CONIC SECTIONS/ORTHOGONAL MATRICES

When an arbitrarily aligned plane intersects with two cones joined at the tip, the result is a **Conic section.**

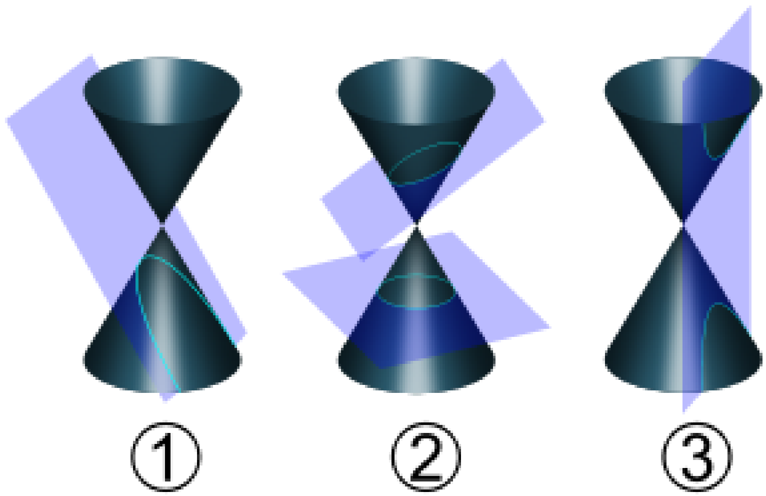

Figure 1: Three types of conic sections

There are three types of conic sections:

1. Parabola
2. Ellipse (where circle is just a special case)
3. Hyperbola

Later we will look at how conic sections relate to ellipticity and orbital mechanics, but for now we will see how to sketch these from their standard forms.

Sketching rotated conic sections

We are familiar with the standard equations for conic sections, whose axes align with the x and y axes.

Ellipse

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a \geq b > 0$. Here a is the semi-major axis and b is the semi-minor axis. The Foci are located at $(\pm\sqrt{a^2 - b^2}, 0)$
2. $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ where $a \geq b > 0$. Here a is the semi-major axis and b is the semi-minor axis. The Foci are located at $(0, \pm\sqrt{a^2 - b^2})$

Note: If the foci coincide, we are dealing with a circle.

Hyperbola

1. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a \geq b > 0$. Here the asymptotes are $y = \pm\frac{b}{a}x$. The Foci are located at $(\pm\sqrt{a^2+b^2}, 0)$
2. $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ where $a \geq b > 0$. Here the asymptotes are $y = \pm\frac{a}{b}x$. The Foci are located at $(0, \pm\sqrt{b^2+a^2})$

However, the general form for a conic whose axes are ***NOT*** parallel with the x and y axes is:

$$\boxed{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0}$$

Where D and E non-zero for conics which are not centred at (0,0). Recall that we complete the square for both x and y to get the location of the centre.

Sketching conic sections from general form

The following algorithm is one way to sketch both skewed ellipses and skewed hyperbolae from their general forms.

1. Set up the LHS as a system of matrices: $\underline{\mathbf{x}}^T M \underline{\mathbf{x}} = -F$ for $\underline{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $M = \begin{bmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{bmatrix}$
2. Orthogonally diagonalize matrix M as $PDP^{-1} = PDP^T$
   (a) Find eigenvalues of M, λ_1 and λ_2.
   (b) Construct D, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
   (c) Find the eigenvectors, hence eigenspace of M. Call these $\underline{\mathbf{v_1}}$ and $\underline{\mathbf{v_2}}$, corresponding to λ_1 and λ_2 respectively. $P = \begin{bmatrix} \underline{\mathbf{v_1}} & \underline{\mathbf{v_2}} \end{bmatrix}$ (a 2×2)
   (d) Construct P.
3. Introduce a new co-ordinate system, $\underline{\mathbf{x}'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$. These are defined as $\underline{\mathbf{x}'}^T D \underline{\mathbf{x}'} = -F'$. Since the cross terms in D are both 0, the cross term in the standard form vanishes and we can work from the simpler general forms that are more familiar. In primed co-ordinates it should be of the form:
$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$
4. Find the direction of the x' and y' axes in terms of x and y.
5. Sketch x' and y' relative to x and y
6. Sketch the conic section on the x' and y' axes using the simpler canonical forms.

Co-ordinate transforms from (x, y) to (x', y')

Say we have simplified a skewed conic from $Ax^2+Bxy+Cy^2+Dx+Ey+F = 0$ to $A'(x')^2+C'(y')^2+D'x'+E'y'+F' = 0$ by using eigenvalues and eigenvectors as outlined above. These two ways to represent the same conic have the following invariance properties:

1. $F = F'$: This preserves the size of the semi-major and semi-minor axes for ellipses, and the smallest radius of curvature for hyperbolae. In otherwords, it preserves the shape and size of the conic.
2. $A + C = A' + C' = tr(M) = tr(M')$
3. $B^2 - 4AC = B'^2 - 4A'C'$: This is called the **discriminant** of the conic.

You can convince yourself of these invariants during the exercises where you sketch conics.

Discriminant

Invariance of a quantity over a co-ordinate transform suggests an essential property in the system. The discriminant tells us what type of conic we are dealing with. Specifically:

- $B^2 - 4AC < 0$: ellipse/circle
- $B^2 - 4AC = 0$: parabola
- $B^2 - 4AC > 0$: hyperbola

In orbital mechanics, ellipses and circles form 'bound' systems, while parabolae and hyperbolae form 'unbound' systems.
It will be an exercise to show how these cases relate to a property known as **eccentricity**

Orthogonal matrices

Definitions

- An $n \times n$ matrix M is said to be **symmetric** if it satisfies $M^T = M$.
- An $n \times n$ matrix Q is said to be **orthogonal** if it satisfies $P^T P = I$, that is: $P^{-1} = P^T$
(**NOTE:** For this to be true, $det(P) = det(P^{-1}) = \pm 1$. Always check this once you have found a P)
- Orthogonal matrices are formed using mutually orthogonal sets.

Orthogonally diagonalizing matrices

The principle axis theorem states the following:

Every symmetric matrix can be diagonalized via an orthogonal matrix P

If M is symmetric, we can find an orthogonal matrix P such that

$$A = PDP^{-1} = PDP^T$$

Since we demanded that P be orthogonal.
Decomposing M in terms of P and D is extremely useful for raising the M to some power k.
If you have ever tried raising a matrix to a large power, say $k = 100$ using graphics software, it can take a considerable number of seconds, sometimes minutes.
Solving by hand will no doubt take much longer, and by the hundredth iteration you've probably ended up with the wrong answer anyway.
But look what happens when we raise M, a symmetric matrix to the power k.

$$A^k = (PDP^T)^k = (PDP^{-1})^k)$$

$$= (PDP^{-1})(PDP^{-1})(PDP^{-1}) \dots (PDP^{-1})(PDP^{-1})(PDP^{-1})$$

$$= PD(P^{-1}P)D(P^{-1}P)D(P^{-1} \dots P)D(P^{-1}P)D(P^{-1}P)DP^{-1}$$

$$Now\ (P^{-1}P) = I,\ So\dots$$

$$M^k = PD^kP^{-1}$$

Since D is a diagonal matrix, say $\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$,

$$\boxed{M^k = P \begin{bmatrix} A^k & 0 \\ 0 & C^k \end{bmatrix} P^{-1}}$$

Sketching a skewed conic

eg. Sketch the following conic: $8x^2 + 4xy + 5y^2 = 36$

Answer: Following the procedure outlined on page (2) ...

1. $$8x^2 + 4xy + 5y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 36$$

2. Orthogonally diagonalize matrix M as $PDP^{-1} = PDP^T$

   (a) **Find the eigenvalues of** M

   $$det(\lambda I - M) = 0$$

   $$\begin{vmatrix} \lambda - 8 & -2 \\ -2 & \lambda - 5 \end{vmatrix} = 0 \implies (\lambda - 8)(\lambda - 5) - 4 = 0$$

   $$\lambda^2 - 13\lambda + 40 - 4 = 0$$

   $$\lambda^2 - 13\lambda + 36 = 0$$

   $$\implies \lambda_1 = 9,\ \lambda_2 = 4$$

   (b)

   $$D = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

   (c) **Find the eigenvectors of M**: Solve the nullspace/kernel of $(\lambda I - M)$

   λ = 9 case:

   $$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\mathbf{0}}$$

   $$\implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \underline{\mathbf{v}}_1$$

   λ = 4 case:

   $$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\mathbf{0}}$$

   $$\implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \underline{\mathbf{v}}_2$$

   (d)

   $$P' = \begin{bmatrix} \underline{\mathbf{v_1}} & \underline{\mathbf{v_2}} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

   The reason this is P' (not P) is because P is defined as an orthogonal matrix, where $det(P) = det(P^{-1}) = \pm 1$. So we need to scale P' for this to be true.

   $$det(P') = -4 - 1 = -5$$

$$\implies P' = \pm 5P$$

$$\implies P = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

So M has been decomposed into orthogonal and diagonal matrices:

$$M = PDP^{-1} = PDP^T$$

$$\begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

$$\begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

3.

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 36$$

$$\implies 9(x')^2 + 4(y')^2 = 36$$

$$\boxed{\frac{(x')^2}{4} + \frac{(y')^2}{9} = 1}$$

This is the form of an ellipse/circle. Since $a \neq b$ it is an ellipse. This could also have been determined from the discriminant. So we have the equation for our ellipse relative to (x', y'). It is now the task to see where the primed co-ordinate axes (x', y') hence the ellipse lie relative to (x, y).

4. Find the direction of the x' and y' axes in terms of x and y.
   Let P act on the x and y axes.

x' axis

$$x' : \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

y' axis

$$y' : \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The primed co-ordinate axes are just the column vectors of P. They are orthogonal, which is expected since P was constructed as an orthogonal matrix.

5. Sketch x' and y' relative to x and y

6. Sketch the conic section on the x' and y' axes using the simpler canonical forms.

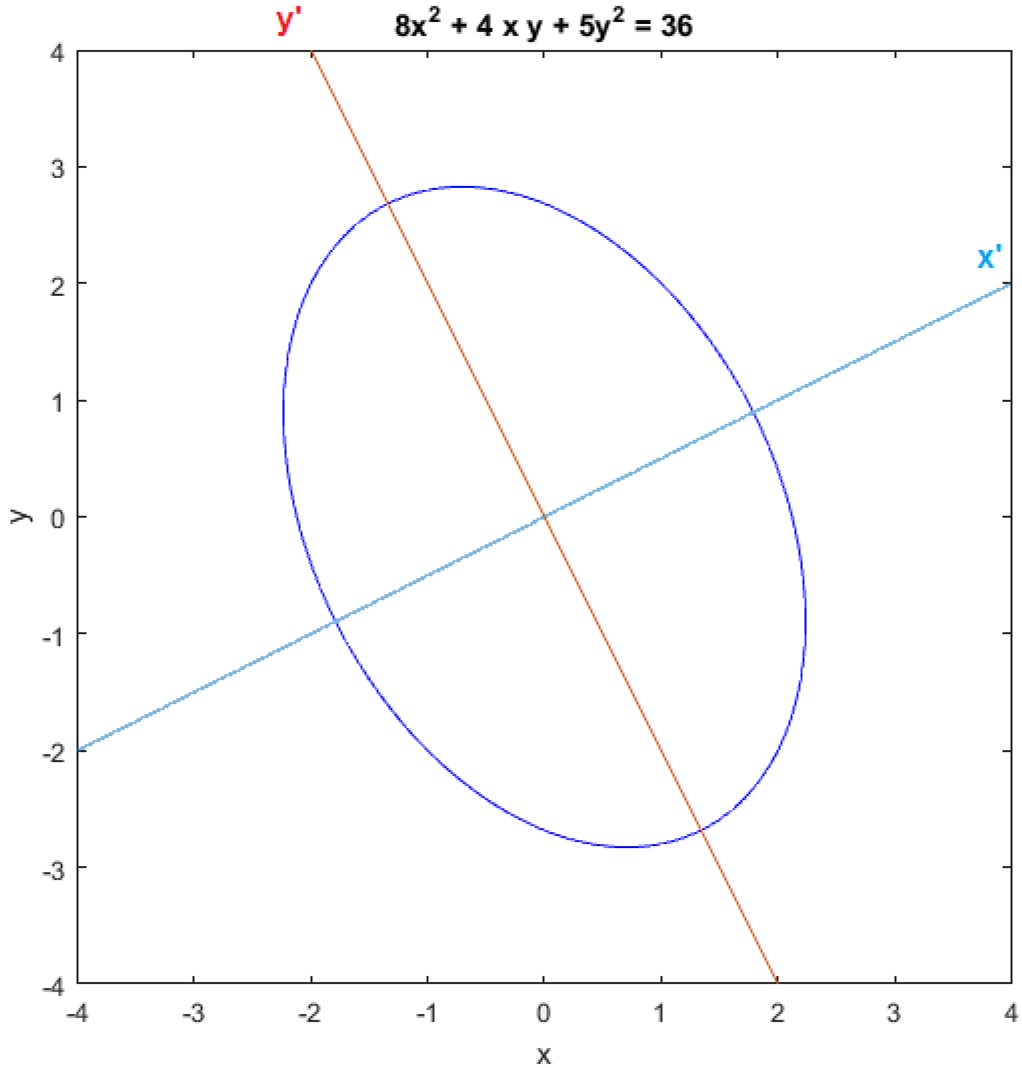

Figure 2: Final sketch

Kepler's Laws

The following is true for gravitationally bound, two-body systems:

1. A planet orbits its star in an ellipse, with the star at one focus of the ellipse.
2. If there is a chord joining the planet and sun, the planet 'sweeps out' equal areas in equal time from all starting points on the ellipse.
3. $P^2 \propto a^3$ (proportionality factor: $\frac{4\pi^2}{G(m_1+m_2)}$)
   Where a is the semi-major axis and P is the orbital period.

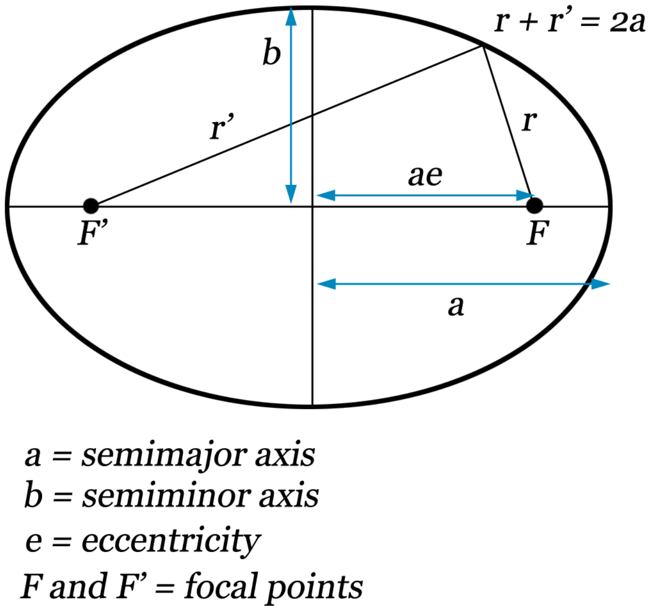

Figure 3: An ellipse http://hyperphysics.phy-astr.gsu.edu/hbase/math/ellipse.html

So ellipses are really just the set of points which satisfy $r + r' = 2a$ as defined in figure 3. We can see that $ae \leq a$ **everywhere** on the ellipse, so we expect e to take values between 0 and 1.

EXERCISES

1. Sketch the following conics:
   (a) $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$
   (b) $-2x^2 + 2\sqrt{3}xy = 3$

2. **CHALLENGE**: Orbital mechanics

   We consider the case where the planet is parallel to b in Figure 3. Here $r = r'$. Since b and ae are perpendicular, we can use them to find an expression for r using Pythagoras:

$$r^2 = b^2 + (ae)^2$$

   But e is an essential property of an ellipse, so r needs to be arbitrary. Now, $r+r' = 2a$. But we have constructed the problem so that $r = r'$.

$$\implies r + r = 2a$$

$$\boxed{r = a}$$

   So,

$$a^2 = b^2 + (ae)^2$$

$$b^2 = a^2 - (ae)^2$$

$$= a^2(1 - e^2)$$

$$\implies e^2 = 1 - \frac{b^2}{a^2}$$

   Which is how eccentricity is defined. Given that $a > b$ by their definitions, $1 > e > 0$. This is consistent with the geometric definition of ae in figure 3.

   Defining the eccentricity of an ellipse

   (a) Using Figure 3, express the length of r' for a planet located *anywhere* on the ellipse, in terms of r, a, e and θ. Then find an expression for r in terms of only a, e and θ. (*Hint: recall that* $r + r' = 2a$)
   (b) What happens when $\theta = 0$? What about $\theta = \frac{\pi}{2}$? Is this consistent with the set up in Figure 3?

REFERENCES

1. Geyling, F., Westerman, H. (1971) The Conic sections. In *'Introduction to orbital mechanics'* Bell Telephone Laboratories Inc. Murray Hill, NJ USA. 12-25 pp.

2. Carrol, B., Ostlie, D. (1996) Celestial Mechanics. In *'An Introduction to Modern Astrophysics'*, Addison-Wesley Publishing Company Inc. Boston, MA USA, 25-53pp.