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***Electricity Trading Between Smart Nano-Grids:  
Matching Supply and Demand in the Face of Unpredictable  
Supply***

Brian McNestry

B.A. (Mod.) Integrated Computer Science

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Supervisor: Dr. Donal O'Mahony

School of Computer Science and Statistics

O'Reilly Institute, Trinity College, Dublin 2, Ireland

# Declaration

I hereby declare that this project is entirely my own work and that it has not been submitted as an exercise for a degree at this or any other university.

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## **Abstract**

This is the abstract

# Acknowledgements

Acknowledge the various people here

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## **Part I**

# **Introduction**



## **Part II**

# **Background**

# Chapter 1

## Decentralised Grid

At present in Ireland and in many other countries, the national electric grid infrastructure is controlled by a central body, namely the ESB. While there are several electricity providers in Ireland, such as Bord Gáis Energy, SSE Airtricity and Energy Ireland, each of them use the same distribution network as one another. Essentially the power is provided from each of the different providers and then routed into the same centralised hub belonging to the ESB. From there, each consumer (a household) receives the energy that they pay for accordingly at a fixed rate through that same infrastructure belonging to the ESB. This is much the same system as any other country, where there is a centralised grid.

This system has been in place for decades and lends itself very well to the situation where large companies can provide a steady supply of energy by way of electricity plants that use both renewable and non-renewable energy sources. Non-renewable energy sources, also known as fossil fuels, include resources such as coal, gas and oil. While these are finite resources, at present they can be burned at a steady rate in order to meet the demands of customers. Electricity from renewable sources can also be produced at a fairly steady rate by placing large farms in areas that are particularly well suited to the type of renewable energy being produced. For example, large wind farms are set up in windy regions far removed from residential or urban areas and solar panels can be placed in regions that typically enjoy clearer skies than other areas.

However in the future, perhaps the very near future, with the ongoing depletion of non-renewable resources, more and more people will turn to deploying solar panels and local wind farms in their locale, regardless of whether or not they are living in a particularly sunny or windy area. At the moment there are a few houses out there that use a solar panel to heat their water or other smaller tasks but soon more and more people will become more and more dependent on what they can produce either within their own home, or in a more collective sense in their own neighbourhood to power their houses.

The issue that then arises in these areas that aren't as sunny or windy is that supply of electricity is no longer steady. The current system could not be maintained as the energy produced on a local level would be

small enough that it would not be worth it to pass this energy upstream to the central grid. The energy would instead be used at a local level to try to cover the demand for electricity of the house or business with which that particular device is associated.

The model of infrastructure that would then be required is that of a decentralised grid. This model would need a massive infrastructure overhaul in order to implement so it would not exist in the world until it is needed and accepted by the major companies who would then go about implementing it. In this case necessity would be the mother of invention, at least on a practical level. The rough idea of a distributed grid is described in figure 1.1. Throughout the rest of this report distributed grid and decentralised grid are used interchangeably.

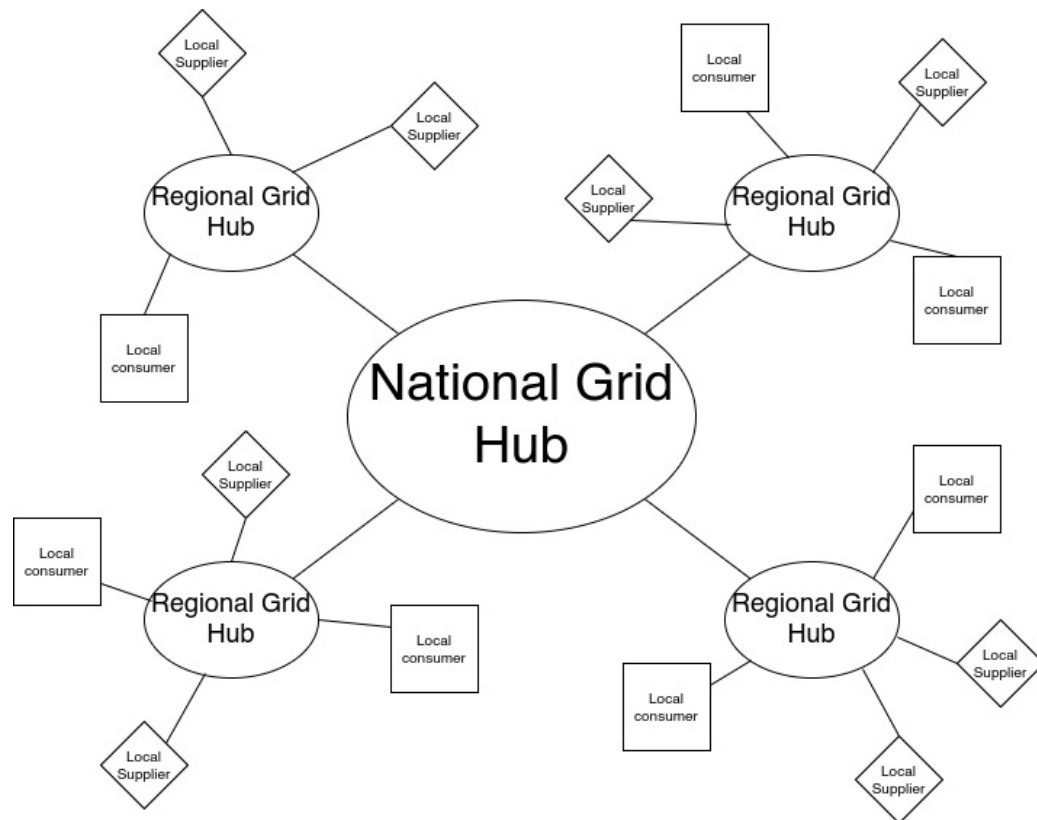


Figure 1.1: Each local consumer and supplier is attached to a regional grid hub which manages the allocation of electricity between suppliers and consumers. This is just a simple overview of the idea but conceivably a consumer or a supplier could be connected to two or more different regional grid hubs.

## Chapter 2

# Smart Grid

### 2.1 Overview

Due to advancements in networking technologies as well as in the field of sophisticated decision making technologies, the idea of a smart grid has become increasingly popular. The idea of the smart grid is that actors within a grid, be they individual consumers or suppliers, or groups thereof, can be fitted with small computers that perceive changes in the grid and then these actors can then react accordingly. Several different types of management systems have been constructed in order to successfully, fairly and efficiently allocate resources for each of these different types of actors. The two primary types of management systems that were examined as part of this final year project were Auctions and Game Theory which will both be discussed in detail later on.

The smart grid is not only used in this regard but in fact has many other potential applications, some of which have been implemented already in several cities and regions throughout the world. Other applications of the technology include energy consumption or production prediction, scheduling the use of consumers in order to reduce costs or operation and smart reaction and response to disruptions or blackout within the grid to reduce the damage that occurs as a result.

In this project it is assumed that the consumers within the system are outfitted with some kind of prediction technology. An example of such a system has been proposed by Garcia et al [1] where a device tries to time its own operation within a certain time-frame in accordance with when the price of energy is cheapest. It also tries to predict how much energy it will consume based off its own knowledge of previous experiences in buying power at that particular time of day, allowing the system to learn over time and make smarter decisions as time goes on.

## 2.2 Microgrids and Nanogrids

At present smart grids have generally been implemented on the level of microgrids. Microgrids are generally thought of in terms of having a consumer be a single house, or perhaps a group of houses, and a supplier being a small wind farm or solar farm, or perhaps a group of these together. In the case of a microgrid, actors within the system are defined in similar to the units involved in a centralised grid system meaning that the transition from a centralised grid to the microgrid scheme was a relatively easy one.

An example of such a real world implementation is that of the system in place in Japan. This system was mostly implemented following the disaster of Fukushima, where it became clear that a reliance on a single power source and a centralised power distribution network left the country vulnerable following the disaster [2]. Several regions were cut off from power as a result of the disaster which hampered the relief efforts as well as making the lives of ordinary Japanese citizens more difficult. Had a microgrid system been in place then not so many hospitals and homes would have been left without power following the disaster. The company ENEL has also introduced a smart grid system in the region of Apulia in southern Italy [3].

The nanogrid system is very similar to that of the microgrid system conceptually but is concerned with a much smaller scale. A nanogrid is one that operates within the confines of a single building, generally where each consumer is a single appliance such as a washing machine or an electronic vehicles (EV). Suppliers would also be very small scale perhaps a set of solar panels or a small wind turbine. A nanogrid system could also be adapted to aggregate a number of devices to act as one as a single actor within the nanogrid system, for example all the lights on one floor of a house could act as a single consumer and draw on a shared reserve of power.

Another extension of the nanogrid system, which will be discussed in further detail in the conclusion section of this paper, would be to incorporate a nanogrid as a sub-node of a microgrid. This would create a hierarchy of distributed grids. This tree could also be adapted into a graph where a parent node in the tree could have multiple children and a child could have multiple parents. This will be discussed more in the conclusion.

## Chapter 3

# REFIT Scheme

The REFIT scheme (Renewable Energy Feed In Tariff) [4] is one of the most common ways in which countries around the world, including Germany, Spain and the state of Hawaii, try to incentivise renewable energy sources and suppliers to sell energy into the main grid for consumption by consumers. The primary tenet of the REFIT scheme is to guarantee a fixed price for energy provided at particular times of the day. The prices are offered in a non-discriminatory fashion for every kWh produced by the supplier. The price can be lower or higher based on the type of energy being produced, for example in Germany the price is higher for suppliers of solar energy than for suppliers of wind energy, according to the EU at the time of the writing of this report [5].

The main advantage of this type of a scheme is that it first of all incentivises companies to invest in renewable energy because they know they'll receive a good return on their investment but also incentivises landowners and home owners to invest, thereby creating a large infrastructure of renewable energy resources in a relatively small space of time and this has worked effectively in Germany. The payment also easily covers the cost of creating the solar panels or wind turbine.

The downside to the REFIT scheme however is that because it is a fixed amount based primarily on the type of energy produced and for how long it is being provided, it means that it can mean that it is not worth it for a supplier to sell if it has a poor supply in reserve for example. In this case, the incentive to sell energy is quite low as selling any energy would drain the supplier of most of its power. Therefore a scheme of a dynamic price model might be better that would incentivise all suppliers at all times.

## Chapter 4

# Auctions

### 4.1 Overview

The first type of node management systems considered as part of this report was that of auctions. Auctions have a number of different types of properties generally and as such, can be classified into different groupings.

- Single- or multi-dimensional
- One- or two-sided
- Open-cry or sealed-bid
- First- or  $k$  th-price
- Single- or multi-unit
- Single- or multi-item

While all of these are discussed in detail in the book by Simon Parsons [6], only one of these will be discussed here as it the only type of auction that was considered, as well as the decision as to why this was the only type considered. The type of auction investigated was a continuous double auction.

### 4.2 Continuous Double Auction

A continuous double auction was discussed by the paper by Ramachandran [7] among others and was therefore a popular candidate by several potential energy management systems. The idea of a double auction is a simple one. Instead of trying to match multiple bidders to a single seller or multiple sellers to a single buyer,

a double auction is where there are multiple sellers and multiple bidders. By combining the buy-side and the sell-side of an auction into a single process, we then have a two-sided or double action.

A continuous double auction is an extension and a refinement of a double auction where multiple rounds are conducted until as many bidders and sellers have been satisfied as is possible. The first stage attempts to match up as many bidders and sellers as possible who have compatible bids. After that both the sellers and the bidders attempt to adjust their respective ask and bid prices and then another round begins. This process continues iteratively until either all actors involved in the auction are satisfied or until all remaining actors have reached their thresholds of how much they are willing to sell for or buy for.

The reason why this particular style of auction was chosen to be investigated was that it matches the real world scenario of having multiple consumers within the nanogrid environment as well as multiple suppliers. It is also reasonable to assume that some kind of memory might be built into the consumers and suppliers so that they might remember what each other offered on previous occasions and submit bids in order to be accepted quicker. The iterative style of the continuous auction was also appealing and realistic due to the nature of having to manage the bids and sales of so many different actors within one given system. Most of the auctions investigated as part of this project required the central controller having access to all the private information of all the other nodes. This, among other reasons, led to auctions not being implemented for this project and this will be discussed in further detail later.



## Chapter 5

# Game Theory

### 5.1 Overview

The field of game theory has been one that has many different facets and versions depending on the type of situation required. In this section of the report the nomenclature and jargon of game theory will be discussed, as will a short explanation about the decision of selection of the type of game implemented as part of this final year project. First the two primary types of interactions between players in a game will be discussed and after that the two primary types of playing styles will be discussed. However first of all there are certain traits that are universal for any type of game that must first be explained in order to grasp the concept of game theory enough to understand some of the implementation decisions later in this report as well as to grasp the general concept of game theory itself.

In game theory, players within a game compete for a finite resource with the objective of maximising their own utility within the scope of that game. Each player within the game has an associated utility function that is generally the same for all players within that game. The utility function generally results in some scalar value that is trying to reach some max value, whether on an individual or collective level. There is also generally some kind of manager node that helps to conduct the game between all of the players involved. Within any particular game, the players are all trying to maximise their own utility, however in different types of games they may also be conscious of the utilities of all the other players involved and try to react accordingly, whether to further their own goal or to further the goals of the collective group.

A well defined game also has some form of state of equilibrium. This state of equilibrium is when the sum of utilities of all the players within the game reaches a maximum. The central managing node, if there is one, generally decides whether or not this state has been reached. This state is the success state of the game. In a well-designed game the utility function must be designed such that the state of equilibrium not only can be reached but also that reaching that state is appealing to all players within the game.

## 5.2 Non-Cooperative Game Theory

Non-Cooperative games are the simplest types of games to both understand and design. As previously stated, each player is trying to maximise its own utility but the core component of a non-cooperative game is that all of the players are operating purely independently. Each player within the game knows the best strategy to take in order to maximise its own utility. Because each player in a game has the same moves open for them to take and therefore the same strategy that each other player will take to maximise their own respective utilities.

This is where the concept of Nash Equilibrium comes into play. Nash Equilibrium is the state in which there is the least disparity between the best player and the worst player, that is that each player performs the best that it can with the knowledge that all other players are similarly going to try to maximise their own utilities. With this knowledge, each player is then able to pick the strategy that maximises its own utility, taking into consideration that all other players are trying to do the exact same thing and therefore it picks an appropriate strategy. In a well designed game, there should also be no incentive for a player to change their strategy to try to undercut other players. If made correctly, such an action would have an adverse effect on the player in the game. In this case all other players would then be aware that this players strategy had changed and would then react accordingly in order to maximise their own utility and decrease that player's utility.

## 5.3 Cooperative Game Theory

Cooperative game theory shares many similar traits with that of non-cooperative game theory as outlined in the overview section of this part on game theory in this report. However the key aspect of cooperative game theory is that players within the game will form coalitions based on threats and incentives that occur between each other. The key component of cooperative game theory is the analysis of which coalitions are likely to form within any given game and what the projected outcomes are based upon these permutations of coalitions. In this way the study of cooperative games have two main facets. First of all they are concerned with what might cause different groups of players to act together in unison. Secondly they are concerned with the outcomes from the most likely of each of these games that happen when different groups form.

In this project, the nodes involved in the game are all energy suppliers who are each trying to maximise their own profit based on the amount of energy that they are able to sell. The utility functions of the nodes and other such details will be discussed later in the Implementation section of this report. The desired outcome of each player is therefore entirely selfish and because they are all trying to compete for a finite price, they each want to obtain as much of that money as possible. Therefore it does not make sense to design this game in such a way that these players should be able to form coalitions, as any coalition would involve compromising and receiving less money which doesn't make sense in this game. Similarly due to the lack of communication

between the players in the game, they can also never know if other players could change their strategies so are unable to even realise that cooperation is even possible at any given stage.

## 5.4 Cournot and Stackelberg Games

Cournot and Stackelberg games are two manners in which players participate in the game, in other words they constitute the structure of the game as opposed to how players react to one another and strategise within the game. Both of these are relatively easy concepts to understand so this section should be quite short. Because these different structures of games effect the way in which a player interacts with the other players in the game, different strategies can be better or worse based on whether the game is a Cournot game or a Stackelberg game and in some cases some strategies may not even be possible within different game structures.

A Cournot game is simply where all the players make their moves at the same time. For example, all players may submit their moves separately to a central manager node who then reveals all of the different moves at the same time and tries to work out and resolve all the different collisions and determine what exactly the outcome of the game was on that particular turn. In a Cournot game, the players all have to predict what the most likely turn of all the other players are and react accordingly for every round of the game.

A Stackelberg game is where there is a leader within the game who plays first, attempting to maximise its own utility first and then all other players in the game play in turn after that and are able to see the moves of all other players before them. Obviously in this kind of a game, where players are competing over a finite resource, whoever plays first has an immediate advantage over the over players in the game. This trickles down through the game, so that while any given player has a disadvantage compared the whoever had the preceding turn, they have a distinct advantage over all players who come afterwards.

The reasoning behind choosing a Stackelberg game over a Cournot game for this project will be discussed later in the Implementation section of this report.

## Chapter 6

# Optimisation Techniques

### 6.1 Overview

Optimisation techniques are an important part of the field of mathematics and are reasonably simple to understand, but can be extremely difficult to formulate. Optimisation problems concern themselves with a key problem that is relevant to many different fields of engineering and computer programming.

For a function  $f: A \rightarrow \mathbb{R}^n$  for a particular set  $A$ , an optimisation problem is concerned with finding an element  $x_o$  of  $A$  where  $f(x_o) \leq f(x)$  for a minimisation problem or  $f(x_o) \geq f(x)$  for a maximisation problem,  $\forall x \in A$ . These optimisation problems manifest themselves in countless fields from economics [8], civil engineering [9] and of course as part of the smart grid [10]. The optimisation techniques involved in this particular project are used on each of the two utility functions involved in the process namely that of each of the game players and then the moderator actor process involved in the system. This will of course be discussed in more detail later on.

One of the main benefits of an optimisation technique is that it is often obtainable using linear algebraic methods which means that a computer can figure out the solution to the optimisation problem in polynomial time. Another benefit of this is that an optimisation technique can be used in tandem with any other problem solving technique in order to find a better solution much faster. If any problem fits the parameters of the optimisation as defined above then different optimisation techniques can be applied or at least the same one in multiple places.

While the basic premise and motivation behind every optimisation technique is the same, different types of sets of values can be used for the set  $A$  and as a result. Fortunately, different types of optimisation techniques have been developed in order to more efficiently solve problems in each of these areas. In some cases, the type of values in the set such as in a convex set, actually make other optimisation methods useless. In this project, two main optimisation methods were used, namely Convex Optimisation and Hyperplane Projection

Optimisation. Both techniques are involved with quickly and accurately solving for a maximum in the case of two different utility functions but operate with different types of sets, each one being suitable for the relevant type of problem.

## 6.2 Convex Optimisation

Convex optimisation is defined as the solving of minimisation problems that involve convex functions being applied to convex sets [11]. Due to the nature of the convexity of the sets involved in these sorts of problems, a term that I will discuss momentarily, the local minimum that is discovered is actually a global minimum. Basically this means that the curve of the graphed outputs from mapping the values of a convex set through a convex function, only has a single minimum as opposed to a situation where the curve could have multiple minimums or values that can be converged on which are not the true minimum of the curve. This property of a convex optimisation problem as well as the property of general optimisation problems of being able to solve the problem in polynomial time means that the true solution can be discovered relatively quickly.

A convex set is simply a region in which, if you draw a line between any two arbitrary points in the region, then all points on the line are also inside the region as outlined in the left side of Fig 6.1. The right side shows a non-convex set where there is a hollow section to the region.

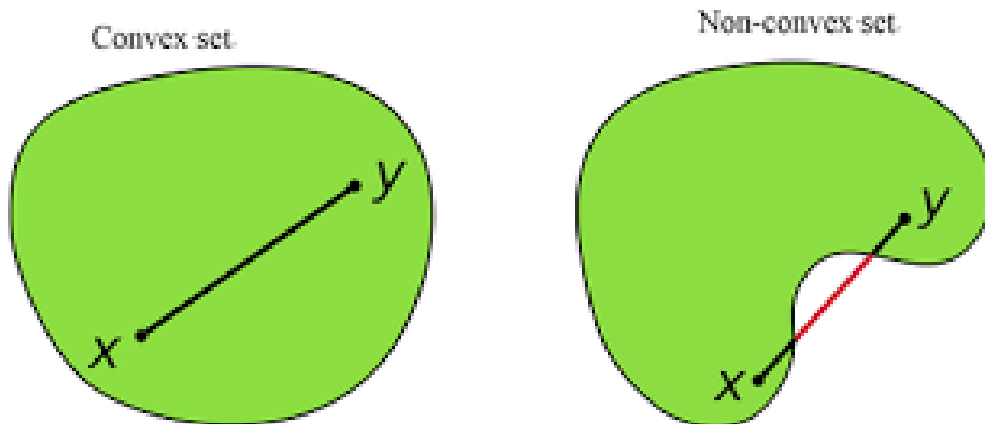


Figure 6.1: A convex set (gtMath March 2016) [12]

A convex function on the other hand is simply a function where the entire line segment between any two points on the graph is above the or on the graph. This is the part of convex optimisation that determines the fact that the local minimum is a global minimum. Convex functions are extremely common in the field of mathematics such as the quadratic function  $x^2$  and the exponential function  $e^x$ .

Convex optimisation is therefore a relatively simple concept to understand and is clearly seen to be a very useful and efficient method of accurately and quickly finding solutions to minimisation problems.

## 6.3 Hyperplane Projection

### 6.3.1 Variational Inequality Problem

The hyperplane projection method is a tool for solving problems that suit the criteria of a variational inequality problem so first that must be explained before moving onto the concept of the solution to such a problem.

A variational inequality is an inequality that involves a functional that must be solved for all variables in a set, usually a convex set. As a side note, although this problem also involves a convex set like the convex optimisation problem, the functional is not a convex function and therefore convex optimisation does not apply in this instance. A functional is a function that maps a vector space onto its underlying field of scalars. Often this vector space can be a series of functions, meaning that the functional takes a function as an argument and can be interpreted as a function of functions. This is similar to the Haskell idea of higher order functions, where a single higher order function can be used to operate on multiple functions and perhaps capture some other important piece of data for a given system.

The origin of, and primary application of, variational inequality problems is in the field of finding solutions of equilibrium in a given system. As we'll see later on in the implementation section of this report, finding the state of Nash Equilibrium between the different suppliers that take part in the game requires a state of equilibrium. Therefore it can be easily inferred that the variational inequality problem is applicable and the problem can be solved as such using a method appropriate for such a problem.

The hyperplane projection method defined here also stipulates that the underlying functional involved in the problem must meet a certain monotonicity criteria. Monotonicity is a property of a function that says that the function must either be non-decreasing or non-increasing. The function does not have to be constantly increasing or decreasing but for example if it is increasing then it cannot decrease or vice versa in order to be deemed monotonic. This can be represented mathematically as  $f(x) \leq f(y) \forall x \leq y$  or  $f(x) \geq f(y) \forall x \leq y$ . Functions that cleave to this mould are called monotonically increasing and monotonically decreasing respectively.

### 6.3.2 Hyperplane Projection Method

Having covered a number of the prerequisites for using a hyperplane projection method, the method itself can be explained. The version I looked at was developed by Solodov and Svaiter and is called the Solodov and Svaiter Hyperplane Projection Method (SSHPPM) [13]. Figure 6.2 will be referred to as a part of the explanation.

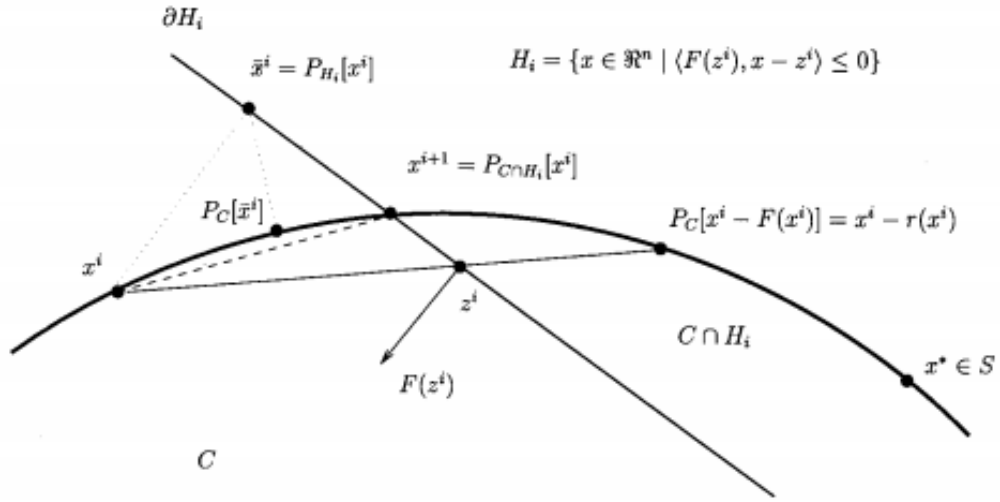


Figure 6.2: Solodov and Svaiter Hyperplane Projection Method

The curve in the figure describes the functional in the variational inequality (VE) problem. This method uses the projection operator  $P_C[x] := \operatorname{argmin}_{y \in C} \|y - x\|$  where  $y \in C$ . Suppose we have a point  $x^i$  which is the current approximation of the solution to the VE problem involving the set  $C$  and the functional  $F$ . First we calculate a projection point  $P_C[x^i - F(x^i)]$ . The segment between  $x^i$  and  $P_C[x^i - F(x^i)]$  is searched for a point  $z^i$ , using a linesearch method like the Armijo linesearch method [14], such that a hyperplane  $\delta H_i$  (using the definition of  $H_i$  as defined in figure 6.2) strictly separates  $x^i$  from any solution  $x^*$  of the problem. The next approximation to the solution  $x^{i+1}$  is calculated by projecting  $x^i$  onto the intersection of the set  $C$  and the halfspace  $H^i$  that contains the solution set using  $P_{C \cap H_i}$ .

The benefit of this solution is that each iteration of the method only requires two projections which makes it computationally efficient, the first to calculate the hyperplane  $H_i$  and another onto the intersection  $C \cap H^i$  to find the next iterate in finding the solution. Later on in the Implementation section, the application of this method will be discussed in further detail.

## **Part III**

# **Implementation**



## **Chapter 7**

# **Design**

## **Chapter 8**

# **Framework**

## **Chapter 9**

# **Processes**

## **Part IV**

# **Conclusion**

## **Chapter 10**

## **Assessment**

## **Chapter 11**

# **Future Work and Continuations**

# Bibliography

- [1] A.-H. Mohsenian-Rad and A. Leon-Garcia, "Optimal residential load control with price prediction in real-time electricity pricing environments," *IEEE transactions on Smart Grid*, vol. 1, no. 2, pp. 120–133, 2010.
- [2] T. Harbert. (Feb. 18, 2015). Post-fukushima, japanese companies build microgrids, [Online]. Available: <http://spectrum.ieee.org/energy/the-smarter-grid/postfukushima-japanese-companies-build-microgrids>.
- [3] G. Sapienza, G. Bianco, G. Di Lembo, L. D. Carpinì, G. Scrosati, and P. Paulon, "The enel smart grid test system: A real-time digital simulator-based infrastructure," 2013.
- [4] T. Couture and Y. Gagnon, "An analysis of feed-in tariff remuneration models: Implications for renewable energy investment," *Energy policy*, vol. 38, no. 2, pp. 955–965, 2010.
- [5] S.-L. Nicola. (Feb. 20, 2017). Feed-in tariff (eeg feed-in tariff), [Online]. Available: <http://www.res-legal.eu/search-by-country/germany/single/s/res-e/t/promotion/aid/feed-in-tariff-eeg-feed-in-tariff/lastp/135/>.
- [6] S. Parsons, J. A. Rodriguez-Aguilar, and M. Klein, "Auctions and bidding: A guide for computer scientists," *ACM Computing Surveys (CSUR)*, vol. 43, no. 2, p. 10, 2011.
- [7] B. Ramachandran, S. K. Srivastava, C. S. Edrington, and D. A. Cartes, "An intelligent auction scheme for smart grid market using a hybrid immune algorithm," *IEEE Transactions on Industrial Electronics*, vol. 58, no. 10, pp. 4603–4612, 2011.
- [8] A. K. Dixit, *Optimization in economic theory*. Oxford University Press on Demand, 1990.
- [9] S. M. Pirayonesi and M. Tavakolan, "A mathematical programming model for solving cost-safety optimization (cso) problems in the maintenance of structures," *KSCE Journal of Civil Engineering*, pp. 1–10, 2017.
- [10] M. Ahat, S. B. Amor, M. Bui, A. Bui, G. Guérard, and C. Petermann, "Smart grid and optimization," 2013.

- [11] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.
- [12] G. Troderman. (Mar. 5, 2016). Convexity and jensen's inequality, [Online]. Available: <http://www.gtmath.com/2016/03/convexity-and-jensens-inequality.html>.
- [13] M. V. Solodov and B. F. Svaiter, "A new projection method for variational inequality problems," *SIAM Journal on Control and Optimization*, vol. 37, no. 3, pp. 765–776, 1999.
- [14] L. Armijo, "Minimization of functions having lipschitz continuous first partial derivatives," *Pacific Journal of mathematics*, vol. 16, no. 1, pp. 1–3, 1966.