)([-e^{-t}t^{x-3}]_{t=0}^{t=\infty} - \int_0^{\infty}(x-3)t^{x-4}\cdot -e^{-t}dt))$$

$$= [-e^{-t}t^{x-1}]_{t=0}^{t=\infty} + (x-1)([-e^{-t}t^{x-2}]_{t=0}^{t=\infty} + (x-2)([-e^{-t}t^{x-3}]_{t=0}^{t=\infty} - \int_0^{\infty}(x-3)t^{x-4}\cdot -e^{-t}dt))$$

$$= [-e^{-t}t^{x-1}]_{t=0}^{t=\infty} + (x-1)([-e^{-t}t^{x-2}]_{t=0}^{t=\infty} + (x-2)([-e^{-t}t^{x-3}]_{t=0}^{t=\infty} + \ldots (x-n)([-e^{-t}t^{x-n-1}]_{t=0}^{t=\infty}$$

$$-\int_0^{\infty} t^{x-n}\cdot -e^{-t}dt)))$$

And so on. Given the recursive fashion in which we derived the expression, it can be seen that

$$\Gamma(x+1) = x\Gamma(x) \tag{10}$$

EXERCISES

1. (a)Using the first few terms in the asymptotic expansion of $\Gamma(x)$, convince yourself of result (4)
   (b)Evaluate the Gamma function using the recursive relationship (4), down to $n = 1$. What is the definition of the gamma function?

2. Find the Taylor series expansion for $\text{erf}(x)$ around $x \to 0$ (3). Compare this to the asymptotic expansion as $x \to \infty$ (8). Which do you think is more accurate and why? (Look at their respective remainder terms)

3. Express the exponential integral

   $$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt$$

   as an asymptotic expansion.

4. **CHALLENGE**
   Consider the Euler integral,

   $$I(x) = \int_{0}^{\infty} \frac{e^{-t}}{1 + xt} dt \tag{11}$$

   (a) Find the Taylor series expansion around $x = 0$ by setting $x \to xt$.
   (b) Using integration by parts, find the asymptotic expansion of (11).

   Does the answer for (a) always converge? Over what region?
   Does the answer for (b) always converge? Over what region?

REFERENCES

Erdlyi, A. (1956), *'Asymptotic expansions'*, Dover publications, Mineola NY, USA.