Graph Clustering and Community Detection

Working with distance graphs

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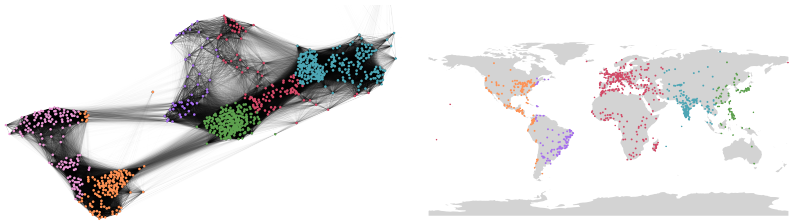


Figure 1: World cities distance graph clustered using the EM wERMG algorithm. (Q = 6)

Abstract

In this report we are interested in extending *graph clustering* methods designed in [2] with a probabilistic graphical model, to weighted graphs. This will allow us to define a graphical model for clustering on manifold-like graphs, at the cost of a bit of complexity and an added free parameter.

Matthieu designed the wERMG model, derived the proofs for its properties, and wrote the report.

Martin implemented the model, wrote the EM algorithm for ERMG and wERMG, and prepared the poster.

Introduction

The problem of graph clustering, also known as community detection, concerns partitions of the vertices of a graph into groups (clusters or communities) such that vertices within the same group are more densely connected to each other than to vertices in other groups. This fundamental problem arises across diverse domains: identifying functional modules in biological networks, detecting communities in social networks, discovering related documents in information retrieval systems, and analyzing interaction patterns in complex systems.

Traditional approaches to graph clustering often rely on heuristic methods or spectral techniques. However, these approaches typically lack a probabilistic foundation that would enable principled statistical inference, model selection, and uncertainty quantification. The mixture model framework that we extend below offers a compelling alternative by providing a rigorous statistical foundation for understanding and detecting community structure in networks.

The paper by Daudin, Picard and Robin ([2]) introduces a mixture of probability distributions used to model the classical Erdös-Rényi model of random graphs, allowing for the traditional EM algorithm to do statistical inference on the parameters of the graph, as well as a better model for networks which appear most importantly in biology. In this report, we suggest a possible method of providing the model to possibly take into account less well defined connections and especially weighted graphs.

We refer to [6] for a more detailed overview on graph clustering.

Related Work

This work was only based on[1] [2], although other people have worked on the matter. One could for example cite [3] which present a gaussian mixture model based on the extension of [2]'s Bernoulli by modifying those to use beta laws, [5] which use laws in the exponential family to model weight distributions on counts of interactions. Although we will in the end present a similar solution to simplify our inference procedure, it is important to note that our model and algorithm is more general. In a similar fashion, [4] propose a gaussian mixture model for the weights and apply the EM to optimize on the parameters.

1 Mixture models for clustering of weighted graphs

The base random graph model we work on is the following: Given a set V of nodes (usually $V = \{1, \dots, n\}$), we generate, at random, edges between nodes based on a probability distribution to mimic clustering behaviours that might be found in real life. The goal is to find a parametric law for edge generation which allows us to correctly generate clusters.

The *Erdös-Rényi Mixture for Graphs* model, as defined in [2], is a probabilistic graphical model which proposes distributions of edges in a graph depending on the number of clusters it exhibits, and how *distinct* those clusters are. Their goal is to find a law for the *between group connectivity* and *aggregation factor* (how much is the graph composed of clusters), which reproduces the distribution of degrees in biological data. In the following paragraphs, we will (re-)present the notations introduced in [2], and modify those to

*Both authors contributed equally. All figures are our own.

[1] Our goal was to expand [2] by making our own model, to consider a specific class of problems. As such, we decided to not read any literature on the subject before completing the mathematical form of the model.

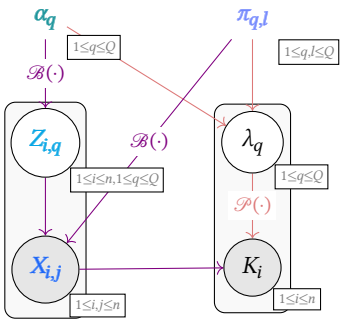


Figure 2: Graphical model for ERMG

extend their usage to *weighted* graphs in our *wERMG* model, trying to account for a better representation.

1.1 ERMG model

The ERMG model admits the graphical model representation in Figure 2 and is such that the diagram formed by violet and pink paths is commutative.

In the model by [2]:

- $Z_{i,q} \sim \mathscr{B}(\alpha_q)$ is the probability that node i is a member of cluster q;
- $X_{i,j} \sim \sum_{q,l} Z_{i,q} Z_{j,l} \mathscr{B}\left(\pi_{q,l}\right)$ is the probability that nodes i and j are connected with an edge;
- $\lambda_q = (n-1)\sum_l \alpha_l \pi_{q,l}$ is the right parameter for approximation in degree definitions;
- K_i is the degree of node i;

The edge in the graph model between K_i and λ_q comes from the Poisson approximation of the binomial law:

$$K_i \sim \mathscr{B}\left(n-1, \sum_l \alpha_l \pi_{q,l}\right) \xrightarrow[n\to\infty]{\text{law}} \mathscr{P}\left(\lambda_q\right)$$

The probability distribution for K_i is not actually defined in that sense, but is actually defined through the $X_{i,j}$, with the parameters for degrees observed being the one that need fit for actual data. This fact will be the corner stone of our extension to weighted graphs, but first, let us remind the version of the EM algorithm used for optimization of the parameters α_q and $\pi_{q,l}$ based on observed data $X_{i,j} = \mathscr{X}$:

E Step. First, we approximate that the joint distribution of the $Z_{i,q}$ is the product of conditional distributions given the other coordinates. Let $\mathscr{Z}_i = \{Z_{l,q}\}_{1\leq q\leq Q}$ and $\mathscr{Z}^i = \mathscr{Z} \setminus \mathscr{Z}_i$ where $\mathscr{Z} = \{Z_{i,q}\}_{i,q}$.

$$\mathbb{P}\left(Z_{i,q} \mid \mathscr{X}\right) = \prod_i \mathbb{P}\left(\mathscr{Z}_i \mid \mathscr{X}, \mathscr{Z}^i\right).$$

We then iterate until convergence to define $\mathbb{P}(Z_{i,q})$ over the fact that:

$$\begin{aligned}\widehat{\tau_{i,q}} &= \mathbb{P}\left(Z_{i,q} = 1 \,\middle|\, \mathscr{Z}^i\right) \\ &\propto \alpha_q \prod_m b\left(\sum_k Z_{k,m} X_{i,k}; \sum_{j\neq i} Z_{j,m}, \pi_{q,m}\right)\end{aligned}$$

where $b(C, N, \pi) = \pi^C (1-\pi)^{N-C}$ is the Bernoulli likelihood, a continuous version of the binomial law. This is actually an approximation, which holds when n goes to infinity.

M Step. We modify the values of the parameters as follows to maximize the complete-data log-likelihood:

$$\widehat{\alpha_q} = \sum_i \frac{\widehat{\tau_{i,q}}}{n} \tag{M-α}$$

$$\widehat{\pi_{q,l}} = \frac{\sum_i \sum_j \widehat{\tau_{i,q}} \widehat{\tau_{j,l}} X_{i,j}}{\sum_i \sum_j \widehat{\tau_{i,q}} \widehat{\tau_{j,l}}} \tag{M-π}$$

1.2 wERMG model

The main issue with the above model is that, in real life, graphs edges often model continuous similarities/proximities instead of purely binary settings. For example a network of all social media posts with edges weighted by the content similarity, or a transportation network with edges weighted by the time it takes to go from a point to another using a single mode of transportation. In turn, this means that the distribution chosen for weights of edges will actually be meaningful only when considering a setting defining the edges and must be adapted to the edges. In the definition of the model, we choose that two points whose similarity or proximity is 0 are not connected in the graph, even when considering edge weights as a measure of distance between nodes. This is done without loss of generality up to post-composition on the weights by a bijection on $\mathbb{R}$ which sends 0 to infinity, and we can always consider weights as similarities instead of distances by the same reasoning.

In the *weighted Erdös-Rényi Mixture for Graphs* model that we propose here, we simply modify the definition of the usage of α_q and $\pi_{q,l}$:

- $Z_{i,q}$, the *clustering probability* of node i being a member of cluster q still follows a Bernoulli law of parameter α_q;
- $X_{i,j} \sim \sum_{q,l} Z_{i,q} Z_{j,l} \mathscr{L}(\pi_{q,l})$ where $\mathscr{L}(\pi_{q,l})$ is any probability distribution such that the collection of $X_{i,j}$s verifies *Lyapunov's condition*[2] is the *weight label of edge* (i,j);
- K_i becomes $\sum_j X_{i,j}$ the *weighted degree* of node i.

Note that again, we should have $\sum_q \alpha_q = 1$.

Proposition 1.1. *The law of K_i converges to a normal law when n goes to infinity.*

Proof. We have the following conditional distribution for K_i:

$$\begin{aligned}(K_i \,|\, i \in Q) &= \sum_j \left(X_{i,j} \,|\, i \in q\right) \\ &= \sum_j \sum_l \left(X_{i,j} \,|\, i \in q \wedge j \in l\right) Z_{j,l} \\ &\sim \sum_j \sum_l \alpha_l \mathscr{L}\left(\pi_{q,l}\right) \\ &= \sum_l n \alpha_l \mathscr{L}\left(\pi_{q,l}\right)\end{aligned}$$

Here, we will make the same approximation as for the ERMG that the $X_{i,j}$ actually are independent when the number of vertices

[2] See [1] for an overview of the condition (and a presentation of ERMG's Poisson approximation in this setting).

goes to infinity (see the next paragraph for more details). Defining the *empirical standard deviation of the model*:

$$\sigma_n = \sqrt{\sum_j \sum_l \mathbb{V}\left[\alpha_l \mathscr{L}\left(\pi_{q,l}\right)\right]} \overset{\text{def}}{=} \sigma_q \sqrt{n},$$

The above assumption is the only one we need to make, as it implies the correct definition of σ_n and σ_q without having to consider covariances. The Lyapunov central limit theorem then gives :

$$\begin{aligned}
&\frac{1}{\sigma_n} \sum_j \sum_l \left(\alpha \mathscr{L}(\pi) - \mathbb{E}\left[\alpha \mathscr{L}(\pi)\right]\right) \\
=&\frac{1}{\sigma_n} \sum_j \underbrace{\left(\sum_l \alpha \mathscr{L}(\pi) - \underbrace{\sum_l \mathbb{E}\left[\alpha \mathscr{L}(\pi)\right]}_{\overset{\text{def}}{=} \mu_q} \right)}_{\text{independent of } j} \\
=&\frac{1}{\sigma_n} n \left(\sum_l \alpha \mathscr{L}(\pi) - \mu_q \right) \\
&\xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0,1)
\end{aligned}$$

where $n\mu_q$ is the *empirical mean of the model* and we have used the shorthands $\pi = \pi_{q,l}$ and $\alpha = \alpha_l$.

$$\frac{\sqrt{n}}{\sigma_q}\left(K_i - n\mu_q\right) \xrightarrow[n\to+\infty]{\text{law}} \mathcal{N}(0,1)$$

which, equivalently can be rewritten as:

$$K_i \sim \mathcal{N}\left(n\mu_q, \sqrt{n}\sigma_q\right) = \mathcal{N}\left(\lambda_q\right)$$

□

This, again, should be considered as exactly true only when n is large enough for the independence of the $X_{i,j}$ to be true. Moreover, the actual parameters of the proof provide us with a way to get the result of ERMG by replacing them with parameters that arise from $\mathscr{L} = \mathscr{B}$.

Main differences with ERMG. The main difference that one should consider when comparing our wERMG model to [2]'s ERMG, is that there is no reason that $\pi_{q,l} = \pi_{l,q}$. Indeed, the graph can now be freely oriented, for example considering shortest routes from a town to another which might differ based on the train schedules.

However we still should have a condition on the probability distributions of edges between clusters:

$$\begin{aligned}
\forall q, l, \mathbb{P}\left(\mathscr{L}\left(\pi_{q,l}\right) = 0 \,\middle|\, \mathscr{L}\left(\pi_{l,q}\right) \neq 0\right) = 0 \\
\text{or equivalently } \mathbb{P}\left(\mathscr{L}\left(\pi_{q,l}\right) \neq 0 \,\middle|\, \mathscr{L}\left(\pi_{l,q}\right) \neq 0\right) = 1
\end{aligned}$$

This is saying that (almost surely), there are no one-way potential barriers, and there is (almost surely) always a way to go from i to j if there is a way to go from j to i. Really this assumption is purely an assumption to improve the realism of the model in that case, and is not needed for the model to make sense mathematically[3].

[3]Even worse, it actually adds the need for the approximation of independence on the degrees that would not be needed in a purely directed setting.

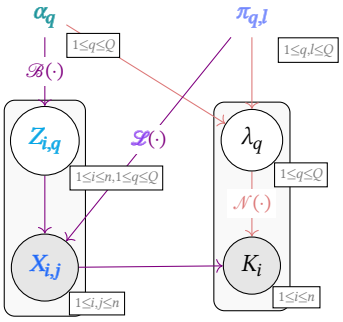


Figure 3: Graphical Model for wERMG

The other main difference with ERMG comes with the notion of degree and clusters. In ERMG, the clustering coefficient of i is defined as

$$C_i = \mathbb{E}_{j,k}\left[\mathbb{P}\left(X_{j,k} = 1 \,\middle|\, X_{i,j}X_{i,k} = 1\right)\right]$$

where $j \neq k$ are taken uniformly among the set of vertices.

However, in the above example considering that our graph is modelling travel times, most vertices will be connected but with much higher[4] values for vertices (stations) in different clusters (towns). See Figure 4 for a visual comparison of the two models in this regard.

This means that clustering coefficients and cluster interconnectivity cannot be defined in the same way as for ERMG: lower[5] values are present when clusters are tighter, but even if all points in q and all points in l are connected, they might be with very large distances and thus q and l might be different. To account for this, we suggest the following definition for the clustering coefficient:

$$C_i = \mathbb{E}_{w,j,k}\left[\mathbb{P}\left(X_{j,k} \le w \,\middle|\, X_{i,j} \le w \wedge X_{i,k} \le w\right)\right]$$

where $j \neq k$ are taken uniformly among the set of vertices and w takes values along $\sum \alpha_l \mathscr{L}(\pi_{q,l})$ and has to indicate a connection. Note that we get back ERMG's definition when $X_{j,k}$ follows a Bernoulli law. This definition only makes sense when considering the $\mathscr{L}(\pi_{q,l})$ can be taken on the same physical scale (or compared to a common physical scale), e.g. when we have travel times or cosine similarities in $[0,1]$.

On the graphical side we have the representation in Figure 3, where again, the diagram formed by violet and pink paths is commutative:

$$\begin{array}{ccc}
\alpha_q \otimes \pi_{q,l} & \xrightarrow{n\mu_q(\cdot)\otimes\sqrt{n}\sigma_q(\cdot)} & \lambda_q \\
\downarrow{\scriptstyle \pi_1(\cdot)\times\mathscr{L}(\pi_2(\cdot))} & & \downarrow{\scriptstyle \mathcal{N}(\cdot)} \\
X_{i,j} & \xrightarrow[\sum_j \cdot]{} & K_i
\end{array}$$

with $\otimes$ the cartesian product and p_1, p_2 the associated projections[6] and μ_q and σ_q as defined in the proof of Theorem 1.1.

We now have the following version of the EM algorithm, adapted for probability distributions:

[4]We can always say, without loss of generality, that $X_{i,j} = 0$ means that direct travel is impossible when $i \neq j$ and thus distance is infinite.

[5]Again, without loss of generality, one could consider higher edge weights as *tighter* connections.

[6]Usually denoted with π, which would be more confusing than anything here.

E Step. First, we make the same approximation as in ERMG that the joint distribution of the $Z_{i,q}$ is the product of conditional distributions given the other coordinates:

$$\mathbb{P}\left(Z_{i,q} \mid \mathcal{X}\right) = \prod_i \mathbb{P}\left(Z_i \mid \mathcal{X}, \mathcal{Z}^i\right).$$

We then iterate until convergence to define $\mathbb{P}(Z_{i,q})$ over the following fact:

Proposition 1.2. *The posterior probabilities τ satisfies the following fixed point relation:*

$$\begin{aligned}\widehat{\tau_{i,q}} &= \mathbb{P}\left(Z_{i,q} = 1 \,\middle|\, \mathcal{X}, \widehat{\mathcal{Z}^i}\right) \\ &\propto \alpha_q \prod_{j\neq i}\prod_l \left[\mathscr{L}\left(\pi_{q,l}\right)\left(X_{i,j}\right)\right]^{Z_{j,l}} \qquad \text{(wE-}\tau\text{)}\end{aligned}$$

We actually need to approximate the variables in $\mathcal{Z}^i$ as equal to their conditional expectations. Unlike for ERMG, there is no way to simplify more the result of this computation until we actually define $\mathscr{L}$.

Proof. To better understand the formula in (wE-τ), see that $x^{Z_{j,l}}$ is equal to 1 (the multiplicative unit) if $j \notin l$ and x if $j \in l$. As such, the product in Equation wE-τ should be understood as the product for all vertices distinct from i of the probability that the edge $X_{i,j}$ as the value it has knowing that j is in cluster l. What this means is that the posterior probability of i being in cluster q is equal (up to renormalization) to the product of the posterior probabilities that i is in q, and that $X_{i,j}$ actually has the observed value knowing that j is in l. To better understand the formula in Equation (wE-τ), see that $x^{Z_{j,l}}$ is equal to 1 (the multiplicative unit) if $j \notin l$ and x if $j \in l$. As such, the product in Equation (wE-τ) should be understood as the product for all vertices distinct from i of the probability that the edge $X_{i,j}$ as the value it has knowing that j is in cluster l. What this means is that the posterior probability of i being in cluster q is equal (up to renormalization) to the product of the posterior probabilities that i is in q, and that $X_{i,j}$ actually has the observed value knowing that j is in l. This is of course true for the posterior explaining why we have a fixed-point relationship. □

M Step. The complete data log-likelihood is written here as

$$\begin{aligned}Q(\mathcal{X}) &= \mathbb{E}[\log p(\mathcal{X}, \mathcal{Z}) \mid \mathcal{X}] \\ &= \sum_i \sum_q \tau_{i,q} \log \alpha_q \\ &\quad + \sum_{i<j}\sum_{q,l} \theta_{iqjl} \log(\mathscr{L}(\pi_{q,l})(X_{i,j})),\end{aligned}$$

where $\theta_{iqjl} = \tau_{i,q}\tau_{j,l}$. We can then derive the M step for the EM algorithm:

Proposition 1.3. *The following updates to the values of the parameters maximize $Q(\mathcal{X})$ subject to the normalization constraint $\sum \alpha = 1$:*

$$\widehat{\alpha_q} = \sum_i \frac{\widehat{\tau_{i,q}}}{n} \qquad \text{(wM-}\alpha\text{)}$$

$$\widehat{\pi_{q,l}} = \operatorname*{argmax}_{\pi} \sum_{i<j} \widehat{\theta_{iqjl}} \log \mathscr{L}(\pi)(X_{i,j}) \qquad \text{(wM-}\beta\text{)}$$

Proof. The proof is done by seeing that each of the pairs of sums above can be exchanged and that each of the returned sums depend only on one of the α_q or $\pi_{q,l}$. Maximizing Q then amounts to maximizing each of the individual sums, thus giving the result presented above. □

Again, we cannot go further in the computation for the updated value of π without defining $\mathscr{L}$ with a mathematical expression, which is why we suggest that the following proposition gives a good starting point for the models.

Proposition 1.4. *If $(\mathscr{L}(\pi))_\pi$ is an exponential family such that*

$$\mathscr{L}(\pi)(x) = h(x)\exp\left(\eta(\pi)T(x) - A(\eta(\pi))\right),$$

where A is the normalization function in the true parameter $\eta(\pi)$, h is a normalization function in x and $T(x)$ is the actual data transformation, then, necessarily, the maximization condition (when deriving) leads to

$$A'(\eta(\pi_{q,l})) = \frac{\sum_{i,j} \widehat{\theta_{iqjl}} T(X_{i,j})}{\sum_{i,j} \widehat{\theta_{iqjl}}}. \qquad \text{(}\pi\text{-exp-update)}$$

This shape of the update state can more easily be solved when $A' \circ \eta$ is a bijection.

The exponential family of probability distributions functions chosen as examples for $\mathscr{L}$ might seem arbitrary, but it actually encompasses everything we need to prove the result for Bernoulli[7], Poisson[8], Exponential[9] and Gaussian[10] distributions, showing *en passant* the coherence of our results and model with ERMG.

Proof. To prove statement (π-exp-update), start from the definition of Q in our setting (and in particular only the terms of Q depending on $\pi_{q,l}$ as in Theorem 1.3)

$$\begin{aligned}Q_{q,l}(\eta) &= \sum_{i,j} \theta_{iqjl} \log \mathscr{L}(\eta)\left(X_{i,j}\right) \\ &= \sum_{i,j} \theta_{iqjl}\left(\log h\left(X_{i,j}\right) + \eta T\left(X_{i,j}\right) - A(\eta)\right) \\ &= \underbrace{\sum_{i,j} \theta_{iqjl} \log h\left(X_{i,j}\right)}_{\text{constant in } \eta} \\ &\qquad + \eta \sum_{i,j} \theta_{iqjl} T\left(X_{i,j}\right) - W A(\eta)\end{aligned}$$

where $W = \sum \theta_{iqjl}$. Differentiating with respect to η gives

$$\frac{\mathrm{d}}{\mathrm{d}\eta} Q_{q,l}(\eta) = \sum_{i,j} \theta_{iqjl} T\left(X_{i,j}\right) - W A'(\eta),$$

which, given the extremum condition on the vanishing of the derivative rewrites as

$$A'(\eta) \sum_{i,j} \theta_{iqjl} = \sum_{i,j} \theta_{iqjl} T\left(X_{i,j}\right).$$

□

[7] Take $\eta(\pi) = \log \frac{\pi}{1-\pi}$ and $T = \mathrm{id}$.
[8] Take $\eta(\lambda) = \log \lambda$, $h(x) = \frac{1}{x!}$ and $T = \mathrm{id}$.
[9] Take $\eta(\lambda) = -\lambda$, h, $T = \mathrm{id}$
[10] Given variance σ, take $\eta(\mu) = \frac{\mu}{\sigma^2}$, $h(x)$ the pdf for centered normal distributions and $T = \mathrm{id}$.

2 Implementation and evaluation of weighted clustering models

In this section we describe the different experiments we made to test our wERMG model, and especially keeping in mind our goal of applying it to a distance graph.

2.1 General Implementation

We implemented [2]'s method as well as our own method in Python 3.13.19, using the libraries NumPy 2.2.6, NetworkX 3.5, SciPy 1.16.3, and pandas 2.2.3. The code is available on GitHub.

The wERMG EM algorithm was implemented specifically for the exponential distribution. We will explain this in more details in section 2.2, but it came from our belief that the exponential law is a proper model for the edge weights of our real-data experiments, where we built distance graphs on major world cities. This choice allowed us to significantly simplify the general formulation, leading to a more optimized version of the algorithm suited for the large graphs processed.

2.2 Experiments

We began by experimenting on toy examples, randomly generated through some variation of the Erdös-Rényi process. Mostly, we will describe those variations as being ERMG or wERMG graphs, as we used those models to generate random graphs. Here, when talking about wERMG, we will be talking about wERMG with $\mathscr{L}$ designing the exponential law of parameter π.

ERMG-wERMG Model Comparison. In this first experiment, we present a visual comparison of the expressiveness of both ERMG and wERMG. To do so, we present in Figure 4 a random graph generated with the ERMG model, and a graph generated with wERMG, using the same parameters α and π for all clusters and cross-cluster parameters.

Coherence of Lyapunov's central limit theorem. In Figure 5 we present an example of fit of the observed degree distribution when generating a graph through a pre-defined wERMG with $n = 3000$ vertices, three clusters with parameters $\alpha = \begin{bmatrix}0.5 & 0.4 & 0.1\end{bmatrix}$ and the following matrix for the probability parameters of $\mathscr{L}(\pi) = \lambda x.\frac{1}{\pi}e^{-x\pi}$

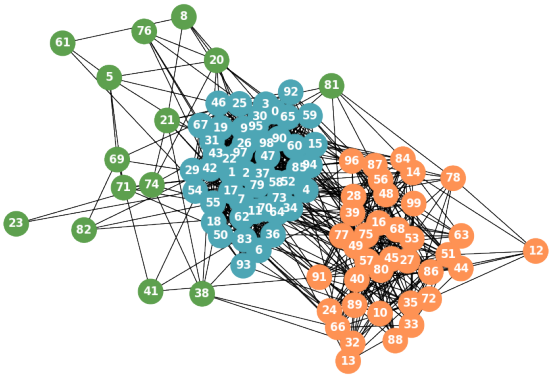
$$\pi_{q,l} = \begin{bmatrix} 0.5 & 0.05 & 0.05 \\ 0.05 & 0.3 & 0.05 \\ 0.05 & 0.05 & 0.2 \end{bmatrix}.$$

EM algorithm reconstruction of clusters. In this section we present an attempt at reconstructing with (w)ERMG a graph that was generated using a (w)ERMG model (again, ERMG is just wERMG with $\mathscr{L}(\pi) = \mathscr{B}(\pi)$). The parameters used were

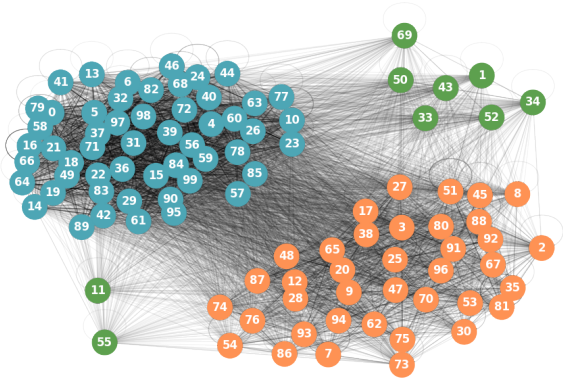
$$n = 200, \quad \alpha = \begin{bmatrix}0.4 & 0.3 & 0.2 & 0.1\end{bmatrix}, \tag{1}$$

$$\pi = \begin{bmatrix} 0.6 & 0.05 & 0.02 & 0.03 \\ 0.05 & 0.5 & 0.04 & 0.01 \\ 0.02 & 0.04 & 0.4 & 0.05 \\ 0.03 & 0.01 & 0.05 & 0.3 \end{bmatrix}. \tag{2}$$

We present two examples of the process in Figure 7. Figure 6b presents the clusters computed by the EM algorithm on the graph of Figure 6a generated by the ERMG model, while Figure 7b similarly



(a) Synthetic ERMG graph example



(b) Synthetic wERMG graph example

Figure 4: Visual comparison of the two mixture models for graphs ERMG and wERMG, using the same defining parameters but using an exponential distribution for the weight model.

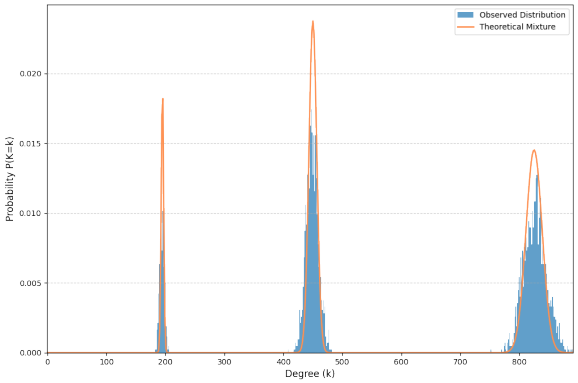
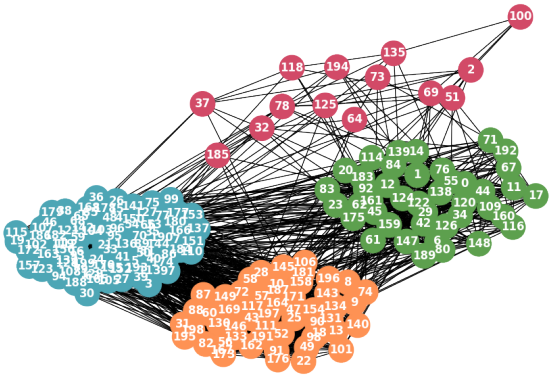
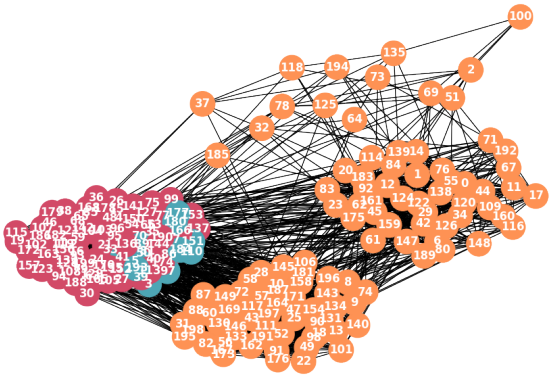


Figure 5: Degree distribution for a synthetic wERMG graph along with its normal approximation

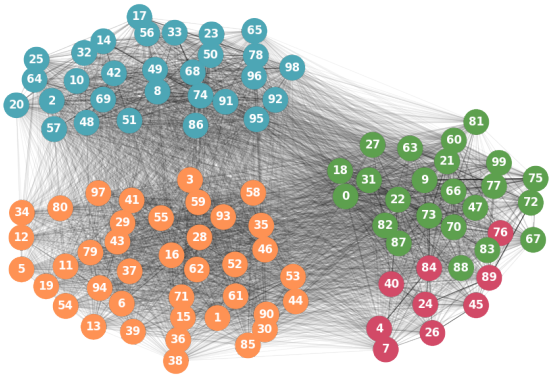


(a) Objective ERMG Graph

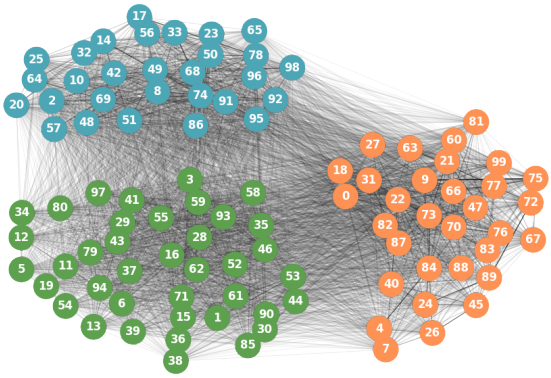


(b) Clustered ERMG graph using the EM algorithm.

Figure 6: Reconstruction of graphs generated by the ERMG models with the parameters in Equation (2), using the EM algorithm with a base of 4 objective clusters.



(a) Objective wERMG Graph



(b) Clustered wERMG graph using the EM algorithm.

Figure 7: Reconstruction of graphs generated by the wERMG models with the parameters in Equation (2), using the EM algorithm with a base of 4 objective clusters.

presents the clusters suggested by EM on the graph of Figure 7a generated by the wERMG model. Notably, we see that some of the clusters will collapse into one another when they are not so well connected and too small. What this means is that, even though the generating process chose certain clusters, in practice, another generating process with less clusters might better model the actual representation. We will see another example of such over-clustering in section 2.2.

We will not go into much details comparing the two algorithms as the two are tailored to different usages: ERMG works for graph where the *connection* is the important information, whereas wERMG necessitates much more information to be well applied.

Clustering world cities. From the dataset in [7], we built distance graphs using the Haversine formula for geodesic distances. The free version of this dataset consists of approximately 48,000 large cities across the world, including their names, countries, longitudes, and latitudes. The distances were then inverted to obtain a weighted graph with weights between 0 and 1, where higher weights correspond to cities that are geographically closer.

Using all 48,000 data points would yield a distance graph with over 2 billion edges. This is impractical and would require excessive RAM. Since our independence approximation for the Lyapunov Central Limit Theorem 1.1 is empirically verified for $n \geq 500$ vertices, we considered a subset of 1,000 cities chosen randomly from the dataset.

After examining the weight distribution of the distance graph, we selected an exponential law for our wERMG model. More precisely, we defined

$$\mathscr{L}(\pi)(x) = \frac{1}{\pi} e^{-\frac{x}{\pi}},$$

basing ourselves on Figure 8, which, at first glance seems to present a mixture of two types of exponential distribution: groups of close distances, and a wider flatter group of larger distances, which justifies why this is the model we chose for $\mathscr{L}$ in wERMG all through the experiments.

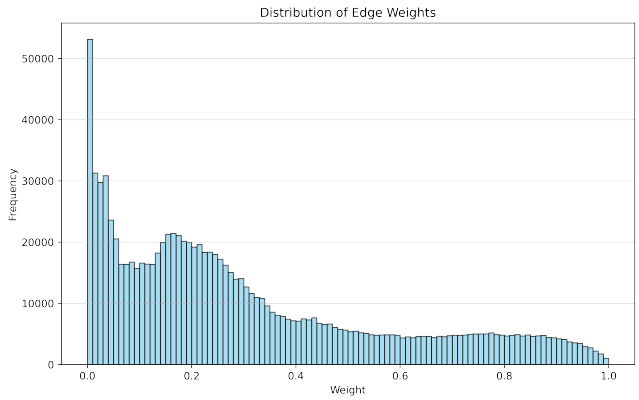


Figure 8: Weight distribution of the real life distance graph

In Figure 10 we will now present what clusters our method computed when considering the distance graph defined above. The code was run for particular values of Q to try and find the smallest number of clusters that does not lose too much information. The degree distribution is indeed better fitted when running the algorithm for $Q = 20$ (as seen in Figure 11b), but the fit with $Q = 6$ is already good[11] (see Figure 11a). We found the following parameters:

$$\alpha = \begin{bmatrix} 0.11 & 0.29 & 0.28 & 0. & 0.18 & 0.13 \end{bmatrix}$$

$$\pi = \begin{bmatrix} 0.62 & 0.02 & 0.17 & 0.00 & 0.51 & 0.07 \\ 0.02 & 0.68 & 0.16 & 0.00 & 0.00 & 0.38 \\ 0.17 & 0.16 & 0.69 & 0.00 & 0.13 & 0.37 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.51 & 0.00 & 0.13 & 0.00 & 0.81 & 0.03 \\ 0.07 & 0.38 & 0.37 & 0.00 & 0.03 & 0.50 \end{bmatrix}$$

It appears from Figure 10 that the algorithm found clusters based on their longitude, even though it was only provided information on the distances between points and not their position in space.

Conclusion

In this report, we generalized a somewhat used probabilistic model to a wide class of random graphs for clustering and generating purposes. This allows us to compute clusters on real life graphs, on a larger class of problems than what [2] proposed, by adding the ability to consider weighted directed graphs, without loss of precision since we can retrieve their model with no added algorithmic complexity.

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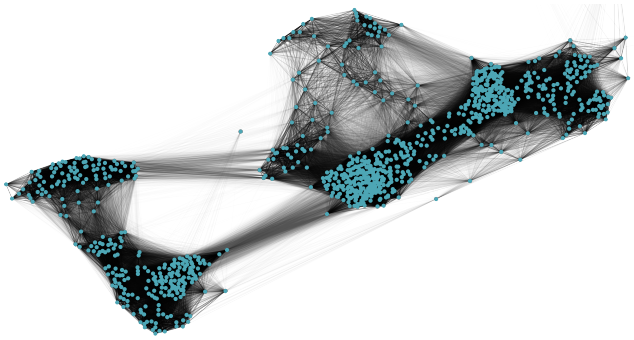
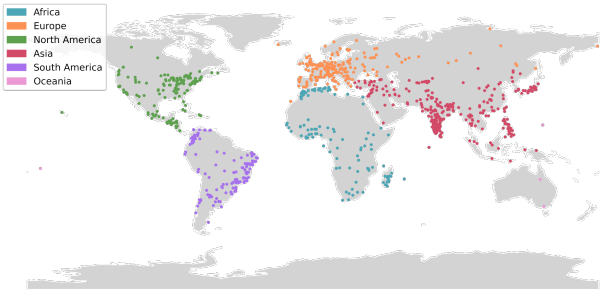
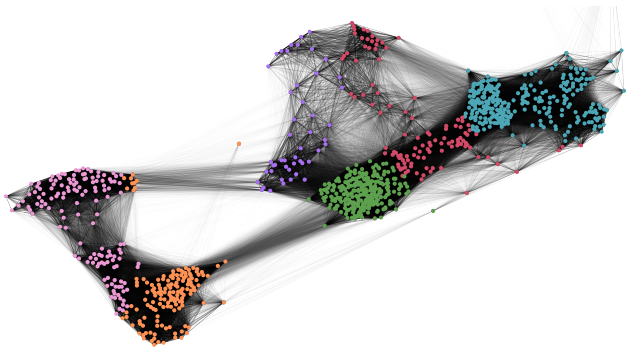
[11] Albeit unstable on repetition.



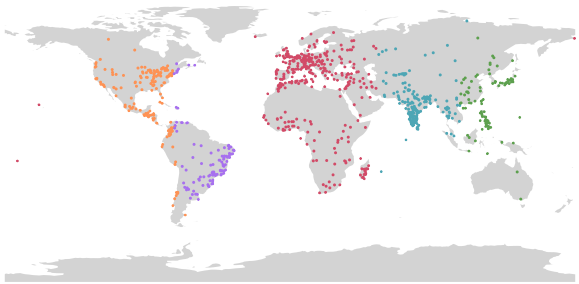
(a) Distance graph of major world cities.



(b) World map of the different world cities and their continents.

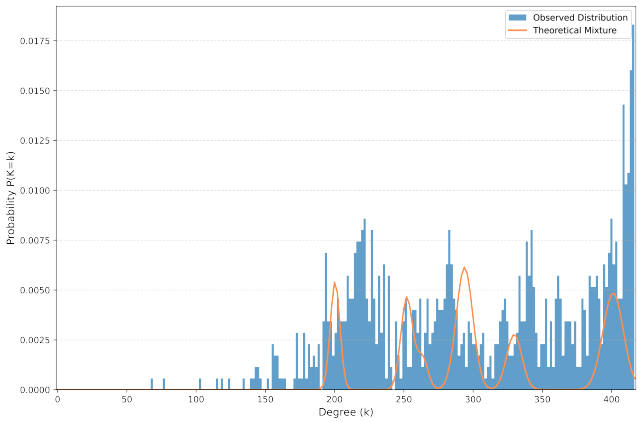


(a) Clustered version of the distance graph of major world cities.

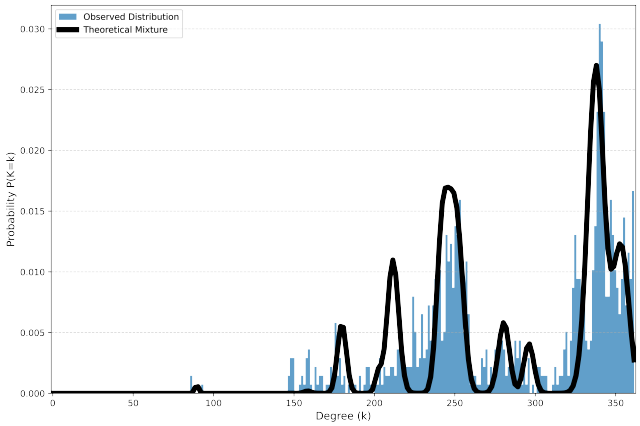


(b) Clusters of the distance graphs on the world map.

Figure 10: Generation of $Q = 6$ clusters for the geodesic distance graph on a perfectly spherical Earth with vertices in the thousand most populated cities on the planet.



(a) Fit of the degree distribution obtained for $Q = 6$ clusters



(b) Fit of the degree distribution obtained for $Q = 20$ clusters

Figure 11: Comparison of the degree distribution fits obtained for $Q = 6$ and $Q = 20$ clusters when running the EM algorithm.

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